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Tensor product of torsion free C(n)-modules of finite degree and finite dimensional modules.

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Tensor Product of Torsion Free C_n-Modules of Finite Degree and Finite Dimensional Modules

by

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Abstract

It is known that every torsion free C_n-module of finite degree is completely reducible. In this thesis, we provide a formula for the decomposition of the tensor product of any simple torsion free C_n-module of finite degree with any simple finite dimensional C_n-module.

Dedication

To my wife Chun Ngan

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Contents

1 Introduction

Let L be a finite dimensional simple Lie algebra over the complex numbers $\mathbb{C}$, and let H be a Cartan subalgebra of L. An L-module V is said to be a weight module if and only if $V = \bigoplus_{\lambda \in H^*} V_\lambda$, where

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in H\}$$

Every simple finite dimensional L-module is a weight module and is completely determined by its highest weight. However, in the case of simple infinite dimensional L-modules, the story is quite different. In fact Lemire [13] showed the existence of simple infinite dimensional modules which are not weight modules and due to this, the classification of arbitrary simple modules seems unreachable. Therefore, the theory has focused on the classification of simple weight L-modules having finite dimensional weight spaces.

Early work by Lemire [12], Lemire and Pap [14], and Britten and Lemire [3], [4] classifies all simple infinite dimensional modules having at least one 1-dimensional weight space.

A big break through in the general problem of classifying all simple L-modules having finite dimensional weight spaces came when Suren Fernando [7] reduced the classification to classifying all simple torsion free modules of finite degree. A weight L-module is torsion free provided all root vectors in L act injectively on V. Such a module has, as its set of weights, a complete integral root lattice coset, and each weight space has the same dimension, called the degree of the module. Fernando goes on to show that the only simple Lie algebras over $\mathbb{C}$ admitting torsion free modules are those of type A and C.

Mathieu [15] classifies and provides a realization of all simple torsion free weight modules of finite degree. However, encouraged by Mathieu, Britten and Lemire continued on to obtain an elementary and explicit realization of simple torsion free modules using the notion of tensor products. Georgia Benkart, in a private communication, showed that the tensor product of any torsion free module of finite degree and a finite dimensional module produces a torsion free module. Motivated by this, and their own result that explicitly constructs all simple torsion free modules of degree 1, Britten and Lemire [5] used the machinery established by Mathieu to prove that every simple torsion free module of finite degree is a submodule of the tensor product of a simple degree 1 torsion free module and a finite dimensional module.

The proof of this result was complicated by the fact that torsion free A_n-modules are not, in general, completely reducible. However, Britten, Khomenko, Lemire, and Marzorchuk [2] established the complete reducibility of torsion free C_n-modules of finite degree. The tragic flaw of their theorem is that they did not give the constituents of the decomposition. In this thesis, we help diminish this gap by giving the actual tensor product decomposition when a simple torsion free C_n-module of finite degree is tensored with a simple finite dimensional module.

We begin by reviewing several known concepts that will be used throughout this thesis. In sections 2 and 3, we provide some general properties of associative algebras and Lie algebras, in particular, we give a more detailed illustration of semisimple Lie algebras. The work in section 3 largely follows Humphreys [8].

Continuing with a review of known concepts, sections 4 and 5 give an overview of the representation theory concepts used in this thesis. In particular, we give several results (most of which are found in [8]), relating to the specific types of modules of interest, those being finite dimensional modules, admissible modules, Verma modules, and torsion free modules. Also in section 5, we introduce the main tools used in obtaining the tensor product decomposition formulas given in sections 8 and 9, in particular, the formal character and the central character.

The crucial result used in obtaining the formula in section 9 is Mathieu's classification of simple torsion free modules of finite degree [15]. In sections 6 and 7, we review Mathieu's construction, giving his results in the setting needed in this thesis.

In section 8, in particular 8.3, we give our first previously unpublished results. Theorem 8.2 gives a formula for the decomposition of the tensor product of any simple admissible highest weight C_n-module and any simple finite dimensional module. Finally, in section 9, we give Theorem 9.1, which is the main result of this thesis. This theorem provides a formula for the decomposition of any simple torsion free

C_n-module of finite degree and any finite dimensional C_n-module.

2 Algebras

In this section, the algebraic structures and basic results that will be studied in this thesis are briefly reviewed. It is assumed that the reader is familiar with vector space theory, linear algebra, and some concepts from group theory.

2.1 Associative Algebras

Definition 2.1. *An* **associative algebra** *A is a vector space over a field $\mathbb{F}$ endowed with an operation $* : A \times A \longrightarrow A$ having the following properties:*

i) There exists $1 \in A$ such that $a = a * 1 = 1 * a$ for all $a \in A$

ii) $(ax + by) * z = a(x * z) + b(y * z), \; x * (ay + bz) = a(x * y) + b(x * z)$

iii) $x * (y * z) = (x * y) * z$

for all $a, b \in \mathbb{F}$, and $x, y, z \in A$. A **multiplicative subset** *of A is a subset $S \subseteq A$ with the property that $x * y \in S$ for all $x, y \in S$. A* **subalgebra** *$B \leq A$ is a sub-vector space of A with the property that $1 \in B$ and $x * y \in B$ for all $x, y \in B$.*

Definition 2.2. *Let A and B be associative algebras, with products $*$ and $\star$ respectively. Let $\varphi : A \longrightarrow B$ be a linear map from A to B with the property that $\varphi(x * y) = \varphi(x) \star \varphi(y)$ for all $x, y \in A$. Then φ is called an* **algebra homomorphism**. *If φ is bijective then φ is called an* **isomorphism**. *In this case, A and B are said to be* **isomorphic**, *denoted $A \cong B$. When φ is bijective, and $A = B$, we call φ an* **automorphism**.

Definition 2.3. *Let A be an associative algebra, and $I \subseteq A$ be a sub-vector space of A. Then I is a* **left ideal** *of A provided $y * x \in I$ for all $x \in I$ and $y \in A$. I is a* **right ideal** *of A provided $x * y \in I$ for all $x \in I$ and $y \in A$. I is a* **two sided ideal** *if I is both a left and right ideal.*

Definition 2.4. *Let A be an associative algebra, and $S \subseteq A$. Let*

$$\mathcal{I} = \{I \subseteq A \mid I \text{ is an ideal of } A \text{ and } S \subseteq I\}$$

Then $\bigcap_{I \in \mathcal{I}} I$ is the **ideal generated by** *S.*

Definition 2.5. *Let A be an associative algebra, and I be a two sided ideal of A. The* **quotient algebra** *is the associative algebra $A/I = \{x + I \mid x \in A\}$ with addition and scalar multiplication given by*

$$a(x + I) + b(y + I) = (ax + by) + I$$

and product

$$(x + I) * (y + I) = x * y + I$$

for any $x, y \in A$ and $a, b \in \mathbb{F}$.

Definition 2.6. *Let A be an associative algebra. Then*

$$Z(A) = \{x \in A \mid xa = ax \text{ for all } a \in A\}$$

is called the **centre** *of A.*

$Z(A)$ is a commutative subalgebra of A.

2.2 Localization of Algebras

Definition 2.7. *Let A be an associative algebra, and $S \subseteq A$ be a multiplicative subset of A. Then S satisfies* **Ore's localizability condition** *provided*

i) $1 \in S$

ii) S does not contain any zero divisors in A

iii) For any $s \in S$ and $a \in A$ there exist $s' \in S$ and $a' \in A$ such that $as' = sa'$

iv) For any $s \in S$ and $a \in A$ there exist $s'' \in S$ and $a'' \in A$ such that $s''a = a''s$

We call conditions *iii)* and *iv)* the right and respectively left Ore conditions. Further, if $S \subseteq A$ is a multiplicative subset of A, then an element $s \in S$ is said to satisfy the right Ore condition provided for any $a \in A$ there exist $s' \in S$ and $a' \in A$ such that $as' = sa'$. Likewise for the left Ore condition. Thus a multiplicative subset containing 1, and not containing any zero divisors, satisfies Ore's localizability condition provided all of its elements satisfy both the left and right Ore conditions.

We now show that if two elements satisfy the left and right Ore conditions, then their product also satisfies the left and right Ore conditions.

Property 2.1. *Let A be an associative algebra, and $S \subseteq A$ be a multiplicative subset of A. Let $s_1, s_2 \in S$ be two elements with the property that for any $a \in A$, there exist $s_1', s_2' \in S$ and $a_1', a_2' \in A$ such that $as_1' = s_1a_1'$ and $as_2' = s_2a_2'$. Then for any $a \in A$ there exist $s' \in S$ and $a' \in A$ such that $as' = s_1s_2a'$. Likewise, if $s_1, s_2 \in S$ have the property that for any $a \in A$, there exist $s_1'', s_2'' \in S$ and $a_1'', a_2'' \in A$ such that $s_1''a = a_1''s_1$ and $s_2''a = a_2''s_2$. Then for any $a \in A$ there exist $s'' \in S$ and $a'' \in A$ such that $s''a = a''s_1s_2$.*

Proof. We prove only the statement about the right Ore condition, since proof for the left Ore condition is similar. Let $a \in A$, and choose $s_1' \in S$ and $a_1' \in A$ such that $as_1' = s_1a_1'$. Next, choose $s_2' \in S$ and $a_2' \in A$ such that $a_1's_2' = s_2a_2'$. Then

$$as_1's_2' = s_1a_1's_2' = s_1s_2a_2'$$

setting $a' = a_2'$ and $s' = s_1's_2'$ gives the desired result. □

In particular, the previous property implies that if $S \subseteq A$ is a set of vectors satisfying the left and right Ore conditions, which doesn't contain any zero divisors, then the multiplicative subset generated by $\{1\} \cup S$ satisfies Ore's localizability condition.

Theorem 2.1. *Let A be an associative algebra, and $S \subseteq A$ be a multiplicative subset of A. If S satisfies Ore's localizability condition, then there exists an associative algebra B with the following properties:*

i) There exists an injective algebra homomorphism $\phi : A \longrightarrow B$.

ii) If $s \in S$ then $\phi(s)$ is invertible in B.

iii) If $b \in B$ then $b = \phi(s)^{-1}\phi(a)$ for some $s \in S$ and $a \in A$.

Proof. (See 3.6.2 and 3.6.4 in [6]) □

The following shows that such an algebra is unique.

Proposition 2.1. *Let A be an associative algebra, and $S \subseteq A$ be a multiplicative subset of A satisfying Ore's localizability condition. If B_1 and B_2 are associative algebras satisfying properties i), ii) and iii) in the above theorem, then $B_1 \cong B_2$.*

Proof. Let $\phi_1 : A \longrightarrow B_1$ and $\phi_2 : A \longrightarrow B_2$ be injective homomorphisms of A into B_1 and B_2 respectively. Define $\psi : B_1 \longrightarrow B_2$ by

$$\psi(\phi_1(s)^{-1}\phi_1(a)) = \phi_2(s)^{-1}\phi_2(a)$$

by property *iii)*, ϕ is surjective. We claim that for $i = 1, 2$, we have $\phi_i(s_1)^{-1}\phi_i(a_1) = \phi_i(s_2)^{-1}\phi_i(a_2)$ if and only if there exist $x \in S$ and $y \in A$ such that $xa_1 = ya_2$ and $xs_1 = ys_2$, and hence that ψ is both well defined, and injective. Indeed, if there exist such and x and y, then

$$\begin{aligned}
\phi_i(s_1)^{-1}\phi_i(a_1) &= \phi_i(s_1)^{-1}\phi_i(x)^{-1}\phi_i(x)\phi_i(a_1)\\
&= \phi_i(xs_1)^{-1}\phi_i(xa_1)\\
&= \phi_i(ys_2)^{-1}\phi_i(ya_2)\\
&= \phi_i(s_2)^{-1}\phi_i(y)^{-1}\phi_i(y)\phi_i(a_2)\\
&= \phi_i(s_2)^{-1}\phi_i(a_2)
\end{aligned}$$

Conversely, if $\phi_i(s_1)^{-1}\phi_i(a_1) = \phi_i(s_2)^{-1}\phi_i(a_2)$, then since S satisfies Ore's localizability condition, we can choose $x \in S$ and $y \in A$ such that $xs_1 = ys_2$. Therefore,

$$\begin{aligned}
\phi_i(xa_1) = \phi_i(x)\phi_i(a_1) &= \phi_i(x)\phi_i(s_1)\phi_i(s_1)^{-1}\phi_i(a_1)\\
&= \phi_i(x)\phi_i(s_1)\phi_i(s_2)^{-1}\phi_i(a_2)\\
&= \phi_i(y)\phi_i(s_2)\phi_i(s_2)^{-1}\phi_i(a_2)\\
&= \phi_i(y)\phi_i(a_2) = \phi_i(ya_2)
\end{aligned}$$

Since ϕ_i is injective, we have $xa_1 = ya_2$, which proves our claim. It remains to show that ψ is linear, and a homomorphism. Let $b_1, b_2 \in B_1$, we will show that $\psi(b_1 + b_2) = \psi(b_1) + \psi(b_2)$, and $\psi(b_1 b_2) = \psi(b_1)\psi(b_2)$. To this end, choose $s_1, s_2 \in S$ and $a_1, a_2 \in A$ such that

$$b_1 = \phi_1(s_1)^{-1}\phi_1(a_1) \qquad \text{and} \qquad b_2 = \phi_1(s_2)^{-1}\phi_1(a_2)$$

Applying Ore's localizability condition, choose $t \in S$ and $c \in A$ such that $ts_1 = cs_2$, set $s = ts_1 = cs_2$. Then,

$$\begin{aligned}
\psi(b_1 + b_2) &= \psi(\phi_1(s_1)^{-1}\phi_1(a_1) + \phi_1(s_2)^{-1}\phi_1(a_2)) \\
&= \psi(\phi_1(s)^{-1}(\phi_1(t)\phi_1(a_1) + \phi_1(c)\phi_1(a_2))) \\
&= \psi(\phi_1(s)^{-1}(\phi_1(ta_1 + ca_2))) \\
&= \phi_2(s)^{-1}(\phi_2(ta_1 + ca_2)) \\
&= \phi_2(s)^{-1}(\phi_2(t)\phi_2(a_1) + \phi_2(c)\phi_2(a_2)) \\
&= \phi_2(s_1)^{-1}\phi_2(a_1) + \phi_2(s_2)^{-1}\phi_2(a_2) \\
&= \psi(b_1) + \psi(b_2)
\end{aligned}$$

Next, choose $u \in S$ and $d \in A$ such that $ua_1 = ds_2$. Therefore

$$\phi_1(a_1)\phi_1(s_2)^{-1} = \phi_1(u)^{-1}\phi_1(d) \qquad \text{and} \qquad \phi_2(a_1)\phi_2(s_2)^{-1} = \phi_2(u)^{-1}\phi_2(d)$$

The following calculation completes the proof:

$$\begin{aligned}
\psi(b_1 b_2) &= \psi(\phi_1(s_1)^{-1}\phi_1(a_1)\phi_1(s_2)^{-1}\phi_1(a_2)) \\
&= \psi(\phi_1(s_1)^{-1}\phi_1(u)^{-1}\phi_1(d)\phi_1(a_2)) \\
&= \psi(\phi_1(us_1)^{-1}\phi_1(da_2)) \\
&= \phi_2(us_1)^{-1}\phi_2(da_2) \\
&= \phi_2(s_1)^{-1}\phi_2(u)^{-1}\phi_2(d)\phi_2(a_2) \\
&= \phi_2(s_1)^{-1}\phi_2(a_1)\phi_2(s_2)^{-1}\phi_2(a_2) \\
&= \psi(\phi_1(s_1)^{-1}\phi_1(a_1))\psi(\phi_1(s_2)^{-1}\phi_1(a_2)) \\
&= \psi(b_1)\psi(b_2)
\end{aligned}$$

□

In practice, when A is embedded in B, we will simply denote the image of an element $a \in A$ by a itself, thus considering A as a subalgebra of B. Therefore, for an associative algebra A, and a multiplicative subset S satisfying Ore's condition, if B is an associative algebra satisfying properties *i)* *ii)* and *iii)* in Theorem 2.1, we will consider A as a subalgebra of B, and denote the elements of B by $s^{-1}a$, with $a \in A$ and $s \in S$.

Definition 2.8. *Let A be an associative algebra, and $S \subseteq A$ be a multiplicative subset of A, satisfying Ore's localizability condition. The* **localization** *of A relative to S is the unique associative algebra, denoted A_S, satisfying*

i) $A \leq A_S$.
ii) Every element of S is invertible in A_S.
iii) Every element of A_S can be written in the form $s^{-1}a$ for some $s \in S$ and $a \in A$.

2.3 Lie Algebras

The following sections review the structure of Lie algebras, and in particular semisimple Lie algebras. The majority of results in this section are taken from [8], generally following the notation of that source.

Definition 2.9. *A* **Lie algebra** *L is a vector space over a field $\mathbb{F}$ endowed with an operation $[\cdot,\cdot] : L \times L \to L$ having the following properties:*

i) $[ax + by, z] = a[x, z] + b[y, z], \; [x, ay + bz] = a[x, y] + b[x, z]$
ii) $[x, x] = 0$
iii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

for all $a, b \in \mathbb{F}$, and $x, y, z \in L$. A **Lie subalgebra** *$K \leq L$ is a sub-vector space of L with the property that $[x, y] \in K$ for all $x, y \in K$.*

Notice that conditions *i)* and *ii)* give us $0 = [x-y, x-y] = [x,x] - [y,x] - [x,y] + [y,y] = -[y,x] - [x,y]$, and hence $[x,y] = -[y,x]$ for all $x, y \in L$. Also, notice that given any associative algebra A, we can create a Lie algebra A' by setting $A = A'$ as vector spaces, and defining $[\cdot,\cdot] : A' \times A' \to A'$ by $[x,y] = x*y - y*x$ for all $x, y \in A'$.

For an example of a Lie algebra, let V be a vector space over a field $\mathbb{F}$, and consider $End\ V$, the set of all linear transformations on V. Define $[\cdot,\cdot] : End\ V \times End\ V \to End\ V$ by $[x,y] = xy - yx$ where xy denotes the composition of maps x and y. Then $End\ V$ with this operation, and the usual addition, is a Lie algebra. We call this the **general linear algebra**, denoted $gl(V)$. In the case where V is finite dimensional, $End\ V \cong M_n(\mathbb{F})$, the $n \times n$ matrices over $\mathbb{F}$. In this case, the operation $[\cdot,\cdot]$ is $[x,y] = xy - yx$ where xy is the usual matrix multiplication of x and y. We denote the general linear algebra of $n \times n$ matrices of $\mathbb{F}$ by $gl_n(\mathbb{F})$.

Definition 2.10. *Let K and L be Lie algebras. Let $\varphi : K \to L$ be a linear map with the property that $\varphi([x,y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in K$. Then φ is called a* **Lie algebra homomorphism**. *As with associative algebras, if φ is bijective then φ is called an* **isomorphism**, *and K and L are said to be* **isomorphic**, *denoted $K \cong L$. Likewise, when φ is bijective, and $K = L$, we call φ an* **automorphism**.

Definition 2.11. *Let L be a Lie algebra. Then*

$$[L, L] = span_{\mathbb{F}}\{[x,y] \mid x, y \in L\}$$

is called the **derived algebra** *of L. If $[L, L] = (0)$ then L is said to be* **abelian**.

For subalgebras K_1 and K_2 of a Lie algebra L, we also make use of the notation:

$$[K_1, K_2] = span_{\mathbb{F}}\{[x,y] \mid x \in K_1, y \in K_2\}$$

Definition 2.12. *Let L be a Lie algebra, and $I \subseteq L$ be a sub-vector space of L. Then I is an* **ideal** *of L, denoted $I \trianglelefteq L$, provided $[x,y] \in I$ for all $x \in I$ and $y \in L$. We say L is* **simple** *provided L is not abelian and the only ideals of L are (0) and L itself.*

In the definition of simple, we require that L be non-abelian because if L is abelian, then every sub-vector space of L is an ideal. Hence if L is abelian and the only ideals of L are (0) and L itself, then $L = (0)$ or L is one dimensional. Therefore the added condition that L is not abelian simply guarantees that L is neither trivial nor one dimensional.

Proposition 2.2. *Let L be a Lie algebra, and $I, J \trianglelefteq L$ be ideals of L. Then $[I, J]$ is an ideal of L.*

Proof. Let $x = [x_I, x_J]$ for some $x_I \in I$ and $x_J \in J$. Then,

$$[x, y] = [[x_I, x_J], y] = [x_I, [x_J, y]] + [x_J, [y, x_I]]$$

Since $I \trianglelefteq L$, we have that $[y, x_I] \in I$, hence

$$[x_J, [y, x_I]] = -[[y, x_I], x_J] \in [I, J]$$

Likewise, since $J \trianglelefteq L$, we have that $[x_J, y] \in J$, hence

$$[x_I, [x_J, y]] \in [I, J]$$

Therefore $[x, y] \in [I, J]$. □

3 Semisimple Lie Algebras

Though some of the results in this section are true over an arbitrary field, for simplicity, we will restrict our attention to the case when $\mathbb{F} = \mathbb{C}$. Hence all Lie algebras and Vector spaces throughout this section are assumed to be over the complex numbers. Further, from this point on, we only consider finite dimensional Lie algebras. Hence, in all results, the Lie algebra L is assumed to be finite dimensional.

Definition 3.1. *Let L be a Lie algebra. Let $L^{(0)} = L$, and for each $i \geq 1$ let $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$. Then the sequence $(L^{(i)})$ is called the* **derived series** *of L. L is called* **solvable** *if $L^{(n)} = (0)$ for some n. If $I \trianglelefteq L$ then I is said to be a* **solvable ideal** *of L provided I is solvable as a Lie algebra.*

Notice that due to Proposition 2.2, we have that for each i, $L^{(i)}$ is an ideal of L.

Proposition 3.1. *Let L be a Lie algebra. If I and J are solvable ideals of L, then $I + J$ is a solvable ideal of L.*

Proof. (See Proposition 3.1 in [8]) □

Proposition 3.2. *If L is a Lie algebra then L has a unique maximal solvable ideal, denoted Rad L.*

Proof. Notice that $(0) \trianglelefteq L$, and (0) is solvable. Therefore L contains at least one solvable ideal. Due to the finite dimensionality of L, we therefore have the existence of a maximal solvable ideal. It only remains to show uniqueness. To this end, let M_1 and M_2 be maximal solvable ideals of L. The previous proposition implies that $M = M_1 + M_2$ is a solvable ideal of L. By maximality of M_1, we have that $M = M_1$ and likewise, maximality of M_2 gives us $M = M_2$. Thus $M_1 = M_2$ □

Definition 3.2. *Let L be a Lie algebra. L is called* **semisimple** *provided Rad $L =$ (0). i.e. L is semisimple provided L has no non-trivial solvable ideals.*

Proposition 3.3. *If L is a non-trivial semisimple Lie algebra, then there exist ideals $L_1, \ldots, L_k \trianglelefteq L$ such that each L_i is simple as a Lie algebra, and*

$$L = \bigoplus_{i=1}^{k} L_i$$

Proof. (See Theorem 5.3 in [8]) □

3.1 Root Space Decomposition

Definition 3.3. *Let L be a Lie algebra. For each $x \in L$, let $ad_x : L \longrightarrow L$ be the linear map on L given by $ad_x(y) = [x, y]$ for all $y \in L$. Define $ad : L \longrightarrow End\ L$ by $ad(x) = ad_x$ for all $x \in L$. For each $x \in L$, ad_x is called the* **adjoint** *map of x.*

Definition 3.4. *Let V be a vector space, and $\phi \in End\ V$. Then ϕ is said to be* **semisimple** *provided all of the roots of the minimal polynomial of ϕ are distinct. Let L be a Lie algebra. An element $x \in L$ is called* **semisimple** *provided ad_x is semisimple.*

Notice that since $\mathbb{C}$ is algebraically closed, ϕ is semisimple if and only if ϕ is diagonalizable. i.e. there exists a basis $\mathfrak{B}$ of V such that the matrix of ϕ with respect to $\mathfrak{B}$ is a diagonal matrix.

The following is a standard result from linear algebra, and as such, we will omit the proof.

Proposition 3.4. *Let V be a vector space with $\dim V = n < \infty$, and $\phi_1, \ldots, \phi_m \in$ End V be commuting diagonalizable endomorphisms. Then $\phi_1, \ldots, \phi_m$ are simultaneously diagonalizable.*

Proof. (Omitted.) □

Recall that an element x in a Lie algebra L is called semisimple provided $ad_x \in$ *End V* is semisimple.

Definition 3.5. *Let L be a Lie algebra. A subalgebra $T \leq L$ is called* **toral** *provided $T \neq (0)$, and every $x \in T$ is semisimple.*

Proposition 3.5. *Let L be a Lie algebra, and $T \leq L$ be a toral subalgebra of L, then $[T, T] = (0)$.*

Proof. (See Lemma 8.1 in [8]) □

This implies that if L is a Lie algebra, and T is a toral subalgebra of L, then for every $x, y \in T$ we have

$$(ad_x ad_y - ad_y ad_x)(z) = [x, [y, z]] - [y, [x, z]] = -[z, [x, y]] = 0$$

for all $z \in L$, and hence $ad_x ad_y = ad_y ad_x$. Therefore, the collection of all ad_x such that $x \in T$ is a commuting family of diagonalizable endomorphisms on L. Due to Proposition 3.4, we can find a basis $\mathfrak{B} = \{y_1, \ldots, y_n\}$ of L consisting of simultaneous eigenvectors of the endomorphisms ad_x for all $x \in T$. Hence, for each $1 \leq i \leq n$, and each $x \in T$ we have $ad_x(y_i) = \gamma_{ix} y_i$ for some $\gamma_{ix} \in \mathbb{F}$. Therefore, each eigenvector y_i defines a function of eigenvalues $\gamma_i \in T^*$ given by $\gamma_i(x) = \gamma_{ix}$, for all $x \in T$, where T^* denotes the vector space of all linear functions of T into $\mathbb{C}$.

Proposition 3.6. *If L is a non-trivial semisimple Lie algebra, then there exists an $x \in L$ such that x is semisimple.*

Proof. (See Section 8.1 in [8]) □

In particular, the previous proposition implies that every non-trivial semisimple Lie algebra contains a toral subalgebra. Further, due to finite dimensionality, we have that every non-trivial semisimple Lie algebra contains a maximal toral subalgebra.

Definition 3.6. *Let L be a semisimple Lie Algebra. A* **Cartan subalgebra** *$H \leq L$ is a maximal toral subalgebra of L.*

Definition 3.7. *Let L be a semisimple Lie algebra, and fix a Cartan subalgebra H. For each $\alpha \in H^*$, set*

$$L_\alpha = \{x \in L \mid ad_h(x) = \alpha(h)x \text{ for all } h \in H\}$$

Set

$$\Delta = \{\alpha \in H^* \mid \alpha \neq 0 \text{ and } L_\alpha \neq \{0\}\}$$

Then the elements $\alpha \in \Delta$ are called the **roots** *of L with respect to H, and the sets L_α for $\alpha \in \Delta$ are called the* **root spaces.**

i.e. L_α is the collection of all simultaneous eigenvectors of $ad(H)$ with corresponding eigenvalue function α. Since there exists a basis of L consisting of simultaneous eigenvectors of $ad\, H$, we have that

$$L = L_0 \oplus \bigoplus_{\alpha \in \Delta} L_\alpha$$

Notice that $L_0 = \{x \in L \mid ad_h(x) = 0 \text{ for all } h \in H\}$ is $C_L(H)$, the centralizer of H in L.

Proposition 3.7. *If L is a semisimple Lie algebra, and $H \leq L$ is a Cartan subalgebra of L then $C_L(H) = H$.*

Proof. (See Proposition 8.2 in [8]) □

This gives us the following **root space decomposition**:

$$L = H \oplus \bigoplus_{\alpha \in \Delta} L_\alpha$$

3.2 The Special Linear Algebra $sl(2,\mathbb{C})$

We introduce the simple Lie algebra $sl(2,\mathbb{C})$, which is the subalgebra of $gl(2,\mathbb{C})$ consisting of all 2×2 traceless matrices over $\mathbb{C}$. $sl(2,\mathbb{C})$ is called the **special linear algebra** of rank 1. Here, the rank refers to the dimension of the Cartan subalgebra. A basis for $sl(2,\mathbb{C})$ is given by $\{x_1, x_2, h\}$ where

$$x_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Notice that $H = span_{\mathbb{C}}\{h\}$ is a Cartan subalgebra. Further,

$$[h, x_1] = 2x_1 \quad \text{and} \quad [h, x_2] = -2x_2$$

therefore taking $\alpha \in H^*$ given by $\alpha(h) = 2$ gives us $L_\alpha = span_{\mathbb{C}}\{x_1\}$ and $L_{-\alpha} = span_{\mathbb{C}}\{x_2\}$. Notice, also, that in this case $[x_1, x_2] = h$.

3.3 The Euclidean Space of Linear Functionals on H

Definition 3.8. *Let L be a Lie algebra. Define $\kappa : L \times L \to \mathbb{C}$ by*

$$\kappa(x,y) = Tr(ad_x ad_y)$$

for all $x, y \in L$. κ is called the **Killing form** *of L.*

Clearly, κ is a symmetric bilinear form on L. Further, κ has the following associative property:

$$\kappa([x,y],z) = \kappa(x,[y,z])$$

for all $x, y, z \in L$. To see this, notice that for any $x, y \in L$,

$$ad_{[x,y]}(z) = [[x,y],z] = [x,[y,z]] - [y,[x,z]] = ad_x ad_y(z) - ad_y ad_x(z)$$

and hence $ad_{[x,y]} = ad_x ad_y - ad_y ad_x$. Therefore

$$\begin{aligned} Tr(ad_{[x,y]} ad_z) &= Tr(ad_x ad_y ad_z - ad_y ad_x ad_z) \\ &= Tr(ad_x ad_y ad_z - ad_x ad_z ad_y) \\ &= Tr(ad_x ad_{[y,z]}) \end{aligned}$$

Proposition 3.8. *Let L be a semisimple Lie algebra, with Cartan subalgebra H. Then κ is non-degenerate on H. i.e. If $\kappa(x, y) = 0$ for all $y \in L$, then $x = 0$.*

Proof. (See Corollary 8.2 in [8]) □

Proposition 3.9. *Let L be a semisimple Lie algebra, with Cartan subalgebra H, and root system Δ with respect to H. For each $\gamma \in H^*$, there exists a unique element $t_\gamma \in H$ such that $\gamma(h) = \kappa(t_\gamma, h)$, for every $h \in H$.*

Proof. For each $t \in H$, we can define a map $\gamma_t : H \to \mathbb{C}$ by $\gamma_t(h) = \kappa(t, h)$, for all $h \in H$. Of course, since κ is a bilinear form, each γ_t is a linear map. We now consider the map $\varphi : H \to H^*$ given by $\varphi(t) = \gamma_t$. Again, since κ is bilinear, the map φ is linear. Also, since κ is nondegenerate on H, the map φ is injective. Therefore, φ is an injective vector space homomorphism of H into H^*. However, since $\dim H = \dim H^*$, we must have that φ is surjective as well. Hence, φ is invertible. i.e. for each $\gamma \in H^*$, we can choose a unique $t_\gamma = \varphi^{-1}(\gamma)$ such that $\gamma(h) = \kappa(t_\gamma, h)$ for all $h \in H$. □

In particular, for each $\alpha \in \Delta$, there exists a unique t_α such that

$$\alpha(h) = \kappa(t_\alpha, h)$$

for all $h \in H$. Notice that in the above proposition, due to the linearity of φ^{-1}, we have that if $\gamma_1, \gamma_2 \in H^*$ then $t_{\gamma_1+\gamma_2} = t_{\gamma_1} + t_{\gamma_2}$.

Proposition 3.10. *Let L be a semisimple Lie algebra, with Cartan subalgebra H, and root system Δ with respect to H. Then the following properties hold:*

i) Δ spans H^*.

ii) For every $\alpha \in \Delta$, $\alpha(t_\alpha) \neq 0$.

iii) If $\alpha \in \Delta$ then $-\alpha \in \Delta$.

Proof. (See Proposition 8.3 in [8]) □

Definition 3.9. *Let L be a semisimple Lie algebra, with Cartan subalgebra H, and root system Δ with respect to H. Define $(\cdot, \cdot) : H^* \times H^* \to \mathbb{C}$ by*

$$(\gamma, \lambda) = \kappa(t_\gamma, t_\lambda)$$

for each $\gamma, \lambda \in H^*$.

Proposition 3.11. *Let L be a semisimple Lie algebra, with root system Δ. Let $E = span_{\mathbb{R}}(\Delta)$ be the real span of the roots in Δ. Then $(\cdot, \cdot)$ is an inner product on E. Hence E is a Euclidean space.*

Proof. (See Section 8.5 in [8]) □

3.4 Simple Roots

Definition 3.10. *Let L be a semisimple Lie algebra with Cartan subalgebra H and root system Δ. A* **base** *$\Delta^{++} \subseteq \Delta$ for the root system Δ is a basis for the vector space H^* with the property that for each $\beta \in \Delta$,*

$$\beta = \sum_{\alpha \in \Delta^{++}} a_\alpha \alpha$$

for some a_α, with either all $a_\alpha \in \mathbb{Z}_{\geq 0}$ or all $a_\alpha \in \mathbb{Z}_{\leq 0}$. If Δ^{++} is a base for Δ, then the roots $\alpha \in \Delta^{++}$ are called **simple roots.**

Let $E = span_{\mathbb{R}}(\Delta)$ denote the real span of the roots in Δ. For each $\gamma \in E$, let $\Delta^+(\gamma) = \{\alpha \in \Delta \mid (\gamma, \alpha) > 0\}$. For each $\alpha \in \Delta$, let

$$P_\alpha = \{\lambda \in E \mid (\alpha, \lambda) = 0\}$$

be the hyperplane perpendicular to α. Since the union of finitely many hyperplanes cannot cover the entire space E, we have that

$$E \setminus \bigcup_{\alpha \in \Delta} P_\alpha \neq \emptyset$$

We call an element $\gamma \in E$ **regular** if $\gamma \in E \setminus \bigcup_{\alpha \in \Delta} P_\alpha$, and **singular** if $\gamma \in P_\alpha$ for some $\alpha \in \Delta$.

If $\gamma \in E$ is regular, then for every $\alpha \in \Delta$, we have $(\gamma, \alpha) \neq 0$, hence $\alpha \in \Delta^+(\gamma)$ or $\alpha \in -\Delta^+(\gamma)$. Thus $\Delta = \Delta^+(\gamma) \cup -\Delta^+(\gamma)$.

For each $\alpha \in \Delta^+(\gamma)$, we say that α is **decomposable** with respect to $\Delta^+(\gamma)$ if there exist $\alpha_1, \alpha_2 \in \Delta^+(\gamma)$ with $\alpha = \alpha_1 + \alpha_2$. We say that $\alpha \in \Delta^+(\gamma)$ is **indecomposable** if α is not decomposable.

Theorem 3.1. *Let L be a semisimple Lie algebra, with root system Δ. Let $E = span_{\mathbb{R}}(\Delta)$. If $\gamma \in E$ is regular, then*

$$\Delta^{++}(\gamma) = \{\alpha \in \Delta^{+}(\gamma) \mid \alpha \text{ is indecomposable}\}$$

is a base for Δ. Further, if Δ^{++} is any base for Δ, then $\Delta^{++} = \Delta^{++}(\gamma)$ for some regular $\gamma \in E$.

Proof. (See Theorem 10.1.2 in [8]) □

In particular, the previous theorem implies that every root system Δ, of a semi-simple Lie algebra, has a base.

Definition 3.11. *Let L be a semisimple Lie algebra, with root system Δ, and base Δ^{++}. Let $\beta \in \Delta$. If $\beta = \sum_{\alpha \in \Delta^{++}} a_\alpha \alpha$, we say that β is a* **positive root** *if all $a_\alpha \in \mathbb{Z}_{\geq 0}$, and that β is a* **negative root** *if all $a_\alpha \in \mathbb{Z}_{\leq 0}$. Set*

$$\Delta^{+} = \{\alpha \in \Delta \mid \alpha \text{ is positive}\}$$

and

$$\Delta^{-} = \{\alpha \in \Delta \mid \alpha \text{ is negative}\}$$

Clearly, by the definition of a base, $\Delta = \Delta^{+} \cup \Delta^{-}$. Notice that since $\alpha \in \Delta$ implies $-\alpha \in \Delta$, we have that $\Delta^{-} = -\Delta^{+}$. Also, notice that if $\Delta^{++} = \Delta^{++}(\gamma)$, with γ regular, then Δ^{+} coincides with our earlier definition of $\Delta^{+}(\gamma)$. Indeed, let $\Delta^{++} = \{\alpha_1, \ldots, \alpha_n\}$, and let $\alpha \in \Delta^{+}$. Then

$$\alpha = \sum_{i=1}^{n} a_i \alpha_i$$

for some $a_i \in \mathbb{Z}_{\geq 0}$. Therefore

$$(\gamma, \alpha) = \sum_{i=1}^{n} a_i (\gamma, \alpha_i)$$

since $(\gamma, \alpha_i) > 0$ for all i, we have that $(\gamma, \alpha) > 0$, hence $\alpha \in \Delta^{+}(\gamma)$. Conversely, if $\alpha \in \Delta^{+}(\gamma)$, then

$$(\gamma, \alpha) = \sum_{i=1}^{n} a_i (\gamma, \alpha_i) > 0$$

and since either all a_i are non-negative, or all a_i are non-positive, we must have the former. i.e. $a_i \geq 0$ for all i. Therefore $\alpha \in \Delta^{+}$.

