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Multivariate moments and Cochran theorems.

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MULTIVARIATE MOMENTS AND COCHRAN THEOREMS

by

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To my wife Jialin and my children Ying and Xi

MULTIVARIATE MOMENTS AND COCHRAN THEOREMS

BY

TONGHUI (TONY) WANG

Abstract

This thesis is divided into two related parts:

(I) **Moments.** For a multivariate elliptically contoured random matrix $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$, formulae for finding the higher order moments of both Y and its quadratic forms are obtained in terms of μ, Σ_Y and ϕ, where Σ_Y is not required to have the form $A \otimes \Sigma$. These results are so general that they are new even for the normal setting. Specific worked out examples on moments are given for both normal and certain non-normal settings such as multivariate uniform distributions, symmetric multivariate Pearson Type VII distributions, generalized Wishart distributions, multivariate components of variance models and MANOVA models.

The proofs involve linear operators in inner product spaces, Kronecker products, multilinear differential forms and adjoint operators of the linear functions.

(II) **Cochran Theorems.** For a family of quadratic forms, $\{Q_i(Y)\}_{i=1}^{\ell}$, of Y with $Q_i(Y) = Y'W_iY + B_i'Y + Y'C_i + D_i$, W_i symmetric and $Y \sim N_{n\times p}(\mu, \Sigma_Y)$, necessary and sufficient conditions are obtained under which

$\{Q_i(Y)\}$ is an independent family of Wishart $W_p(m_i, \Sigma, \lambda_i)$ random matrices. (*)

Such a result is referred to as a Cochran theorem. The Cochran theorems just mentioned are general in that the covariance matrix Σ_Y need not take the form $A \otimes \Sigma$ and need not be positive definite. Some of these results are extended further to the case where either (i) $W_p(m_i, \Sigma, \lambda_i)$ in (*) is replaced by $DW_p(m_{1i}, m_{2i}, \Sigma, \lambda_{1i}, \lambda_{2i})$, the distribution of the difference of two independent Wishart random matrices Q_{1i} and Q_{2i} with $Q_{1i} \sim W_p(m_{1i}, \Sigma, \lambda_{1i})$ and $Q_{2i} \sim W_p(m_{2i}, \Sigma, \lambda_{2i})$, or (ii) Y is multivariate elliptically contoured distributed.

The proofs involve linear operators in inner product spaces, Moore-Penrose inverses, projections, inclusion maps, spectra, invariant measures and conditional expectations.

PREFACE

Multivariate moments and Cochran Theorems were discussed by many scholars. In the Summer of 1989, we began to study these topics. This thesis is the result of our investigation in the last three years.

We would like to express our gratitude to Dr. Joe Masaro of Acadia University for his kindness in allowing me to quote certain results in our joint papers.

The author wishes to express his sincerest gratitude and appreciation to Professor Chi Song Wong for his help, constant support and supervision at every stage especially near the end of this work, without which, this thesis would never have been written. Many ideas and results from his lectures and our joint papers are included in this thesis.

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CONTENTS

CHAPTER ONE

INTRODUCTION

This thesis deals with multuvariate moments and Cochran theorems; all results and applications are of a theoretical nature. In this thesis, all inner product spaces will be finite dimensional and over the real field $\Re$. Those who prefer the matrix approach may read inner product spaces as Euclidean spaces and treat linear operators as matrices. Also in this thesis, we shall use E, V to denote certain n-, p-dimensional inner product spaces over $\Re$, use $\mathcal{L}(V, E)$ to denote the linear space of all linear maps of V into E, use $M_{n\times p}$ to denote the set of all $n \times p$ matrices over $\Re$, use $\Re^n$ to denote $M_{n\times 1}$, use $\mathcal{S}_E$ $(\mathcal{N}_E)$ to denote the set of all self-adjoint (nonnegative definite) $T \in \mathcal{L}(E, E)$, and use $\mathcal{S}_n$ $(\mathcal{N}_n)$ to denote the set of all symmetric (nonnegative definite) $T \in M_{n\times n}$.

1.1. Motivation and goals

Until 1990, standard textbooks on multivariate analysis mainly concentrated on multivariate normal distributions, see, e.g., Muirhead (1981), Eaton (1983) and Anderson (1984). This was due not only to the fact that the multivariate observations are, often, approximately normally distributed because of the central limit theorem effect, but also to the fact that multivariate normal distributions and the corresponding sampling distributions are mathematically tractable and "nice" results can be obtained. In reality, samples obtained by researchers are not always from normal populations. So paying attention to samples from nonnormal populations is very important in multivariate analysis. In the past two decades, numerous papers and several books have been focused on a class of nonnormal distributions, called the elliptically contoured distributions. Many properties of the elliptically contoured distributions are similar to those of the multivariate normal distributions. This class of distributions contains both multivariate normal distributions and certain nonnormal distributions, such as multivariate t-distributions, multivariate Cauchy distributions, multivariate Laplace distributions and multivariate uniform distributions, see, e.g., Fang and Anderson (1990), Fang, Kotz and Ng (1990) and Fang and Zhang (1990). The distributions discussed in this thesis are also referred to as multivariate elliptically contoured distributions [or vector-elliptically contoured distributions, as in Fang and Zhang (1990)], and they form a subclass of the above "elliptically contoured distributions".

Another major assumption in the standard multivariate analysis is that the covariance, Σ_Y, of a normal random matrix Y, is either of the form $A \otimes \Sigma$ with positive definite A and Σ, or of the form $\sum_{j=1}^{k} A_j \otimes \Sigma_j$ with nonsingular $\sum_{j=1}^{k} A_j \otimes \Sigma_j$, see, e.g., Anderson (1984, 1987), Rao and Kleffe (1988) and Mathew (1989). To meet the requirements both in theory and in practice, the above restrictions on Σ_Y will be relaxed in this thesis.

As shown by its title, our goals in this thesis are:

Part I: to generalize the existing results on moments from normal settings to multivariate elliptically contoured settings without any restriction on Σ_Y, and

Part II: to extend the existing Cochran theorems for normal as well as for multivariate elliptically contoured settings.

Before further describing Part I and Part II in Section 1.3 and Section 1.4, we shall first introduce multivariate elliptically contoured distributions in Section 1.2.

1.2. Multivariate elliptically contoured distributions

As a natural extension of multivariate normal distributions, we shall define multivariate elliptically contoured (MEC) distributions:

Definition 1.2.1. *Let E, V be, respectively, n-, p-dimensional inner product spaces over $\Re$, $\mu \in \mathcal{L}(V, E)$ and ϕ be a function of $[0, \infty)$ into the complex field C. Then a random operator Y in $\mathcal{L}(V, E)$ is said to be multivarite elliptically contoured distributed [written as $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$] if the characteristic function (c.f.), $\hat{Y}$, of Y is given by*

$$\hat{Y}(T) = e^{i\langle T, \mu\rangle}\phi(u), \quad u = \langle T, \Sigma_Y(T)\rangle, \quad T \in \mathcal{L}(V, E). \tag{1.2.1}$$

where μ is the mean of Y, $c\Sigma_Y \equiv Cov(Y) \in \mathcal{L}(\mathcal{L}(V, E), \mathcal{L}(V, E))$ is the covariance of Y, c is a constant to be given Theorem 2.2.3, and $\langle\ ,\ \rangle$ is the trace inner product defined by $\langle B, C\rangle = tr(B'C) = tr(C'B)$.

A special case of the above definition was given by Anderson and Fang (1982a), where Σ_Y is of the form $A \otimes \Sigma$, or $\Sigma_Y = \mathrm{diag}(\Sigma_1, \Sigma_2, \cdots, \Sigma_n)$. Note that in (1.2.1), if

$$\phi(u) = e^{-u/2}, \qquad u \geq 0, \tag{1.2.2}$$

then $Y \sim N_{n\times p}(\mu, \Sigma_Y)$, the multivariate normal distribution with mean μ and covariance Σ_Y. When $p = 1$ and $\Sigma_Y = A$, $MEC_{n\times p}(\mu, \Sigma_Y, \phi)$ is referred to as the univariate elliptically contoured distribution with parameter (μ, A, ϕ) and is given by written as $EC_n(\mu, A, \phi)$. For this case, the c.f., $\hat{y}$, of $y \equiv Y$ is

$$\hat{y}(t) = e^{it'\mu}\phi(t'At), \qquad t,\ \mu \in E. \tag{1.2.3}$$

Another special case of $MEC_{n\times p}(\mu, \Sigma_Y, \phi)$, called the multivariate components of variance model, was discussed by Anderson et al. (1986), Anderson (1987), Rao and Kleffe (1988) and Mathew (1989), where $\mu = XB$, $\Sigma_Y = \sum_{j=1}^{k} A_j \otimes \Sigma_j$ and $\phi(u) = e^{-u/2}$.

Basic properties of $MEC_{n\times p}(\mu, \Sigma_Y, \phi)$ will be discussed in Section 6.2, see, e.g., Schoenberg (1938), Kelker (1970), Kariya and Eaton (1977), Dawid (1978), Chmielswski (1980), Muirhead (1980), Jensen and Good (1981), Cambanis, Huang, and Simons (1981), Anderson and Fang (1982a), Fang and Zhang (1990), Fang, Kotz, and Ng (1990), and numerous references in Fang and Anderson (1990).

1.3 Moments

Calculations of moments are very important in statistical inference and have received considerable attention in the past twenty years. But most papers on this topic have been restricted to normally distributed random variables, see, e.g., Searle (1971), Drygas (1984, 1985), Jinadasa (1986), Kleffe (1978), Magnus and Neudecker (1979), Neudecker (1985) and Neudecker and Wansbeek (1983, 1987). Von Rosen (1988) adopted an approach based on differentiating the moment generating function of a normally distributed random matrix by the aid of matrix derivatives. He obtained closed expressions for the first four order moments of $Y \sim N_{n\times p}(\mu, A\otimes\Sigma)$ and gave a recursive formula for the mth order moment, $E(\otimes^m Y)$, of Y for the case where $\mu = 0$.

For $y \in EC_n(\mu, A, \phi)$, formulae for evaluating first four order moments of y can be found in Fang and Xu (1986) and Anderson and Fang (1990). Li (1987) obtained a formula for finding the first four order moments of y and gave expressions for $E(\otimes^L(y\otimes y'))$ and $E(\otimes^L(y\otimes y')\otimes y)$.

Based on the matrix differential methods presented in Wong (1985, 1986), Wong and Wang (1991) obtained a formula for evaluating $E(\otimes^m Y)$ for $Y \sim MEC_{n\times p}(\mu, A\otimes \Sigma, \phi)$. Even for the normal setting, this formula generalizes the corresponding results of Jinadasa (1986), Neudecker and Wansbeek (1983, 1987) and von Rosen (1988).

The first and second order moments of quadratic forms of $Y \in N_{n\times p}(\mu, A\otimes\Sigma)$ were discussed by Neudecker (1985, 1990), Jinadasa (1986), Browne and Neudecker

(1988) and von Rosen (1988). For $y \sim EC_n(\mu, A, \phi)$, Li (1987) obtained a formula for $\mathrm{Cov}(y'C_1y, y'C_2y)$ with symmetric C_1 and C_2; he also gave expressions for higher order moments of its quadratic forms. Wong and Wang (1991) obtained a formula for $\mathrm{Cov}(Y'WY)$ for the case where $Y \sim MEC_{n\times p}(\mu, A\otimes\Sigma, \phi)$ and W is symmetric.

In Part I, we shall obtain formulae, Theorem 2.3.4 and Theorem 3.2.1, for evaluating higher order moments of both $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$ and its quadratic forms where Σ_Y need not be of the form $A\otimes\Sigma$. We shall give some specific worked out examples on moments for both normal and certain non-normal settings such as multivariate components of variance models, multivariate uniform distributions, generalized Wishart distributions and symmetric multivariate Pearson Type VII distributions. For the case where Σ_Y is not of the form $A\otimes\Sigma$, examples will also be given in Part I. These results generalize the corresponding results of Li (1987), von Rosen (1988) and Wong and Wang (1991).

1.4 Cochran theorems

It is well known that Cochran theorems play an important role in regression analysis, analysis of variance, covariance analysis, MINQUE theory, etc. A brief review will be given as follows.

Suppose that $y \sim N_n(0, I_n)$. Let $i \in \{1, 2, \ldots, \ell\}$ and W_i be a symmetric matrix of rank m_i such that $\sum_{i=1}^{\ell} W_i = I_n$. Cochran (1934) proved that $\{y'W_iy\}_{i=1}^{\ell}$ is an independent family of $\chi^2_{m_i}$ random variables with m_i degrees of freedom if and only if

$$\sum_{i=1}^{\ell} m_i = n. \tag{1.4.1}$$

This result is referred to as Cochran's theorem. Note that (1.4.1) is equivalent to the following condition: for any distinct $i,\ j \in \{1, 2, \cdots, \ell\}$,

$$W_i = W_i^2, \qquad W_i W_j = 0. \tag{1.4.2}$$

For $y \sim N_n(\mu,\ I_n)$, Craig (1943) proved that $y'W_1 y$ and $y'W_2 y$ are independent if and only if $W_1 W_2 = 0$. Ogasawra and Takehashi (1951) extended Cochran's theorem to include the case where $y \sim N_n(\mu,\ \Sigma_Y)$ with $\Sigma_Y = A$. They proved that " $y'W_1 y$ and $y'W_2 y$ are independent $\chi^2_{r_1}(\lambda_1)$ and $\chi^2_{r_2}(\lambda_2)$ random variables if and only if (a) $AW_iAW_iA = AW_iA$, $r(AW_iA) = r_i$, $i = 1,\ 2$, (b) $AW_1AW_2A = 0$, (c) $AW_1AW_2\mu = AW_2AW_1\mu = 0$, and (d) $\mu'W_1AW_2\mu = 0$, where W_1 and W_2 are symmetric and A is n.n.d. Since then, various versions of Cochran's theorem were obtained by Rao (1962, 1973), Khatri (1963, 1968, 1977), Good (1963, 1969), Chipman and Rao (1964), Hogg and Craig (1958), Rayner and Livingstone (1965), Shanbhag (1966, 1969), Styan (1970), Nagase and Banerjee (1976), Tan (1977), Wong (1982), Sik and DeGunst (1985), and many others mentioned in Anderson and Styan (1982). All of these authors dealt with the situation where each W_i is symmetric and/or Σ_Y may be singular.

Khatri (1962, 1963) extended Cochran's theorem from the univariate case to the multivariate case. His papers (1977, 1980, 1982) dealt with the case where $\Sigma_Y = A \otimes \Sigma$. He obtained a necessary and sufficient condition under which

$$\{\, Q_i(Y)\,\}_{i=1}^{\ell} \text{ is an independent family of } W_p(m_i,\ \Sigma,\ \lambda_i) \text{ random operators } Q_i(Y), \tag{1.4.3}$$

where $Q_i(Y) = Y'W_iY + B_i'Y + Y'C_i + D_i$, $W_i \in S_n$, B_i, $C_i \in M_{n\times p}$ and $D_i \in M_{p\times p}$, and $W_p(m,\ \Sigma,\ \lambda)$ denotes the distribution of $Z'Z$ with $Z \sim N_{m\times p}(\mu,\ I_m \otimes \Sigma)$ and $\lambda = \mu'\mu$. [In the sequel, $W_p(m,\ \Sigma,\ \lambda)$ is referred to as a Wishart distribution and $W_p(m, \Sigma, 0)$ is denoted by $W_p(m, \Sigma)$.] Later DeGunst (1987) obtained a Cochran theorem, which, however, was essentially included in the above result of Khatri (1980). On the other hand, Pavur (1987) obtained a Cochran theorem that no longer requires that Σ_Y has the form $A \otimes \Sigma$. He proved, in his Theorem 1, that " $\{Y'W_iY\}_{i=1}^{\ell}$ is an independent family of $W_p(r(W_i), \Sigma)$ random matrices $Y'W_iY$ " if and only if for any distinct $i, j \in \{1, 2, \cdots, \ell\}$,

$$(W_i \otimes I_p)\Sigma_Y(W_i \otimes I_p) = W_i \otimes \Sigma \tag{1.4.4}$$

and

$$(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) = 0, \tag{1.4.5}$$

where $Y \sim N_{n\times p}(0, \Sigma_Y)$, Σ is positive definite, $W_i \in \mathcal{N}_n$ and $r(W_i)$ denotes the rank of W_i. This result was extended by Wong, Masaro and Wang (1991) to the case where Σ may be singular, each W_i is self-adjoint, $\mathrm{Im}\,\Sigma_Y = S_1 \square S_2$ and S_1, S_2 are, respectively, nontrivial linear subspaces of E, V, see Theorem 4.6.3. For the general case where no condition whatever is imposed on (μ, Σ_Y), Wong, Masaro and Wang (1991) also gave a necessary and sufficient condition in terms of self-adjoint operators under which

$$\{Y'W_iY\}_{i=1}^{\ell} \text{ is an independent family of } W_p(m_i, \Sigma, \lambda_i) \text{ random operators } Y'W_iY, \tag{1.4.6}$$

where each $W_i \in S_E$. Although this Cochran theorem, Corollary 4.4.8, is very general, verification of its conditions is not always easy. To remedy this, Wong and Wang (1992) obtained a Cochran theorem for the case where all W_i's are nonnegative definite. This result, Corollary 5.3.5, is also an extension of Pavur's result mentioned above. The above two Cochran theorems of Wong et al. (1991) and Wong and Wang (1992) are further generalized in this thesis, see Theorem 4.4.7 and Theorem 5.3.4.

Let $DW_p(m_1, m_2, \Sigma, \lambda_1, \lambda_2)$ denote the distribution of $Q_1 - Q_2$, where Q_1 and Q_2 are independent $W_p(m_1, \Sigma, \lambda_1)$ and $W_p(m_2, \Sigma, \lambda_2)$ random operators. With

$Q_i(Y)$ and $W_p(m_i, \Sigma, \lambda_i)$ in (1.4.3) being replaced by $Q_i(Y) = Y'W_iY + B_i'Y + Y'B_i + D_i$ and $DW_p(m_1, m_2, \Sigma, \lambda_1, \lambda_2)$, Tan (1975, 1976) obtained a necessary and sufficient condition under which (1.4.3) holds. Also with $W_p(m_i, \Sigma, \lambda_i)$ in (1.4.6) being replaced by $DW_p(m_{1i}, m_{2i}, \Sigma, \lambda_{1i}, \lambda_{2i})$, Wong (1992) obtained a necessary and sufficient condition under which (1.4.6) holds. This result is further generalized in this thesis, see Theorem 4.5.6. To relate this result to standard Cochran theorems, we note that when $\lambda_2 = 0$ and $m_2 = 0$, $DW_p(m_1, m_2, \Sigma, \lambda_1, \lambda_2)$ is nothing but the Wishart distribution $W_p(m_1, \Sigma, \lambda_1)$.

All Cochran theorems mentioned above are for the normal setting. Cochran theorems for univariate elliptically contoured distributions can be found in Kelker (1970), Anderson and Fang (1982a, 1982b), Fang and Wu (1984), and Fan (1986).

For the multivariate case where $X \sim MEC_{n\times p}(0, I_n \otimes \Sigma, \phi)$ with positive definite Σ and $Pr(X = 0) = 0$, Anderson and Fang (1982a) obtained a necessary and sufficient condition under which $(X'W_1X, \cdots, X'W_\ell X)$ is a generalized Wishart distributed random matrix with parameter $(m_1, \cdots, m_\ell; m_{\ell+1}; \Sigma; \phi)$, where $m_{\ell+1} \geq 1$ and $\sum_{i=1}^{\ell+1} m_i = n$. In Wang and Wong (1991), this result was extended to Theorem 6.3.3, where $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$, $Pr(Y = \mu) < 1$, $\Sigma \in \mathcal{N}_V$, and $W_i \in \mathcal{N}_E$ $(i = 1, \cdots, \ell)$.

PART I

MOMENTS

CHAPTER TWO

MOMENTS OF AN MULTIVARIATE ELLIPTICALLY CONTOURED RANDOM OPERATOR

2.1 Introduction

In this chapter, we shall obtain a formula for evaluating the mth order moment, $E(\otimes^m Y)$, of $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$ in terms of μ, Σ_Y and ϕ. This formula is general in that Σ_Y need not be of the form $A \otimes \Sigma$.

In Section 2.2, we shall first introduce the notions of Kronecker products and commutation operators and cite their basic properties. Then by using the differential method presented in Wong (1985, 1986), we shall obtain formulae for finding the first four order moments of Y in detail. Even for the normal setting, these formulae generalize the corresponding results of Magnus and Neudecker (1979), Neudecker and Wansbeck (1983, 1987), Jinadasa (1986) and von Rosen (1988).

In Section 2.3, with the ideas and results given in Section 2.2, we shall "conjecture" a formula for evaluating $E(\otimes^m Y)$ in terms of μ, Σ_Y and ϕ and prove it by induction. Special cases of this formula were given in Wong and Wang (1991) with $\Sigma_Y = A \otimes \Sigma$ and von Rosen (1988) with $Y \sim N_{n\times p}(0, A \otimes \Sigma)$.

In Section 2.4, we shall rewrite our general formula for evaluating the mth order moment of Y with small m in conventional forms. We then give some specific worked out examples on moments for normal and certain non-normal settings, such as multivariate uniform distributions and symmetric multivariate Pearson Type VII distributions.

2.2 First four order moments

For this section and other sections of this thesis, we need the following notions of outer products, Kronecker products and commutation operators.

For any $x \in E$ and $y \in V$, the outer product, $x\Box y$, of x and y is defined as the element in $\mathcal{L}(V, E)$ such that

$$(x\Box y)(z) = \langle y, z\rangle x \qquad \text{for all } z \in V, \tag{2.2.1}$$

where $\langle , \rangle$ is the inner product on V. If $x \in \Re^n$ and $y \in \Re^p$, then with the usual bases for $\Re^n$ and $\Re^p$, $x\Box y = xy' \in M_{n\times p}$. For any linear subspaces S_1 and S_2 of E and V, $S_1\Box S_2$ will denote the linear span of $\{x\Box y : x \in S_1,\ y \in S_2\}$.

For any $A \in \mathcal{L}(E_1, E_2)$ and $B \in \mathcal{L}(V_1, V_2)$, the Kronecker product, $A \otimes B$, of A and B is defined as the element in $\mathcal{L}(\mathcal{L}(V_1, E_1), \mathcal{L}(V_2, E_2))$ such that

$$(A \otimes B)(C) = ACB' \qquad \text{for all} \quad C \in \mathcal{L}(V_1, E_1), \tag{2.2.2}$$

where E_1, E_2, V_1 and V_2 are all finite dimensional inner product spaces over $\Re$. The space $\mathcal{L}(\mathcal{L}(V_1, E_1), \mathcal{L}(V_2, E_2))$ will be written as $\mathcal{L}(E_1, E_2) \otimes \mathcal{L}(V_1, V_2)$. If $A = (a_{ij}) \in M_{n\times p}$ and $B = (b_{k\ell}) \in M_{r\times s}$, then with the usual bases for $M_{n\times p}$ and $M_{r\times s}$, $A \otimes B = (a_{ij}B) = (a_{ij}b_{k\ell})_{((i,k),(j,\ell))}$, where the (i,k)'s and (j,ℓ)'s can be ordered in any fixed way or not at all.

The commutation operator, $K_{p,n}$, on $\mathcal{L}(V, E)$ is defined by

$$K_{p,n}(T) = T', \qquad T \in \mathcal{L}(V, E). \tag{2.2.3}$$

With the usual bases for $E = \Re^n$ and $V = \Re^p$, $K_{pn} \in M_{np\times np}$ is called the commutation matrix of order (np, np) and given by

$$K_{pn}\text{vec}\,T = \text{vec}(T'), \qquad T \in M_{n\times p},$$

where $\text{vec}\,T$ is the columnized vector of T.

For easy citation, we list the following properties of Kronecker products and commutation matrices in Henderson and Searle (1981), Magnus and Neudecker (1979), and Neudecker and Wansbeek (1983):

Lemma 2.2.1. *(a) For any* $A \in M_{m\times n}$, $B \in M_{p\times q}$,

(i) $K_{pm}\,(A \otimes B)K_{nq} = B \otimes A$,

(ii) $\mathrm{vec}(A \otimes B) = (I_n \otimes K_{qm} \otimes I_p)(\mathrm{vec}A \otimes \mathrm{vec}B)$,

(b) $K'_{pn} = K_{np}, \quad K_{pn}K_{np} = I_{np}$.

(c) $K_{pr,m} = (I_p \otimes K_{rm})(K_{pm} \otimes I_r)$.

(d) For any $A, B \in M_{n\times p}$, $tr(K_{nn}(A' \otimes B)) = tr(A'B) = (\mathrm{vec}A)'\,\mathrm{vec}B$.

(e) For any $A \in M_{n\times p}$, $B \in M_{p\times q}$ *and* $C \in M_{q\times r}$, $\mathrm{vec}(ABC) = (A \otimes C')\ \mathrm{vec}B$.

Lemma 2.2.2. *For any* A, $B \in \mathcal{L}(E, E)$,

(i) $(K_{n^2,n} \otimes I_n)\mathrm{vec}(A \otimes B) = \mathrm{vec}(K_{n,n}(A' \otimes B))$

and

(ii) $(K_{n,n^2} \otimes I_n)(\mathrm{vec}\,A \otimes \mathrm{vec}\,B) = \mathrm{vec}(K_{n,n}(A \otimes B))$.

Proof. We shall merely prove (i). (ii) can be proved similarly. Let $\{e_i\}_{i=1}^n$ be the orthonormal basis of E. Then A and B can be written as

$$A = \sum_{i,i'=1}^{n} a_{ii'} e_i \Box e_{i'}, \qquad B = \sum_{\ell,\ell'=1}^{n} b_{\ell\ell'} e_\ell \Box e_{\ell'}. \tag{2.2.4}$$

Thus by Lemma 2.2.1,

$$\begin{aligned}(K_{n^2,n} \otimes I_n)\mathrm{vec}(A \otimes B) &= (K_{n^2,n} \otimes I_n)(I_n \otimes K_{n,n} \otimes I_n)(\mathrm{vec}A \otimes \mathrm{vec}B)\\ &= \sum_{i,i'=1}^{n}\sum_{\ell,\ell'=1}^{n} a_{ii'}b_{\ell\ell'}(K_{n^2,n} \otimes I_n)(I_n \otimes K_{n,n} \otimes I_n)(\mathrm{vec}(e_i \Box e_{i'}) \otimes \mathrm{vec}(e_\ell \Box e_{\ell'}))\end{aligned} \tag{2.2.5}$$

and

$$\text{vec}(K_{n,n}(A' \otimes B)) = (K_{n,n} \otimes I_{n^2})\,\text{vec}(A' \otimes B)$$

$$= (K_{n,n} \otimes I_{n^2})(I_n \otimes K_{n,n} \otimes I_n)((\text{vec}A') \otimes \text{vec}B)$$

$$= \sum_{i,i'=1}^{n} \sum_{\ell,\ell'=1}^{n} a_{ii'} b_{\ell\ell'} (K_{n,n^2} \otimes I_n)(K_{n,n} \otimes I_{n^2})(\text{vec}(e_i \Box e_{i'}) \otimes \text{vec}(e_\ell \Box e_{\ell'})). \tag{2.2.6}$$

Note that $\text{vec}(e_i \Box e_\ell) = e_i \otimes e_\ell$. So by (2.2.5) and (2.2.6), it suffices to show that

$$(K_{n^2,n} \otimes I_n)(I_n \otimes K_{n,n} \otimes I_n)(e_i \otimes e_{i'} \otimes e_\ell \otimes e_{\ell'})$$

$$= (K_{n,n^2} \otimes I_n)(K_{n,n} \otimes I_{n^2})(e_i \otimes e_{i'} \otimes e_\ell \otimes e_{\ell'}).$$

Since

$$(K_{n^2,n} \otimes I_n)(I_n \otimes K_{n,n} \otimes I_n)(e_i \otimes e_{i'} \otimes e_\ell \otimes e_{\ell'})$$

$$= (K_{n^2,n} \otimes I_n)(e_i \otimes e_\ell \otimes e_{i'} \otimes e_{\ell'})$$

$$= e_\ell \otimes e_{i'} \otimes e_i \otimes e_{\ell'}$$

and

$$(K_{n,n^2} \otimes I_n)(K_{n,n} \otimes I_{n^2})(e_i \otimes e_{i'} \otimes e_\ell \otimes e_{\ell'})$$

$$= (K_{n,n^2} \otimes I_n)(e_{i'} \otimes e_i \otimes e_\ell \otimes e_{\ell'}) = e_\ell \otimes e_{i'} \otimes e_i \otimes e_{\ell'},$$

the desired result follows. □

Now let $\{e_\ell\}_{\ell=1}^n$ and $\{f_j\}_{j=1}^p$ be respectively the orthonormal bases of E and V. Then $\{e_\ell \Box f_j\}$, $\{e_\ell \Box e_{\ell'}\}$, $\{f_j \Box f_{j'}\}$, and $\{(e_\ell \Box e_{\ell'}) \otimes (f_j \Box f_{j'})\}$ are orthonormal bases of $\mathcal{L}(V,E)$, $\mathcal{L}(E,E)$, $\mathcal{L}(V,V)$, and $\mathcal{L}(E,E) \otimes \mathcal{L}(V,V)$ respectively. Thus Σ_Y in $\mathcal{L}(E,E) \otimes \mathcal{L}(V,V)$ can be written as

$$\Sigma_Y = \sum_{\ell,\ell'=1}^{n} \sum_{j,j'=1}^{p} \sigma_{\ell\ell' jj'} (e_\ell \Box e_{\ell'}) \otimes (f_j \Box f_{j'}), \tag{2.2.7}$$

where $\sigma_{\ell\ell' jj'} \equiv \langle e_\ell \Box f_j, \Sigma_Y(e_{\ell'} \Box f_{j'}) \rangle = \sigma_{\ell'\ell j' j}$.

Theorem 2.2.3. *Suppose that $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$ and the first four order moments of Y exist. Then*

(a) $E(Y) = \mu$,

(b) $E(Y \odot Y) = \mu \odot \mu - 2\phi'(0)\bar{V}$, $\quad$ $\text{Cov}(Y) = -2\phi'(0)\Sigma_Y$,

(c) $E(\odot^3 Y) = \odot^3 \mu - 2\phi'(0)[\bar{V} \odot \mu + \mu \odot \bar{V} + (I_n \odot K_{n,n})(\bar{V} \odot \mu)(I_p \odot K_{p,p})]$.

and

(d) $E(\odot^4 Y) = \odot^4 \mu - 2\phi'(0)\Delta_1 + 4\phi''(0)\Delta_2$,

where

$$\bar{V} = \sum_{\ell,\ell'=1}^{n} \sum_{j,j'=1}^{p} \sigma_{\ell\ell' jj'}(e_\ell \otimes e_{\ell'})(f_j \otimes f_{j'})', \tag{2.2.8}$$

$$\begin{aligned}\Delta_1 =& \bar{V} \odot \mu \odot \mu + \mu \odot \mu \odot \bar{V} + \mu \odot \bar{V} \odot \mu \\ &+ (I_n \odot K_{n,n} \odot I_n)(\bar{V} \odot \mu \odot \mu + \mu \odot \mu \odot \bar{V})(I_p \odot K_{p,p} \odot I_p) \\ &+ (K_{n,n^2} \odot I_n)(\mu \otimes \mu \odot \bar{V})(K_{p^2,p} \otimes I_p),\end{aligned} \tag{2.2.9}$$

$$\begin{aligned}\Delta_2 =& \bar{V} \odot \bar{V} + (I_n \odot K_{n,n} \odot I_n)(\bar{V} \otimes \bar{V})(I_p \otimes K_{p,p} \otimes I_p) \\ &+ (K_{n,n^2} \odot I_n)(\bar{V} \otimes \bar{V})(K_{p^2,p} \otimes I_p),\end{aligned} \tag{2.2.10}$$

and $\sigma_{\ell\ell' jj'}$, e_ℓ, $e_{\ell'}$, f_j, *and* $f_{j'}$ *are given in (2.2.7).*

Proof. To $\hat{Y}(T)$ in (1.2.1), we shall apply the differential methods presented in Wong (1985, 1986). Let $dT_1 \in \mathcal{L}(V, E)$. Then by differentiating $\hat{Y}(T)$ with respect to T, we obtain

$$E\left(e^{i\langle T,Y\rangle} i\langle dT_1, Y\rangle\right) = \hat{Y}(T) i\langle dT_1, \mu\rangle + 2e^{i\langle T,\mu\rangle}\phi'(u)\langle dT_1, \Sigma_Y(T)\rangle. \tag{2.2.11}$$

By letting $T = 0$ in (2.2.11),

$$iE(\langle dT_1, Y\rangle) = i\langle dT_1, E(Y)\rangle = i\langle dT_1, \mu\rangle.$$

Thus by varying $dT_1 \in \mathcal{L}(V, E)$, (a) follows. By treating dT_1 as a constant operator, we shall differentiate (2.2.11) with respect to T. Let $dT_2 \in \mathcal{L}(V, E)$. Then

$$\begin{aligned}E\left(e^{i\langle T,Y\rangle} i^2 \langle dT_1, Y\rangle \langle dT_2, Y\rangle\right) =& \hat{Y}(T) i^2 \langle dT_1, \mu\rangle \langle dT_2, \mu\rangle + 2e^{i\langle T,\mu\rangle}\phi'(u) \\ &\times \{i [\langle dT_1, \mu\rangle \langle dT_2, \Sigma_Y(T)\rangle + \langle dT_2, \mu\rangle \langle dT_1, \Sigma_Y(T)\rangle] + \langle dT_1, \Sigma_Y(dT_2)\rangle\} \\ &+ 4e^{i\langle T,\mu\rangle}\phi''(u) \langle dT_1, \Sigma_Y(T)\rangle \langle dT_2, \Sigma_Y(T)\rangle.\end{aligned} \tag{2.2.12}$$

Letting $T = 0$ in (2.2.12), we obtain

$$i^2 E(\langle dT_1, Y\rangle \langle dT_2, Y\rangle) = i^2 \langle dT_1, \mu\rangle \langle dT_2, \mu\rangle + 2\phi'(0)\langle dT_1, \Sigma_Y(dT_2)\rangle. \qquad (2.2.13)$$

By the basic formulae $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$, $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$, and Lemma 2.2.1, we obtain

$$\langle dT_1, \mu\rangle\langle dT_2, \mu\rangle = \langle dT_1 \otimes dT_2, \mu \otimes \mu\rangle = \langle \mathrm{vec}\, \mu(\mathrm{vec}\, \mu)', \mathrm{vec}\, dT_1(\mathrm{vec}\, dT_2)'\rangle, \qquad (2.2.14)$$

and

$$\langle dT_1, \Sigma_Y(dT_2)\rangle = \langle \bar{V}, dT_1 \otimes dT_2\rangle = \langle \Sigma_Y, \mathrm{vec}\, dT_1(\mathrm{vec}\, dT_2)'\rangle, \qquad (2.2.15)$$

where $\bar{V}$ is given in (2.2.8). By (2.2.14) and (2.2.15), (2.2.13) becomes

$$\langle E(Y \otimes Y), dT_1 \otimes dT_2\rangle = \langle \mu \otimes \mu - 2\phi'(0)\bar{V},\ dT_1 \otimes dT_2\rangle$$

and

$$\langle E(\mathrm{vec}\, Y(\mathrm{vec}\, Y)'), \mathrm{vec}\, dT_1(\mathrm{vec}\, dT_2)'\rangle = \langle \mathrm{vec}\, \mu(\mathrm{vec}\, \mu)' - 2\phi'(0)\Sigma_Y, \mathrm{vec}\, dT_1(\mathrm{vec}\, dT_1)'\rangle.$$

So (b) follows from varying dT_1 and dT_2 in $\mathcal{L}(V, E)$.

Similarly, differentiating (2.2.12) with respect to T and letting $dT_3 \in \mathcal{L}(V, E)$, we obtain

$$E(e^{i\langle T,Y\rangle} i^3 \prod_{\ell=1}^{3} \langle dT_\ell, Y\rangle) = E_0 + E_1 + E_2 + E_3. \qquad (2.2.16)$$

where

$$E_0 = \hat{Y}(T)\prod_{\ell=1}^{3}\langle \mu, dT_\ell\rangle,$$

$$\begin{aligned}E_1 =& 2ie^{i(T,\mu)}\phi'(u)\{i[\langle dT_1,\mu\rangle\langle dT_2,\mu\rangle\langle dT_3,\Sigma_Y(T)\rangle\\ &+\langle dT_1,\mu\rangle\langle dT_3,\mu\rangle\langle dT_2,\Sigma_Y(T)\rangle+\langle dT_2,\mu\rangle\langle dT_3,\mu\rangle\langle dT_1,\Sigma_Y(T)\rangle]\\ &+\langle dT_1,\mu\rangle\langle dT_2,\Sigma_Y(dT_3)\rangle+\langle dT_2,\mu\rangle\langle dT_1,\Sigma_Y(dT_3)\rangle\\ &+\langle dT_3,\mu\rangle\langle dT_1,\Sigma_Y(dT_2)\rangle\},\end{aligned}$$

$$\begin{aligned}E_2 =& 4\phi''(u)e^{i(T,\mu)}\{i[\langle dT_1,\mu\rangle\langle dT_2,\Sigma_Y(T)\rangle\langle dT_3,\Sigma_Y(T)\rangle\\ &+\langle dT_2,\mu\rangle\langle dT_1,\Sigma_Y(T)\rangle\langle dT_3,\Sigma_Y(T)\rangle+\langle dT_3,\mu\rangle\langle dT_1,\Sigma_Y(T)\rangle\langle dT_2,\Sigma_Y(T)\rangle]\\ &+\langle dT_1,\Sigma_Y(dT_2)\rangle\langle dT_3,\Sigma_Y(T)\rangle+\langle dT_1,\Sigma_Y(dT_3)\rangle\langle dT_2,\Sigma_Y(T)\rangle\\ &+\langle dT_2,\Sigma_Y(dT_3)\rangle\langle dT_1,\Sigma_Y(T)\rangle\},\end{aligned}$$

and

$$E_3 = 8\phi''(u)e^{i(T,\mu)}\prod_{\ell=1}^{3}\langle dT_\ell,\Sigma_Y(T)\rangle.$$

Letting $T=0$ in (2.2.16), we obtain, upon simplifying,

$$\begin{aligned}&\langle i^3E(Y\otimes Y\otimes Y),\, dT_1\otimes dT_2\otimes dT_3\rangle = \langle i^3\mu\otimes\mu\otimes\mu,\, dT_1\otimes dT_2\otimes dT_3\rangle\\ &\quad+i\phi'(0)\langle\mu\otimes\bar{V},\, dT_1\otimes dT_2\otimes dT_3+dT_3\otimes dT_1\otimes dT_2+dT_2\otimes dT_1\otimes dT_3\rangle\\ &=\langle i^3\mu\otimes\mu\otimes\mu,\, \bigotimes_{\ell=1}^{3} dT_\ell\rangle\\ &\quad+2i\phi'(0)\left[\mu\otimes\bar{V}+\bar{V}\otimes\mu+(I_n\otimes K_{n,n})(\bar{V}\otimes\mu)(I_p\otimes K_{p,p})\right],\, \bigotimes_{\ell=1}^{3} dT_\ell\rangle\end{aligned}$$

and hence (c) follows from varying dT_1, dT_2 and dT_3 in $\mathcal{L}(V,E)$.

Differentiating (2.2.16) with respect to T again and letting $dT_4\in\mathcal{L}(V,E)$,

$$E\left(e^{i(T,Y)}i^4\prod_{\ell=1}^{4}\langle dT_\ell,Y\rangle\right)\Bigg|_{T=0} = d(E_0+E_1+E_2+E_3)(dT_4)\Big|_{T=0}, \tag{2.2.17}$$

where

$$dE_0(dT_4)|_{T=0} = i^4\prod_{\ell=1}^{4}\langle dT_\ell,\mu\rangle = i^4\langle\otimes^4\mu,\, \bigotimes_{\ell=1}^{4} dT_\ell\rangle, \tag{2.2.18}$$

$$dE_1(dT_4)|_{T=0} = 2i^2\phi'(0)\{\langle dT_1,\mu\rangle[\langle dT_2,\mu\rangle\ \langle dT_3,\Sigma_Y(dT_4)\rangle$$

$$+ \langle dT_3,\mu\rangle\ \langle dT_2,\Sigma_Y(dT_4)\rangle + \langle dT_4,\mu\rangle\ \langle dT_2,\Sigma_Y(dT_3)\rangle]$$

$$+ \langle dT_2,\mu\rangle\ \langle dT_3,\mu\rangle\ \langle dT_1,\Sigma_Y(dT_4)\rangle + \langle dT_2,\mu\rangle\ \langle dT_4,\mu\rangle\ \langle dT_1,\Sigma_Y(dT_3)\rangle$$

$$+ \langle dT_3,\mu\rangle\ \langle dT_4,\mu\rangle\ \langle dT_1,\Sigma_Y(dT_2)\rangle\},$$

$$dE_2(dT_4)|_{T=0} = 4\phi''(0)\,[\langle dT_1,\Sigma_Y(dT_2)\rangle\ \langle dT_3,\Sigma_Y(dT_4)\rangle$$

$$+ \langle dT_1,\Sigma_Y(dT_3)\rangle\ \langle dT_2,\Sigma_Y(dT_4)\rangle + \langle dT_1,\Sigma_Y(dT_4)\rangle\ \langle dT_2,\Sigma_Y(dT_3)\rangle],$$

and

$$dE_3(dT_4)|_{T=0} = 0.$$

By Lemma 2.2.1, (2.2.14) and (2.2.15), we obtain

$$\begin{aligned} dE_1(dT_4)|_{T=0} =& -2\phi'(0)\Big[\langle \mu\otimes\mu\otimes\tilde{V},\ \bigotimes_{\ell=1}^{4} dT_\ell + dT_1\otimes dT_3\otimes dT_2\otimes dT_4 \\ &+ dT_1\otimes dT_4\otimes dT_2\otimes dT_3 + dT_2\otimes dT_3\otimes dT_1\otimes dT_4 \\ &+ dT_2\otimes dT_4\otimes dT_1\otimes dT_3 + dT_3\otimes dT_4\otimes dT_1\otimes dT_2\rangle\Big] \\ =&\langle -2\phi'(0)\Delta_1,\ \bigotimes_{\ell=1}^{4} dT_\ell\rangle \end{aligned} \tag{2,2.19}$$

and

$$\begin{aligned} dE_2(dT_4)|_{T=0} =& 4\phi''(0)\langle \tilde{V}\otimes\tilde{V},\ \bigotimes_{\ell=1}^{4} dT_\ell + dT_1\otimes dT_3\otimes dT_2\otimes dT_4 \\ &+ dT_1\otimes dT_4\otimes dT_2\otimes dT_3\rangle \\ =&\langle 4\phi''(0)\Delta_2,\ \bigotimes_{\ell=1}^{4} dT_\ell\rangle, \end{aligned} \tag{2.2.20}$$

where Δ_1 and Δ_2 are given in (2.2.9) and (2.2.10) respectively. Substituting (2.2.18) - (2.2.29) into (2.2.17), we obtain

$$\langle E(\otimes^4 Y),\ \bigotimes_{j=1}^{4} dT_j\rangle = \langle \otimes^4\mu - 2\phi'(0)\Delta_1 + 4\phi''(0)\Delta_2,\ \bigotimes_{j=1}^{4} dT_j\rangle,$$

which yields (d) through varying $dT_j \in \mathcal{L}(V,E)$, $j = 1,2,3,4$. □

Note that if $Y \sim N_{n\times p}(\mu, A\otimes\Sigma)$, then $\phi'(0) = -1/2$ and $\phi''(0) = 1/4$; so (d) of Theorem 2.2.3 yields the corresponding results in Neudecker and Wansbeek (1983, 1987) and von Rosen (1988).

Corollary 2.2.4. *(Li (1987)) Suppose that $y \sim EC_n(\mu, A, \phi)$. Then*

(i) $E(yy') = \mu\mu' - 2\phi'(0)A$ *and*

(ii) $E(yy' \otimes yy') = \mu\mu' \otimes \mu\mu' - 2\phi'(0)\Delta_1^* + 4\phi''(0)\Delta_2^*$,

where

$$\Delta_1^* = (I_{n^2} + K_{n,n})(\mu\mu' \otimes A + A \otimes \mu\mu') + (\mu \otimes \mu)(\mathrm{vec}\, A)' + \mathrm{vec}\, A(\mu \otimes \mu)' \quad (2.2.21)$$

and

$$\Delta_2^* = (I_{n^2} + K_{n,n})(A \otimes A) + \mathrm{vec}\, A(\mathrm{vec}\, A)'. \quad (2.2.22)$$

Proof. Since

$$y \otimes y = \mathrm{vec}\,(yy'), \qquad y \in E,$$

by applying vec^{-1} to Theorem 2.2.3(b), we obtain (i). Note that

$$\begin{aligned}\mathrm{vec}\,(yy' \otimes yy') &= (I_n \otimes K_{nn} \otimes I_n)(\mathrm{vec}\,(yy') \otimes \mathrm{vec}\,(yy')) \\ &= (I_n \otimes K_{nn} \otimes I_n)(y \otimes y \otimes y \otimes y).\end{aligned} \quad (2.2.23)$$

By Theorem 2.2.3(d),

$$\begin{aligned}E(yy' \otimes yy') &= \mathrm{vec}^{-1}\{(I_n \otimes K_{nn} \otimes I_p)E(y \otimes y \otimes y \otimes y)\} \\ &= \mathrm{vec}^{-1}\{(I_n \otimes K_{nn} \otimes I_p)[\otimes^4 \mu - 2\phi'(0)\Delta_1 + 4\phi''(0)\Delta_2]\},\end{aligned} \quad (2.2.24)$$

where Δ_1 and Δ_2 are given in (2.2.9) and (2.2.10) respectively. By Lemma 2.2.1 and 2.2.2, we obtain

$$(I_n \otimes K_{nn} \otimes I_n)(\mathrm{vec}\, A \otimes \mu \otimes \mu + \mu \otimes \mu \otimes \mathrm{vec}\, A) = \mathrm{vec}\,(A \otimes \mu\mu' + \mu\mu' \otimes A), \quad (2.2.25)$$

$$\mathrm{vec}\, A \otimes \mu \otimes \mu + \mu \otimes \mu \otimes \mathrm{vec}\, A = \mathrm{vec}\,[\mathrm{vec}\, A(\mathrm{vec}\,(\mu\mu'))' + \mathrm{vec}\,(\mu\mu')(\mathrm{vec}\, A)'], \quad (2.2.26)$$

and

$$\begin{aligned}&(I_n \otimes K_{nn} \otimes I_n)(K_{n,n^2} \otimes I_p)(\mathrm{vec}\, A \otimes \mu \otimes \mu + \mu \otimes \mu \otimes \mathrm{vec}\, A) \\ &= (K_{n^2,n} \otimes I_p)(I_n \otimes K_{nn} \otimes I_n)[\mathrm{vec}\, A \otimes \mathrm{vec}\,(\mu\mu') + \mathrm{vec}\,(\mu\mu') \otimes \mathrm{vec}\, A] \\ &= (K_{n^2,n} \otimes I_p)\mathrm{vec}\,(A \otimes \mu\mu' + \mu\mu' \otimes A) \\ &\mathrm{vec}\,[K_{nn}(A \otimes \mu\mu' + \mu\mu' \otimes A)].\end{aligned} \quad (2.2.27)$$

By (2.2.23),

$$\mathrm{vec}^{-1}\{(I_n \otimes K_{nn} \otimes I_p)(\otimes^4 \mu)\} = \mu\mu' \otimes \mu\mu'. \qquad (2.2.28)$$

Also by (2.2.25) - (2.2.27),

$$\begin{aligned}\mathrm{vec}^{-1}\{(I_n \otimes K_{nn} \otimes I_p)\Delta_1\} &= A \otimes \mu\mu' + \mu\mu' \otimes A \\ &+ \mathrm{vec}\, A(\mathrm{vec}\,(\mu\mu'))' + \mathrm{vec}\,(\mu\mu')(\mathrm{vec}\, A)' + K_{nn}(A \otimes \mu\mu' + \mu\mu' \otimes A) \qquad (2.2.29) \\ &= \Delta_1^*.\end{aligned}$$

Similarly we can obtain that

$$\mathrm{vec}^{-1}\{(I_n \otimes K_{nn} \otimes I_p)\Delta_2\} = \Delta_2^*. \qquad (2.2.30)$$

Now (ii) follows by substituting (2.2.28) - (2.2.30) into (2.2.24). □

2.3 Higher order moments

In the last section, we obtained the first four order moments of Y. Now we shall use a similar approach to obtain formulae for evaluating the higher order moments of Y.

Let $T, dT_1, \cdots, dT_m \in \mathcal{L}(V, E)$. Note that $d\hat{Y}(T)(dT_1)$ is the value of the differential $d\hat{Y}(T)$ at dT_1. For $\hat{Y}$ in (1.2.1), we define, inductively,

$$d^m\hat{Y}(T)((dT_j)_{j=1}^m) = dw_{m-1}(T)(dT_m), \qquad (2,3.1)$$

where

$$w_{m-1}(T) = d^{m-1}\hat{Y}(T)((dT_j)_{j=1}^{m-1}), \quad m \geq 2, \qquad (2.3.2)$$

treating $(dT_j)_{j=1}^{m-1}$ as a constant. For brevity, the higher order moments are assumed to exist whenever they are used. Thus $d^m\hat{Y}(T)$ is a multilinear function on the

Cartesian product $(\mathcal{L}(V,E))^m \equiv \mathcal{L}(V,E) \times \mathcal{L}(V,E) \times \cdots \times \mathcal{L}(V,E)$ of m copies of $\mathcal{L}(V,E)$. Since $\hat{Y}(T) = E(e^{i\langle T,Y\rangle})$, by (2.3.1) and (2.3.2), we obtain

$$\begin{aligned} d^m\hat{Y}(T)((dT_j)_{j=1}^m)|_{T=0} &= E\left(e^{i\langle T,Y\rangle} i^m \prod_{j=1}^m \langle dT_j, Y\rangle \right)\Bigg|_{T=0} \\ &= \langle i^m E(\otimes^m Y), \bigotimes_{j=1}^m dT_j\rangle, \end{aligned} \tag{2.3.3}$$

where $\otimes^m Y = Y \otimes Y \otimes \cdots \otimes Y$ for m times. Thus the finding of $E(\otimes^m Y)$ is reduced to finding the multilinear differential form $d^m\hat{Y}(T)((dT_j)_{j=1}^m)$. Let S_m be the set of all permutations of σ on $\{1, 2, \cdots, m\}$ (σ is a one-to-one function of $\{1, 2, \cdots, m\}$ onto itself). Let $\ell \in \{1, 2, \cdots\}$ with $2\ell \geq m \geq \ell$ and

$$\begin{aligned} S_{m,\ell} = \{\sigma \in S_m : \sigma(1) < \sigma(2) < \cdots < \sigma(m-\ell), \sigma(k) < \sigma(m-k+1), \\ \text{for} \quad k \leq m-\ell, \quad \sigma(m-\ell+1) < \sigma(m-\ell+2) < \cdots < \sigma(\ell)\}\,. \end{aligned} \tag{2.3.4}$$

Note that $S_{m,\ell}$ has $m!/(2^{m-\ell}(m-\ell)!(2\ell-m)!)$ elements.

Theorem 2.3.1. *Suppose that* $Y \sim MEC_{n\times p}(0, \Sigma_Y, \phi)$. *Then*

$$\begin{aligned} d^m\hat{Y}(T)((dT_j)_{j=1}^m) = \sum_{\ell \geq \frac{m}{2}}^{m} 2^\ell \phi^{(\ell)}(u) \sum_{\sigma \in S_{m,\ell}} \\ \times \prod_{k=1}^{m-\ell} \langle \bar{V}, dT_{\sigma(k)} \otimes dT_{\sigma(m-k+1)}\rangle \prod_{k=m-\ell+1}^{\ell} \langle \bar{V}, dT_{\sigma(k)} \otimes T\rangle, \end{aligned} \tag{2.3.5}$$

where $\bar{V}$ *is given in (2.2.8).*

Proof. By (1.2.1), $\hat{Y}(T) = \phi(u)$, $u = \langle T, \Sigma_Y(T)\rangle$. Since

$$du(dT_1) = 2\langle dT_1, \Sigma_Y(T)\rangle$$

and

$$d^2u(dT_1, dT_2) = 2\langle dT_1, \Sigma_Y(dT_2)\rangle,$$

we obtain from (2.2.15),

$$du(dT_1) = 2\langle \bar{V}, dT_1 \otimes T\rangle$$

and

$$d^2 u(dT_1, dT_2) = 2\langle \bar{V}, dT_1 \otimes dT_2\rangle. \qquad (2.3.6)$$

Note that from (2.3.6),

$$d^r u((dT_j)_{j=1}^r) = 0 \quad \text{for} \quad r > 2. \qquad (2.3.7)$$

By using (2.3.6) and (2.3.7),

$$d\hat{Y}(T)(dT_1) = \phi'(u)du(dT_1) = 2\phi'(u)\langle \bar{V}, dT_1 \otimes T\rangle,$$

$$d^2\hat{Y}(T)(dT_1, dT_2) = 2^2\phi''(u)\prod_{j=1}^{2}\langle \bar{V}, dT_j \otimes T\rangle + 2\phi'(u)\langle \bar{V}, dT_1 \otimes dT_2\rangle,$$

$$d^3\hat{Y}(T)((dT_j)_{j=1}^3) = 2^2\phi''(u)\sum_{\sigma(1)<\sigma(2)}\langle \bar{V}, dT_{\sigma(1)} \otimes dT_{\sigma(2)}\rangle\,\langle \bar{V}, dT_{\sigma(3)} \otimes T\rangle$$
$$+ 2^3\phi^{(3)}(u)\prod_{j=1}^{3}\langle \bar{V}, dT_j \otimes T\rangle,$$

and

$$d^4\hat{Y}(T)((dT_j)_{j=1}^4) = 2^2\phi''(u)\sum_{\substack{\sigma(1)<\sigma(2)\\ \sigma\in S_3}}\langle \bar{V}, dT_{(1)} \otimes dT_{\sigma(2)}\rangle\,\langle \bar{V}, dT_{\sigma(3)} \otimes dT_4\rangle$$
$$+ 2^3\phi^{(3)}(u)\sum_{\substack{\sigma(1)<\sigma(2)\\ \sigma\in S_3}}\langle \bar{V}, dT_{\sigma(1)} \otimes dT_{\sigma(2)}\rangle\prod_{j=3}^{4}\langle \bar{V}, dT_j \otimes T\rangle$$
$$+ 2^4\phi^{(4)}(u)\prod_{j=1}^{4}\langle \bar{V}, dT_j \otimes T\rangle$$
$$= 2^2\phi''(u)\sum_{\sigma\in S_{4,2}}\prod_{k=1}^{2}\langle \bar{V}, dT_{\sigma(k)} \otimes dT_{\sigma(m-k+1)}\rangle$$
$$+ 2^3\phi^{(3)}(u)\sum_{\sigma\in S_{4,3}}\langle \bar{V}, dT_{\sigma(1)} \otimes dT_{\sigma(4)}\rangle\prod_{k=2}^{3}\langle \bar{V}, dT_{\sigma(k)} \otimes T\rangle$$
$$+ 2^4\phi^{(4)}(u)\prod_{j=1}^{4}\langle \bar{V}, dT_{\sigma(j)} \otimes T\rangle.$$

Therefore (2.3.5) is true for $m = 1, 2, 3, 4$. In fact, these special cases were used to conjecture (2.3.5). We shall now prove (2.3.5) by induction. Suppose that (2.3.5) holds. Then by (2.3.1) and (2.3.2),

$$d^{m+1}\hat{Y}(T)((dT_j)_{j=1}^{m+1}) = \sum_{\ell \geq \frac{m}{2}}^{m} (\Delta_{\ell+1} + \Delta_\ell), \tag{2.3.8}$$

where

$$\Delta_{\ell+1} = 2^{\ell+1}\phi^{(\ell+1)}(u) \sum_{\sigma \in S_{m,\ell}} \prod_{k=1}^{m-\ell} \langle \bar{V}, dT_{\sigma(k)} \otimes dT_{\sigma(m-\ell+1)} \rangle$$
$$\times \prod_{k=m-\ell+1}^{\ell} \langle \bar{V}, dT_{\sigma(k)} \otimes T \rangle \langle \bar{V}, dT_{m+1} \otimes T \rangle,$$

and

$$\Delta_\ell = 2^\ell \phi^{(\ell)}(u) \sum_{\sigma \in S_{m,\ell}} \prod_{k=1}^{m-\ell} \langle \bar{V}, dT_{\sigma(k)} \otimes dT_{\sigma(m-k+1)} \rangle$$
$$\times \sum_{t=m-\ell+1}^{\ell} \prod_{\substack{k=m-\ell+1 \\ k \neq t}}^{\ell} \langle \bar{V}, dT_{\sigma(k)} \otimes T \rangle \langle \bar{V}, dT_{\sigma(t)} \otimes dT_{m+1} \rangle.$$

It suffices to show that (2.3.8) is (2.3.5) with m being replaced by $m+1$. Note that if $m = 2s$, then Δ_s in (2.3.8) is 0 and $\ell = s+1 > (m+1)/2$; if $m = 2s+1$, then $\ell = s+1 = (m+1)/2$. Thus reordering (2.3.8) in terms of $\phi^{(\ell)}$, we have

$$d^{m+1}\hat{Y}(T)((dT_j)_{j=1}^{m+1}) = 2^{m+1}\phi^{(m+1)}(u) \sum_{\sigma \in S_{m,m}} \prod_{k=1}^{m} \langle \bar{V}, dT_{\sigma(k)} \otimes T \rangle \langle \bar{V}, dT_{m+1} \otimes T \rangle$$
$$+ \sum_{\ell \geq \frac{m+1}{2}}^{m} 2^\ell \phi^{(\ell)}(u) \{\Delta_1^* + \Delta_2^*\}, \tag{2.3.9}$$

where

$$\Delta_1^* \equiv \sum_{\sigma \in S_{m,\ell-1}} \prod_{k=1}^{m-\ell+1} \langle \bar{V}, dT_{\sigma(k)} \otimes dT_{\sigma(m-k+1)} \rangle \prod_{k=m-\ell+2}^{\ell-1} \langle \bar{V}, dT_{\sigma(k)} \otimes T \rangle \langle \bar{V}, dT_{m+1} \otimes T \rangle$$

and

$$\Delta_2^* \equiv \sum_{\sigma \in S_{m,\ell}} \prod_{k=1}^{m-\ell} \langle \bar{V}, dT_{\sigma(k)} \otimes dT_{\sigma(m-k+1)} \rangle$$
$$\times \sum_{t=m-\ell+1}^{\ell} \prod_{\substack{k=m-\ell+1 \\ k \neq t}}^{\ell} \langle \bar{V}, dT_{\sigma(k)} \otimes T \rangle \langle \bar{V}, dT_{m+1} \otimes dT_{\sigma(t)} \rangle.$$

Since

$$\sum_{\sigma \in S_{m,m}} \prod_{k=1}^{m} \langle \bar{V}, dT_{\sigma(k)} \otimes T \rangle \langle \bar{V}, dT_{m+1} \otimes T \rangle = \sum_{\sigma \in S_{m+1,m+1}} \prod_{k=1}^{m+1} \langle \bar{V}, dT_{\sigma(k)} \otimes T \rangle,$$

it suffices to show, from (2,3.9), that for $(m+1)/2 \leq \ell \leq m$,

$$\sum_{\sigma \in S_{m+1,\ell}} \prod_{k=1}^{m+1-\ell} \langle \bar{V}, dT_{\sigma(k)} \otimes dT_{\sigma(m+1-k+1)} \rangle \prod_{k=m+1-\ell+1}^{\ell} \langle \bar{V}, dT_{\sigma(k)} \otimes T \rangle = \Delta_1^* + \Delta_2^*. \tag{2.3.10}$$

The remaining argument given below is adopted from Wong and Liu (1992):

For Δ_1^*, define σ^* on $\{1, 2, \cdots, m, m+1\}$ such that

(i) $\sigma^*(k) = \sigma(k)$, $\sigma^*(m-k+1) = \sigma(m-k+1)$ for $k \leq m-\ell$,

(ii) $\sigma^*(s) = \sigma(s)$ for $m-\ell+2 \leq s \leq \ell-1$, and

(iii) $\sigma^*(\ell) = m+1$.

Then

$$\sigma^* \in S_{m+1,\ell}. \tag{2.3.11}$$

Similarly for Δ_2^*, define $\tilde{\sigma}_t$ on $\{1, 2, \cdots, m, m+1\}$ such that

(iv) $\tilde{\sigma}_t(s)$ is the sth smallest element in $\{\sigma(1), \cdots, \sigma(m-\ell), \sigma(t)\}$, $s = 1, 2, \cdots,$ $m-\ell, m-\ell+1$,

(v) for s in (iv), $\tilde{\sigma}_t(m+1-s+1) = \sigma(m-k+1)$ if $\tilde{\sigma}_t(s) = \sigma(k)$ and $k \neq t$, and $\tilde{\sigma}_t(m+1-s+1) = m+1$ if $\tilde{\sigma}_t(s) = \sigma(t)$, and

(vi) $\tilde{\sigma}_t(s)$ is the sth smallest element in $\{\sigma(m-\ell+1), \sigma(m-\ell+2), \cdots, \sigma(\ell)\} \setminus$ $\{\sigma(t)\}$, $s = m-\ell+3, \cdots, \ell$.

Then

$$\bar{\sigma}_\ell \in S_{m+1,\ell}. \tag{2.3.12}$$

From (2.3.11) and (2.3.12), we know that any term in Δ_1^* and Δ_2^* is a term of the left hand side of (2.3.10). Thus it suffices to show that the numbers of the terms in both sides of (2.3.10) are equal. Note that $dT_{\sigma(\ell)}$ in Δ_2^* can be $dT_{\sigma(m-\ell+1)}, \cdots, dT_{\sigma(\ell)}$ and the number of elements, $|S_{m,\ell}|$, of $S_{m,\ell}$ is $m!/(2^{m-\ell}(m-\ell)!(2\ell-m)!)$. So the sum of the number of terms of Δ_1^* and Δ_2^* is

$$\begin{aligned}
&|S_{m,\ell-1}| + (\ell-(m-\ell))|S_{m,\ell}| \\
&= \frac{m!}{2^{m-\ell+1}(m-\ell+1)!(2\ell-2-m)!} + (2\ell-m)\frac{m!}{2^{m-\ell}(m-\ell)!(2\ell-m)!} \\
&= \frac{(m+1)!}{2^{m+1-\ell}(m+1-\ell)!(2\ell-(m+1))!} = |S_{m+1,\ell}|.
\end{aligned}$$

Therefore (2.3.10) holds. By induction, we conclude that (2.3.5) holds for any integer m. □

Corollary 2.3.2. *Suppose that $Y \sim MEC_{n\times p}(0, \Sigma_Y, \phi)$. Then*

$$E(\otimes^{2s-1}Y) = 0, \quad s = 1, 2, \cdots, \tag{2.3.13}$$

and

$$\begin{aligned}
\langle E \otimes^{2s} Y), \bigotimes_{j=1}^{2s} dT_j\rangle &= (-2)^s\phi^{(s)}(0) \sum_{\sigma\in S_{2s,s}} \prod_{k=1}^{s} \langle \bar{V}, dT_{\sigma(k)} \otimes dT_{\sigma(2s-k+1)}\rangle \\
&= (-2)^s\phi^{(s)}(0) \sum_{\sigma\in S_{2s,s}} \langle \otimes^s\bar{V}, \bigotimes_{k=1}^{s}(dT_{\sigma(k)} \otimes dt_{\sigma(2s-k+1)})\rangle.
\end{aligned} \tag{2.3.14}$$

Let $\omega_{2s,\sigma}^0$ be the linear function on $\mathcal{L}(V,E) \times \cdots \times \mathcal{L}(V,E)$ ($2s$ times) determined by

$$\omega_{2s,\sigma}^0(\bigotimes_{j=1}^{2s} dT_j) = \sum_{\sigma\in S_{2s,s}} \bigotimes_{k=1}^{s}(dT_{\sigma(k)} \otimes dT_{\sigma(2s-k+1)}) \tag{2.3.15}$$

and let $(\omega^0_{2s,\sigma})^t$ be its adjoint operator of $\omega^0_{2s,\sigma}$. Then for $Y \sim MEC_{n\times p}(0, \Sigma_Y, \phi)$, we obtain from (2.3.14)

$$E(\otimes^{2s}Y) = (-2)^s \phi^{(s)}(0)(\omega^0_{2s,\sigma})^t(\otimes^s \bar{V}). \qquad (2.3.16)$$

For example, if we let $s = 2$, then $S_{4,2} = \{\sigma \in S_4 : \sigma(1) < \sigma(2), \sigma(1) < \sigma(4), \sigma(2) < \sigma(3)\} = \{(1,2,3,4),(1,3,4,2), (1,2,4,3)\}$. Thus (2.3.14) becomes

$$\begin{aligned}\langle E(\otimes^4 Y), \bigotimes_{j=1}^{4} dT_j\rangle = 4\phi''(0)\langle \bar{V} \otimes \bar{V}, dT_1 \otimes dT_4 \otimes dT_2 \otimes dT_3 \\ + \bigotimes_{j=1}^{4} dT_j + dT_1 \otimes dT_4 \otimes dT_2 \otimes dT_3\rangle,\end{aligned}$$

which , by varying $dT_1, \cdots, dT_4$ in $\mathcal{L}(V, E)$, is Theorem 2.2.1 (d) with $\mu = 0$.

Now for $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$ with $\mu \neq 0$, let

$$D_{m,k} = \{\sigma \in S_m : \sigma(1) < \cdots < \sigma(k),\ \sigma(k+1) < \cdots < \sigma(m)\}. \qquad (2.3.17)$$

Then the number of elements, $|D_{m,k}|$, of $D_{m,k}$ is given by

$$|D_{m,k}| = m!/(k!(m-k)!).$$

Theorem 2.3.3. *Suppose that $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$. Then*

$$d^m\hat{Y}(T)((dT_j)_{j=1}^m) = \sum_{k=0}^{m} \sum_{\sigma \in D_{m,k}} d^k\phi(u)((dT_{\sigma(\ell)})_{\ell=1}^k) \prod_{j=k+1}^{m} i\langle dT_{\sigma(j)}, \mu\rangle.$$

Theorem 2.3.3 can be proved by using an argument similar to the proof of Theorem 2.3.1.

Let $S^*_{\sigma,2s}$ be the set of all permutations σ_* on $\{\sigma(j), j = 1, \cdots 2s\}$ with

$$\sigma_*(\sigma(1)) < \cdots < \sigma_*(\sigma(s)), \quad \sigma_*(\sigma(k)) < \sigma_*(\sigma(2s-k+1)) \quad \text{for} \quad k \le s. \quad (2.3.18)$$

Then by (2.3.3), (2.3.18), Theorem 2.3.1 and Theorem 2.3.3, we obtain

Theorem 2.3.4. *Suppose that $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$. Then*

$$\langle E(\otimes^m(Y)), \overset{m}{\underset{j=1}{\otimes}} dT_j\rangle = \sum_{s=0}^{[\frac{m}{2}]} \phi^{(s)}(0) \sum_{\sigma\in D_{m,2s}}$$

$$\times \sum_{\sigma_*\in S^*_{\sigma,2s}} \langle \otimes^s \tilde{V}, \overset{s}{\underset{k=1}{\otimes}} (dT_{\sigma_*(\sigma(k))} \otimes dT_{\sigma_*(\sigma(2s-k+1))})\rangle \prod_{j=2s+1}^{m} \langle \mu, dT_{\sigma(j)}\rangle$$

$$= \sum_{s=0}^{[\frac{m}{2}]} (-2)^s \phi^{(s)}(0) \sum_{\sigma\in D_{m,2s}} \langle (\otimes^s \tilde{V}) \otimes (\otimes^{m-2s}\mu),$$

$$\times \sum_{\sigma_*\in S^*_{\sigma,2s}} \overset{s}{\underset{k=1}{\otimes}} (dT_{\sigma_*(\sigma(k))} \otimes dT_{\sigma_*(\sigma(2s-k+1))}) \otimes (\overset{m}{\underset{j=2s+1}{\otimes}} dT_{\sigma(j)})\rangle,$$

where $\tilde{V}$ is given in (2.2.8) and $[\frac{m}{2}]$ denotes the integral part of $\frac{m}{2}$.

As in Wong and Wang (1991), let $s, \ell \in \{0, 1, \cdots\}$, $\sigma \in S_{2s+\ell}$, $\omega_{2s,\ell,\sigma}$ be the linear function on $\mathcal{L}(V, E) \times \cdots \times \mathcal{L}(V, E)$ $(2s + \ell$ times) determined by

$$\omega_{2s,\ell,\sigma}((dT_j)_{j=1}^{2s+\ell}) = \sum_{\sigma_*\in S^*_{\sigma,2s}} \overset{s}{\underset{k=1}{\otimes}} (dT_{\sigma_*(\sigma(k))} \otimes dT_{\sigma_*(\sigma(2s-k+1))}) \otimes (\overset{\ell}{\underset{j=2s+1}{\otimes}} dT_{\sigma(j)}), \tag{2.3.19}$$

and

$$\omega_{0,\ell,\sigma}(\overset{\ell}{\underset{j=1}{\otimes}} dT_{\sigma(j)}) = \overset{\ell}{\underset{j=1}{\otimes}} dT_j, \quad \text{where} \quad dT_j \in \mathcal{L}(V, E), \quad j = 1, \cdots, \ell. \tag{2.3.20}$$

Then by Theorem 2.3.4,

$$E(\otimes^m Y) = \sum_{s=0}^{[m/2]} (-2)^s \phi^{(s)}(0) \Omega_{m,2s}((\otimes^s \tilde{V}) \otimes (\otimes^{m-2s}\mu)), \tag{2.3.21}$$

where

$$\Omega_{m,2s} = \sum_{\sigma\in D_{m,2s}} (\omega_{2s,m-2s,\sigma})^t \tag{2.3.22}$$

and $(\omega_{2s,m-2s,\sigma})^t$ is the adjoint operator of $\omega_{2s,m-2s,\sigma}$.

The formula for mth order moment of Y in (2.3.21) is more general than the results given in von Rosen (1988) and Wong and Wang (1991) in that Σ_Y need not have the form $A \otimes \Sigma$, where $A \in \mathcal{N}_E$ and $\Sigma \in \mathcal{N}_V$.

Corollary 2.3.5. *For the multivariate components of variance model* $Y \sim MEC_{n\times p}(\mu, \sum_{j=1}^{k} A_j \otimes \Sigma_j, \phi)$ *with* $A_j \in \mathcal{N}_E$ *and* $\Sigma_j \in \mathcal{N}_V$, $j = 1, \cdots, k$, *if the mth order moments of Y exist, then $E(\otimes^m Y)$ is given by (2.3.21) with* $\bar{V} = \sum_{j=1}^{k} \text{vec}A_j(\text{vec}\Sigma_j)'$.

Corollary 2.3.6. *In Theorem 2.3.4, let $p = 1$ and $\Sigma_Y = A \in \mathcal{N}_n$, i.e., $y \equiv Y_{n\times 1} \sim MEC_{n\times 1}(\mu, A, \phi) \equiv EC_n(\mu, A, \phi)$, the univariate elliptically contoured distribution. Then for $L \in \{1, 2, \cdots\}$,*

$$E(\otimes^L \text{vec}(yy')) = \sum_{s=0}^{L}(-2)^s \phi^{(s)}(0) \left\{\Omega_{2L,2s}(\otimes^s \text{vec} A) \otimes (\otimes^{L-s} \text{vec}(\mu\mu'))\right\} \quad (2.3.23)$$

and

$$E[(\otimes^L \text{vec}(yy')) \otimes y] = \sum_{s=0}^{L}(-2)^s \phi^{(s)}(0) \left\{\Omega_{2L+1,2s}((\otimes^s \text{vec} A) \otimes (\otimes^{L-s} \text{vec}(\mu\mu')) \otimes \mu)\right\}. \quad (2.3.24)$$

Proof. Note that $y \otimes y = \text{vec}(yy')$, $y \in \Re^n$. From (2.3.21)

$$E(\otimes^L \text{vec}(yy')) = E(\otimes^{2L} y) = \sum_{s=0}^{L}(-2)^s \phi^{(s)}(0)\Omega_{2L,2s}((\otimes^s \text{vec} A) \otimes (\otimes^{L-s} \text{vec}(\mu\mu')))$$

and

$$E[(\otimes^L \text{vec}(yy')) \otimes y] = E(\otimes^{2L+1} y) = \sum_{s=0}^{L}(-2)^s \phi^{(s)}(0)\Omega_{2L+1,2s}((\otimes^s \text{vec} A) \otimes (\otimes^{L-s} \text{vec}(\mu\mu') \otimes \mu),$$

proving (2.3.23) and (2.3.24). □

2.4 Examples

For small m, we shall first show how to rewrite our general formula (2.3.21) for evaluating $E(\odot^m Y)$ in the conventional forms as we did in Theorem 2.2.1.

Example 2.4.1. *Let m=3 and m=4 in (2.3.21). Then*

$$\begin{aligned} E(\odot^3 Y) &= \Omega_{3,0}(\otimes^3 \mu) - 2\phi'(0)\Omega_{3,2}(\bar{V}\otimes\mu) \\ &= \odot^3\mu - 2\phi'(0)[\bar{V}\otimes\mu + \mu\otimes\bar{V} + (I_n\otimes K_{n,n})(\bar{V}\otimes\mu)(I_p\otimes K_{p,p})] \end{aligned} \tag{2.4.1}$$

and

$$\begin{aligned} E(\odot^4 Y) &= \Omega_{4,0}(\otimes^4\mu) - 2\phi'(0)\Omega_{4,2}(\bar{V}\otimes\mu\otimes\mu) + 4\phi''(0)(\bar{V}\otimes\bar{V}) \\ &= \odot^4\mu - 2\phi'(0)\Delta_1 + 4\phi''(0)\Delta_2, \end{aligned} \tag{2.4.2}$$

where $\bar{V}$, Δ_1 and Δ_2 are given in (2.2.8), (2.2.9) and (2.2.10) respectively.

Proof. We shall merely prove (2.4.1) and

$$\Omega_{4,2}(\bar{V}\otimes\mu\otimes\mu) = \Delta_1. \tag{2.4.3}$$

By (2.3.22) and (2.3.20),

$$\Omega_{3,0}(\otimes^3\mu) = \omega^t_{0,3,(1,2,3)}(\otimes^3\mu) = \otimes^3\mu. \tag{2.4.4}$$

Also by (2.3.17),

$$D_{3,2} = \{\sigma\in S_3 : \sigma(1) < \sigma(2), \sigma(3)\} = \{(1,2,3),(1,3,2),(2,3,1)\}.$$

Thus we obtain from (2.3.22),

$$\Omega_{3,2} = \sum_{\sigma\in D_{3,2}} \omega^t_{2,1,\sigma} = \omega^t_{2,1,(1,2,3)} + \omega^t_{2,1,(1,3,2)} + \omega^t_{2,1,(2,3,1)}. \tag{2.4.5}$$

By (2.3.18),

$$S^{*}_{(1,2,3),2} = \{(1,2)\}, \quad S^{*}_{(1,3,2),2} = \{(1,3)\}, \quad S^{*}_{(2,3,1),2} = \{(2,3)\}.$$

Let $T_1, T_2, T_3 \in \mathcal{L}(V, E)$. Then obviously,

$$\omega^{t}_{2,1,(1,2,3)}(\bar{V} \otimes \mu) = \bar{V} \otimes \mu. \tag{2.4.6}$$

Since

$$\begin{aligned}
&\langle \omega^{t}_{2,1,(1,3,2)}(\bar{V} \otimes \mu), \bigotimes_{j=1}^{3} T_j \rangle = \langle \bar{V} \otimes \mu, \omega_{2,1,(1,3,2)}(\bigotimes_{j=1}^{3} T_j) \rangle \\
&= \langle \bar{V} \otimes \mu, T_1 \otimes T_3 \otimes T_2 \rangle = \langle \bar{V} \otimes \mu, (I_n \otimes K_{n,n})(T_1 \otimes T_2 \otimes T_3)(I_p \otimes K_{p,p}) \rangle \\
&= \langle (I_n \otimes K_{n,n})(\bar{V} \otimes \mu)(I_p \otimes K_{p,p}), \bigotimes_{j=1}^{3} T_j \rangle,
\end{aligned}$$

by varying $T_1, T_2, T_3 \in \mathcal{L}(V, E)$, we obtain

$$\omega^{t}_{2,1,(1,3,2)}(\bar{V} \otimes \mu) = (I_n \otimes K_{n,n})(\bar{V} \otimes \mu)(I_p \otimes K_{p,p}). \tag{2.4.7}$$

Similarly, since

$$\begin{aligned}
&\langle \omega^{t}_{2,1,(2,3,1)}(\bar{V} \otimes \mu), \bigotimes_{j=1}^{3} T_j \rangle = \langle \bar{V} \otimes \mu, \omega_{2,1,(2,3,1)}(\bigotimes_{j=1}^{3} T_j) \rangle \\
&= \langle \bar{V} \otimes \mu, T_2 \otimes T_3 \otimes T_1 \rangle = \langle \bar{V} \otimes \mu, K_{n^2,n}(T_1 \otimes T_2 \otimes T_3) K_{p,p^2} \rangle \\
&= \langle K_{n,n^2}(\bar{V} \otimes \mu) K_{p^2,p}, \bigotimes_{j=1}^{3} T_j \rangle = \langle \mu \otimes \bar{V}, \bigotimes_{j=1}^{3} T_j \rangle,
\end{aligned}$$

$$\omega^{t}_{2,1,(2,3,1)}(\bar{V} \otimes \mu) = \mu \otimes \bar{V}. \tag{2.4.8}$$

Therefore (2.4.1) follows from (2.4.4), (2.4.6), (2.4.7) and (2.4.8).