Definition 3.12. *Let L be a semisimple Lie algebra, with Cartan subalgebra H, and root system Δ with respect to H. For each $\alpha \in \Delta$, choose $t_\alpha \in H$ as in Proposition 3.9, and define*

$$h_\alpha = 2\frac{t_\alpha}{\kappa(t_\alpha, t_\alpha)} = 2\frac{t_\alpha}{(\alpha, \alpha)}$$

Proposition 3.12. *Let L be a semisimple Lie algebra, with Cartan subalgebra H, and root system Δ with respect to H. If Δ^{++} is a base for Δ, then $\{h_\alpha \mid \alpha \in \Delta^{++}\}$ is a basis for H.*

Proof. Set $\Delta^{++} = \{\alpha_1, \ldots, \alpha_n\}$. Since $\dim H = \dim H^*$, and Δ^{++} is a basis for H^*, we need only show that $\{h_{\alpha_1}, \ldots, h_{\alpha_n}\}$ spans H. To this end, let $t \in H$, and choose $\gamma \in H^*$ such that $\gamma(h) = \kappa(t, h)$ for all $h \in H$. Since Δ^{++} is a basis for H^*, we have that $\gamma = \sum_{i=1}^n a_i \alpha_i$ for some $a_i \in \mathbb{C}$. Therefore for every $h \in H$, we have

$$\kappa(t, h) = \gamma(h) = \sum_{i=1}^n a_i \alpha_i(h) = \sum_{i=1}^n a_i \kappa(t_{\alpha_i}, h)$$

Therefore

$$\kappa(t - \sum_{i=1}^n a_i t_{\alpha_i}, h) = 0$$

for all $h \in H$. Since κ is non-degenerate on H, we must have

$$t = \sum_{i=1}^n a_i t_{\alpha_i} = \sum_{i=1}^n a_i \frac{\kappa(t_{\alpha_i}, t_{\alpha_i})}{2} h_{\alpha_i}$$

□

For a semisimple Lie algebra L with root system Δ with respect to the Cartan subalgebra H, we now fix a base $\Delta^{++} = \{\alpha_1, \ldots, \alpha_n\}$, and hence we have a basis $\mathfrak{H} = \{h_1, \ldots, h_n\}$ of H, where $h_i = h_{\alpha_i}$. We call $\mathfrak{H}$ the **simple basis** of H with respect to Δ^{++}. We can also obtain the dual basis for H^* relative to $\mathfrak{H}$ by choosing, for each i, $\omega_i \in H^*$ given by

$$\omega_i(h_j) = \delta_{ij}$$

and extending linearly. We call

$$\mathfrak{F} = \{\omega_1, \ldots, \omega_n\}$$

the **fundamental basis** for H^*, and the elements $\omega_i \in \mathfrak{F}$ are called the **fundamental weights**.

Definition 3.13. *Let L be a semisimple Lie algebra, with Cartan subalgebra H. Define the product $\langle \cdot, \cdot \rangle : E \times \Delta \to \mathbb{C}$ by*

$$\langle \gamma, \alpha \rangle = 2\frac{(\gamma, \alpha)}{(\alpha, \alpha)}$$

for all $\gamma \in E$ and $\alpha \in \Delta$.

Notice that this definition is possible, since by Proposition 3.10,

$$(\alpha, \alpha) = \kappa(t_\alpha, t_\alpha) = \alpha(t_\alpha) \neq 0$$

Also, notice that $\langle \cdot, \cdot \rangle$ is linear in the first coordinate, but not in the second, and for any $\alpha \in \Delta$, and $\gamma \in E$, we have that

$$\gamma(h_\alpha) = \kappa(t_\gamma, h_\alpha) = 2\frac{\kappa(t_\gamma, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = 2\frac{(\gamma, \alpha)}{(\alpha, \alpha)} = \langle \gamma, \alpha \rangle$$

In particular, if $\Delta^{++} = \{\alpha_1, \ldots, \alpha_n\}$, we have

$$\langle \omega_i, \alpha_j \rangle = \delta_{ij}$$

Proposition 3.13. *Let L be a semisimple Lie algebra, with root system Δ. Then the following properties hold:*

i) $\dim L_\alpha = 1$ *for every $\alpha \in \Delta$.*

ii) If $\alpha, \beta, \alpha + \beta \in \Delta$ then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.

iii) For each $\alpha \in \Delta$, $\dim[L_\alpha, L_{-\alpha}] = 1$, and $\{h_\alpha\}$ is a basis for $[L_\alpha, L_{-\alpha}]$.

iv) For each $\alpha \in \Delta$ and each $x_\alpha \in L_\alpha$ there exists an $x_{-\alpha} \in L_{-\alpha}$ such that $[x_\alpha, x_{-\alpha}] = h_\alpha$ and $span_{\mathbb{C}}\{x_\alpha, x_{-\alpha}, h_\alpha\} \cong sl(2, \mathbb{C})$.

v) L is generated by $\bigcup_{\alpha \in \Delta} L_\alpha$.

Proof. (See Proposition 8.3 and Proposition 8.4 in [8]) □

Due to the previous proposition, we can choose a set of elements

$$\{x_\alpha \mid \alpha \in \Delta\} \subseteq L$$

with the following properties:

i) For each $\alpha \in \Delta$ we have $L_\alpha = span_{\mathbb{C}}\{x_\alpha\}$

ii) $[x_\alpha, x_{-\alpha}] = h_\alpha$, for each $\alpha \in \Delta$

iii) $span_{\mathbb{C}}\{x_\alpha, x_{-\alpha}, h_\alpha\} \cong sl(2, \mathbb{C})$ for each $\alpha \in \Delta$.

Notice that since $span_{\mathbb{C}}\{x_\alpha, x_{-\alpha}, h_\alpha\} \cong sl(2, \mathbb{C})$, we must have

$$[h_\alpha, x_\alpha] = 2x_\alpha$$

and

$$[h_\alpha, x_{-\alpha}] = -2x_{-\alpha}$$

Definition 3.14. *Let L be a semisimple Lie algebra with Cartan subalgebra H. Let Δ be the root system of L, with base $\Delta^{++} = \{\alpha_1, \ldots, \alpha_n\}$. A* **Chevalley basis** *of L is a basis*

$$\{x_\alpha \mid \alpha \in \Delta\} \cup \{h_{\alpha_i} \mid 1 \leq i \leq n\}$$

satisfying the properties

i) $[x_\alpha, x_{-\alpha}] = h_\alpha$ *for all* $\alpha \in \Delta$.

ii) If, whenever $\alpha, \beta, \alpha + \beta \in \Delta$ we have $[x_\alpha, x_\beta] = c_{\alpha,\beta} x_{\alpha+\beta}$, then $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$.

Lemma 3.1. *Let L be a semisimple Lie algebra with Cartan subalgebra H. Let Δ be the root system of L. For each $\alpha \in \Delta$ choose $x_\alpha \in L_\alpha$ and $x_{-\alpha} \in L_{-\alpha}$ such that $[x_\alpha, x_{-\alpha}] = h_\alpha$. Then there exists an automorphism σ of L such that $\sigma(x_\alpha) = -x_{-\alpha}$ for all $\alpha \in \Delta$ and $\sigma(h) = -h$ for all $h \in H$.*

Proof. (See Proposition 14.3 in [8]) □

Proposition 3.14. *Let L be a semisimple Lie algebra with Cartan subalgebra H. Let Δ be the root system of L with base $\Delta^{++} = \{\alpha_1, \ldots, \alpha_n\}$. Choose $\{x_\alpha \mid \alpha \in \Delta\}$ such that $L_\alpha = span_{\mathbb{C}}\{x_\alpha\}$ and $[x_\alpha, x_{-\alpha}] = h_\alpha$ for all $\alpha \in \Delta$. Then*

$$\{x_\alpha \mid \alpha \in \Delta\} \cup \{h_{\alpha_i} \mid 1 \leq i \leq n\}$$

is a Chevalley basis of L.

Proof. We need only show that if, whenever $\alpha, \beta, \alpha + \beta \in \Delta$ we have

$$[x_\alpha, x_\beta] = c_{\alpha,\beta} x_{\alpha+\beta},$$

then $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$. Let $\alpha, \beta, \alpha+\beta \in \Delta$. By Proposition 3.13, since $[L_\alpha, L_\beta] = L_{\alpha+\beta}$, we have $[x_\alpha, x_\beta] = c_{\alpha,\beta} x_{\alpha+\beta}$ for some $c_{\alpha,\beta} \in \mathbb{C}$ and $[x_{-\alpha}, x_{-\beta}] = c_{-\alpha,-\beta} x_{-\alpha-\beta}$ for some $c_{-\alpha,-\beta} \in \mathbb{C}$. Let $\sigma : L \to L$ be an automorphism of L, as in Lemma 3.1. The following calculation gives us our result

$$\begin{aligned} c_{-\alpha,-\beta} x_{-\alpha-\beta} &= [x_{-\alpha}, x_{-\beta}] \\ &= [-x_{-\alpha}, -x_{-\beta}] \\ &= \sigma([x_\alpha, x_\beta]) \\ &= \sigma(c_{\alpha,\beta} x_{\alpha+\beta}) \\ &= -c_{\alpha,\beta} x_{-\alpha-\beta} \end{aligned}$$

□

3.5 Root Strings

Definition 3.15. *Let L be a semisimple Lie algebra over $\mathbb{C}$, with root system Δ. If $\alpha, \beta \in \Delta$, then the* **root string** *of α through β is defined to be*

$$\{\beta + i\alpha \in \Delta \mid i \in \mathbb{Z}\}$$

The following proposition gives a characterization of the root strings occurring in Δ.

Proposition 3.15. *Let L be a semisimple Lie algebra, with root system Δ. Then the following properties hold:*

i) for all $\alpha \in \Delta$, $i\alpha \in \Delta$ if and only if $i = \pm 1$.
ii) For all $\alpha, \beta \in \Delta$, $\beta(h_\alpha) \in \mathbb{Z}$.

iii) For all $\alpha, \beta \in \Delta$, $\beta - \beta(h_\alpha)\alpha \in \Delta$.

iv) For each $\alpha, \beta \in \Delta$, *if* m *and* n *are the largest non-negative integers such that* $\beta - m\alpha, \beta + n\alpha \in \Delta$ *then* $\beta - i\alpha \in \Delta$ *for all* $-m \leq i \leq n$ *and* $\beta(h_\alpha) = m - n$.

Proof. (See Proposition 8.4 in [8]) □

Notice that for each $\alpha, \beta \in \Delta$, since

$$\beta(h_\alpha) = \langle \beta, \alpha \rangle = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}$$

we have that

$$\beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in \Delta$$

which, in the Euclidean space E, is the reflection of the root β in the hyperplane perpendicular to α.

3.6 The Weyl Group

Definition 3.16. *Let* L *be a semisimple Lie algebra, with root system* Δ. *Let* E *be the Euclidean space spanned by* Δ. *For each* $\alpha \in \Delta$, *let* $\sigma_\alpha : E \longrightarrow E$ *denote the reflection in the hyperplane perpendicular to* α. *i.e.*

$$\sigma_\alpha(\gamma) = \gamma - 2\frac{(\gamma, \alpha)}{(\alpha, \alpha)}\alpha = \gamma - \langle \gamma, \alpha \rangle \alpha$$

for all $\gamma \in E$. *Define the* **Weyl group**, *denoted* $\mathcal{W}$, *to be the group generated by* $\{\sigma_\alpha \mid \alpha \in \Delta\}$.

Proposition 3.16. *Let* L *be a semisimple Lie algebra, with root system* Δ. *Let* Δ^{++} *be a base for* Δ. *Then* $\mathcal{W}$ *is generated by the set* $\{\sigma_\alpha \mid \alpha \in \Delta^{++}\}$.

Proof. (See Theorem 10.3 in [8]) □

Hence any element $\sigma \in \mathcal{W}$ can be written as a product of reflections in the simple roots.

Definition 3.17. *Let L be a semisimple Lie algebra, with root system Δ. Let Δ^{++} be a base for Δ, and Δ^+ be the set of positive roots with respect to Δ^{++}. Define*

$$\rho = \frac{1}{2}\sum_{\beta\in\Delta^+}\beta$$

Proposition 3.17. *Let L be a semisimple Lie algebra, with root system Δ. Let Δ^{++} be a base for Δ. Then for each $\alpha \in \Delta^{++}$, if $\beta \in \Delta^+$ with $\beta \neq \alpha$ then $\sigma_\alpha(\beta) \in \Delta^+$.*

Proof. (See Lemma 10.2 B in [8]) □

Corollary 3.1. *Let L be a semisimple Lie algebra, with root system Δ. Let Δ^{++} be a base for Δ. Then for each $\alpha \in \Delta^{++}$, we have $\sigma_\alpha(\rho) = \rho - \alpha$.*

Proof. Since σ_α permutes the $\beta \in \Delta^+$ with $\beta \neq \alpha$, we have that

$$\sigma_\alpha(\rho) = \sigma_\alpha\left(\frac{1}{2}\alpha + \frac{1}{2}\sum_{\substack{\beta\in\Delta^+\\ \beta\neq\alpha}}\beta\right) = -\frac{1}{2}\alpha + \frac{1}{2}\sum_{\substack{\beta\in\Delta^+\\ \beta\neq\alpha}}\beta = \rho - \alpha$$

□

Corollary 3.2. *Let L be a semisimple Lie algebra, with Cartan subalgebra H and root system Δ. Let $\omega_1, \ldots, \omega_n$ be the fundamental weights with respect to a fixed base $\Delta^{++} = \{\alpha_1, \ldots, \alpha_n\}$. Then*

$$\rho = \sum_{i=1}^{n}\omega_i$$

Proof. Since $\{\omega_1, \ldots, \omega_n\}$ is a basis for H^*, we can write $\rho = \sum_{i=1}^{n} a_i\omega_i$ for some $a_i \in \mathbb{C}$. For each k, due to Corollary 3.1, we have

$$\begin{aligned}\rho - \alpha_k = \sigma_{\alpha_k}(\rho) &= \rho - \langle\rho, \alpha_k\rangle\alpha_k \\ &= \rho - \langle\sum_{i=1}^{n} a_i\omega_i, \alpha_k\rangle\alpha_k \\ &= \rho - \sum_{i=1}^{n} a_i\langle\omega_i, \alpha_k\rangle\alpha_k \\ &= \rho - a_k\alpha_k\end{aligned}$$

Therefore $a_k = 1$ for all k. □

We now define another useful action of the Weyl group. It is simply the usual action under a translation by ρ.

Definition 3.18. *Let L be a semisimple Lie algebra, with root system Δ and fixed base Δ^{++}. Let E be the Euclidean space spanned by Δ, and $\mathcal{W}$ be the Weyl group of L. Define the* **affine action** *of $\mathcal{W}$ on E to be $\cdot : \mathcal{W} \times E \to E$ given by*

$$\sigma \cdot \gamma = \sigma(\gamma + \rho) - \rho$$

Definition 3.19. *Let L be a semisimple Lie algebra, with root system Δ and Weyl group $\mathcal{W}$. Let Δ^+ and Δ^- be the sets of positive and, respectively, negative roots with respect to a fixed base Δ^{++}. For each $\sigma \in \mathcal{W}$ define the* **length** *of σ, denoted l_σ, to be*

$$l_\sigma = |\{\alpha \in \Delta^+ \mid \sigma(\alpha) \in \Delta^-\}|$$

We say σ is **even** *if l_σ is even, and σ is* **odd** *if l_σ is odd.*

Notice that Proposition 3.17 implies that each simple reflection σ_α with $\alpha \in \Delta^{++}$ has length equal to 1. Therefore the reflections in the simple roots are all odd.

Definition 3.20. *Let L be a semisimple Lie algebra, with Weyl group $\mathcal{W}$. Define $sgn : \mathcal{W} \to \mathbb{Z}_2$ by*

$$sgn(\sigma) = \begin{cases} 1 & if\ \sigma\ is\ even \\ -1 & if\ \sigma\ is\ odd \end{cases}$$

for each $\sigma \in \mathcal{W}$.

Proposition 3.18. *Let L be a semisimple Lie algebra, with Weyl group $\mathcal{W}$. Then the map $sgn : \mathcal{W} \to \mathbb{Z}_2$ is a group homomorphism.*

Proof. (See page 54 in [8]) □

Corollary 3.3. *Let L be a semisimple Lie algebra with root system Δ and base Δ^{++} for Δ. Let $\mathcal{W}$ be the Weyl group of L. Then for each $\sigma \in \mathcal{W}$, σ is even if and only if*

$$\sigma = \prod_{i=1}^{k} \sigma_{\alpha_i}$$

for some $\alpha_i \in \Delta^{++}$ with k even, and σ is odd if and only if

$$\sigma = \prod_{i=1}^{k} \sigma_{\alpha_i}$$

for some $\alpha_i \in \Delta^{++}$ with k odd.

Proof. By Proposition 3.16, we have

$$\sigma = \prod_{i=1}^{k} \sigma_{\alpha_i}$$

for some $\alpha_i \in \Delta^{++}$. Since sgn is a homomorphism, and every simple reflection is odd, we have that

$$sgn(\sigma) = \prod_{i=1}^{k} sgn(\sigma_{\alpha_i}) = (-1)^k$$

Therefore σ is even if and only if k is even, and σ is odd if and only if k is odd. □

3.7 Type C Lie algebras

The main focus of this thesis will be the **symplectic algebras**, denoted $sp(2n, \mathbb{C})$ for $n \in \mathbb{Z}_{>0}$. These are also called **type C Lie algebras**, and $sp(2n, \mathbb{C})$ is simply denoted C_n. $sp(2n, \mathbb{C})$ is defined as follows: Let

$$f : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$$

be the non-degenerate skew-symmetric form on $\mathbb{C}^n$ given by

$$f(v, w) = v^T \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} w$$

where I_n denotes the $n \times n$ identity matrix. Define $sp(2n, \mathbb{C})$ to be the subalgebra of $gl(2n, \mathbb{C})$ consisting of all endomorphisms $x \in gl(2n, \mathbb{C})$ such that $f(x(v), w) = -f(v, x(w))$.

The algebras C_n for $n \in \mathbb{Z}_{>0}$ are simple, and therefore semisimple Lie algebras. We now give a realization of the Lie algebra C_n. We further specialize the general concepts from Sections 3.1-3.6 to this special case.

C_n can be viewed as the subalgebra of $gl(2n, \mathbb{C})$ consisting of all matrices of the form $X = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{bmatrix}$ where each Γ_i is an $n \times n$ matrix such that $\Gamma_1^T = -\Gamma_4$, $\Gamma_2^T = \Gamma_2$ and $\Gamma_3^T = \Gamma_3$.

We fix a Cartan subalgebra $\mathcal{H}$, equal to the set of all diagonal matrices in C_n. i.e.

$$\mathcal{H} = \left\{ \begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix} \middle| \; D \text{ is a diagonal } n \times n \text{ matrix over } \mathbb{C} \right\}$$

Define the linear maps $\epsilon_1, \ldots, \epsilon_{2n} \in \mathcal{H}^*$ by

$$\epsilon_i \left(\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & d_{2n,2n} \end{bmatrix} \right) = d_{ii}$$

for any diagonal matrix $(d_{ii}) \in \mathcal{H}$.

Notice that for each i, with $1 \leq i \leq n$, we have $\epsilon_i = -\epsilon_{n+i}$, and that $\{\epsilon_1, \ldots, \epsilon_n\}$ is a basis for $\mathcal{H}^*$.

Definition 3.21. *Define the* **epsilon basis** *of* $\mathcal{H}^*$ *to be*

$$\mathfrak{S} = \{\epsilon_1, \ldots, \epsilon_n\}$$

The root system Δ of C_n with respect to $\mathcal{H}$ is given by:

$$\Delta = \{\pm\epsilon_i \pm \epsilon_j \mid i < j\} \cup \{\pm 2\epsilon_i\}$$

A base for Δ is given by

$$\Delta^{++} = \{\epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n-1\} \cup \{2\epsilon_n\}$$

We identify the simple roots with respect to this base as $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = 2\epsilon_n$, hence $\Delta^{++} = \{\alpha_1, \ldots, \alpha_n\}$. The positive roots with respect to Δ^{++} are given by

$$\Delta^+ = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{2\epsilon_i \mid 1 \leq i \leq n\}$$

and the negative roots by $\Delta^- = -\Delta^+$. The following is a Chevalley basis for C_n:

$$x_{\epsilon_i - \epsilon_j} = e_{i,j} - e_{n+j,n+i} \quad \textit{for } 1 \le i < j \le n$$

$$x_{\epsilon_i + \epsilon_j} = e_{i,n+j} + e_{j,n+i} \quad \textit{for } 1 \le i < j \le n$$

$$x_{2\epsilon_i} = e_{i,n+i} \quad 1 \le i \le n$$

$$x_{-\alpha} = x_\alpha^T \quad \textit{for each } \alpha \in \Delta^+$$

$$h_{\alpha_i} = e_{i,i} - e_{i+1,i+1} - (e_{n+i,n+i} - e_{n+i+1,n+i+1}) \quad \textit{for } 1 \le i \le n-1$$

$$h_{\alpha_n} = e_{n,n} - e_{2n,2n}$$

Where $e_{i,j}$ is the unit matrix, with a one in the i, j position, and zeros elsewhere.

Lemma 3.2. *The epsilon basis $\{\epsilon_1, \ldots, \epsilon_n\}$ is orthogonal with respect to the inner product given in Section 3.3. Further, $(\epsilon_i, \epsilon_i) = (\epsilon_j, \epsilon_j)$ for all i, j.*

Proof. For $1 \le i \le n$, we have

$$h_{2\epsilon_i} = [x_{2\epsilon_i}, x_{-2\epsilon_i}] = e_{i,n+i}e_{n+i,i} - e_{n+i,i}e_{i,n+i} = e_{i,i} - e_{n+i,n+i}$$

Therefore, for $1 \le i, j \le n$, we have

$$2\frac{(2\epsilon_i, 2\epsilon_j)}{(2\epsilon_j, 2\epsilon_j)} = 2\epsilon_i(h_{2\epsilon_j}) = 2\epsilon_i(e_{j,j} - e_{n+j,n+j}) = 2\delta_{i,j}$$

In particular, if $i \ne j$ then $(\epsilon_i, \epsilon_j) = 0$. Therefore the elements ϵ_i for $1 \le i \le n$ are pairwise orthogonal. It only remains to show that for any i, j, we have $(\epsilon_i, \epsilon_i) = (\epsilon_j, \epsilon_j)$. To this end, select i, with $1 \le i < n$. Then

$$2\epsilon_i(h_{\alpha_i}) = 2\epsilon_i(e_{i,i} - e_{i+1,i+1} - e_{n+i,n+i} + e_{n+i+1,n+i+1}) = 2$$

Therefore

$$2 = 2\frac{(2\epsilon_i, \epsilon_i - \epsilon_{i+1})}{(\epsilon_i - \epsilon_{i+1}, \epsilon_i - \epsilon_{i+1})} = 4\frac{(\epsilon_i, \epsilon_i)}{(\epsilon_i, \epsilon_i) + (\epsilon_{i+1}, \epsilon_{i+1})}$$

Solving this equation gives us $(\epsilon_i, \epsilon_i) = (\epsilon_{i+1}, \epsilon_{i+1})$, and hence $(\epsilon_i, \epsilon_i) = (\epsilon_j, \epsilon_j)$ for all i, j. □

Proposition 3.19. *Let $(\cdot, \cdot)_\kappa$ denote the inner product given in section 3.3, and $(\cdot, \cdot)$ denote the inner product with respect to which $\{\epsilon_1, \ldots, \epsilon_n\}$ is an orthonormal basis for $\mathcal{H}^*$. Then there is some $k \in \mathbb{C}$ such that*

$$(\gamma_1, \gamma_2)_\kappa = k(\gamma_1, \gamma_2)$$

for all $\gamma_1, \gamma_2 \in span_{\mathbb{R}}\Delta$. *In particular, we still have* $\gamma(h_\alpha) = 2\frac{(\gamma,\alpha)}{(\alpha,\alpha)}$ *for all* $\gamma \in span_{\mathbb{R}}\Delta$ *and* $\alpha \in \Delta$.

Proof. Set $k = (\epsilon_1, \epsilon_1)_\kappa$. Due to the previous lemma, we have $k = (\epsilon_i, \epsilon_i)_\kappa$ for all i. Let $\gamma_1, \gamma_2 \in span_{\mathbb{R}}\Delta$. Then

$$\gamma_1 = \sum_{i=1}^{n} a_i \epsilon_i \quad \text{and} \quad \gamma_2 = \sum_{i=1}^{n} b_i \epsilon_i$$

for some $a_i, b_i \in \mathbb{R}$.

$$(\gamma_1, \gamma_2)_\kappa = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j (\epsilon_i, \epsilon_j)_\kappa = \sum_{i=1}^{n} a_i b_i k = k \sum_{i=1}^{n} a_i b_i (\epsilon_i, \epsilon_i) = k(\gamma_1, \gamma_2)$$

□

For ease of computation, in C_n we will use the inner product with respect to which the ϵ_i are orthonormal. Due to the previous proposition, the formulas

$$\gamma(h_\alpha) = 2\frac{(\gamma, \alpha)}{(\alpha, \alpha)}$$

and

$$\sigma_\alpha(\gamma) = \gamma - 2\frac{(\gamma, \alpha)}{(\alpha, \alpha)}\alpha$$

remain unchanged under this new inner product.

We can define a norm on the Euclidean space $E = span_{\mathbb{R}}\Delta$, by

$$|\gamma| = \sqrt{(\gamma, \gamma)}$$

Notice that the roots of C_n come in only two different magnitudes. Indeed, for $1 \leq i < j \leq n$, we have that

$$(\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_j) = (\epsilon_i + \epsilon_j, \epsilon_i + \epsilon_j) = (-\epsilon_i + \epsilon_j, -\epsilon_i + \epsilon_j) = (-\epsilon_i - \epsilon_j, -\epsilon_i - \epsilon_j) = 2$$

and for $1 \leq i \leq n$ we have

$$(2\epsilon_i, 2\epsilon_i) = (-2\epsilon_i, -2\epsilon_i) = 4$$

Therefore roots of the form $\pm\epsilon_i \pm \epsilon_j$ have length equal to $\sqrt{2}$ and roots of the form $\pm 2\epsilon_i$ have length equal to 2.

Definition 3.22. *Let $\alpha \in \Delta$. Then α is called a* **short root** *if $|\alpha| = \sqrt{2}$ and α is called a* **long root** *if $|\alpha| = 2$. Denote by $\widetilde{\Delta}$ the set of all short roots, and by $\widetilde{\Delta}^+$ and $\widetilde{\Delta}^-$ the positive, and respectively negative, short roots. i.e.*

$$\widetilde{\Delta}^+ = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\}$$

$$\widetilde{\Delta}^- = -\widetilde{\Delta}^+$$

and

$$\widetilde{\Delta} = \widetilde{\Delta}^+ \cup \widetilde{\Delta}^-$$

The fundamental weights of C_n are given by

$$\omega_i = \sum_{k=1}^{i} \epsilon_k$$

Indeed, for $1 \leq i \leq n$ and $1 \leq j < n$, we have

$$\left\langle \sum_{k=1}^{i} \epsilon_k, \alpha_j \right\rangle = 2\frac{(\sum_{k=1}^{i} \epsilon_k, \epsilon_j - \epsilon_{j+1})}{(\epsilon_j - \epsilon_{j+1}, \epsilon_j - \epsilon_{j+1})}$$
$$= \sum_{k=1}^{i} \delta_{kj} - \sum_{k=1}^{i} \delta_{kj+1}$$
$$= \begin{cases} 0 & if\ j > i \\ 0 & if\ j < i \\ 1 & if\ j = i \end{cases} = \delta_{ij}$$

and

$$\left\langle \sum_{k=1}^{i} \epsilon_k, \alpha_n \right\rangle = 2\frac{(\sum_{k=1}^{i} \epsilon_k, 2\epsilon_n)}{(2\epsilon_n, 2\epsilon_n)} = \frac{1}{2}\sum_{k=1}^{i} 2\delta_{kn} = \delta_{in}$$

The fundamental basis for C_n is

$$\mathfrak{F} = \{\omega_1, \ldots, \omega_n\}$$

The element $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha = \sum_{i=1}^{n} \omega_i$ is given, with respect to the epsilon basis, by

$$\rho = \sum_{i=1}^{n} (n - i + 1)\epsilon_i$$

Let $\mathcal{W}$ denote the Weyl group of C_n, and denote by σ_α the reflection in the hyperplane perpendicular to α for each $\alpha \in \Delta$. We now illustrate the action of these reflections on elements of the Euclidean space E, with respect to the epsilon basis. Notice that $(\epsilon_k, \epsilon_i - \epsilon_j) = 0$, provided $k \neq i$ or j. This means that if $k \neq i$ or j then ϵ_k lies on the hyperplane perpendicular to $\epsilon_i - \epsilon_j$, and hence $\sigma_{\epsilon_i - \epsilon_j}(\epsilon_k) = \epsilon_k$. Further,

$$\sigma_{\epsilon_i - \epsilon_j}(\epsilon_i) = \epsilon_i - 2\frac{(\epsilon_i, \epsilon_i - \epsilon_j)}{(\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_j)}(\epsilon_i - \epsilon_j) = \epsilon_j$$

and

$$\sigma_{\epsilon_i - \epsilon_j}(\epsilon_j) = \epsilon_j - 2\frac{(\epsilon_j, \epsilon_i - \epsilon_j)}{(\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_j)}(\epsilon_i - \epsilon_j) = \epsilon_i$$

Viewed as permutations on the subscripts of the elements $\{\epsilon_1, \dots, \epsilon_n\}$, the maps $\sigma_{\epsilon_i - \epsilon_j}$ where $i < j$ are the two cycles $(i \quad j)$. This implies that $\mathcal{W}$ contains all two cycles, and hence contains all permutations on the subscripts of the elements $\{\epsilon_1, \dots, \epsilon_n\}$.

Also, $(\epsilon_j, 2\epsilon_i) = 0$ provided $j \neq i$. Hence, if $i \neq j$ then $\sigma_{2\epsilon_i}(\epsilon_j) = \epsilon_j$. Further,

$$\sigma_{2\epsilon_i}(\epsilon_i) = \epsilon_i - \langle \epsilon_i, 2\epsilon_i \rangle 2\epsilon_i = -\epsilon_i$$

Thus, $\sigma_{2\epsilon_i}$ is the map that changes the sign of ϵ_i. Defining $\epsilon_{-i} = -\epsilon_i$, we have that $\mathcal{W}$ contains any number of sign changes on the subscripts of the elements $\{\epsilon_1, \dots \epsilon_n\}$.

Since $\mathcal{W}$ is generated by the reflections σ_{α_i} for $\alpha_i \in \Delta^{++}$, and

$$\Delta^{++} \subseteq \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\} \cup \{2\epsilon_i \mid 1 \leq i \leq n\}$$

we have that $\mathcal{W}$ is the group of all permutations and sign changes on the subscripts of $\{\epsilon_1, \dots, \epsilon_n\}$.

Definition 3.23. *Define $\widetilde{\mathcal{W}}$ to be the subgroup of $\mathcal{W}$ generated by the reflections in the hyperplanes perpendicular to the short roots. i.e. $\widetilde{\mathcal{W}}$ is generated by $\{\sigma_{\epsilon_i \pm \epsilon_j} \mid 1 \leq i < j \leq n\}$.*

We can simplify calculations by noticing that for any $\gamma \in E$ and any short root $\alpha \in \Delta$, we have

$$\langle \gamma, \alpha \rangle = 2\frac{(\gamma, \alpha)}{(\alpha, \alpha)} = (\gamma, \alpha)$$

Since $\sigma_{\epsilon_i - \epsilon_j} \in \widetilde{\mathcal{W}}$ for $1 \leq i < j \leq n$, we have that $\widetilde{\mathcal{W}}$ still contains all permutations on the subscripts of the elements $\{\epsilon_1, \dots, \epsilon_n\}$. The elements $\sigma_{\epsilon_i + \epsilon_j}$ generate all even sign changes. Indeed,

$$\sigma_{\epsilon_i + \epsilon_j}(\epsilon_i) = \epsilon_i - (\epsilon_i, \epsilon_i + \epsilon_j)(\epsilon_i + \epsilon_j) = -\epsilon_j$$

$$\sigma_{\epsilon_i+\epsilon_j}(\epsilon_j) = \epsilon_j - (\epsilon_j, \epsilon_i + \epsilon_j)(\epsilon_i + \epsilon_j) = -\epsilon_i$$

and

$$\sigma_{\epsilon_i+\epsilon_j}(\epsilon_k) = \epsilon_k$$

whenever $k \neq i, j$. Therefore $\sigma_{\epsilon_i-\epsilon_j}\sigma_{\epsilon_i+\epsilon_j}$ is the map given by $\epsilon_i \mapsto \epsilon_{-i}$ and $\epsilon_j \mapsto \epsilon_{-j}$. We therefore have that $\widetilde{\mathcal{W}}$ is the group of all permutations and even sign changes on the subscripts of the elements $\{\epsilon_1, \ldots, \epsilon_n\}$.

4 Representations

4.1 Representations of Lie Algebras and Associative Algebras

Recall that for any vector space V, the general linear algebra $gl(V)$ is the Lie algebra formed by taking the vector space $End\ V$, of all endomorphisms on V, together with the commutator product $[x, y] = xy - yx$ for $x, y \in End\ V$.

Definition 4.1. *Let L be a Lie algebra. A* **representation** *of L is a pair (ϕ, V) where V is a vector space over $\mathbb{C}$, and $\phi : L \longrightarrow gl(V)$ is a Lie algebra homomorphism, where $gl(V)$ denotes the general linear algebra. In this case, the vector space V is called an L-***module***. For $x \in L$, we can define the* **action** *of x on V by*

$$xv = \phi(x)(v)$$

for each $v \in V$. If $W \leq V$ is a sub-vector space of V with the property that $\phi(x)w \in W$ for all $x \in L$ and $w \in W$, then W is said to be a **submodule** *of V, and $(\phi|_W, W)$ is called a* **sub-representation***. In the case where V has no non-trivial, proper submodules, we say that the module V is* **simple***, and that the representation (ϕ, V) is* **irreducible**.

We have already, in a sense, made use of one Lie algebra representation, that being the **adjoint representation** which is by definition

$$ad : L \longrightarrow gl(L)$$

given by $ad(x) = ad_x$ for any $x \in L$. This is indeed a representation, since for any $x, y, z \in L$, we have

$$ad_{[x,y]}(z) = [[x,y],z] = [x,[y,z]] - [y,[x,z]] = ad_x ad_y(z) - ad_y ad_x(z)$$

and hence $ad([x,y]) = ad(x)ad(y) - ad(y)ad(x)$. Under the adjoint representation, L is itself, an L-module.

Definition 4.2. *Let L be a Lie algebra, and U and V be L-modules, with action given by $\phi_U : L \longrightarrow gl(U)$ and $\phi_V : L \longrightarrow gl(V)$ respectively. The L-modules U and V are said to be* **equivalent**, *denoted $U \simeq V$ provided there exists a vector space isomorphism $\theta : U \longrightarrow V$ such that*

$$\theta(\phi_U(x)(u)) = \phi_V(x)(\theta(u))$$

for all $x \in L$ and $u \in U$.