For (2.4.3), we obtain from (2.3.17),

$$D_{4,2} = \{(1,2,3,4), (1,3,2,4), (1,4,2,3), (2,3,1,4), (2,4,1,3), (3,4,1,2)\}. \tag{2.4.9}$$

So by (2.3.22),

$$\begin{aligned}
\Omega_{4,2} =& \omega^{t}_{2,2,(1,2,3,4)} + \omega^{t}_{2,2,(1,3,2,4)} + \omega^{t}_{2,2,(1,4,2,3)} \\
&+ \omega^{t}_{2,2,(2,3,1,4)} + \omega^{t}_{2,2,(2,4,1,3)} + \omega^{t}_{2,2,(3,4,1,2)},
\end{aligned} \tag{2.4.10}$$

By (2.3.18),

$$S^{\bullet}_{(1,2,3,4),2} = \{(1,2)\}, S^{\bullet}_{(1,3,2,4),2} = \{(1,3)\}, S^{\bullet}_{(1,4,2,3),2} = \{(1,4)\},$$

$$S^{\bullet}_{(2,3,1,4),2} = \{(2,3)\}, S^{\bullet}_{(2,4,1,3),2} = \{(2,4)\}, S^{\bullet}_{(3,4,1,2),2} = \{(3,4)\}.$$

Let $T_1, T_2, T_3, T_4 \in \mathcal{L}(V, E)$. Then

$$\omega^{\ell}_{2,2,(1,2,3,4)}(V \otimes \mu \otimes \mu) = V \otimes \mu \otimes \mu. \qquad (2.4.11)$$

Since

$$\begin{aligned}
&\langle \omega^{\ell}_{2,2,(1,3,2,4)}(\bar{V} \otimes \mu \otimes \mu), \bigotimes_{j=1}^{4} T_j \rangle \\
&= \langle \bar{V} \otimes \mu \otimes \mu, \omega_{2,2,(1,3,2,4)}(\bigotimes_{j=1}^{4} T_j) \rangle \\
&= \langle \bar{V} \otimes \mu \otimes \mu, T_1 \otimes T_3 \otimes T_2 \otimes T_4 \rangle \\
&= \langle \bar{V} \otimes \mu \otimes \mu, (I_n \otimes K_{nn} \otimes I_n)(\otimes_{j=1}^{4} T_j)(I_p \otimes K_{pp} \otimes I_p) \rangle \\
&= \langle (I_n \otimes K_{nn} \otimes I_n)'(\bar{V} \otimes \mu \otimes \mu)(I_p \otimes K_{pp} \otimes I_p)', \bigotimes_{j=1}^{4} T_j \rangle \\
&= \langle (I_n \otimes K_{nn} \otimes I_n)(\bar{V} \otimes \mu \otimes \mu)(I_p \otimes K_{pp} \otimes I_p), \bigotimes_{j=1}^{4} T_j \rangle,
\end{aligned}$$

by varying the T_j's in $\mathcal{L}(V, E)$, we obtain

$$\omega^{\ell}_{2,2,(1,3,2,4)}(\bar{V} \otimes \mu \otimes \mu) = (I_n \otimes K_{nn} \otimes I_n)(\bar{V} \otimes \mu \otimes \mu)(I_p \otimes K_{pp} \otimes I_p). \qquad (2.4.12)$$

Since

$$\begin{aligned}
&\langle \omega^{\ell}_{2,2,(1,4,2,3)}(\bar{V} \otimes \mu \otimes \mu), \bigotimes_{j=1}^{4} T_j \rangle \\
&= \langle \bar{V} \otimes \mu \otimes \mu, \omega_{2,2,(1,4,2,3)}(\bigotimes_{j=1}^{4} T_j) \rangle \\
&= \langle \bar{V} \otimes \mu \otimes \mu, T_1 \otimes T_4 \otimes T_2 \otimes T_3 \rangle \\
&= \langle \bar{V} \otimes \mu \otimes \mu, K_{n^2,n^2}(T_2 \otimes T_3 \otimes T_1 \otimes T_4) K_{p^2,p^2} \rangle \\
&= \langle \bar{V} \otimes \mu \otimes \mu, K_{n^2,n^2}(K_{n^2,n} \otimes I_n)(\otimes_{j=1}^{4} T_j)(K_{p,p^2} \otimes I_p) K_{p^2,p^2} \rangle
\end{aligned}$$

$$= \langle [K_{n^2,n^2}(K_{n^2,n} \otimes I_n)]'(\bar{V} \otimes \mu \otimes \mu)[(K_{p,p^2} \otimes I_p)K_{p^2,p^2}]', \overset{4}{\underset{j=1}{\otimes}} T_j \rangle$$

$$= \langle (K_{n,n^2} \otimes I_n)K_{n^2,n^2}(\bar{V} \otimes \mu \otimes \mu)K_{p^2,p^2}(K_{p^2,p} \otimes I_p), \overset{4}{\underset{j=1}{\otimes}} T_j \rangle$$

$$= \langle (K_{n,n^2} \otimes I_n)(\mu \otimes \mu \otimes \bar{V})(K_{p^2,p} \otimes I_p), \overset{4}{\underset{j=1}{\otimes}} T_j \rangle.$$

$$\omega^t_{2,2,(1,4,2,3)}(\bar{V} \otimes \mu \otimes \mu) = (K_{n,n^2} \otimes I_n)(\mu \otimes \mu \otimes \bar{V})(K_{p^2,p} \otimes I_p). \tag{2.4.13}$$

Similarly,

$$\omega^t_{2,2,(2,3,1,4)}(\bar{V} \otimes \mu \otimes \mu) = \mu \otimes \bar{V} \otimes \mu, \tag{2.4.14}$$

$$\omega^t_{2,2,(2,4,1,3)}(\bar{V} \otimes \mu \otimes \mu) = (I_n \otimes K_{nn} \otimes I_n)(\mu \otimes \mu \otimes \bar{V})(I_p \otimes K_{pp} \otimes I_p). \tag{2.4.15}$$

and

$$\omega^t_{2,2,(3,4,1,2)}(\bar{V} \otimes \mu \otimes \mu) = \mu \otimes \mu \otimes \bar{V}. \tag{2.4.16}$$

By (2.4.10) and (2.4.11) - (2.4.16), we obtain (2.4.3). □

Example 2.4.2. *The symmetric multivariate Pearson Type VII distribution, $MPVII_n(\mu, A, \phi)$, is $EC_n(\mu, A, \phi)$ with*

$$\phi(u^2) = \frac{2\Gamma\left(N - \frac{n-1}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(N - \frac{n}{2}\right)} \int_0^\infty \cos(m^{\frac{1}{2}} xu)(1 + x^2)^{-N+\frac{n-1}{2}} dx, \tag{2.4.17}$$

where $N > \frac{n}{2}$ and $m > 0$. See Fang, Kotz and Ng (1990). Suppose that $N > \frac{n}{2} + L$ and $y \sim MPVII_n(\mu, A, \phi)$. Then $E(\otimes^L vec(yy'))$ is given by (2.3.23) with

$$\phi^{(s)}(0) = (-m)^s 2^{-2s}\Gamma(N - \frac{n}{2} - s)/\Gamma(N - \frac{n}{2}), \quad s = 0, 1, \cdots, L. \tag{2.4.18}$$

Proof. By differentiating both sides of (2.4.17) $2L$ times with respect to u, we obtain for $s = 1, \cdots, L$,

$$\left.\frac{d^{2s}\phi(u^2)}{du^{2s}}\right|_{u=0} = 2^s\phi^{(s)}(0)|S_{2s,s}| = 2^s\phi^{(s)}(0)\Gamma(2s+1)/\Gamma(s+1) \tag{2.4.19}$$

and

$$\left.\frac{d^{2s}(u^2)}{du^{2s}}\right|_{u=0} = \frac{2\Gamma\left(N-\frac{n-1}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(N-\frac{n}{2}\right)}\int_0^\infty (-m)^s x^{2s}(1+x^2)^{-N+\frac{n-1}{2}}dx$$
$$= (-m)^s \frac{\Gamma\left(N-\frac{n-1}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(N-\frac{n}{2}\right)}\cdot\frac{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(N-\frac{n}{2}-s\right)}{\Gamma\left(N-\frac{n-1}{2}\right)} \tag{2.4.20}$$
$$= \frac{(-m)^s\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(N-\frac{n}{2}-s\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(N-\frac{n}{2}\right)}.$$

Thus (2.4.18) follows from (2.4.19) and (2.4.20). □

Note that the multivariate t-distribution, $Mt_n(m,\mu,A)$, is a special case of $MPVII_n(\mu,A,\phi)$ with $N=(m+n)/2$. So we can use Corollary 2.3.6, (2.3.23) and (2.3.24) to obtain higher order moments of $Mt_n(m,\mu,A)$. In particular,

$$\mathrm{Cov}(y) = \frac{m}{m-2}A \qquad \text{for } m>2$$

and

$$E(yy'\otimes yy') = \mu\mu'\otimes\mu\mu' + \frac{m}{m-2}\Delta_1^* + \frac{m^2}{(m-2)(m-4)}\Delta_2^* \quad \text{for } m>4,$$

where Δ_1^* and Δ_2^* are given in (2.2.21) and (2.2.22).

Example 2.4.3. *Let U be an random operator such that* $\mathrm{vec}\,U \stackrel{d}{=} u^{(np)}$, *a uniform random vector over the np-dimensional unit sphere in $\Re^{np}$. Then*

(i) $E(U)=0, \quad E(U\otimes U\otimes U)=0,$

(ii) $E(U\otimes U) = \bar{V}/np,$

and

(iii)$E(U\otimes U\otimes U\otimes U)$

$$= \frac{1}{np(np+2)}\left[\bar{V}\otimes\bar{V} + (I_n\otimes K_{nn}\otimes I_n)(\bar{V}\otimes\bar{V})(I_p\otimes K_{pp}\otimes I_p) + (K_{n,n^2}\otimes I_n)(\bar{V}\otimes\bar{V})(K_{p^2,p}\otimes I_p)\right],$$

where $\bar{V} = \text{vec}\, I_n (\text{vec}\, I_p)'$.

Proof. (i) follows from (2.3.13). Let $Z \sim N(0, I_n \otimes I_p)$. Then by Theorem 2.2.1, we have $E(Z \otimes Z) = \bar{V}$ and

$$E(Z \otimes Z \otimes Z \otimes Z) = \bar{V} \otimes \bar{V} + (I_n \otimes K_{nn} \otimes I_n)(\bar{V} \otimes \bar{V})(I_p \otimes K_{pp} \otimes I_p) + (K_{n,n^2} \otimes I_n)(\bar{V} \otimes \bar{V})(K_{p^2,p} \otimes I_p).$$

Recall that the stochastic representation of Z is $Z \stackrel{d}{=} RU$, where $R^2 \sim \chi^2_{np}$, $\text{vec}\, U \stackrel{d}{=} u^{(np)}$, and R is independent of U. So

$$E(U \otimes U) = E(Z \otimes Z)/E(R^2) = \bar{V}/(np)$$

and

$$E(U \otimes U \otimes U \otimes U) = E(Z \otimes Z \otimes Z \otimes Z)/E(R^4)$$
$$= \frac{1}{np(np+2)} \left[\bar{V} \otimes \bar{V} + (I_n \otimes K_{nn} \otimes I_n)(\bar{V} \otimes \bar{V})(I_p \otimes K_{pp} \otimes I_p) + (K_{n,n^2} \otimes I_n)(\bar{V} \otimes \bar{V})(K_{p^2,p} \otimes I_p)\right],$$

proving (ii) and (iii). □

Note that for large m, we may, as we did in Example 2.4.1, rewrite $E(\otimes^m Y)$ in (2.3.21) by using the notion of commutation operators. But, as it will be shown below, the latter is less general and more complicated and also involves a lot of zero entries.

Example 2.4.4. *Suppose that $Y \sim MEC_{n\times p}(0, \Sigma_Y, \phi)$. Let $m = 6$. Then by (2.3.16),*

$$E(\otimes^6 Y) = -8\phi^{(3)}(0)(\omega^0_{6,\sigma})'(\otimes^3 \bar{V}).$$

By (2.3.4),

$$S_{6,3} = \{\sigma \in S_6 : \sigma(1) < \sigma(2) < \sigma(3),\ \sigma(1) < \sigma(6),\ \sigma(2) < \sigma(5),\ \sigma(3) < \sigma(4)\},$$

which has $6!/(2^3 3!) = 15$ elements. Thus for any $T_j \in \mathcal{L}(V, E)$, $j = 1, \cdots, 6$,

$$\langle (\omega^0_{6,\sigma})^t(\otimes^3 \bar{V}), \overset{6}{\underset{j=1}{\otimes}} T_j \rangle = \langle \otimes^3 \bar{V}, \omega^0_{6,\sigma}(\overset{6}{\underset{j=1}{\otimes}} T_j) \rangle. \qquad (2.4.21)$$

By (2.3.15) and (2.4.21),

$$\begin{aligned}
\omega^0_{6,\sigma}(\overset{6}{\underset{j=1}{\otimes}} T_j) =& T_1 \otimes T_2 \otimes T_3 \otimes [T_4 \otimes T_5 \otimes T_6 + T_5 \otimes T_4 \otimes T_6 + T_6 \otimes T_4 \otimes T_5] \\
&+ T_1 \otimes T_3 \otimes T_2 \otimes [T_4 \otimes T_5 \otimes T_6 + T_5 \otimes T_4 \otimes T_6 + T_6 \otimes T_4 \otimes T_5] \\
&+ T_1 \otimes T_4 \otimes T_2 \otimes [T_3 \otimes T_5 \otimes T_6 + T_5 \otimes T_3 \otimes T_6 + T_6 \otimes T_3 \otimes T_5] \\
&+ T_1 \otimes T_5 \otimes T_2 \otimes [T_3 \otimes T_4 \otimes T_6 + T_4 \otimes T_3 \otimes T_6 + T_6 \otimes T_3 \otimes T_4] \\
&+ T_1 \otimes T_6 \otimes T_2 \otimes [T_3 \otimes T_4 \otimes T_5 + T_4 \otimes T_3 \otimes T_5 + T_5 \otimes T_3 \otimes T_4].
\end{aligned}$$

Thus by (2.3.16), we obtain, upon simplifying,

$$\begin{aligned}
E(\otimes^6 Y) =& -8\phi^{(3)}(0)\,[\otimes^3 \bar{V} + (I_{n^3} \otimes K_{n,n} \otimes I_n)(\otimes^3 \bar{V})(I_{p^3} \otimes K_{p,p} \otimes I_p) \\
&+ (I_{n^3} \otimes K_{n^2,n})(\otimes^3 \bar{V})(I_{p^3} \otimes K_{p,p^2}) + (I_n \otimes K_{n,n} \otimes I_{n^3})(\otimes^3 \bar{V})(I_p \otimes K_{p,p} \otimes I_{p^3}) \\
&+ (I_n \otimes K_{n,n} \otimes K_{n,n} \otimes I_n)(\otimes^3 \bar{V})(I_p \otimes K_{p,p} \otimes K_{p,p} \otimes I_p) \\
&+ (I_n \otimes K_{n,n} \otimes K_{n^2,n})(\otimes^3 \bar{V})(I_p \otimes K_{p,p} \otimes K_{p,p^2}) \\
&+ (I_n \otimes K_{n^2,n} \otimes I_{n^2})(\otimes^3 \bar{V})(I_p \otimes K_{p,p^2} \otimes I_{p^2}) \\
&+ (I_n \otimes K_{n^2,n} \otimes I_{n^2})(I_{n^3} \otimes K_{n,n} \otimes I_n)(\otimes^3 \bar{V})(I_{p^3} \otimes K_{p,p} \otimes I_p)(I_p \otimes K_{p,p^2} \otimes I_{p^2}) \\
&+ (I_n \otimes K_{n^2,n} \otimes I_{n^2})(I_{n^3} \otimes K_{n^2,n})(\otimes^3 \bar{V})(I_{p^3} \otimes K_{p,p^2})(I_p \otimes K_{p,p^2} \otimes I_{p^2}) \\
&+ (I_n \otimes K_{n^3,n} \otimes I_n)(\otimes^3 \bar{V})(I_p \otimes K_{p,p^3} \otimes I_p) \\
&+ (I_n \otimes K_{n^3,n} \otimes I_n)(I_{n^3} \otimes K_{n,n} \otimes I_n)(\otimes^3 \bar{V})(I_{p^3} \otimes K_{p,p} \otimes I_p)(I_p \otimes K_{p,p^3} \otimes I_p) \\
&+ (I_n \otimes K_{n^3,n} \otimes I_n)(I_{n^3} \otimes K_{n^2,n})(\otimes^3 \bar{V})(I_{p^3} \otimes K_{p,p^2})(I_p \otimes K_{p,p^3} \otimes I_p) \\
&+ (I_n \otimes K_{n^4,n})(\otimes^3 \bar{V})(I_p \otimes K_{p,p^4}) \\
&+ (I_n \otimes K_{n^4,n})(I_{n^3} \otimes K_{n,n} \otimes I_n)(\otimes^3 \bar{V})(I_{p^3} \otimes K_{p,p} \otimes I_p)(I_p \otimes K_{p,p^4}) \\
&+ (I_n \otimes K_{n^4,n})(I_{n^3} \otimes K_{n^2,n})(\otimes^3 \bar{V})(I_p \otimes K_{p,p^2})(I_p \otimes K_{p,p^4}).
\end{aligned}$$

Note that the above expression for writing $E(\otimes^6 Y)$ in terms of $Y \otimes Y \otimes Y$ and commutation matrices is by no means unique.

CHAPTER THREE

MOMENTS OF QUADRATIC FORMS

3.1 Introduction

In Chapter 2, we obtained formulae for evaluating the higher order moments of $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$ without assuming that Σ_Y has the form $A \otimes \Sigma$. In this chapter, these results will be used to obtain expressions for higher order moments of quadratic forms . For comparison, the second order moment of these quadratic forms and the covariance of the second degree polynomial of Y are written in conventional forms. Even for the normal setting, these results generalize the corresponding results of Browne and Neudecker (1988), Jinadasa (1986), Neudecker (1985, 1990), Neudecker and Wansbeck (1987), and von Rosen (1988). For illustration, our results are applied to multivariate components of variance models, ANOVA models and generalized Wishart distributions. These models and distributions were investigated by Pavur (1987), Mathew (1989), Wong, Masaro and Wang (1991), Wong and Wang (1992) and others.

3.2 Moments of quadratic forms

Theorem 3.2.1. *Suppose that $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$. Let $W_1, W_2, \cdots, W_L \in \mathcal{L}(E, E)$. Then*

$$E(\overset{L}{\underset{j=1}{\otimes}}(Y'W_jY)) = \sum_{s=0}^{L}(-2)^s\phi^{(s)}(0)vec^{-1}\left\{\left[\Omega_{2L,2s}((\otimes^s\bar{V})\otimes(\otimes^{2(L-s)}\mu))\right]' vec(\overset{L}{\underset{j=1}{\otimes}}W_j)\right\} \quad (3.2.1)$$

and when $\mu = 0$,

$$E(\overset{L}{\underset{j=1}{\otimes}}(Y'W_jY)) = (-2)^L\phi^{(L)}(0)vec^{-1}\left\{\left[(\omega^0_{2L,\sigma})'(\otimes^L\bar{V})\right]' vec(\overset{L}{\underset{j=1}{\otimes}}W_j)\right\}, \quad (3.2.2)$$

where $\bar{V}$, $\Omega_{2L,2s}$ and $\omega^0_{2L,\sigma}$ are given in (2.2.8), (2.3.21) and (2.3.16) respectively and vec^{-1} is the inverse mapping of vec.

Proof. Since $\overset{L}{\underset{j=1}{\otimes}}(Y'W_jY) = (\otimes^L Y)'(\overset{L}{\underset{j=1}{\otimes}} W_j)(\otimes^L Y)$,

$$\begin{aligned}\text{vec}\left[E\left(\overset{L}{\underset{j=1}{\otimes}}(Y'W_jY)\right)\right] &= E\left[((\otimes^L Y)' \otimes (\otimes^L Y)')\right] \text{vec}(\overset{L}{\underset{j=1}{\otimes}} W_j) \\ &= \left[E(\otimes^{2L} Y)'\right] \text{vec}(\overset{L}{\underset{j=1}{\otimes}} W_j).\end{aligned} \tag{3.2.3}$$

By (2.3.21), (3.2.3) becomes

$$\begin{aligned}&\text{vec}\left[E(\overset{L}{\underset{j=1}{\otimes}}(Y'W_jY))\right] \\ &= \sum_{s=0}^{L}(-2)^s\phi^{(s)}(0)\left\{\Omega_{2L,2s}\left((\otimes^s\bar{V})\otimes(\times^{2(L-s)}\mu)\right)\right\}' \text{vec}(\overset{L}{\underset{j=1}{\otimes}} W_j).\end{aligned} \tag{3.2.4}$$

Thus by applying vec^{-1} to both sides of (3.2.4), we obtain (3.2.1). Similarly, (3.2.2) can be obtained from (3.2.3) and (2.3.16). □

Corollary 3.2.2. *In Theorem 3.2.1, let $p = 1$, $y = Y$ and $\Sigma_Y = A$. Then*

$$\begin{aligned}&E[\prod_{j=1}^{L}(y'W_jy)] \\ &= \sum_{s=0}^{L}(-2)^s\phi^{(s)}(0)\left\{\Omega_{2L,2s}((\otimes^s \text{vec}A)\otimes(\otimes^{L-s}\text{vec}(\mu\mu')))\right\}' \text{vec}(\overset{L}{\underset{j=1}{\otimes}} W_j)\end{aligned}$$

and when $\mu = 0$,

$$E[\prod_{j=1}^{L}(y'W_jy)] = (-2)^L\phi^{(L)}(0)\left\{(\omega^0_{2L,\sigma})'(\otimes^L \text{vec}A)\right\}' \text{vec}(\overset{L}{\underset{j=1}{\otimes}} W_j)$$

Proof. Note that

$$\begin{aligned}E[\prod_{j=1}^{L}(y'W_jy)] &= E[(\otimes^L y)'(\overset{L}{\underset{j=1}{\otimes}} W_j)] \\ &= E(\otimes^{2L} y)'\text{vec}(\overset{L}{\underset{j=1}{\otimes}} W_j)\end{aligned} \tag{3.2.5}$$

The desired results follow from (2.3.23) and (3.2.5). □

Corollary 3.2.3. *In Theorem 3.2.1, let $L = 2$ and consider the multivariate components of variance model Y where $\Sigma_Y = \sum_{j=1}^{k} A_j \otimes \Sigma_j$, $A_j \in \mathcal{N}_E$ and $\Sigma_j \in \mathcal{N}_V$. Then*

(i) $E[(Y'W_1Y) \otimes (Y'W_2Y)] = \mu'W_1\mu \otimes \mu'W_2\mu - 2\phi'(0)\nabla_1 + 4\phi''(0)\nabla_2$,

where

$$\begin{aligned}\nabla_1 = \sum_{j=1}^{k} \{ & tr(A_jW_1)(\Sigma_j \otimes \mu'W_2\mu) + tr(A_jW_2)(\mu'W_1\mu \otimes \Sigma_j) \\ & + vec\Sigma_j(vec(\mu'W_1'A_jW_2\mu))' + vec(\mu'W_1A_jW_2'\mu)(vec\Sigma_j)' \\ & + K_{p,p}(\Sigma_j \otimes (\mu'W_1A_jW_2\mu) + (\mu'W_2A_jW_1\mu) \otimes \Sigma_j)\}\end{aligned} \tag{3.2.6}$$

and

$$\begin{aligned}\nabla_2 = \sum_{j,\ell=1}^{k} \{ & tr(A_jW_1A_\ell W_2')vec\Sigma_j(vec\Sigma_\ell)' + tr(A_jW_1A_\ell W_2)K_{p,p}(\Sigma_j \otimes \Sigma_\ell) \\ & + tr(A_jW_1)tr(A_\ell W_2)(\Sigma_j \otimes \Sigma_\ell)\}\,.\end{aligned} \tag{3.2.7}$$

(ii) $Cov(Y'W_1Y, Y'W_2Y) = 4\phi''(0)\nabla_3 + 4[\phi''(0) - (\phi'(0))^2]\nabla_4 - 2\phi'(0)\nabla_5$,

where

$$\nabla_3 = \sum_{j,\ell=1}^{k} \{tr(A_jW_1A_\ell W_2')(\Sigma_j \otimes \Sigma_\ell) + tr(A_jW_1A_\ell W_2)K_{p,p}(\Sigma_j \otimes \Sigma_\ell)\}\,,$$

$$\nabla_4 = \sum_{j,\ell=1}^{k} tr(A_jW_1)tr(A_\ell W_2)vec\Sigma_j(vec\Sigma_\ell)',$$

and

$$\begin{aligned}\nabla_5 = \sum_{j=1}^{k} \{ & \Sigma_j \otimes (\mu'W_1'A_jW_2\mu) + (\mu'W_1A_jW_2'\mu) \otimes \Sigma_j \\ & + K_{p,p}[(\mu'W_2'V_jW_1'\mu) \otimes \Sigma_j + \Sigma_j \otimes (\mu'W_2A_jW_1\mu)]\}\,.\end{aligned}$$

Proof. By Theorem 3.2.1,

$$E(Y'W_1Y \otimes Y'W_2Y) = vec^{-1}\{E(Y \otimes Y \otimes Y \otimes Y)'vec(W_1 \otimes W_2)\}\,. \tag{3.2.8}$$

By Theorem 2.2.3 and Corollory 2.3.5,

$$[E(\otimes^4 Y)]'\text{vec}(W_1 \otimes W_2) = \left[\otimes^4\mu - 2\phi'(0)\Delta_1 + 4\phi''(0)\Delta_2\right]'\text{vec}(W_1 \otimes W_2),$$

where Δ_1 and Δ_2 are given in (2.2.9) and (2.2.10) respectively. By Lemma 2.2.1, we obtain

$$(\otimes^4\mu)'\text{vec}(W_1 \otimes W_2) = \text{vec}(\mu' W_1 \mu \otimes \mu' W_2 \mu), \tag{3.2.9}$$

$$\begin{aligned}
&(\bar{V} \otimes \mu \otimes \mu + \mu \otimes \mu \otimes \bar{V})'\text{vec}(W_1 \otimes W_2) \\
&= \text{vec}\left[\bar{V}'(W_1 \otimes W_2)(\mu \otimes \mu) + (\mu \otimes \mu)'(W_1 \otimes W_2)\bar{V}\right] \\
&= \text{vec}\left\{\sum_{j=1}^{k}[\text{vec}\Sigma_j(\text{vec}(\mu' W_1' A_j W_2 \mu))' + \text{vec}(\mu' W_1 A_j W_2' \mu)(\text{vec}\Sigma_j)']\right\},
\end{aligned} \tag{3.2.10}$$

and

$$\begin{aligned}
&(I_p \otimes K_{p,p} \otimes I_p)(\bar{V} \otimes \mu \otimes \mu + \mu \otimes \mu \otimes \bar{V})'(I_n \otimes K_{n,n} \otimes I_n)\text{vec}(W_1 \otimes W_2) \\
&= (I_p \otimes K_{p,p} \otimes I_p)(\bar{V} \otimes \mu \otimes \mu + \mu \otimes \mu \otimes \bar{V})'(\text{vec}W_1 \otimes \text{vec}W_2) \\
&= (I_p \otimes K_{p,p} \otimes I_p)[\bar{V}'\text{vec}W_1 \otimes \text{vec}(\mu' W_2 \mu) + \text{vec}(\mu' W_1 \mu) \otimes (\bar{V}'\text{vec}W_2)] \\
&= \text{vec}\left\{\sum_{j=1}^{k}[\text{tr}(A_j W_1)(\Sigma_j \otimes \mu' W_2 \mu) + \text{tr}(A_j W_2)(\mu' W_1 \mu \otimes \Sigma_j)]\right\}.
\end{aligned} \tag{3.2.11}$$

Also by Lemma 2.2.1 and Lemma 2.2.2,

$$\begin{aligned}
&\left[\mu \otimes \bar{V} \otimes \mu + (K_{n,n^2} \otimes I_n)(\mu \otimes \mu \otimes \bar{V})(K_{p^2,p} \otimes I_p)\right]'\text{vec}(W_1 \otimes W_2) \\
&= (K_{p,p^2} \otimes I_p)(\bar{V} \otimes \mu \otimes \mu + \mu \otimes \mu \otimes \bar{V})'(K_{n^2,n} \otimes I_n)\text{vec}(W_1 \otimes W_2) \\
&= (K_{p,p^2} \otimes I_p)(\bar{V} \otimes \mu \otimes \mu + \mu \otimes \mu \otimes \bar{V})'\text{vec}(K_{n,n}(W_1' \otimes W_2)) \\
&= (K_{p,p^2} \otimes I_p)\left[\text{vec}(\bar{V}' K_{n,n}(W_1' \otimes W_2)(\mu \otimes \mu) + (\mu \otimes \mu)' K_{n,n}(W_1' \otimes W_2)\bar{V})\right] \\
&= (K_{p,p^2} \otimes I_p)\left[\sum_{j=1}^{k}(\text{vec}\Sigma_j \otimes \text{vec}(\mu' W_1 A_j W_2 \mu) + \text{vec}(\mu' W_2 A_j W_1 \mu) \otimes \text{vec}\Sigma_j)\right] \\
&= \text{vec}\left\{\sum_{j=1}^{k} K_{pp}[\Sigma_j \otimes (\mu' W_1 A_j W_2 \mu) + (\mu' W_2 A_j W_1 \mu) \otimes \Sigma_j]\right\}.
\end{aligned} \tag{3.2.12}$$

Thus by (3.2.10) - (3.2.12),

$$\begin{aligned}
&\mathrm{vec}^{-1}\left[(\Delta_1)'\mathrm{vec}(W_1 \otimes W_2)\right] \\
&\quad = \sum_{j=1}^{k} \{\mathrm{tr}(A_j W_1)(\Sigma_j \otimes \mu' W_2 \mu) + \mathrm{tr}(A_j W_2)(\mu' W_1 \mu \otimes \Sigma_j) \\
&\quad + \mathrm{vec}\Sigma_j(\mathrm{vec}(\mu' W_1' A_j W_2 \mu))' + \mathrm{vec}(\mu' W_1 A_j W_2' \mu)(\mathrm{vec}\Sigma_j)' \\
&\quad + K_{p,p}(\Sigma_j \otimes (\mu' W_1 A_j W_2 \mu) + (\mu' W_2 A_j W_1 \mu) \otimes \Sigma_j)\},
\end{aligned}$$

proving (3.2.6). Similarly, we obtain

$$(\bar{V} \otimes \bar{V})'\mathrm{vec}(W_1 \otimes W_2) = \sum_{j,\ell=1}^{k} \mathrm{tr}(A_j W_1 A_\ell W_2')(\mathrm{vec}\Sigma_j \otimes \mathrm{vec}\Sigma_\ell), \tag{3.2.13}$$

$$\begin{aligned}
&(I_p \otimes K_{p,p} \otimes I_p)(\bar{V} \otimes \bar{V})'(I_n \otimes K_{n,n} \otimes I_n)\mathrm{vec}(W_1 \otimes W_2) \\
&\quad = \mathrm{vec}\left\{ \sum_{j,\ell=1}^{k} \mathrm{tr}(A_j W_1)\mathrm{tr}(A_\ell W_2)(\Sigma_j \otimes \Sigma_\ell) \right\},
\end{aligned} \tag{3.2.14}$$

and

$$\begin{aligned}
&(K_{p,p^2} \otimes I_p)(\bar{V} \otimes \bar{V})'(K_{n^2,n} \otimes I_n)\mathrm{vec}(W_1 \otimes W_2) \\
&\quad = \mathrm{vec}\left\{ \sum_{j,\ell=1}^{k} \mathrm{tr}(A_j W_1 A_\ell W_2) K_{p,p}(\Sigma_j \otimes \Sigma_\ell) \right\}.
\end{aligned} \tag{3.2.15}$$

By applying vec^{-1} to (3.2.13) - (3.2.15), ∇_2 in (3.2.7) follows. Therefore (i) follows from (3.2.9), (3.2.6) and (3.2.7).

(ii) Since

$$\begin{aligned}
&\mathrm{vec}(\mathrm{Cov}(Y'W_1Y, Y'W_2Y)) \\
&= E(\mathrm{vec}(Y'W_1Y) \otimes \mathrm{vec}(Y'W_2Y)) - \mathrm{vec}(E(Y'W_1Y)) \otimes \mathrm{vec}(E(Y'W_2Y)) \\
&= (I_p \otimes K_{p,p} \otimes I_p)\mathrm{vec}\left[E(Y'W_1Y \otimes Y'W_2Y) - E(Y'W_1Y) \otimes E(Y'W_2Y)\right],
\end{aligned} \tag{3.2.16}$$

we obtain from (i),

$$
\begin{aligned}
&E(Y'W_1Y \otimes Y'W_2Y) - E(Y'W_1Y) \otimes E(Y'W_2Y) \\
&= -2\phi'(0) \sum_{j=1}^{k} \{\mathrm{vec}\Sigma_j(\mathrm{vec}(\mu' W_1' A_j W_2 \mu))' \otimes \mathrm{vec}(\mu' W_1 A_j W_2' \mu)(\mathrm{vec}\Sigma_j)' \\
&\qquad + K_{p,p}(\mu' W_1 A_j W_2 \mu \otimes \Sigma_j + \Sigma_j \otimes \mu' W_2 A_j W_1 \mu)\} \\
&+ 4\phi''(0) \sum_{j,\ell=1}^{k} \{\mathrm{tr}(A_j W_1 A_\ell W_2')\mathrm{vec}\Sigma_j(\mathrm{vec}\Sigma_\ell)' + \mathrm{tr}(A_j W_1 A_\ell W_2) K_{p,p}(\Sigma_j \otimes \Sigma_\ell)\} \\
&+ 4[\phi''(0) - (\phi'(0))^2] \sum_{j,\ell=1}^{k} \mathrm{tr}(A_j W_1)\mathrm{tr}(A_\ell W_2)(\Sigma_j \otimes \Sigma_\ell).
\end{aligned} \tag{3.2.17}
$$

Substituting (3.2.17) into (3.2.16), we obtain, upon simplification,

$$
\begin{aligned}
&\mathrm{vec}\,[\mathrm{cov}(Y'W_1Y, Y'W_2Y)] \\
&= \mathrm{vec}\Bigg\{ 4\phi''(0) \sum_{j,\ell=1}^{k} [\mathrm{tr}(A_j W_1 A_\ell W_2')(\Sigma_j \otimes \Sigma_\ell) + \mathrm{tr}(A_j W_1 A_\ell W_2) K_{p,p}(\Sigma_j \otimes \Sigma_\ell)] \\
&\qquad + 4[\phi''(0) - (\phi'(0))^2] \sum_{j,\ell=1}^{k} \mathrm{tr}(A_j W_1)\mathrm{tr}(A_\ell W_2)\mathrm{vec}\Sigma_j(\mathrm{vec}\Sigma_\ell)' \\
&\qquad - 2\phi'(0) \sum_{j=1}^{k} [\Sigma_j \otimes (\mu' W_1' A_j W_2 \mu) + (\mu' W_1 A_j W_2' \mu) \otimes \Sigma_j \\
&\qquad\qquad + K_{p,p}((\mu' W_2' A_j W_1' \mu) \otimes \Sigma_j + \Sigma_j \otimes (\mu' W_2 A_j W_1 \mu))]\Bigg\}.
\end{aligned} \tag{3.2.18}
$$

Thus, by applying vec^{-1} to (3.2.18), (ii) follows . □

The above multivariate components of variance model Y was investigated by Mathew (1989) and Wong and Wang (1992) for the normal setting. For k=1, von Rosen (1988) and Neudecker (1990) obtained the above corollary for the case where $Y \sim N_{n \times p}(\mu, A \otimes \Sigma)$ and $\bar{V} = \mathrm{vec}\, A(\mathrm{vec}\Sigma)'$.

Corollary 3.2.4. *In Theorem 3.2.1, let* $L = 2$, $\mu = 0$ *and* $W_1 = W_2 = W \in S_E$

and write

$$\Sigma_Y = \sum_{i,i'=1}^{n} \sum_{j,j'=1}^{p} \sigma_{ii'jj'} (e_i \Box e_{i'}) \otimes (f_j \Box f_{j'}) = \sum_{i,i'=1}^{n} (e_i \Box e_{i'}) \otimes \Sigma_{ii'},$$

where $\Sigma_{ii'} = \sum_{j,j'=1}^{p} \sigma_{ii'jj'} f_j \Box f_{j'}$. Then

(*i*) $E(Y'WY) = -2\phi'(0) \sum_{i,i'=1}^{n} (e_i' W e_i) \Sigma_{ii'}$

and

(*ii*) $E[(Y'WY \otimes (Y'WY)] = 4\phi''(0)\Delta,$

where

$$\begin{aligned}\Delta = \sum_{i_1,i_1',i_2,i_2'=1}^{n} &\Big\{ (e_{i_1}' W e_{i_2})(e_{i_1'}' W e_{i_2'}) \text{vec}\, \Sigma_{i_1 i_1'} (\text{vec}\, \Sigma_{i_2 i_2'})' \\ &+ (e_{i_1}' W e_{i_1'})(e_{i_2}' W e_{i_2'}) \Sigma_{i_1 i_1'} \otimes \Sigma_{i_2 i_2'} \\ &+ (e_{i_1}' W e_{i_2'})(e_{i_1'}' W e_{i_2}) K_{p,p} (\Sigma_{i_1 i_1'} \otimes \Sigma_{i_2 i_2'}) \Big\},\end{aligned}$$

and hence

$$\begin{aligned}Cov(Y'WY) = \sum_{i_1,i_1',i_2,i_2'=1}^{n} &\Big\{ 4\phi''(0) \Big[(e_{i_1}' W e_{i_2})(e_{i_1'}' W e_{i_2'}) \Sigma_{i_1 i_1'} \otimes \Sigma_{i_2 i_2'} \\ &+ (e_{i_1}' W e_{i_2'})(e_{i_1'}' W e_{i_2}) K_{p,p} (\Sigma_{i_1' i_1} \otimes \Sigma_{i_2 i_2'}) \Big] \\ &+ 4[\phi''(0) - (\phi'(0))^2] (e_{i_1}' W e_{i_1'})(e_{i_2}' W e_{i_2'}) \text{vec}\, \Sigma_{i_1 i_1'} \left(\text{vec}\, \Sigma_{i_2 i_2'} \right)' \Big\}.\end{aligned}$$

Proof. By Theorem 3.2.1 and Theoren 2.2.3,

$$E[(Y'WY \otimes (Y'WY)] = 4\phi''(0) \text{vec}^{-1} [\Delta_2]' \text{vec}\, (W \otimes W), \tag{3.2.19}$$

where Δ_2 is given in (2.2.10). By (2.2.10) and (3.2.19), we obtain

$$\begin{aligned}&(\bar{V} \otimes \bar{V})' \text{vec}\, (W \otimes W) = \text{vec}\, [\bar{V}'(W \otimes W)\bar{V}] \\ &= \sum_{i_1,i_1',i_2,i_2'=1}^{n} (e_{i_1}' W e_{i_2})(e_{i_1'}' W e_{i_2'}) (\text{vec}\, \Sigma_{i_1 i_1'} \otimes \text{vec}\, \Sigma_{i_2 i_2'}),\end{aligned} \tag{3.2.20}$$

$$(I_p \otimes K_{p,p} \otimes I_p)(\bar V \otimes \bar V)'(I_n \otimes K_{n,n} \otimes I_n)\text{vec}\,(W \otimes W)$$

$$= (I_p \otimes K_{p,p} \otimes I_p)(\bar V'\text{vec}\,W \otimes \bar V'\text{vec}\,W)$$

$$= \sum_{i_1,i_1',i_2,i_2'=1}^{n} (e_{i_1}' W e_{i_1'})(e_{i_2}' W e_{i_2'})(I_p \otimes K_{p,p} \otimes I_p)[\text{vec}\,\Sigma_{i_1 i_1'} \otimes \text{vec}\,\Sigma_{i_2 i_2'}] \quad (3.2.21)$$

$$= \sum_{i_1,i_1',i_2,i_2'=1}^{n} (e_{i_1}' W e_{i_1'})(e_{i_2}' W e_{i_2'})\text{vec}\,(\Sigma_{i_1 i_1'} \otimes \Sigma_{i_2 i_2'}),$$

and

$$(K_{p,p^2} \otimes I_p)(\bar V \otimes \bar V)'(K_{n^2,n} \otimes I_n)\text{vec}\,(W \otimes W)$$

$$= (K_{p,p^2} \otimes I_p)(\bar V \otimes \bar V)'\text{vec}\,(K_{n,n}(W \otimes W))$$

$$= (K_{p,p^2} \otimes I_p)\text{vec}\,\left[\bar V' K_{n,n}(W \otimes W)\bar V\right]$$

$$= \sum_{i_1,i_1',i_2,i_2'=1}^{n} (e_{i_1}' W e_{i_2'})(e_{i_1'}' W e_{i_2})(K_{p,p^2} \otimes I_p)\left[\text{vec}\,\Sigma_{i_1 i_1'} \otimes \text{vec}\,\Sigma_{i_2 i_2'}\right] \quad (3.2.22)$$

$$= \sum_{i_1,i_1',i_2,i_2'=1}^{n} (e_{i_1}' W e_{i_2'})(e_{i_1'}' W e_{i_2})\text{vec}\,\left[K_{p,p}(\Sigma_{i_1 i_1'} \otimes \Sigma_{i_2 i_2'})\right].$$

Thus (ii) follows by substiuting (3.2.20) - (3.2.22) into (3.2.19). □

Example 3.2.1. *Suppose that* $Y \equiv (Y_1', Y_2')' \sim MEC_{n\times p}(0, I_n \otimes \Sigma, \phi)$ *with* $Y_1 \in M_{m\times p}$. *Then the distribution of* $\tilde W = Y_1'Y_1$ *is referred to as the generalized Wishart distribution* $GW_p(m; n-m; \Sigma; \phi)$. *By Corollary 3.2.3,*

(i) $E(\tilde W) = -2\phi'(0)m\Sigma$ *and*

(ii) $Cov(\tilde W) = 4\phi''(0)m(I_{p^2} + K_{pp})(\Sigma \otimes \Sigma) + 4m^2[\phi''(0) - (\phi'(0))^2]vec\Sigma(vec\Sigma)'$.

Note that if $Y \sim N_{n\times p}(0, I_n \otimes \Sigma)$, then $\tilde W \sim GW_p(m; n-m; \Sigma; \phi) = W_p(m, \Sigma)$, and $\phi'(0) = -\frac{1}{2}$, $\phi''(0) = \frac{1}{4}$. Thus $E(\tilde W) = m\Sigma$ and $\text{cov}(\tilde W) = m(I_{p^2} + K_{pp})(\Sigma \otimes \Sigma)$.

Example 3.2.2. *In ANOVA models with balanced subsample sizes, the following properties are given for matrices* $W_1, \cdots, W_k$ *in* $\mathcal{N}_n$,

$$W_i W_j = \delta_{ij} W_i, \qquad \sum_{i=1}^{k} W_i = I_n - J_n, \quad (3.2.23)$$

where $J_n \in M_{n\times n}$ *with each component equal to* $1/n$ *(see, e.g. Pavur (1987) and Wong, Masaro and Wang (1991)). Suppose that* $Y \sim MEC_{n\times p}(0, \Sigma_Y, \phi)$ *with*

$$\Sigma_Y = \sum_{i=1}^{k} W_i \otimes \Sigma + (J_n \otimes I_p)H + H'(J_n \otimes I_p),$$

where $H \in M_{np\times np}$. *Let* $Q_\ell(Y) = Y'W_\ell Y$, $\ell = 1, \cdots, k$. *Then*

(i) $E(Q_\ell(Y)) = -2\phi'(0)r(W_\ell)\Sigma$.