Definition 4.3. *Let L be a Lie algebra, and V be an L-module. A subset $S \subseteq V$ of V is said to* **generate** *V provided whenever $W \leq V$ is a submodule of V with $S \subseteq W$, we have $W = V$. i.e. there are no proper submodules of V containing S. We say an element $v \in V$ generates V if $\{v\}$ generates V.*

Definition 4.4. *Let L be a Lie algebra, and V be an L-module. V is said to be* **completely reducible** *if for every submodule $U \leq V$, there exists a submodule $U' \leq V$ such that*

$$V = U \oplus U'$$

Proposition 4.1. *If L is a Lie algebra, and V is a completely reducible L-module, then any submodule $W \leq V$ is also completely reducible.*

Proof. Let W be a submodule of V. If $U \leq W$ is any submodule of W, then U is a submodule of V, hence $V = U \oplus U'$ for some submodule $U' \leq V$. Set $U'_W = U' \cap W$. For any $w \in W$, we have $w = u_1 + u_2$ where $u_1 \in U$ and $u_2 \in U'$. Since $u_2 = w - u_1 \in W$, we must have $u_2 \in U'_W$. Therefore $w = u_1 + u_2$ with $u_1 \in U$ and $u_2 \in U'_W$. Hence

$$W = U + U'_W$$

Since both U' and W are submodules of V, we must have that U'_W is a submodule of V. Hence U'_W is a submodule of W. Further, since $U \cap U' = (0)$, we must have $U \cap U'_W = (0)$. Therefore

$$W = U \oplus U'_W$$

□

Definition 4.5. *Let L be a Lie algebra, and V be an L-module. V is called a* **semisimple** *module if there exist simple submodules $W_1, \ldots, W_k$ such that*

$$V = \bigoplus_{i=1}^{k} W_i$$

Proposition 4.2. *Let L be a Lie algebra, and V be a semisimple L-module. Then V is completely reducible.*

Proof. Let $U \leq V$ be a submodule of V. Choose simple modules $W_1, \ldots, W_k$ such that $V = \bigoplus_{i=1}^{k} W_i$. For each i, since $U \cap W_i$ is a submodule of the simple module W_i, we must have $U \cap W_i = (0)$ or $U \cap W_i = W_i$. If $U \neq V$, then there exists i_1 with $U \cap W_{i_1} = (0)$. Choose $\{i_1, \ldots, i_m\}$ maximal such that $U, W_{i_1}, \ldots, W_{i_m}$ are linearly independent. i.e. $U \cap (W_{i_1} \oplus \cdots \oplus W_{i_m}) = 0$. Set $U' = W_{i_1} \oplus \cdots \oplus W_{i_m}$. Clearly $U \oplus U'$ is a submodule of V. Thus, for any $i \notin \{i_1, \ldots, i_m\}$ we have $W_i \cap (U \oplus U')$ is a submodule of W_i. Since W_i is simple, this implies that either $W_i \cap (U \oplus U') = (0)$, or $W_i \subseteq U \oplus U'$. The former contradicts maximality of the set $\{i_1, \ldots, i_m\}$, and hence we must have $W_i \subseteq U \oplus U'$ for all $1 \leq i \leq k$. Therefore $V = U \oplus U'$. □

The converse of the above proposition is not always true. However, it is true in the case where V is finite dimensional.

Proposition 4.3. *Let L be a Lie algebra, and V be a finite dimensional completely reducible L-module. Then V is semisimple.*

Proof. We apply induction on $\dim V$. If $\dim V = 1$ then V is simple, hence semisimple. Suppose the proposition is true for any completely reducible module W with $\dim W < \dim V$. If V is not simple, choose a proper submodule $W \leq V$. Since

V is completely reducible, there is a submodule $W' \leq V$ such that $V = W \oplus W'$. By Proposition 4.1, the modules W and W' are completely reducible. Further, since $\dim W < \dim V$ and $\dim W' < \dim V$, we can find simple modules $W_1, \ldots, W_k \leq W$ and $W'_1, \ldots, W'_m \leq W'$ such that

$$W = \bigoplus_{i=1}^{k} W_i$$

and

$$W' = \bigoplus_{i=1}^{m} W'_i$$

Hence

$$V = \bigoplus_{i=1}^{k} W_i \oplus \bigoplus_{i=1}^{m} W'_i$$

□

Since for any vector space V, the endomorphisms *End* V form an associative algebra under the operation of composition, we can give a similar definition for representations of associative algebras.

Definition 4.6. *Let A be an associative algebra. A* **representation** *of the algebra A is a pair (ϕ, V), where V is a vector space over $\mathbb{C}$, and*

$$\phi : L \longrightarrow End\, V$$

*is an associative algebra homomorphism. Once again, the vector space V is called an A-***module***. The action of A on V, submodules, sub-representations, simple modules, irreducible representations and completely reducible representations are defined for associative algebras analogous to their definitions for Lie algebras.*

Definition 4.7. *Let A be an associative algebra, and V be an A-module. An* **ascending chain** *of submodules is a finite sequence $\mathcal{C} = (W_0, \ldots, W_k)$ consisting of submodules of V such that*

$$W_0 \subset W_1 \subset \cdots \subset W_k$$

where all inclusions are proper. The number k is called the length of the ascending chain $\mathcal{C}$, and is denoted by $\mathfrak{l}(\mathcal{C})$.

Definition 4.8. *Let A be an associative algebra, and V be an A-module. Define the length of V to be the (possibly infinite) value*

$$\text{Length}(V) = \sup\{k \in \mathbb{Z}_{>0} \mid \mathfrak{l}(\mathcal{C}) = k \text{ for some ascending chain } \mathcal{C} \text{ of submodules of } V\}$$

Theorem 4.1. *(Jordan-Hölder) Let A be an associative algebra, and V be an A-module. If $\text{Length}(V) = k < \infty$ then there exists an ascending chain*

$$W_0 \subset W_1 \subset \cdots \subset W_k$$

such that $W_0 = (0)$, $W_k = V$ and for each $1 \leq i \leq k$ the module W_i/W_{i-1} is simple. Such a sequence is called a **composition series** *of V. Further, if $W_0 \subset \cdots \subset W_k$ and $U_0 \subset \cdots \subset U_k$ are two composition series of V, then the semisimple modules*

$$U = \bigoplus_{i=1}^{k} U_i/U_{i-1} \quad \text{and} \quad W = \bigoplus_{i=1}^{k} W_i/W_{i-1}$$

are equivalent.

Proof. (See Theorem 3.5, and page 156 in [11]) □

4.2 The Universal Enveloping Algebra

Definition 4.9. *Let L be a Lie algebra. A* **universal enveloping algebra** *of L is an associative algebra $\mathfrak{U}$, with a map $\sigma : L \longrightarrow \mathfrak{U}$ satisfying*

$$\sigma([x,y]) = \sigma(x)\sigma(y) - \sigma(y)\sigma(x)$$

such that for any associative algebra A having a map $\psi : L \longrightarrow A$ satisfying

$$\psi([x,y]) = \psi(x)\psi(y) - \psi(y)\psi(x)$$

there exists a unique associative algebra homomorphism $\psi' : \mathfrak{U} \longrightarrow A$ such that $\psi = \psi' \circ \sigma$.

The following proposition shows uniqueness of the universal enveloping algebra.

Proposition 4.4. *Let L be a Lie algebra. If $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are both universal enveloping algebras, with maps $\sigma_1 : L \longrightarrow \mathfrak{U}_1$ and $\sigma_2 : L \longrightarrow \mathfrak{U}_2$ respectively, then there is an associative algebra isomorphism $\varphi : \mathfrak{U}_1 \longrightarrow \mathfrak{U}_2$ such that $\sigma_2 = \varphi \circ \sigma_1$. Hence $\mathfrak{U}_1 \cong \mathfrak{U}_2$ and σ_2 is simply the image of σ_1 under this isomorphism.*

Proof. Since $\mathfrak{U}_1$ is a universal enveloping algebra, we can find an algebra homomorphism $\varphi_1 : \mathfrak{U}_1 \longrightarrow \mathfrak{U}_2$ such that

$$\sigma_2 = \varphi_1 \circ \sigma_1$$

We need only show that φ_1 is a bijection. Since $\mathfrak{U}_2$ is also a universal enveloping algebra, we choose the algebra homomorphism $\varphi_2 : \mathfrak{U}_2 \longrightarrow \mathfrak{U}_1$ such that

$$\sigma_1 = \varphi_2 \circ \sigma_2$$

Then

$$\varphi_1 \circ \varphi_2 \circ \sigma_2 = \varphi_1 \circ \sigma_1 = \sigma_2$$

and

$$\varphi_2 \circ \varphi_1 \circ \sigma_1 = \varphi_2 \circ \sigma_2 = \sigma_1$$

However, $1_{\mathfrak{U}_1} : \mathfrak{U}_1 \longrightarrow \mathfrak{U}_1$ is the unique homomorphism such that

$$1_{\mathfrak{U}_1} \circ \sigma_1 = \sigma_1$$

likewise, $1_{\mathfrak{U}_2} : \mathfrak{U}_2 \longrightarrow \mathfrak{U}_2$ is the unique homomorphism such that

$$1_{\mathfrak{U}_2} \circ \sigma_2 = \sigma_2$$

Thus $\varphi_2 \circ \varphi_1 = 1_{\mathfrak{U}_1}$ and $\varphi_1 \circ \varphi_2 = 1_{\mathfrak{U}_2}$. Therefore $\varphi_2 = \varphi_1^{-1}$, and hence φ_1 is an isomorphism. □

Definition 4.10. *Let V and W be vector spaces over $\mathbb{C}$. The* **tensor product** *of V and W, denoted $V \otimes W$ is the vector space spanned by all vectors of the form $v \otimes w$ with $v \in V$ and $w \in W$, such that the following properties hold:*

i) $(av_1 + bv_2) \otimes w = av_1 \otimes w + bv_2 \otimes w$

ii) $v \otimes (aw_1 + bw_2) = av \otimes w_1 + bv \otimes w_2$

for all $a, b \in \mathbb{C}$, $v_1, v_2 \in V$ *and* $w_1, w_2 \in W$. *We can extend this definition to the tensor product of any finite number of vector spaces, by associativity. i.e. If* V_1, V_2, V_3 *are vector spaces, then*

$$V_1 \otimes V_2 \otimes V_3 = V_1 \otimes (V_2 \otimes V_3) = (V_1 \otimes V_2) \otimes V_3$$

Notice that the linear property of the tensor product implies that whenever $\mathfrak{B}_V$ is a basis for V, and $\mathfrak{B}_W$ is a basis for W, then

$$\mathfrak{B}_{V\otimes W} = \{v \otimes w \mid v \in \mathfrak{B}_V \text{ and } w \in \mathfrak{B}_W\}$$

is a basis for the tensor product $V \otimes W$. In particular, we have that if $\dim V = n$ and $\dim W = m$, then $\dim V \otimes W = nm$.

Definition 4.11. *Let* V *be a vector space over* $\mathbb{C}$. *Let* $T^n = V \otimes V \otimes \cdots \otimes V$ *(n times), with the convention that* $T^0 = \mathbb{C}$. *The* **tensor algebra** *of* V *is defined to be*

$$T = \bigoplus_{n=0}^{\infty} T^n$$

where the product in T *is tensor multiplication. i.e.* $vw = v \otimes w$ *for all* $v, w \in T$.

T is an associative algebra, due to the associative property of the tensor product.

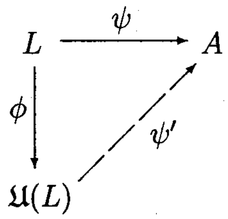
Definition 4.12. *Let* L *be a Lie algebra, and* T *be the tensor algebra of* L. *Let* I *be the two sided ideal of* T *generated by* $\{x \otimes y - y \otimes x - [x, y] \mid x, y \in L\}$. *Define*

$$\mathfrak{U}(L) = T/I$$

Proposition 4.5. *(Universal Mapping Property) Let* L *be a Lie algebra, and* A *be an associative algebra. Let* $\phi : L \longrightarrow \mathfrak{U}(L)$ *be the canonical embedding of* L *into* $\mathfrak{U}(L)$. *If* $\psi : L \longrightarrow A$ *is a linear map satisfying the property*

$$\psi([x, y]) = \psi(x)\psi(y) - \psi(y)\psi(x)$$

for all $x, y \in L$, then there exists a unique associative algebra homomorphism ψ' : $\mathfrak{U}(L) \to A$ such that $\psi'(1) = 1$, and $\psi = \psi' \circ \phi$. i.e.



Proof. (See Lemma 2.1.3 in [6]) □

Hence $\mathfrak{U}(L)$ is a universal enveloping algebra of L. Since such an algebra is unique up to isomorphism, from here on we will call $\mathfrak{U}(L)$ **the universal enveloping algebra** of L.

To simplify notation, when working with the universal enveloping algebra, we will neglect the tensor signs. i.e. $x \otimes y$ will be denoted simply by xy.

Theorem 4.2. *(Poincaré-Birkhoff-Witt) Let L be a Lie algebra, and $\mathfrak{U}(L)$ be the universal enveloping algebra of L. If $\{x_1, x_2, \ldots, x_n\}$ is an ordered basis for the vector space L, then $\{x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \mid m_i \in \mathbb{Z}_{\geq 0}\}$ is a basis of $\mathfrak{U}(L)$.*

Proof. (See Theorem 2.1.11 in [6]) □

Notice that the universal mapping property implies that if (ψ, V) is a Lie algebra representation, i.e. $\psi : L \to gl(V)$ is a Lie algebra homomorphism, then there is an associative algebra representation $\psi' : \mathfrak{U}(L) \to End\ V$, extending ψ. Hence every L-module V is also a $\mathfrak{U}(L)$-module, where the action of $\mathfrak{U}(L)$ on V is an extension of the action of L on V. Conversely, if $\phi : \mathfrak{U}(L) \to End\ V$ is a representation of $\mathfrak{U}(L)$ then, considering the restriction of ϕ to L, we have for any $x, y \in L$,

$$\phi|_L([x,y]) = \phi(xy - yx) = \phi(x)\phi(y) - \phi(y)\phi(x) = \phi|_L(x)\phi|_L(y) - \phi|_L(y)\phi|_L(x)$$

and hence $\phi|_L$ is a Lie algebra representation of L. Therefore any $\mathfrak{U}(L)$-module V is also an L-module under the same action.

We will also make use of two representations under which $\mathfrak{U}(L)$ is itself an L-module. The first being the **left regular representation**, denoted $R : L \longrightarrow gl(\mathfrak{U}(L))$ and given by

$$R(x)u = xu$$

for any $x \in L$, and $u \in \mathfrak{U}(L)$. This is in fact a representation, since for any $x, y \in L$ and $u \in \mathfrak{U}(L)$ we have

$$R([x,y])u = [x,y]u = (xy - yx)u = xyu - yxu = R(x)R(y)u - R(y)R(x)u$$

hence $R([x,y]) = R(x)R(y) - R(y)R(x)$. The second representation of interest is the **adjoint representation**, again denoted $ad : L \longrightarrow gl(\mathfrak{U}(L))$, and given by

$$ad(x)u = xu - ux$$

This has already been shown to be a representation, and further, for each $x, y \in L$ we have $ad(x)|_L(y) = xy - yx = [x,y] = ad_x(y)$. Therefore when the action is restricted to L, this definition coincides with our previous definition of the adjoint action of L on itself. We can thus continue to denote $ad(x)$ simply by ad_x, without fear of ambiguity.

The following proposition shows that this correspondence between L-modules and $\mathfrak{U}(L)$-modules also preserves simplicity and complete reducibility.

Proposition 4.6. *Let L be a Lie algebra. Then V is a simple L-module if and only if V is a simple $\mathfrak{U}(L)$-module, and V is a completely reducible L-module if and only if V is a completely reducible $\mathfrak{U}(L)$-module.*

Proof. Let $\mathfrak{B}_L = \{x_1, x_2, \ldots, x_n\}$ be an ordered basis for L, and

$$\mathfrak{B}_{\mathfrak{U}(L)} = \{x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \mid m_i \in \mathbb{Z}_{\geq 0}\}$$

be the corresponding Poincaré-Birkhoff-Witt basis for $\mathfrak{U}(L)$. Let (ψ, V) be a representation of L. Then (ψ', V) is a representation of $\mathfrak{U}(L)$, where ψ' is an extension of ψ. If $W \leq V$ is a submodule of V under ψ, then for any $x_1^{m_1} \ldots x_n^{m_n} \in \mathfrak{B}_{\mathfrak{U}(L)}$ and

$w \in W$, we have

$$\psi'(x_1^{m_1} \dots x_n^{m_n})(w) = \psi'(x_1)^{m_1} \dots \psi'(x_n)^{m_n}(w)$$
$$= \psi(x_1)^{m_1} \dots \psi(x_n)^{m_n}(w) \in W$$

Therefore W is a submodule of V under ψ'. Conversely, if W is a submodule of V under ψ', then for any $x \in L$ and $w \in W$, we have

$$\psi(x)(w) = \psi'(x)(w) \in W$$

Therefore W is a submodule of V under ψ if and only if W is a submodule of V under ψ'. It follows immediately that V is simple under ψ if and only if V is simple under ψ'. Complete reducibility follows as well. Indeed, if V is completely reducible under the action of L, then for any submodule W of V under the action of $\mathfrak{U}(L)$, W is also submodule of V under the action of L. In this case, there exists a submodule W' of V under the action of L and hence also under $\mathfrak{U}(L)$ such that $V = W \oplus W'$. The argument for the converse is identical. □

4.3 Induced Representations

We now give a useful method for constructing a representation of an associative algebra A, given a representation of a subalgebra $B \leq A$.

Let A be an associative algebra, and $B \leq A$ be a subalgebra of A. Let V be a B-module. Let W be the sub-vector space of the vector space $A \otimes V$ spanned by the set $\{(ab) \otimes v - a \otimes (bv) \mid a \in A, b \in B \textit{ and } v \in V\}$. Define the vector space $A \otimes_B V$ by

$$A \otimes_B V = (A \otimes V)/W$$

Let $\mathfrak{B}_{A \otimes_B V}$ be a basis of $A \otimes_B V$ consisting of cosets of the form $a \otimes v + W$, with $a \in A$ and $v \in V$.

Definition 4.13. *Let A be an associative algebra, and $B \leq A$ be a subalgebra of A. Let V be a B-module with action given by $\phi : B \longrightarrow End V$. Define the* **induced representation** *of V from B to A to be the pair $(\phi_B^A, A \otimes_B V)$ where*

$$\phi_B^A(x)(a \otimes v + W) = (xa) \otimes v + W$$

for all $a \otimes v + W \in \mathfrak{B}_{A \otimes_B V}$*, and extending linearly. In this case,* $A \otimes_B V$ *is called the* **induced module** *of* V *from* B *to* A*.*

For simplicity, we denote the cosets $a \otimes v + W \in A \otimes_B V$ by any choice of representative $a \otimes v$, with $a \in A$ and $v \in V$, under the condition that for any $a \in A$, $b \in B$ and $v \in V$, we have $(ab) \otimes v = a \otimes (bv)$.

5 Representations of Semisimple Lie Algebras

In this section, we restrict our attention to representations of semisimple Lie algebras over $\mathbb{C}$. Unless otherwise mentioned any Lie algebra L in this section is assumed to be semisimple, and over the complex numbers. Most of the results given in this section can be found in [8].

5.1 Weight Space Decomposition

Definition 5.1. *Let* L *be a semisimple Lie algebra with Cartan subalgebra* H*. Let* V *be an* L*-module. For each* $\lambda \in H^*$*, define the* λ-**weight space** V_λ *of* V *to be*

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in H\}$$

The elements $v \in V_\lambda$ *are called* **weight vectors** *having* **weight** *equal to* λ*. The* **support** *of the module* V*, denoted* Supp V *is defined to be*

$$\text{Supp } V = \{\lambda \in H^* \mid V_\lambda \neq (0)\}$$

i.e. the set of all linear functionals corresponding to non-zero weight spaces in V*.*

Proposition 5.1. *Let* L *be a semisimple Lie algebra with Cartan subalgebra* H*. Let* V *be an* L*-module. If* $v_1, \ldots, v_n \in V$ *are non-zero weight vectors having distinct weights* $\lambda_1, \ldots, \lambda_n \in H^*$ *respectively, then* $v_1, \ldots, v_n$ *are linearly independent.*

Proof. We apply induction on n. The result is trivial if $n = 1$. Assume $v_1, \ldots, v_{n-1}$ are linearly independent, and suppose

$$v_n = \sum_{i=1}^{n-1} a_i v_i$$

for some $a_i \in \mathbb{C}$. Then, for every $h \in H$, we have that

$$\sum_{i=1}^{n-1} a_i \lambda_i(h) v_i = \sum_{i=1}^{n-1} a_i h v_i = h v_n = \lambda_n(h) v_n = \sum_{i=1}^{n-1} \lambda_n(h) v_i$$

Therefore

$$\sum_{i=1}^{n-1} a_i (\lambda_i - \lambda_n)(h) v_i = 0$$

due to linear independence of $v_1, \ldots, v_{n-1}$, we must have $a_i(\lambda_i - \lambda_n)(h) = 0$ for all $1 \leq i \leq n-1$ and all $h \in H$. Thus for each i, either $a_i = 0$ or $\lambda_i = \lambda_n$. Since the λ_i were assumed to be distinct, we must have that $a_i = 0$ for all $1 \leq i \leq n-1$. This implies that $v_n = 0$, which is a contradiction. □

In particular, the previous proposition implies that for any L-module V, the sum

$$\sum_{\lambda \in \text{Supp } V} V_\lambda$$

is in fact a direct sum.

Definition 5.2. *Let L be a semisimple Lie algebra, and V be an L-module. Then V is said to admit a* **weight space decomposition** *provided*

$$V = \bigoplus_{\lambda \in \text{Supp } V} V_\lambda$$

Proposition 5.2. *Let L be a semisimple Lie algebra with root system Δ. Let V be an L-module. Then the following hold:*

i) For each $\alpha \in \Delta$, if $v \in V_\lambda$ then $x_\alpha v \in V_{\lambda+\alpha}$.
ii) The sum $\bigoplus_{\lambda \in \text{Supp } V} V_\lambda$ is a submodule of V.
iii) If V is finite dimensional, then V admits a weight space decomposition.

Proof. (See Lemma 20.1 in [8]) □

Proposition 5.3. *Let L be a semisimple Lie algebra with Cartan subalgebra H and root system Δ. Then $\mathfrak{U}(L)$, under the adjoint representation, admits a weight space decomposition, and* $\text{Supp}\,\mathfrak{U}(L) = \text{span}_{\mathbb{Z}}\Delta$.

Proof. If $v_1, v_2 \in \mathfrak{U}(L)$ are weight vectors with weights γ_1 and γ_2 respectively, then for any $h \in H$ we have

$$\begin{aligned} ad_h(v_1v_2) &= hv_1v_2 - v_1v_2h \\ &= hv_1v_2 - v_1hv_2 + v_1hv_2 - v_1v_2h \\ &= ad_h(v_1)v_2 + v_1ad_h(v_2) \\ &= (\gamma_1(h) + \gamma_2(h))v_1v_2 \end{aligned}$$

Therefore v_1v_2 is a weight vector with weight $\gamma_1+\gamma_2$. Applying induction, we see that for any sequence $v_1, \ldots, v_k \in \mathfrak{U}(L)$ of weight vectors with respective weights $\gamma_1, \ldots, \gamma_k$, the vector $\prod_{i=1}^k v_i$ is a weight vector with weight $\sum_{i=1}^k \gamma_i$. Let $\Delta^+ = \{\beta_1, \ldots, \beta_m\}$ be the positive roots of Δ with respect to $\Delta^{++} = \{\alpha_1, \ldots, \alpha_n\}$. Then by the Poincaré-Birkhoff-Witt theorem,

$$\mathfrak{B}_{\mathfrak{U}(L)} = \{x_{-\beta_1}^{r_1} \ldots x_{-\beta_m}^{r_m} h_{\alpha_1}^{s_1} \ldots h_{\alpha_n}^{s_n} x_{\beta_1}^{t_1} \ldots x_{\beta_m}^{t_m} \mid r_i, s_j, t_i \in \mathbb{Z}_{\geq 0} \text{ for all } i, j\}$$

is a basis of $\mathfrak{U}(L)$. For any $h \in H$, and any $\beta \in \Delta$, we have $ad_h(x_\beta) = \beta(h)x_\beta$, and hence each x_β for $\beta \in \Delta$ is a weight vector with weight β. Further, since H is abelian, the vectors h_{α_i} are weight vectors with weight equal to 0. Therefore, for any choice of $r_i, s_j, t_i \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, the vector $x_{-\beta_1}^{r_1} \ldots x_{-\beta_m}^{r_m} h_{\alpha_1}^{s_1} \ldots h_{\alpha_n}^{s_n} x_{\beta_1}^{t_1} \ldots x_{\beta_m}^{t_m}$ is a weight vector with weight

$$\sum_{i=1}^{m} (t_i - r_i)\beta_i$$

It is clear that any possible $\mathbb{Z}$-linear combination of roots $\beta \in \Delta$ can be formed by such a sum. Further, if $v \in \mathfrak{U}(L)$ is a weight vector with weight $\gamma \notin span_{\mathbb{Z}}\Delta$ then by Proposition 5.1, v is linearly independent of all vectors in $\mathfrak{B}_{\mathfrak{U}(L)}$, which contradicts the fact that $\mathfrak{B}_{\mathfrak{U}(L)}$ is a basis of $\mathfrak{U}(L)$. □

We now introduce the integral root lattice which, as we have just seen, is the set of weights occurring in the universal enveloping algebra.

Definition 5.3. *Let L be a semisimple Lie algebra with root system Δ, having positive roots Δ^+. The* **integral root lattice**, *denoted Q, is defined to be*

$$Q = \{\sum_{\alpha \in \Delta} k_\alpha \alpha \mid k_\alpha \in \mathbb{Z}\}$$

Define

$$Q^+ = \{\sum_{\alpha\in\Delta^+} k_\alpha \alpha \mid k_\alpha \in \mathbb{Z}_{\geq 0}\}$$

and

$$Q^- = \{\sum_{\alpha\in\Delta^+} k_\alpha \alpha \mid k_\alpha \in \mathbb{Z}_{\leq 0}\}$$

Definition 5.4. *Let L be a semisimple Lie algebra with root system Δ, and Δ^+ be the positive roots with respect to a fixed base. Let V be an L-module. A weight vector $v^+ \in V_\lambda$ is called a* **maximal vector** *if $x_\alpha v^+ = 0$ for all $\alpha \in \Delta^+$. The module V is called a* **highest weight module** *of weight λ if V is generated by v^+. In this case, λ is called the* **highest weight** *of V.*

Notice that this definition depends on the choice of base Δ^{++}.

Proposition 5.4. *Let L be a semisimple Lie algebra with root system Δ, and $\Delta^+ = \{\beta_1, \dots, \beta_m\}$ be the positive roots with respect to a fixed base $\Delta^{++} = \{\alpha_1, \dots, \alpha_n\}$. Let V be a highest weight L-module of weight λ, with maximal vector $v^+ \in V_\lambda$. Then the following hold:*

i) $V = span_{\mathbb{C}}\{x^{k_1}_{-\beta_1} \cdots x^{k_m}_{-\beta_m} v^+ \mid k_i \in \mathbb{Z}_{\geq 0} \text{ for } 1 \leq i \leq m\}$.
ii) If $\mu \in Supp\ V$ then $\mu = \lambda - \sum_{i=1}^n k_i\alpha_i$ for some $k_i \in \mathbb{Z}_{\geq 0}$.
iii) $\dim V_\lambda = 1$, *and for each* $\mu \in \operatorname{Supp} V$ *we have* $\dim V_\mu < \infty$.

Proof. (See Theorem 20.2 in [8]) □

In particular, if v^+ has weight λ, then each vector of the form $x^{k_1}_{-\beta_1} \dots x^{k_m}_{-\beta_m} v^+$ is a weight vector, with weight equal to $\lambda - \sum_{i=1}^m k_i\beta_i$. Thus, the previous proposition implies that every highest weight module admits a weight space decomposition. Further, all weights lie in the coset $\lambda + Q^-$.

Proposition 5.5. *Let L be a semisimple Lie algebra with Cartan subalgebra H. If V is a simple L-module admitting a weight space decomposition, then* $\operatorname{Supp} V \subseteq \lambda + Q$ *for some $\lambda \in H^*$.*

Proof. Let Δ be the root system of L. Let $\lambda \in \operatorname{Supp} V$ and let

$$U = \bigoplus_{\gamma \in \operatorname{Supp} V \cap (\lambda + Q)} V_\gamma$$

We will show that U is a submodule of V. To this end, choose a basis $\mathfrak{B}_U$ of U consisting of weight vectors. Let $u \in \mathfrak{B}_U$ have weight γ, hence $\gamma \in \lambda + Q$. Then for any $\alpha \in \Delta$ we have that either $x_\alpha u = 0$ or $x_\alpha u$ is a weight vector with weight $\gamma + \alpha$. Indeed, for any $h \in H$ we have

$$hx_\alpha u = ad_h(x_\alpha)u + x_\alpha hu = \alpha(h)x_\alpha u + \gamma(h)x_\alpha u = (\alpha + \gamma)(h)x_\alpha u$$

Since $\alpha + \gamma \in \gamma + Q = \lambda + Q$, we have $x_\alpha u \in U$. Since $\{x_\alpha \mid \alpha \in \Delta\}$ generates L, we have that U is a submodule. Since V is simple, we must have $V = U$. □

Proposition 5.6. *Let L be a semisimple Lie algebra with Cartan subalgebra H. Let Δ^{++} be a base for the root system of L with respect to H. For every $\lambda \in H^*$ there exists a unique simple highest weight L-module of weight λ, with respect to Δ^{++}.*

Proof. (See Theorem 20.3A, and Theorem 20.3B in [8]) □

The previous proposition allows us to make the following definition.

Definition 5.5. *Let L be a semisimple Lie algebra with Cartan subalgebra H, and fixed base Δ^{++} for the root system of L. For each $\lambda \in H^*$, denote the simple highest weight L-module of weight λ by $L(\lambda)$.*

Notice that the previous definition depends on the choice of base Δ^{++}. When the base is implicit, we will denote simple λ-highest weight L-module by $L(\lambda)$, however, if we wish to specify a particular base for Δ, say B, we will denote the simple λ-highest weight module relative to B by $L_B(\lambda)$.

Proposition 5.7. *Let L be a semisimple Lie algebra with Cartan subalgebra H. Let $\Delta^{++} = \{\alpha_1, \ldots, \alpha_n\}$ be a base for the root system of L. If V is a finite dimensional simple L-module then $V = L(\lambda)$ for some $\lambda \in H^*$.*

Proof. Let Δ^+ be the positive roots with respect Δ^{++}. By Proposition 5.2 , since V is finite dimensional, V admits a weight space decomposition. Also, since V is finite dimensional, we must have that $\operatorname{Supp} V$ is a finite set. If $\lambda_0 \in \operatorname{Supp} V$, then the set

$$\{\lambda_0 + \sum_{i=1}^{n} k_i \alpha_i \in \operatorname{Supp} V \mid k_i \in \mathbb{Z}_{\geq 0} \text{ for each } i\}$$

is also finite. We can therefore choose $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}$ such that

$$\lambda = \lambda_0 + \sum_{i=1}^{n} m_i \alpha_i \in \operatorname{Supp} V$$

and for any sequence $(k_1, \ldots, k_n) \in \mathbb{Z}_{\geq 0}$ with $(k_1, \ldots, k_n) \neq (m_1, \ldots, m_n)$ and $k_i \geq m_i$ for all i, we have

$$\lambda_0 + \sum_{i=1}^{n} k_i \alpha_i \notin \operatorname{Supp} V$$

Let $v^+ \in V_\lambda$ with $v^+ \neq 0$. Let $\beta \in \Delta^+$. Then $\beta = \sum_{i=1}^{n} b_i \alpha_i$ for some $b_i \in \mathbb{Z}_{\geq 0}$. Therefore $x_\beta v^+$ has weight equal to $\lambda_0 + \sum_{i=1}^{n} (m_i + b_i)\alpha_i$. Since $\beta \neq 0$ we have $(m_1 + b_1, \ldots, m_n + b_n) \neq (m_1, \ldots, m_n)$. Further, for each i, $m_i + b_i \geq m_i$ and hence

$$\lambda_0 + \sum_{i=1}^{n} (m_i + b_i)\alpha_i \notin \operatorname{Supp} V$$

Therefore $x_\beta v^+ = 0$, which implies v^+ is a maximal vector. Since the highest weight module generated by v^+ is a submodule of V, and V is simple, we must have that V is itself generated by v^+. Therefore $V = L(\lambda)$. □

5.2 Finite Dimensional Modules

Theorem 5.1. *(Weyl) If L be a semisimple Lie algebra, and V is a non-zero, finite dimensional L-module, then V is completely reducible.*

Proof. (See Theorem 6.3 in [8]) □

Definition 5.6. *Let L be a semisimple Lie algebra, with Cartan subalgebra H having simple basis $\mathfrak{H} = \{h_1, \ldots, h_n\}$. Let $\mu \in H^*$ such that $\mu(h_i) \in \mathbb{Z}_{\geq 0}$ for all i. Then μ is called a* **dominant integral** *weight.*

Theorem 5.2. *Let L be a semisimple Lie algebra, with Cartan subalgebra H. For each $\lambda \in H^*$, the simple highest weight L-module $L(\lambda)$ is finite dimensional if and only if λ is a dominant integral weight.*

Proof. (See Theorem 21.1 and Theorem 21.2 in [8]) □

Corollary 5.1. *Let L be a semisimple Lie algebra, with Cartan subalgebra H. Every finite dimensional simple L-module is some $L(\lambda)$ where λ is a dominant integral weight.*

Proof. If V is any finite dimensional simple L-module, then by Proposition 5.7, $V = L(\lambda)$ for some $\lambda \in H^*$. Due to the previous theorem, λ must be a dominant integral weight. □

For calculation purposes, we introduce the following characterizations of dominant integral weights.

Property 5.1. *Let L be a semisimple Lie algebra, with Cartan subalgebra H having simple basis $\mathfrak{H} = \{h_1, \ldots, h_n\}$. Let $\mathfrak{F} = \{\omega_1, \ldots, \omega_n\}$ be the fundamental basis of H^*. Then $\lambda \in H^*$ is a dominant integral weight if and only if*

$$\lambda = \sum_{i=1}^{n} b_i \omega_i$$

with each $b_i \in \mathbb{Z}_{\geq 0}$.

Proof. Set $\lambda = \sum_{i=1}^{n} b_i \omega_i$ for some $b_i \in \mathbb{C}$. Then

$$\lambda(h_i) = \sum_{j=1}^{n} b_j \omega_j(h_i) = \sum_{j=1}^{n} b_j \delta_{ij} = b_i$$

Hence λ is a dominant integral weight if and only if $b_i \in \mathbb{Z}_{\geq 0}$ for all i. □

In the case where the algebra is C_n, it will also be useful for us to consider dominant integral weights using the epsilon basis $\mathfrak{E} = \{\epsilon_1, \ldots, \epsilon_n\}$, for which we give the following characterization:

Property 5.2. *Let $\mathcal{H}$ be the Cartan subalgebra of C_n given in Section 3.7, and $\mathfrak{S} = \{\epsilon_1, \ldots, \epsilon_n\}$ be the epsilon basis of $\mathcal{H}^*$. Let $\mu \in \mathcal{H}^*$ such that $\mu = \sum_{i=1}^n a_i\epsilon_i$, with $a_i \in \mathbb{Z}$. Then μ is dominant integral if and only if $a_i \geq a_{i+1}$ for $1 \leq i \leq n-1$, and $a_n \geq 0$.*

Proof. Let $b_n = a_n$, and let $b_i = a_i - a_{i+1}$ for $1 \leq i \leq n-1$. Then

$$\sum_{i=1}^{n} b_i\omega_i = \sum_{i=1}^{n}\sum_{j=1}^{i} b_i\epsilon_j = \sum_{j=1}^{n}\left(\sum_{i=j}^{n} b_i\right)\epsilon_j = \sum_{j=1}^{n} a_j\epsilon_j$$

since for each j, we have $\sum_{i=j}^n b_i = a_n + \sum_{i=j}^{n-1} a_i - \sum_{i=j}^{n-1} a_{i+1} = a_j$. Therefore $\mu = \sum_{i=1}^n b_i\omega_i$, and hence μ is dominant integral if and only if $b_i \geq 0$ for all i. □

Proposition 5.8. *Let L be a semisimple Lie algebra, with Weyl group $\mathcal{W}$. Let $L(\mu)$ be a finite dimensional simple L-module, hence μ is dominant integral. Then for every $\sigma \in \mathcal{W}$, and every $\nu \in \operatorname{Supp} L(\mu)$ we have*

$$\sigma(\nu) \in \operatorname{Supp} L(\mu)$$

and

$$\dim L(\mu)_\nu = \dim L(\mu)_{\sigma(\nu)}$$

Proof. (See Theorem 21.2 in [8]) □

The previous proposition implies that for any $\sigma \in \mathcal{W}$ we have

$$\{\sigma(\nu) \mid \nu \in \operatorname{Supp} L(\mu)\} \subseteq \operatorname{Supp} L(\mu)$$

Further, if $\nu \in \operatorname{Supp} L(\mu)$ then

$$\nu = \sigma(\sigma^{-1}(\nu)) \in \{\sigma(\nu) \mid \nu \in \operatorname{Supp} L(\mu)\}$$

Therefore

$$\{\sigma(\nu) \mid \nu \in \operatorname{Supp} L(\mu)\} = \operatorname{Supp} L(\mu)$$

i.e. for every $\sigma \in \mathcal{W}$ we have $\sigma(\operatorname{Supp} L(\mu)) = \operatorname{Supp} L(\mu)$.

Recall that the algebra $sl(2, \mathbb{C})$ is spanned by $\{x_\alpha, h, x_{-\alpha}\}$ where $H = span_{\mathbb{C}}\{h\}$ is a Cartan subalgebra, and $\alpha \in H^*$ is given by $\alpha(h) = 2$. The following lemma will be used to show that in a finite dimensional module V, for any weight $\gamma \in \operatorname{Supp} V$, and any root $\alpha \in \Delta$, the weights of the form $\gamma + n\alpha$ for $n \in \mathbb{Z}$ form a connected string.

Lemma 5.1. *Let L be a semisimple Lie algebra with root system Δ, and V be a finite dimensional weight L-module. Let $\gamma \in \operatorname{Supp} V$. Let $\alpha \in \Delta$. Choose $k, l \in \mathbb{Z}_{\geq 0}$ minimal such that $\gamma + (k+1)\alpha \notin \operatorname{Supp} V$ and $\gamma - (l+1)\alpha \notin \operatorname{Supp} V$. Then $(\gamma + k\alpha)(h_\alpha) \geq 0$ and $(\gamma - l\alpha)(h_\alpha) \leq 0$.*

Proof. Let $v_0 \in V_{\gamma+k\alpha}$. Then $x_\alpha v_0 = 0$, since $\gamma + (k+1)\alpha \notin \operatorname{Supp} V$. Set $\lambda = \gamma + k\alpha$, i.e. v_0 has weight λ. Let $v_n = \frac{1}{n!}x_{-\alpha}^n v_0$ for all $n > 0$. Then for any $h \in H$, we have

$$hv_n = \frac{1}{n!}((ad_h)(x_{-\alpha}^n)v_0 + x_{-\alpha}^n hv_0) = (\lambda - n\alpha)(h)\frac{1}{n!}x_{-\alpha}^n v_0 = (\lambda - n\alpha)(h)v_n$$

Hence v_n has weight $\lambda - n\alpha$.

$$x_{-\alpha}v_n = \frac{1}{n!}x_{-\alpha}^{n+1}v_0 = (n+1)v_{n+1}$$

We claim that $x_\alpha v_n = (\lambda(h_\alpha) - n + 1)v_{n-1}$ with the convention that $v_n = 0$ for $n < 0$. Applying induction on n, we notice that

$$x_\alpha v_0 = 0 = (\lambda(h_\alpha) + 1)v_{-1}$$

assuming $x_\alpha v_{n-1} = (\lambda(h_\alpha) - n + 2)v_{n-2}$, we have that

$$\begin{aligned}
x_\alpha v_n &= \frac{1}{n!}(ad_{x_\alpha}(x_{-\alpha})x_{-\alpha}^{n-1}v_0) + \frac{1}{n!}(x_{-\alpha}x_\alpha x_{-\alpha}^{n-1}v_0) \\
&= \frac{1}{n}h_\alpha v_{n-1} + \frac{1}{n}x_{-\alpha}x_\alpha v_{n-1} \\
&= \frac{1}{n}((\lambda(h_\alpha) - 2(n-1))v_{n-1} + (\lambda(h_\alpha) - n + 2)x_{-\alpha}v_{n-2}) \\
&= \frac{1}{n}((\lambda(h) - 2(n-1))v_{n-1} + (\lambda(h_\alpha) - n + 2)(n-1)v_{n-1}) \\
&= (\lambda(h_\alpha) - n + 1)v_{n-1}
\end{aligned}$$

Notice that since $\lambda - (k+l+1)\alpha = \gamma - (l+1)\alpha \notin \operatorname{Supp} V$, we must have $v_{k+l+1} = 0$. Choose m maximal such that $v_m \neq 0$. Then $m \leq k + l$. Notice that $0 = x_\alpha v_{m+1} = (\lambda(h_\alpha) - m)v_m$ and hence $\lambda(h_\alpha) = m$ Therefore $(\gamma + k\alpha)(h_\alpha) = m \geq 0$. Further,

$$(\gamma - l\alpha)(h_\alpha) = (\lambda - (k+l)\alpha)(h_\alpha) = m - 2(k+l) \leq m - 2m = -m$$

□

Proposition 5.9. *Let L be a semisimple Lie algebra with root system Δ. Let $L(\mu)$ be a finite dimensional simple L-module, hence μ is dominant integral. Then for every $\nu \in \operatorname{Supp} L(\mu)$ and every $\alpha \in \Delta$, if $\nu + n\alpha \in \operatorname{Supp} L(\mu)$ then for every m with $0 \leq m \leq n$, we have $\nu + m\alpha \in \operatorname{Supp} L(\mu)$.*

Proof. Suppose $0 \leq m \leq n$ and $\nu + m\alpha \notin \operatorname{Supp} L(\mu)$. Since $\nu \in \operatorname{Supp} L(\mu)$, we can choose $k \in \mathbb{Z}_{\geq 0}$ minimal such that $\nu + (k+1)\alpha \notin \operatorname{Supp} L(\mu)$. Then $k < m$. Also, since $\nu + n\alpha \in \operatorname{Supp} L(\mu)$, we can choose $l \in \mathbb{Z}$ minimal such that $\nu + (n-l-1)\alpha \notin \operatorname{Supp} L(\mu)$. Then $l < n - m$. By Lemma 5.1, we have that $(\nu + k\alpha)(h_\alpha) \geq 0$ and $(\nu + n\alpha - l\alpha)(h_\alpha) \leq 0$. Thus

$$\nu(h_\alpha) + 2m > \nu(h_\alpha) + 2k = (\nu + k\alpha)(h_\alpha) \geq 0$$

and

$$\nu(h_\alpha) + 2m < \nu(h_\alpha) + 2(n-l) = (\nu + n\alpha - l\alpha)(h_\alpha) \leq 0$$

which is a contradiction. □

5.3 Admissible Modules

Definition 5.7. *Let L be a semisimple Lie algebra, and V be an L-module admitting a weight space decomposition. For each $\nu \in \operatorname{Supp} V$, we define the* **multiplicity** *of ν in V, denoted $m_V(\nu)$ to be the dimension of the ν weight space in V. i.e.*

$$m_V(\nu) = \dim V_\nu$$

In the case where $V = L(\lambda)$ for some weight λ, we will denote the multiplicity of ν in $L(\lambda)$ simply by $m_\lambda(\nu)$.

Notice that Proposition 5.4 implies that if V is a highest weight module with highest weight equal to λ, then $m_V(\lambda) = 1$ and for all $\nu \in \operatorname{Supp} V$ we have $m_V(\nu) < \infty$. Also, If μ is a dominant integral weight then Proposition 5.8 implies that for any $\nu \in \operatorname{Supp} L(\mu)$ and any $\sigma \in \mathcal{W}$, we have

$$m_\mu(\nu) = m_\mu(\sigma(\nu))$$

Definition 5.8. *Let L be a semisimple Lie algebra, and V be an L-module admitting a weight space decomposition. We say that V is* **admissible** *provided V is infinite dimensional, and there exists an $N \in \mathbb{Z}_{\geq 0}$ such that for all $\nu \in \operatorname{Supp} V$ we have $m_V(\nu) \leq N$. i.e. the dimensions of the weight spaces in V are bounded.*

Definition 5.9. *Let L be a semisimple Lie algebra, and V be an admissible L-module. Define the* **degree** *of V, denoted $\deg V$ to be the least upper bound of the weight spaces occurring in V. i.e.*

$$\deg V = \max\{m_V(\nu) \mid \nu \in \operatorname{Supp} V\}$$

Lemma 5.2. *(Mathieu) Let L be a finite dimensional simple Lie algebra, and V be an admissible L-module. Then V has finite length.*

Proof. (See Lemma 3.3 in [15]) □

Proposition 5.10. *Let L be a finite dimensional simple Lie algebra, and V be a completely reducible admissible L-module. Then V is semisimple.*

Proof. Due to the previous lemma, the length of V is finite. We apply induction on the length of V. If $\text{Length}(V) = 1$, then V is simple, and hence semisimple. Assume the result is true for any admissible module W with $\text{Length}(W) < \text{Length}(V)$. If V is not simple, choose $W < V$ to be a proper submodule of V, and choose a submodule $W' < V$ such that $V = W \oplus W'$. For any ascending chain $W_0 \subset W_1 \subset \cdots \subset W_k$ of submodules of W, we have that

$$W_0 \subset W_1 \subset \cdots \subset W_k \subset V$$

is an ascending chain of submodules of V. Hence $\text{Length}(W) < \text{Length}(V)$. Similarly, $\text{Length}(W') < \text{Length}(V)$. We can therefore find simple modules $W_1, \ldots, W_k \leq W$ and $W'_1, \ldots, W'_m \leq W'$ such that

$$W = \bigoplus_{i=1}^{k} W_i$$

and

$$W' = \bigoplus_{i=1}^{m} W'_i$$

Hence

$$V = \bigoplus_{i=1}^{k} W_i \oplus \bigoplus_{i=1}^{m} W_i'$$

□

In Section 7, following the work of Mathieu in [15], we give a complete characterization of all simple admissible highest weight C_n-modules.

5.4 Verma Modules

For a semisimple Lie algebra L, with Cartan subalgebra H and root system Δ, with positive roots Δ^+ and negative roots Δ^-, we let

$$L^+ = \bigoplus_{\alpha \in \Delta^+} L_\alpha$$

and

$$L^- = \bigoplus_{\alpha \in \Delta^-} L_\alpha$$

Hence $L = H \oplus L^+ \oplus L^-$. Further, since $[L_\alpha, L_\beta] = L_{\alpha+\beta}$ whenever α, β and $\alpha+\beta \in \Delta$, it is clear that L^+ and L^- are Lie subalgebras of L. Let $\hat{L}^+ = H \oplus L^+$. Then $\hat{L}^+$ is also a Lie subalgebra of L.

For each $\lambda \in H^*$, define the one dimensional $\hat{L}^+$ representation $(\Psi_\lambda, \mathbb{C})$ where $\Psi_\lambda : \hat{L}^+ \to gl_1(\mathbb{C})$ is the linear map defined as follows: For all $c \in \mathbb{C}$,

$$\Psi_\lambda(h)(c) = \lambda(h)c$$

for all $h \in H$, and

$$\Psi_\lambda(x)(c) = 0$$

for all $x \in L^+$.

Ψ_λ indeed defines a representation, since if $y_1, y_2 \in \hat{L}^+$ then $y_1 = h_1 + x_1$ and $y_2 = h_2 + x_2$ for some $h_1, h_2 \in H$ and $x_1, x_2 \in L^+$. Therefore, for all $c \in \mathbb{C}$, we have

$$\begin{aligned}\Psi_\lambda([y_1, y_2])(c) &= \Psi_\lambda([h_1 + x_1, h_2 + x_2])(c) \\ &= \Psi_\lambda([h_1, h_2])(c) + \Psi_\lambda([h_1, x_2])(c) + \Psi_\lambda([x_1, h_2])(c) + \Psi_\lambda([x_1, x_2])(c) \\ &= 0\end{aligned}$$

since $[h_1, h_2] = 0$, and $[h_1, x_2], [x_1, h_2], [x_1, x_2] \in L^+$. Also,

$$\begin{aligned}(\Psi_\lambda(y_1)\Psi_\lambda(y_2) - \Psi_\lambda(y_2)\Psi_\lambda(y_1))(c) &= \Psi_\lambda(h_1 + x_1)\Psi_\lambda(h_2 + x_2)(c) \\ &\quad - \Psi_\lambda(h_2 + x_2)\Psi_\lambda(h_1 + x_1)(c) \\ &= \Psi_\lambda(h_1)\Psi_\lambda(h_2)(c) + \Psi_\lambda(h_1)\Psi_\lambda(x_2)(c) \\ &\quad + \Psi_\lambda(x_1)\Psi_\lambda(h_2)(c) + \Psi_\lambda(x_1)\Psi_\lambda(x_2)(c) \\ &\quad - \Psi_\lambda(h_2)\Psi_\lambda(h_1)(c) - \Psi_\lambda(x_2)\Psi_\lambda(h_1)(c) \\ &\quad - \Psi_\lambda(h_2)\Psi_\lambda(x_1)(c) - \Psi_\lambda(x_2)\Psi_\lambda(x_1)(c) \\ &= \lambda(h_1)\lambda(h_2)c + 0 + \lambda(h_2)(0) + 0 \\ &\quad - \lambda(h_2)\lambda(h_1)c - \lambda(h_1)(0) - 0 - 0 \\ &= 0\end{aligned}$$

Thus $\Psi_\lambda([y_1, y_2]) = \Psi_\lambda(y_1)\Psi_\lambda(y_2) - \Psi_\lambda(y_2)\Psi_\lambda(y_1)$ for all $y_1, y_2 \in \hat{L}^+$. Due to the universal mapping property, the $\hat{L}^+$-module V is also a $\mathfrak{U}(\hat{L}^+)$-module under the same action. This enables us to induce the following λ-highest weight L-module.

Definition 5.10. *Let L be a semisimple Lie algebra with Cartan subalgebra H. For each $\lambda \in H^*$, define the* **Verma module** *with highest weight λ, denoted $M(\lambda)$ to be the induced module*

$$M(\lambda) = \mathfrak{U}(L) \otimes_{\mathfrak{U}(\hat{L}^+)} \mathbb{C}$$

where $\mathfrak{U}(\hat{L}^+)$ acts on $\mathbb{C}$ according to Ψ_λ.

Notice that $M(\lambda)$ is a λ-highest weight module, with maximal vector $1 \otimes 1$.

Proposition 5.11. *Let L be a semisimple Lie algebra with Cartan subalgebra H. Then for each $\lambda \in H^*$, viewing $M(\lambda)$ as a $\mathfrak{U}(L^-)$ module, we have*

$$M(\lambda) \simeq \mathfrak{U}(L^-)$$

where $\mathfrak{U}(L^-)$ is under the left regular representation.

Proof. Let $\Delta^{++} = \{\alpha_1, \ldots, \alpha_n\}$ be a base for the roots system Δ of L. Let $\Delta^+ = \{\beta_1, \ldots, \beta_m\}$ be the set of positive roots with respect to Δ^{++}. By the Poincaré-Birkhoff-Witt theorem, we can choose a basis of $\mathfrak{U}(L^-)$ given by

$$\mathfrak{B}_{L^-} = \{x_{-\beta_1}^{r_1} \ldots x_{-\beta_m}^{r_m} \mid r_i \in \mathbb{Z}_{\geq 0} \text{ for } 1 \leq i \leq m\}$$

and a basis of $\mathfrak{U}(L)$ given by

$$\mathfrak{B}_{\mathfrak{U}(L)} = \{x_{-\beta_1}^{r_1} \dots x_{-\beta_m}^{r_m} h_{\alpha_1}^{s_1} \dots h_{\alpha_n}^{s_n} x_{\beta_1}^{t_1} \dots x_{\beta_m}^{t_m} \mid r_i, s_j, t_i \in \mathbb{Z}_{\geq 0} \text{ for all } i, j\}$$

Define the map $\psi : \mathfrak{U}(L^-) \to M(\lambda)$ by

$$\psi(x_{-\beta_1}^{r_1} \dots x_{-\beta_m}^{r_m}) = x_{-\beta_1}^{r_1} \dots x_{-\beta_m}^{r_m} \otimes 1$$

for each choice of $r_1, \dots, r_m \in \mathbb{Z}_{>0}$, and extending linearly. ψ is clearly injective, and for any choice of $r_1, \dots, r_m, s_1, \dots, s_n, t_1 \dots, t_m \in \mathbb{Z}$ we have

$$\begin{aligned} &x_{-\beta_1}^{r_1} \dots x_{-\beta_m}^{r_m} h_{\alpha_1}^{s_1} \dots h_{\alpha_n}^{s_n} x_{\beta_1}^{t_1} \dots x_{\beta_m}^{t_m} \otimes 1 \\ &= \begin{cases} 0 & \text{if } t_i \neq 0 \text{ for some } i \\ x_{-\beta_1}^{r_1} \dots x_{-\beta_m}^{r_m} \otimes \prod_{i=1}^n \lambda(h_{\alpha_i}) & \text{otherwise} \end{cases} \\ &= \begin{cases} \psi(0) & \text{if } t_i \neq 0 \text{ for some } i \\ \psi\left((\prod_{i=1}^n \lambda(h_{\alpha_i}))x_{-\beta_1}^{r_1} \dots x_{-\beta_m}^{r_m}\right) & \text{otherwise} \end{cases} \end{aligned}$$

Since the elements $x_{-\beta_1}^{r_1} \dots x_{-\beta_m}^{r_m} h_{\alpha_1}^{s_1} \dots h_{\alpha_n}^{s_n} x_{\beta_1}^{t_1} \dots x_{\beta_m}^{t_m} \otimes 1$ span $M(\lambda)$, we have that ψ is surjective as well. Finally, since $\mathfrak{U}(L^-)$ acts on both $\mathfrak{U}(L^-)$ and $M(\lambda)$ by left multiplication, it is clear that ψ satisfies the condition

$$\psi(xv) = x\psi(v)$$

for any choice of $x, v \in \mathfrak{U}(L^-)$. □

Notice that for each $\lambda \in H^*$, we have $\operatorname{Supp} M(\lambda) = \lambda + Q^- = \lambda - Q^+$. We can also give a formula for the dimensions of the weight spaces of $M(\lambda)$, in the following way:

Definition 5.11. *Let L be a semisimple Lie algebra, with Cartan subalgebra H. Let Δ be the root system of L, and Δ^+ be the set of positive roots. Define the* **Kostant partition function** *$K : Q \to \mathbb{Z}_{\geq 0}$ in the following way: For each $\nu \in Q$, set $K(\nu)$ equal to the number of sequences $(k_\alpha)_{\alpha \in \Delta^+} \subseteq \mathbb{Z}_{\geq 0}$ for which $\nu = \sum_{\alpha \in \Delta^+} k_\alpha \alpha$.*

Due to the restriction that the sequences $(k_\alpha)_{\alpha \in \Delta^+}$ must contain only non-negative integers, we have that $K(\nu) = 0$ for any $\nu \in Q^-$.

Proposition 5.12. *Let L be a semisimple Lie algebra, with Cartan subalgebra H. Let $\lambda \in H^*$, and $M(\lambda)$ be the Verma module with highest weight λ. Then for each $\nu \in \operatorname{Supp} M(\lambda)$, we have $\dim M(\lambda)_\nu = K(\lambda - \nu)$.*

Proof. Let $\nu \in \operatorname{Supp} M(\lambda)$. Let $\gamma = \lambda - \nu$. Let $\Delta^+ = \{\beta_1, \ldots, \beta_m\}$ be the set of positive roots of L. Due to Proposition 5.11, we have that

$$\{x_{-\beta_1}^{k_1} \cdots x_{-\beta_m}^{k_m} \otimes 1 \mid k_i \in \mathbb{Z}_{\geq 0} \text{ for } 1 \leq i \leq m\}$$

is a basis for $M(\lambda)$. For any $h \in H$, we have that

$$\begin{aligned} hx_{-\beta_1}^{k_1} \ldots x_{-\beta_m}^{k_m} \otimes 1 &= ad_h(x_{-\beta_1}^{k_1} \ldots x_{-\beta_m}^{k_m}) \otimes 1 + x_{-\beta_1}^{k_1} \ldots x_{-\beta_m}^{k_m} \otimes h(1) \\ &= \left(\lambda(h) - \sum_{i=1}^{m} k_i \beta_i(h)\right) x_{-\beta_1}^{k_1} \ldots x_{-\beta_m}^{k_m} \otimes 1 \end{aligned}$$

Hence $x_{-\beta_1}^{k_1} \ldots x_{-\beta_m}^{k_m} \otimes 1$ is a weight vector of weight $\lambda - \sum_{i=1}^m k_i\beta_i$. Therefore $x_{-\beta_1}^{k_1} \ldots x_{-\beta_m}^{k_m} \otimes 1$ has weight ν if and only if $\gamma = \sum_{i=1}^m k_i\beta_i$. Thus the dimension of the ν weight space in $M(\lambda)$ is the number of sequences $(k_1, \ldots, k_m) \subset \mathbb{Z}_{\geq 0}$ for which $\gamma = \sum_{i=1}^m k_i\beta_i$, which is precisely $K(\gamma)$. □

5.5 Torsion Free Modules

Definition 5.12. *Let L be a semisimple Lie algebra with root system Δ, and V be an L-module admitting a weight space decomposition. We say that V is* **torsion free** *provided for every $\alpha \in \Delta$, the action of x_α on V is injective.*

We restrict our attention to those torsion free modules V in which all weight spaces of V are finite dimensional. Hence, from this point forward, when referring to a torsion free module V, it is assumed that $\dim V_\lambda < \infty$ for all $\lambda \in \operatorname{Supp} V$.

Proposition 5.13. *(Fernando) Let L be a semisimple Lie algebra with Cartan subalgebra H, and V be a simple L-module admitting a weight space decomposition. Then V is torsion free if and only if $\operatorname{Supp} V = \lambda + Q$ for some $\lambda \in H^*$.*

Proof. (See Corollary 1.4 in [15]) □

Proposition 5.14. *Let L be a semisimple Lie algebra, and V be a simple torsion free L-module. Then there exists an $N \in \mathbb{Z}_{\geq 0}$ such that $\dim V_\nu = N$ for all $\nu \in \operatorname{Supp} V$. In particular, V is admissible.*

Proof. Let H be a Cartan subalgebra of L, and let Δ be the root system of L with respect to H. Let $\phi : L \to gl(V)$ be the map defining the action of L on V. By the previous proposition, we have that $\operatorname{Supp} V = \lambda + Q$ for some $\lambda \in H^*$. Let $\nu, \gamma \in \operatorname{Supp} V$. Then $\gamma - \nu \in Q$, and hence

$$\gamma = \nu + \sum_{\beta \in \Delta} k_\beta \beta = \nu + \sum_{\beta \in \Delta^+} k_\beta \beta + \sum_{\beta \in \Delta^-} k_\beta \beta$$

for some $k_\beta \in \mathbb{Z}_{\geq 0}$. Let $\Delta^+ = \{\beta_1, \ldots, \beta_m\}$, and hence $\Delta^- = \{-\beta_1, \ldots, -\beta_m\}$. Therefore,

$$\gamma = \nu + \sum_{i=1}^{m} k_i \beta_i - \sum_{j=1}^{m} l_i \beta_i$$

for some $k_1, \ldots, k_m, l_1, \ldots, l_m \in \mathbb{Z}_{\geq 0}$. Set

$$\sigma = \phi(x_{\beta_1})^{k_1} \ldots \phi(x_{\beta_m})^{k_m} \phi(x_{-\beta_1})^{l_1} \ldots \phi(x_{-\beta_m})^{l_m}$$

then $\sigma \in gl(V)$ is an injective linear map. Further, for any $v \in V_\nu$ we have that $\sigma(v) \in V_\gamma$. We can therefore find a injective linear map between any two weight spaces of V. Thus all weight spaces of V must have the same dimension. Since torsion free modules are assumed to have finite dimensional weight spaces, we have our result. □

5.6 Tensor Products of Modules

Definition 5.13. *Let L be a Lie algebra, and V, W be L-modules. Let $\mathfrak{B}_V$ and $\mathfrak{B}_W$ be bases for V and W respectively. We define the* **tensor product L-module** *to be the vector space $V \otimes W$ under the following action:*

$$x(v \otimes w) = (xv) \otimes w + v \otimes (xw)$$

for each $x \in L$, $v \in \mathfrak{B}_V$ and $w \in \mathfrak{B}_W$, and extending linearly. Viewed as representations, we have that if $\phi : L \to gl(V)$ and $\psi : L \to gl(W)$ define representations of

L on V and W respectively, then the **tensor product representation** *is the pair* $(\phi \otimes \psi, V \otimes W)$ *where* $\phi \otimes \psi : L \to gl(V \otimes W)$ *is the linear map given by*

$$\phi \otimes \psi(x)(v \otimes w) = (\phi(x)v) \otimes w + v \otimes (\psi(x)w)$$

For all $x \in L$, $v \in \mathfrak{B}_V$ *and* $w \in \mathfrak{B}_W$.

The following calculation shows that the tensor product representation is indeed a representation of L. For simplicity, denote the map $\phi \otimes \psi$ by θ. Then, for any $x, y \in L$ and $v \otimes w \in V \otimes W$, we have

$$\begin{aligned}
(\theta(x)\theta(y) - \theta(y)\theta(x))(v \otimes w) &= \phi(x)\phi(y)(v) \otimes w + \phi(y)(v) \otimes \psi(x)(w) \\
&\quad + \phi(x)(v) \otimes \psi(y)(w) + v \otimes \psi(x)\psi(y)(w) \\
&\quad - \phi(y)\phi(x)(v) \otimes w - \phi(x)(v) \otimes \psi(y)(w) \\
&\quad - \phi(y)(v) \otimes \psi(x)(w) - v \otimes \psi(y)\psi(x)(w) \\
&= \phi(x)\phi(y)(v) \otimes w - \phi(y)\phi(x)(v) \otimes w \\
&\quad + v \otimes \psi(x)\psi(y)(w) - v \otimes \psi(y)\psi(x)(w) \\
&= (\phi(x)\phi(y) - \phi(y)\phi(x))\,(v) \otimes w \\
&\quad - v \otimes (\psi(x)\psi(y) - \psi(y)\psi(x))\,(w) \\
&= \phi([x,y])(v) \otimes w - v \otimes \psi([x,y])(w) \\
&= \theta([x,y])(v \otimes w)
\end{aligned}$$

Proposition 5.15. *Let L be a semisimple Lie algebra, with Cartan subalgebra H. If* $\mu_1, \mu_2 \in H^*$ *are both dominant integral weights then*

$$L(\mu_1) \otimes L(\mu_2) = \bigoplus_{\nu \in H^*} a_\nu L(\nu)$$

for some $a_\nu \in \mathbb{Z}_{\geq 0}$, *where if* $a_\nu \neq 0$ *then* ν *is a dominant integral weight.*

Proof. Since $L(\mu_1)$ and $L(\mu_2)$ are both finite dimensional, the tensor product $L(\mu_1) \otimes L(\mu_2)$ is also finite dimensional. By Weyl's theorem, we have that $L(\mu_1) \otimes L(\mu_2)$ is completely reducible, hence by Proposition 4.3, $L(\mu_1) \otimes L(\mu_2)$ is a semisimple module. Therefore there exist simple modules $V_1, \ldots V_n$ such that

$$L(\mu_1) \otimes L(\mu_2) = \bigoplus_{i=1}^{n} a_i V_i$$

for some $a_i \in \mathbb{Z}_{\geq 0}$. Each V_i is finite dimensional, being a submodule of the finite dimensional module $L(\mu_1) \otimes L(\mu_2)$. Corollary 5.1, therefore implies that each $V_i = L(\nu_i)$ for some dominant integral ν_i. □

Proposition 5.16. *Let L be a semisimple Lie algebra with root system Δ. Let V be a torsion free L-module, and W be a finite dimensional weight L-module. Then $V \otimes W$ is torsion free.*

Proof. Let $\alpha \in \Delta$. Let $\mathfrak{B}_V$ be a basis for V, and $\mathfrak{B}_W$ be a basis for W consisting of weight vectors. Let $u \in V \otimes W$, with $u \neq 0$. Then

$$u = \sum_{v \in \mathfrak{B}_V} \sum_{w \in \mathfrak{B}_W} a_{vw} v \otimes w$$

for some $a_{vw} \in \mathbb{C}$ with all but finitely many $a_{vw} = 0$. Suppose $x_\alpha u = 0$. Then

$$\sum_{v \in \mathfrak{B}_V} \sum_{w \in \mathfrak{B}_W} a_{vw} x_\alpha v \otimes w = - \sum_{v \in \mathfrak{B}_V} \sum_{w \in \mathfrak{B}_W} a_{vw} v \otimes x_\alpha w \qquad (1)$$

Let

$$M = \{w \in \mathfrak{B}_W \mid a_{vw} \neq 0 \text{ for some } v \in \mathfrak{B}_V\}$$

Set

$$\overline{M} = \{\gamma \in \operatorname{Supp} W \mid w \text{ has weight } \gamma \text{ for some } w \in M\}$$

Since $\overline{M}$ is a finite set, we can choose $\gamma_0 \in \overline{M}$ such that $\gamma_0 - \alpha \notin \overline{M}$. Choose $w_0 \in M$ such that w_0 has weight γ_0. Notice that there is no w appearing with some $a_{vw} \neq 0$ for which $x_\alpha w$ has weight γ_0. Therefore w_0 cannot appear in any basic tensor on the right hand side of (1). This implies that

$$\sum_{v \in \mathfrak{B}_V} a_{vw_0} x_\alpha v \otimes w_0 = 0$$

Therefore $x_\alpha \sum_{v \in \mathfrak{B}_V} a_{vw_0} v = 0$. Since V is torsion free, x_α acts injectively on V, and hence $\sum_{v \in \mathfrak{B}_V} a_{vw_0} v = 0$. This implies that $a_{vw_0} = 0$ for all $v \in \mathfrak{B}_V$, which contradicts our choice of w_0. □

5.7 The Formal Character

Definition 5.14. *Let L be a semisimple Lie algebra with Cartan subalgebra H. Define the group $\mathcal{E}$ to be the abelian group consisting of the formal expressions e^λ for each $\lambda \in H^*$, with product given by*

$$e^{\lambda_1} e^{\lambda_2} = e^{\lambda_1+\lambda_2}$$

for all $\lambda_1, \lambda_2 \in H^*$.

We now introduce one of the main tools that will be used for computations involving the tensor product of two modules.

Definition 5.15. *Let L be a semisimple Lie algebra. Let V be an L-module admitting a weight space decomposition, with finite dimensional weight spaces. Define the* **formal character** *of V, denoted $\operatorname{ch} V$, to be the element in the group algebra $\mathbb{Z}[\mathcal{E}]$, given by*

$$ch\, V = \sum_{\nu \in \operatorname{Supp} V} m_V(\nu) e^\nu$$

Property 5.3. *Let L be a semisimple Lie algebra with Cartan subalgebra H. Then for any $\lambda \in H^*$, the formal character of the Verma module $M(\lambda)$ is given by*

$$\operatorname{ch} M(\lambda) = \sum_{\gamma \in Q^+} K(\gamma) e^{\lambda-\gamma}$$

where K is the Kostant partition function.

Proof. Due to Proposition 5.11, we know that $\operatorname{Supp} M(\lambda) = \lambda - Q^+$, and by Proposition 5.12, we have that $m_{M(\lambda)}(\lambda - \gamma) = K(\gamma)$ for each $\gamma \in Q^+$, which gives our result. □

Lemma 5.3. *Let L be a semisimple Lie algebra. If U and V are both L-modules admitting weight space decompositions, then*

$$\operatorname{ch}(U \otimes V) = (\operatorname{ch} U)(\operatorname{ch} V)$$

Proof. Choose bases $\mathfrak{B}_U$ and $\mathfrak{B}_V$ of U and V respectively, each consisting of weight vectors. Then $\{u \otimes v \mid u \in \mathfrak{B}_U \text{ and } v \in \mathfrak{B}_V\}$ is a basis of $U \otimes V$. Further, if $u \in \mathfrak{B}_U$

has weight $\nu \in \operatorname{Supp} U$ and $v \in \mathfrak{B}_V$ has weight $\gamma \in \operatorname{Supp} V$ then for any $h \in H$, we have

$$h(u \otimes v) = (hu) \otimes v + u \otimes (hv) = (\nu + \gamma)(h)(u \otimes v)$$

Hence $u \otimes v$ is a weight vector with weight $\lambda + \nu$. Since such vectors form a basis for $U \otimes V$, we have that $\operatorname{Supp}(U \otimes V) = \operatorname{Supp} U + \operatorname{Supp} V$, and further,

$$\operatorname{ch}(U \otimes V) = \sum_{\substack{\nu \in \operatorname{Supp} U \\ \gamma \in \operatorname{Supp} V}} m_U(\nu) m_V(\gamma) e^{\gamma+\nu} = (\operatorname{ch} U)(\operatorname{ch} V)$$

□

Proposition 5.17. *Let L be a semisimple Lie algebra. Let V be a finite dimensional L-module, and U be an admissible L-module. Then $U \otimes V$ is admissible. In particular,*

$$m_{U\otimes V}(\nu) \leq (\deg U)(\dim V)$$

for all $\nu \in \operatorname{Supp}(U \otimes V)$.

Proof. Due to the previous lemma,

$$\operatorname{ch}(U \otimes V) = (\operatorname{ch} U)(\operatorname{ch} V) = \sum_{\gamma \in \operatorname{Supp} U} \sum_{\nu \in \operatorname{Supp} V} m_U(\gamma) m_V(\nu) e^{\gamma+\nu}$$

Let $\lambda \in \operatorname{Supp}(U \otimes V)$. We see that $\lambda = \gamma + \nu$ for some $\gamma \in \operatorname{Supp} U$ and $\nu \in \operatorname{Supp} V$. Define

$$S_\lambda = \{\nu \in \operatorname{Supp} \mathrm{V} \mid \lambda - \nu \in \operatorname{Supp} U\}$$

Since $\operatorname{Supp} V$ is a finite set, we can choose weights $\nu_1, \ldots, \nu_k \in \operatorname{Supp} V$ such that

$$S_\lambda = \{\nu_1, \ldots, \nu_k\}$$

For each $1 \leq i \leq k$ set $\gamma_i = \lambda - \nu_i$. Then the dimension of the λ weight space in $U \otimes V$ must be

$$\begin{aligned} m_{U\otimes V}(\lambda) = \sum_{i=1}^{k} m_U(\gamma_i) m_V(\nu_i) &\leq (\deg U) \sum_{i=1}^{k} m_V(\nu_i) \\ &\leq (\deg U) \left(\sum_{\nu \in \operatorname{Supp} V} m_V(\nu) \right) \\ &= (\deg U)(\dim V) \end{aligned}$$

Since this gives a bound for all weight spaces occurring in $U \otimes V$, we must have that $U \otimes V$ is admissible. □

5.8 The Central Character

Proposition 5.18. *(Schur's Lemma) Let L be a semisimple Lie algebra, and V be a simple L-module, with action given by $\phi : L \to gl(V)$. If $\pi \in gl(V)$ such that $[\pi, \phi(x)] = 0$ for all $x \in L$, then there exists $c \in \mathbb{C}$ such that $\pi(v) = cv$ for all $v \in V$. i.e. π acts as scalar multiplication.*

Proof. (See Lemma 6.1 in [8]) □

Definition 5.16. *Let L be a Lie algebra, and $\mathfrak{U}(L)$ be the universal enveloping algebra of L. The* **centre** *of $\mathfrak{U}(L)$, denoted $Z(\mathfrak{U}(L))$, is defined to be*

$$Z(\mathfrak{U}(L)) = \{z \in \mathfrak{U}(L) \mid xz - zx = 0 \text{ for all } x \in \mathfrak{U}(L)\}$$

Definition 5.17. *Let L be a semisimple Lie algebra, and $Z(\mathfrak{U}(L))$ be the centre of the universal enveloping algebra of L. A function $\chi : Z(\mathfrak{U}(L)) \to \mathbb{C}$ that is an algebra homomorphism is called a* **central character**. *If V is a $\mathfrak{U}(L)$-module with the property that there exists a central character χ_V for which $zv = \chi_V(z)v$ for all $z \in Z(\mathfrak{U}(L))$ and all $v \in V$, then V is said to* **admit a central character**, *and χ_V is called the* **central character of** V.