(ii) $Cov(Q_\ell(Y)) = 4\phi''(0)r(W_\ell)(I_{p^2} \otimes K_{p,p})(\Sigma\otimes\Sigma) + 4[\phi'' - (\phi'(0))^2]\text{vec}\,\Sigma(\text{vec}\,\Sigma)'$.

and for any distinct ℓ, $\ell' = 1, 2, \cdots, k$,

(iii) $Cov(Q_\ell(Y), Q_{\ell'}(Y)) = 0$,

where $r(W_\ell)$ *denotes the rank of* W_ℓ.

Proof. As we did in Section 2.2, let $\{e_i\}_{i=1}^{n}$ and $\{f_j\}_{j=1}^{p}$ be the orthonormal bases of $\Re^n$ and $\Re^p$ respectively, $E_{ii'} = e_ie_{i'}'$ and $F_{jj'} = f_jf_{j'}'$. Then $H \in M_{np\times np}$ can be written as

$$H = \sum_{ii'}^{n}\sum_{jj'}^{p} h_{ii'jj'}E_{ii'} \otimes F_{jj'},$$

where $h_{ii'jj'} = \langle \text{vec}E_{ii'}(\text{vec}F_{jj'})', H\rangle$. Thus by (3.2.2),

$$E(Q_\ell(Y)) = E(Y'W_\ell Y) = -2\phi'(0)\text{vec}^{-1}\left\{[(\omega_{2,\sigma}^{0})^{t}(\bar{V})]'\,\text{vec}W_\ell\right\}, \tag{3.2.24}$$

where

$$\begin{aligned}\bar{V} &= \sum_{i=1}^{k}\text{vec}W_i(\text{vec}\Sigma)' \\ &+ \sum_{i,i'=1}^{n}\sum_{j,j'=1}^{p} h_{ii'jj'}\left[\text{vec}(J_nE_{ii'})(\text{vec}F_{jj'})' + (\text{vec}E_{i'i}J_n)(\text{vec}F_{j'j})'\right].\end{aligned} \tag{3.2.25}$$

Since $(\omega_{2,\sigma}^{0})^{t}(\bar{V}) = \bar{V}$, we obtain from (3.2.24) and (3.2.25) that

$$\begin{aligned}E(Q_\ell(Y)) = -2\phi'(0)\text{vec}^{-1}\{\bar{V}'\text{vec}W_\ell\} = -2\phi'(0)\Bigg\{&\sum_{i=1}^{k}\text{tr}(W_iW_\ell)\Sigma \\ &+ \sum_{i,i'=1}^{n}\sum_{j,j'=1}^{p} h_{ii'jj'}\left[\text{tr}(J_nE_{ii'}W_\ell)F_{jj'} + \text{tr}(E_{i'i}J_nW_\ell)F_{j'j}\right]\Bigg\}.\end{aligned} \tag{3.2.26}$$

Thus by (3.2.23), (3.2.26) becomes

$$E(Q_\ell(Y)) = -2\phi'(0)\mathrm{tr}(W_\ell^2)\Sigma = -2\phi'(0)r(W_\ell)\Sigma,$$

proving (i).

For (ii) and (iii), note that from Theorem 2.2.3 and (3.2.2), we obtain

$$\begin{aligned} &E(Y'W_\ell Y) \otimes Y'W_{\ell'}Y) \\ &= 4\phi''(0)\mathrm{vec}^{-1}\left\{\left[\tilde{V}\otimes\tilde{V} + (K_{n,n^2}\otimes I_n)(\tilde{V}\otimes\tilde{V})(K_{p^2,p}\otimes I_p)\right.\right. \\ &\quad \left.\left. + (I_n\otimes K_{nn}\otimes I_n)(\tilde{V}\otimes\tilde{V})(I_p\otimes K_{pp}\otimes I_p)\right]'\mathrm{vec}(W_\ell\otimes W_{\ell'})\right\}. \end{aligned} \tag{3.2.27}$$

Similarly, as in the proof of Corollary 3.2.3, we obtain from (3.2.23) and (3.2.25),

$$\mathrm{vec}^{-1}\left[(\tilde{V}\otimes\tilde{V})'\mathrm{vec}(W_\ell\otimes W_{\ell'})\right] = \delta_{\ell\ell'}r(W_\ell)\mathrm{vec}\Sigma(\mathrm{vec}\Sigma)', \tag{3.2.28}$$

$$\begin{aligned} &\mathrm{vec}^{-1}\left[(K_{p,p^2}\otimes I_p)(\tilde{V}\otimes\tilde{V})'(K_{n^2,n}\otimes I_n)\mathrm{vec}(W_\ell\otimes W_{\ell'})\right] \\ &\quad = \delta_{\ell\ell'}r(W_\ell)K_{pp}(\Sigma\otimes\Sigma), \end{aligned} \tag{3.2.29}$$

and

$$\begin{aligned} &\mathrm{vec}^{-1}\left[(I_p\otimes K_{pp}\otimes I_p)(\tilde{V}\otimes\tilde{V})'(I_n\otimes K_{nn}\otimes I_n)\mathrm{vec}(W_\ell\otimes W_{\ell'})\right] \\ &\quad = r(W_\ell)r(W_{\ell'})(\Sigma\otimes\Sigma). \end{aligned} \tag{3.2.30}$$

Substituting (3.2.28) - (3.2.30) into (3.2.27), we obtain

$$\begin{aligned} E(Q_\ell(Y)\otimes Q_{\ell'}(Y)) = 4\phi''(0)r(W_\ell)\{&\delta_{\ell\ell'}[\mathrm{vec}\Sigma(\mathrm{vec}\Sigma)' + K_{pp}(\Sigma\otimes\Sigma)] \\ &+ r(W_\ell)r(W_{\ell'})(\Sigma\otimes\Sigma)\}. \end{aligned} \tag{3.2.31}$$

Thus by (i) and (3.2.31),

$$\mathrm{Cov}(Q_\ell(Y)) = 4\phi''(0)r(W_\ell)(I_{p^2}+K_{pp})(\Sigma\otimes\Sigma) + 4[\phi''(0) - (\phi'(0))^2]\mathrm{vec}\,\Sigma(\mathrm{vec}\,\Sigma)',$$

proving (ii). Note that if $\ell \neq \ell'$, then (3.2.27) becomes

$$E(Q_\ell(Y)\otimes Q_{\ell'}(Y)) = r(W_\ell)r(W_\ell)(\Sigma\otimes\Sigma) = E(Q_\ell(Y))\otimes E(Q_{\ell'}(Y))$$

and hence (iii) follows. $\square$

3.3 Further applications

For simplicity, in this section we shall assume that $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$ with $\Sigma_Y = \sum_{j=1}^{k} A_j \otimes \Sigma_j$.

First we shall find the expectation of

$$Q_1(Y) = (Y'W_1Y)W(Y'W_2Y), \tag{3.3.1}$$

where $W_1,\ W_2 \in \mathcal{L}(E, E)$ and $W \in \mathcal{L}(V, V)$.

Proposition 3.3.1. *Suppose that $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$. Let $Q_1(Y)$ be given in (3.3.1). Then*

$$E(Q_1(Y)) = (\mu' W_1 \mu)W(\mu' W_2 \mu) - 2\phi'(0)\Box_1 + 4\phi''(0)\Box_2, \tag{3.3.2}$$

where

$$\begin{aligned}\Box_1 = \sum_{j=1}^{k} [& tr(A_jW_1)(\Sigma_j W \mu' W_2 \mu) + tr(A_jW_2)(\mu' W_1 \mu W \Sigma_j) \\ & + tr(\mu' W_2 A_j W_1 \mu W)\Sigma_j + tr(\Sigma_j W)(\mu' W_1 A_j W_2 \mu) \\ & + \mu' W_1 A_j W_2' \mu W' \Sigma_j + \Sigma_j W' \mu' W_1' A_j W_2 \mu]\end{aligned}$$

and

$$\begin{aligned}\Box_2 = \sum_{j,\ell=1}^{k} & [tr(A_jW_1A_\ell W_2)tr(W\Sigma_\ell)\Sigma_j + tr(A_jW_1A_\ell W_2')(\Sigma_\ell W' \Sigma_j) \\ & + tr(A_jW_1)tr(A_\ell W_2)(\Sigma_j W \Sigma_\ell)].\end{aligned}$$

Proof. Since

$$\text{vec}(Q_1(Y)) = \text{vec}[(Y'W_1Y)W(Y'W_2Y)][(Y'W_1Y) \otimes (Y'W_2'Y)]\text{vec}W,$$

$$\text{vec}(E(Q_1(Y)) = E[(Y'W_1Y) \otimes (Y'W_2'Y)]\text{vec}W. \tag{3.3.3}$$

By Corollary 3.2.3,

$$E[(Y'W_1Y) \otimes (Y'W_2'Y) = (\mu' W_1 \mu) \otimes (\mu' W_2' \mu) - 2\phi'(0)\nabla_1^* + 4\phi''(0)\nabla_2^*, \tag{3.3.4}$$

where ∇_1^* and ∇_2^* are given in (3.2.6) and (3.2.7) with W_2 being replaced by W_2'.
Since

$$[(\mu' W_1 \mu) \otimes (\mu' W_2' \mu)]\text{vec}W = \text{vec}[(\mu' W_1 \mu) W (\mu' W_2 \mu)], \tag{3.3.5}$$

$$\begin{aligned}
\nabla_1^* \text{vec}W = \sum_{j=1}^{k} \{ & \text{tr}(A_j W_1)(\Sigma_j \otimes \mu' W_2' \mu) + \text{tr}(A_j W_2)(\mu' W_1 \mu \otimes \Sigma_j) \\
& + \text{vec}\Sigma_j (\text{vec}(\mu' W_1' A_j W_2' \mu))' + \text{vec}(\mu' W_1 A_j W_2 \mu)(\text{vec}\Sigma_j)' \\
& + K_{p,p}(\Sigma_j \otimes (\mu' W_1 A_j W_2' \mu) + (\mu' W_2' A_j W_1 \mu) \otimes \Sigma_j)\} \text{vec}W \\
= \text{vec}\{\sum_{j=1}^{k} & [\text{tr}(A_j W_1)(\Sigma_j W \mu' W_2 \mu) + \text{tr}(A_j W_2)(\mu' W_1 \mu W \Sigma_j) \\
& + \text{tr}(\mu' W_2 A_j W_1 \mu W)\Sigma_j + \text{tr}(\Sigma_j W)(\mu' W_1 A_j W_2 \mu) \\
& + \mu' W_1 A_j W_2' \mu W' \Sigma_j + \Sigma_j W' \mu' W_1' A_j W_2 \mu]\},
\end{aligned} \tag{3.3.6}$$

and

$$\begin{aligned}
\nabla_2^* \text{vec}W = \sum_{j,\ell=1}^{k} & \{\text{tr}(A_j W_1 A_\ell W_2)\text{vec}\Sigma_j (\text{vec}\Sigma_\ell)' + \text{tr}(A_j W_1 A_\ell W_2') K_{p,p}(\Sigma_j \otimes \Sigma_\ell) \\
& + \text{tr}(A_j W_1)\text{tr}(A_\ell W_2)(\Sigma_j \otimes \Sigma_\ell)\} \text{vec}W \\
= \text{vec}\{ \sum_{j,\ell=1}^{k} & [\text{tr}(A_j W_1 A_\ell W_2)\text{tr}(W \Sigma_\ell)\Sigma_j + \text{tr}(A_j W_1 A_\ell W_2')(\Sigma_\ell W' \Sigma_j) \\
& + \text{tr}(A_j W_1)\text{tr}(A_\ell W_2)(\Sigma_j W \Sigma_\ell)]\}.
\end{aligned} \tag{3.3.7}$$

Thus (3.3.2) follows by substituting (3.3.4)-(3.3.7) into (3.3.3) and by applying vec^{-1} to both sides of (3.3.3). □

For the special case where $Y \sim N_{n\times p}(\mu, A \otimes \Sigma)$, the above result was obtained by Neudecker and Wansbeek (1987).

Example 3.3.1. *In Proposition 3.3.1, let $Y \sim N_{n\times p}(\mu, I_n \otimes \Sigma)$, $k = 1$, $A_1 = I_n$, $\Sigma_1 \equiv \Sigma$. Suppose that W_1 be idempotent of rank m and $W_1 \mu = 0$. Then*

$$E[(Y' W_1 Y) W (Y' W_1 Y)] = 4\phi''(0)\{m\, tr(\Sigma W)\Sigma + m \Sigma W' \Sigma + m^2 \Sigma W \Sigma\}.$$

Example 3.3.2. *Consider the multivariate elliptically contoured distribution given in Anderson and Fang (1982a) where* $Y \sim MEC_{n\times p}(\mu, diag(\Sigma_1, \Sigma_2, \cdots, \Sigma_n), \phi)$. *Let* $W_1 \in S_n$ *and* $W_1\mu = 0$. *Then by Proposition 3.3.1,*

$$E[(Y'W_1Y)W(Y'W_1Y)] = 4\phi''(0)\{\sum_{j,\ell=1}^{n} w_{1j\ell}^2(tr(W\Sigma_\ell)\Sigma_j + \Sigma_\ell W'\Sigma_j) + w_{1jj}w_{1\ell\ell}\Sigma_j W\Sigma_\ell\},$$

where $W_1 = (w_{1j\ell})$.

Proof. Let $\{e_j\}_{j=1}^n$ be an orthonormal basis of $\Re^n$. Then

$$\mathrm{diag}(\Sigma_1, \Sigma_2, \cdots, \Sigma_n) = \sum_{j=1}^{n}(e_j \Box e_j) \otimes \Sigma_j.$$

Thus with $k = n$ and $A_j = e_j \Box e_j (\equiv e_j e_j')$, we obtain

$$\mathrm{tr}(A_j W_1 A_\ell W_1) = \sum_{j,\ell=1}^{n} w_{1j\ell}^2, \quad \mathrm{tr}(A_j W_1) = \sum_{j=1}^{n} w_{1jj}.$$

So by Proposition 3.3.1, the desired result follows. □

Now let E_1 be the q-dimensional inner product space over $\Re$ and consider the matrix second degree polynomial

$$Q_2(Y) = D'Y'WYD + LYB + B'Y'L' + C, \tag{3.3.8}$$

where $W \in S_E$, D, $B \in \mathcal{L}(E_1, V)$, $L \in \mathcal{L}(E_1, E)$ and $C \in S_{E_1}$. Let $X = Y - \mu$. Then $X \sim MEC_{n\times p}(0, \Sigma_Y, \phi)$ and

$$\begin{aligned} Q_*(X) \equiv Q_2(Y) &= D'(X+\mu)'W(X+\mu)D + L(X+\mu)B + B'(X+\mu)'L' + C \\ &= D'XWXD + D'(X'W\mu + \mu'WX)D + LXB + B'X'L' + Q_2(\mu), \end{aligned} \tag{3.3.9}$$

where $Q_2(\mu) = D'\mu'W\mu D + L\mu B + B'\mu'L' + C$.

Proposition 3.3.2. *Suppose that* $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$ *with* $\Sigma_Y = \sum_{j=1}^{k} A_j \otimes \Sigma_j$. *Let* $Q_2(Y)$ *be given in (3.3.8). Then*

(i) $E(Q_2(Y)) = -2\phi'(0)\sum_{j=1}^{k} \mathrm{tr}(WA_j)D'\Sigma_j D + Q_2(\mu)$

and

(ii) $\mathrm{Cov}(Q_2(Y)) = 4\phi''(0)\sum_{j,\ell=1}^{k} \mathrm{tr}(WA_jWA_\ell)(I_{q^2}+K_{q,q})[(D'\Sigma_jD)\otimes(D'\Sigma_\ell D)]$

$$+4[\phi''(0)-(\phi'(0))^2]\sum_{j,\ell=1}^{k}\mathrm{tr}(WA_j)\mathrm{tr}(WA_\ell)\mathrm{vec}(D'\Sigma_jD)(\mathrm{vec}(D'\Sigma_\ell D))'$$

$$-2\phi'(0)\sum_{j=1}^{k}(I_{q^2}+K_{q,q})(L\otimes B'+(D'\mu'W)\otimes D')(A_j\otimes\Sigma_j)$$
$$\times(L'\otimes B+(W\mu D)\otimes D)(I_q+K_{q,q}).$$

Proof. By (2.3.13) and (3.3.9),

$$\begin{aligned} E(Q_2(Y)) &= D'E(X'WX)D + Q_2(\mu) \\ &= D'\mathrm{vec}^{-1}\{E(X\otimes X)'\mathrm{vec}W\}D + Q_2(\mu) \\ &= -2\phi'(0)\sum_{j=1}^{k}\mathrm{tr}(A_jW)D'\Sigma_jD + Q_2(\mu), \end{aligned}$$

proving (i). From Corollary 2.3.2, we know that the odd order moments of X are zero. Thus

$$\begin{aligned} \mathrm{Cov}(Q_2(Y)) =& \mathrm{Cov}(D'X'WXD) \\ &+ \mathrm{Cov}(D'(X'W\mu+\mu'WX)D+LXB+B'X'L'). \end{aligned} \tag{3.3.10}$$

Note that

$$\begin{aligned} &\mathrm{vec}(D'(X'W\mu+\mu'WX)D+LXB+B'X'L') \\ &= (I_{q^2}+K_{q,q,})(L\otimes B'+(D'\mu'W)\otimes D')\mathrm{vec}X. \end{aligned} \tag{3.3.11}$$

$$\begin{aligned} &\mathrm{Cov}(D'(X'W\mu+\mu'WX)D+LXB+B'X'L) = (I_{q^2}+K_{q,q}) \\ &\times[L\otimes B'+(D'\mu'W)\otimes D']\mathrm{Cov}(X)[L'\otimes B+(W\mu D)\otimes D](I_{q^2}+K_{q,q}) \\ &= -2\phi'(0)\sum_{j=1}^{k}(I_{q^2}+K_{q,q})(L\otimes B'+(D'\mu W)\otimes D')(A_j\otimes\Sigma_j) \\ &\quad (L'\otimes B+(W\mu D)\otimes D)(I_{q^2}+K_{q,q}). \end{aligned} \tag{3.3.12}$$

Let $X_\bullet = XD$. Then $X_\bullet \sim MEC_{n\times q}(0, \sum_{j=1}^k A_j \otimes \Sigma_{\bullet j}, \phi)$ with $\Sigma_{\bullet j} = D'\Sigma_j D$. Thus by Corolllary 3.2.3(ii),

$$
\begin{aligned}
\mathrm{Cov}(D'X'WXD) &= \mathrm{Cov}(X_\bullet' W X_\bullet) \\
&= 4\phi''(0) \sum_{j,\ell=1}^{k} \mathrm{tr}(WA_jWA_\ell)(I_{p^2} + K_{p,p})(\Sigma_{\bullet j} \otimes \Sigma_{\bullet \ell}) \\
&\quad + 4[\phi''(0) - (\phi'(0))^2] \sum_{j,\ell=1}^{k} \mathrm{tr}\,(WA_j)\mathrm{tr}\,(WA_\ell)\mathrm{vec}\,\Sigma_{\bullet j}(\mathrm{vec}\,\Sigma_{\bullet \ell})' \\
&= 4\phi''(0) \sum_{j,\ell=1}^{k} \mathrm{tr}(WA_jWA_\ell)(I_{p^2} + K_{p,p})(D'\Sigma_j D) \otimes (D'\Sigma_\ell D) \\
&\quad + 4[\phi''(0) - (\phi'(0))^2] \sum_{j,\ell=1}^{k} \mathrm{tr}\,(WA_j)\mathrm{tr}\,(WA_\ell)\mathrm{vec}\,(D'\Sigma_j D)(\mathrm{vec}\,(D'\Sigma_\ell D))'.
\end{aligned}
\tag{3.3.13}
$$

Substituting (3.3.12) and (3.3.13) into (3.3.10), we obtain (ii). □

Jinadasa (1986) and Browne and Neudecker (1988) obtained the above theorem for the case where $k = 1$ and $Y \sim N_{n\times p}(\mu, A \otimes \Sigma)$.

Example 3.3.3. *Suppose that $y \sim EC_n(\mu, A, \phi)$. Let $Q_2(y) = y'Wy + 2b'y + c$ with $W = W' \in M_{n\times n}$, $b \in \Re^n$ and $c \in \Re$. Then from Proposition 3.3.2, we obtain*

$$E(Q_2(y)) = -2\phi'(0)tr(WA) + \mu'W\mu + 2b'\mu + c,$$

and

$$
\begin{aligned}
Cov(Q_2(y)) &= 8\phi''(0)tr(AWAW) + 4[\phi''(0) - (\phi'(0))^2](tr(AW))^2 \\
&\quad - 8\phi'(0)\,[(\mu'W + b')A(W\mu + b)]\,.
\end{aligned}
$$

PART II

COCHRAN THEOREMS

CHAPTER FOUR

MULTIVARIATE VERSIONS OF COCHRAN'S THEOREMS

4.1. Introduction

As mentioned in Section 1.4, it is well-known that Cochran theorems play an important role in regression analysis and analysis of variance. In this chapter, we shall discuss the moment generating functions, the independence, and the distributions of quadratic forms of a normally distributed random operator Y in $\mathcal{L}(V, E)$ and give several multivariate versions of Cochran's theorems for the normal setting.

In section 4.2, we shall obtain an expression for the joint moment generating function of quadratic functions $\{ Q_i(Y) \}_{i=1}^{\ell}$ with $Q_i(Y) = Y'W_iY + B_i'Y + Y'C_i + D_i$ and W_i's symmetric, and list some special cases for later use.

By using the formula given in Section 4.2, we shall, in Section 4.3, obtain a necessary and sufficient condtion under which $\{ Q_i(Y) \}$ is an independent family of random operators $Q_i(Y)$. This result generalizes the corresponding result of Khatri (1980).

In Section 4.4, we shall obtain a necessary and sufficient condition under which $\{ Q_i(Y) \}$ is an independent family of Wishart $W_p(m_i, \Sigma, \lambda_i)$ random operators $Q_i(Y)$. This result generalizes the corresponding results of DeGunst (1987) and Khatri (1980).

In Section 4.5, we shall extend the Cochran theorem given in Section 4.4 to the case where $W_p(m_i, \Sigma, \lambda_i)$ is replaced by $DW_p(m_{1i}, m_{2i}, \Sigma, \lambda_{1i}, \lambda_{2i})$. This generalizes the corresponding results of Tan (1975, 1976) and Wong (1992).

We shall, in the last section, obtain a more applicable Cochran theorem for the case where $\mu = 0$ and $Im\,\Sigma_Y = S_1 \Box S_2 (\neq 0)$ with S_1 and S_2 being linear subspaces of E and V respectively. For practical use, we shall give some conditions that imply (1.4.6) but not vice versa.

4.2. Moment generating functions of quadratic forms

Let $T \in \mathcal{L}(V,V)$ and $\alpha > 0$. We shall use T^+ to denote the Moore-Penrose inverse of T, use T^α to denote the αth nonnegative definite (n.n.d.) root of T, use $T^{-\alpha}$ to denote the αth n.n.d. root of T^+ and use T^0 to denote T^+T. For $T \in \mathcal{L}(V,V)$, $r_\sigma(T)$ will denote the spectral radius of T, i.e. $r_\sigma(T) = \max\{|\nu| : \nu \text{ is an eigenvalue of } T\}$.

Theorem 4.2.1. *Let Y be a $N_{n\times p}(\mu, \Sigma_Y)$ random operator of a probability space $(\Omega, \mathcal{A}, P)$ into $\mathcal{L}(V,E)$, $i \in \{1,2,\ldots,\ell\}$, $W_i \in \mathcal{S}_E$, B_i, $C_i \in \mathcal{L}(V,E)$, $D_i \in \mathcal{L}(V,V)$, $y \in \mathcal{L}(V,E)$ and*

$$Q_i(y) = y'W_iy + B_i'y + y'C_i + D_i. \tag{4.2.1}$$

the joint moment generating function, $M_{Q(Y)}$, of $Q(Y) \equiv (Q_1(Y), Q_2(Y), \cdots, Q_\ell(Y))$ is given by

$$M_{Q(Y)}(T) = |I_{np} - 2\Psi|^{-\frac{1}{2}} exp\{\langle T,\, Q(\mu)\rangle + \langle L, [\Sigma_Y^{\frac{1}{2}}(I-2\Psi)^{-1}\Sigma_Y^{\frac{1}{2}}](L)\rangle/2\}, \tag{4.2.2}$$

where

$$T = (T_1, T_2, \cdots, T_\ell), \quad T_i^\circ = (T_i + T_i')/2, \quad T_i \in \mathcal{L}(V,V),$$

$$\Psi_i = \Sigma_Y^{\frac{1}{2}}(W_i \otimes T_i^\circ)\Sigma_Y^{\frac{1}{2}}, \quad \Psi = \sum_{i=1}^{\ell} \Psi_i, \quad r_\sigma(\Psi) < 1/2,$$

and

$$L_i = B_iT_i + C_iT_i' + 2W_i\mu T_i^\circ, \quad L = \sum_{i=1}^{\ell} L_i.$$

Proof. Let $Z \sim N_{n\times p}(0, I_{np})$, where I_{np} is the identity map on $\mathcal{L}(E,E) \otimes \mathcal{L}(V,V)$. Let $Y_* = \mu + \Sigma_Y^{\frac{1}{2}}(Z)$. Then Y and Y_* have the same distribution. So we may assume that $Y = Y_* = \mu + \Sigma_Y^{\frac{1}{2}}(Z)$. Let $i \in \{1, 2, \ldots, \ell\}$, $z \in \mathcal{L}(V, E)$, and $Q_i^*(z) = Q_i(y)$. Then

$$\begin{aligned} Q_i^*(z) &= (\mu + \Sigma_Y^{\frac{1}{2}}(z))' W_i (\mu + \Sigma_Y^{\frac{1}{2}}(z)) + B_i'(\mu + \Sigma_Y^{\frac{1}{2}}(z)) \\ &\quad + (\mu + \Sigma_Y^{\frac{1}{2}}(z))' C_i + D_i \\ &= (\Sigma_Y^{\frac{1}{2}}(z))' W_i (\Sigma_Y^{\frac{1}{2}}(z)) + (B_i' + \mu' W_i)(\Sigma_Y^{\frac{1}{2}}(z)) \\ &\quad + (\Sigma_Y^{\frac{1}{2}}(z))'(W_i \mu + C_i) + Q_i(\mu) \end{aligned}$$

and therefore

$$\begin{aligned} \langle T_i, Q_i^*(z)\rangle &= tr(T_i' Q_i^*(z)) = \langle T_i, (\Sigma_Y^{\frac{1}{2}}(z))' W_i (\Sigma_Y^{\frac{1}{2}}(z))\rangle \\ &\quad + \langle T_i, (B_i' + \mu' W_i)(\Sigma_Y^{\frac{1}{2}}(z)) + (\Sigma_Y^{\frac{1}{2}}(z))'(W_i \mu + C_i)\rangle + \langle T_i, Q_i(\mu)\rangle \\ &= \langle z, \Psi_i(z)\rangle + \langle z, \Sigma_Y^{\frac{1}{2}}(L_i)\rangle + \langle T_i, Q_i(\mu)\rangle. \end{aligned} \tag{4.2.3}$$

Now

$$\begin{aligned} \langle T, Q(y)\rangle &= \sum_{i=1}^{\ell} \langle T_i, Q_i(y)\rangle = \sum_{i=1}^{\ell} \langle T_i, Q_i^*(z)\rangle \\ &= \langle z, \Psi(z)\rangle + \langle z, \Sigma_Y^{\frac{1}{2}}(L)\rangle + \langle T, Q(\mu)\rangle. \end{aligned}$$

Since $Z \sim N_{n \times p}(0, I_{np})$, the probability density function of Z with respect to the Lebesgue measure for $\mathcal{L}(V, E)$ is f with

$$f(z) = (2\pi)^{-pn/2} \exp\{-\langle z, z\rangle/2\}, \qquad z \in \mathcal{L}(V, E).$$

So

$$\begin{aligned} M_{Q(Y)}(T) &\equiv E(exp\{\langle T, (Q_i(Y))\rangle\}) \\ &= \int_{\mathcal{L}(V,E)} exp\{\langle T, Q(\mu)\rangle + \langle z, \Psi(z)\rangle + \langle z, \Sigma_Y^{\frac{1}{2}}(L)\rangle\} f(z) dz, \end{aligned}$$

Through factorization,

$$M(T) = |I - 2\Psi|^{-\frac{1}{2}} e^{\langle T, Q(\mu)\rangle} M_X(\Sigma_Y^{\frac{1}{2}}(L)),$$

where $X \sim N(0, (I-2\Psi)^{-1})$, requiring that $I_{np} - 2\Psi$ is positive definite. Hence for $r_\sigma(\Psi) < 1/2$,

$$M_{Q(Y)}(T) = |I_{np} - 2\Psi|^{-\frac{1}{2}} exp\{\langle T, Q(\mu)\rangle + \langle L, [\Sigma_Y^{\frac{1}{2}}(I-2\Psi)^{-1}\Sigma_Y^{\frac{1}{2}}](L)\rangle/2\}.$$

□

Corollary 4.2.2. *In Theorem 4.2.1, let* $\ell = 1$ *and* $Q(y) = y'Wy + B'y + y'C + D$ *with* $W \in S_E$, $B, C \in \mathcal{L}(V, E)$ *and* $D \in \mathcal{L}(V, V)$. *Then the moment generating function (mgf) of* $Q(Y)$ *is*

$$M_{Q(Y)}(T) = |I_{np} - 2\Psi|^{-\frac{1}{2}} exp\{\langle T, Q(\mu)\rangle + \langle L, \Sigma_Y^{\frac{1}{2}}(I_{np}-2\Psi)^{-1}\Sigma_Y^{\frac{1}{2}}(L)\rangle/2\},$$
$$r_\sigma(\Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}}) < \frac{1}{2}, \qquad (4.2.4)$$

where $T \in \mathcal{L}(V, V)$, $T^\circ = (T + T')/2$,

$$\Psi = \Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}}, \qquad (4.2.5)$$

and

$$L = BT + CT' + 2W\mu T^\circ. \qquad (4.2.6)$$

Khatri (1980) obtained Corollary 4.2.2 for the case where $\Sigma_Y = A \otimes \Sigma$. Note that due to the self-adjoint property of $W \otimes T^\circ$ in (4.2.4),

$$|I_{np} - \Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}}| = |I_{np} - \Sigma_Y(W \otimes T^\circ)| = |I_{np} - (W \otimes T^\circ)\Sigma_Y| \qquad (4.2.7)$$

and

$$\Sigma_Y^{\frac{1}{2}}(I_{np} - \Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}})^{-1}\Sigma_Y^{\frac{1}{2}} = \Sigma_Y(I_{np} - (W \otimes T^\circ)\Sigma_Y)^{-1}$$
$$= (I_{np} - \Sigma_Y(W \otimes T^\circ))^{-1}\Sigma_Y. \qquad (4.2.8)$$

So we can express $M_{Q(Y)}(T)$ in (4.2.4) without involving $\Sigma_Y^{\frac{1}{2}}$. But in theory, it is more convenient to use (4.2.4) because $(W \otimes T^\circ)\Sigma_Y$, or $\Sigma_Y(W \otimes T^\circ)$, may not even be diagonalizable; see Wong (1982).

Corollary 4.2.3. *In Corollary 4.2.2, if $W = I_n$, $B = C = 0$, $D = 0$, and $\Sigma_Y = I_n \otimes \Sigma$, then $Q(Y) = Y'Y \sim W_p(n, \Sigma, \lambda)$, where $\lambda = \mu'\mu$. Thus by (4.2.4), the mgf of $W_p(n, \Sigma, \lambda)$ is given by*

$$M_W(T) = |I_p - 2\Sigma^{\frac{1}{2}}T\Sigma^{\frac{1}{2}}|^{-\frac{n}{2}} exp\{\langle T, \lambda\rangle + 2\langle T\lambda T, \Sigma^{\frac{1}{2}}(I_p - 2\Sigma^{\frac{1}{2}}T\Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}}\rangle\},$$
$$T \in \mathcal{S}_V, \quad r_\sigma(\Sigma^{\frac{1}{2}}T\Sigma^{\frac{1}{2}}) < \frac{1}{2}. \quad (4.2.9)$$

Example 4.2.1. *Let $y \sim N_n(\mu, A)$ and let $W \in \mathcal{S}_E$, $b \in \Re^n$, $d \in \Re$ and $q(y) = y'Wy + 2b'y + d$. Then by Corollary 4.2.2, the mgf of $q(y)$ is given by*

$$M_q(t) = |I_n - 2tA^{\frac{1}{2}}WA^{\frac{1}{2}}|^{-\frac{1}{2}} exp\{t(\mu'W\mu + 2b'\mu + d)$$
$$+ 2t^2(W\mu + b)'A^{\frac{1}{2}}(I_n - 2tA^{\frac{1}{2}}WA^{\frac{1}{2}})^{-1}A^{\frac{1}{2}}(W\mu + b)\},$$

where $t \in \Re$ and $r_\sigma(tA^{\frac{1}{2}}WA^{\frac{1}{2}}) < \frac{1}{2}$. Moreover if $W = A = I_n$, $b = 0$ and $d = 0$, then M_q is the mgf of $\chi^2_n(\lambda)$:

$$M_q(t) = (1 - 2t)^{\frac{n}{2}} exp\{(t + \frac{2t^2}{1 - 2t})\lambda\},$$

where $\lambda = \mu'\mu$ and $t < 1/2$.

4.3. Independence of quadratic forms

Suppose that $Y \sim N_{n\times p}(\mu, \Sigma_Y)$. Consider the second degree polynomial operators $Q_1(Y), Q_2(Y), \cdots, Q_\ell(Y)$, where $Q_i(Y)$ are given in (4.2.1). For establishing the independence of $Q_i(Y)$'s, we need the following lemmas:

Lemma 4.3.1. *(Laha (1956)). Suppose that $x \sim N_m(0, I_m)$. Let $H_1, H_2 \in \mathcal{S}_m$, $h_1, h_2 \in \Re^m$, $q_1(x) = x'H_1x + h_1'x$ and $q_2(x) = x'H_2x + h_2'x$. Then $q_1(x)$ and $q_2(x)$ are independent if and only if*

$$(i)\ H_1H_2 = 0, \quad (ii)\ H_1h_2 = 0, \quad (iii)\ H_2h_1 = 0, \quad (iv)\ h_1'h_2 = 0.$$

Lemma 4.3.2. *(Craig (1943)). Let $A, B \in S_m$. Then $AB = 0$ if and only if*

$$|I_m - sA|\,|I_m - tB| = |I_m - sA - tB|$$

holds for all s and t in $\Re$.

Lemma 4.3.3. *Let P, Q, R, and S be polynomials in s and t with rational coefficients such that*

$$P(s,t)/Q(s,t) = exp\{R(s,t)/S(s,t)\}$$

holds for all $s, t \in \Re$ where $P(0,0)/Q(0,0) = 1$ and $R(0,0)/S(0,0) = 0$. Then $P(s,t) = Q(s,t)$ and $R(s,t) = S(s,t)$.

This lemma can be found in Laha (1956) and Dricoll and Gundberg (1986).

Theorem 4.3.4. *Suppose that $Y \sim N_{n\times p}(\mu, \Sigma_Y)$. Let $i \in \{1, 2, \cdots, \ell\}$ and $Q_i(Y)$ be given in (4.2.1). Then $\{Q_i(Y)\}_{i=1}^{\ell}$ is an independent family of operators $Q_i(Y)$ if and only if for any distinct $i, j \in \{1, 2, \cdots, \ell\}$ and any $T_i, T_j \in \mathcal{L}(V, V)$,*

(a) $\Sigma_Y(W_i \otimes T_i^\circ)\Sigma_Y(W_j \otimes T_j^\circ)\Sigma_Y = 0$,

(b) $\Sigma_Y(W_i \otimes T_i^\circ)\Sigma_Y(L_j) = 0$,

and

(c) $\langle L_i, \Sigma_Y(L_j)\rangle = 0$,

where $T_i^\circ = (T_i + T_i')/2$ and $L_i = B_iT_i + C_iT_i' + 2W_i\mu T_i^\circ$. Hence $\{Q_i(Y)\}$ is independent if and only if $\{Q_i(Y)\}$ is pairwise independent.

Proof. Suppose that $\{Q_i(Y)\}$ is independent. Let $i, j \in \{1, 2, \cdots, \ell\}$ with $i \neq j$. Then $Q_i(Y)$ and $Q_j(Y)$ are independent. Let $Q(Y) = (Q_i(Y), Q_j(Y))$ and $T = (T_i, T_j)$. Recall that $Q_i(Y)$ and $Q_j(Y)$ are independent if and only if

$$M_{Q(Y)}(T) = M_{Q_i(Y)}(T_i)\, M_{Q_j(Y)}(T_j). \tag{4.3.1}$$

By Theorem 4.2.1, (4.3.1) is reduced to

$$\begin{aligned} &|I_{np} - 2\Psi|^{-\frac{1}{2}} exp\{\langle L, [\Sigma_Y^{\frac{1}{2}}(I_{np} - 2\Psi)^{-1}\Sigma_Y^{\frac{1}{2}}](L)\rangle\} \\ &= |I_{np} - 2\Psi_i|^{-\frac{1}{2}} exp\{\langle L_i, [\Sigma_Y^{\frac{1}{2}}(I_{np} - 2\Psi_i)^{-1}\Sigma_Y^{\frac{1}{2}}](L_i)\rangle\} \\ &\quad + |I_{np} - 2\Psi_j|^{-\frac{1}{2}} exp\{\langle L_j, [\Sigma_Y^{\frac{1}{2}}(I_{np} - 2\Psi_j)^{-1}\Sigma_Y^{\frac{1}{2}}](L_j)\rangle\}, \\ &\quad r_\sigma(\Psi) < 1/2, \quad r_\sigma(\Psi_i) < 1/2, \quad r_\sigma(\Psi_j) < 1/2, \end{aligned} \tag{4.3.2}$$

where $\Psi_i = \Sigma_Y^{\frac{1}{2}}(W_i \otimes T_i^o)\Sigma_Y^{\frac{1}{2}}$, $\Psi = \Psi_i + \Psi_j$ and $L = L_i + L_j$. By Lemma 4.3.3, (4.3.2) implies that

$$|I_{np} - 2(\Psi_i + \Psi_j)| = |I_{np} - 2\Psi_i||I_{np} - 2\Psi_j| \tag{4.3.3}$$

and

$$\begin{aligned} &\langle L_i + L_j, [\Sigma_Y^{\frac{1}{2}}[I_{np} - 2(\Psi_i + \Psi_j)]^{-1}\Sigma_Y^{\frac{1}{2}}](L_i + L_j)\rangle \\ &= \langle L_i, [\Sigma_Y^{\frac{1}{2}}(I_{np} - 2\Psi_i)^{-1}\Sigma_Y^{\frac{1}{2}}](L_i)\rangle + \langle L_j, [\Sigma_Y^{\frac{1}{2}}(I_{np} - 2\Psi_j)^{-1}\Sigma_Y^{\frac{1}{2}}](L_j)\rangle. \end{aligned} \tag{4.3.4}$$

Let c_i and c_j be the real values in the neighborhood of the origin such that

$$c_i < 1/r_\sigma(\Psi_i), \quad c_j < 1/r_\sigma(\Psi_j). \tag{4.3.5}$$

Replacing $2T_i$ and $2T_j$ in (4.3.3) and (4.3.4) by c_iT_i and c_jT_j respectively, we obtain

$$|I_{np} - c_i\Psi_i - c_j\Psi_j| = |I_{np} - c_i\Psi_i||I_{np} - c_j\Psi_j| \tag{4.3.6}$$

and

$$\begin{aligned} &\langle c_iL_i + c_jL_j, [\Sigma_Y^{\frac{1}{2}}(I_{np} - c_i\Psi_i - c_j\Psi_j)^{-1}\Sigma_Y^{\frac{1}{2}}](c_iL_i + c_jL_j)\rangle \\ &= \langle c_iL_i, [\Sigma_Y^{\frac{1}{2}}(I_{np} - c_i\Psi_i)^{-1}\Sigma_Y^{\frac{1}{2}}](c_iL_i)\rangle \\ &\quad + \langle c_jL_j, [\Sigma_Y^{\frac{1}{2}}(I_{np} - c_j\Psi_j)^{-1}\Sigma_Y^{\frac{1}{2}}](c_jL_j)\rangle. \end{aligned} \tag{4.3.7}$$

Note that Ψ_i and Ψ_j are self-adjoint operators in $\mathcal{L}(E, E) \otimes \mathcal{L}(V, V)$ and (4.3.6) holds for all c_i and c_j in $\Re$ with restriction (4.3.5). By Lemma 4.3.2, (4.3.6) implies that

$$\Psi_i\Psi_j = 0. \tag{4.3.8}$$

i.e.

$$\Sigma_Y^{\frac{1}{2}}(W_i \otimes T_i^\circ)\Sigma_Y(W_j \otimes T_j^\circ)\Sigma_Y^{\frac{1}{2}} = 0,$$

proving (a). By (4.3.5) and (4.3.8),

$$\begin{aligned}(I_{np} - c_i\Psi_i - c_j\Psi_j)^{-1} &= (I_{np} - c_i\Psi_i)^{-1}(I_{np} - c_j\Psi_j)^{-1} \\ &= \left(\sum_{k=0}^{\infty}(c_i\Psi_i)^k\right)\left(\sum_{k=0}^{\infty}(c_j\Psi_j)^k\right) \\ &= I_{np} + \sum_{k=1}^{\infty}(c_i\Psi_i)^k + \sum_{k=1}^{\infty}(c_j\Psi_j)^k \\ &= (I_{np} - c_i\Psi_i)^{-1} + (I_{np} - c_j\Psi_j)^{-1} - I_{np}.\end{aligned} \tag{4.3.9}$$

Substituting (4.3.9) into (4.3.7) and denoting $\Sigma_Y^{\frac{1}{2}}(L_i)$ by L_i^*, we obtain, upon simplification,

$$\begin{aligned}&c_j^2\langle L_i^*, (I_{np} - c_i\Psi_i)^{-1}(L_j^*)\rangle + 2c_ic_j\langle L_i^*, (I_{np} - c_i\Psi_i)^{-1}(L_j^*)\rangle \\ &+ c_i^2\langle L_i^*, (I_{np} - c_j\Psi_j)^{-1}(L_i^*)\rangle + 2c_ic_j\langle L_i^*, (I_{np} - c_j\Psi_j)^{-1}(L_j^*)\rangle \\ &- c_i^2\langle L_i^*, L_i^*\rangle - c_j^2\langle L_j^*, L_j^*\rangle - 2c_ic_j\langle L_i^*, L_j^*\rangle \\ &= 0.\end{aligned} \tag{4.3.10}$$

By (4.3.5), we have the power series expansions:

$$(I_{np} - c_i\Psi_i)^{-1} = I_{np} + c_i\Psi_i + c_i^2\Psi_i^2 + \cdots \tag{4.3.11}$$

and

$$(I_{np} - c_j\Psi_j)^{-1} = I_{np} + c_j\Psi_j + c_j^2\Psi_j^2 + \cdots. \tag{4.3.12}$$

Substituting the expressions on the right-hand sides of (4.3.11) and (4.3.12) into (4.3.10) and collecting the coefficients of c_ic_j and $c_i^2c_j^2$, we obtain

$$\langle L_i^*, L_j^*\rangle = 0 \tag{4.3.13}$$

and

$$\langle L_i^*, \Psi_j\Psi_j(L_i^*)\rangle + \langle L_j^*, \Psi_i\Psi_i(L_j^*)\rangle = 0. \tag{4.3.14}$$

Note that

$$\langle L_i^*, L_j^*\rangle = \langle \Sigma_Y^{\frac{1}{2}}(L_i), \Sigma_Y^{\frac{1}{2}}(L_j)\rangle = \langle L_i, \Sigma_Y(L_j)\rangle \tag{4.3.15}$$

and

$$\langle L_i^*, \Psi_j\Psi_j(L_i^*)\rangle = \langle \Psi_i(L_j^*), \Psi_i(L_j^*)\rangle \geq 0. \tag{4.3.16}$$

Thus (c) follows from (4.3.13) and (4.3.15), and (b) follows from (4.3.14) and (4.3.16).

Now suppose that (a) - (c) hold. It suffices to show that

$$M_{Q(Y)}(T) = \prod_{i=1}^{\ell} M_{Q_i(Y)}(T_i)$$

for all $T = (T_1, T_2, \cdots, T_\ell)$ in N_0, where N_0 is a neighborhood of 0 in $\prod^\ell \mathcal{L}(V, V)$. By Theorem 4.2.1, $\{Q_i(Y)\}$ is independent if

(i) $|I_{np} - 2\Psi| = \prod_{i=1}^{\ell} |I_{np} - 2\Psi_i|$

and

(ii) $\langle L, [\Sigma_Y^{\frac{1}{2}}(I_{np} - 2\Psi)^{-1}\Sigma_Y^{\frac{1}{2}}](L)\rangle = \sum_{i=1}^{\ell} \langle L_i, [\Sigma_Y^{\frac{1}{2}}(I_{np} - 2\Psi_i)^{-1}\Sigma_Y^{\frac{1}{2}}](L_i)\rangle$.

where $\Psi_i = \Sigma_Y^{\frac{1}{2}}(W_i \odot T_i^\circ)\Sigma_Y^{\frac{1}{2}}$, $L_i = B_iT_i + C_iT_i' + 2W_i\mu T_i^\circ$, $\Psi = \sum_{i=1}^{\ell} \Psi_i$ and $L = \sum_{i=1}^{\ell} L_i$. By (a), $\Psi_i\Psi_j = 0$ for all distinct $i, j \in \{1, 2, \cdots, \ell\}$. So by Lemma 4.3.2, (i) follows. For the same reason, we have

$$(I_{np} - 2\Psi)^{-1} = \prod_{i=1}^{\ell} (I - 2\Psi_i)^{-1}.$$

Since $L_i = B_iT_i + C_iT_i' + 2W_i\mu T_i^\circ$, for (ii), it suffices to show that with

$$\Delta_{ij} \equiv \langle (L_i, [\Sigma_Y^{\frac{1}{2}}(I - 2\Psi)^{-1}\Sigma_Y^{\frac{1}{2}}(L_j)\rangle,$$

$\Delta_{ij} = 0$ for $i \neq j$ and

$$\Delta_{ii} = \langle L_i, [\Sigma_Y^{\frac{1}{2}}(I - 2\Psi_i)^{-1}\Sigma_Y^{\frac{1}{2}}](L_i)\rangle.$$

Since $\Psi_i\Psi_j = 0$ for $i \neq j$,

$$(I_{np} - 2\Psi)^{-1} = \prod_{s=1}^{\ell}(I - 2\Psi_s)^{-1} = \prod_{s=1}^{\ell}\sum_{k=0}^{\infty}(2\Psi_s)^k$$
$$= I_{np} + \sum_{s=1}^{\ell}\sum_{k=1}^{\infty}(2\Psi_s)^k.$$

Since each $\Psi_s = \Sigma_Y^{\frac{1}{2}}(W_s \otimes T_s^\circ)\Sigma_Y^{\frac{1}{2}}$, by (b),

$$\Delta_{ij} = \langle L_i,\, [\Sigma_Y^{\frac{1}{2}}(I_{np} + \sum_{s=1}^{\ell}\sum_{k=1}^{\infty}(2\Psi_s)^k)\Sigma_Y^{\frac{1}{2}}](L_j)\rangle$$
$$= \langle L_i,\, \Sigma_Y(L_j)\rangle + \langle L_i,\, [\sum_{s=1}^{\ell}\sum_{k=1}^{\infty}[(2\Sigma_Y(W_s \otimes T_s^\circ)^k)\Sigma_Y](L_j)\rangle \qquad (4.3.17)$$
$$= \langle L_i,\, \Sigma_Y(L_j)\rangle + \langle L_i,\, [\sum_{k=1}^{\infty}[(2\Sigma_Y(W_j \otimes T_j^\circ)^k)\Sigma_Y](L_j)\rangle.$$

By (b) again, for distinct i, j,

$$\Delta_{ij} = \langle L_i,\, \Sigma_Y(L_j)\rangle.$$

So by (c), $\Delta_{ij} = 0$ for $i \neq j$. Now by (4.3.17),

$$\Delta_{ii} = \langle L_i,\, \Sigma_Y(L_i)\rangle + \langle L_i,\, [\sum_{k=1}^{\infty}[(2\Sigma_Y(W_i \otimes T_i^\circ)^k)\Sigma_Y](L_i)\rangle$$
$$= \langle L_i,\, [\Sigma_Y^{\frac{1}{2}}\sum_{k=0}^{\infty}(2\Psi_i)^k\Sigma_Y^{\frac{1}{2}}](L_i)\rangle$$
$$= \langle L_i,\, [\Sigma_Y^{\frac{1}{2}}(I_{np} - \Psi_i)^{-1}\Sigma_Y^{\frac{1}{2}}](L_i)\rangle.$$

□

Now we are going to give an alternative proof of 'only if part' of Theorem 4.3.4: Let $i, j \in \{1, 2, \cdots, \ell\}$ with $i \neq j$ and T_i, $T_j \in \mathcal{L}(V, V)$. Since $\{Q_i(Y)\}$ is independent, $\mathrm{tr}(T_i'Q_i(Y))$ and $\mathrm{tr}(T_j'Q_j(Y))$ are independent. Similarly, as in the proof of Theorem 4.2.1, let $Z \sim N_{n\times p}(0, I_n \otimes I_p)$. Then Y and $\mu + \Sigma_Y^{\frac{1}{2}}(Z)$ have the same distribution. Since Q_i is a Borel function, $Q_i(Y)$ and $Q_i(\mu + \Sigma_Y^{\frac{1}{2}}(Z))$ have the same distribution. Thus $\mathrm{tr}[T_i' Q_i(\mu + \Sigma_Y^{\frac{1}{2}}(Z))]$ and $\mathrm{tr}[T_j' Q_i(\mu + \Sigma_Y^{\frac{1}{2}}(Z))]$ are independent. By (4.2.3),

$$\begin{aligned} q_i \equiv \mathrm{tr}[T_i' Q_i(\mu + \Sigma_Y^{\frac{1}{2}}(Z))] &= \langle T_i, Q_i(\mu + \Sigma_Y^{\frac{1}{2}}(Z))\rangle \\ &= \langle Z, \Psi_i(Z)\rangle + \langle Z, \Sigma_Y^{\frac{1}{2}}(L_i)\rangle + \langle T_i, Q_i(\mu)\rangle \qquad (4.3.18) \\ &= (\mathrm{vec}Z)'\Psi_i \mathrm{vec}Z + (\mathrm{vec}(\Sigma_Y^{\frac{1}{2}}(L_i))'\mathrm{vec}Z + \langle T_i, Q_i(\mu)\rangle, \end{aligned}$$

where $\Psi_i = \Sigma_Y^{\frac{1}{2}}(W_i \otimes T_i^\circ)\Sigma_Y^{\frac{1}{2}}$. Since $\mathrm{vec}Z \sim N_{np}(0, I_{np})$, by Lemma 4.3.1, q_i and q_j are independent, which implies that

$$\text{(i) } \Psi_i\Psi_j = 0, \qquad \text{(ii) } \Psi_i \mathrm{vec}(\Sigma_Y^{\frac{1}{2}}(L_j)) = 0,$$

$$\text{(iii) } \Psi_j \mathrm{vec}(\Sigma_Y^{\frac{1}{2}}(L_i)) = 0, \qquad \text{(iv) } [\mathrm{vec}(\Sigma_Y^{\frac{1}{2}}(L_i))]'\mathrm{vec}(\Sigma_Y^{\frac{1}{2}}(L_j)) = 0.$$

Now (a) follows from (i). Note that

$$\Psi_i \mathrm{vec}(\Sigma_Y^{\frac{1}{2}}(L_j)) = \mathrm{vec}[\Sigma_Y^{\frac{1}{2}}(W_i \otimes T_i^\circ)\Sigma_Y(L_j)] = 0. \qquad (4.3.19)$$

By (ii), (iii) and (4.3.19),

$$\Psi_j \mathrm{vec}(\Sigma_Y^{\frac{1}{2}}(L_i)) = \mathrm{vec}[\Sigma_Y^{\frac{1}{2}}(W_j \otimes T_j^\circ)\Sigma_Y(L_i)] = 0,$$

i.e.

$$\Sigma_Y^{\frac{1}{2}}(W_i \otimes T_i^\circ)\Sigma_Y(L_j) = 0 \quad \text{for } i \neq j,$$

proving (b). Similarly from (iv),

$$[\mathrm{vec}(\Sigma_Y^{\frac{1}{2}}(L_i))]'\mathrm{vec}(\Sigma_Y^{\frac{1}{2}}(L_j)) = \langle \Sigma_Y^{\frac{1}{2}}(L_i), \Sigma_Y^{\frac{1}{2}}(L_j)\rangle = \langle L_i, \Sigma_Y(L_j)\rangle = 0,$$

proving (c). □

Corollary 4.3.5. *In Theorem 4.3.4, if $B = C = 0$ and $D = 0$. Then $\{Y'W_iY\}$ is independent if and only if for any distinct $i, j \in \{1, 2, \cdots, \ell\}$ and any $T_i, T_j \in S_V$,*

(a1) $\Sigma_Y(W_i \otimes T_i)\Sigma_Y Y(W_j \otimes T_j)\Sigma_Y = 0.$

(b1) $[\Sigma_Y(W_i \otimes T_i)\Sigma_Y(W_j \otimes T_j)](\mu) = 0.$

and

(c1) $\langle \mu, (W_i \otimes T_i)\Sigma_Y(W_j \otimes T_j)(\mu)\rangle = 0.$

Hence $\{Y'W_iY\}$ is independent if and only if $\{Y'W_iY\}$ is pairwise independent.

Corollary 4.3.6. *In Corollary 4.3.5, if each W_i is n.n.d., then $\{Y'W_iY\}$ is independent if and only if for any distinct $i, j \in \{1, 2, \cdots, \ell\}$,*

$$(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) = 0. \tag{4.3.20}$$

Proof. By Corollary 4.3.5, it suffices to show that (a1) - (c1) are equivalent to (4.3.20). Suppose that (a1) - (c1) hold. Let $T_i = T_j = I_p$. Then (a1) is reduced to

$$\Sigma_Y(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p)\Sigma_Y = 0,$$

which is equivalent to

$$(W_i^{\frac{1}{2}} \otimes I_p)\Sigma_Y(W_j^{\frac{1}{2}} \otimes I_p) = 0.$$

So (4.3.20) follows.

The 'if part' is obvious. □

Corollary 4.3.7. *In Corollary 4.3.5, suppose that $\mu = 0$. Then $\{Y'W_iY\}$ is independent if and only if for any distinct $i, j \in \{1, 2, \cdots, \ell\}$, $\Sigma_Y(W_i \otimes T_i)\Sigma_Y(W_j \otimes T_j)\Sigma_Y = 0$ for all $T_i, T_j \in S_V$. Hence if $\mu = 0$ and $\Sigma_Y = A \otimes \Sigma$ for some $A \in \mathcal{N}_E$ and $\Sigma \in \mathcal{N}_V$ with $\Sigma \neq 0$, then $\{Y'W_iY\}$ is independent if and only if $AW_iAW_jA = 0$ for all distinct $i, j \in \{1, 2, \cdots, \ell\}$.*

Example 4.3.1. *Let $y \sim N_n(\mu, A)$ and $q_i(y) = y'W_iy + 2b_i'y + d_i$, $i = 1, 2$, where W_1, $W_2 \in S_E$. Then $q_1(y)$ and $q_2(y)$ are independent if and only if*

(i) $AW_1AW_2A = 0$,

(ii) $AW_2A(W_1\mu + b_1) = AW_1A(W_2\mu + b_2) = 0$,

and

(iii) $(W_1\mu + b_1)'A(W_2\mu + b_2) = 0$.

Moreover if $(W_i\mu + b_i) = W_iAe_i$ for some vector e_i, $i = 1, 2$, then $q_1(y)$ and $q_2(y)$ are independent if and only if $AW_1AW_2A = 0$.

4.4. Cochran Theorems

Before we prove our generalized versions of Cochran's theorems, we need the following lemma:

Lemma 4.4.1. *Let $\Sigma \in \mathcal{S}_V$ with $\Sigma \neq 0$. Then*

(a) $\mathcal{S}_V = \langle\{T\Sigma T : T \in \mathcal{S}_V\}\rangle$,

(b) $\mathcal{S}_V = \langle\{T\Sigma T\Sigma T : T \in \mathcal{S}_V\}\rangle$,

and

(c) $\mathcal{S}_V = \langle\{T\Sigma T\Sigma T\Sigma T : T \in \mathcal{S}_V\}\rangle$,

where $\langle S\rangle$ denotes the linear span of a given set S.

Proof. We shall merely prove (a) and (c).

(a) Since $\Sigma \in \mathcal{S}_V$ and $\Sigma \neq 0$, there exists an orthonormal basis $\{f_j\}_{j=1}^p$ of V such that

$$\Sigma = \sum_{j=1}^{r} \sigma_j f_j \Box f_j, \quad \sigma_j \neq 0,\ j \leq r,\ \sigma_j = 0,\ j > r,\ j = 1, \cdots, p, \qquad (4.4.1)$$

where $r = r(\Sigma) > 0$. Let

$$\mathcal{B} = \{f_j \Box f_j : j = 1, \cdots, p\} \cup \{f_i \Box f_j + f_j \Box f_i : i < j,\ i, j = 1, \cdots, p\}.$$

Then $\mathcal{B}$ is a basis for $\mathcal{S}_V$. Let

$$C_1 = < \{T\Sigma T : T \in \mathcal{S}_V\} > .$$

Then it suffices to show that C_1 contains all $f_i \Box f_i$ and all $f_i \Box f_j + f_j \Box f_i$ with $i < j$. Let $T = f_i \Box f_i$, $i \le r$. Then $T\Sigma T = \sigma_i f_i \Box f_i$. Since $\sigma_i \ne 0$, $f_i \Box f_i \in C_1$. Let $T = f_1 \Box f_i + f_i \Box f_1$, $i > r$. Then $T\Sigma T = \sigma_1 f_i \Box f_i$. Since $\sigma_1 \ne 0$, $f_i \Box f_i \in C_1$. Thus $f_i \Box f_i \in C_1$ for all $i = 1, \ldots, p$. Now let $T = f_i \Box f_1 + f_1 \Box f_i + f_k \Box f_1 + f_1 \Box f_k$. Then

$$T\Sigma T = \sigma_1 f_i \Box f_i + \sigma_i f_1 \Box f_1 + \sigma_1 f_k \Box f_k + \sigma_k f_1 \Box f_1 + \sigma_1 f_i \Box f_k + \sigma_1 f_k \Box f_i.$$

Since $f_i \Box f_i \in C_1$, $T\Sigma T \in C_1$, and $\sigma_1 \ne 0$, we conclude that $f_i \Box f_k + f_k \Box f_i \in C_1$.

(c) Let $C = \langle \{T\Sigma T\Sigma T\Sigma T : T \in \mathcal{S}_V\} \rangle$. Similarly as in the proof of (a), it suffices to prove that $\mathcal{B} \subset C$. Let $T = f_i \Box f_i$. Then $T(\Sigma T)^3 = \sigma_i^3 f_i \Box f_i$. Thus

$$f_i \Box f_i \in C \quad \text{for} \quad i \le r. \tag{4.4.2}$$

Let α, $\beta \in \Re$ and $T = \alpha f_i \Box f_i + \beta(f_i \Box f_k + f_k \Box f_i)$. Then

$$\begin{aligned} T(\Sigma T)^3 =& [(\alpha^2\sigma_i + \beta^2\sigma_k)^2\sigma_i + \alpha^2\beta^2\sigma_i^2\sigma_k] f_i \Box f_i \\ & + \alpha\beta\sigma_i^2(\alpha^2\sigma_i + 2\beta^2\sigma_k)(f_i \Box f_k + f_i \Box f_k) \\ & + \beta^2\sigma_i^2(\alpha^2\sigma_i + \beta^2\sigma_k) f_k \Box f_k. \end{aligned} \tag{4.4.3}$$

By (4.4.2) and (4.4.3) with $\alpha = \beta = 1$,

$$f_i \Box f_k + f_k \Box f_i \in C \quad \text{for} \quad i,\ k \le r. \tag{4.4.4}$$

By (4.4.2) and (4.4.3) again,

$$\alpha^3\beta(f_i \Box f_k + f_k \Box f_i) + \alpha^2\beta^2 f_k \Box f_k \in C \quad \text{for} \quad i \le r, \quad k > r,$$

whence by varying $\alpha, \beta \in \Re$, we obtain

$$f_i \Box f_k + f_k \Box f_i \in C, \qquad f_k \Box f_k \in C \quad \text{for} \quad i \le r, \quad k > r. \tag{4.4.6}$$

Let $T = f_1\Box f_1 + f_1\Box f_i + f_i\Box f_1 + f_1\Box f_k + f_k\Box f_1$, $i, k > r$. Then

$$T(\Sigma T)^3 = \sigma_1^3(f_1\Box f_1 + f_k\Box f_1 + f_1\Box f_k + f_i\Box f_1 + f_1\Box f_i + f_i\Box f_i + f_k\Box f_k + f_k\Box f_i + f_i\Box f_k).$$

Thus by (4.4.2) and (4.4.6),

$$f_i\Box f_k + f_k\Box f_i \in C \quad \text{for} \quad i,\ k > r. \tag{4.4.7}$$

Hence $\mathcal{B} \subset \mathcal{C}$. $\Box$

We shall begin our versions of Cochran's theorems by assuming that $\ell = 1$ and $Q(Y)$, the one given in Corollary 4.2.2.

Theorem 4.4.2. *Suppose that $Y \sim N_{n\times p}(\mu, \Sigma_Y)$. Let $W \in \mathcal{S}_E$, $\Sigma \in \mathcal{N}_V$ with $\Sigma \neq 0$, B, $C \in \mathcal{L}(V, E)$, $D \in \mathcal{L}(V, V)$, $y \in \mathcal{L}(V, E)$ and $Q(y) = y'Wy + B'y + y'C + D$. Then*

$$Q(Y) \sim W_p(m, \Sigma, \lambda)$$

if and only if for any T in a neighborhood, $\mathcal{N}_0$, of 0 in $\mathcal{L}(V, V)$,

(a) $tr(\Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}})^k = m\,tr(\Sigma^{\frac{1}{2}}T\Sigma^{\frac{1}{2}})^k, \quad k = 1, 2, \ldots,$

(b) $\lambda = Q(\mu)$,

(c) $\Sigma_Y(BT^\bullet) = \Sigma_Y(CT^\bullet)$,

and

(d) $\langle (B + C + 2W\mu)T^\circ,\ \Sigma_Y^{\frac{1}{2}}[I_{np} - 2\Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}}]^{-1}\Sigma_Y^{\frac{1}{2}}((B + C + 2W\mu)T^\circ)\rangle$
$= 4\langle \lambda,\ T^\circ\Sigma^{\frac{1}{2}}[I_p - 2\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}}]^{-1}\Sigma^{\frac{1}{2}}T^\circ\rangle$,

where $T = T^\circ + T^\bullet$, $T^\circ = (T + T')/2$ and $T^\bullet = (T - T')/2$. Moreover, if $Q(Y) \sim W_p(m, \Sigma, \lambda)$ then

(e) $m = tr[\Sigma_Y(W \otimes \Sigma^+)]/r(\Sigma)$.

Proof. Suppose that $Q(Y) \sim W_p(m, \Sigma, \lambda)$. Then

$$M_{Q(Y)}(T) = M_W(T^\circ), \tag{4.4.8}$$

where $M_{Q(Y)}(T)$ and $M_W(T^\circ)$ are given, respectively, in (4.2.4) and (4.2.9). Thus by Lemma 4.3.3, (4.4.8) is equivalent to

(i) $|I_{np} - 2\Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}}| = |I_p - 2\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}}|^m$

and

(ii) $\langle T, Q(\mu)\rangle + \langle L, \Sigma_Y^{\frac{1}{2}}(I_{np} - 2\Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}})^{-1}\Sigma_Y^{\frac{1}{2}}(L)\rangle/2$

$$= \langle \lambda, T^\circ\rangle + 2\langle \lambda, T^\circ\Sigma^{\frac{1}{2}}(I_p - 2\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}}T^\circ\rangle,$$

where $L = BT + CT' + 2W\mu T^\circ$ and $T \in \mathcal{N}_0$, a neighborhood of 0 in $\mathcal{L}(V, V)$. By analytic continuation, (i) amounts to

(i$_1$) $|I - \Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}}| = |I - P \otimes \Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}}|$,

where P is an idempotent in $\mathcal{L}(E, E)$ of rank m. Replacing $2T^\circ$ in (i$_1$) by T°/c with nonzero $c \in \Re$, we conclude that (i$_1$) amounts to "$\Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}}$ and $P \otimes (\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})$ have the same characteristic polynomial", i.e.

(i$_2$) $\Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}}$ and $P \otimes (\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})$ have the same spectrum $\{\nu_j\}_{j=1}^{np}$.

Since

$$tr(P \otimes (\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}}))^k = tr(P^k \otimes (\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^k)$$
$$= tr(P)tr(\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^k = m\,tr(\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^k,$$

(i$_2$) amounts to (a). By letting $T^\circ = \Sigma^+$ in (a), (c) follows. Similarly, replacing $2T$ in (ii) by cT with $c < 1/r_\sigma(\Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}})$ and collecting the coefficients of c and c^2, we obtain

(ii$_1$) $\langle Q(\mu), T\rangle = \langle \lambda, T^\circ\rangle$

and

(ii$_2$) $\langle BT+CT'+2W\mu T^\circ, \Sigma_Y^{\frac{1}{2}}(I_{np}-c\Sigma_Y^{\frac{1}{2}}(W\otimes T^\circ)\Sigma_Y^{\frac{1}{2}})^{-1}\Sigma_Y^{\frac{1}{2}}(BT+CT'+2W\mu T^\circ)\rangle$

$$= 4\langle \lambda, T^\circ\Sigma^{\frac{1}{2}}(I_p - c\Sigma^{\frac{1}{2}}T^\circ)^{-1}\Sigma^{\frac{1}{2}}T^\circ\rangle.$$

By choosing $T = T^*$ in (ii$_1$) and (ii$_2$), we obtain $T^\circ = 0$,

(iii$_1$) $\langle Q(\mu), T^*\rangle = 0$,

and

(iii$_2$) $\langle (B-C)T^*, \Sigma_Y((B-C)T^*)\rangle = 0$.

Since:

$$\langle Q(\mu), T^*\rangle = \mathrm{tr}[(Q(\mu))'T^*] = -\mathrm{tr}[Q(\mu)T^*],$$

(iii$_1$) implies that

$$\langle Q(\mu) - (Q(\mu))', T^*\rangle = 0,$$

i.e. $Q(\mu)$ is a self-adjoint operator in $\mathcal{L}(V,V)$. Thus by (ii$_1$),

$$\begin{aligned}\langle Q(\mu), T\rangle - \langle \lambda, T^\circ\rangle &= \langle Q(\mu), T^\circ + T^*\rangle - \langle \lambda, T^\circ\rangle \\ &= \langle Q(\mu) - \lambda, T^\circ\rangle = 0.\end{aligned} \tag{4.4.9}$$

By choosing $T^\circ = Q(\mu) - \lambda$ in (4.4.9), we obtain (b). Also by (iii$_2$),

$$\Sigma_Y^{\frac{1}{2}}((B-C)T^*) = 0,$$

proving (c). By using (iii$_2$) and substituting $T = T^\circ + T^*$ into (ii$_2$), we obtain

$$\begin{aligned}&\langle (B+C+2W\mu)T^\circ, \Sigma_Y^{\frac{1}{2}}[I_{np} - c\Sigma_Y^{\frac{1}{2}}(W\otimes T^\circ)\Sigma_Y^{\frac{1}{2}}]^{-1}\Sigma_Y^{\frac{1}{2}}((B+C+2W\mu)T)\rangle \\ &= 4\langle \lambda, \Sigma^{\frac{1}{2}}[I_p - c\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}}]^{-1}\Sigma^{\frac{1}{2}}T^\circ\rangle,\end{aligned}$$

proving (d).

Now suppose that (a) - (d) hold. Then it suffices to show that (i) and (ii) above hold. By the above argument for proving 'necessity', we know that (a) is equivalent to (i). By (b), $Q(\mu)$ is self-adjoint in $\mathcal{S}_V$. Thus for any $T \in \mathcal{L}(V,V)$,

$$\begin{aligned}\langle Q(\mu), T^*\rangle &= \frac{1}{2}\langle Q(\mu), T - T'\rangle \\ &= \frac{1}{2}[\langle Q(\mu), T\rangle - \langle Q(\mu), T'\rangle] \\ &= \frac{1}{2}[\langle Q(\mu), T\rangle - \langle Q(\mu), T\rangle] = 0.\end{aligned}$$

So by (b),

$$\langle Q(\mu), T\rangle = \langle Q(\mu), T^\circ + T^*\rangle = \langle \lambda, T^\circ\rangle. \tag{4.4.10}$$

By (c)

$$\Sigma_Y^{\frac{1}{2}}((B-C)T^*) = 0 \tag{4.4.11}$$

Let $\Psi = \Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}}$. Then by (4.4.11),

$$\begin{aligned}
&\langle BT + CT' + 2W\mu T^\circ, \Sigma_Y^{\frac{1}{2}}(I_{np} - 2\Psi)^{-1}\Sigma_Y^{\frac{1}{2}}(BT + CT' + 2W\mu T^\circ)\rangle \\
&= \langle \Sigma_Y^{\frac{1}{2}}((B + C + 2W\mu)T^\circ + (B - C)T^*), (I_{np} - 2\Psi)^{-1} \\
&\qquad \times \Sigma_Y^{\frac{1}{2}}((B + C + 2W\mu)T^\circ + (B - C)T^*)\rangle \qquad (4.4.12) \\
&= \langle \Sigma_Y^{\frac{1}{2}}((B + C + 2W\mu)T^\circ), (I_{np} - 2\Psi)^{-1}\Sigma_Y^{\frac{1}{2}}((B + C + 2W\mu)T^\circ)\rangle.
\end{aligned}$$

Now (ii) follows from (4.4.10), (4.4.12) and (d). □

¿From Theorem 4.4.2 and its proof, we have the following two results:

Corollary 4.4.3. *In Theorem 4.4.2, if $B = C = 0$ and $D = 0$ then $Q(Y) = Y'WY \sim W(m, \Sigma, \lambda)$ if and only if for any $T \in \mathcal{S}_V$,*

(a1) $tr(\Sigma_Y^{\frac{1}{2}}(W \otimes T)\Sigma_Y^{\frac{1}{2}})^k = m\,tr(\Sigma^{\frac{1}{2}}T\Sigma^{\frac{1}{2}})^k, \quad k = 1, 2, \ldots,$

and for T in a neighborhood N_0 of $\mathcal{S}_V$,

(b1) $\lambda = \mu' W\mu$

and

(c1) $\langle \mu, [(W \otimes T)\Sigma_Y^{\frac{1}{2}}(I_{np} - 2\Sigma_Y^{\frac{1}{2}}(W \otimes T)\Sigma_Y^{\frac{1}{2}})^{-1}\Sigma_Y^{\frac{1}{2}}(W \otimes T)](\mu)\rangle$
$= \langle \lambda, T\Sigma^{\frac{1}{2}}(I_p - 2\Sigma^{\frac{1}{2}}T\Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}}T\rangle.$

Moreover, if $Q(Y) \sim W(m, \Sigma, \lambda)$, then

(d1) $m = tr(\Sigma_Y(W \otimes \Sigma^+))/r(\Sigma).$

Corollary 4.4.4. *In Theorem 4.4.2, suppose that $\mu = B = C = 0$, $D = 0$, and $P \in \mathcal{L}(E, E)$ with $P^2 = P$ and $r(P) = m$. Then the following conditions are equivalent:*

(a2) $Q(Y) \sim W_p(m, \Sigma)$.

(b2) $|I - \Sigma_Y^{\frac{1}{2}}(W \otimes T)\Sigma_Y^{\frac{1}{2}}| = |I - \Sigma^{\frac{1}{2}}T\Sigma^{\frac{1}{2}}|^m, \quad T \in \mathcal{S}_V$.

(c2) $|I - \Sigma_Y^{\frac{1}{2}}(W \otimes T)\Sigma_Y^{\frac{1}{2}}| = |I - P \otimes (\Sigma^{\frac{1}{2}} T \Sigma^{\frac{1}{2}})|, \quad T \in \mathcal{S}_V,$

(d2) $\Sigma_Y^{\frac{1}{2}}(W \otimes T)\Sigma_Y^{\frac{1}{2}}$ and $P \otimes (\Sigma^{\frac{1}{2}} T \Sigma^{\frac{1}{2}})$ are similar, $\quad T \in \mathcal{S}_V$.

(e2) $tr(\Sigma_Y^{\frac{1}{2}}(W \otimes T)\Sigma_Y^{\frac{1}{2}})^k = m tr(\Sigma^{\frac{1}{2}} T \Sigma^{\frac{1}{2}})^k, \quad k = 1, 2, \ldots; \quad T \in \mathcal{S}_V.$

The involvement of T in Theorem 4.4.2 is caused by the reality that Σ_Y is not assumed to have the form $A \otimes \Sigma$ with $A \in \mathcal{N}_E$.

Corollary 4.4.5. *In Theorem 4.4.2, if $\Sigma_Y = A \otimes \Sigma$ with $r(\Sigma) > 1$. then $Q(Y) \sim W_p(m, \Sigma, \lambda)$ if and only if*

(a3) $AWAWA = AWA, \qquad tr(AW) = m,$

(b3) $AB = AC,$

(c3) $\lambda = Q(\mu) = (B + W\mu)'A(B + W\mu) = (B + W\mu)'AWA(B + W\mu).$

Hence if $B = C = 0$ and $D = 0$, then $Y'WY \sim W_p(m, \Sigma, \lambda)$ if and only if (a3) and (d3) hold, where

(d3) $\lambda = \mu' W \mu = \mu' WAW \mu = \mu' WAWAW \mu.$

Proof. By Theorem 4.4.2, it suffices to show that (a) - (d) in Theorem 4.4.2 are equivalent to (a3) - (c3) here.

Suppose that (a) - (d) hold. Since $\Sigma_Y = A \otimes \Sigma$,

$$\begin{aligned} \mathrm{tr}(\Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}})^k &= \mathrm{tr}(A^{\frac{1}{2}} W A^{\frac{1}{2}} \otimes \Sigma^{\frac{1}{2}} T^\circ \Sigma^{\frac{1}{2}})^k \\ &= \mathrm{tr}(A^{\frac{1}{2}} W A^{\frac{1}{2}})^k \mathrm{tr}(\Sigma^{\frac{1}{2}} T^\circ \Sigma^{\frac{1}{2}})^k, \quad k = 1, 2, \ldots, \end{aligned} \tag{4.4.13}$$

Let $T^\circ = \Sigma^+$. Then by (4.4.13), (a) is reduced to

$$\mathrm{tr}(A^{\frac{1}{2}} W A^{\frac{1}{2}})^k \mathrm{tr}(\Sigma^0)^k = m \mathrm{tr}(\Sigma^0)^k, \quad k = 1, 2, \cdots,$$

Since $\mathrm{tr}(\Sigma^0)^k = \mathrm{tr}(\Sigma^0) = r(\Sigma^0) = r(\Sigma) \neq 0$, we obtain

$$\mathrm{tr}(A^{\frac{1}{2}} W A^{\frac{1}{2}})^k = m, \qquad k = 1, 2, \cdots, \tag{4.4.14}$$

and hence

$$\mathrm{tr}(A^{\frac{1}{2}}WA^{\frac{1}{2}}) = \mathrm{tr}(AW) = m. \tag{4.4.15}$$

Let $\nu_1, \nu_2, \cdots, \nu_s$ be the nonzero eigenvalues of $A^{\frac{1}{2}}WA^{\frac{1}{2}}$. Then by (4.4.14),

$$\mathrm{tr}(A^{\frac{1}{2}}WA^{\frac{1}{2}})^6 - 2\mathrm{tr}(A^{\frac{1}{2}}WA^{\frac{1}{2}})^4 + \mathrm{tr}(A^{\frac{1}{2}}WA^{\frac{1}{2}})^2 = 0,$$

i.e.,

$$\sum_{i=1}^{s}(\nu_i^6 - 2\nu_i^4 + \nu_i^2) = \sum_{i=1}^{s}\nu_i^2(1-\nu_i^2)^2 = 0. \tag{4.4.16}$$

Since $\nu_i \neq 0$, we obtain from (4.4.16),

$$\nu_i = 1 \quad \text{or} \quad -1, \qquad i = 1, 2, \cdots, s. \tag{4.4.17}$$

But by (4.4.14) and (4.4.17),

$$s = \sum_{i=1}^{s}\nu_i^2 = \mathrm{tr}(A^{\frac{1}{2}}WA^{\frac{1}{2}})^2 = m.$$

Thus by (4.4.15), all $\nu_i = 1$, i.e. $A^{\frac{1}{2}}WA^{\frac{1}{2}}$ is idempotent of rank m, proving (a3). By (c),

$$ABT^*\Sigma = ACT^*\Sigma, \quad \text{for all } T^* = (T - T')/2. \tag{4.4.18}$$

Since $r(\Sigma) > 1$, we may choose an orthonormal basis, $\{f_j\}_{j=1}^{p}$, of V such that

$$\Sigma = \sum_{j=1}^{r(\Sigma)}\sigma_j f_j \Box f_j, \quad \sigma_j \neq 0, \quad j = 1, 2, \cdots, r(\Sigma).$$

Thus $T^* \in \mathcal{L}(V, V)$ can be written as

$$T^* = \sum_{j<j'}^{p} t_{jj'}(f_j \Box f_{j'} - f_{j'} \Box f_j).$$

Let $t = f_1$ and f_2. Then by (4.4.18),

$$A(B-C)T^*\Sigma f_1 = -\sigma_1 A(B-C)\sum_{j'=2}^{p} t_{1j'}f_{j'} = 0$$

and

$$A(B-C)T^*\Sigma f_2 = \sigma_2 A(B-C)[t_{12}f_1 - \sum_{j'=3}^{p} t_{2j'}f_{j'}] = 0.$$

Thus by letting $t_{2j} = 0$ and varying $t_{1j}, \in \Re$, $j = 2, \cdots, p$, we obtain

$$A(B-C)f_j = 0, \qquad j = 1, 2, \cdots, p. \tag{4.4.19}$$

Since $\{f_j\}_{j=1}^{p}$ is an orthonormal basis of V, (4.4.19) implies (b3). Replacing $2T^\circ$ by cT° with $c < 1/r_\sigma(\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})$ and substituting (a3) and (b3) into (d), we have

$$\begin{aligned}
&\langle (B+C+2W\mu)T^\circ, (A^{\frac{1}{2}} \otimes \Sigma^{\frac{1}{2}})(I_{np} - 2A^{\frac{1}{2}}WA^{\frac{1}{2}} \otimes \Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{-1} \\
&\qquad\qquad \times (A^{\frac{1}{2}} \otimes \Sigma^{\frac{1}{2}})((B+C+2W\mu)T^\circ)\rangle \\
&= c^2\langle (B+W\mu)T^\circ, \sum_{k=0}^{\infty}\left[A^{\frac{1}{2}}(A^{\frac{1}{2}}WA^{\frac{1}{2}})^k A^{\frac{1}{2}} \otimes \Sigma^{\frac{1}{2}}(c\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^k\Sigma^{\frac{1}{2}}\right]((B+W\mu)T^\circ)\rangle \\
&= c^2\langle (B+W\mu)T^\circ, A(B+W\mu)T^\circ\Sigma\rangle \\
&\quad + c^2\langle (B+W\mu)T^\circ, AWA(B+W\mu)T^\circ \sum_{k=1}^{\infty}\Sigma^{\frac{1}{2}}(c\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^k\Sigma^{\frac{1}{2}}\rangle \\
&= c^2\langle (B+W\mu)'A(B+W\mu), T^\circ\Sigma T^\circ\rangle \\
&\quad + c^2\langle (B+W\mu)'AWA(B+W\mu), T^\circ \sum_{k=1}^{\infty}\Sigma^{\frac{1}{2}}(c\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^k\Sigma^{\frac{1}{2}}T^\circ\rangle
\end{aligned} \tag{4.4.20}$$

and

$$4\langle\lambda, T^\circ\Sigma^{\frac{1}{2}}(I_p - 2\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}}T^\circ\rangle = c^2\langle\lambda, T^\circ\Sigma^{\frac{1}{2}}\sum_{k=0}^{\infty}(c\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^k\Sigma^{\frac{1}{2}}T^\circ\rangle. \tag{4.4.21}$$

By (4.4.20) and (4.4.21), (d) reduces to

$$\begin{aligned}
&\langle (B+W\mu)'A(B+W\mu), c^2T^\circ\Sigma T^\circ\rangle \\
&\quad + \langle (B+W\mu)'AWA(B+W\mu), c^2T^\circ\sum_{k=1}^{\infty}\Sigma^{\frac{1}{2}}(c\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^k\Sigma^{\frac{1}{2}}T^\circ\rangle \\
&= \langle\lambda, c^2T^\circ\Sigma^{\frac{1}{2}}T^\circ\rangle + \langle\lambda, c^2T^\circ\Sigma^{\frac{1}{2}}\sum_{k=1}^{\infty}(c\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^k\Sigma^{\frac{1}{2}}T^\circ\rangle.
\end{aligned} \tag{4.4.22}$$

By comparing the coefficients of $c^2, c^3, \cdots$ on both sides of (4.4.22), we obtain that for all $T^c \in \mathcal{S}_V$,

$$\langle (B + W\mu)' A (B + W\mu), T^c \Sigma T^c \rangle \tag{4.4.23}$$

and

$$\langle (B + W\mu)' AWA (B + W\mu), T^c \Sigma T^c \Sigma T^c \rangle. \tag{4.4.24}$$

Thus (c3) follows from (b), (4.4.23), (4.4.24) and Lemma 4.4.1.