Proposition 5.19. *Let L be a semisimple Lie algebra with Cartan subalgebra H. Let V be a simple L-module, then V admits a central character. In particular, for any $\lambda \in H^*$, the simple highest weight module $L(\lambda)$ admits a central character, which we will denote by χ_λ.*

Proof. Suppose the action of V on L is given by the map $\phi : L \to gl(V)$. Let $z \in Z(\mathfrak{U}(L))$. Then for any $x \in L$, we have

$$[\phi(z), \phi(x)] = \phi(z)\phi(x) - \phi(x)\phi(z) = \phi(zx - xz) = \phi(0) = 0$$

By Schur's lemma, we have that for each $z \in Z(\mathfrak{U}(L))$ $\phi(z)(v) = c_z v$ for some $c_z \in \mathbb{C}$ and all $v \in V$. Define $\chi : Z(\mathfrak{U}(L)) \to \mathbb{C}$ by $\chi(z) = c_z$. Clearly, since ϕ is an algebra homomorphism, we have that χ is an algebra homomorphism. Hence χ is the central character of V. □

Theorem 5.3. *(Harish-Chandra) Let L be a semisimple Lie algebra, with Cartan subalgebra H, and let $\lambda, \mu \in H^*$. Then $\chi_\lambda = \chi_\mu$ if and only if there exists $\sigma \in \mathcal{W}$ such that $\sigma(\lambda + \rho) - \rho = \mu$.*

Proof. (See Theorem 23.3 in [8]) □

Lemma 5.4. *If χ_1 and χ_2 are central characters with $\ker \chi_1 \subseteq \ker \chi_2$ then $\chi_1 = \chi_2$.*

Proof. Let $z \in Z(\mathfrak{U}(L))$, and $z_0 = z - \chi_1(z)$. Then $z_0 \in \ker \chi_1 \subseteq \ker \chi_2$, and hence $0 = \chi_2(z_0) = \chi_2(z) - \chi_1(z)$. Therefore $\chi_1(z) = \chi_2(z)$. □

Proposition 5.20. *Let L be a semisimple Lie algebra, and V be an L-module. If $V_1, \dots, V_n$ are submodules of V with distinct non-zero central characters $\chi_1, \dots, \chi_n$ then $V_1, \dots, V_n$ are linearly independent.*

Proof. Suppose, to the contrary, that $v_1, \dots, v_n \in V$ with each $v_i \in V_i$, and $v_1 = \sum_{i=2}^{n} a_i v_i$, for some $a_i \in \mathbb{C}$. Since the χ_i are distinct, the previous lemma implies that $\ker \chi_i \setminus \ker \chi_1 \neq \emptyset$ for each $i \neq 1$. Therefore, for each $2 \leq i \leq n$, we can choose $z_i \in Z(\mathfrak{U}(L))$ such that $z_i \in \ker \chi_i$ and $z_i \notin \ker \chi_1$. Then

$$\begin{aligned} 0 \neq \chi_1(z_2)\chi_1(z_3)\dots\chi_1(z_n)v_1 &= z_2 z_3 \dots z_n v_1 \\ &= \sum_{i=2}^{n} a_i z_2 z_3 \dots z_n v_i \\ &= \sum_{i=2}^{n} a_i \chi_i(z_2)\chi_i(z_3)\dots\chi_i(z_n) v_i \\ &= 0 \end{aligned}$$

which is a contradiction. □

Proposition 5.21. *Let L be a semisimple Lie algebra with Cartan subalgebra H, and $Z(\mathfrak{U}(L))$ be the centre of the universal enveloping algebra of L. If $\chi : Z(\mathfrak{U}(L)) \to \mathbb{C}$ is an algebra homomorphism then $\chi = \chi_\lambda$ for some $\lambda \in H^*$.*

Proof. (See Proposition 7.4.8 in [6]) □

Proposition 5.22. *(Kostant) Let L be a semisimple Lie algebra, with Cartan subalgebra H, and let $\lambda, \mu \in H^*$ with $L(\lambda)$ admissible, and $L(\mu)$ finite dimensional. Let $Supp\, L(\mu) = \{\nu_1, \dots, \nu_k\}$. If $z \in Z(\mathfrak{U}(L))$ then for any $v \in L(\lambda) \otimes L(\mu)$ we have*

$$\prod_{i=1}^{k}(z - \chi_{\lambda+\nu_i}(z))v = 0$$

Proof. (See Theorem 5.1 in [10]) □

Corollary 5.2. *Let L be a semisimple Lie algebra, with Cartan subalgebra H, and let $\lambda, \mu \in H^*$ with $L(\lambda)$ admissible, and $L(\mu)$ finite dimensional. Let $Supp\, L(\mu) = \{\nu_1, \dots, \nu_k\}$. If V is a submodule of $L(\lambda) \otimes L(\mu)$, having central character χ_V, then $\chi_V = \chi_{\lambda+\nu_i}$ for some $\nu_i \in Supp\, L(\mu)$.*

Proof. Assume $\chi_V \neq \chi_{\lambda+\nu_i}$ for any $\nu_i \in Supp\, L(\mu)$. Then by Lemma 5.4, for each $1 \leq i \leq k$ we can choose $z_i \in \ker \chi_{\lambda+\nu_i} \setminus \ker \chi_V$. Setting $z = z_1 \dots z_k$, and applying the previous proposition, we obtain for any $v \in V$,

$$\prod_{i=1}^{k}(z - \chi_{\lambda+\nu_i}(z))v = \prod_{i=1}^{k} zv = \chi_V(z)^k v = 0$$

Therefore, $z \in \ker \chi_V$. Yet, since $\chi_V(z) = \chi_V(z_1) \dots \chi_V(z_k)$, we must have $z_i \in \ker \chi_V$ for some i, which is a contradiction. □

6 Construction of Simple Torsion Free Modules

In this section, we follow the work of Mathieu in [15] giving a characterization of all simple torsion free modules.

6.1 Some Useful Computational Identities

We begin this section with three formulas, for computations in $\mathfrak{U}(L)$.

Lemma 6.1. *Let L be a Lie algebra. Let $x \in L$ and $y \in \mathfrak{U}(L)$. Then for every $N \in \mathbb{Z}_{>0}$,*

$$(ad_x)^N(y) = \sum_{n=0}^{N}(-1)^{N-n}\begin{pmatrix} N \\ n \end{pmatrix} x^n y x^{N-n}$$

Proof. We apply induction on N. When $N = 1$, this formula becomes $ad_x(y) = xy - yx$, which is simply the definition of the adjoint action on the universal enveloping algebra. Assume the formula is true for $N < k$. Then

$$\begin{aligned}
(ad_x)^k(y) &= ad_x((ad_x)^{k-1}(y)) \\
&= ad_x\left(\sum_{n=0}^{k-1}(-1)^{k-1-n}\begin{pmatrix} k-1 \\ n \end{pmatrix} x^n y x^{k-1-n}\right) \\
&= \sum_{n=0}^{k-1}(-1)^{k-1-n}\begin{pmatrix} k-1 \\ n \end{pmatrix} x^{n+1} y x^{k-1-n} \\
&\quad - \sum_{n=0}^{k-1}(-1)^{k-1-n}\begin{pmatrix} k-1 \\ n \end{pmatrix} x^n y x^{k-n} \\
&= \sum_{n=1}^{k}(-1)^{k-n}\begin{pmatrix} k-1 \\ n-1 \end{pmatrix} x^n y x^{k-n} \\
&\quad + \sum_{n=0}^{k-1}(-1)^{k-n}\begin{pmatrix} k-1 \\ n \end{pmatrix} x^n y x^{k-n} \\
&= (-1)^k x y^k + x^k y + \sum_{n=1}^{k-1}(-1)^{k-n}\left(\begin{pmatrix} k-1 \\ n-1 \end{pmatrix} + \begin{pmatrix} k-1 \\ n \end{pmatrix}\right) x^n y x^{k-n} \\
&= (-1)^k x y^k + x^k y + \sum_{n=1}^{k-1}(-1)^{k-n}\begin{pmatrix} k \\ n \end{pmatrix} x^n y x^{k-n} \\
&= \sum_{n=0}^{k}(-1)^{k-n}\begin{pmatrix} k \\ n \end{pmatrix} x^n y x^{k-n}
\end{aligned}$$

□

Lemma 6.2. *Let L be a Lie algebra. Let $x \in L$ and $y \in \mathfrak{U}(L)$. Then for every*

$N \in \mathbb{Z}_{>0}$,

$$x^N y = \sum_{n=0}^{N} \binom{N}{n} (ad_x)^n(y)x^{N-n}$$

Proof. (The proof is similar to that of 6.1) □

Lemma 6.3. *Let L be a Lie algebra. Let $x \in L$ and $y_1, y_2 \in \mathfrak{U}(L)$. Then for every $N \in \mathbb{Z}_{>0}$,*

$$(ad_x)^N(y_1y_2) = \sum_{n=0}^{N} \binom{N}{n} (ad_x)^n(y_1)(ad_x)^{N-n}(y_2)$$

Proof. (The proof is similar to that of 6.1) □

6.2 A Commuting Set of Roots

Definition 6.1. *Let L be a Lie algebra, and (ϕ, V) be a representation of L. An element $x \in L$ is said to be* ϕ**-nilpotent** *if there exist an $N \in \mathbb{Z}_{\geq 0}$ such that $\phi(x)^N v = 0$ for all $v \in V$. $x \in L$ is called* **locally** ϕ**-nilpotent** *provided the action of x is ϕ-nilpotent on every finite subset of V. Equivalently, $x \in L$ is locally ϕ-nilpotent if for each $v \in V$ there exists an $N \in \mathbb{Z}_{\geq 0}$ such that $\phi(x)^N v = 0$. If the representation is implicit, we will simply say x is nilpotent, or locally nilpotent.*

Proposition 6.1. *Let L be a finite dimensional simple Lie algebra with root system Δ. Then for each $\alpha \in \Delta$, the element x_α is locally ad-nilpotent on $\mathfrak{U}(L)$.*

Proof. Let $\alpha \in \Delta$. Let

$$M_\alpha = \{y \in \mathfrak{U}(L) \mid (ad_{x_\alpha})^N(y) = 0 \text{ for some } N \in \mathbb{Z}_{>0}\}$$

If $x_\beta \in L$ is another root vector, then $(ad_{x_\alpha})^n(x_\beta) \in L_{\beta+n\alpha}$. Yet, since L is finite dimensional, $\{n \in \mathbb{Z}_{\geq 0} \mid L_{\beta+n\alpha} \neq (0)\}$ is a finite set, hence $x_\beta \in M_\alpha$. Clearly, M_α is closed under addition, and therefore $L \subseteq M_\alpha$, due to the root space decomposition of L. Further, if $y_1, y_2 \in M_\alpha$ then we can choose N_1 and N_2 such that $(ad_{x_\alpha})^{N_1}(y_1) = (ad_{x_\alpha})^{N_2}(y_2) = 0$. Let $N = N_1 + N_2$. Then, by Lemma 6.3 we have

$$(ad_{x_\alpha})^N(y_1y_2) = \sum_{n=0}^{N} \binom{N}{n} (ad_{x_\alpha})^n(y_1)(ad_x)^{N-n}(y_2)$$

yet, since for every n, either $n \geq N_1$ or $N - n > N_2$, we have

$$(ad_{x_\alpha})^N(y_1y_2) = 0$$

Therefore, whenever $y_1, y_2 \in M_\alpha$, $y_1y_2 \in M_\alpha$, hence $\mathfrak{U}(L) \subseteq M_\alpha$. $\square$

Proposition 6.2. *If L is a finite dimensional simple Lie algebra, with root system Δ, and V is a simple L-module. Then for each $\alpha \in \Delta$, the action of x_α on V is either injective or locally nilpotent.*

Proof. Let $M = \{v \in V \mid x_\alpha^N v = 0 \text{ for some } N \in \mathbb{Z}_{>0}\}$. We intend to show that M is a submodule of V. To this end, let $v \in M$, $x \in L$, and choose $N_1, N_2 \in \mathbb{Z}_{>0}$ such that $x_\alpha^{N_1} v = 0$ and $(ad_{x_\alpha})^{N_2}(x) = 0$. Notice that the latter is possible due to Proposition 6.1. Letting $N = N_1 + N_2$, by Lemma 6.2 we have

$$x_\alpha^N x v = \sum_{n=0}^{N} \begin{pmatrix} N \\ n \end{pmatrix} (ad_{x_\alpha})^n(x) x_\alpha^{N-n} v$$

yet, since for every n, either $n \geq N_2$ or $N-n > N_1$, we have $x_\alpha^N x v = 0$ hence $xv \in M$. Therefore M is a submodule of V. Since V is simple, this implies that $M = (0)$ or $M = V$. If $M = (0)$, then the action of x_α on V is injective, and if $M = V$, the action is, of course, locally nilpotent. $\square$

Corollary 6.1. *If L is a finite dimensional simple Lie algebra, with root system Δ, and V is a simple L-module, then $\Delta = \Delta_V^I \uplus \Delta_V^0 \uplus \Delta_V^+ \uplus \Delta_V^-$, where*

$$\Delta_V^I = \{\alpha \in \Delta \mid x_\alpha \textit{ and } x_{-\alpha} \textit{ are injective}\}$$

$$\Delta_V^0 = \{\alpha \in \Delta \mid x_\alpha \textit{ and } x_{-\alpha} \textit{ are locally nilpotent}\}$$

$$\Delta_V^+ = \{\alpha \in \Delta \mid x_\alpha \textit{ is locally nilpotent and } x_{-\alpha} \textit{ is injective}\}$$

$$\Delta_V^- = \{\alpha \in \Delta \mid x_\alpha \textit{ is injective and } x_{-\alpha} \textit{ is locally nilpotent}\}$$

Proof. Let $\alpha \in \Delta$, then x_α is either injective or locally nilpotent, and $x_{-\alpha}$ is either injective or locally nilpotent. Therefore α is in one of Δ_V^I, Δ_V^0, Δ_V^+ or Δ_V^-. Also, since x_α cannot be both injective and locally nilpotent, the sets Δ_V^I, Δ_V^0, Δ_V^+ and Δ_V^- must be disjoint. $\square$

Definition 6.2. *Let L be a finite dimensional simple Lie algebra with root system Δ. Let $\Sigma \subseteq \Delta$ such that $[x_{\beta_1}, x_{\beta_2}] = 0$ for all $\beta_1, \beta_2 \in \Sigma$. Then Σ is called a* **commuting set of roots**.

Notice that $[x_{\beta_1}, x_{\beta_2}] = 0$ if and only if $L_{\beta_1+\beta_2} = (0)$, which happens if and only if $L_{-\beta_1-\beta_2} = (0)$. Thus $\Sigma \subseteq \Delta$ is a commuting set of roots if and only if $[x_{-\beta_1}, x_{-\beta_2}] = 0$ for all $\beta_1, \beta_2 \in \Sigma$.

Proposition 6.3. *(Mathieu) Let L be a finite dimensional simple Lie algebra and V be a simple admissible L-module. Then there exists a set $\Sigma_V \subseteq \Delta$ of commuting roots which is a basis for Q such that $\Sigma_V \subseteq \Delta_V^I \cup \Delta_V^+$.*

Proof. (See Lemma 4.4 in [15]) □

Notice that the condition $\Sigma_V \subseteq \Delta_V^I \cup \Delta_V^+$ is equivalent to $x_{-\beta}$ acting injectively on V for all $\beta \in \Sigma_V$.

Definition 6.3. *Let L be a finite dimensional simple Lie algebra and V be a simple admissible L-module. Define a* **basis of commuting roots** *with respect to V to be a set $\Sigma_V \subseteq \Delta$ of commuting roots which is a basis for Q such that $x_{-\beta}$ acts injectively on V for all $\beta \in \Sigma_V$.*

Recall that the usual base for the root system of C_n is given by

$$\Delta^{++} = \{\alpha_1, \dots, \alpha_n\}$$

where $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \le i < n$ and $\alpha_n = 2\epsilon_n$. For $1 \le i \le n$, set $\beta_i = \sum_{k=i}^{n} \alpha_k$. Clearly $\beta_i + \beta_j \notin \Delta$ for any i and j, and hence $[x_{\beta_i}, x_{\beta_j}] = 0$ for all i and j. Thus

$$\Sigma_n = \{\beta_1, \dots, \beta_n\}$$

is a commuting set of roots. Further, Σ_n is a basis for Q. Also, one can show that the elements $x_{-\beta_i}$ act injectively on any simple admissible highest weight C_n-module $L(\lambda)$. Thus, Σ_n is a basis of commuting roots with respect to any simple admissible highest weight C_n-module.

Proposition 6.4. *Let L be a finite dimensional simple Lie algebra. Let V be a simple admissible L-module, and $\Sigma_V = \{\beta_1, \ldots, \beta_n\} \subseteq \Delta$ be a basis of commuting roots with respect to V. Let $S(\Sigma_V)$ be the multiplicative subset of $\mathfrak{U}(L)$ generated by $\{1, x_{-\beta_1}, \ldots, x_{-\beta_n}\}$. Then $S(\Sigma_V)$ satisfies Ore's Localizability condition.*

Proof. By definition, $1 \in S(\Sigma_V)$. $S(\Sigma_V)$ cannot contain any zero-divisors, because $S(\Sigma_V) \subseteq \mathfrak{U}(L)$, and $\mathfrak{U}(L)$ does not contain any zero-divisors. Due to Property 2.1, it suffices to show that the generators $x_{-\beta}$ for $\beta \in \Sigma_V$ satisfy the left and right Ore conditions. Let $\beta \in \Sigma_V$ and $u \in \mathfrak{U}(L)$. Applying Proposition 6.1, we can choose $N \in \mathbb{Z}_{>0}$ such that $ad^N_{x_{-\beta}}(u) = 0$. For simplicity, choose N to be even. Applying Lemma 6.1, we have

$$\sum_{0 \leq n \leq N} (-1)^{N-n} \begin{pmatrix} N \\ n \end{pmatrix} x^n_{-\beta} u x^{N-n}_{-\beta} = 0$$

Therefore

$$x_{-\beta} \left(\sum_{1 \leq n \leq N} (-1)^{N-n+1} \begin{pmatrix} N \\ n \end{pmatrix} x^{n-1}_{-\beta} u x^{N-n}_{-\beta} \right) = u x^N_{-\beta} \qquad (1)$$

and

$$\left(\sum_{0 \leq n \leq N-1} (-1)^{N-n+1} \begin{pmatrix} N \\ n \end{pmatrix} x^n_{-\beta} u x^{N-n-1}_{-\beta} \right) x_{-\beta} = x^N_{-\beta} u \qquad (2)$$

Setting $u' = \sum_{1 \leq n \leq N} (-1)^{N-n+1} \begin{pmatrix} N \\ n \end{pmatrix} x^{n-1}_{-\beta} u x^{N-n}_{-\beta}$ and $s' = x^N_{-\beta}$ in (1), and $u'' = \sum_{0 \leq n \leq N-1} (-1)^{N-n+1} \begin{pmatrix} N \\ n \end{pmatrix} x^n_{-\beta} u x^{N-n-1}_{-\beta}$ and $s'' = x^N_{-\beta}$ in (2) gives us $x_{-\beta} u' = us'$ and $u'' x_{-\beta} = s'' u$. □

We now have that for a finite dimensional simple Lie algebra L, given any simple admissible L-module, V, we can find a basis of commuting roots, Σ, with respect to V. Further, the multiplicative subset $S(\Sigma)$ generated by $\{1\} \cup \{x_{-\beta} \mid \beta \in \Sigma\}$ satisfies Ore's localizability condition. We can therefore form the localization algebra of $\mathfrak{U}(L)$ with respect to $S(\Sigma)$, which we will denote by $\mathfrak{U}_\Sigma(L)$. We now form the induced module

$$V^\Sigma = \mathfrak{U}_\Sigma(L) \otimes_{\mathfrak{U}(L)} V$$

Recall that every element of $\mathfrak{U}_\Sigma(L)$ can be written in the form $s^{-1}u$ with $s \in S(\Sigma)$ and $u \in \mathfrak{U}(L)$. Hence, if $\Sigma = \{\beta_1, \ldots, \beta_n\}$, and $\mathfrak{B}_V$ is any basis for V, we have that

$$\mathfrak{B}_{V^\Sigma} = \{x_{-\beta_1}^{k_1} x_{-\beta_2}^{k_2} \ldots x_{-\beta_n}^{k_n} \otimes v \mid k_1, \ldots, k_n \in \mathbb{Z}_{\leq 0},\ v \in \mathfrak{B}_V\}$$

is a basis for V^Σ.

Since $\mathfrak{U}(L)$ is embedded in $\mathfrak{U}_\Sigma(L)$, we have that V^Σ is also a $\mathfrak{U}(L)$-module under the action

$$u(\frac{1}{s} \otimes v) = \frac{u}{1}(\frac{1}{s} \otimes v)$$

for any $u \in \mathfrak{U}(L)$ and $s^{-1} \otimes v \in \mathfrak{B}_{V^\Sigma}$.

We now give some useful properties of V^Σ.

Proposition 6.5. *(Mathieu) Let L be a finite dimensional simple Lie algebra. Let V be a simple admissible L-module, and $\Sigma \subseteq \Delta$ be a basis of commuting roots with respect to V. Then V^Σ is a weight module with the following properties:*

i) $V \leq V^\Sigma$

ii) $\operatorname{Supp} V^\Sigma = \operatorname{Supp} V + Q$

iii) $\dim V_\mu^\Sigma = \deg V$ *for all* $\mu \in \operatorname{Supp} V^\Sigma$

Proof. (See Lemma 4.4 in [15]) □

Recall Proposition 5.19, that all simple modules of a semisimple Lie algebra admit a central character.

Proposition 6.6. *Let L be a finite dimensional simple Lie algebra. Let V be a simple admissible L-module, and $\Sigma \subseteq \Delta$ be a basis of commuting roots with respect to V. Then V^Σ admits a central character, and the central character of V^Σ is that of V.*

Proof. Let $\mathfrak{B}_V$ be a basis for V, and

$$\mathfrak{B}_{V^\Sigma} = \{x_{-\beta_1}^{k_1} x_{-\beta_2}^{k_2} \ldots x_{-\beta_n}^{k_n} \otimes v \mid k_1, \ldots, k_n \in \mathbb{Z}_{\leq 0},\ v \in \mathfrak{B}_V\}$$

be a basis for V^{Σ}. Let $z \in Z(\mathfrak{U}(L))$ and $s^{-1} \otimes v \in \mathfrak{B}_{V^{\Sigma}}$, where $s = x_{-\beta_1}^{k_1} \dots x_{-\beta_n}^{k_n}$ for some non-negative integers $k_1, \dots, k_n$, and $v \in \mathfrak{B}_V$. Since $zs = sz$, we have that

$$\frac{z}{1}\frac{1}{s} = \frac{z}{s} = \frac{1}{s}\frac{z}{1}$$

hence

$$z(s^{-1} \otimes v) = s^{-1}z \otimes v = s^{-1} \otimes zv = \chi(z)(s^{-1} \otimes v)$$

where χ is the central character of V. □

6.3 Some Automorphisms

First, notice that if V is a simple admissible L-module, and Σ_V is a basis of commuting roots with respect to V, then the adjoint representation, and the left regular representation can be extended to all of $\mathfrak{U}_{\Sigma_V}(L)$ acting on itself. We define for each $x \in \mathfrak{U}_{\Sigma_V}(L)$ the map $ad_x : \mathfrak{U}_{\Sigma_V}(L) \to \mathfrak{U}_{\Sigma_V}(L)$, given by

$$ad_x(y) = xy - yx$$

for all $y \in \mathfrak{U}_{\Sigma_V}(L)$, and the left regular action of x on $\mathfrak{U}_{\Sigma_V}(L)$, given by

$$x(y) = xy$$

for all $y \in \mathfrak{U}_{\Sigma_V}(L)$. These definitions, of course, coincide with the previous definitions of the adjoint and left regular actions of L on $\mathfrak{U}(L)$. Also, notice that the formulas given in section 6.1 still hold for the action of $\mathfrak{U}_{\Sigma_V}(L)$ on itself.

Proposition 6.7. *Let L be a finite dimensional simple Lie algebra. Let V be a simple admissible L-module, and $\Sigma_V \subseteq \Delta$ be a basis of commuting roots with respect to V. Then for each $\beta \in \Sigma_V$, the action of $ad_{x_{-\beta}}$ on $\mathfrak{U}_{\Sigma_V}(L)$ is locally nilpotent.*

Proof. Since $S(\Sigma_V)$ is generated by $\{x_{-\beta} \mid \beta \in \Sigma_V\}$, and Σ_V is a commuting set of roots, we must have that for each $\beta \in \Sigma_V$ and $s \in S(\Sigma_V)$, $ad_{x_{-\beta}}(s) = 0$. Let $\frac{u}{s} \in \mathfrak{U}_{\Sigma_V}(L)$, then

$$ad_{x_{-\beta}}\left(\frac{u}{s}\right) = x_{-\beta}\left(\frac{u}{s}\right) - \left(\frac{u}{s}\right)x_{-\beta}$$

since $x_{-\beta}s = sx_{-\beta}$, we have that

$$ad_{x_{-\beta}}\left(\frac{u}{s}\right) = \frac{x_{-\beta}u}{s} - \frac{ux_{-\beta}}{s} = \frac{ad_{x_{-\beta}}(u)}{s}$$

Therefore, for any $N \in \mathbb{Z}_{\geq 0}$ we have

$$ad_{x_{-\beta}}^{N}\left(\frac{u}{s}\right) = \frac{ad_{x_{-\beta}}^{N}(u)}{s}$$

Since by Proposition 6.1, the action of $ad_{x_{-\beta}}$ on $\mathfrak{U}(L)$ is locally nilpotent, the result follows. □

Proposition 6.8. *Let L be a finite dimensional simple Lie algebra. Let V be a simple admissible L-module, and $\Sigma_V \subseteq \Delta$ be a basis of commuting roots with respect to V. Then for each $\beta \in \Sigma_V$, $x_{-\beta}^{-1}$ is locally ad-nilpotent on $\mathfrak{U}_{\Sigma_V}(L)$. Further, for each $y \in \mathfrak{U}_{\Sigma_V}(L)$, and each $\beta \in \Sigma_V$, we have $ad_{x_{-\beta}^{-1}}^{k}(y) = 0$ if and only if $ad_{x_{-\beta}}^{k}(y) = 0$.*

Proof. Let $y \in \mathfrak{U}_{\Sigma_V}(L)$. Then for each $\beta \in \Sigma_V$ we have

$$ad_{x_{-\beta}^{-1}}^{k}y = \sum_{n=0}^{k}(-1)^{k-n}\begin{pmatrix} k \\ n \end{pmatrix}(x_{-\beta}^{-1})^{n}y(x_{-\beta}^{-1})^{k-n}$$

Therefore

$$\begin{aligned} x^{k}ad_{x_{-\beta}^{-1}}^{k}yx^{k} &= \sum_{n=0}^{k}(-1)^{k-n}\begin{pmatrix} k \\ n \end{pmatrix} x_{-\beta}^{k-n}yx_{-\beta}^{n} \\ &= (-1)^{k}\sum_{n=0}^{k}(-1)^{k-n}\begin{pmatrix} k \\ n \end{pmatrix} x_{-\beta}^{n}yx_{-\beta}^{k-n} \\ &= (-1)^{k}ad_{x_{-\beta}}^{k}y \end{aligned}$$

Therefore

$$ad_{x_{-\beta}^{-1}}^{k}y = (-1)^{k}x^{-k}ad_{x_{-\beta}}^{k}yx^{-k}$$

and

$$ad_{x_{-\beta}}^{k}y = (-1)^{k}x^{k}ad_{x_{-\beta}^{-1}}^{k}yx^{k}$$

for all $y \in \mathfrak{U}_{\Sigma_V}(L)$. Thus $ad_{x_{-\beta}^{-1}}^{k}(y) = 0$ if and only if $ad_{x_{-\beta}}^{k}(y) = 0$. Finally, since $x_{-\beta}$ acts locally ad-nilpotent on $\mathfrak{U}_{\Sigma_V}(L)$, we have that $x_{-\beta}^{-1}$ also acts locally ad-nilpotent on $\mathfrak{U}_{\Sigma_V}(L)$. □

Definition 6.4. *For any $a \in \mathbb{C}$, and $n \in \mathbb{Z}_{\geq 0}$, define*

$$\begin{pmatrix} a \\ n \end{pmatrix} = \frac{a(a-1)\dots(a-n+1)}{n!}$$

with the usual convention that $\begin{pmatrix} a \\ 0 \end{pmatrix} = 1$.

Notice that in the case where $a \in \mathbb{Z}_{>0}$ and $n \leq a$, this coincides with the usual binomial coefficient.

Definition 6.5. *Let L be a finite dimensional simple Lie algebra. Let V be a simple admissible L-module, and $\Sigma_V = \{\beta_1, \dots, \beta_n\} \subseteq \Delta$ be a basis of commuting roots with respect to V. For each $\bar{a} = (a_1, \dots, a_n) \in \mathbb{C}^n$, define $\Phi_{\bar{a}} : \mathfrak{U}_{\Sigma_V}(L) \to \mathfrak{U}_{\Sigma_V}(L)$ as follows: For each $y \in \mathfrak{U}_{\Sigma_V}(L)$ choose $N \in \mathbb{Z}_{\geq 0}$ such that for every $k > N$ we have $ad^k_{x_{-\beta_i}}(y) = 0$ for all i, and set*

$$\Phi_{\bar{a}}(y) = \sum_{k_1=0}^{N} \cdots \sum_{k_n=0}^{N} \begin{pmatrix} a_1 \\ k_1 \end{pmatrix} \cdots \begin{pmatrix} a_n \\ k_n \end{pmatrix} (ad_{x_{-\beta_1}})^{k_1} \dots (ad_{x_{-\beta_n}})^{k_n}(y) x_{-\beta_n}^{-k_n} \dots x_{-\beta_1}^{-k_1}$$

Definition 6.6. *Let L be a finite dimensional simple Lie algebra. Let V be a simple admissible L-module, and $\Sigma_V = \{\beta_1, \dots, \beta_n\} \subseteq \Delta$ be a basis of commuting roots with respect to V. For each $\bar{a} = (a_1, \dots, a_n) \in \mathbb{C}^n$, define $\Phi'_{\bar{a}} : \mathfrak{U}_{\Sigma_V}(L) \to \mathfrak{U}_{\Sigma_V}(L)$ as follows: For each $y \in \mathfrak{U}_{\Sigma_V}(L)$ choose $N \in \mathbb{Z}_{\geq 0}$ such that for every $k > N$ we have $ad^k_{x_{-\beta_i}}(y) = 0$ for all i, and set*

$$\Phi'_{\bar{a}}(y) = \sum_{k_1=0}^{N} \cdots \sum_{k_n=0}^{N} \begin{pmatrix} a_1 \\ k_1 \end{pmatrix} \cdots \begin{pmatrix} a_n \\ k_n \end{pmatrix} (ad_{x^{-1}_{-\beta_1}})^{k_1} \dots (ad_{x^{-1}_{-\beta_n}})^{k_n}(y) x_{-\beta_n}^{k_n} \dots x_{-\beta_1}^{k_1}$$

Lemma 6.4. *Let L be a finite dimensional simple Lie algebra. Let V be a simple admissible L-module, and Σ_V be a basis of commuting roots with respect to V. Let $x_1, \dots, x_n \in \mathfrak{U}_{\Sigma_V}(L)$ be commuting elements, which act locally ad-nilpotent on $\mathfrak{U}_{\Sigma_V}(L)$. Let $y \in \mathfrak{U}_{\Sigma_V}(L)$ and choose $N \in \mathbb{Z}_{\geq 0}$ such that for every $k > N$ we have*

$ad^k_{x_i}(y) = 0$ *for all* i*. Then for any* $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}$ *we have*

$$x_1^{m_1} \ldots x_n^{m_n} y x_n^{-m_n} \ldots x_1^{-m_1} = \sum_{k_1=0}^{N} \cdots \sum_{k_n=0}^{N} \binom{m_1}{k_1} \cdots \binom{m_n}{k_n} \times (ad_{x_1})^{k_1} \ldots (ad_{x_n})^{k_n}(y) x_n^{-k_n} \ldots x_1^{-k_1}$$

Proof. By Lemma 6.2, we have

$$x_n^{m_n} y x_n^{-m_n} = \sum_{k_n=0}^{m_n} \binom{m_n}{k_n} (ad_{x_n})^{k_n}(y) x_n^{-k_n}$$

If $m_n \geq N$, then for all $k_n > m_n$, we have $ad^{k_n}_{x_n}(y) = 0$. Hence

$$\sum_{k_n=0}^{N} \binom{m_n}{k_n} (ad_{x_n})^{k_n}(y) x_n^{-k_n} = \sum_{k_n=0}^{m_n} \binom{m_n}{k_n} (ad_{x_n})^{k_n}(y) x_n^{-k_n} = x_n^{m_n} y x_n^{-m_n}$$

Also, if $m_n < N$, then for any $k_n > m_n$, we have

$$\binom{m_n}{k_n} = \frac{m_n(m_n-1)\ldots(m_n-k_n+1)}{k_n!}$$

Since $k_n > m_n$, the term $m_n - (m_n+1) + 1 = 0$ must appear in the numerator of the above expression. Thus $\binom{m_n}{k_n} = 0$. We therefore still have

$$x_n^{m_n} y x_n^{-m_n} = \sum_{k_n=0}^{N} \binom{m_n}{k_n} (ad_{x_n})^{k_n}(y) x_n^{-k_n}$$

Finally, notice that since the x_i commute, we must also have that ad_{x_i} commutes with ad_{x_j} for all i and j. Thus, for any $i < n$, we have

$$ad^N_{x_i}\left(\sum_{k_{i+1}=0}^{N} \cdots \sum_{k_n=0}^{N} \binom{m_{i+1}}{k_{i+1}} \cdots \binom{m_n}{k_n} \times (ad_{x_{i+1}})^{k_{i+1}} \ldots (ad_{x_n})^{k_n}(y) x_n^{-k_n} \ldots x_{i+1}^{-k_{i+1}} \right) = 0$$

The result follows by induction. □

Corollary 6.2. *Let L be a finite dimensional simple Lie algebra. Let V be a simple admissible L-module, and $\Sigma_V = \{\beta_1, \dots, \beta_n\} \subseteq \Delta$ be a basis of commuting roots with respect to V. Then for any $\bar{m} = (m_1, \dots, m_n)$ with all $m_i \in \mathbb{Z}_{\geq 0}$, we have*

$$\Phi_{\bar{m}}(y) = x_{-\beta_1}^{m_1} \dots x_{-\beta_n}^{m_n} y x_{-\beta_n}^{-m_n} \dots x_{-\beta_1}^{-m_1}$$

and

$$\Phi'_{\bar{m}}(y) = x_{-\beta_1}^{-m_1} \dots x_{-\beta_n}^{-m_n} y x_{-\beta_n}^{m_n} \dots x_{-\beta_1}^{m_1}$$

for all $y \in \mathfrak{U}_{\Sigma_V}(L)$.