The 'if part' is obvious. □

Note that in Corollary 4.4.5, if $B = C$, then the assumption $r(\Sigma) > 1$ can be reduced to $\Sigma \neq 0$. A similar result of Corollary 4.4.5 was obtained by Khatri (1980) as follows:

Corollary 4.4.6. *In Corollary 4.4.5, conditions (a3)-(c3) are equivalent to (a3), (b3), (e3) and (f3), where*

(e3) $\lambda = Q(\mu) = (B + W\mu)' A (B + W\mu)$.

and

(f3) $A(B + W\mu) = AWAM$ *for some* $M \in \mathcal{L}(V, E)$.

Proof. Suppose that (a3)-(c3) hold. Let $A_* = A^{\frac{1}{2}} W A^{\frac{1}{2}}$. By (a3), A_* is an idempotent of rank m. Thus $I_n - A_*$ is also an idempotent. By (c3), we obtain (e3) and

$$(B + W\mu)' A (B + W\mu) - (B + W\mu)' AWA (B + W\mu) = 0,$$

i.e.,

$$(A^{\frac{1}{2}}(B + W\mu))'(I_n - A_*)\left(A^{\frac{1}{2}}(B + W\mu)\right) = 0. \tag{4.4.25}$$

Since $I_n - A_* = (I_n - A_*)^2$, we obtain from (4.4.25) that

$$(I_n - A_*)\left(A^{\frac{1}{2}}(B + W\mu)\right) = 0,$$

i.e.,

$$A(B + W\mu) = AWA(B + W\mu).$$

Therefore (f3) follows from choosing $M = B + W\mu$.

Suppose that (a3), (b3), (c3) and (f3) hold. By (f3) and (a3),

$$\begin{aligned}(B + W\mu)'A(B + W\mu) &= (B + W\mu)'AWAM \\ &= (B + W\mu)'AWAWAM = (B + W\mu)'AWA(B + W\mu).\end{aligned} \tag{4.4.26}$$

So by (c3) and (4.4.26), we obtain (c3). □

Now combining Theorem 4.3.4 and Theorem 4.4.2, we obtain the following version of Cochran's theorem:

Theorem 4.4.7. *Suppose that $Y \sim N_{n\times p}(\mu, \Sigma_Y)$. Let $i \in \{1, 2, \cdots, \ell\}$, $W_i \in \mathcal{S}_E$, B_i, $C_i \in \mathcal{L}(V, E)$, $D_i \in \mathcal{L}(V, V)$, and $Q_i(Y) = Y'W_iY + B_i'Y + Y'C_i + D_i$. Then $\{Q_i(Y)\}$ is an independent family of $W_p(m_i, \Sigma, \lambda_i)$ random operators if and only if for any distinct i, $j \in \{1, 2, \cdots, \ell\}$ and $T_i \in \mathcal{L}(V, V)$,*

(a) $tr(\Sigma_Y(W_i \otimes T_i^\circ)^k = m_i tr(\Sigma T_i^\circ)^k$ *for all* $k = 1, 2, \cdots$,

(b) $\lambda_i = Q_i(\mu)$,

(c) $\Sigma_Y(B_iT_i^*) = \Sigma_Y(C_iT_i^*)$,

(d) $\langle L_i, (I_{np} - 2\Sigma_Y(W_i \otimes T_i^\circ))^{-1}\Sigma_Y(L_i)\rangle = 4\langle \lambda_i, T_i^\circ(I_p - 2\Sigma T_i^\circ)^{-1}\Sigma T_i^\circ\rangle$,

(e) $\Sigma_Y(W_i \otimes T_i^\circ)\Sigma_Y(W_j \otimes T_j^\circ)\Sigma_Y = 0$,

(f) $\Sigma_Y(W_i \otimes T_i^\circ)\Sigma_Y(L_j) = 0$,

and

(g) $\langle L_i, \Sigma_Y(L_j)\rangle = 0$,

where $T_i^\circ = (T_i + T_i')/2$, $T_i^* = (T_i - T_i')/2$ and $L_i = (B_i + C_i + 2W_i\mu)T_i^\circ$.

Corollary 4.4.8. *In Theorem 4.4.7, let $B_i = C_i = 0$ and $D = 0$. Then $\{Y'W_iY\}$ is an independent family of $W_p(m_i, \Sigma, \lambda_i)$ random operators if and only if for any i, $j \in \{1, 2, \cdots, \ell\}$ and $T_i \in \mathcal{S}_V$,* (a1) $tr(\Sigma_Y(W_i \otimes T_i)^k = m_i tr(\Sigma T_i)^k$ for all $k = 1, 2, \cdots$,

(b1) $\langle \mu, [(W_i \otimes T_i)(I_{np} - 2\Sigma_Y(W_i \otimes T_i))^{-1}\Sigma_Y(W_i \otimes T_i)](\mu)\rangle$

$$= \langle \lambda_i, T_i(I_p - 2\Sigma T_i)^{-1}\Sigma T_i\rangle.$$

(c1) $\Sigma_Y(W_i \otimes T_i)\Sigma_Y(W_j \otimes T_j)\Sigma_Y = 0$.

(d1) $[\Sigma_Y(W_i \otimes T_i)\Sigma_Y(W_j \otimes T_j](\mu) = 0$.

and

(e1) $\langle \mu, [(W_i \otimes T_i)\Sigma_Y(W_j \otimes T_j)](\mu)\rangle$.

Hence if $\mu = 0$, then $\{Y'W_iY\}$ is an independent family of $W_p(m_i, \Sigma)$ random operators if and only if (a1) and (c1) hold for any $i, j \in \{1, 2, \cdots, \ell\}$ and $T_i \in S_Y$.

Corollary 4.4.9. *In Theorem 4.4.7, let $\Sigma_Y = A \otimes \Sigma$ with $r(\Sigma) > 1$. Then $\{Q_i(Y)\}$ is an independent family of $W_p(m_i, \Sigma, \lambda_i)$ random operators if and only if for any $i, j \in \{1, 2, \cdots, \ell\}$,*

(a2) $AW_iAW_iA = AW_iA, \qquad r(AW_i) = m_i$,

(b2) $AB_i = AC_i$,

(c2) $\lambda_i = Q_i(\mu) = (B_i + W_i\mu)'A(B_i + W_i\mu) = (B_i + W_i\mu)'AWA(B_i + W_i\mu)$,

(d2) $AW_iAW_jA = 0$,

(e2) $AW_iA(B_j + W_j\mu) = 0$,

and

(f2) $(B_i + W_i\mu)'A(B_j + W_j\mu) = 0$.

Example 4.4.1. *In Corollary 4.4.9, let*

$$A = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}, \qquad W = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

and $Y \sim N(0, A \otimes \Sigma)$. Then A is singular and W is symmetric but not n.n.d. It is easy to verify that the conditions of Corollary 4.4.9 are satisfied and so $Y'WY \sim W(m, \Sigma)$ with $m = 2$.

In Theorem 4.4.7, we obtained a necessary and sufficient condition under which $\{Q_i(Y)\}$ is an independent family of $W_p(m_i, \Sigma, \lambda_i)$ random operators. In this section, we shall give some further extensions of this Cochran theorem to the case where $\{Q_i(Y)\}$ is an independent family of the difference of two independent Wishart random operators $W_p(m_{1i}, \Sigma, \lambda_{1i})$ and $W_p(m_{2i}, \Sigma, \lambda_{2i})$. Let $y \sim N_n(0, I_n)$ and $W \in S_E$. Then Graybill (1969, p.352) proved that $y'Wy$ is distributed as the difference of two independently distributed χ^2-variates if and only if $W^3 = W$. This result was extended by Tan (1975, 1976) and Wong (1992).

Lemma 4.5.1. *Let Q_1, Q_2 be independent $W_p(m_1, \Sigma, \lambda_1)$, $W_p(m_2, \Sigma, \lambda_2)$ random operators in $\mathcal{L}(V, V)$ and let $U = Q_1 - Q_2$. Then the mgf of U is M_U with*

$$\begin{aligned} M_U(T) =& |I_p - 2\Sigma^{\frac{1}{2}}T\Sigma^{\frac{1}{2}}|^{-\frac{m_1}{2}}|I_p + 2\Sigma^{\frac{1}{2}}T\Sigma^{\frac{1}{2}}|^{-\frac{m_2}{2}} \\ & \times \exp\Big\{\langle\lambda_1 - \lambda_2, T\rangle + 2\langle\lambda_1, T\Sigma^{\frac{1}{2}}(I_p - 2\Sigma^{\frac{1}{2}}T\Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}}T\rangle \\ & +2\langle\lambda_2, T\Sigma^{\frac{1}{2}}(I_p + 2\Sigma^{\frac{1}{2}}T\Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}}T\rangle\Big\}, \\ & T \in S_V \quad r_\sigma(\Sigma^{\frac{1}{2}}T\Sigma^{\frac{1}{2}}) < \frac{1}{2}. \end{aligned} \tag{4.5.1}$$

We shall use $DW_p(m_1, m_2, \Sigma, \lambda_1, \lambda_2)$ to denote the distribution of the above U and use $DW_p(m_1, m_2, \Sigma)$ to denote $DW_p(m_1, m_2, \Sigma, 0, 0)$. Note that when $m_2 = 0$ and $\lambda_2 = 0$, $DW_p(m_1, m_2, \Sigma, \lambda_1, \lambda_2)$ is nothing but the Wishart distribution $W_p(m_1, \Sigma, \lambda_1)$. To avoid the nuisance of treating $DW_p(m_1, m_2, \Sigma)$ or $W_p(m_1, \Sigma)$ with $m_1 = 0$, we assume, in the following sections, that $m_1 > 0$ and $m_2 \geq 0$.

For $Y \sim N_{n\times p}(\mu, \Sigma_Y)$, let

$$Q(Y) = Y'WY + B'Y + Y'C + D \tag{4.5.2}$$

and

$$Q_i(Y) = Y'W_iY + B_i'Y + Y'C_i + D_i, \qquad i = 1, 2, \cdots, \ell, \tag{4.5.3}$$

where W, $W_i \in \mathcal{S}_E$, D, $D_i \in \mathcal{S}_V$, B, B_i, C, $C_i \in \mathcal{L}(V, E)$, and D, $D_i \mathcal{L}(V, V)$. Several proofs in this section are adopted from Wong (1992), where $B = C$ and $B_i = C_i$, $i = 1, 2, \ldots, \ell$.

Theorem 4.5.2. *Suppose that $Y \sim N_{n \times p}(\mu, \Sigma_Y)$. Let $Q(Y)$ be given in (4.5.2). Then $Q(Y) \sim DW_p(m_1, m_2, \Sigma, \lambda_1, \lambda_2)$ if and only if there exists a neighborhood, N_0, of $0 \in \mathcal{L}(V, V)$ such that for all $T \in N_0$,*

(a) $|I_{np} - \Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}}| = |I_p - \Sigma^{\frac{1}{2}} T^\circ \Sigma^{\frac{1}{2}}|^{m_1} |I_p + \Sigma^{\frac{1}{2}} T^\circ \Sigma^{\frac{1}{2}}|^{m_2}$,

(b) $\Sigma_Y(BT^\bullet) = \Sigma_Y(CT^\bullet)$,

(c)
$$\begin{aligned}&\langle L, [\Sigma_Y^{\frac{1}{2}}(I_{np} - \Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}})^{-1}\Sigma_Y^{\frac{1}{2}}](L)\rangle/2 \\ &= 2\langle \lambda_1, T^\circ \Sigma^{\frac{1}{2}}(I_p - \Sigma^{\frac{1}{2}} T^\circ \Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}} T^\circ\rangle \\ &\quad + 2\langle \lambda_2, T^\circ \Sigma^{\frac{1}{2}}(I_p + \Sigma^{\frac{1}{2}} T^\circ \Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}} T^\circ\rangle,\end{aligned}$$

and

(d) $\lambda_1 - \lambda_2 = Q(\mu)$,

where $T^\circ = (T + T')/2$, $T^\bullet = (T - T')/2$, and $L = (B + C + 2W\mu)T^\circ$.

Proof. Suppose that $Q(Y) \sim DW_p(m_1, m_2, \Sigma, \lambda_1, \lambda_2)$. Then by (4.2.4) and (4.5.1), we know that for some neighborhood, N_0, of 0 in $\mathcal{L}(V, V)$,

$$M_{Q(Y)}(T) = M_U(T) \quad \text{for all } T \in N_0, \tag{4.5.4}$$

which, by Lemma 4.3.3, is equivalent to: for all $T \in N_0$,

$$|I_{np} - 2\Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}}|^{-\frac{1}{2}} = |I_p - 2\Sigma^{\frac{1}{2}} T^\circ \Sigma^{\frac{1}{2}}|^{-\frac{m_1}{2}} |I_p + 2\Sigma^{\frac{1}{2}} T^\circ \Sigma^{\frac{1}{2}}|^{-\frac{m_2}{2}} \tag{4.5.5}$$

and

$$\begin{aligned}&\langle Q(\mu), T\rangle + \langle BT + CT' + 2W\mu T^\circ, [\Sigma_Y^{\frac{1}{2}}(I_{np} - 2\Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}})^{-1} \\ &\qquad \times \Sigma_Y^{\frac{1}{2}}](BT + CT' + 2W\mu T^\circ)\rangle/2 \\ &= \langle \lambda_1 - \lambda_2, T^\circ\rangle + 2\langle \lambda_1, T^\circ \Sigma^{\frac{1}{2}}(I_p - 2\Sigma^{\frac{1}{2}} T^\circ \Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}} T^\circ\rangle \\ &\quad + 2\langle \lambda_2, T^\circ \Sigma^{\frac{1}{2}}(I_p + 2\Sigma^{\frac{1}{2}} T^\circ \Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}} T^\circ\rangle.\end{aligned} \tag{4.5.6}$$

Since $2T^\circ \in \mathcal{N}_0$, (4.5.5) becomes

$$|I_{np} - \Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}}| = |I_p - \Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}}|^{m_1}|I_p + \Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}}|^{m_2}$$

and therefore (a) follows. Similarly, replacing $2T$ in (4.5.6) by $cT \in \mathcal{N}_0$ with $c \in \Re$ and equating the coefficients of c and c^2, we obtain that for any $T \in \mathcal{N}_0$,

$$\langle Q(\mu), T\rangle = \langle \lambda_1 - \lambda_2, T^\circ\rangle \tag{4.5.7}$$

and

$$\begin{aligned}\langle BT+CT' + 2W\mu T^\circ, \Sigma_Y^{\frac{1}{2}}(I_{np} - c\Sigma_Y^{\frac{1}{2}}(W \otimes T^\circ)\Sigma_Y^{\frac{1}{2}})^{-1} \\ \times \Sigma_Y^{\frac{1}{2}}((BT + CT' + 2W\mu T^\circ))/2 \\ =2\langle\lambda_1, T^\circ\Sigma^{\frac{1}{2}}(I_p - c\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}}T^\circ\rangle \\ + 2\langle\lambda_2, T^\circ\Sigma^{\frac{1}{2}}(I_p + c\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}}T^\circ\rangle.\end{aligned} \tag{4.5.8}$$

Let $T = T^\bullet$. Then $T^\circ = 0$ and (4.5.8) reduces to

$$\langle BT^\bullet - CT^\bullet, \Sigma_Y(BT^\bullet - CT^\bullet)\rangle = 0,$$

i.e.

$$\Sigma_Y^{\frac{1}{2}}(BT^\bullet - CT^\bullet) = 0,$$

proving (b). Substituting (b) into (4.5.8), we obtain (c). By the same arguement as the proof of Theorem 4.4.2, we obtain from (4.5.7) that $Q(\mu) = \lambda_1 - \lambda_2$, proving (d).

The 'if part' is obvious. □

Corollary 4.5.3. *In Theorem 4.5.2, let $\Sigma_Y = A\otimes\Sigma$ with $r(\Sigma) > 1$. Then $Q(Y) \sim DW_p(m_1, m_2, \Sigma, \lambda_1, \lambda_2)$ if and only if*

(i) $AWAWAWA = AWA$,

(ii) $tr(AW) = m_1 - m_2, \qquad tr(AW)^2 = m_1 + m_2,$

(iii) $AB = AC$.

(iv) $\lambda_1 - \lambda_2 = Q(\mu) = (B + W\mu)'AWA(B + W\mu)$.

and

(v) $\lambda_1 + \lambda_2 = (B + W\mu)'A(B + W\mu) = (B + W\mu)'AWAWA(B + W\mu)$.

Moreover if $Q(Y) \sim DW_p(m_1, m_2, \Sigma, \lambda_1, \lambda_2)$, *then*

$$m_1 = \frac{1}{2}(tr(AW) + tr(AW)^2), \qquad m_2 = \frac{1}{2}(tr(AW)^2) - tr(AW)),$$
$$\lambda_1 = \frac{1}{2}(B + W\mu)'(AWAWA + AWA)(B + W\mu),$$
$$\lambda_2 = \frac{1}{2}(B + W\mu)'(AWAWA - AWA)(B + W\mu).$$

Proof. By Theorem 4.5.2, it suffices to show that (a) - (d) there are equivalent to (i) - (v) here.

Suppose that (a) - (d) hold. Let $T = c\Sigma^+ \in N_0$ with $c \in \Re$ and $\Sigma^0 = \Sigma^+\Sigma$. Then (a) reduces to

$$|I_{np} - cA^{\frac{1}{2}}WA^{\frac{1}{2}} \otimes \Sigma^0| = |I_p - c\Sigma^0|^{m_1}|I_p + c\Sigma^0|^{m_2}. \tag{4.5.9}$$

Let $\nu_1, \nu_2, \cdots, \nu_s$ be the nonzero eigenvalues of $A^{\frac{1}{2}}WA^{\frac{1}{2}}$. Then we obtain from (4.5.9),

$$\prod_{j=1}^{s}(1 - c\nu_j)^r = (1 - c)^{rm_1}(1 + c)^{rm_2}, \tag{4.5.10}$$

where $r = r(\Sigma)$. By taking the logarithms, expanding and equating the coefficients of $c, c^2, \cdots$ on both sides of (4.5.10), we obtain

$$\nu_j = 1 \quad \text{or} \quad \nu_j = -1, \quad j = 1, 2, \cdots, s \tag{4.5.11}$$

and

$$\sum_{j=1}^{s} \nu_j = m_1 - m_2, \qquad \sum_{j=1}^{s} \nu_j^2 = s = m_1 + m_2. \tag{4.5.12}$$

Now (i) and (ii) follow from (4.5.11) and (4.5.12). By the proof of Corollary 4.4.5, we know that (b) implies (iii). Note that

$$\begin{aligned}&\langle\lambda_1, T^\circ\Sigma^{\frac{1}{2}}(I_p-\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}}T^\circ\rangle+\langle\lambda_2, T^\circ\Sigma^{\frac{1}{2}}(I_p-\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}}T^\circ\rangle.\\ &=\langle\lambda_1+\lambda_2, \sum_{k=0}^{\infty}T^\circ\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{2k}\Sigma^{\frac{1}{2}}T^\circ\rangle\\ &\quad+\langle\lambda_1-\lambda_2, \sum_{k=0}^{\infty}T^\circ\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{2k+1}\Sigma^{\frac{1}{2}}T^\circ\rangle\end{aligned} \tag{4.5.13}$$

and

$$\begin{aligned}&\langle L, [\Sigma_Y^{\frac{1}{2}}(I_{np}-\Sigma_Y^{\frac{1}{2}}(W\otimes T^\circ\Sigma_Y^{\frac{1}{2}})^{-1}\Sigma_Y^{\frac{1}{2}}](L)\rangle/2\\ &=\langle A^{\frac{1}{2}}L\Sigma^{\frac{1}{2}}, [(I_{np}-A^{\frac{1}{2}}WA^{\frac{1}{2}}\otimes\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{-1}](A^{\frac{1}{2}}L\Sigma^{\frac{1}{2}})\rangle/2\\ &=\sum_{k=0}^{\infty}\{\langle A^{\frac{1}{2}}L\Sigma^{\frac{1}{2}}, (A^{\frac{1}{2}}WA^{\frac{1}{2}})^kA^{\frac{1}{2}}L\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^k\rangle\}/2\\ &=\sum_{k=0}^{\infty}\{\langle(B+C+2W\mu)'A^{\frac{1}{2}}(A^{\frac{1}{2}}WA^{\frac{1}{2}})^kA^{\frac{1}{2}}(B+C+2W\mu),\\ &\qquad T^\circ\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^k\Sigma^{\frac{1}{2}}T^\circ\rangle\}/2,\end{aligned} \tag{4.5.14}$$

where $L=(B+C+2W\mu)T^\circ$. By (i),

$$(A^{\frac{1}{2}}WA^{\frac{1}{2}})^{2k}=(A^{\frac{1}{2}}WA^{\frac{1}{2}})^2,\quad k=1,2,\cdots$$

and

$$(A^{\frac{1}{2}}WA^{\frac{1}{2}})^{2k+1}=A^{\frac{1}{2}}WA^{\frac{1}{2}},\quad k=0,1,2,\cdots.$$

Thus (4.5.14) with (iii) reduces to

$$\begin{aligned}&\langle L. [\Sigma_Y^{\frac{1}{2}}(I_{np}-\Sigma_Y^{\frac{1}{2}}(W\otimes T^\circ\Sigma_Y^{\frac{1}{2}})^{-1}\Sigma_Y^{\frac{1}{2}}](L)\rangle/2\\ &=2\langle(B+W\mu)'A(B+W\mu), T^\circ\Sigma T^\circ\rangle\\ &\quad+2\langle(B+W\mu)'AWA(B+W\mu), \sum_{k=0}^{\infty}T^\circ\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{2k+1}\Sigma^{\frac{1}{2}}T^\circ\rangle\\ &\quad+2\langle(B+W\mu)'AWAWA(B+W\mu), \sum_{k=1}^{\infty}T^\circ\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{2k}\Sigma^{\frac{1}{2}}T^\circ\rangle.\end{aligned} \tag{4.5.15}$$

From (b), (4.5.13) and (4.5.15), we obtain

$$\begin{aligned}
&\langle Q(\mu), T^\circ\rangle + 2\langle (B+W\mu)'A(B+W\mu), T^\circ\Sigma T^\circ\rangle \\
&\quad + 2\langle (B+W\mu)'AWA(B+W\mu), \sum_{k=0}^{\infty} T^\circ\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{2k+1}\Sigma^{\frac{1}{2}}T^\circ\rangle \\
&\quad + 2\langle (B+W\mu)'AWAWA(B+W\mu), \sum_{k=1}^{\infty} T^\circ\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{2k}\Sigma^{\frac{1}{2}}T^\circ\rangle \qquad (4.5.16)\\
&= \langle \lambda_1 - \lambda_2, T^\circ\rangle + 2\langle \lambda_1 - \lambda_2, \sum_{k=0}^{\infty} T^\circ\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{2k+1}\Sigma^{\frac{1}{2}}T^\circ\rangle \\
&\quad + 2\langle \lambda_1 + \lambda_2, \sum_{k=1}^{\infty} T^\circ\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}})^{2k}\Sigma^{\frac{1}{2}}T^\circ\rangle.
\end{aligned}$$

By comparing the cofficients of T°, $(T^\circ)^2, \cdots$ on both sides of (4.5.16), we obtain that for all $T^\circ \in N_0$,

$$\langle Q(\mu), T^\circ\rangle = \langle \lambda_1 - \lambda_2, T^\circ\rangle, \qquad (4.5.17)$$

$$\langle (B+W\mu)'A(B+W\mu), T^\circ\Sigma T^\circ\rangle = \langle \lambda_1 + \lambda_2, T^\circ\Sigma T^\circ\rangle, \qquad (4.5.18)$$

$$\langle (B+W\mu)'AWA(B+W\mu), T^\circ\Sigma T^\circ\Sigma T^\circ\rangle = \langle \lambda_1 - \lambda_2, T^\circ\Sigma T^\circ\Sigma T^\circ\rangle, \qquad (4.5.19)$$

and

$$\langle (B+W\mu)'AWAWA(B+W\mu), T^\circ(\Sigma T^\circ)^3\rangle = \langle \lambda_1 + \lambda_2, T^\circ(\Sigma T^\circ)^3\rangle. \qquad (4.5.20)$$

Thus (iv) and (v) follow from (4.5.17) - (4.5.20) and Lemma 4.4.1.

Now suppose that (i) - (v) hold. By (i) and (ii), there exists an orthonormal basis, $\{e_i\}_{i=1}^n$, of E such that $A^{\frac{1}{2}}WA^{\frac{1}{2}} = \sum_{i=1}^{m_1} e_i\square e_i - \sum_{i=m_1+1}^{m_1+m_2} e_i\square e_i$. Thus

$$\begin{aligned}
|I_{np} - A^{\frac{1}{2}}WA^{\frac{1}{2}} \otimes \Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}}| &= |I_{np} - (\sum_{i=1}^{m_1} e_i\square e_i - \sum_{i=m_1+1}^{m_1+m_2} e_i\square e_i) \otimes \Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}}| \\
&= |I_p - \Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}}|^{m_1}|I_p + \Sigma^{\frac{1}{2}}T^\circ\Sigma^{\frac{1}{2}}|^{m_2},
\end{aligned}$$

proving (a). (b) follows directly from (iii) and (c) can be proved by substituting (i)-(v) into (4.5.14). □

Corollary 4.5.4. *In Corollary 4.5.3, conditions (i)-(v) are equivalent to (i)-(iv), (vi) and (vii), where*

(vi) $\lambda_1 + \lambda_2 = (B + W\mu)'A(B + W\mu)$

and

(vii) $(A - AWAWA)(B + W\mu) = 0.$

Proof. Let $A_* = (A^{\frac{1}{2}}WA^{\frac{1}{2}})^2$. By (i) and (ii), A_* is an idempotent of rank $m_1 + m_2$. Thus $I - A_*$ is also an idempotent. By (v),

$$(B + W\mu)'A^{\frac{1}{2}}(I_n - A_*)A^{\frac{1}{2}}(B + W\mu) = 0,$$

which implies that $(I_n - A_*)A^{\frac{1}{2}}(B + W\mu) = 0$. So

$$(A^{\frac{1}{2}} - A^{\frac{1}{2}}WAWA)(B + W\mu) = 0,$$

proving (vii).

'(i) - (iv), (vi) and (vii)' $\Rightarrow$ '(i) - (v)' is obvious. □

By Corollary 4.5.4, we can rewrite Corollary 4.5.3 as follows:

Corollary 4.5.5. *In Theorem 4.5.2, let* $\Sigma_Y = A \otimes \Sigma$ *with* $r(\Sigma) > 1$. *Then*

$$Q(Y) \sim DW_p(m_1, m_2, \Sigma, \lambda_1, \lambda_2),$$

where

$$\lambda_1 = \frac{1}{2}(B + W\mu)'(AWAWA + AWA)(B + W\mu)$$

and

$$\lambda_2 = \frac{1}{2}(B + W\mu)'(AWAWA - AWA)(B + W\mu)$$

if and only if

(i) $AWAWAWA = AWA,$

(ii) $tr(AW) = m_1 - m_2, \qquad tr(AW)^2 = m_1 + m_2,$

(iii) $AB = AC,$

(iv) $Q(\mu) = (B + W\mu)'AWA(B + W\mu),$

and

(v) $(A - AWAWA)(B + W\mu) = 0.$

Corollary 4.5.5 was obtained by Tan (1975, 1976) for the case where $B = C$. Corollary 4.5.3 was given by Khatri (1962, 1963) for the case where $\lambda_2 = 0$ and $m_2 = 0$. We now arrive at our second version of Cochran's theorem.

Theorem 4.5.6. *Suppose that* $Y \sim N(\mu, \Sigma_Y)$. *Let* $i \in \{1, 2, \cdots, \ell\}$, $W_i \in S_E$, $\Sigma \in \mathcal{N}_V$ *with* $\Sigma \neq 0$, *and* $Q_i(Y)$ *be given in (4.5.2). Then* $\{Q_i(Y)\}$ *is an independent family of* $DW_p(m_{1i}, m_{2i}, \Sigma, \lambda_{1i}, \lambda_{2i})$ *random operators if and only if there exists a neighborhood* N_0 *of* 0 *in* $\mathcal{L}(V, V)$ *such that for any distinct* $i, j \in \{1, 2, \cdots, \ell\}$ *and for all* $T_i \in N_0$,

(a) $|I_{np} - \Sigma_Y(W_i \otimes T_i^\circ)| = |I_p - \Sigma T_i^\circ|^{m_{1i}} |I_p + \Sigma T_i^\circ|^{m_{2i}},$

(b) $\Sigma_Y(B_i T_i^*) = \Sigma_Y(C_i T_i^*),$

(c) $$\langle (B_i + C_i + 2W_i\mu)T_i^\circ, [(I_{np} - \Sigma_Y(W_i \otimes T_i^\circ))^{-1}\Sigma_Y]((B_i + C_i + 2W_i\mu)T_i^\circ)\rangle$$
$$= 4[\langle \lambda_{1i}, T_i^\circ(I_p - \Sigma T_i^\circ)^{-1}\Sigma T_i^\circ\rangle + \langle \lambda_{2i}, T_i^\circ(I_p + \Sigma T_i^\circ)^{-1}\Sigma T_i^\circ\rangle],$$

(d) $\lambda_{1i} - \lambda_{2i} = Q_i(\mu),$

(e) $\Sigma_Y(W_i \otimes T_i^\circ)\Sigma_Y(W_j \otimes T_j^\circ)\Sigma_Y = 0,$

(f) $[\Sigma_Y(W_i \otimes T_i^\circ)\Sigma_Y]((B_j + C_j + 2W_j\mu)T_j^\circ) = 0,$

and

(g) $\langle (B_i + C_i + 2W_i\mu)T_i^\circ, \Sigma_Y((B_j + C_j + 2W_j\mu)T_j^\circ)\rangle = 0,$

where $T_i^\circ = (T_i + T_i')/2$ and $T_i^* = (T_i - T_i')/2$.

Theorem 4.5.6 can be proved by Theorem 4.5.2, Theorem 4.3.4, (4.2.7) and (4.2.8).

Corollary 4.5.7. *In Theorem 4.5.6, let $\Sigma_Y = A \otimes \Sigma$ with $r(\Sigma) > 1$. Then $\{Q_i(Y)\}$ is an independent family of $DW_p(m_{1i}, m_{2i}, \Sigma, \lambda_{1i}, \lambda_{2i})$ random operators if and only if for any distinct $i, j \in \{1, 2, \cdots, \ell\}$,*

(i) $AW_iA = AW_iAW_iAW_iA$,

(ii) $tr(AW_i) = m_{1i} - m_{2i}, \qquad tr(AW_i)^2 = m_{1i} + m_{2i}$,

(iii) $AB_i = AC_i$,

(iv) $\lambda_{1i} - \lambda_{2i} = Q_i(\mu) = (B_i + W_i\mu)'AW_iA(B_i + W_i\mu)$,

(v) $\lambda_{1i} + \lambda_{2i} = (B_i + W_i\mu)'A(B_i + W_i\mu) = (B_i + W_i\mu)'AW_iAW_iA(B_i + W_i\mu)$,

(vi) $AW_iAW_jA = 0$.

(vii) $AW_iA(B_j + W_j\mu) = 0$, *and*

(viii) $(B_i + W_i\mu)'A(B_j + W_j\mu) = 0$.

Corollary 4.5.7 can be proved by Theorem 4.5.6 and Corollary 4.5.2. Corollary 4.5.7 was discussed by Tan (1975, 1976) for the case where all $B_i = C_i$. If all $m_{2i} = 0$ and all $\lambda_{2i} = 0$, then (i) and (ii) are equivalent to " $AW_iAW_iA = AW_iA$ and $r(AW_i) = m_{1i}$ " and Corollary 4.5.7 is, therefore, reduced to the standard multivariate Cochran theorem.

Example 4.5.1. *Let $y \sim N_n(\mu, A)$, $i \in \{1, 2, \cdots, \ell\}$, $Q_i(y) = y'W_iy + 2b_i'y + c_i$ with $W_i \in S_E$, $Q(y) = \sum_{i=1}^{\ell} Q_i(y)$, $A = \sum_{i=1}^{\ell} A_i$, $b = \sum_{i=1}^{\ell} b_i$ and $c = \sum_{i=1}^{\ell} c_i$. Consider the following conditions:*

(a) $Q_i(y) \sim D\chi^2_{m_{1i}, m_{2i}}(\lambda_{1i}, \lambda_{2i}) \equiv \chi^2_{m_{1i}}(\lambda_{1i}) - \chi^2_{m_{2i}}(\lambda_{2i})$, *where $\chi^2_{m_{1i}}(\lambda_{1i})$ and $\chi^2_{m_{2i}}(\lambda_{2i})$ are independent noncentral chi-square random variables, and where*

$$m_{1i} = \frac{1}{2}(tr(AW_i)^2 + tr(AW_i)), \qquad m_{2i} = \frac{1}{2}(tr(AW_i)^2 - tr(AW_i)),$$

$$\lambda_{1i} = \frac{1}{2}(W_i\mu + b_i)'(AW_iAW_iA + AW_iA)(W_i\mu + b_i),$$

and

$$\lambda_{2i} = \frac{1}{2}(W_i\mu + b_i)'(AW_iAW_iA - AW_iA)(W_i\mu + b_i);$$

(b) $Q_1(y), Q_2(y), \cdots, Q_\ell(y)$ *are independently distributed;*

(c) $Q(y) \sim D\chi^2_{m_1,m_2}(\lambda_1, \lambda_2)$, *where*

$$m_1 = \frac{1}{2}(tr(AW)^2 + tr(AW)), \qquad m_2 = \frac{1}{2}(tr(AW)^2 - tr(AW)),$$

$$\lambda_1 = \frac{1}{2}(W\mu + b)'(AWAWA + AWA)(W\mu + b),$$

and

$$\lambda_2 = \frac{1}{2}(W\mu + b)'(AWAWA - AWA)(W\mu + b);$$

(d) $r(A^{\frac{1}{2}}WA^{\frac{1}{2}}) = \sum_{i=1}^{\ell} r(A^{\frac{1}{2}}W_iA^{\frac{1}{2}})$ *and* $tr(AW)^2 = \sum_{i=1}^{\ell} tr(AW_i)^2$;

(e) $Q_i(\mu) = (W_i\mu + b_i)'AW_iA(W_i\mu + b_i)$, *and* $(A - AW_iAW_iA)(W_i\mu + b_i) = 0$;

and

(f) $Q(\mu) = (W\mu + b)'AWA(W\mu + b)$, *and* $(A - AWAWA)(W\mu + b) = 0$.

Then

(i). (a), (b) and (f) imply (c), (d) and (e);

(ii). (b), (c) and (e) imply (a), (d) and (f);

(iii). (c), (d) and (e) imply (a), (b) and (f).

Proof. We shall merely prove (iii). (i) and (ii) can be proved easily by using Corollary 4.5.4, Corollary 4.5.5 and Corollary 4.5.7. Suppose that (c), (d) and (e) hold. Then by (c), we obtain (f) and

$$A^{\frac{1}{2}}WAWAWA^{\frac{1}{2}} = A^{\frac{1}{2}}WA^{\frac{1}{2}}. \tag{4.5.21}$$

By Theorem 2.1 of Tan (1975), (4.5.21) and (d) imply that for any distinct i, $j \in \{1, 2, \cdots, \ell\}$,

$$A^{\frac{1}{2}}W_iAW_iAW_iA^{\frac{1}{2}} = A^{\frac{1}{2}}W_iA^{\frac{1}{2}}, \tag{4.5.22}$$

and

$$A^{\frac{1}{2}}W_iAW_jA^{\frac{1}{2}} = 0. \tag{4.5.23}$$

Thus by (4.5.23) and (c),

$$\begin{aligned}&(W_i\mu + b_i)'A(W_j\mu + b_j)\\ &= (W_i\mu + b_i)'AW_jAW_jA((W_j\mu + b_j) \qquad (4.5.24)\\ &= (W_i\mu + b_i)AW_iAW_iAW_jAW_jA(W_j\mu + b_j) = 0.\end{aligned}$$

and

$$AW_iA(W_j\mu + b_j) = AW_iAW_jAW_j(W_j\mu + b_j) = 0. \tag{4.5.25}$$

Now (b) follows from (4.5.23)-(4.5.25) and (a) follows from (4.5.22) and (c). □

4.6. Cochran theorems for $\{Y'W_iY\}$ with $Im\Sigma_Y = S_1\Box S_2$

Although the Cochran theorems obtained in Section 4.4 and 4.5 are very general, verifications of the conditions there are not always easy because of the involvement of variables T_i 's in $\mathcal{L}(V, V)$. The involvement of T_i's is caused by the fact that either Σ_Y may not be of the form $A \otimes \Sigma$ or W_i's are not assumed to be n.n.d. In order to set up some easily verifiable Cochran theorems, we may consider some restrictions either on Σ_Y or on W_i's. In this section, we shall improve some restrictions on Σ_Y to the extent that Σ_Y is not required to have the form $A \otimes \Sigma$. Restrictions on W_i's will be discussed in the next chapter.

Let S_1, S_2 be linear subspaces of E, V respectively and $S_1\Box S_2$ be the linear span of $\{x\Box y : x \in S_1, y \in S_2\}$. Suppose that $Y \sim N_{n\times p}(0, \Sigma_Y)$ with restriction $Im\Sigma_Y = S_1\Box S_2$. Assume also that Σ is n.n.d. We shall show that, in the Cochran theorem, this S_2 is the place in which Σ lies: $Im\Sigma = S_2$. Note that $\Sigma_Y = A \otimes \Sigma$ is the special case of the above restriction where $S_1 = ImA$ and $S_2 = Im\Sigma$.

First, we introduce the notion of inclusion maps. For any subset H of a given set K, the inclusion map of H into K, denoted by $I_{H,K}$ or I_H, is defined by

$$I_H(x) = x, \qquad x \in H. \tag{4.6.1}$$

Note that $I_H(x)$ is used to show how H is embedded in K, see Wong (1986). Suppose that E_1, E_2 are n_1-, n_2-dimensional inner product spaces over $\Re$. Recall that for any $T \in \mathcal{L}(E_1, E_2)$, the adjoint $T' \in \mathcal{L}(E_2, E_1)$ is defined by

$$\langle T'(y), x\rangle = \langle y, T(x)\rangle, \quad x \in E_1, \quad y \in E_2. \tag{4.6.2}$$

Lemma 4.6.1. *Let E, V be n-, p-dimensional inner product spaces over $\Re$, let S, T be linear subspaces of E, V respectively, and let I_S, I_T be the inclusion maps of S, T into E, V respectively. Then*

(a) As functions, $I_S \in \mathcal{L}(S, E)$ and $I_S' = P_S$. Hence as operators $I_S I_S' = P_S$ in $\mathcal{L}(E, E)$.

(b) Let $L \in S \square T$ and $L_* = I_S' L I_T$. Then (i) $P_S L = L = L P_T$, (ii) $L = I_S L_* I_T'$, and (iii) $L_*^+ = I_T' L^+ I_S$. Hence if $dim S = dim T = r(L)$, then L_* is nonsingular and $L_*^{-1} = I_T' L^+ I_S$.

Proof. We shall merely prove (b)(iii): Let $B = I_T' L^+ I_S$. Then

$$L_* B = I_S' L I_T I_T' L^+ I_S = I_S' L P_T L^+ I_S = I_S' L L^+ I_S.$$

Since LL^+ is self-adjoint, so is $L_* B$. Similarly, $BL_* = I_T' L^+ L I_T$ and BL_* is self-adjoint. Now

$$L_* B L_* = (I_S' L I_T)(I_T' L^+ L I_T) = I_S' L P_T L^+ L I_T = I_S' L L^+ L I_T = I_S' L I_T = L_*.$$

Similarly, $BL_*B = B$. Hence $B = L_*^+$. □

Theorem 4.6.2. *Let Y be a random vector of a probability space $(\Omega, \mathcal{A}, P)$ into $\mathcal{L}(V, E)$ such that $Y \sim N(0, \Sigma_Y)$ and for some linear subspaces S_1, S_2 of E, V respectively, $Im\Sigma_Y = S_1 \Box S_2 \neq \{0\}$. Let $W \in \mathcal{S}_E$, $y \in \mathcal{L}(V, E)$, $\Sigma \neq 0$ and $Q(y) = y'Wy$. Then $Q(Y)$ has a $W_p(m, \Sigma)$ distribution for some positive integer m if and only if*

(a) $\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y = \Sigma_Y(W \otimes \Sigma^+)\Sigma_Y$,

(b) $S_2 = Im\Sigma$,

and

(c) $tr(\Sigma_Y(W \otimes \Sigma^+)) = mr(\Sigma) \neq 0$.