Proof. This is simply Lemma 6.4, in the special cases where our set of commuting elements are $x_{-\beta_1}, \dots, x_{-\beta_n}$ and respectively, $x_{-\beta_1}^{-1}, \dots, x_{-\beta_n}^{-1}$. □

Lemma 6.5. *Let $p(x_1, \dots, x_n)$ be a polynomial over $\mathbb{C}$, in n variables. If there exists an infinite set $D \subseteq \mathbb{C}$, for which $p(x_1, \dots, x_n) = 0$ whenever $x_1, \dots, x_n \in D$, then $p(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in \mathbb{C}$.*

Proof. We apply induction on n. If $n = 1$, then either $p = 0$, or

$$p(x) = \prod_{i=1}^{k} (x - a_i)$$

for some $k \in \mathbb{Z}_{>0}$, and some $a_i \in \mathbb{C}$. If $p(x) = 0$ whenever $x \in D$, then

$$D \subseteq \{a_1, \dots, a_k\}$$

Since D is infinite, this is impossible, and hence $p = 0$. Assume the result is true for any polynomial in $n - 1$ variables. Let $d_n \in D$, and let $q_{d_n}(x_1, \dots, x_{n-1}) = p(x_1, \dots, x_{n-1}, d_n)$. Then $q_{d_n}(x_1, \dots, x_{n-1}) = 0$ whenever $x_1, \dots, x_{n-1} \in D$, and hence $q_{d_n}(x_1, \dots, x_{n-1}) = 0$ whenever $x_1, \dots, x_n \in \mathbb{C}$. Therefore, $p(x_1, \dots, x_{n-1}, x_n) = 0$ whenever $x_1, \dots, x_n \in \mathbb{C}$, and $x_n \in D$. Let $c_1, \dots, c_{n-1} \in \mathbb{C}$, and let $q(x_n) = p(c_1, \dots, c_{n-1}, x_n)$. Then $q(x_n) = 0$ whenever $x_n \in D$, and hence $q(x_n) = 0$ for all $x_n \in \mathbb{C}$. Thus, $p(x_1, \dots, x_n) = 0$ for any choice of $x_1, \dots, x_n \in \mathbb{C}$. □

Proposition 6.9. *Let L be a finite dimensional simple Lie algebra. Let V be a simple admissible L-module, and $\Sigma_V = \{\beta_1, \dots, \beta_n\} \subseteq \Delta$ be a basis of commuting roots with respect to V. Then for each $\bar{a} = (a_1, \dots, a_n) \in \mathbb{C}^n$, $\Phi_{\bar{a}}$ is a $\mathfrak{U}_{\Sigma_V}(L)$ automorphism.*

Proof. Let $y_1, y_2 \in \mathfrak{U}_{\Sigma_V}(L)$, and choose N_1, N_2 such that for any $k_1 > N_1$ and any $k_2 > N_2$, we have $ad^{k_1}_{x_{-\beta_i}}(y_1) = 0$ and $ad^{k_2}_{x_{-\beta_i}}(y_2) = 0$ for all i. Setting $N = \max\{N_1, N_2\}$ we have, for any $k > N$,

$$ad^{k}_{x_{-\beta_i}}(y_1) = ad^{k}_{x_{-\beta_i}}(y_2) = ad^{k}_{x_{-\beta_i}}(y_1 + y_2) = 0$$

for all i and hence by linearity of the $ad_{x_{-\beta_i}}$ we have that

$$\Phi_{\bar{a}}(y_1 + y_2) = \Phi_{\bar{a}}(y_1) + \Phi_{\bar{a}}(y_2)$$

for any $\bar{a} \in \mathbb{C}^n$. We now intend to show that $\Phi_{\bar{a}}(y_1 y_2) = \Phi_{\bar{a}}(y_1)\Phi_{\bar{a}}(y_2)$ for all $\bar{a} \in \mathbb{C}^n$. As in the proof of Proposition 6.1, we have that, for any $k > N_1 + N_2$, $ad^{k}_{x_{-\beta_i}}(y_1 y_2) = 0$ for all i. Hence

$$\Phi_{\bar{a}}(y_1 y_2) = \sum_{k_1=0}^{N_1+N_2} \cdots \sum_{k_n=0}^{N_1+N_2} \binom{a_1}{k_1} \cdots \binom{a_n}{k_n} u_{k_1 \ldots k_n}$$

and

$$\Phi_{\bar{a}}(y_1)\Phi_{\bar{a}}(y_2) = \left(\sum_{k_1=0}^{N_1} \cdots \sum_{k_n=0}^{N_1} \binom{a_1}{k_1} \cdots \binom{a_n}{k_n} v_{k_1 \ldots k_n} \right) \times \left(\sum_{k_1=0}^{N_2} \cdots \sum_{k_n=0}^{N_2} \binom{a_1}{k_1} \cdots \binom{a_n}{k_n} w_{k_1 \ldots k_n} \right)$$

Where $u_{k_1 \ldots k_n}, v_{k_1 \ldots k_n}, w_{k_1 \ldots k_n} \in \mathfrak{U}_{\Sigma_V}(L)$ for all k_i, and do not depend on $\bar{a}$. Consider $\Phi_{\bar{a}}(y_1 y_2) - \Phi_{\bar{a}}(y_1)\Phi_{\bar{a}}(y_2)$. Let $\mathfrak{B}$ be a basis for $\mathfrak{U}_{\Sigma_V}(L)$. We can write

$$\Phi_{\bar{a}}(y_1 y_2) - \Phi_{\bar{a}}(y_1)\Phi_{\bar{a}}(y_2) = \sum_{v \in \mathfrak{B}} c_v v$$

for some $c_v \in \mathbb{C}$. Further, since the coefficients appearing in $\Phi_{\bar{a}}(y_1 y_2)$ and $\Phi_{\bar{a}}(y_1)\Phi_{\bar{a}}(y_2)$ are all polynomials in $a_1, \ldots a_n$, we have that

$$\Phi_{\bar{a}}(y_1 y_2) - \Phi_{\bar{a}}(y_1)\Phi_{\bar{a}}(y_2) = \sum_{v \in \mathfrak{B}} p_v(a_1, \ldots, a_n) v$$

where each $p_v(a_1, \ldots, a_n)$ is a polynomial in variables $a_1, \ldots, a_n$. Further, whenever $a_i \in \mathbb{Z}_{\geq 0}$ for all i, due to Corollary 6.2, we have

$$\Phi_{\bar{a}}(y_1 y_2) - \Phi_{\bar{a}}(y_1)\Phi_{\bar{a}}(y_2) = \sum_{v \in \mathfrak{B}} p_v(a_1, \ldots, a_n) v = 0$$

and hence $p_v(a_1, \ldots, a_n) = 0$ for each $v \in \mathfrak{B}$. By Lemma 6.5, we have that $p_v(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in \mathbb{C}$. Thus

$$\Phi_{\bar{a}}(y_1 y_2) - \Phi_{\bar{a}}(y_1)\Phi_{\bar{a}}(y_2) = 0$$

for all $\bar{a} \in \mathbb{C}^n$. It only remains to show that each $\Phi_{\bar{a}}$ is a bijection. We claim that $\Phi'_{\bar{a}}$ is an inverse for $\Phi_{\bar{a}}$. To see this, let $y \in \mathfrak{U}_{\Sigma_V}(L)$, and consider

$$\Phi'_{\bar{a}}\Phi_{\bar{a}}(y) - y$$

Due to Corollary 6.2, we have

$$\Phi'_{\bar{a}}\Phi_{\bar{a}}(y) - y = 0$$

whenever $a_i \in \mathbb{Z}_{\geq 0}$ for all i. As before, since

$$\Phi'_{\bar{a}}\Phi_{\bar{a}}(y) - y = \sum_{v \in \mathfrak{B}} q_v(a_1, \ldots, a_n)v$$

for some polynomials q_v in variables $a_1, \ldots, a_n$, and each $q_v(a_1, \ldots, a_n) = 0$ whenever $a_1, \ldots, a_n \in \mathbb{Z}$, by Lemma 6.5, we must have that $q_v(a_1, \ldots, a_n) = 0$ for all choices of $a_1, \ldots, a_n \in \mathbb{C}$. Hence

$$\Phi'_{\bar{a}}\Phi_{\bar{a}}(y) - y = 0$$

for all $\bar{a} \in \mathbb{C}^n$ and all $y \in \mathfrak{U}_{\Sigma_V}(L)$. □

6.4 Characterization of Simple Torsion Free Modules

Definition 6.7. *Let L be a finite dimensional simple Lie algebra with Cartan subalgebra H. A* **coherent family** $\mathcal{M}$ *is an admissible L-module of degree d such that*

i) $\operatorname{Supp} \mathcal{M} = H^*$.

ii) $\dim \mathcal{M}_\lambda = d$ *for all* $\lambda \in H^*$.

iii) For any $u \in \mathfrak{U}(L)_0$ there exists a polynomial $p(x)$ such that $p(\lambda) = Tr\, u|_{\mathcal{M}_\lambda}$ for all $\lambda \in H^*$.

Where $\mathfrak{U}(L)_0$ denotes the zero weight space of $\mathfrak{U}(L)$ with respect to the adjoint representation of L on $\mathfrak{U}(L)$. We say $\mathcal{M}$ is **irreducible** *provided there exists $\lambda \in H^*$ such that the $\mathfrak{U}(L)_0$ module $\mathcal{M}_\lambda$ is simple.*

Recall that Σ is a basis for Q, and hence Σ is a basis for H^*.

Definition 6.8. *Let L be a finite dimensional simple Lie algebra. Let V be a simple admissible L-module, and $\Sigma = \{\beta_1, \dots, \beta_n\} \subseteq \Delta$ be a basis of commuting roots with respect to V. Suppose the action of L on V is given by $\phi : L \to gl(V)$. Let*

$$\phi' : \mathfrak{U}_\Sigma(L) \to gl(V^\Sigma)$$

be the extension of ϕ to $\mathfrak{U}_\Sigma(L)$. Let $\nu \in H^*$, i.e.

$$\nu = \sum_{i=1}^{n} a_i \beta_i$$

for some $a_i \in \mathbb{C}$. Define the L-module $V^\Sigma[\nu]$ to be the vector space V^Σ under the action given by $\phi' \circ \Phi_{\bar{a}}$ restricted to L, where $\bar{a} = (a_1, \dots, a_n)$.

Proposition 6.10. *Let L be a finite dimensional simple Lie algebra with Cartan subalgebra H, and V be a simple admissible L-module. Let $\Sigma = \{\beta_1, \dots, \beta_n\} \subseteq \Delta$ be a basis of commuting roots with respect to V. If*

$$\nu = \sum_{i=1}^{n} a_i \beta_i \in H^*$$

and

$$\kappa = \sum_{i=1}^{n} b_i \beta_i \in H^*$$

such that $b_i - a_i \in \mathbb{Z}$ for each i, then $V^\Sigma[\nu] \simeq V^\Sigma[\kappa]$.

Proof. Let the action of $\mathfrak{U}_\Sigma(L)$ on V^Σ be given by the map

$$\phi : \mathfrak{U}_\Sigma(L) \to gl(V^\Sigma)$$

Let $m_i = b_i - a_i$ for each $1 \le i \le n$. Define the map $\sigma : V^\Sigma \to V^\Sigma$ by

$$\sigma(v) = \phi(x_{-\beta_1})^{m_1} \dots \phi(x_{-\beta_n})^{m_n}(v)$$

for all $v \in V^{\Sigma}$. Then σ is also a vector space isomorphism of $V^{\Sigma}[\nu]$ onto $V^{\Sigma}[\kappa]$. Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$. We claim that for all $y \in L$ and all $v \in V^{\Sigma}$,

$$\sigma((\phi \circ \Phi_{\bar{a}})(y)(v)) = (\phi \circ \Phi_{\bar{b}})(y)(\sigma(v)) \qquad (1)$$

and hence that $V^{\Sigma}[\nu] \simeq V^{\Sigma}[\kappa]$. Let $y \in L$. On the left hand side of (1), we have

$$\begin{aligned}
\sigma((\phi\circ\Phi_{\bar{a}})(y)(v)) &= \sigma\left(\phi\left(\sum_{k_1=0}^{N}\cdots\sum_{k_n=0}^{N}\binom{a_1}{k_1}\cdots\binom{a_n}{k_n}u_{k_1\ldots k_n}\right)(v)\right)\\
&= \phi(x_{-\beta_1}^{m_1}\ldots x_{-\beta_n}^{m_n})\phi\left(\sum_{k_1=0}^{N}\cdots\sum_{k_n=0}^{N}\binom{a_1}{k_1}\cdots\binom{a_n}{k_n}u_{k_1\ldots k_n}\right)(v)\\
&= \phi\left(\sum_{k_1=0}^{N}\cdots\sum_{k_n=0}^{N}\binom{a_1}{k_1}\cdots\binom{a_n}{k_n}v_{k_1\ldots k_n}\right)(v)
\end{aligned}$$

Where

$$u_{k_1,\ldots,k_n} = (ad_{x_{-\beta_1}})^{k_1}\ldots(ad_{x_{-\beta_n}})^{k_n}(y)x_{-\beta_n}^{-k_n}\ldots x_{-\beta_1}^{-k_1} \in \mathfrak{U}_{\Sigma}(L)$$

and

$$v_{k_1,\ldots,k_n} = x_{-\beta_1}^{m_1}\ldots x_{-\beta_n}^{m_n}u_{k_1,\ldots,k_n} \in \mathfrak{U}_{\Sigma}(L)$$

Letting $\mathfrak{B}_{\mathfrak{U}_{\Sigma}(L)}$ be a basis of $\mathfrak{U}_{\Sigma}(L)$, we have that

$$\sigma((\phi \circ \Phi_{\bar{a}})(y)(v)) = \phi\left(\sum_{v\in\mathfrak{B}_{\mathfrak{U}_{\Sigma}(L)}} p_x(a_1,\ldots,a_n)x\right)(v)$$

where the $p_x(a_1, \ldots, a_n)$ are polynomials in variables $a_1, \ldots, a_n$. Likewise, on the right hand side of (1), we have

$$\begin{aligned}
(\phi\circ\Phi_{\bar{b}})(y)(\sigma(v)) &= \phi\left(\sum_{k_1=0}^{N}\cdots\sum_{k_n=0}^{N}\binom{b_1}{k_1}\cdots\binom{b_n}{k_n}u_{k_1\ldots k_n}\right)(\sigma(v))\\
&= \phi\left(\sum_{k_1=0}^{N}\cdots\sum_{k_n=0}^{N}\binom{b_1}{k_1}\cdots\binom{b_n}{k_n}u_{k_1\ldots k_n}x_{-\beta_1}^{m_1}\ldots x_{-\beta_n}^{m_n}\right)(v)\\
&= \phi\left(\sum_{k_1=0}^{N}\cdots\sum_{k_n=0}^{N}\binom{b_1}{k_1}\cdots\binom{b_n}{k_n}w_{k_1\ldots k_n}\right)(v)
\end{aligned}$$

Where

$$w_{k_1,\ldots,k_n} = u_{k_1,\ldots,k_n} x_{-\beta_1}^{m_1} \cdots x_{-\beta_n}^{m_n} \in \mathfrak{U}_\Sigma(L)$$

Therefore

$$(\phi \circ \Phi_{\bar{b}})(y)(\sigma(v)) = \phi\left(\sum_{x \in \mathfrak{B}_{\mathfrak{U}_\Sigma(L)}} q_x(b_1, \ldots, b_n)x\right)(v)$$

where the $q_x(a_1, \ldots, a_n)$ are polynomials in variables $b_1, \ldots, b_n$. Hence

$$\sigma((\phi \circ \Phi_{\bar{a}})(y)(v)) - (\phi \circ \Phi_{\bar{b}})(y)(\sigma(v)) = \phi\left(\sum_{x \in \mathfrak{B}_{\mathfrak{U}_\Sigma(L)}} f_x(a_1, \ldots, a_n, b_1, \ldots, b_n)x\right)(v)$$

where the $f_x(a_1, \ldots, a_n, b_1, \ldots, b_n)$ are polynomials in variables $a_1, \ldots, a_n, b_1, \ldots, b_n$. If $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Z}_{\geq 0}$, Corollary 6.2 implies that

$$\begin{aligned}
\sigma((\phi \circ \Phi_{\bar{a}})(y)(v)) &= \phi(x_{-\beta_1}^{m_1} \cdots x_{-\beta_n}^{m_n})\phi\left(x_{-\beta_1}^{a_1} \cdots x_{-\beta_n}^{a_n} y x_{-\beta_n}^{-a_n} \cdots x_{-\beta_1}^{-a_1}\right)(v) \\
&= \phi\left(x_{-\beta_1}^{m_1} \cdots x_{-\beta_n}^{m_n} x_{-\beta_1}^{a_1} \cdots x_{-\beta_n}^{a_n} y x_{-\beta_n}^{-a_n} \cdots x_{-\beta_1}^{-a_1}\right)(v) \\
&= \phi\left(x_{-\beta_1}^{a_1+m_1} \cdots x_{-\beta_n}^{a_n+m_n} y x_{-\beta_n}^{-a_n-m_n} \cdots x_{-\beta_1}^{-a_1-m_1} x_{-\beta_1}^{m_1} \cdots x_{-\beta_n}^{m_n}\right)(v) \\
&= \phi\left(x_{-\beta_1}^{b_1} \cdots x_{-\beta_n}^{b_n} y x_{-\beta_n}^{-b_n} \cdots x_{-\beta_1}^{-b_1}\right)\phi(x_{-\beta_1}^{m_1} \cdots x_{-\beta_n}^{m_n})(v) \\
&= (\phi \circ \Phi_{\bar{b}})(y)(\sigma(v))
\end{aligned}$$

Thus, by Lemma 6.5, the polynomials $f_x(a_1, \ldots, a_n, b_1, \ldots, b_n) = 0$ for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}$. This proves (1), which gives the desired result. □

The previous proposition implies that for each coset $\bar{\nu} = \nu + Q \in H^*/Q$, where $\nu \in H^*$, we can define, up to equivalence, the module $V^\Sigma[\bar{\nu}]$ by

$$V^\Sigma[\bar{\nu}] = V^\Sigma[\nu]$$

Proposition 6.11. *Let L be a finite dimensional simple Lie algebra with Cartan subalgebra H, and V be a simple admissible L-module. Let Σ be a basis of commuting roots with respect to V. If $\bar{\nu} \in H^*/Q$ then* $\operatorname{Supp} V^\Sigma[\bar{\nu}] = \bar{\nu} + \operatorname{Supp} V$.

Proof. Let ϕ be the map defining the action of L on V^Σ. Let $\Sigma = \{\beta_1, \ldots, \beta_n\}$. Choose $\nu \in H^*$ with $\bar{\nu} = \nu + Q$. Then

$$\nu = \sum_{i=1}^{n} a_i \beta_i$$

for some $a_i \in \mathbb{C}$. Set $\bar{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$. Then by definition, $V^\Sigma[\bar{\nu}]$ is the vector space V^Σ under the action given by $\phi \circ \Phi_{\bar{a}}$. Let $h \in H$. Notice that for any $\beta_i \in \Sigma$, we have $ad^2_{x_{-\beta_i}}(h) = 0$. Therefore

$$\Phi_{\bar{a}}(h) = \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} \begin{pmatrix} a_1 \\ k_1 \end{pmatrix} \cdots \begin{pmatrix} a_n \\ k_n \end{pmatrix} (ad_{x_{-\beta_1}})^{k_1} \ldots (ad_{x_{-\beta_n}})^{k_n}(h) x_{-\beta_n}^{-k_n} \ldots x_{-\beta_1}^{-k_1}$$

Further, since $ad_{x_{-\beta_i}}(x_{-\beta_j}) = 0$ for all i, j, the above equation reduces to

$$\begin{aligned} \Phi_{\bar{a}}(h) &= h + \sum_{i=1}^{n} \begin{pmatrix} a_i \\ 1 \end{pmatrix} ad_{x_{-\beta_i}}(h) x_{-\beta_i}^{-1} \\ &= h - \sum_{i=1}^{n} \begin{pmatrix} a_i \\ 1 \end{pmatrix} ad_h(x_{-\beta_i}) x_{-\beta_i}^{-1} \\ &= h + \sum_{i=1}^{n} a_i \beta_i(h) \\ &= h + \nu(h) \end{aligned}$$

Let $v \in V^\Sigma[\bar{\nu}]$, such that v is a weight vector of V^Σ. i.e. $\phi(h)(v) = \gamma(h)v$ for some $\gamma \in \operatorname{Supp} V^\Sigma$ and all $h \in H$. Then for any $h \in H$ we have

$$(\phi \circ \Phi_{\bar{a}})(h)(v) = \phi(h + \nu(h))v = (\gamma + \nu)(h)v$$

Hence v is a weight vector of $V^\Sigma[\bar{\nu}]$ with weight $\gamma + \nu$. Therefore

$$\operatorname{Supp} V^\Sigma[\bar{\nu}] = \nu + \operatorname{Supp} V^\Sigma$$

Since by Proposition 6.5 $\operatorname{Supp} V^\Sigma = \operatorname{Supp} V + Q$, we have our result. □

Corollary 6.3. *Let L be a finite dimensional simple Lie algebra with Cartan subalgebra H. Let $\bar{\nu} \in H^*/Q$. If V is a simple admissible L-module, then* $\operatorname{Supp} V^\Sigma[\bar{\nu}] = \bar{\nu} + \bar{\lambda}$ *for some $\lambda \in H^*$, where Σ is a basis of commuting roots with respect to V.*

Proof. By Proposition 5.5, we have that $\operatorname{Supp} V \subseteq \bar{\lambda}$ for some $\lambda \in H^*$. The previous proposition implies that

$$\operatorname{Supp} V^\Sigma[\bar{\nu}] \subseteq \bar{\nu} + \bar{\lambda}$$

Conversely, since $\{\lambda\} \subseteq \operatorname{Supp} V$, we have that

$$\{\lambda\} + \bar{\nu} \subseteq \operatorname{Supp} V^{\Sigma}[\bar{\nu}]$$

Since, $\{\lambda\} + \bar{\nu} = \lambda + \nu + Q = \bar{\lambda} + \bar{\nu}$, we have our result. □

Definition 6.9. *Let L be a finite dimensional simple Lie algebra with Cartan subalgebra H, and V be a simple admissible L-module. Let $\Sigma \subseteq \Delta$ be a basis of commuting roots with respect to V. Define the module*

$$\mathcal{M}_{\Sigma}(V) = \bigoplus_{\bar{\nu} \in H^*/Q} V^{\Sigma}[\bar{\nu}]$$

Proposition 6.12. *(Mathieu) Let L be a finite dimensional simple Lie algebra, and V be a simple admissible L-module. Let Σ be a basis of commuting roots with respect to V. Then $\mathcal{M}_{\Sigma}(V)$ is a coherent family with degree equal to the degree of V. Further, V is a submodule of $\mathcal{M}_{\Sigma}(V)$.*

Proof. (See Lemma 4.5 in [15]) □

Property 6.1. *Let L be a finite dimensional simple Lie algebra with Cartan subalgebra H. Let $\bar{\nu} \in H^*/Q$. If V is a simple admissible L-module then*

$$V^{\Sigma}[\bar{\nu} - \bar{\lambda}] = \bigoplus_{\gamma \in \bar{\nu}} \mathcal{M}_{\Sigma}(V)_{\gamma}$$

where $\lambda \in H^*$ with $\operatorname{Supp} V \subseteq \bar{\lambda}$.

Proof. Clearly $V^{\Sigma}[\bar{\nu} - \bar{\lambda}] \leq \mathcal{M}_{\Sigma}(V)$, hence $V^{\Sigma}[\bar{\nu} - \bar{\lambda}]_{\gamma} \leq \mathcal{M}_{\Sigma}(V)_{\gamma}$ for each $\gamma \in \operatorname{Supp} V^{\Sigma}[\bar{\nu} - \bar{\lambda}]$. By Corollary 6.3, we have $\operatorname{Supp} V^{\Sigma}[\bar{\nu} - \bar{\lambda}] = \bar{\nu}$ and hence $V^{\Sigma}[\bar{\nu} - \bar{\lambda}]_{\gamma} \leq \mathcal{M}_{\Sigma}(V)_{\gamma}$ for all $\gamma \in \bar{\nu}$. However, since by Proposition 6.5,

$$\dim V^{\Sigma}[\bar{\nu} - \bar{\lambda}]_{\gamma} = \deg V = \dim \mathcal{M}_{\Sigma}(V)_{\gamma}$$

we must have

$$V^{\Sigma}[\bar{\nu} - \bar{\lambda}]_{\gamma} = \mathcal{M}_{\Sigma}(V)_{\gamma}$$

for all $\gamma \in \bar{\nu}$. Thus

$$V^{\Sigma}[\bar{\nu} - \bar{\lambda}] = \bigoplus_{\gamma \in \bar{\nu}} V^{\Sigma}[\bar{\nu} - \bar{\lambda}]_{\gamma} = \bigoplus_{\gamma \in \bar{\nu}} \mathcal{M}_{\Sigma}(V)_{\gamma}$$

□

Notice that by Lemma 5.2, since the module $\mathcal{M}_\Sigma(V)$ is admissible, it has finite length. We can therefore apply the Jordan-Hölder theorem to make the following definition:

Definition 6.10. *Let L be a finite dimensional simple Lie algebra with Cartan subalgebra H, and let V be a simple admissible L-module. Let Σ be a basis of commuting roots with respect to V. Define the module $\mathcal{M}_{ss}(V)$ to be the module such that for each $\bar{\nu} \in H^*/Q$, the module*

$$\bigoplus_{\gamma \in \bar{\nu}} \mathcal{M}_{ss}(V)_\gamma$$

is the direct sum of the simple quotients in any composition series of

$$\bigoplus_{\gamma \in \bar{\nu}} \mathcal{M}_\Sigma(V)_\gamma$$

Notice that for each $\bar{\nu} \in H^*/Q$, the module $\bigoplus_{\gamma \in \bar{\nu}} \mathcal{M}_{ss}(V)_\gamma$ is semisimple.

Definition 6.11. *Let L be a finite dimensional simple Lie algebra with Cartan subalgebra H. A coherent family $\mathcal{M}$ of L is said to be* **semisimple** *provided for each $\bar{\nu} \in H^*/Q$, the module $\bigoplus_{\gamma \in \bar{\nu}} \mathcal{M}_\gamma$ is semisimple.*

Lemma 6.6. *(Mathieu) Let L be a finite dimensional simple Lie algebra, and V be a simple admissible L-module with degree d. Then the following hold:*

i) There exists a unique semisimple coherent family $\mathcal{M}$ of degree d such that V is a submodule of $\mathcal{M}$.

ii) Such a coherent family $\mathcal{M}$ is irreducible.

iii) $\mathcal{M} \simeq \mathcal{M}_{ss}(V)$

iv) If V' is any infinite dimensional submodule of $\mathcal{M}$ then V is admissible, and $\deg V' = d$.

v) All simple submodules of $\mathcal{M}$ have the same central character.

Proof. (See Proposition 4.8 in [15]) □

In particular, the previous lemma implies that $\mathcal{M}_{ss}(V)$ is independent of the choice of commuting roots Σ. Further, if V' is any simple infinite dimensional submodule of $\mathcal{M}_{ss}(V)$, then $\mathcal{M}_{ss}(V') \simeq \mathcal{M}_{ss}(V)$. And V' has the same central character as V.

Lemma 6.7. *Let L be a finite dimensional simple Lie algebra with Cartan subalgebra H and root system Δ. Let V be a simple admissible L-module. Then $\mathcal{M}_{ss}(V)$ contains a simple admissible highest weight module $L_B(\lambda)$ for some $\lambda \in H^*$ relative to some base B of Δ.*

Proof. (See Proposition 5.7 in [15]) □

In view of the previous Lemma, for each simple admissible L-module, V, and base B of Δ, we define the following sets:

$$HW_B(\mathcal{M}_{ss}(V)) = \{\lambda \in \mathcal{H}^* \mid L_B(\lambda) \text{ is a submodule of } \mathcal{M}_{ss}(V)\}$$

and

$$\overline{HW}_B(\mathcal{M}_{ss}(V)) = \{\lambda + Q \mid \lambda \in HW_B(\mathcal{M}_{ss}(V))\}$$

Recall Proposition 5.19, that every simple module admits a central character, and Proposition 5.21, that every central character is some χ_λ with $\lambda \in H^*$.

Proposition 6.13. *(Mathieu) Let L be a finite dimensional simple Lie algebra with Cartan subalgebra H, root system Δ and Weyl group $\mathcal{W}$. Let B be a base for Δ, and V be a simple admissible L-module with central character χ_λ for some $\lambda \in H^*$. Then*

$$\overline{HW}_B(\mathcal{M}_{ss}(V)) = \{\sigma(\lambda + \rho) - \rho + Q \mid \sigma \in \mathcal{W}\}$$

Proof. (See Lemma 10.1 in [15]) □

In particular, the previous proposition implies that the set $\overline{HW}_B(\mathcal{M}_{ss}(V))$ is independent of the choice of base B. Therefore if V is any simple admissible L-module, then $\mathcal{M}_{ss}(V)$ contains a simple admissible highest weight module $L(\lambda)$ for some $\lambda \in H^*$, relative to any base of Δ.

Theorem 6.1. *Let L be a finite dimensional simple Lie algebra with Cartan subalgebra H and root system Δ. Let V be a simple torsion free L-module. Then there exist $\lambda \in H^*$ and $\bar{\nu} \in H^*/Q$ such that*

$$V \simeq L(\lambda)^{\Sigma}[\bar{\nu}]$$

where $\Sigma \subseteq \Delta$ is a basis of Q consisting of commuting roots such that $x_{-\beta}$ acts injectively on $L(\lambda)$ for all $\beta \in \Sigma$.

Proof. Let $\deg V = d$. Since V is torsion free, by Proposition 5.13, we have that $\operatorname{Supp} V = \bar{\gamma}$ for some $\bar{\gamma} \in H^*/Q$, and every weight space of V has dimension equal to d. Further, since V is a submodule of $\mathcal{M}_{ss}(V)$, and every weight space of $\mathcal{M}_{ss}(V)$ has dimension equal to d, we must have

$$V = \bigoplus_{\kappa \in \bar{\gamma}} \mathcal{M}_{ss}(V)_\kappa$$

Due to Lemma 6.7 and Proposition 6.13, we can choose $\lambda \in H^*$ such that $L(\lambda)$ is a simple admissible highest weight submodule of $\mathcal{M}_{ss}(V)$. By Lemma 6.6, since $\mathcal{M}_{ss}(V)$ is a semisimple coherent family of degree d, containing $L(\lambda)$, we must have

$$\mathcal{M}_{ss}(V) \simeq \mathcal{M}_{ss}(L(\lambda))$$

Therefore

$$V \simeq \bigoplus_{\kappa \in \bar{\gamma}} \mathcal{M}_{ss}(L(\lambda))_\kappa$$

Recall that, by definition, $\mathcal{M}_{ss}(L(\lambda))$ has the property that the module $\bigoplus_{\kappa \in \bar{\gamma}} \mathcal{M}_{ss}(L(\lambda))_\kappa$ has the same composition series as $\bigoplus_{\kappa \in \bar{\gamma}} \mathcal{M}_\Sigma(L(\lambda))_\kappa$. However, since V is simple, the composition series of V is V itself, and hence

$$V \simeq \bigoplus_{\kappa \in \bar{\gamma}} \mathcal{M}_\Sigma(L(\lambda))_\kappa$$

Finally, Property 6.1 implies that

$$\bigoplus_{\kappa \in \bar{\gamma}} \mathcal{M}_\Sigma(L(\lambda))_\kappa = L(\lambda)^\Sigma[\bar{\gamma} - \bar{\lambda}]$$

Taking $\bar{\nu} = \bar{\gamma} - \bar{\lambda}$ gives the desired result. □

Notice that in the previous theorem, since the simple admissible highest weight module $L(\lambda)$ is a submodule of $\mathcal{M}_{ss}(V)$, due to Lemma 6.6, we must have that the central character of V is χ_λ.

We now have that every simple torsion free module is equivalent to the module $L(\lambda)^\Sigma[\bar{\nu}]$ for some $\bar{\nu} \in H^*/Q$, and some simple admissible highest weight module $L(\lambda)$. In the next section, we give a characterization of all simple admissible highest weight modules for C_n.

7 Simple Admissible Highest Weight C_n-Modules

In this section, as in the previous section, we follow the work of Mathieu in [15], giving several equivalent characterizations of simple admissible highest weight C_n-modules. For the remainder of this thesis, unless otherwise stated, the algebra is assumed to be C_n, with fixed Cartan subalgebra $\mathcal{H}$ as given in Section 3.7. Δ is the root system with respect to $\mathcal{H}$. $\Delta^{++} = \{\alpha_1, \ldots, \alpha_n\}$ is the base for Δ given in Section 3.7, with Δ^+ and Δ^- being the positive and, respectively, negative roots with respect to Δ^{++}. $\mathfrak{H} = \{h_1, \ldots, h_n\}$ denotes the simple basis of $\mathcal{H}$. $\mathfrak{E} = \{\epsilon_1, \ldots, \epsilon_n\}$ and $\mathfrak{F} = \{\omega_1, \ldots, \omega_n\}$ are the epsilon and, respectively, fundamental bases for $\mathcal{H}^*$. $\mathcal{W}$ denotes the Weyl group of C_n, and

$$\rho = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta = \sum_{i=1}^{n} \omega_i$$

Further, the epsilon basis is orthonormal with respect to the inner product $(\cdot, \cdot)$.

Proposition 7.1. *(Mathieu) Let $\lambda \in \mathcal{H}^*$. The simple highest weight C_n-module $L(\lambda)$ is admissible if and only if λ satisfies the following three conditions:*

i) $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ *for all* $i \neq n$

ii) $\lambda(h_n) \in \frac{1}{2} + \mathbb{Z}$

iii) $\lambda(h_{n-1} + 2h_n) \in \mathbb{Z}_{\geq -2}$

Proof. (See Lemma 9.2 in [15]) □

Recall that $\lambda(h_i) = \langle \lambda, \alpha_i \rangle$, and that $\widetilde{\Delta}^+ = \{\epsilon_i \pm \epsilon_j \mid i < j\}$ is the subset of Δ consisting of all positive short roots. We now make the following observation, based on Mathieu's characterization of admissible C_n modules:

Corollary 7.1. *Let $\lambda = \sum_{i=1}^{n} a_i \epsilon_i \in \mathcal{H}^*$. The simple highest weight C_n-module $L(\lambda)$ is admissible if and only if $a_i \in \frac{1}{2} + \mathbb{Z}$ for all i, and $(\lambda + \rho, \alpha) > 0$ for all $\alpha \in \widetilde{\Delta}^+$.*

Proof. First, suppose $L(\lambda)$ is admissible. Condition *ii)* implies that

$$a_n \in \frac{1}{2} + \mathbb{Z}$$

since

$$\lambda(h_n) = \langle \lambda, 2\epsilon_n \rangle = 2\frac{(\lambda, 2\epsilon_n)}{(2\epsilon_n, 2\epsilon_n)} = (\lambda, \epsilon_n) = a_n$$

Further, condition $i)$ implies that for $1 \le i < n$, if $a_{i+1} \in \frac{1}{2} + \mathbb{Z}$ then $a_i \in \frac{1}{2} + \mathbb{Z}$. We therefore have that $a_i \in \frac{1}{2} + \mathbb{Z}$ for all i. If $1 \le i < j \le n$, then

$$(\lambda + \rho, \epsilon_i - \epsilon_j) = (\lambda + \rho, \sum_{k=i}^{j-1} \alpha_k) = j - i + \sum_{k=i}^{j-1} \lambda(h_k) > 0$$

due to condition $i)$. Condition $iii)$ implies that

$$\begin{aligned}(\lambda + \rho, \epsilon_{n-1} + \epsilon_n) &= (\lambda + \rho, \alpha_{n-1} + \alpha_n) \\ &= \langle \lambda, \alpha_{n-1} \rangle + 2\langle \lambda, \alpha_n \rangle + 3 \\ &= \lambda(h_{n-1} + 2h_n) + 3 > 0\end{aligned}$$

Finally, if $1 \le i < j < n$, we have that

$$\begin{aligned}(\lambda + \rho, \epsilon_i + \epsilon_j) = (\lambda + \rho, \epsilon_i - \epsilon_j) &+ 2(\lambda + \rho, \epsilon_j - \epsilon_{n-1}) \\ &+ (\lambda + \rho, \epsilon_{n-1} - \epsilon_n) + (\lambda + \rho, \epsilon_{n-1} + \epsilon_n) \\ > 0 &\end{aligned}$$

Next, suppose that $a_i \in \frac{1}{2} + \mathbb{Z}$ for all i, and that $(\lambda + \rho, \alpha) > 0$ for all $\alpha \in \widetilde{\Delta}^+$. For $i \ne n$, we have $\lambda(h_i) = a_i - a_{i+1} \in \mathbb{Z}$, and

$$\lambda(h_i) = (\lambda + \rho, \alpha_i) - 1 \ge 0$$

since $\alpha_i \in \widetilde{\Delta}^+$. Thus condition $i)$ in Mathieu's characterization holds. $\lambda(h_n) = a_n \in \frac{1}{2} + \mathbb{Z}$, and hence condition $ii)$ holds as well. Finally,

$$\lambda(h_{n-1} + 2h_n) = (\lambda, \epsilon_{n-1} + \epsilon_n) = a_{n-1} + a_n \in \mathbb{Z}$$

and $\lambda(h_{n-1} + 2h_n) = (\lambda + \rho, \epsilon_{n-1} + \epsilon_n) - 3 \ge -2$. Thus condition $iii)$ holds as well. □

Corollary 7.2. *Let $\lambda = \sum_{i=1}^n a_i\epsilon_i \in \mathcal{H}^*$. Let $\lambda + \rho = \sum_{i=1}^n c_i\epsilon_i$. The simple highest weight C_n-module $L(\lambda)$ is admissible if and only if $c_i \in \frac{1}{2} + \mathbb{Z}$ for all i, and $c_i > |c_j|$ for all $i < j$. i.e. $c_1 > c_2 > \cdots > c_{n-1} > |c_n|$.*

Proof. Clearly, $a_i \in \frac{1}{2} + \mathbb{Z}$ if and only if $c_i \in \frac{1}{2} + \mathbb{Z}$. Further, $(\lambda + \rho, \epsilon_i - \epsilon_j) > 0$ if and only if $c_i > c_j$ and $(\lambda + \rho, \epsilon_i + \epsilon_j) > 0$ if and only if $c_i > -c_j$. □

Lemma 7.1. *If $\chi : Z(\mathfrak{U}) \to \mathbb{C}$ is a central character, then there exists at most two non-equivalent simple admissible highest weight C_n-modules having central character χ. Further, if $L(\lambda_1)$ and $L(\lambda_2)$ are non-equivalent simple admissible highest weight C_n-modules having central character χ, then*

$$\lambda_1 - \lambda_2 \notin Q$$

Proof. Suppose $\chi_{\lambda_1} = \chi_{\lambda_2}$ for some $\lambda_1, \lambda_2 \in \mathcal{H}^*$ such that $L(\lambda_1)$ and $L(\lambda_2)$ are admissible. By Theorem 5.3, there exists $\sigma \in \mathcal{W}$ such that $\sigma(\lambda_1 + \rho) = \lambda_2 + \rho$. By Corollary 7.2, we have

$$\lambda_1 + \rho = \sum_{i=1}^{n} c_i \epsilon_i$$

for some $c_i \in \mathbb{C}$ with $c_1 > c_2 > \cdots > c_{n-1} > |c_n|$. Likewise,

$$\lambda_2 + \rho = \sum_{i=1}^{n} d_i \epsilon_i$$

for some $d_i \in \mathbb{C}$ with $d_1 > d_2 > \cdots > d_{n-1} > |d_n|$. Since $\mathcal{W}$ is the group of all permutations and sign changes on the subscripts of the ϵ_i, we see that the only possibility, if $\lambda_1 \neq \lambda_2$, is that

$$\lambda_2 + \rho = \left(\sum_{i=1}^{n-1} c_i \epsilon_i \right) - c_n \epsilon_n$$

Assuming $\lambda_1 \neq \lambda_2$, we have $\lambda_1 - \lambda_2 = 2c_n \epsilon_n$. Since $c_n \in \frac{1}{2} + \mathbb{Z}$, we have that $2c_n \in 1 + 2\mathbb{Z}$. Since $k\epsilon_n \notin Q$ for any odd number k, we have $\lambda_1 - \lambda_2 \notin Q$. □

In particular, the previous lemma implies that if $L(\lambda_1)$ and $L(\lambda_2)$ are non-equivalent simple admissible highest weight C_n-modules with the same central character, then

$$(\operatorname{Supp} L(\lambda_1)) \cap (\operatorname{Supp} L(\lambda_2)) = \emptyset$$

8 Decomposition of $L(\lambda) \otimes L(\mu)$

As in the previous section, our algebra is assumed to be C_n with $\mathcal{H}$, Δ, $\Delta^{++} = \{\alpha_1, \ldots, \alpha_n\}$, Δ^+, Δ^-, $\mathcal{W}$, $\mathfrak{H} = \{h_1, \ldots, h_n\}$, $\mathfrak{E} = \{\epsilon_1, \ldots, \epsilon_n\}$, $\mathfrak{F} = \{\omega_1, \ldots, \omega_n\}$ and ρ all as given in section 3.7. Further, unless otherwise stated, we will let $\lambda, \mu \in \mathcal{H}^*$, with $L(\lambda)$ admissible, and $L(\mu)$ finite dimensional, hence μ is dominant integral.