Proof. Since $Y \sim N(0, \Sigma_Y)$, $Y \in Im\Sigma_Y$ with probability 1. So we may assume that for any $\omega \in \Omega$, $Y(\omega) \in Im\Sigma_Y$.

Suppose that $Q(Y) \sim W_p(m, \Sigma)$. Then by Theorem 4.4.2, for any $T \in \mathcal{S}_V$,

$$\operatorname{tr}(\Sigma_Y^{\frac{1}{2}}(W \otimes T)\Sigma_Y^{\frac{1}{2}})^k = m\operatorname{tr}(\Sigma^{\frac{1}{2}} T \Sigma^{\frac{1}{2}})^k, \quad k = 1, 2, \ldots. \tag{4.6.3}$$

Let $T = \Sigma^+$. Then from (4.6.3),

$$\operatorname{tr}(\Sigma_Y^{\frac{1}{2}}(W \otimes \Sigma^+)\Sigma_Y^{\frac{1}{2}})^k = mr(\Sigma), \quad k = 1, 2, \ldots.$$

So $\Sigma_Y^{\frac{1}{2}}(W \otimes \Sigma^+)\Sigma_Y^{\frac{1}{2}}$ is idempotent; hence (a) follows. Note that

$$\Sigma_Y^0 = P_{Im\Sigma_Y} = P_{S_1 \Box S_2} = P_{S_1} \otimes P_{S_2}. \tag{4.6.4}$$

So

$$\operatorname{tr}(\Sigma_Y^{\frac{1}{2}}(W \otimes T)\Sigma_Y^{\frac{1}{2}}) = \operatorname{tr}(\Sigma_Y^{\frac{1}{2}}(W \otimes (P_{S_2} T P_{S_2}))\Sigma_Y^{\frac{1}{2}})$$

and therefore by (4.6.3),

$$m\operatorname{tr}(\Sigma^{\frac{1}{2}} T \Sigma^{\frac{1}{2}}) = m\operatorname{tr}(\Sigma^{\frac{1}{2}}(P_{S_2} T P_{S_2})\Sigma^{\frac{1}{2}}).$$

Since $m > 0$, $tr(\Sigma T) = tr(P_{S_2}\Sigma P_{S_2}t)$. i.e., $\langle \Sigma, T\rangle = \langle P_{S_2}\Sigma P_{S_2}, T\rangle$. Since Σ, $P_{S_2}\Sigma P_{S_2}$ are self-adjoint, $\Sigma = P_{S_2}\Sigma P_{S_2}$. In particular

$$Im\Sigma \subset Im P_{S_2} = S_2. \tag{4.6.5}$$

Now let $T = I_p - \Sigma^\circ$. Then by (4.6.3), $\Sigma_Y^{\frac{1}{2}}(W \otimes (I_p - \Sigma^\circ))\Sigma_Y^{\frac{1}{2}} = 0$ and therefore $\Sigma_Y^\circ(W \otimes (I_p - \Sigma^\circ))\Sigma_Y^\circ = 0$, i.e.,

$$(P_{S_1} W P_{S_1}) \otimes (P_{S_2}(I_p - \Sigma^\circ)P_{S_2}) = 0. \tag{4.6.6}$$

By Theorem 4.4.2(e),

$$0 < mr(\Sigma) = tr(\Sigma_Y(W \otimes \Sigma^+)) = tr(\Sigma_Y \Sigma_Y^\circ(W \otimes \Sigma^+)\Sigma_Y^\circ)$$
$$= tr(\Sigma_Y((P_{S_1} W P_{S_1}) \otimes (P_{S_2}\Sigma P_{S_2}))).$$

So $P_{S_1} W P_{S_1} \neq 0$. Thus by (4.6.6), $P_{S_2}(I_p - \Sigma^\circ)P_{S_2} = 0$. i.e., $P_{S_2} = \Sigma^\circ P_{S_2}$ and hence $S_2 \subset Im\Sigma$. Therefore by (4.6.5), $S_2 = Im\Sigma$, proving (b). By Theorem 4.4.2(e), $m = tr(\Sigma_Y(W \otimes \Sigma^+))/r(\Sigma)$; hence (c) follows.

Now suppose that (a) - (c) hold. Consider the inclusion maps I_1 and I_2 of S_1 and S_2 into E and V respectively:

$$I_1(x) = x, \quad x \in S_1, \quad I_2(z) = z, \quad z \in S_2. \tag{4.6.7}$$

Let

$$Z(\omega) = I_1' Y(\omega) I_2, \quad \omega \in \Omega. \tag{4.6.8}$$

Then each $Y(\omega) \in S_1 \square S_2$ and $Z(\omega) = (I_1' \otimes I_2')(Y(\omega))$, $\omega \in \Omega$. So $Z \sim N(0, \Sigma_Z)$ with

$$\Sigma_Z = (I_1' \otimes I_2')\Sigma_Y(I_1 \otimes I_2). \tag{4.6.9}$$

Note that $I_1 \otimes I_2$ is a linear map of $\mathcal{L}(S_2, S_1)$ into $\mathcal{L}(V, E)$ and by Lemma 4.6.1,

$$(I_1 \otimes I_2)\Sigma_Z(I_1' \otimes I_2') = \Sigma_Y. \tag{4.6.10}$$

Hence $r(\Sigma_Z) = r(\Sigma_Y)$ and Σ_Z is nonsingular. By (a),

$$\Sigma_Y^\circ(W \otimes \Sigma^+)\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y^\circ = \Sigma_Y^\circ(W \otimes \Sigma^+)\Sigma_Y^\circ. \tag{4.6.11}$$

By (4.6.4) and Lemma 4.6.1, $\Sigma_Y^\circ = (I_1 \otimes I_2)(I_1 \otimes I_2)'$. So by (4.6.11),

$$(I_1 \otimes I_2)'(W \otimes \Sigma^+)\Sigma_Y(W \otimes \Sigma^+)(I_1 \otimes I_2) = (I_1 \otimes I_2)'(W \otimes \Sigma^+)(I_1 \otimes I_2). \tag{4.6.12}$$

Let

$$W_* = I_1'WI_1, \quad \Sigma_* = I_2'\Sigma I_2. \tag{4.6.13}$$

Then $W_* \in \mathcal{S}_{S_1}$ and $\Sigma_* \in \mathcal{N}_{S_2}$. Since $S_2 = Im\Sigma$, by Lemma 4.6.1, Σ_* is nonsingular and

$$\Sigma_*^{-1} = I_2'\Sigma^+ I_2, \quad \Sigma = I_2\Sigma_* I_2'. \tag{4.6.14}$$

By (4.6.9), (4.6.12), and (4.6.14),

$$(W_* \otimes \Sigma_*^{-1})\Sigma_Z(W_* \otimes \Sigma_*^{-1}) = W_* \otimes \Sigma_*^{-1}. \tag{4.6.15}$$

Now, consider the inclusion map I_* of ImW_* into S_1:

$$I_*(x) = x, \quad x \in ImW_*. \tag{4.6.16}$$

Let

$$Z_*(\omega) = I_*'Z(\omega), \quad \omega \in \Omega. \tag{4.6.17}$$

Then $Z_* = (I_*' \otimes I_{S_2})(Z) \sim N(0, \Sigma_{Z_*})$ with

$$\Sigma_{Z_*} = (I_*' \otimes I_{S_2})\Sigma_Z(I_* \otimes I_{S_2}). \tag{4.6.18}$$

Let

$$W_{\bullet\bullet} = I'_{\bullet} W_{\bullet} I_{\bullet}. \tag{4.6.19}$$

Then by Lemma 4.6.1, $W_{\bullet\bullet}$ is nonsingular and

$$W_{\bullet\bullet}^{-1} = I'_{\bullet} W_{\bullet}^{+} I_{\bullet}. \tag{4.6.20}$$

By Lemma 2.1, $W_{\bullet} = W_{\bullet} P_{ImW_{\bullet}} = W_{\bullet} I_{\bullet} I'_{\bullet}$. So by (4.6.15),

$$((W_{\bullet} I_{\bullet}) \otimes \Sigma_{\bullet}^{-1}) \Sigma_{Z_{\bullet}} ((I'_{\bullet} W_{\bullet}) \otimes \Sigma_{\bullet}^{-1}) = W_{\bullet} \otimes \Sigma_{\bullet}^{-1}. \tag{4.6.21}$$

By multiplying $I'_{\bullet} \otimes I_{S_2}$ from the left and multiplying $I_{\bullet} \otimes I_{S_2}$ from the right, we obtain from (4.6.2) that

$$(W_{\bullet\bullet} \otimes \Sigma_{\bullet}^{-1}) \Sigma_{Z_{\bullet}} (W_{\bullet\bullet} \otimes \Sigma_{\bullet}^{-1}) = W_{\bullet\bullet} \otimes \Sigma_{\bullet}^{-1}.$$

Thus $\Sigma_{Z_{\bullet}} = W_{\bullet\bullet}^{-1} \otimes \Sigma_{\bullet}$. Let

$$Z_{\bullet\bullet} = W_{\bullet\bullet}^{\frac{1}{2}} Z_{\bullet}. \tag{4.6.22}$$

Then $Z_{\bullet\bullet} \sim N(0, \Sigma_{Z_{\bullet\bullet}})$ with $\Sigma_{Z_{\bullet\bullet}} = I_{ImW_{\bullet}} \otimes \Sigma_{\bullet}$. By definition, $Z'_{\bullet\bullet} Z_{\bullet\bullet} \sim W(r(W_{\bullet}), \Sigma_{\bullet})$. So $I_2 Z'_{\bullet\bullet} Z_{\bullet\bullet} I'_2 \sim W_p(r(W_{\bullet}), I_2 \Sigma_{\bullet} I'_2)$. By (4.6.14), $I_2 \Sigma_{\bullet} I'_2 = \Sigma$. So it sufficies to show that $Y'WY = I_2 Z'_{\bullet\bullet} Z_{\bullet\bullet} I'_2$. By (4.6.22), (4.6.19), (4.6.17), (4.6.12), (4.6.8), (4.6.7), and Lemma 4.6.1,

$$\begin{aligned} I_2 Z'_{\bullet\bullet} Z_{\bullet\bullet} I'_2 &= I_2 Z'_{\bullet} W_{\bullet\bullet} Z_{\bullet} I'_2 = I_2 Z' I_{\bullet} I'_{\bullet} W_{\bullet} I_{\bullet} I'_{\bullet} Z I'_2 \\ &= I_2 Z' P_{ImW_{\bullet}} W_{\bullet} P_{ImW_{\bullet}} Z I'_2 = I_2 Z' W_{\bullet} Z I'_2 \\ &= I_2 I'_2 Y' I_1 I'_1 W I_1 I'_1 Y I_2 I'_2 = P_{S_2} Y' P_{S_1} W P_{S_1} Y P_{S_2} = Y'WY. \quad \square \end{aligned}$$

The following result is our third version of Cochran's theorem:

Theorem 4.6.3. *Suppose that* $Y \sim N_{n\times p}(0, \Sigma_Y)$ *with* $Im\Sigma_Y = S_1 \boxtimes S_2 \neq \{0\}$, *where* S_1, S_2 *are linear subspaces of* E, V *respectively. Let* $i \in \{1, 2, \cdots, \ell\}$, $W_i \in S_E$, $y \in \mathcal{L}(V, E)$, $Q_i(y) = y'W_iy$, *and* $\Sigma \in \mathcal{N}_V$ *with* $\Sigma \neq 0$. *Then* $\{Q_i(Y)\}_{i=1}^{\ell}$ *is an independent family of* $W_p(m_i, \Sigma)$ *random operators for some* $m_i \in \{1, 2, \ldots\}$ *if and only if for any distinct* $i, j \in \{1, 2, \cdots, \ell\}$,

(a) $\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y = \Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y$,

(b) $S_2 = Im\Sigma$,

(c) $tr(\Sigma_Y(W_i \otimes \Sigma^+)) = m_i r(\Sigma) \neq 0$,

and

(d) $\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y(W_j \otimes \Sigma^+)\Sigma_Y = 0$.

Proof. Suppose that $\{Q_i(Y)\}$ is an independent family of $W_p(m_i, \Sigma)$ random operators $Q_i(Y)$. By Theorem 4.6.2, (a) - (c) hold. By Corollary 4.4.8,

$$\Sigma_Y(W_i \otimes T_i)\Sigma_Y(W_j \otimes T_j)\Sigma_Y = 0. \tag{4.6.23}$$

Let $T_i = T_j = \Sigma^+$. Then (d) follows from (4.6.23).

Now suppose that (a) - (d) hold. Then by Theorem 4.6.2, $\{Q_i(Y)\}$ is a family of $W_p(m_i, \Sigma)$ random operators $Q_i(Y)$. By Corollary 4.3.5, we need only prove that $\{Q_i(Y)\}$ is pairwise independent. Let i, j be distinct elements in $\{1, 2, \cdots, \ell\}$. By (d) and the notations in the proof of Theorem 4.6.2, we obtain

$$(W_{i\bullet} \otimes I)\Sigma_Z(W_{j\bullet} \otimes I) = 0 \tag{4.6.24}$$

in the same way as we obtained (4.6.21). It can be proved that each $W_{i\bullet}$ is n.n.d. Let $Z_i = W_{i\bullet}^{\frac{1}{2}}Z$. Then $Z_i = (W_{i\bullet}^{\frac{1}{2}} \otimes I)(Z)$ and so by (4.6.24),

$$Cov(Z_i, Z_j) = (W_{i\bullet}^{\frac{1}{2}} \otimes I)\Sigma_Z(W_{j\bullet}^{\frac{1}{2}} \otimes I) = 0.$$

Since Z_i, Z_j are jointly normal, Z_i, Z_j and therefore $\Delta_i \equiv I_2 Z_i' Z_i I_2'$, $\Delta_j \equiv I_2 Z_j' Z_j I_2'$ are independent. By Lemma 4.6.1, $\Delta_i = Q_i(Y)$ and $\Delta_j = Q_j(Y)$. So $Q_i(Y)$ and $Q_j(Y)$ are independent. □

Corollary 4.6.4. *In Theorem 4.6.3, suppose that Σ_Y is nonsingular. Then $\{Q_i(Y)\}$ is an independent family of $W_p(r(W_i), \Sigma)$ random operators $Q_i(Y)$ if and only if for any distinct $i, j \in I$,*

(a) $(W_i \odot \Sigma^+)\Sigma_Y(W_i \odot \Sigma^+) = W_i \odot \Sigma^+$

and

(b) $(W_i \odot \Sigma^+)\Sigma_Y(W_j \odot \Sigma^+) = 0.$

Note that $Im\Sigma_Y = E\Box V$, so Corollary 4.6.4 follows from Theorem 4.6.3 directly.

So far we have concentrated on deriving necessary and sufficient conditions under which $\{Q_i(Y)\}$ is an independent family of $W_p(m_i, \Sigma, \lambda_i)$ random operators. But in practice, it is, mostly, the sufficient condition that is used in statistical inference, and often, one can afford an easily verifiable sufficient condition that is not necessary. We shall give some examples in this direction.

Example 4.6.1. *Suppose that $Y \sim N(0, \Sigma_Y)$. Let $W \in S_E$ such that*

$$\Sigma_Y(W \otimes I_p) = P \odot \Sigma \tag{4.6.25}$$

for some $P \in \mathcal{L}(E, E)$ and $\Sigma \in \mathcal{N}_V$ with $\Sigma \neq 0$. Then $Y'WY$ has a $W_p(m, \Sigma)$ distribution if and only if

(i) $P^3 = P^2$

and

(ii) $\operatorname{tr} P = \operatorname{tr}(P^2) = m.$

Proof. Suppose that $Y \sim W_p(m, \Sigma)$. Then by Corollary 4.4.4(e),

$$\operatorname{tr}(\Sigma_Y(W \otimes T))^k = m\operatorname{tr}(\Sigma T)^k, \quad k = 1, 2, \ldots, \quad T \in S_V. \tag{4.6.26}$$

By (4.6.25),

$$tr(\Sigma_Y(W \otimes T))^k = tr(\Sigma_Y(W \otimes I_p)(I_n \otimes T))^k = tr((P \otimes \Sigma)(I_n \otimes T))^k$$
$$= tr(P \otimes (\Sigma T))^k = tr(P^k)tr(\Sigma T)^k.$$

Since $\Sigma \neq 0$, we obtain from (4.6.26) that

$$tr(P^k) = m, \quad k = 1, 2, \ldots. \tag{4.6.27}$$

So (ii) follows and all eigenvalues of P must be either 0 or 1. By using matrix representations, we may assume that P, $W \in M_{n\times n}$, $\Sigma \in M_{p\times p}$, where W is symmetric and Σ is n.n.d. Now by (4.6.27),

$$P = Q_1 \text{diag}(J_1, J_2, \ldots, J_s)Q_1^{-1}, \tag{4.6.28}$$

where $Q_1 \in M_{n\times n}$, each Jordan block J_i of P is of the form $J = \lambda I + N_\ell$, $\lambda \in \{0, 1\}$, $N_1 = \{0\}$, and

$$N_\ell = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in M_{\ell\times\ell} \quad \text{for} \quad \ell \geq 2.$$

Since Σ is n.n.d.,

$$\Sigma = Q_2 \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)Q_2' \tag{4.6.29}$$

for some $p \times p$ orthogonal matrix Q_2 and for some $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0$. By (4.6.25), (4.6.28), and (4.6.29),

$$\Sigma_Y(W \otimes \Sigma^+) = (Q_1 \otimes Q_2)C(Q_1 \otimes Q_2)^{-1}, \tag{4.6.30}$$

where $C = diag(J_1, J_2, \ldots, J_s) \otimes (\delta_{ij}\mu_j)$ and $\mu_j = 1$ for $j = 1, 2, \ldots, r(\Sigma)$; $\mu_j = 0$ for $j > r(\Sigma)$, where δ_{ij}'s are the Kronecker symbols. Recall that $A \otimes B$ and $B \otimes A$ have similar matrix representations. So

$$C \quad \text{is similar to} \quad D,$$

where $D \equiv diag(\mu_1, \ldots, \mu_p) \otimes diag(J_1, J_2, \ldots, J_s)$. Now

$$D = \mathrm{diag}(\mu_1 \mathrm{diag}(J_1, \ldots, J_s), \mu_2 \mathrm{diag}(J_1, \ldots, J_s), \ldots, \mu_p \mathrm{diag}(J_1, \ldots, J_s)).$$

Since $\Sigma \neq 0$, $\lambda_1 > 0$ and $\mu_1 = 1$. By Proposition 2.4 of Wong (1982), each Jordan block of $\Sigma_Y(W \otimes \Sigma^+)$ is $a \in \Re$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_{2\times 2}$. So by (4.6.30), the J_i's must be 0, 1, or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Therefore by (4.6.28), $P^3 = P^2$, proving (i).

Now suppose that (i) and (ii) hold. Then

$$\begin{aligned} \mathrm{tr}\,(\Sigma_Y(W \otimes T))^k &= \mathrm{tr}\,[\Sigma_Y(W \otimes I_p)(I_n \otimes T)]^k \\ &= \mathrm{tr}\,[(P \otimes \Sigma)(I_n \otimes T)]^k = \mathrm{tr}\,(P^k)\mathrm{tr}\,(\Sigma T)^k \quad k = 1, 2, \cdots. \end{aligned}$$

Thus by (i),

$$\mathrm{tr}\,(\Sigma_Y(W \otimes T)) = \mathrm{tr}\,(P)\mathrm{tr}\,(\Sigma T) \tag{4.6.31}$$

and

$$\mathrm{tr}\,(\Sigma_Y(W \otimes T))^k = \mathrm{tr}\,(P^2)\mathrm{tr}\,(\Sigma T)^k, \quad k = 2, 3, \cdots. \tag{4.6.32}$$

Now (4.6.26) follows from (4.6.31), (4.6.32) and (ii). □

Example 4.6.2. *Suppose that $Y \sim N(0, \Sigma_Y)$ and*

$$\Sigma_Y = A \otimes \Sigma + (M \otimes I_p)H + H'(M' \otimes I_p)$$

for some $A \in \mathcal{N}_E$, $\Sigma \in \mathcal{N}_V$, $M \in \mathcal{L}(E, E)$, and $H \in \mathcal{L}(E, E) \otimes \mathcal{L}(V, V)$. Let $W_i \in \mathcal{S}_E, i \in \{1, 2, \cdots, \ell\}$. Suppose also that for all distinct $i, j \in \{1, 2, \cdots, \ell\}$,

(a) $AW_iAW_iA = AW_iA$,

(b) $W_iM = 0$,

and

(c) $W_iAW_j = 0$.

Then $\{Y'W_iY\}$ is an independent family of $W_p(m_i, \Sigma)$ random operators $Y'W_iY$ with $m_i = tr(AW_i)$.

CHAPTER FIVE

COCHRAN THEOREMS WITH NONNEGATIVE DEFINITE W_i'S

5.1. Introduction

In Chapter 4, we obtained three multivariate versions of Cochran's theorems. In practice, for statistical inference in linear models such as MANOVA and regression models, it is often the case that all W_i's involved in quadratic forms $\{Y'W_iY\}$ are not only self-adjoint but also n.n.d. So in this chapter, we shall consider quadratic forms $Q_i(Y) = Y'W_iY + B_i'Y + Y'C_i + D_i$ $(i = 1, 2, \cdots, \ell)$ with n.n.d. W_i's. A general Cochran theorem is then obtained for a normal random operator Y. This result does not require that the covariance, Σ_Y, of Y is nonsingular or is of the form $A \otimes \Sigma$.

This chapter is organized as follows. In Section 5.2, we shall state and prove some preliminary results. In Section 5.3, an easily verifiable Cochran theorem is obtained and it is an extension of the results of Pavur (1987) and Wong and Wang (1992). Examples and applications of the above Cochran theorem will be discussed in the last section.

5.2. Preliminaries

We shall first prove the following lemmas.

Lemma 5.2.1. *Suppose that Y is a random vector of a probability space into $\mathcal{L}(V, E)$ such that the mean of Y is 0 and the covariance of Y is Σ_Y. Let $W \in \mathcal{N}_E$, $T \in \mathcal{S}_V$. Then*

$$tr(\Sigma_Y(W \otimes T))^k = tr(\Sigma_{Y'}(T \otimes W))^k, \quad k = 1, 2, \cdots. \tag{5.2.1}$$

Proof. Recall that Σ_Y in $\mathcal{L}(\mathcal{L}(V,E),\mathcal{L}(V,E))$ is defined by

$$\langle u, \Sigma_Y v\rangle = cov(\langle u, Y\rangle, \langle v, Y\rangle), \quad u, v \in \mathcal{L}(V,E).$$

Let $K_{p,n}$, be the commutation operator on $\mathcal{L}(V,E)$. Then by (2.2.2),

$$Y' = K_{p,n}(Y), \qquad \Sigma_{Y'} = K_{p,n}\Sigma_Y K_{n,p}.$$

So by Lemma 2.2.1(a),

$$\begin{aligned} \mathrm{tr}\,(\Sigma_{Y'}(T \otimes W))^k &= \mathrm{tr}\,[K_{p,n}\Sigma_Y K_{n,p}(T \otimes W)]^k \\ &= \mathrm{tr}\,[K_{p,n}(\Sigma_Y(W \otimes T))^{k-1}\Sigma_Y K_{n,p}(T \otimes W)] \\ &= \mathrm{tr}\,[(\Sigma_Y(W \otimes T))^{k-1}\Sigma_Y K_{n,p}(T \otimes W)K_{p,n}] \\ &= \mathrm{tr}\,[(\Sigma_Y(W \otimes T))^{k-1}\Sigma_Y(W \otimes T)] \\ &= \mathrm{tr}\,(\Sigma_Y(W \otimes T))^k. \end{aligned}$$

□

For any $T \in \mathcal{L}(V,E)$, we shall use $Ker\,T$ to denote the kernel $\{x \in V : Tx = 0\}$ of T, use $(Im\,T)^{\perp}$ to denote the orthogonal complement of $Im\,T$ and use $dim\,(Im\,T)$ to denote the dimension of $Im\,T$.

Lemma 5.2.2. *Let $W \in \mathcal{N}_E$, $\Sigma \in \mathcal{N}_V$, $\Sigma_{Y'} \in \mathcal{N}_{\mathcal{L}(E,V)}$, let m be a positive integer, and let $\{e_i\}_{i=1}^n, \{f_j\}_{j=1}^p$, be respectively orthonormal bases of E and V such that*

$$W = \sum_{i=1}^{s} w_i e_i \Box e_i, \qquad \Sigma = \sum_{j=1}^{r} \sigma_j f_j \Box f_j,$$

where $s = r(W)$, $r = r(\Sigma)$, $w_i > 0$, $i = 1, 2, \cdots, s$, and $\sigma_j > 0$, $j = 1, \cdots, r$. Then

$$tr(\Sigma_{Y'}(T \otimes W))^2 = m\,tr(\Sigma T)^2 \qquad \text{for all} \quad T \in \mathcal{S}_V \qquad (5.2.2)$$

if and only if

$$tr(W\Sigma_{jj}W\Sigma_{j'j'}) = m\sigma_j\sigma_{j'}, \quad j,j' = 1,2,\cdots,r, \tag{5.2.3}$$

$$\Sigma_{jj} = 0, \quad j = r+1,\cdots,p, \tag{5.2.4}$$

and

$$\Sigma_{jj'} = 0, \quad j \neq j', \quad j,j' = 1,2,\cdots,p, \tag{5.2.5}$$

where $\Sigma_{jj'} = \sum_{i,i'=1}^{s} \sigma_{jj'ii'} c_i \Box c_{i'}$ and $\sigma_{jj'ii'} = \langle f_j \Box e_i, \Sigma_{Y'}(f_{j'} \Box e_{i'})\rangle$ for all $i,i' = 1,2,\cdots,n$, $j,j' = 1,2,\cdots,p$.

Proof. Suppose that for any $T \in \mathcal{S}_V$, (5.2.2) holds. Since $\Sigma_{Y'} \in \mathcal{N}_{\mathcal{L}(E,V)}$,

$$\sigma_{jj'ii'} = \sigma_{j'ji'i}. \tag{5.2.6}$$

Since $\{(f_j \Box f_{j'}) \otimes (e_i \Box e_{i'})\}$ is an orthonormal basis of $\mathcal{L}(V,V) \otimes \mathcal{L}(E,E)$,

$$\begin{aligned} \Sigma_{Y'} &= \sum_{j,j'=1}^{p} \sum_{i,i'=1}^{n} \sigma_{jj'ii'}(f_j \Box f_{j'}) \odot (c_i \Box c_{i'}) \\ &= \sum_{j,j'=1}^{p} \sum_{i,i'=1}^{s} \sigma_{jj'ii'}(f_j \Box f_{j'}) \otimes (c_i \Box c_{i'}) + H, \end{aligned} \tag{5.2.7}$$

where

$$H = \sum_{j,j'=1}^{p} \left[\sum_{i,i'=s+1}^{n} + \sum_{i'=s+1}^{n} \sum_{i=1}^{s} + \sum_{i=s+1}^{n} \sum_{i'=1}^{s} \right] \sigma_{jj'ii'}(f_j \Box f_{j'}) \otimes (c_i \Box c_{i'}). \tag{5.2.8}$$

Therefore

$$\Sigma_{Y'}(T \otimes W) = \sum_{i'=1}^{s} \sum_{i=1}^{n} \sum_{j,j'=1}^{p} w_{i'} \sigma_{jj'ii'}(f_j \Box T f_{j'}) \otimes (c_i \Box c_{i'}).$$

Let j, $j' = 1,2,\cdots,p$ and $\delta_{ii'}$'s be Kronecker symbols. Then

$$\begin{aligned} \operatorname{tr}(\Sigma_{Y'}(T \otimes W))^2 = \operatorname{tr} \Bigg\{ & \sum_{i'_1,i'_2=1}^{s} \sum_{i_1,i_2=1}^{n} \sum_{j_1,j'_1,j_2,j'_2=1}^{p} w_{i'_1} w_{i'_2} \sigma_{j_1 j'_1 i_1 i'_1} \sigma_{j_2 j'_2 i_2 i'_2} \\ & \times \left[(f_{j_1} \Box T f_{j'_1}) \otimes (e_{i_1} \Box e_{i'_1})\right] \left[(f_{j_2} \Box T f_{j'_2}) \otimes (e_{i_2} \Box e_{i'_2})\right] \Bigg\} \\ = & \sum_{i,i'=1}^{s} \sum_{j_1,j'_1,j_2,j'_2=1}^{p} w_i w_{i'} \sigma_{j_1 j'_1 i i'} \sigma_{j_2 j'_2 i' i} \langle f_{j_1}, T f_{j'_2}\rangle \langle f_{j'_1}, T f_{j_2}\rangle. \end{aligned}$$

Since

$$\Sigma T = \sum_{j=1}^{r} \sigma_j (f_j \Box f_j) T = \sum_{j=1}^{r} \sigma_j (f_j \Box T f_j).$$

$$tr(\Sigma T)^2 = \sum_{j_1, j_2 = 1}^{r} \sigma_{j_1} \sigma_{j_2} (\langle f_{j_1}, T f_{j_2} \rangle)^2.$$

Therefore (5.2.2) becomes

$$\sum_{i,i'=1}^{s} \sum_{j_1, j_1', j_2, j_2' = 1}^{p} w_i w_{i'} \sigma_{j_1 j_1' i i'} \sigma_{j_2 j_2' i' i} \langle f_{j_1}, T f_{j_2'} \rangle \langle f_{j_1'}, T f_{j_2} \rangle$$

$$= \sum_{j_1, j_2 = 1}^{r} \sigma_{j_1} \sigma_{j_2} (\langle f_{j_1}, T f_{j_2} \rangle)^2 \quad \text{for all} \quad T \in \mathcal{S}_V. \tag{5.2.9}$$

Let $T = f_j \Box f_j$. Then

$$\langle f_{j_1}, T f_{j_2'} \rangle \langle f_{j_1'}, T f_{j_2} \rangle = \delta_{j_1 j} \delta_{j_1' j} \delta_{j_2 j} \delta_{j_2' j}$$

and

$$(\langle f_{j_1}, T f_{j_2} \rangle)^2 = \delta_{j_1 j} \delta_{i_2 j} \delta_{j_1 j} \delta_{j_2 j}.$$

Thus from (5.2.9), we obtain

$$\sum_{i,i'=1}^{s} w_i w_{i'} \sigma_{jjii'}^2 = m \sigma_j^2 \quad \text{for} \quad j = 1, 2, \cdots, r, \tag{5.2.10}$$

and

$$\sum_{i,i'=1}^{s} w_i w_{i'} \sigma_{jjii'}^2 = 0 \quad \text{for } j = r+1, \cdots, p. \tag{5.2.11}$$

For distinct j and j', let $T = f_j \Box f_j + f_{j'} \Box f_{j'}$. Then

$$\langle f_{j_1}, T f_{j_2'} \rangle \langle f_{j_1'}, T f_{j_2} \rangle = (\delta_{j_1 j} \delta_{j_2' j} + \delta_{j_1 j'} \delta_{j_2' j'})(\delta_{j_1' j} \delta_{j_2 j} + \delta_{j_1' j'} \delta_{j_2 j'})$$

$$= \delta_{j_1 j} \delta_{j_2' j} \delta_{j_1' j} \delta_{j_2 j} + \delta_{j_1 j'} \delta_{j_2' j'} \delta_{j_1' j} \delta_{j_2 j} + \delta_{j_1 j} \delta_{j_2' j} \delta_{j_1' j'} \delta_{j_2 j'} + \delta_{j_1 j'} \delta_{j_2' j'} \delta_{j_1' j'} \delta_{j_2 j'}$$

and

$$(\langle f_{j_1}, T f_{j_2} \rangle)^2 = (\delta_{j_1 j} \delta_{j_2 j} + \delta_{j_1 j'} \delta_{j_2 j'})^2.$$

Thus (5.2.9) becomes

$$\sum_{i,i'=1}^{s} w_i w_{i'}(\sigma_{jjii'}^2 + \sigma_{j'j'ii'}^2 + \sigma_{jj'ii'}\sigma_{j'ji'i} + \sigma_{j'jii'}\sigma_{jj'i'i}) = m(\sigma_j^2 + \sigma_{j'}^2).$$

which, by (5.2.10) and (5.2.11), implies that

$$\sum_{i,i'=1}^{s} w_i w_{i'}(\sigma_{jj'ii'}^2 + \sigma_{jj'i'i}^2) = 0. \qquad (5.2.12)$$

Since $w_i > 0$ for $i = 1, 2, \cdots, s$, (5.2.11) and (5.2.12) become

$$\sigma_{jjii'} = 0 \quad \text{for} \quad i, i' = 1, 2, \cdots, s, \quad j = r+1, \cdots, p,$$

and

$$\sigma_{jj'ii'} = \sigma_{jj'i'i} = 0 \quad \text{for} \quad j \neq j', \quad j, j' = 1, 2, \cdots, p, \quad i, i' = 1, 2, \cdots, s. \quad (5.2.13)$$

Thus

$$\Sigma_{jj} = \sum_{i,i'=1}^{s} \sigma_{jjii'} e_i \Box e_{i'} = 0 \quad \text{for} \quad j = r+1, \cdots, p$$

and

$$\Sigma_{jj'} = \sum_{i,i'=1}^{s} \sigma_{jj'ii'} e_1 \Box e_{i'} = 0 \quad \text{for} \quad j \neq j', \quad j, j' = 1, 2, \cdots, p,$$

proving (5.2.4) and (5.2.5).

Now let $T = f_j \Box f_{j'} + f_{j'} \Box f_j$ with $j \neq j'$. Then

$$\langle f_{j_1}, T f_{j_2'} \rangle \langle f_{j_1'}, T f_{j_2} \rangle = (\delta_{j_1 j}\delta_{j_2' j'} + \delta_{j_1 j'}\delta_{j_2' j})(\delta_{j_1' j}\delta_{j_2 j'} + \delta_{j_1' j'}\delta_{j_2 j})$$

$$= \delta_{j_1 j}\delta_{j_2' j'}\delta_{j_1' j}\delta_{j_2 j'} + \delta_{j_1 j'}\delta_{j_2' j}\delta_{j_1' j}\delta_{j_2 j'} + \delta_{j_1 j}\delta_{j_2' j'}\delta_{j_1' j'}\delta_{j_2 j} + \delta_{j_1 j'}\delta_{j_2' j}\delta_{j_1' j'}\delta_{j_2 j}$$

and

$$(\langle f_{j_1}, T f_{j_2} \rangle)^2 = (\delta_{j_1 j}\delta_{j_2 j'} + \delta_{j_1 j'}\delta_{j_2 j})^2.$$

Thus (5.2.9) becomes

$$\sum_{i,i'=1}^{s} w_i w_{i'}(\sigma_{jjii'}\sigma_{j'j'i'i} + \sigma_{j'jii'}\sigma_{j'ji'i} + \sigma_{jj'ii'}\sigma_{jj'i'i} + \sigma_{j'j'ii'}\sigma_{jji'i}) = m(\sigma_j\sigma_{j'} + \sigma_{j'}\sigma_j),$$

which, by (5.2.13), implies that

$$\sum_{i,i'=1}^{s} w_i w_{i'} \sigma_{jjii'} \sigma_{j'j'i'i} = m\sigma_j \sigma_{j'}, \quad j \neq j', \quad j, j' = 1, \cdots, r. \tag{5.2.14}$$

Combining (5.2.10) and (5.2.14), we obtain

$$\sum_{i,i'=1}^{s} w_i w_{i'} \sigma_{jjii'} \sigma_{j'j'i'i} = m\sigma_j \sigma_{j'}, \quad j, j = 1, 2, \cdots, r.$$

i.e.

$$tr(W\Sigma_{jj} W \Sigma_{j'j'}) = m\sigma_j \sigma_{j'}, \quad j, j' = 1, 2, \cdots, r,$$

proving (5.2.3).

Now suppose that (5.2.3) - (5.2.5). Then from (5.2.7) and (5.2.8), we obtain

$$\begin{aligned}
\Sigma_{Y'} &= \sum_{j,j'=1}^{p} (f_j \Box f_{j'}) \otimes \left(\sum_{i,i'=1}^{s} \sigma_{jj'ii'} (e_i \Box e_{i'}) \right) + H \\
&= \sum_{j,j'=1}^{p} (f_j \Box f_{j'}) \otimes \Sigma_{jj'} + H.
\end{aligned}$$

By (5.2.4) and (5.2.5),

$$\Sigma_{Y'} = \sum_{j=1}^{r} ((f_j \Box f_j) \otimes \Sigma_{jj}) + H.$$

For $i, i' = s+1, \cdots, n$, since $e_i, e_{i'} \in \ker W$, we have $(I_p \otimes W) H (I_p \otimes W) = 0$. So $(T \otimes W) H (T \otimes W) = 0$ and

$$\begin{aligned}
\operatorname{tr}(\Sigma_{Y'}(T \otimes W))^2 &= \operatorname{tr} \left[\left(\sum_{j=1}^{r} (f_j \Box f_j) \otimes \Sigma_{jj}\right)(T \otimes W) + H(T \otimes W) \right]^2 \\
&= \operatorname{tr} \left[\sum_{j=1}^{r} (f_j \Box T f_j) \otimes \Sigma_{jj} W + H(T \otimes W) \right]^2 \\
&= \operatorname{tr} \left[\sum_{j=1}^{r} (f_j \Box T f_j) \otimes \Sigma_{jj} W \right]^2 \\
&= \sum_{j,j'=1}^{r} (f_j \Box T f_j)(f_{j'} \Box T f_{j'}) \operatorname{tr}(\Sigma_{jj} W \Sigma_{j'j'} W).
\end{aligned}$$

Thus by (5.2.3),

$$\begin{aligned}\operatorname{tr}(\Sigma_{Y'}(T\otimes W))^2 &= m\sum_{j,j'=1}^{r}(f_j\Box Tf_j)(f_{j'}\Box Tf_{j'})\sigma_j\sigma_{j'}\\ &= m\operatorname{tr}(\sum_{j=1}^{r}\sigma_j(f_j\Box Tf_j))^2\\ &= m\operatorname{tr}[\sum_{j=1}^{r}\sigma_j(f_j\Box f_j)T]^2,\end{aligned}$$

i.e.,

$$\operatorname{tr}(\Sigma_{Y'}(T\otimes W))^2 = m\operatorname{tr}(\Sigma T)^2,$$

proving (5.2.7). □

The above random vector Y' is introduced for convenience. Indeed, the matrix representation, $[\Sigma_{Y'}]$, with respect to usual bases in $\Re^n$ and $\Re^p$, can be written in the form:

$$[\Sigma_{Y'}] = \operatorname{diag}(\Sigma_{11},\Sigma_{22},\cdots,\Sigma_{ss},0,\cdots,0)+[H]$$

with $(I_p\otimes W)H(I_p\otimes W)=0$. The same cannot be said for Σ_Y.