8.1 Complete Reducibility

We first confirm that the tensor product $L(\lambda) \otimes L(\mu)$ decomposes into a direct sum of simple admissible highest weight modules.

Proposition 8.1. *(Britten-Hooper-Lemire) Let $\mu = \sum_{i=1}^n a_i\omega_i$ be a dominant integral weight, and let*

$$\mathcal{T}_\mu = \left\{ \mu - \sum_{i=1}^n d_i\epsilon_i \;\middle|\; d_i \in \mathbb{Z}_{\geq 0},\ 0 \leq d_n \leq 2a_n + 1, \right.$$
$$\left. 0 \leq d_i \leq a_i \text{ for } 1 \leq i \leq n-1, \text{ and } \sum_{i=1}^n d_i \in 2\mathbb{Z} \right\}$$

then $L(-\frac{1}{2}\omega_1) \otimes L(\mu)$ is completely reducible, with decomposition

$$L(-\frac{1}{2}\omega_n) \otimes L(\mu) \simeq \bigoplus_{\nu \in \mathcal{T}_\mu} L(-\frac{1}{2}\omega_n + \nu)$$

Proof. (See Theorem 5.5 in [1]) □

Lemma 8.1. *If $\lambda \in \mathcal{H}^*$ with $L(\lambda)$ admissible, then there exists a dominant integral weight μ_0 such that $L(\lambda) \leq L(-\frac{1}{2}\omega_n) \otimes L(\mu_0)$*

Proof. Due to Corollary 7.2, we can write $\lambda = \sum_{i=1}^n b_i\epsilon_i$ with $b_i \in \frac{1}{2} + \mathbb{Z}$, and $\lambda + \rho = \sum_{i=1}^n c_i\epsilon_i$, where $c_i = b_i + n - i + 1$, and $c_1 > c_2 > \cdots > c_{n-1} > |c_n|$. Let $k_i = b_i + \frac{1}{2}$ for each i, and let

$$\mu_0 = \lambda + \frac{1}{2}\omega_n + \left(\sum_{i=1}^{n-1} 2\epsilon_i\right) + (|k_n| - k_n)\epsilon_n$$

hence

$$\mu_0 = \left(\sum_{i=1}^{n-1}(k_i + 2)\epsilon_i\right) + |k_n|\epsilon_n$$

We first show that μ_0 is dominant integral by making use of the characterization given in Property 5.2. For simplicity, denote the i^{th} coefficient in μ_0 by a_i. i.e. set $a_i = k_i + 2$ for $1 \leq i \leq n-1$ and $a_n = |k_n|$, hence

$$\mu_0 = \sum_{i=1}^{n} a_i \epsilon_i$$

Since $k_i \in \mathbb{Z}$ for all i, we have that $a_i \in \mathbb{Z}$ for all i. Further, for $1 \leq i \leq n-1$, we have that

$$0 \leq c_i - c_{i+1} - 1 = b_i + n - i + 1 - (b_{i+1} + n - i) - 1 = b_i - b_{i+1} = k_i - k_{i+1}$$

and hence $k_i \geq k_{i+1}$ for $1 \leq i \leq n-1$. Also,

$$0 \leq c_n + c_{n-1} - 1 = b_n + 1 + (b_{n-1} + 2) - 1 = b_n + b_{n-1} + 2 = k_n + k_{n-1} + 1$$

In particular, for $1 \leq i \leq n-2$, we have $k_i \geq k_{i+1}$ and hence

$$a_i = k_i + 2 \geq k_{i+1} + 2 = a_{i+1}$$

Also, $a_n = |k_n| \geq 0$. It only remains to show that $a_{n-1} \geq a_n$. Since $k_{n-1} \geq k_n$, we have that $k_{n-1} + 2 \geq k_n$, and since $k_{n-1} \geq -k_n - 1$, we have that $k_{n-1} + 2 \geq -k_n$. Therefore $a_{n-1} = k_{n-1} + 2 \geq |k_n| = a_n$. Thus μ_0 is dominant integral.

We now show that $L(\lambda) \leq L(-\frac{1}{2}\omega_n) \otimes L(\mu_0)$. First, notice that since $k_{n-1} \geq k_n$ and $k_{n-1} \geq -k_n - 1$, we have that $2k_{n-1} \geq -1$, and hence $k_{n-1} \geq 0$. Further, since $k_i \geq k_{i+1}$ for $1 \leq i \leq n-1$, we have that $k_i \geq 0$ for $1 \leq i \leq n-1$. With Proposition 8.1 in mind, we set $d_i = 2$ for $1 \leq i \leq n-1$, and $d_n = |k_i| - k_i$. Clearly $d_i \in \mathbb{Z}_{\geq 0}$ for all i, and since $d_n = 0$ or $2|k_n|$, we have that $d_i \in 2\mathbb{Z}$ for all i, hence

$$\sum_{i=1}^{n} d_i \in 2\mathbb{Z}$$

Further, since $k_i \geq 0$ for $1 \leq i \leq n-1$, we have that $0 \leq 2 \leq k_i + 2$, i.e.

$$0 \leq d_i \leq a_i$$

Finally, since $0 \leq |k_n| - k_n \leq 2|k_n|$, we have that

$$0 \leq d_n \leq 2a_n + 1$$

Thus

$$\lambda + \frac{1}{2}\omega_2 = \mu_0 - \sum_{i=1}^{n} d_i\epsilon_i \in \mathcal{T}_{\mu_0}$$

Therefore

$$L(\lambda) \leq \bigoplus_{\nu \in \mathcal{T}_{\mu_0}} L(-\frac{1}{2}\omega_n + \nu)$$

□

Theorem 8.1. *If $\lambda, \mu \in \mathcal{H}^*$, with $L(\lambda)$ admissible, and μ dominant integral, then $L(\lambda) \otimes L(\mu)$ is completely reducible.*

Proof. Choose μ_0 such that $L(\lambda) \leq L(-\frac{1}{2}\omega) \otimes L(\mu_0)$. Then

$$L(\lambda) \otimes L(\mu) \leq L(-\frac{1}{2}\omega) \otimes (L(\mu_0) \otimes L(\mu))$$

Since $L(\mu_0) \otimes L(\mu)$ decomposes into a direct some of simple modules with dominant integral highest weights, and the tensor product of $L(-\frac{1}{2}\omega)$ with any such module is semisimple, hence completely reducible, we must have that $L(-\frac{1}{2}\omega) \otimes (L(\mu_0) \otimes L(\mu))$ is completely reducible. Therefore
$L(\lambda) \otimes L(\mu)$ is a submodule of a completely reducible module, and is thus, itself, completely reducible. □

Corollary 8.1. *If $\lambda, \mu \in \mathcal{H}^*$, with $L(\lambda)$ admissible, and μ dominant integral, then*

$$L(\lambda) \otimes L(\mu) \simeq \bigoplus_{\nu \in Supp\, L(\mu)} a_\nu L(\lambda + \nu)$$

for some $a_\nu \in \mathbb{Z}_{\geq 0}$, where $a_\nu = 0$ if $L(\lambda + \nu)$ is not admissible.

Proof. By Proposition 5.16, we have that $L(\lambda) \otimes L(\mu)$ is admissible. Since we have just shown it is completely reducible, Proposition 5.10 implies that $L(\lambda) \otimes L(\mu)$ is semisimple. Further, since $\mathrm{Supp}\,(L(\lambda) \otimes L(\mu)) = \lambda + \mu + Q^-$, the weights on any α-string, for any $\alpha \in \Delta^+$ are bounded in the positive direction. This implies that any simple submodule of $L(\lambda) \otimes L(\mu)$ is a highest weight module. Thus, we have

$$L(\lambda) \otimes L(\mu) \simeq \bigoplus_{\gamma \in \lambda + \mu + Q^-} a_\gamma L(\gamma)$$

for some $a_\gamma \in \mathbb{Z}_{\geq 0}$. Suppose $\gamma \in \lambda + \mu + Q^-$ with $a_\gamma \neq 0$. The dimensions of the weight spaces of $L(\lambda) \otimes L(\mu)$ are bounded, and hence so are the dimensions of the weight spaces of $L(\gamma)$. Thus $L(\gamma)$ is admissible. Further, $L(\gamma)$ has central character χ_γ. By Corollary 5.2, we have $\chi_\gamma = \chi_{\lambda+\nu}$ for some $\nu \in \operatorname{Supp} L(\mu)$. Due to Lemma 7.1, we have either $\gamma = \lambda + \nu$, or

$$(\operatorname{Supp} L(\gamma)) \cap (\operatorname{Supp} L(\lambda + \nu)) = \emptyset$$

Since $\operatorname{Supp} L(\lambda + \nu) \subseteq \lambda + \mu + Q$ is in the same Q-coset as $\operatorname{Supp}(L(\lambda) \otimes L(\mu))$, we must have $\gamma = \lambda + \nu$. □

8.2 Kac-Wakimoto Character Formula

For each $\lambda \in \mathcal{H}^*$, let $\Delta_\lambda = \{\alpha \in \Delta \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}\}$, and let $\Delta_\lambda^+ = \Delta_\lambda \cap \Delta^+$. Let $\mathcal{W}_\lambda = \langle \sigma_\alpha \mid \alpha \in \Delta_\lambda \rangle$ be the subgroup of the Weyl group $\mathcal{W}$ generated by the reflections σ_α for $\alpha \in \Delta_\lambda$. The following proposition is proven in [9] for Kac-Moody algebras. Since every finite dimensional simple Lie algebra is a Kac-Moody algebra, the result is true for finite dimensional simple Lie algebras. Further, in [9] the result is given for only those $\lambda \in \mathcal{H}^*$ such that $\langle \lambda + \rho, \alpha \rangle \geq 0$ for all but finitely many $\alpha \in \Delta^+$. However, since for any finite dimensional simple Lie algebra, Δ^+ is finite, we have that, in the case of finite dimensional simple Lie algebras, the result is true for all $\lambda \in \mathcal{H}^*$. Since C_n is finite dimensional and simple, we quote the result in our setting, in which the algebra is C_n, with Cartan $\mathcal{H}$, and root system Δ. Recall that for each $\gamma \in \mathcal{H}^*$, $M(\gamma)$ denotes the Verma module with highest weight γ.

Proposition 8.2. *(Kac-Wakimoto) Let $\lambda \in \mathcal{H}^*$ be such that $\langle \lambda + \rho, \alpha \rangle > 0$ for all $\alpha \in \Delta_\lambda^+$. Then*

$$\operatorname{ch} L(\lambda) = \sum_{\sigma \in \mathcal{W}_\lambda} sgn(\sigma) \operatorname{ch} M(\sigma \cdot \lambda)$$

Proof. (See Theorem 1 in [9]) □

Notice that if $\lambda = \sum_{i=1}^n a_i \epsilon_i$ is such that $a_i \in \frac{1}{2} + \mathbb{Z}$ then $\Delta_\lambda = \widetilde{\Delta}$, and hence $\Delta_\lambda^+ = \widetilde{\Delta}^+$ and $\mathcal{W}_\lambda = \widetilde{\mathcal{W}}$. This gives us the following corollary:

Corollary 8.2. *Let $\lambda \in \mathcal{H}^*$. If $L(\lambda)$ is admissible then*

$$\operatorname{ch} L(\lambda) = \sum_{\sigma \in \widetilde{W}} sgn(\sigma) \operatorname{ch} M(\sigma \cdot \lambda)$$

Proof. For each $\alpha \in \widetilde{\Delta}^+$ we have $\langle \lambda + \rho, \alpha \rangle = (\lambda + \rho, \alpha) > 0$ by Corollary 7.1, and thus the conditions of Proposition 8.2 are satisfied. □

8.3 Multiplicity Formula

In this section, we provide a formula for the coefficients a_ν occurring in the decomposition of $L(\lambda) \otimes L(\mu)$ given in Corollary 8.1. Recall that $m_\mu(\nu)$ denotes the multiplicity of the weight ν in $L(\mu)$, with $m_\mu(\nu) = 0$ when $\nu \notin \operatorname{Supp} L(\mu)$.

Lemma 8.2. *If $\lambda, \mu \in \mathcal{H}^*$, with $L(\lambda)$ admissible, and $L(\mu)$ finite dimensional, then*

$$\operatorname{ch}(L(\lambda) \otimes L(\mu)) = \sum_{\nu \in \operatorname{Supp} L(\mu)} m_\mu(\nu) \sum_{\sigma \in \widetilde{W}} sgn(\sigma) \operatorname{ch} M(\sigma \cdot (\lambda + \nu))$$

Proof. Recall that

$$\operatorname{ch} M(\lambda) = \sum_{\gamma \in Q^+} K(\gamma) e^{\lambda - \gamma}$$

where $K : Q \to \mathbb{Z}_{\geq 0}$ is the Kostant partition function. The following calculation gives us our result.

$$\begin{aligned}
\operatorname{ch}(L(\lambda)\otimes L(\mu)) &= \left(\sum_{\sigma\in\widetilde{W}} sgn(\sigma)\operatorname{ch} M(\sigma\cdot\lambda)\right)(\operatorname{ch} L(\mu))\\
&= \left(\sum_{\sigma\in\widetilde{W}} sgn(\sigma)\operatorname{ch} M(\sigma\cdot\lambda)\right)\left(\sum_{\nu\in\operatorname{Supp} L(\mu)} m_\mu(\nu)e^{\nu}\right)\\
&= \sum_{\sigma\in\widetilde{W}} sgn(\sigma)\sum_{\nu\in\operatorname{Supp} L(\mu)} m_\mu(\nu)\sum_{\gamma\in Q^+} K(\gamma)e^{\sigma\cdot\lambda-\gamma}e^{\nu}\\
&= \sum_{\sigma\in\widetilde{W}} sgn(\sigma)\sum_{\nu\in\operatorname{Supp} L(\mu)} m_\mu(\nu)\sum_{\gamma\in Q^+} K(\gamma)e^{\sigma\cdot\lambda+\nu-\gamma}\\
&= \sum_{\sigma\in\widetilde{W}} sgn(\sigma)\sum_{\nu\in\operatorname{Supp} L(\mu)} m_\mu(\nu)\operatorname{ch} M(\sigma\cdot\lambda+\nu)\\
&= \sum_{\sigma\in\widetilde{W}} sgn(\sigma)\sum_{\nu\in\operatorname{Supp} L(\mu)} m_\mu(\sigma(\nu))\operatorname{ch} M(\sigma\cdot\lambda+\sigma(\nu))
\end{aligned}$$

By Proposition 5.8

$$= \sum_{\sigma\in\widetilde{W}} sgn(\sigma)\sum_{\nu\in\operatorname{Supp} L(\mu)} m_\mu(\nu)\operatorname{ch} M(\sigma\cdot(\lambda+\nu))$$

Also by Proposition 5.8

□

Clearly, for each $\nu\in\operatorname{Supp} L(\mu)$, we have that $\lambda+\nu=\sum_{i=1}^n a_i\epsilon_i$ where $a_i\in\frac{1}{2}+\mathbb{Z}$. With this in mind, we define the following two subsets of $\operatorname{Supp} L(\mu)$. Set

$$\mathcal{A}_\lambda(\mu)=\{\nu\in\operatorname{Supp} L(\mu)\mid L(\lambda+\nu)\text{ is admissible}\}$$

$\mathcal{A}_\lambda(\mu)$ is thus the set of all weights $\nu\in\operatorname{Supp} L(\mu)$ such that

$$\lambda+\nu+\rho=\sum_{i=1}^n b_i\epsilon_i$$

with $b_1>b_2>\cdots>b_{n-1}>|b_n|$. Set

$$\Pi_\lambda(\mu)=\{\nu\in\operatorname{Supp} L(\mu)\mid(\lambda+\nu+\rho,\alpha)=0\text{ for some }\alpha\in\widetilde{\Delta}^+\}$$

Hence, $\Pi_\lambda(\mu)$ is the set of all weights $\nu\in\operatorname{Supp} L(\mu)$ such that

$$\lambda+\nu+\rho=\sum_{i=1}^n b_i\epsilon_i$$

with $|b_i| = |b_j|$ for some $i < j$.

Lemma 8.3. *Let $\lambda, \mu \in \mathcal{H}^*$, such that $L(\lambda)$ is admissible, and $L(\mu)$ is finite dimensional. Let $\kappa \in \operatorname{Supp} L(\mu)$. If $\kappa \notin \Pi_\lambda(\mu)$ then there exists a unique pair $(\sigma, \nu) \in \widetilde{\mathcal{W}} \times \mathcal{A}_\lambda(\mu)$ such that $\sigma \cdot (\lambda + \nu) - \lambda = \kappa$.*

Proof. Let $\lambda + \rho = \sum_{i=1}^{n} a_i \epsilon_i$. Let $\kappa \in \operatorname{Supp} L(\mu) \setminus \Pi_\lambda(\mu)$, and

$$\kappa + \lambda + \rho = \sum_{i=1}^{n} b_i \epsilon_i$$

We must show that there exists $\nu \in \operatorname{Supp} L(\mu)$ such that $\nu + \lambda$ is admissible and $\nu = \sigma \cdot (\kappa + \lambda) - \lambda$ for some $\sigma \in \widetilde{\mathcal{W}}$. For each $\nu \in \operatorname{Supp} L(\mu) \setminus \Pi_\lambda(\mu)$, set

$$B^+(\nu) = \{(k, l) \mid k < l \text{ and } d_k + d_l < 0\}$$
$$B^-(\nu) = \{(k, l) \mid k < l \text{ and } d_k - d_l < 0\}$$

where $\nu + \lambda + \rho = \sum_{k=1}^{n} d_k \epsilon_k$. Now define $\phi \colon \Pi(\nu) \setminus \Pi_\lambda(\mu) \to \mathbb{Z}_{\geq 0}$ by

$$\phi(\nu) = |B^+(\nu)| + |B^-(\nu)|$$

By Corollary 7.1, if $\phi(\kappa) = 0$ then $\kappa \in \mathcal{A}_\lambda(\mu)$. Claim that if $\phi(\kappa) > 0$ then there exists $\nu \in \operatorname{Supp} L(\mu) \setminus \Pi_\lambda(\mu)$ such that $\nu = \sigma \cdot (\kappa + \lambda) - \lambda$ for some $\sigma \in \widetilde{\mathcal{W}}$ and $\phi(\nu) < \phi(\kappa)$. Indeed, if there exists $i < j$ such that $b_i + b_j < 0$ then set

$$\nu = \sigma_{\epsilon_i + \epsilon_j}(\kappa + \lambda + \rho) - (\lambda + \rho)$$

Let $\nu + \lambda + \rho = \sum_{k=1}^{n} c_k \epsilon_k$, i.e., $c_i = -b_j$, $c_j = -b_i$, and $c_k = b_k$ for $k \neq i, j$. Then,

$$\begin{aligned}
\nu &= \kappa + \lambda + \rho - (\kappa + \lambda + \rho, \epsilon_i + \epsilon_j)(\epsilon_i + \epsilon_j) - (\lambda + \rho) \\
&= \kappa - (b_i + b_j)(\epsilon_i + \epsilon_j) \\
&= \kappa + (c_i + c_j)(\epsilon_i + \epsilon_j)
\end{aligned}$$

and,

$$\begin{aligned}
\sigma(\kappa) &= \kappa - (\kappa, \epsilon_i + \epsilon_j)(\epsilon_i + \epsilon_j) \\
&= \kappa - (b_i - a_i + b_j - a_j)(\epsilon_i + \epsilon_j) \\
&= \kappa + (c_i + c_j + a_i + a_j)(\epsilon_i + \epsilon_j)
\end{aligned}$$

Since $\sigma(\kappa) \in \operatorname{Supp} L(\mu)$, and $0 \leq c_i + c_j \leq c_i + c_j + a_i + a_j$, by Proposition 5.9, we have that $\nu \in \operatorname{Supp} L(\mu)$.

We now show that $|B^-(\nu)| - |B^-(\kappa)| < |B^+(\kappa)| - |B^+(\nu)|$, and hence $\phi(\nu) < \phi(\kappa)$. To see this, partition $S = \{(k,l) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq k < l \leq n\}$ into the following sets:

$$S_0 = \{(k,l) \in S \mid k, l \neq i, j\}$$
$$S_- = \{(k,i) \in S \mid k < i\} \cup \{(k,j) \in S \mid k < j \text{ and } k \neq i\}$$
$$S_+ = \{(i,l) \in S \mid i < l \text{ and } l \neq j\} \cup \{(j,l) \in S \mid j < l\}$$
$$S_{i,j} = \{(i,j)\}$$

Clearly, $|B^-(\nu) \cap S_0| - |B^-(\kappa) \cap S_0| = |B^+(\kappa) \cap S_0| - |B^+(\nu) \cap S_0| = 0$. Since $c_i > b_i$, and $c_j > b_j$, we have $|B^-(\nu) \cap S_+| - |B^-(\kappa) \cap S_+| \leq 0$, and $|B^+(\kappa) \cap S_+| - |B^+(\nu) \cap S_+| \geq 0$. Hence,

$$|B^-(\nu) \cap S_+| - |B^-(\kappa) \cap S_+| \leq |B^+(\kappa) \cap S_+| - |B^+(\nu) \cap S_+|$$

If $k < i$ then we have $c_k - c_i < 0$ and $b_k - b_i > 0$ if and only if $b_k + b_j < 0$ and $c_k + c_j > 0$. Similarly for $k < j$ with $k \neq i$. Thus,

$$|B^-(\nu) \cap S_-| - |B^-(\kappa) \cap S_-| = |B^+(\kappa) \cap S_-| - |B^+(\nu) \cap S_-|$$

Finally, since $c_i - c_j = b_i - b_j$, we have $|B^-(\nu) \cap S_{i,j}| - |B^-(\kappa) \cap S_{i,j}| = 0$, and $|B^+(\kappa) \cap S_{i,j}| - |B^-(\nu) \cap S_{i,j}| = 1$, which gives us the desired result.

If $|B^+(\kappa)| = 0$ and there exists $i < j$ such that $b_i - b_j < 0$ then set $\nu = \sigma_{\epsilon_i - \epsilon_j}(\kappa + \lambda + \rho) - (\lambda + \rho)$. Again, let $\nu + \lambda + \rho = \sum_{k=1}^{n} c_k \epsilon_k$, i.e., $c_i = b_j$, $c_j = b_i$, and $c_k = b_k$ for $k \neq i, j$. Then, $\nu = \kappa + (c_i - c_j)(\epsilon_i - \epsilon_j)$, and $\sigma(\kappa) = \kappa + (c_i - c_j + a_i - a_j)(\epsilon_i - \epsilon_j)$. Since $0 \leq c_i - c_j \leq c_i - c_j + a_i - a_j$, we again have that $\nu \in \operatorname{Supp} L(\mu)$.

Clearly, if $|B^+(\kappa)| = 0$ then $|B^+(\nu)| = 0$. We now show that

$$|B^-(\nu)| < |B^-(\kappa)|$$

and hence that $\phi(\nu) < \phi(\kappa)$. Partition the set $S = \{(k,l) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq k < l \leq n\}$

into the following sets:

$$S_0 = \{(k,l) \in S \mid k,l \neq i,j\}$$
$$S_{i-} = \{(k,i) \in S \mid k < i\}$$
$$S_{i++} = \{(i,l) \in S \mid j < l\}$$
$$S_{i+} = \{(i,l) \in S \mid i < l < j\}$$
$$S_{j+} = \{(j,l) \in S \mid j < l\}$$
$$S_{j--} = \{(k,j) \in S \mid k < i\}$$
$$S_{j-} = \{(k,j) \in S \mid i < k < j\}$$
$$S_{i,j} = \{(i,j)\}$$

Clearly, $|B^-(\nu) \cap S_0| = |B^-(\kappa) \cap S_0|$. If $k < i$, then $c_k - c_i < 0$ if and only if $b_k - b_j < 0$, therefore

$$|B^-(\nu) \cap S_{i-}| = |B^-(\kappa) \cap S_{j--}|$$

Similarly,

$$|B^-(\nu) \cap S_{j--}| = |B^-(\kappa) \cap S_{i-}|$$

$$|B^-(\nu) \cap S_{j+}| = |B^-(\kappa) \cap S_{i++}|$$

and

$$|B^-(\nu) \cap S_{i++}| = |B^-(\kappa) \cap S_{j+}|$$

If $i < k < j$, then since $c_i > b_i$ and $c_j < b_j$, we have $c_i - c_k \geq b_i - b_k$ and $c_k - c_j \geq b_k - b_j$, hence

$$|B^-(\nu) \cap S_{i+}| \leq |B^-(\kappa) \cap S_{i+}|$$

and

$$|B^-(\nu) \cap S_{j-}| \leq |B^-(\kappa) \cap S_{j-}|$$

Finally, since $|B^-(\nu) \cap S_{i,j}| = 0$, and $|B^-(\kappa) \cap S_{i,j}| = 1$, we have our result. □

Here, we note that all reflections in the hyperplanes perpendicular to the roots in $\widetilde{\Delta}^+$ are in fact odd. i.e. $sgn(\sigma_\alpha) = -1$ for all $\alpha \in \widetilde{\Delta}^+$. This can be seen with following calculations: If $\alpha = \epsilon_i - \epsilon_j$ for some $i < j$ then $\sigma_\alpha = \sigma_{\alpha_{j-1}}\sigma_{\alpha_{j-2}} \cdots \sigma_{\alpha_{i+1}}\sigma_{\alpha_i}\sigma_{\alpha_{i+1}} \cdots \sigma_{\alpha_{j-2}}\sigma_{\alpha_j-1}$.

Thus, $\sigma_{\epsilon_i-\epsilon_j}$ can be written as the product of $2(j-i+2)+1$ simple reflections. If $\alpha=\epsilon_i+\epsilon_j$, with $1\le i<j<n$ then $\sigma_\alpha=\sigma_{\epsilon_j-\epsilon_n}\sigma_{\alpha_n}\sigma_{\epsilon_i-\epsilon_n}\sigma_{\alpha_n}\sigma_{\epsilon_j-\epsilon_n}$. Since by the previous calculation we know $\sigma_{\epsilon_i-\epsilon_n}$ and $\sigma_{\epsilon_j-\epsilon_n}$ are odd, σ_α is also odd. Finally, if $\alpha=\epsilon_i+\epsilon_n$ then $\sigma_\alpha=\sigma_{\alpha_n}\sigma_{\epsilon_i-\epsilon_n}\sigma_{\alpha_n}$.

The following proposition shows that, in the character formula given in Lemma 8.2, we may neglect the weights $\nu\in\Pi_\lambda(\mu)$. i.e. the initial sum may be taken over $\operatorname{Supp}L(\mu)\setminus\Pi_\lambda(\mu)$.

Proposition 8.3. *Let $\lambda,\mu\in\mathcal{H}^*$, with $L(\lambda)$ admissible and $L(\mu)$ finite dimensional. If $\nu\in\Pi_\lambda(\mu)$ then*

$$\sum_{\sigma\in\widetilde{\mathcal{W}}} sgn(\sigma)\operatorname{ch}M(\sigma\cdot(\lambda+\nu))=0$$

Proof. Choose $\alpha\in\widetilde{\Delta}^+$ such that $(\lambda+\nu+\rho,\alpha)=0$. Then

$$\begin{aligned}\sigma_\alpha\cdot(\lambda+\nu)&=\sigma_\alpha(\lambda+\nu+\rho)-\rho\\&=\lambda+\nu+\rho-2\frac{(\lambda+\nu+\rho,\alpha)}{(\alpha,\alpha)}\alpha-\rho\\&=\lambda+\nu\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{\sigma\in\widetilde{\mathcal{W}}} sgn(\sigma)\operatorname{ch}M(\sigma\cdot(\lambda+\nu))&=\sum_{\sigma\in\widetilde{\mathcal{W}}} sgn(\sigma)\operatorname{ch}M(\sigma\sigma_\alpha\cdot(\lambda+\nu))\\&=\sum_{\sigma\in\widetilde{\mathcal{W}}} sgn(\sigma\sigma_\alpha)\operatorname{ch}M(\sigma\sigma_\alpha\sigma_\alpha\cdot(\lambda+\nu))\\&=-\sum_{\sigma\in\widetilde{\mathcal{W}}} sgn(\sigma)\operatorname{ch}M(\sigma\cdot(\lambda+\nu))\end{aligned}$$

□

Theorem 8.2. *Let $\lambda,\mu\in\mathcal{H}^*$ such that $L(\lambda)$ is admissible, and $L(\mu)$ is finite dimensional. Then*

$$L(\lambda)\otimes L(\mu)\simeq\bigoplus_{\nu\in\mathcal{A}_\lambda(\mu)}\left(\sum_{\sigma\in\widetilde{\mathcal{W}}} sgn(\sigma)m_\mu(\sigma\cdot(\lambda+\nu)-\lambda)\right)L(\lambda+\nu)$$

Proof. From Lemma 8.3 we have a one to one correspondence between $\operatorname{Supp} L(\mu) \setminus \Pi_\lambda(\mu)$ and all pairs $(\sigma, \nu) \in \widetilde{\mathcal{W}} \times \mathcal{A}_\lambda(\mu)$ with

$$\sigma \cdot (\lambda + \nu) - \lambda \in \operatorname{Supp} L(\mu) \setminus \Pi_\lambda(\mu)$$

Further, since

$$m_\mu(\nu) \sum_{\sigma \in \widetilde{\mathcal{W}}} sgn(\sigma) chM(\sigma \cdot (\lambda + \nu)) = 0$$

if $\nu \notin \operatorname{Supp} L(\mu) \setminus \Pi_\lambda(\mu)$, we may rewrite the formula in Lemma 8.2 as

$$\begin{aligned}
\operatorname{ch}(L(\lambda) \otimes L(\mu)) &= \sum_{\nu \in \mathcal{A}_\lambda(\mu)} \sum_{\sigma \in \widetilde{\mathcal{W}}} m_\mu(\sigma \cdot (\lambda + \nu) - \lambda) \\
&\quad \times \sum_{\sigma' \in \widetilde{\mathcal{W}}} sgn(\sigma') \operatorname{ch} M(\sigma' \cdot (\lambda + \sigma \cdot (\lambda + \nu) - \lambda)) \\
&= \sum_{\nu \in \mathcal{A}_\lambda(\mu)} \sum_{\sigma \in \widetilde{\mathcal{W}}} m_\mu(\sigma \cdot (\lambda + \nu) - \lambda) \\
&\quad \times \sum_{\sigma' \in \widetilde{\mathcal{W}}} sgn(\sigma') \operatorname{ch} M(\sigma'\sigma \cdot (\lambda + \nu)) \\
&= \sum_{\nu \in \mathcal{A}_\lambda(\mu)} \sum_{\sigma \in \widetilde{\mathcal{W}}} sgn(\sigma) m_\mu(\sigma \cdot (\lambda + \nu) - \lambda) \\
&\quad \times \sum_{\sigma' \in \widetilde{\mathcal{W}}} sgn(\sigma') \operatorname{ch} M(\sigma' \cdot (\lambda + \nu)) \\
&= \sum_{\nu \in \mathcal{A}_\lambda(\mu)} \sum_{\sigma \in \widetilde{\mathcal{W}}} sgn(\sigma) m_\mu(\sigma \cdot (\lambda + \nu) - \lambda) \operatorname{ch} L(\lambda + \nu)
\end{aligned}$$

By Corollary 8.1 we have that $L(\lambda) \otimes L(\mu) = \sum_{\nu \in \mathcal{A}_\lambda(\mu)} a_\nu L(\lambda + \nu)$ for some $a_\nu \in \mathbb{Z}_{\geq 0}$. It therefore, only remains to show the linear independence of the characters $\operatorname{ch} L(\lambda + \nu)$ for $\nu \in \mathcal{A}_\lambda(\mu)$.

Suppose $\sum_{\nu \in \mathcal{A}_\lambda(\mu)} b_\nu \operatorname{ch} L(\lambda + \nu) = 0$ for some $b_\nu \in \mathbb{Z}$. Let

$$\Gamma = \{\nu \in \mathcal{A}_\lambda(\mu) \mid b_\nu \neq 0\}$$

Assuming $\Gamma \neq \emptyset$, we may choose $\nu_0 \in \Gamma$ such that $\nu_0 + \sum_{i=1}^n c_i \alpha_i \notin \Gamma$ for any choice of $c_i \in \mathbb{Z}_{\geq 0}$ with not all $c_i = 0$. Since $a_{\nu_0} \neq 0$, and the term $e^{\lambda + \nu_0}$ occurs in $\operatorname{ch} L(\lambda + \nu_0)$, we must have that $e^{\lambda + \nu_0}$ occurs in $\operatorname{ch} L(\lambda + \nu)$ for some $\nu \in \Gamma$, $\nu \neq \nu_0$. Hence $\lambda + \nu_0 = \lambda + \nu - \sum_{i=1}^n c_i \alpha_i$ for some $c_i \in \mathbb{Z}_{\geq 0}$. Therefore, $c_i = 0$ for all i, and hence $\nu = \nu_0$, which is a contradiction. Therefore, $\Gamma = \emptyset$, which completes the proof. □

8.4 Example

We give an example of the formula in Theorem 8.2, for the algebra C_2. Let $\lambda = \frac{1}{2}\epsilon_1 - \frac{1}{2}\epsilon_2$. Then $\lambda + \rho = \frac{5}{2}\epsilon_1 + \frac{1}{2}\epsilon_2$, and hence λ is admissible. Let $\mu = \omega_1 + \omega_2 = 2\epsilon_1 + \epsilon_2$.

$$\begin{aligned}\operatorname{Supp} L(\mu) = \{&2\epsilon_1 + \epsilon_2,\ 2\epsilon_1 - \epsilon_2,\ -2\epsilon_1 + \epsilon_2,\ -2\epsilon_1 - \epsilon_2,\ \epsilon_1 + 2\epsilon_2,\\ &\epsilon_1 - 2\epsilon_2,\ -\epsilon_1 + 2\epsilon_2,\ -\epsilon_1 - 2\epsilon_2,\ \epsilon_1,\ \epsilon_2,\ -\epsilon_1,\ -\epsilon_2\}\end{aligned}$$

where the weights in the $2\epsilon_1 + \epsilon_2$ orbit have multiplicity equal to 1, and the weights in the ϵ_1 orbit have multiplicity equal to 2. We have

$$\mathcal{A}_\lambda(\mu) = \{2\epsilon_1 + \epsilon_2,\ 2\epsilon_1 - \epsilon_2,\ \epsilon_1 + 2\epsilon_2,\ \epsilon_1 - 2\epsilon_2,\ \epsilon_1,\ \epsilon_2,\ -\epsilon_1,\ -\epsilon_2\}$$

$$\Pi_\lambda(\mu) = \{-2\epsilon_1 - \epsilon_2, -\epsilon_1 - 2\epsilon_2\}$$

The only remaining weights in $\operatorname{Supp} L(\mu)$ are

$$-2\epsilon_1 + \epsilon_2 \quad \text{and} \quad -\epsilon_1 + 2\epsilon_2$$

For these, applying Lemma 8.3, we see that

$$\sigma_{\epsilon_1 - \epsilon_2} \cdot (\lambda + -2\epsilon_1 + \epsilon_2) - \lambda = -\epsilon_1$$

and

$$\sigma_{\epsilon_1 - \epsilon_2} \cdot (\lambda - \epsilon_1 + 2\epsilon_2) - \lambda = \epsilon_2$$

Due to the uniqueness in Lemma 8.3, each weight in $\operatorname{Supp} L(\mu)$ can appear as some $\sigma \cdot (\lambda + \nu) - \lambda$ in the formula

$$L(\lambda) \otimes L(\mu) = \sum_{\nu \in \mathcal{A}_\lambda(\mu)} \left(\sum_{\sigma \in \widetilde{\mathcal{W}}} sgn(\sigma) m_\mu(\sigma \cdot (\lambda + \nu) - \lambda) \right) L(\lambda + \nu)$$

for at most one $\nu \in \mathcal{A}_\lambda(\mu)$ and one $\sigma \in \mathcal{W}$. Further, the weights in $\Pi_\lambda(\mu)$ will not equal $\sigma \cdot (\lambda + \nu) - \lambda$ for any choice of $\sigma \in \mathcal{W}$ and $\nu \in \mathcal{A}_\lambda(\mu)$. The above formula is thus

$$\begin{aligned}
&m_\mu(2\epsilon_1+\epsilon_2)L(\lambda+2\epsilon_1+\epsilon_2)\\
&\oplus m_\mu(2\epsilon_1-\epsilon_2)L(\lambda+2\epsilon_1-\epsilon_2)\\
&\oplus m_\mu(\epsilon_1+2\epsilon_2)L(\lambda+\epsilon_1+2\epsilon_2)\\
&\oplus m_\mu(\epsilon_1-2\epsilon_2)L(\lambda+\epsilon_1-2\epsilon_2)\\
&\oplus m_\mu(\epsilon_1)L(\lambda+\epsilon_1)\\
&\oplus (m_m u(\epsilon_2)-m_\mu(-\epsilon_1+2\epsilon_2))L(\lambda+\epsilon_2)\\
&\oplus (m_\mu(-\epsilon_1)-m_\mu(-2\epsilon_1+\epsilon_2))L(\lambda-\epsilon_1)\\
&\oplus m_\mu(-\epsilon_2)L(\lambda-\epsilon_2)
\end{aligned}$$

Therefore

$$\begin{aligned}
L(\lambda)\otimes L(\mu)\simeq{}& L(\lambda+2\epsilon_1+\epsilon_2)\oplus L(\lambda+2\epsilon_1-\epsilon_2)\oplus L(\lambda+\epsilon_1+2\epsilon_2)\\
&\oplus L(\lambda+\epsilon_1-2\epsilon_2)\oplus 2L(\lambda+\epsilon_1)\oplus L(\lambda+\epsilon_2)\\
&\oplus L(\lambda-\epsilon_1)\oplus 2L(\lambda-\epsilon_2)
\end{aligned}$$

9 Decomposition of $V_{\bar{a}}(\lambda)\otimes L(\mu)$

Recall that $\Sigma=\{\beta_1,\dots,\beta_n\}$ where $\beta_i=\sum_{j=i}^{n}\alpha_j$ is a commuting set of roots, which is a basis for Q, and the elements $x_{-\beta_i}$ act injectively on any simple admissible highest weight C_n-module $L(\lambda)$. Let $\mathfrak{U}_\Sigma$ denote the localization of the universal enveloping algebra $\mathfrak{U}$ with respect to the multiplicative subset generated by $\{1,x_{-\beta_1},\dots,x_{-\beta_n}\}$. For any simple admissible C_n-module V, the module V^Σ is the induced module $\mathfrak{U}_\Sigma\otimes_{\mathfrak{U}} V$.

Definition 9.1. *Let $\lambda\in\mathcal{H}^*$ such that $L(\lambda)$ is admissible, and let $\Sigma=\{\beta_1,\dots,\beta_n\}$ be as above. Let $\bar{a}=(a_1,\dots,a_n)\in\mathbb{C}^n$. Define the module $V_{\bar{a}}(\lambda)$ to be the vector space $L(\lambda)^\Sigma$ under the action through the automorphism $\Phi_{\bar{a}}$, where $\Phi_{\bar{a}}$ is as in Definition 6.5 i.e. If the action of $\mathfrak{U}_\Sigma$ on $L(\lambda)^\Sigma$ is defined by the map $\phi:\mathfrak{U}_\Sigma\to gl(L(\lambda)^\Sigma)$, then for all $x\in C_n$ and all $v\in V_{\bar{a}}(\lambda)$, we have*

$$xv=\phi(\Phi_{\bar{a}}(x))(v)$$

Notice that setting $\nu = \sum_{i=1}^{n} a_i\beta_i$ gives us $V_{\bar{a}}(\lambda) = L(\lambda)^{\Sigma}[\bar{\nu}]$, where $\bar{\nu}$ is the coset $\nu + Q$. We have simply changed the notation in the case where the module is derived from a simple admissible highest weight C_n-module, since in this case, the set Σ no longer depends on the module $L(\lambda)$. Notice that if $\bar{a} = (a_1, \dots, a_n)$ and $\bar{b} = (b_1, \dots, b_n)$ with $a_i - b_i \in \mathbb{Z}$, then by Proposition 6.10, we have $V_{\bar{a}}(\lambda) \simeq V_{\bar{b}}(\lambda)$.

Further, notice that if V is any simple torsion free module, then by Theorem 6.1, we have that $V \simeq V_{\bar{a}}(\lambda)$ for some $\lambda \in \mathcal{H}^*$, and some $\bar{a} \in \mathbb{C}^n$.

Finally, notice that since both $V_{\bar{a}}(\lambda)$ and $L(\lambda)$ are submodules of $\mathcal{M}_{ss}(L(\lambda))$, in the case where $V_{\bar{a}}(\lambda)$ is simple, Lemma 6.6 implies that $V_{\bar{a}}(\lambda)$ has central character χ_λ. In particular, if V is a simple torsion free module, then
$V = V_{\bar{a}}(\lambda)$ for some $\lambda \in \mathcal{H}^*$ and $\bar{a} \in \mathbb{C}^n$ where χ_λ is the central character of V.

Lemma 9.1. *Let $\lambda, \mu \in \mathcal{H}^*$ such that $L(\lambda)$ is admissible and $L(\mu)$ is finite dimensional. Set $\bar{m} = (m_1, \dots, m_n)$ with each $m_i \in \mathbb{Z}$. Then $L(\lambda) \otimes L(\mu)$ is equivalent to a submodule of $V_{\bar{m}}(\lambda) \otimes L(\mu)$. In particular, the $\mathfrak{U}_0$-module $L(\lambda) \otimes L(\mu)$ is equivalent to a sub-$\mathfrak{U}_0$-module of $V_{\bar{m}}(\lambda) \otimes L(\mu)$.*

Proof. Consider the case $\bar{m} = \bar{0}$. We have $V_{\bar{0}}(\lambda) = \mathfrak{U}_\Sigma \otimes_{\mathfrak{U}} L(\lambda)$. Let M be the sub-C_n-module $1 \otimes_{\mathfrak{U}} L(\lambda) \leq V_{\bar{0}}(\lambda)$. Then $M \simeq L(\lambda)$, and hence $M \otimes L(\mu) \simeq L(\lambda) \otimes L(\mu)$, and $M \otimes L(\mu)$ is a submodule of $V_{\bar{0}}(\lambda) \otimes L(\mu)$. Proposition 6.10 implies that $V_{\bar{m}}(\lambda) \simeq V_{\bar{0}}(\lambda)$ whenever $m_i \in \mathbb{Z}$ for all i. The result follows. □

Lemma 9.2. *Let $\lambda, \mu \in \mathcal{H}^*$ such that $L(\lambda)$ is admissible and $L(\mu)$ is finite dimensional. There exists a weight $\gamma \in \operatorname{Supp} V_{\bar{0}}(\lambda) \otimes L(\mu)$ such that $\dim(V_{\bar{0}}(\lambda) \otimes L(\mu))_\gamma = (\deg L(\lambda))(\dim L(\mu))$, and for all $\bar{m} = (m_1, \dots, m_n)$ with each $m_i \in \mathbb{Z}_{\leq 0}$, the weight space $(V_{\bar{m}}(\lambda) \otimes L(\mu))_{\gamma + \sum_{i=1}^{n} m_i\beta_i}$ is equivalent as a $\mathfrak{U}_0$-module to some weight space of $L(\lambda) \otimes L(\mu)$.*

Proof. Let $d = \deg L(\lambda)$. Let γ_0 be a weight of $L(\lambda)$ such that

$$\dim L(\lambda)_{\gamma_0} = d$$

Since the $x_{-\beta_i}$ act injectively on $L(\lambda)$, we have that

$$\dim L(\lambda)_{\gamma_0 + \sum_{i=1}^{n} k_i\beta_i} = d$$

for any choice of $k_1, \ldots, k_n \in \mathbb{Z}_{\leq 0}$. Let $\operatorname{Supp} L(\mu) = \{\nu_1, \ldots, \nu_l\}$, where each

$$\nu_j = \mu + \sum_{i=1}^{n} p_{ij}\beta_i$$

with the $p_{ij} \in \mathbb{Z}_{\leq 0}$. For each $1 \leq i \leq n$, set $\hat{p}_i = \min\{p_{i1}, \ldots, p_{il}\}$. Set $\gamma = \mu + \gamma_0 + \sum_{i=1}^{n} \hat{p}_i\beta_i$. Let $\bar{m} = (m_1, \ldots, m_n)$ with each $m_i \in \mathbb{Z}_{\leq 0}$. Set $\gamma_1 = \gamma_0 + \sum_{i=1}^{n}(\hat{p}_i + m_i)\beta_i$. We claim that the $\gamma + \sum_{i=1}^{n} m_i\beta_i$ weight space of $L(\lambda) \otimes L(\mu)$ has dimension equal to $(\deg L(\lambda))(\dim L(\mu))$. Indeed, for each $1 \leq j \leq l$, choose a basis $\mathfrak{B}_{\nu_j}$ for the ν_j weight space of $L(\mu)$. Also, for each $1 \leq j \leq l$, choose a basis $\mathfrak{B}_{\gamma_1 j}$ for the $\gamma_1 - \sum_{i=1}^{n} p_{ij}\beta_i$ weight space of $L(\lambda)$. Notice that since

$$\gamma_1 - \sum_{i=1}^{n} p_{ij}\beta_i = \gamma_0 + \sum_{i=1}^{n}(m_i + \hat{p}_i - p_{ij})\beta_i$$

and $m_i + \hat{p}_i - p_{ij} \leq 0$ for all i, we have that the dimension of the $\gamma_1 - \sum_{i=1}^{n} p_{ij}\beta_i$ weight space of $L(\lambda)$ is equal to d, for all j. We now have that a basis for the $\gamma + \sum_{i=1}^{n} m_i\beta_i = \gamma_1 + \mu$ weight space of $L(\lambda) \otimes L(\mu)$ is given by

$$\{v_j \otimes w_j \mid v_j \in \mathfrak{B}_{\gamma_1 j},\ w_j \in \mathfrak{B}_{\nu_j},\ 1 \leq j \leq l\}$$

and hence the $\gamma + \sum_{i=1}^{n} m_i\beta_i$ weight space of $L(\lambda) \otimes L(\mu)$ has dimension equal to $(\deg L(\lambda))(\dim L(\mu))$, which proves our claim. Since $L(\lambda) \otimes L(\mu)$ is equivalent to a submodule of $V_{\bar{0}}(\lambda) \otimes L(\mu)$, we have that

$$(L(\lambda) \otimes L(\mu))_{\gamma+\sum_{i=1}^{n} m_i\beta_i} \lesssim (V_{\bar{0}}(\lambda) \otimes L(\mu))_{\gamma+\sum_{i=1}^{n} m_i\beta_i}$$

where the equivalence is as $\mathfrak{U}_0$-modules. However, by Proposition 5.17, since $\deg V_{\bar{0}}(\lambda) \otimes L(\mu) \leq (\deg L(\lambda))(\dim L(\mu))$, we must have, in fact,

$$(L(\lambda) \otimes L(\mu))_{\gamma+\sum_{i=1}^{n} m_i\beta_i} \simeq (V_{\bar{0}}(\lambda) \otimes L(\mu))_{\gamma+\sum_{i=1}^{n} m_i\beta_i}$$

where the equivalence is as $\mathfrak{U}_0$-modules. For any such choice of $\bar{m}$, we also have

$$V_{\bar{0}}(\lambda) \otimes L(\mu) \simeq V_{\bar{m}}(\lambda) \otimes L(\mu)$$

and hence

$$(L(\lambda) \otimes L(\mu))_{\gamma+\sum_{i=1}^{n} m_i\beta_i} \simeq (V_{\bar{m}}(\lambda) \otimes L(\mu))_{\gamma+\sum_{i=1}^{n} m_i\beta_i}$$

as $\mathfrak{U}_0$ modules. □

Recall that for $\lambda \in \mathcal{H}^*$, the vector space $V_{\bar{a}}(\lambda) = V_{\bar{0}}(\lambda)$ for all $\bar{a} \in \mathbb{C}^n$. However, the action of L on these vector spaces is different for each $\bar{a}$. With this in mind, we make the following observation:

Lemma 9.3. *Let $\lambda, \mu \in \mathcal{H}^*$ such that $L(\lambda)$ is admissible and $L(\mu)$ is finite dimensional. Choose $\gamma \in \operatorname{Supp}(V_{\bar{0}}(\lambda) \otimes L(\mu))$ as in Lemma 9.2. There exists a set of vectors $\mathfrak{B}$, which is a basis for the $\gamma + \sum_{i=1}^n a_i\beta_i$ weight space of $V_{\bar{a}}(\lambda) \otimes L(\mu)$, for all $\bar{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$. Further, the action of $\mathfrak{U}_0$ on $\mathfrak{B}$ in $V_{\bar{a}}(\lambda) \otimes L(\mu)$ yields elements whose coefficients with respect to $\mathfrak{B}$ are polynomials in variables $a_1, \ldots, a_n$.*

Proof. Let

$$\operatorname{Supp} L(\mu) = \{\nu_1, \ldots, \nu_l\}$$

where each

$$\nu_j = \mu - \sum_{i=1}^n p_{ij}\beta_i$$

with the $p_{ij} \in \mathbb{Z}_{\geq 0}$. For each $1 \leq j \leq l$, choose a basis $\mathfrak{B}_{\nu_j}$ for the ν_j weight space of $L(\mu)$. Let $\{v_1, \ldots, v_d\}$ be a basis for the $\gamma - \mu$ weight space of $V_{\bar{0}}(\lambda)$. Then since the $x_{-\beta_i}$ act injectively, and the dimensions of all weight spaces of $V_{\bar{0}}(\lambda)$ are equal, we have for each $1 \leq j \leq l$,

$$\{x_{-\beta_1}^{-p_{1j}} \ldots x_{-\beta_2}^{-p_{nj}} v_k \mid 1 \leq k \leq d\}$$

is a basis for the $\gamma - (\mu - \sum_{i=1}^n p_{ij}\beta_i)$ weight space of $V_{\bar{0}}(\lambda)$. Thus a basis for the γ weight space of $V_{\bar{0}}(\lambda) \otimes L(\mu)$ is given by

$$\mathfrak{B}_{\bar{0}} = \{x_{-\beta_1}^{-p_{1j}} \ldots x_{-\beta_2}^{-p_{nj}} v_k \otimes w_j \mid 1 \leq k \leq d, w_j \in \mathfrak{B}_{\nu_j}, 1 \leq j \leq l\}$$

The same set of vectors $\mathfrak{B}_{\bar{0}}$ is hence also a basis for the $\gamma + \sum_{i=1}^n a_i\beta_i$ weight space of $V_{\bar{a}}(\lambda) \otimes L(\mu)$, where $\bar{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$. Denote this basis by $\mathfrak{B}_{\bar{a}}$, when in the module $V_{\bar{a}}(\lambda) \otimes L(\mu)$. Notice that for any $v \in \mathfrak{B}_{\bar{a}}$, and any $x \in \mathfrak{U}_0$, we have

$$xv = \sum_{u \in \mathfrak{B}_{\bar{a}}} f_{v,u}(a_1, \ldots, a_n)u$$

where the $f_{v,u}(a_1, \ldots, a_n)$ are polynomials in variables $a_1, \ldots, a_n$. □

Proposition 9.1. *Let $\lambda, \mu \in \mathcal{H}^*$ such that $L(\lambda)$ is admissible, and $L(\mu)$ is finite dimensional. Let $\chi_1, \dots \chi_k$ be the distinct central characters occurring in $L(\lambda) \otimes L(\mu)$. Then for any element $z \in Z(\mathfrak{U})$, we have*

$$\prod_{i=1}^{k} (z - \chi_i(z))v = 0$$

for all $v \in V_{\bar{a}}(\lambda) \otimes L(\mu)$, and all $\bar{a} = (a_1, \dots, a_n) \in \mathbb{C}^n$.

Proof. Choose $\gamma \in (\operatorname{Supp} V_{\bar{0}} \otimes L(\mu))$ as in Lemma 9.2. Applying Lemma 9.3, for each $\bar{a} = (a_1, \dots, a_n) \in \mathbb{C}^n$ let $\mathfrak{B}_{\bar{a}}$ be a basis for the $\gamma + \sum_{i=1}^{n} a_i\beta_i$ weight space of $V_{\bar{a}}(\lambda) \otimes L(\mu)$, such that $\mathfrak{B}_{\bar{a}} = \mathfrak{B}_{\bar{b}}$ as sets of vectors, for all $\bar{a}$ and $\bar{b}$. Let $z \in Z(\mathfrak{U})$, and set

$$z_0 = \prod_{i=1}^{k} (z - \chi_i(z))$$

For any $v \in \mathfrak{B}_{\bar{a}}$, we have

$$z_0 v = \sum_{u \in \mathfrak{B}_{\bar{a}}} f_{v,u}(a_1, \dots, a_n)u$$

where the $f_{v,u}(a_1, \dots, a_n)$ are polynomials in variables $a_1, \dots, a_n$. Clearly $z_0 v = 0$ for any $v \in L(\lambda) \otimes L(\mu)$. Let $\bar{m} = (m_1, \dots, m_n)$, with each $m_i \in \mathbb{Z}_{\leq 0}$. Since $\mathfrak{B}_{\bar{m}}$ is a basis for the $\gamma + \sum_{i=1}^{n} m_i\beta_i$ weight space of $V_{\bar{m}}(\lambda) \otimes L(\mu)$, due to Lemma 9.2, we have $z_0 v = 0$ for any $v \in \mathfrak{B}_{\bar{m}}$. Thus, by Lemma 6.5, the polynomials $f_{v,u}(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in \mathbb{C}$. Therefore, for all $\bar{a}$, we have $z_0 v = 0$ for all $v \in \mathfrak{B}_{\bar{a}}$. Next, let $\bar{a}$ be arbitrary, and let γ' be any weight of $V_{\bar{a}}(\lambda) \otimes L(\mu)$. Set

$$\gamma - \gamma' = \sum_{i=1}^{n} b_i\beta_i$$

for some $b_i \in \mathbb{C}$. Since the support of $V_{\bar{a}}(\lambda) \otimes L(\mu)$ is a single Q-coset, we must have

$$\sum_{i=1}^{n} a_i\beta_i - \sum_{i=1}^{n} b_i\beta_i = \sum_{i=1}^{n} a_i\beta_i + \gamma - \gamma' \in Q$$

Therefore

$$V_{\bar{a}}(\lambda) \otimes L(\mu) \simeq V_{\bar{b}}(\lambda) \otimes L(\mu)$$

in particular, these modules are equivalent $\mathfrak{U}_0$-modules. Hence

$$(V_{\bar{a}}(\lambda) \otimes L(\mu))_{\gamma'} \simeq (V_{\bar{b}}(\lambda) \otimes L(\mu))_{\gamma'}$$

as $\mathfrak{U}_0$-modules. Since $\mathfrak{B}_{\bar{b}}$ is a basis for the $\gamma + \sum_{i=1}^{n} b_i\beta_i = \gamma'$ weight space of $V_{\bar{b}}(\lambda) \otimes L(\mu)$, we have that $z_0 v = 0$ for all $v \in (V_{\bar{b}}(\lambda) \otimes L(\mu))_{\gamma'}$, and hence for all $v \in (V_{\bar{a}}(\lambda) \otimes L(\mu))_{\gamma'}$. Thus for all $\bar{a}$ and all $\gamma' \in \operatorname{Supp} V_{\bar{a}}(\lambda) \otimes L(\mu)$, we have $z_0 v = 0$ for all $v \in (V_{\bar{a}}(\lambda) \otimes L(\mu))_{\gamma'}$, which completes the proof. □

Lemma 9.4. *Let $\chi_1, \ldots, \chi_k$ be distinct non-zero central characters. Then there exists $z \in Z(\mathfrak{U})$ such that $\chi_i(z) \neq \chi_j(z)$ for all $i \neq j$.*

Proof. We apply induction on k. If $k = 1$, the result is trivial. Choose z_0 such that $\chi_i(z_0) \neq \chi_j(z_0)$ for all $1 \leq i \neq j \leq k-1$. For each $i < k$, let $x_i \in \ker \chi_i \setminus \ker \chi_k$. Set $x = \prod_{i \neq k} x_i$. Then $x \in \ker \chi_i \setminus \ker \chi_k$ for all $i < k$. For each $n \in \mathbb{Z}_{\geq 0}$, set $z_n = z_0 + nx$. Then for all $n \in \mathbb{Z}_{\geq 0}$, and all $i < k$, we have $\chi_i(z_n) = \chi_i(z_0)$. Therefore for all $n \in \mathbb{Z}_{\geq 0}$ we have $\chi_i(z_n) \neq \chi_j(z_n)$ for all $1 \leq i \neq j \leq k-1$. Further, since $\chi_k(x) \neq 0$, we have $\chi_k(z_n) \neq \chi_k(z_m)$ for all $n \neq m \in \mathbb{Z}_{\geq 0}$. Since

$$\{\chi_k(z_n) \mid n \in \mathbb{Z}_{\geq 0}\}$$

is an infinite set, and

$$\{\chi_i(z_n) \mid 1 \leq i < k \text{ and } n \in \mathbb{Z}_{\geq 0}\} = \{\chi_i(z_0) \mid 1 \leq i < k\}$$

is a finite set, we can choose $m \in \mathbb{Z}_{\geq 0}$ such that

$$\chi_k(z_m) \notin \{\chi_i(z_n) \mid 1 \leq i < k \text{ and } n \in \mathbb{Z}_{\geq 0}\}$$

which gives the desired result. □

Proposition 9.2. *Let $\lambda, \mu \in \mathcal{H}^*$ such that $L(\lambda)$ is admissible, and $L(\mu)$ is finite dimensional. Set $L(\lambda) \otimes L(\mu) = \sum_{i=1}^{k} M_i L(\lambda_i)$, as in Theorem 8.2. Let $\bar{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$, and for each $1 \leq i \leq k$, set*

$$V_i^{\bar{a}} = \{v \in V_{\bar{a}}(\lambda) \otimes L(\mu) \mid zv = \chi_{\lambda_i}(z)v \text{ for all } z \in Z(\mathfrak{U})\}$$

Then each $V_i^{\bar{a}}$ is a submodule of $V_{\bar{a}}(\lambda) \otimes L(\mu)$, and

$$V_{\bar{a}}(\lambda) \otimes L(\mu) = \bigoplus_{i=1}^{k} V_i^{\bar{a}}$$

Further, for each i, we have $\deg V_i^{\bar{a}} = M_i \deg L(\lambda_i)$.

Proof. For any $x \in \mathfrak{U}$, $v \in V_i^{\bar{a}}$, and $z \in Z(\mathfrak{U})$, we have $zxv = xzv = \chi_{\lambda_i}(z)xv$, and hence $V_i^{\bar{a}}$ is a sub-$\mathfrak{U}$-module of $V_{\bar{a}}(\lambda) \otimes L(\mu)$. By Proposition 5.20, since the χ_{λ_i} are distinct non-zero central characters, we have that $V_1^{\bar{a}}, \ldots, V_k^{\bar{a}}$ are linearly independent. Choose $z_0 \in Z(\mathfrak{U})$ such that $\chi_{\lambda_i}(z_0) \neq \chi_{\lambda_j}(z_0)$ for any $i \neq j$. For each $1 \leq r \leq k$, let $f_r(x)$ be the polynomial

$$f_r(x) = \prod_{i \neq r}(x - \chi_{\lambda_i}(z_0))$$

Since the $\chi_{\lambda_i}(z_0)$ are distinct, we have that the polynomials

$$\{f_r(x) \mid 1 \leq r \leq k\}$$

are relatively prime. Choose polynomials $g_1(x), \ldots, g_k(x)$ such that

$$1 = \sum_{i=1}^{k} f_i(x)g_i(x)$$

For each $1 \leq r \leq k$, set $z_r = f_r(z_0)$, and let $U_r^{\bar{a}} = \{z_r v \mid v \in V_{\bar{a}}(\lambda) \otimes L(\mu)\}$ be the image of z_r on $V_{\bar{a}}(\lambda) \otimes L(\mu)$. Then for any $v \in V_{\bar{a}}(\lambda) \otimes L(\mu)$, we have

$$v = \sum_{i=1}^{k} z_i g_i(z_0)v \in U_1^{\bar{a}} + \cdots + U_k^{\bar{a}}$$

Notice that

$$U_r^{\bar{a}} = \{v \in V_{\bar{a}}(\lambda) \otimes L(\mu)\} \mid z_0 v = \chi_{\lambda_r}(z_0)v\}$$

Indeed, if $z_r v \in U_r^{\bar{a}}$ then

$$(z_0 - \chi_{\lambda_r}(z_0))z_r v = \prod_{i=1}^{k}(z_0 - \chi_{\lambda_i}(z_0))v = 0$$

Conversely, if $v \in V_{\bar{a}}(\lambda) \otimes L(\mu)$ with $z_0 v = \chi_{\lambda_r}(z_0)v$, then

$$z_r \frac{v}{\prod_{i \neq r}(\chi_{\lambda_r}(z_0) - \chi_{\lambda_i}(z_0))} = v$$

and hence $v \in U_r^{\bar{a}}$. Therefore the $U_i^{\bar{a}}$ are the eigenspaces for the action of z_0 on $V_{\bar{a}}(\lambda) \otimes L(\mu)$. Further, $Z(\mathfrak{U})$ is a family of commuting, diagonalizable endomorphisms on $V_{\bar{a}}(\lambda) \otimes L(\mu)$, and is hence simultaneously diagonalizable. Therefore, since the

χ_{λ_i} take distinct values at z_0, we must have that the z_0 eigenspace determines the simultaneous eigenspaces for all $z \in Z(\mathfrak{U})$, hence $U_i^{\bar{a}} = V_i^{\bar{a}}$ for all $1 \leq i \leq k$. Thus,

$$U_1^{\bar{a}} + \cdots + U_k^{\bar{a}} = \bigoplus_{i=1}^{k} V_i^{\bar{a}}$$

and hence

$$V_{\bar{a}}(\lambda) \otimes L(\mu) = \bigoplus_{i=1}^{k} V_i^{\bar{a}}$$

Next, choose $\gamma \in \operatorname{Supp} V_{\bar{0}}(\lambda) \otimes L(\mu)$ according to Lemma 9.2. Applying Lemma 9.3, choose a set of vectors $\mathfrak{B}_{\bar{0}}$ that is a basis for the γ weight space of $V_{\bar{0}}(\lambda) \otimes L(\mu)$, with the same set of vectors, denoted $\mathfrak{B}_{\bar{a}}$, being a basis for the $\gamma + \sum_{i=1}^{n} a_i\beta_i$ weight space of $V_{\bar{a}}(\lambda) \otimes L(\mu)$. Again, we have that the action of $\mathfrak{U}_0$ on $\mathfrak{B}_{\bar{a}}$ yields elements whose coefficients with respect to the basis $\mathfrak{B}_{\bar{a}}$ are polynomials in variables $a_1, \ldots, a_n$. For each $1 \leq r \leq k$, let $[z_r]_{\mathfrak{B}_{\bar{a}}}$ denote the matrix representation of z_r acting on the $\gamma + \sum_{i=1}^{n} a_i\beta_i$ weight space of $V_{\bar{a}}(\lambda) \otimes L(\mu)$ with respect to the basis $\mathfrak{B}_{\bar{a}}$. Thus, the entries in $[z_r]_{\mathfrak{B}_{\bar{a}}}$ are polynomials in variables $a_1, \ldots, a_n$. This implies that the determinant of $[z_r]_{\mathfrak{B}_{\bar{a}}}$, and the determinants of any sub-matrices of $[z_r]_{\mathfrak{B}_{\bar{a}}}$ are all polynomials in variables $a_1, \ldots, a_n$. For any $\bar{m} = (m_1, \ldots, m_n)$ with $m_1, \ldots, m_n \in \mathbb{Z}_{\leq 0}$, Lemma 9.2 implies that the $\gamma + \sum_{i=1}^{n} m_i\beta_i$ weight space of $V_{\bar{m}}(\lambda) \otimes L(\mu)$ is equivalent as a $\mathfrak{U}_0$-module to a weight space of $L(\lambda) \otimes L(\mu)$. Therefore the rank of the matrix $[z_r]_{\mathfrak{B}_{\bar{m}}}$ is at most $M_r \deg L(\lambda_r)$. This implies that for any $q \geq M_r \deg L(\lambda_r)$, the determinant of any $q \times q$ sub-matrix of $[z_r]_{\mathfrak{B}_{\bar{m}}}$ is zero. Let $q \geq M_r \deg L(\lambda_r)$, and let $A_{\bar{a}}$ be any $q \times q$ sub-matrix of $[z_r]_{\mathfrak{B}_{\bar{a}}}$. Then $\det A_{\bar{a}}$ is a polynomial in variables $a_1, \ldots, a_n$, and whenever $a_1, \ldots, a_n \in \mathbb{Z}_{\leq 0}$, we have $\det A_{\bar{a}} = 0$. Thus, by Lemma 6.5, we have $\det A_{\bar{a}} = 0$ for all $\bar{a} \in \mathbb{C}^n$. Therefore the rank of $[z_r]_{\mathfrak{B}_{\bar{a}}}$ is at most $M_r \deg L(\lambda_r)$. However, since $V_r^{\bar{a}}$ is the image of the action of z_r, the rank of $[z_r]_{\mathfrak{B}_{\bar{a}}}$ is equal to the dimension of the $\gamma + \sum_{i=1}^{n} a_i\beta_i$ weight space of $V_r^{\bar{a}}$. Therefore

$$\dim(V_r^{\bar{a}})_{\gamma + \sum_{i=1}^{n} a_i\beta_i} \leq M_r \deg L(\lambda_r)$$

for all $\bar{a}$. Further, if γ' is any other weight of $V_{\bar{a}}(\lambda) \otimes L(\mu)$ then

$$\gamma' = \gamma + \sum_{i=1}^{n} b_i\beta_i$$

for some $b_i \in \mathbb{C}$ with $a_i - b_i \in \mathbb{Z}$. Therefore the γ' weight space of $V_r^{\bar{a}}$ is equivalent to the γ' weight space of $V_r^{\bar{b}}$. Thus

$$\dim(V_r^{\bar{a}})_{\gamma'} = \dim(V_r^{\bar{b}})_{\gamma+\sum_{i=1}^n b_i\beta_i} \leq M_r \deg L(\lambda_r)$$

Since γ' was arbitrary, we have that $\deg V_r^{\bar{a}} \leq M_r \deg L(\lambda_r)$. Since r was arbitrary, the above holds for all $1 \leq r \leq k$. Notice that since $\mathfrak{B}_{\bar{a}}$ is a basis for $(V_{\bar{a}}(\lambda) \otimes L(\mu))_{\gamma+\sum_{i=1}^n a_i\beta_i}$, we must have

$$\dim(V_{\bar{a}}(\lambda) \otimes L(\mu))_{\gamma+\sum_{i=1}^n a_i\beta_i} = (\deg L(\lambda))(\dim L(\mu))$$

Therefore,

$$\begin{aligned}
(\deg L(\lambda))(\dim L(\mu)) &\leq (\deg V_{\bar{a}}(\lambda) \otimes L(\mu)) \\
&= \sum_{i=1}^k \deg V_i^{\bar{a}} \\
&\leq \sum_{i=1}^k M_i \deg L(\lambda_i) \\
&= \deg\left(L(\lambda) \otimes L(\mu)\right) \\
&\leq (\deg L(\lambda))(\dim L(\mu))
\end{aligned}$$

Thus equality holds, and hence $\deg V_i^{\bar{a}} = M_i \deg L(\lambda_i)$ for all $1 \leq i \leq k$. □

Lemma 9.5. *Let V be a simple torsion free C_n-module, with central character χ. If $\lambda \in \mathcal{H}^*$ such that $L(\lambda)$ is admissible, and $\chi = \chi_\lambda$, then $V = V_{\bar{a}}(\lambda)$ for any $\bar{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$ such that $\lambda + \sum_{i=1}^n a_i\beta_i \in \operatorname{Supp} V$.*

Proof. By Theorem 6.1, we have that $V \simeq V_{\bar{b}}(\gamma)$ for some $\gamma \in \mathcal{H}^*$ with $L(\gamma)$ admissible, and some $\bar{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n$. Further, we have $\chi_\gamma = \chi_\lambda$. By Lemma 7.1, if $\gamma \neq \lambda$ then $L(\lambda)$ and $L(\gamma)$ are the only simple admissible highest weight modules with central character χ, and $\lambda + Q \neq \gamma + Q$. Let $\mathcal{M}$ be the semisimple coherent family containing V, with degree equal to the degree of V. By Proposition 6.13, $\mathcal{M}$ contains simple admissible highest weight modules with central character χ, having highest weights in both the $\gamma + Q$ coset, and the $\lambda + Q$ coset. Since there is exactly one such choice in each coset, we must have $L(\lambda) \leq \mathcal{M}$ and $L(\gamma) \leq \mathcal{M}$. Therefore, by Lemma 6.6,

$$\mathcal{M} \simeq \mathcal{M}_{ss}(L(\lambda)) \simeq \mathcal{M}_{ss}(L(\gamma))$$

and hence

$$V_{\bar{b}}(\gamma) = \bigoplus_{\nu \in \gamma + \sum_{i=1}^n b_i\beta_i + Q} \mathcal{M}_\nu \simeq \bigoplus_{\nu \in \lambda + \sum_{i=1}^n a_i\beta_i + Q} \mathcal{M}_\nu = V_{\bar{a}}(\lambda)$$

for any $\bar{a} = (a_1, \dots, a_n) \in \mathbb{C}$ such that $\lambda + \sum_{i=1}^n a_i\beta_i + Q = \operatorname{Supp} V$. □

Proposition 9.3. *(Britten-Khomenko-Lemire-Mazorchuk) Let V be a torsion free C_n-module with finite dimensional weight spaces. Then V is completely reducible, hence semisimple.*

Proof. (See Theorem 1 in [2]) □

Theorem 9.1. *Let $V_{\bar{a}}(\lambda)$ be a simple torsion free C_n-module, and $L(\mu)$ be a simple finite dimensional highest weight C_n-module. Then*

$$V_{\bar{a}}(\lambda) \otimes L(\mu) \simeq \bigoplus_{\nu \in \mathcal{A}_\lambda(\mu)} \left(\sum_{\sigma \in \widetilde{\mathcal{W}}} sgn(\sigma) m_\mu(\sigma \cdot (\lambda + \nu) - \lambda) \right) V_{\bar{a}}(\lambda + \nu)$$

and each $V_{\bar{a}}(\lambda + \nu)$ is a simple torsion free C_n-module.

Proof. For each $\nu \in \mathcal{A}_\lambda(\mu)$, set

$$V_\nu^{\bar{a}} = \{ v \in V_{\bar{a}}(\lambda) \otimes L(\mu) \mid zv = \chi_{\lambda+\nu}(z)v \text{ for all } z \in Z(\mathfrak{U}) \}$$

by Proposition 9.2, we have that

$$V_{\bar{a}}(\lambda) \otimes L(\mu) \simeq \bigoplus_{\nu \in \mathcal{A}_\lambda(\mu)} V_\nu^{\bar{a}}$$

Further, for each $\nu \in \mathcal{A}_\lambda(\mu)$ we have

$$\deg V_\nu^{\bar{a}} = \left(\sum_{\sigma \in \widetilde{\mathcal{W}}} sgn(\sigma) m_\mu(\sigma \cdot (\lambda + \nu) - \lambda) \right) \deg L(\lambda + \nu)$$

Let $\nu \in \mathcal{A}_\lambda(\mu)$. By Proposition 5.16, $V_{\bar{a}}(\lambda) \otimes L(\mu)$ is torsion free, and hence $V_\nu^{\bar{a}}$ is torsion free. Due to Proposition 9.3, $V_\nu^{\bar{a}}$ is therefore semisimple. Thus,

$$V_\nu^{\bar{a}} \simeq \bigoplus_{i=1}^{k} U_i$$

for some simple torsion free modules U_i. Since $V_\nu^{\bar{a}}$ has central character $\chi_{\lambda+\nu}$, and support equal to

$$\lambda + \mu + \sum_{i=1}^{n} a_i\beta_i + Q = \lambda + \nu + \sum_{i=1}^{n} a_i\beta_i + Q$$

we have, by Lemma 9.5, that every U_i is equivalent to $V_{\bar{a}}(\lambda + \nu)$. Therefore

$$V_\nu^{\bar{a}} \simeq kV_{\bar{a}}(\lambda + \nu)$$

Further,

$$\begin{aligned}\left(\sum_{\sigma \in \widetilde{W}} sgn(\sigma) m_\mu(\sigma \cdot (\lambda + \nu) - \lambda)\right) \deg L(\lambda + \nu) &= \deg V_\nu^{\bar{a}} \\ &= k\deg V_{\bar{a}}(\lambda + \nu) \\ &= k\deg L(\lambda + \nu)\end{aligned}$$

and hence

$$V_\nu^{\bar{a}} \simeq \left(\sum_{\sigma \in \widetilde{W}} sgn(\sigma) m_\mu(\sigma \cdot (\lambda + \nu) - \lambda)\right) V_{\bar{a}}(\lambda + \nu)$$

which completes the proof. □

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