Lemma 5.2.3. *In Lemma 5.2.2, let* $i,i'=1,2,\cdots,s$, $j,j'=1,\cdots,r$, $a_{jii'}=\sigma_{jjii'}/\sigma_j$ *and* $A_j=\sum_{i,i'=1}^{s}a_{jii'}e_i\Box e_{i'}=\Sigma_{jj}/\sigma_j$. *Then the following two conditions are equivalent:*

(a) $tr(W\Sigma_{jj}W\Sigma_{j'j'})=m\sigma_j\sigma_{j'},\quad j,j'=1,2,\cdots,r;$

(b) there exists an $A\in\mathcal{N}_E$ *such that*

(i) $A=A_j$ *for* $j=1,2,\cdots,r,$

(ii) $tr(WA)^2=m.$

Proof. Suppose that (a) holds, i.e.

$$\sum_{i,i'=1}^{s}w_iw_{i'}\sigma_{jjii'}\sigma_{j'j'ii'}=m\sigma_j\sigma_{j'},\quad j,j'=1,2,\cdots,r.$$

Then for $j \neq j'$, $j, j' = 1, 2, \cdots, r$,

$$m = \sum_{i,i'=1}^{s} w_i w_{i'} a_{jii'}^2 = \sum_{i,i'=1}^{s} w_i w_{i'} a_{j'ii'}^2 = \sum_{i,i'=1}^{s} w_i w_{i'} a_{jii'} a_{j'ii'}.$$

So

$$\begin{aligned} &\sum_{i,i'=1}^{s} w_i w_{i'} [a_{jii'}^2 - 2a_{jii'} a_{j'ii'} + a_{j'ii'}^2] \\ &= \sum_{i,i'=1}^{s} w_i w_{i'} (a_{jii'} - a_{j'ii'})^2 = 0. \end{aligned} \tag{5.2.15}$$

Since $w_i > 0$, $i = 1, 2, \cdots, s$, we obtain from (5.2.15)

$$a_{jii'} = a_{j'ii'} \quad \text{for } j \neq j', \quad i, i' = 1, 2, \cdots, s, \quad j, j' = 1, 2, \cdots, r.$$

So for $j, j' = 1, 2, \cdots, r$,

$$A_j = \sum_{i,i'=1}^{s} a_{jii'} e_i \Box e_{i'} = \sum_{i,i'=1}^{s} a_{j'ii'} e_i \Box e_{i'} = A_{j'}.$$

Let $A = A_1$. Then it suffices to show that $A \in \mathcal{N}_E$. For any $x = \sum_{\alpha=1}^{n} x_\alpha e_\alpha \in E$, we have

$$\begin{aligned} \langle x, Ax \rangle &= \langle \sum_{\alpha=1}^{n} x_\alpha e_\alpha, \sum_{i,i'=1}^{s} \frac{\sigma_{11ii'}}{\sigma_1} (e_i \Box e_{i'}) (\sum_{\alpha=1}^{n} x_\alpha e_\alpha) \rangle \\ &= \frac{1}{\sigma_1} \sum_{i,i'=1}^{s} x_i x_{i'} \sigma_{11ii'}, \end{aligned} \tag{5.2.16}$$

and

$$\begin{aligned} 0 &\leq \langle f_1 \Box \sum_{\alpha=1}^{s} x_\alpha e_\alpha, \Sigma_{Y'} (f_1 \Box \sum_{\beta=1}^{s} x_\beta e_\beta) \rangle \\ &= \langle \sum_{\alpha=1}^{s} x_\alpha (f_1 \Box e_\alpha), \sum_{i,i'=1}^{n} \sum_{j,j'=1}^{p} \sigma_{jj'ii'} (f_j \Box f_{j'}) \otimes (e_i \Box e_{i'}) (\sum_{\beta=1}^{s} x_\beta (f_1 \Box e_\beta)) \rangle \quad (5.2.16) \\ &= \sum_{i,i'=1}^{s} x_i x_{i'} \sigma_{11ii'}. \end{aligned}$$

Since $\sigma_1 > 0$, from (5.2.16) and (5.2.17), we have $A \in \mathcal{N}_E$ and hence (b)(i) follows. By (a),

$$m = \frac{1}{\sigma_j \sigma_{j'}} tr(W\Sigma_{jj}W\Sigma_{j'j'}) = tr(WA_jWA_{j'}).$$

Thus (b)(ii) follows from (b)(i).

Now assume that (b)(i) and (b)(ii) hold. Then

$$tr(W\Sigma_{jj}W\Sigma_{jj'}) = tr(W\frac{\Sigma_{jj}}{\sigma_j}W\frac{\Sigma_{jj'}}{\sigma_{j'}})\sigma_j\sigma_{j'}$$
$$= tr(WA_jWA_{j'})\sigma_j\sigma_{j'} = tr(WAWA)\sigma_j\sigma_{j'} = m\sigma_j\sigma_{j'}$$

proving (a). □

Lemma 5.2.4. *In Lemma 5.2.2, the following conditions are equivalent:*

(a) For all $T \in \mathcal{S}_V$, $tr(\Sigma_{Y'}(T \otimes W))^k = m\,tr(\Sigma T)^k$, $\quad k = 1, 2, \cdots$.

(b) There exists an $A \in \mathcal{N}_E$ such that

(i) $(I_p \otimes W)(\Sigma_{Y'} - \Sigma \otimes A)(I_p \otimes W) = 0$

and

(ii) $AWAW = AW$, $\quad r(AW) = m$.

Proof. Suppose that (a) holds. Then by letting $k = 2$, we obtain from Lemma 5.2.2 that

$$\mathrm{tr}\,(W\Sigma_{jj}W\Sigma_{j'j'}) = m\sigma_j\sigma_{j'}, \quad j, j' = 1, 2, \cdots, r$$

and

$$\Sigma_{Y'} = \sum_{j=1}^{r}(f_j \Box f_j) \otimes \Sigma_{jj} + H, \tag{5.2.18}$$

where H is given in (5.2.8). Then from Lemma 5.2.3, we know that there exists an $A \in \mathcal{N}_E$ such that $A = A_j = \Sigma_{jj}/\sigma_j$, $j = 1, 2, \cdots, r$. Thus, (5.2.18) becomes

$$\Sigma_{Y'} = \sum_{j=1}^{r}\sigma_j(f_j \Box f_j) \oslash \Sigma_{jj}/\sigma_j + H$$
$$= \sum_{j=1}^{r}\sigma_j(f_j \Box f_j) \oslash A + H = \Sigma \oslash A + H.$$

Since $e_i, e_{i'} \in KerW$ for $i, i' = s+1, \cdots, n$,

$$(I_p \otimes W)(\Sigma_{Y'} - \Sigma \otimes A)(I_p \otimes W) = (I_p \otimes W)H(I_p \otimes W) = 0,$$

proving b(i). Thus by (b)(i),

$$\begin{aligned} \text{tr}\,(\Sigma_{Y'}(T \otimes W))^k &= \text{tr}\,[(\Sigma \otimes A + H)(T \otimes W)]^k \\ &= \text{tr}\,[(\Sigma T) \otimes (AW) + H(T \otimes W)]^k \\ &= \text{tr}\,[(\Sigma T) \otimes (AW)]^k = \text{tr}\,(\Sigma T)^k \text{tr}\,(AW)^k. \end{aligned}$$

Hence for $T = \Sigma^+$,

$$\text{tr}\,(\Sigma_{Y'}(\Sigma^+ \otimes W))^k = \text{tr}\,(\Sigma^0)^k \text{tr}\,(AW)^k = r\text{tr}\,(AW)^k. \qquad (5.2.19)$$

Since $r = r(\Sigma) > 0$, from (a) and (5.2.19), we obtain

$$\text{tr}\,(AW)^k = m, \quad k = 1, 2, \cdots .$$

Also, since $A \in \mathcal{N}_E$, we have

$$\text{tr}\,(AW)^k = \text{tr}\,(A^{\frac{1}{2}}WA^{\frac{1}{2}})^k = m, \quad \text{for} \quad k = 1, 2, \cdots .$$

So $A^{\frac{1}{2}}WA^{\frac{1}{2}}$ is an idempotent of rank m, i.e.,

$$r(A^{\frac{1}{2}}WA^{\frac{1}{2}}) = m$$

and

$$A^{\frac{1}{2}}WA^{\frac{1}{2}}A^{\frac{1}{2}}WA^{\frac{1}{2}} = A^{\frac{1}{2}}WA^{\frac{1}{2}},$$

the latter of which is equivalent to

$$AWAW = AW \quad (\text{or} \quad WAWA = WA).$$

Since $r(AW) = r(A^{\frac{1}{2}}WA^{\frac{1}{2}})$, (b)(ii) follows.

Now we assume that (b) holds. Let $H = \Sigma_{Y'} - \Sigma \otimes A$. Then for all $T \in S_{Y'}$ and $k = 1, 2, \cdots$,

$$\begin{aligned}
\operatorname{tr}(\Sigma_{Y'}(T \otimes W))^k &= \operatorname{tr}[(\Sigma \otimes A + H)(T \otimes W)]^k \\
&= \operatorname{tr}[(\Sigma T) \otimes (AW) + H(T \otimes W)]^k \\
&= \operatorname{tr}(\Sigma T)^k \operatorname{tr}(AW)^k \quad \text{(by (b)(i))} \\
&= \operatorname{tr}(A^{\frac{1}{2}}WA^{\frac{1}{2}})^k \operatorname{tr}(\Sigma T)^k \\
&= m\operatorname{tr}(\Sigma T)^k, \quad \text{(by (b)(ii))}
\end{aligned}$$

proving (a). □

The statement of the following result was given to us by Professor Rong-Lin Fu and Professor Wei-Cai Deng, the proof was given by Professor, Wong and the detail of the proof is listed here:

Lemma 5.2.5. *Let $W_1, W_2, \cdots, W_\ell \in \mathcal{N}_E$ and $W = W_1 + W_2 + \cdots + W_\ell$. Then*

(i) $Ker\, W = Ker\, W_1 \cap Ker\, W_2 \cap \cdots \cap Ker\, W_\ell$

and

(ii) $Im\, W = Im\, W_1 + Im\, W_2 + \cdots + Im\, W_\ell$.

Proof. By induction we may assume that $\ell = 2$.

(i) Let $x \in Ker\, W$. Then

$$Wx = (W_1 + W_2)x = W_1x + W_2x = 0,$$

and hence

$$\langle x, Wx\rangle = \langle x, W_1x\rangle + \langle x, W_2x\rangle = 0. \tag{5.2.20}$$

Since $W_1, W_2 \in \mathcal{N}_E$, (5.2.20) implies that

$$\langle x, W_1x\rangle = 0 \quad \text{and} \quad \langle x, W_2x\rangle = 0,$$

i.e. $W_1x = 0$ and $W_2x = 0$. So

$$x \in Ker\, W_1 \cap Ker\, W_2. \tag{5.2.21}$$

Now let $x \in Ker\, W_1 \cap Ker\, W_2$. Then $W_1x = 0$ and $W_2 = 0$ and hence $W_1x + W_2x = Wx = 0$. Thus

$$x \in Ker\, W. \tag{5.2.22}$$

Combining (5.2.21) and (5.2.22), we obtain (i).

(ii) Let $x \in Im\, W$. Then there exists a $y \in E$ such that $x = Wy = W_1y + W_2y$. Since $W_1y \in Im\, W_1$ and $W_2y \in Im\, W_2$, $x \in Im\, W_1 + Im\, W_2$, i.e.,

$$Im\, W \subset Im\, W_1 + Im\, W_2.$$

Thus it suffices to show that

$$dim\,(Im\, W) = dim\,(Im\, W_2 + Im\, W_2).$$

Let $x \in Ker\, W_1 \cap Ker\, W_2$. Then for all $y, z \in E$,

$$\langle y,\, W_1x\rangle + \langle z,\, W_2x\rangle = 0,$$

i.e.,

$$\langle x,\, W_1y\rangle + \langle x,\, W_2z\rangle = \langle x,\, W_1y + W_2z\rangle = 0.$$

So $x \in (Im\, W_1 + Im\, W_2)^{\perp}$, i.e.,

$$Ker\, W_1 \cap Ker\, W_2 \subset (Im\, W_1 + Im\, W_2)^{\perp}. \tag{5.2.23}$$

Let $x \in (Im\, W_1 + Im\, W_2)^{\perp}$. Then $\langle x,\, W_1y + W_2z\rangle = 0$ for any $y, z \in E$, which is equivalent to

$$\langle y,\, W_1x\rangle = 0 \qquad \langle y,\, W_1x\rangle = 0 \quad \text{for any} \quad y,\, z \in E.$$

So $x \in Ker\, W_1 \cap Ker\, W_2$ and hence by (5.2.23),

$$Ker\, W_1 \cap Ker\, W_2 = (Im\, W_1 + Im\, W_2)^{\perp}. \tag{5.2.24}$$

Now by (i) and (5.2.24),

$$\begin{aligned} dim\,(Im\, W) &= n - dim\,(Ker\, W) = n - dim\,(Ker\, W_1 \cap Ker\, W_2) \\ &= n - dim\,((Im\, W_1 + Im\, W_2)^{\perp}) = n - (n - dim\,(Im\, W_1 + Im\, W_2)) \\ &= dim\,(Im\, W_1 + Im\, W_2). \end{aligned}$$

□

Note that n.n.d. properties of W and W_i's are essential for the above Lemma 5.2.2 - Lemma 5.2.5.

5.3. Cochran theorems with nonnegative definite W_i's

Theorem 5.3.1. *Let Y be a random vector of a probability space into $\mathcal{L}(V, E)$ such that $Y \sim N_{n\times p}(0, \Sigma_Y)$, $W \in \mathcal{N}_E$ and let $\Sigma \in \mathcal{N}_V$ with $\Sigma \neq 0$. Then the following conditions are equivalent:*

(a) $Q(Y) \equiv Y'WY \sim W_p(m, \Sigma)$,

(b) There exists an $A \in \mathcal{N}_E$ such that

(i) $(W \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W \otimes I_p) = 0$

and

(ii) $AWAW = AW, \quad r(AW) = m$.

Proof. By Corollary 4.4.3 with $\mu = 0$, we know that $Q(Y) \sim W_p(m, \Sigma)$ if and only if for any $T \in \mathcal{S}_V$,

$$\text{tr}\,(\Sigma_Y(W \otimes T))^k = m\text{tr}\,(\Sigma T)^k, \quad k = 1, 2, \cdots.$$

Also by Lemma 5.2.1, we have

$$\operatorname{tr}(\Sigma_Y(W \otimes T))^k = \operatorname{tr}(\Sigma_{Y'}(T \otimes W))^k, \quad k = 1, 2, \cdots,$$

and

$$\Sigma_{Y'} = K_{p,n}\Sigma_Y K_{n,p},$$

where $K_{p,n}$ is the commutation operator on $\mathcal{L}(V, E)$ defined by (2.2.2). So by Lemma 5.2.4, it suffices to show that (i) is equivalent to

(iii) $(I_p \otimes W)(\Sigma_{Y'} - \Sigma \otimes A)(I_p \otimes W) = 0.$

Suppose that (i) holds. Let $H = \Sigma_Y - A \otimes \Sigma$. Then by Lemma 2.2.1,

$$\begin{aligned}\Sigma_{Y'} &= K_{p,n}\Sigma_Y K_{n,p} = K_{p,n}[(A \otimes \Sigma) + H]K_{n,p} \\ &= K_{p,n}(A \otimes \Sigma)K_{n,p} + K_{p,n}HK_{n,p} = \Sigma \otimes A + K_{p,n}HK_{n,p}.\end{aligned}$$

Thus by Lemma 2.2.1 again,

$$\begin{aligned}&(I_p \otimes W)(\Sigma_{Y'} - \Sigma \otimes A)(I_p \otimes W) \\ &= (I_p \otimes W)K_{p,n}HK_{n,p}(I_p \otimes W) \\ &= K_{p,n}K_{n,p}(I_p \otimes W)K_{p,n}HK_{n,p}(I_p \otimes W)K_{p,n}K_{n,p} \\ &= K_{p,n}(W \otimes I_p)H(W \otimes I_p)K_{n,p}.\end{aligned} \tag{5.3.1}$$

So by (i) and (5.3.1), (iii) follows.

By a similar argument, we can prove that (iii) $\Rightarrow$ (i). □

Example 5.3.1. *In Theorem 5.3.1, if $W \in \mathcal{N}_E$ is positive definite, then the following conditions are equivalent:*

(a′) $Q(Y) \sim W_p(m, \Sigma)$.

(b′) There exists an A in $\mathcal{N}_E$ such that

(i) $\Sigma_Y = A \otimes \Sigma$

and

(ii) $AWA = A, \quad r(A) = m.$

Proof. Suppose first that (a′) holds. By Theorem 5.3.1, there exists an $A \in \mathcal{N}_E$ such that

$$(W \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W \otimes I_p) = 0.$$

Multiplying both sides by $W^{-1} \otimes I_p$, we obtain $\Sigma_Y - A \otimes \Sigma = 0$, proving (b′)(i). By (b)(ii) and the existence of W^{-1}, we have

$$AWA = A \quad \text{and} \quad r(AW) = r(A) = m,$$

proving (b′).

Also by Theorem 5.3.1, (b′) $\Rightarrow$ (a′). □

Now we shall generalize Theorem 5.3.1 to include the noncentral case, i.e. $Y \sim N_{n\times p}(\mu, \Sigma_Y)$ with $\mu \in \mathcal{L}(V, E)$.

Theorem 5.3.2. *Suppose that $Y \sim N_{n\times p}(\mu, \Sigma_Y)$. Let $W \in \mathcal{N}_E$, $\Sigma \in \mathcal{N}_V$ with $r(\Sigma) > 1$, and $Q(Y) = Y'WY + B'Y + Y'C + D$ with $Im\, B \subset Im\, W$ and $Im\, C \subset Im\, W$. Then $Q(Y) \sim W_p(m, \Sigma, \lambda)$ if and only if there exists an $A \in \mathcal{N}_E$ such that*

(a1) $(W \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W \otimes I_p) = 0,$

(b1) $AWAW = AW, \quad r(AW) = m,$

(c1) $AB = AC,$

and

(d1) $\lambda = Q(\mu) = (B + W\mu)'A(B + W\mu).$

Moreover, if $Q(Y) \sim W_p(m, \Sigma, \lambda)$, then

(e1) $m = tr(\Sigma_Y(W \otimes \Sigma^+))/r(\sigma).$

Proof. By Theorem 4.4.2, we know that $Q(Y) \sim W_p(m, \Sigma, \lambda)$ if and only if for any T in a neighborhood, N_0, of 0 in $\mathcal{L}(V, V)$,

(a) $\operatorname{tr}(\Sigma_Y(W \otimes T^\circ))^k = m\,tr(\Sigma T^\circ)^k, \quad k = 1, 2, \ldots,$

(b) $\lambda = Q(\mu)$,

(c) $\Sigma_Y(BT^\bullet) = \Sigma_Y(CT^\bullet)$,

and

(d) $\langle L, (I_{np} - 2\Sigma_Y(W \otimes T^\circ))^{-1}\Sigma_Y(L)T^\circ)\rangle = 4\langle \lambda, T^\circ(I_p - 2\Sigma T^\circ)^{-1}\Sigma T^\circ\rangle$,

where $T = T^\circ + T^\bullet$, $T^\circ = (T + T')/2$, $T^\bullet = (T - T')/2$ and $L = (B + C + 2W\mu)T^\circ$. Moreover, if $Q(Y) \sim W_p(m, \Sigma, \lambda)$ then

(e) $m = \operatorname{tr}[\Sigma_Y(W \otimes \Sigma^+)]/r(\Sigma)$.

Suppose that $Q(Y) \sim W_p(m, \Sigma, \lambda)$. It suffices to show that (a) - (d) imply (a1)-(d1). By Corollary 5.2.4 and Theorem 5.3.1, we know that (a) implies (a1) and (b1). Since $Im\, B \subset Im\, W$ and $Im\, C \subset Im\, W$, there exist some $B_\bullet$, $C_\bullet \in \mathcal{L}(V, E)$ such that

$$B = WB_\bullet \qquad \text{and} \qquad C = WC_\bullet. \tag{5.3.2}$$

Thus by (c), (a1) and (5.3.2), we obtain

$$\begin{aligned}
&\langle (B - C)T^\bullet, \Sigma_Y((B - C)T^\bullet)\rangle \\
&= \langle (WB_\bullet - WC_\bullet)T^\bullet, \Sigma_Y((WB_\bullet - WC_\bullet)T^\bullet)\rangle \\
&= \langle (W \otimes T^\bullet)(B_\bullet - C_\bullet), [\Sigma_Y(W \otimes T^\bullet)](B_\bullet - C_\bullet)\rangle \\
&= \langle B_\bullet - C_\bullet, [(W \otimes T^\bullet)\Sigma_Y(W \otimes T^\bullet)](B_\bullet - C_\bullet)\rangle \\
&= \langle B_\bullet - C_\bullet, WAW(B_\bullet - C_\bullet)T^\bullet\Sigma T^\bullet\rangle \\
&= \langle BT^\bullet - CT^\bullet, A(BT^\bullet - CT^\bullet)\Sigma\rangle = 0,
\end{aligned}$$

i.e.,

$$A(BT^\bullet - CT^\bullet)\Sigma = 0. \tag{5.3.3}$$

So by the proof from (4.4.18) to (4.4.19), we obtain (c1) from (5.3.3). Note that

$$\begin{aligned}
&(W \otimes T^\circ)(I - 2\Sigma_Y(W \otimes T^\circ))^{-1}\Sigma_Y(W \otimes T^\circ) \\
&= (W \otimes T^\circ)\sum_{k=0}^{\infty}(2\Sigma_Y(W \otimes T^\circ))^k\Sigma_Y(W \otimes T^\circ) \\
&= (W \otimes T^\circ)\Sigma_Y(W \otimes T^\circ) + (W \otimes T^\circ)\sum_{k=1}^{\infty}(2\Sigma_Y(W \otimes T^\circ))^k\Sigma_Y(W \otimes T^\circ) \qquad (5.3.4) \\
&= WAW \otimes T^\circ\Sigma T^\circ + (W \otimes T^\circ)\sum_{k=1}^{\infty}(2\Sigma_Y(W \otimes T^\circ))^k\Sigma_Y(W \otimes T^\circ).
\end{aligned}$$

Let $L_\bullet = B_\bullet + C_\bullet + 2\mu$. Then by (5.3.2),

$$L = (B + C + 2W\mu)T^\circ = (WB_\bullet + WC_\bullet + 2W\mu)^\circ = (W \otimes T^\circ)(L_\bullet). \qquad (5.3.5)$$

Thus replacing $2T^\circ$ by cT° with $c \in \Re$ in (d) and using (5.3.4) and (5.3.5), we obtain from (d),

$$\begin{aligned}
&\langle L_\bullet, [c^2 WAW \otimes T^\circ\Sigma T^\circ + c^2(W \otimes T^\circ)\sum_{k=1}^{\infty}(c\Sigma_Y(W \otimes T^\circ))^k\Sigma_Y(W \otimes T^\circ)](L_\bullet)\rangle \\
&\qquad = 4\langle \lambda, T^\circ(I_p - c\Sigma T^\circ)^{-1}\Sigma T^\circ\rangle \\
&\qquad = 4\langle \lambda,\, c^2 T^\circ\Sigma T^\circ + c^2\sum_{k=1}^{\infty}T^\circ(c\Sigma T^\circ)^k\Sigma T^\circ\rangle.
\end{aligned} \qquad (5.3.6)$$

By comparing the coefficents of c^2, c^3, $\cdots$ on both sides of (5.3.6), we obtain

$$\langle L_\bullet, [WAW \otimes T^\circ\Sigma T^\circ](L_\bullet)\rangle = 4\langle \lambda,\, T^\circ\Sigma T^\circ\rangle,$$

i.e.,

$$\langle L_\bullet' WAWL_\bullet, T^\circ\Sigma T^\circ\rangle = \langle 4\lambda,\, T^\circ\Sigma T^\circ\rangle.$$

So by Lemma 4.4.1, (5.3.2) and (c1),

$$\lambda = \frac{1}{4}L_\bullet' WAWL_\bullet = (B + W\mu)'A(B + W\mu),$$

proving (d1).

Now suppose that (a1) - (a4) hold. Let $Z = WY = (W \otimes I_p)Y$. Then by (a),

$$Z \sim N_{n \times p}(W\mu, (W \otimes I_p)\Sigma_Y(W \otimes I_p) = N_{n \times p}(\mu_*, A_* \otimes \Sigma),$$

where $\mu_* = W\mu$ and $A_* = WAW$. Note that by (5.3.2),

$$\begin{aligned} Q(Y) &= Y'WW^+WY + (WB_*)'Y + Y'WC_* + D \\ &= Z'W^+Z + B_*'Z + Z'C_* + D \equiv Q_*(Z). \end{aligned} \tag{5.3.7}$$

Thus by Corollary 4.4.5, it suffices to show that

(i) $A_*W^+A_*W^+A_* = A_*W^+A_*, \quad \text{tr}(A_*W^+) = m,$

(ii) $A_*B_* = A_*C_*$,

and

(iii) $\lambda = Q_*(\mu_*) = (B_* + W^+\mu_*)'A_*(B_* + W^+\mu_*)$

$$= (B_* + W^+\mu_*)'A_*W^+A_*(B_* + W^+\mu_*).$$

By (a1) and (b1), we obtain

$$A_*W^+A_*W^+A_* = WAWAWAW = WAWW^+WAWA_*W^+A_*$$

and

$$\text{tr}(A_*W^+) = WAWW^+) = \text{tr}(AW) = r(AW) = m,$$

proving (i). By (c1) and (5.3.2),

$$A_*B_* = WAWB_* = WABWAC + WAWC_* = A_*C_*,$$

and hence (ii) follows. Also by (a1), (d1), (5.3.2) and (5.3.7), we have

$$\lambda = Q(\mu) = Q_*(\mu_*) = (B + W\mu)'A(B + W\mu)$$

and

$$(B_* + W^+\mu_*)'A_*W^+A_*(B_* + W^+\mu_*) = (B_* + W^+\mu_*)'WAWAW(B_* + W^+\mu_*)$$

$$= (B_* + W^+\mu_*)'A_*(B_* + W^+\mu_*) = [W(B_* + W^+\mu_*)]'A[W(B_* + W^+\mu_*)]$$

$$= (B + W\mu)'A(B + W\mu),$$

proving (iii). $\square$

Note that in Theorem 5.3.2, if $B = C$, then the condition $r(\Sigma) > 1$ can be replaced by $\Sigma \neq 0$.

Corollary 5.3.3. *Suppose that* $Y \sim N_{n\times p}(\mu, \Sigma_Y)$. *Let* $W \in \mathcal{N}_E$, $\Sigma \in \mathcal{N}_V$ *with* $\Sigma \neq 0$, *and* $Q(Y) = Y'WY$. *Then* $Q(Y) \sim W_p(m, \Sigma, \lambda)$ *if and only if there exists an* $A \in \mathcal{N}_E$ *such that*

(a2) $(W \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W \otimes I_p) = 0$,

(b2) $AWAW = AW$, $\quad r(AW) = m$,

and

(c2) $\lambda = \mu' W \mu = \mu' WAW \mu$.

Example 5.3.2. *(Eaton, 1983). Suppose that* $Y \sim N_{n\times p}(\mu, \Sigma_Y)$ *with* $\Sigma_Y = A \otimes \Sigma$, $A, B \in \mathcal{N}_E$, $W = B^2$, $Q(Y) = Y'WY$ *and* $\Sigma \in \mathcal{N}_V$. *If* BAB *is an orthogonal projection of rank* m *and if* $BAW\mu = B\mu$, *then* $Y'WY \sim W_p(m, \Sigma, \mu' W \mu)$.

Proof. We may assume that $\Sigma \neq 0$. Since $\Sigma_Y = A \otimes \Sigma$, condition (a2) of Corollary 5.3.3 holds. Since $BAB = W^{\frac{1}{2}} A W^{\frac{1}{2}}$ is an orthogonal projection,

$$(W^{\frac{1}{2}} A W^{\frac{1}{2}}) W^{\frac{1}{2}} A W^{\frac{1}{2}} = W^{\frac{1}{2}} AWAW^{\frac{1}{2}} = W^{\frac{1}{2}} A W^{\frac{1}{2}}.$$

Cancelling $W^{\frac{1}{2}} A^{\frac{1}{2}}$, we obtain $A^{\frac{1}{2}} WAW^{\frac{1}{2}} = A^{\frac{1}{2}} W^{\frac{1}{2}}$, and hence

$$AWAW = AW. \tag{5.3.8}$$

Now

$$m = r(BAB) = r((W^{\frac{1}{2}} A^{\frac{1}{2}})(A^{\frac{1}{2}} W^{\frac{1}{2}})) = r(A^{\frac{1}{2}} W^{\frac{1}{2}}) \geq r(AW)$$

and

$$m = r(BAB) = r(BAB^2AB) = r(W^{\frac{1}{2}} AWAW^{\frac{1}{2}}) \leq r(AW).$$

Thus $r(BAB) = r(AW) = m$, proving (b2) of Corollary 5.3.3. Since $BAW\mu = B\mu$, we have $W^{\frac{1}{2}}AW\mu = W^{\frac{1}{2}}\mu$, and hence

$$\mu'WAWAW\mu = (W^{\frac{1}{2}}AW\mu)'(W^{\frac{1}{2}}AW\mu) = (W^{\frac{1}{2}}\mu)'(W^{\frac{1}{2}}\mu) = \mu'W\mu.$$

Therefore by (5.3.8), $\mu'WAW\mu = \mu'W\mu$, proving (c2) of Corollary 5.3.3. Hence by Corollary 5.3.3, $Q(Y) \sim W_p(m, \Sigma, \mu'W\mu)$. □

Now we arrive at our fourth version of Cochran's theorem:

Theorem 5.3.4. *Suppose that* $Y \sim N_{n\times p}(\mu, \Sigma_Y)$. *Let* $i \in \{1, 2, \cdots, \ell\}$, $W_i \in \mathcal{N}_E$, $B_i, C_i \in \mathcal{L}(V, E)$ *with* $Im\, B_i \subset Im\, W_i$ *and* $Im\, C_i \subset Im\, W_i$, $D_i \in \mathcal{L}(V, V)$, $Q_i(Y) = Y'W_iY + B_i'Y + Y'C_i + D_i$, *and* $\Sigma \in \mathcal{N}_V$ *with* $r(\Sigma) > 1$. *Then* $\{Q_i(Y)\}$ *is an independent family of* $W_p(m_i, \Sigma, \lambda_i)$ *random operators if and only if for some* A *in* $\mathcal{N}_E$ *and for any distinct* $i, j \in \{1, 2, \cdots, \ell\}$,

(a) $(W_i \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W_i \otimes I_p) = 0$,

(b) $AW_iAW_i = AW_i, \quad r(AW_i) = m_i$,

(c) $AB_i = AC_i$,

(d) $\lambda_i = Q_i(\mu) = (B_i + W_i\mu)'A(B_i + W_i\mu)$,

and

(e) $(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) = 0$.

Proof. Suppose that $\{Q_i(Y)\}$ is an independent family of $W_p(m_i, \Sigma, \lambda_i)$ random operators. Then by Theorem 5.3.2, for each $i \in \{1, 2, \cdots, \ell\}$, there exists an $A_i \in \mathcal{N}_E$ such that

(a1) $(W_i \otimes I_p)(\Sigma_Y - A_i \otimes \Sigma)(W_i \otimes I_p) = 0$,

(b1) $A_iW_iA_iW_i = A_iW_i, \quad r(A_iW_i) = m_i$,

(c1) $AB_i = AC_i$,

and

(d1) $\lambda_i = Q_i(\mu) = (B_i + W_i\mu)'A_i(B_i + W_i\mu)$.

Let $W = \sum_{i=1}^{\ell} W_i$, $B = \sum_{i=1}^{\ell} B_i$, $C = \sum_{i=1}^{\ell} C_i$, $D = \sum_{i=1}^{\ell} D_i$, $m = \sum_{i=1}^{\ell} m_i$, and $\lambda = \sum_{i=1}^{\ell} \lambda_i$. Then

$$Q(Y) = Y'WY + B'Y + Y'C + D = \sum_{i=1}^{\ell} Q_i(Y) \sim W_p(m, \Sigma, \lambda).$$

Thus by Theorem 5.3.2 again, we obtain that for some $A \in \mathcal{N}_E$,

$$(W \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W \otimes I_p) = 0. \tag{5.3.9}$$

By (a1) and (5.3.9),

$$(W_i \otimes I_p)\Sigma_Y(W_i \otimes I_p) = (W_iA_iW_i) \otimes \Sigma \tag{5.3.10}$$

and

$$(W^0 \otimes I_p)\Sigma_Y(W^0 \otimes I_p) = (W^0AW^0) \otimes \Sigma, \tag{5.3.11}$$

where $W^0 = WW^+$. Multiplying both sides of (5.3.11) by $W_i \otimes I_p$, we obtain

$$(W_iW^0 \otimes I_p)\Sigma_Y(W^0W_i \otimes I_p) = (W_iW^0AW^0W_i) \otimes \Sigma. \tag{5.3.12}$$

By Lemma 5.2.5, we know that $ImW = ImW_1 + \cdots + ImW_\ell$. So $ImW_i \subseteq ImW$ and therefore $W_iW^0 = W^0W_i = W_i$. Thus by (5.3.12),

$$(W_i \otimes I_p)\Sigma_Y(W_i \otimes I_p) = (W_iAW_i) \otimes \Sigma. \tag{5.3.13}$$

By (5.3.10) and (5.3.13), $(W_iA_iW_i) \otimes \Sigma = (W_iAW_i) \otimes \Sigma$. Since $\Sigma \neq 0$,

$$W_iAW_i = W_iA_iW_i, \quad i = 1, \cdots, L. \tag{5.3.14}$$

So (a) follows from (a1) and (5.3.14). Also by (5.3.14) and (b1), we obtain

$$W_iAW_i = W_iA_iW_i = W_iA_iW_iA_iW_i = W_iAW_iAW_i,$$

$$m_i = r(A_i W_i) = r(A_i W_i A_i W_i) \le r(A_i W_i A W_i) \le r(A W_i).$$

and

$$m_i = r(A_i W_i) \ge r(W_i A_i W_i) = r(W_i A W_i) = r(A^{\frac{1}{2}} W_i) \ge r(A W_i).$$

which implies (b). Since $Im\, B_i \subseteq Im\, W_i$ and $Im\, B_i \subseteq Im\, W_i$, there exist B_i^*, $C_i^* \in \mathcal{L}(V, E)$ such that

$$B_i = W_i B_i^*, \qquad C_i = W_i C_i^*. \tag{5.3.15}$$

Thus by (c1), (5.3.14) and (5.3.15),

$$(B_i - C_i)' A_i (B_i - C_i) = (B_i^* - C_i^*)' W_i A_i W_i (B_i^* - C_i^*)$$
$$= (B_i^* - C_i^*)' W_i A W_i (B_i^* - C_i^*) = (B_i - C_i)' A (B_i - C_i) = 0,$$

and hence $A(B_i - C_i) = 0$, proving (c). By (5.3.14) and (5.3.15)

$$\begin{aligned}(B_i + W_i\mu)' A_i (B_i + W_i\mu) &= [W_i(B_i^* + \mu)]' A_i [W_i(B_i^* + \mu)] \\ &= (B_i^* + \mu)' W_i A_i W_i (B_i^* + \mu) \\ &= (B_i^* + \mu)' W_i A W_i (B_i^* + \mu) \\ &= (B_i + W_i\mu)' A((B_i + W_i\mu).\end{aligned} \tag{5.3.16}$$

So (d) follows from (d1) and (5.3.16). Now by Theorem 4.3.4, we obtain

$$\Sigma_Y (W_i \otimes T_i) \Sigma_Y (W_j \otimes T_j) \Sigma_Y = 0. \tag{5.3.17}$$

Let $T_i = T_j = I_p$. Then (5.3.17) becomes

$$\Sigma_Y (W_i \otimes I_p) \Sigma_Y (W_j \otimes I_p) \Sigma_Y = 0. \tag{5.3.18}$$

Since $W_i, W_j \in \mathcal{N}_E$, (e) follows from (5.3.18).

Now suppose that (a) - (e) hold. Then by Theorem 5.3.2. (a) - (d) imply that $Q_i(Y) \sim W_p(m_i, \Sigma, \lambda_i)$ for each $i \in \{1, 2 \cdots, \ell\}$. By (d) and (5.3.15), we obtain that for any distinct i, j and T_i, $T_j \in \mathcal{L}(V, V)$,

$$\begin{aligned}&\Sigma_Y (W_i \otimes T_i^\circ) \Sigma_Y (W_j \otimes T_j^\circ) \Sigma_Y \\ &= \Sigma_Y (I_n \otimes T_i^\circ)(W_i \otimes I_p) \Sigma_Y (W_j \otimes I_p)(I_n \otimes T_j^\circ) \Sigma_Y = 0,\end{aligned} \tag{5.3.19}$$

$$[\Sigma_Y(I_n \otimes T_i^\circ)\Sigma_Y](B_jT_j + C_jT_j' + 2W_i\mu T_j^\circ)$$
$$= [\Sigma_Y(I_n \otimes T_i^\circ)(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p)](B_j^*T_j + C_j^*T_j' + \mu T_j^*) = 0, \tag{5.3.20}$$

and

$$\langle B_jT_j + C_jT_j' + 2W_i\mu T_j^\circ, \Sigma_Y(B_jT_j + C_jT_j' + 2W_i\mu T_j^\circ)\rangle$$
$$= \langle B_i^*T_i + C_i^*T_i' + 2\mu, [(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p)](B_j^*T_j + C_j^*T_j' + 2\mu)\rangle = 0, \tag{5.3.21}$$

where $T_i^\circ = (T_i + T_i')/2$. Thus by Theorem 4.3.4, (5.3.19) - (5.3.21) imply that $\{Q_i(Y)\}$ is independent. Therefore $\{Q_i(Y)\}$ is an independent family of $W_p(m_i, \Sigma, \lambda_i)$ random operators. □

Note that in Theorem 5.3.4, if all $B_i = C_i$, then condition $r(\Sigma) > 1$ can be replaced by $\Sigma \neq 0$.

Corollary 5.3.5. *In Theorem 5.3.4, if $B_i = C_i = 0$ and $D_i = 0$, then $\{Q_i(Y)\}$ is an independent family of $W_p(m_i, \Sigma, \lambda_i)$ random operators if and only if for some A in $\mathcal{N}_E$ and for any distinct $i, j \in \{1, 2, \cdots, L\}$,*

(a1) $(W_i \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W_i \otimes I_p) = 0,$

(b1) $AW_iAW_i = AW_i, \quad r(AW_i) = m_i,$

(c1) $\lambda_i = \mu' W_i \mu = \mu' W_i A W_i \mu,$

and

(d1) $(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) = 0.$

5.4. Applications

As mentioned in Section 4.6, in practice, often it is the sufficient condition that is used in statistical inference. We shall give a sufficient condition under which $\{Q_i(Y)\}$ is an independent family of $W_p(m_i, \Sigma, \lambda_i)$ random operators even without assuming $W_i \in \mathcal{N}_E$.

Proposition 5.4.1. *In Theorem 5.3.4, let $W_i \in \mathcal{S}_E$ instead of $\mathcal{N}_E$. Then $\{Q_i(Y)\}$ is an independent family of $W_p(m_i, \Sigma, \lambda_i)$ random operators if for some $A \in \mathcal{N}_E$ and for any distinct $i, j \in \{1, 2, \cdots, \ell\}$,*

(i) $(W_i \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W_i \otimes I_p) = 0$,

(ii) $AW_iAW_iA = AW_iA, \quad r(AW_i) = m_i$,

(iii) $AB_i = AC_i$,

(iv) $\lambda_i = Q_i(\mu) = (B_i + W_i\mu)'A(B_i + W_i\mu) = (B_i + W_i\mu)'AW_iA(B_i + W_i\mu)$,

and

(v) $(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) = 0$.

Proof. Suppose (i) - (v) hold. Then by the proof of sufficiency of Theorem 5.3.4, (v) implies (5.3.19) - (5.3.21) and hence by Theorem 4.3.4, (v) implies that $\{Q_i(Y)\}$ is an independent family of random operators. Therefore it sufficies to show that for each i, $Q_i(Y) \sim W_p(m_i, \Sigma, \lambda_i)$, which follows from the proof of 'Only if part' of Theorem 5.3.2. □

Note that whether the above (i) - (v) are necessary conditions, under which $\{Q_i(Y)\}$ is an independent family of $W_p(m_i, \Sigma, \lambda_i)$ random operators, is still unknown.

Proposition 5.4.2. *Assume that $Y \sim N_{n \times p}(\mu, \Sigma_Y)$ with $\Sigma_Y = A \otimes \Sigma + H$ for some $A \in \mathcal{N}_E$ and H, such that $ImH \subseteq Im(M \otimes I_p)$ for some M in $\mathcal{N}_E$. Let $i \in \{1, \ldots, \ell\}$, $W_i \in \mathcal{N}_n$ and $Q_i(Y) = Y'W_iY$. Suppose that for all distinct $i, j \in \{1, \ldots, \ell\}$,*

(a) $AW_iAW_i = AW_i, \qquad r(AW_i) = m$,

(b) $W_iM = 0$,

(c) $\lambda_i = \mu'W_i\mu = \mu'W_iAW_i\mu$,

and

(d) $W_i A W_j = 0$.

Then $\{Q_i(Y)\}$ *is an independent family of* $W_p(m_i, \Sigma, \lambda_i)$ *random operators.*

Note that the $A \in \mathcal{N}_E$ in Theorem 5.3.4 can be constructed by Σ, Σ_Y and sum of W_i's. In practice, the given expression of A often does not contain any parameters. The following two results are obtained for this purpose.

Proposition 5.4.3. *In Corollary 5.3.5, suppose that* $\mu = 0$, Σ_Y *is n.n.d. and* $\Sigma \in \mathcal{N}_V$. *Then the following two conditions are equivalent:*

(a) $\{Q_i(Y)\}$ *is an independent family of* $W_p(r(W_i), \Sigma)$ *random operators* $Q_i(Y)$.

(b) For any distinct i, j,

(i) $(W_i \otimes I_p)\Sigma_Y(W_i \otimes I_p) = W_i \otimes \Sigma$,

(ii) $(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) = 0$.

Proof. By Corollary 5.3.5, it suffices to show that with $m_i = r(W_i)$ and $\mu = 0$, conditions (a1) - (c1) in Corollary 5.3.5 are equivalent to condition (b).

Suppose that (a1) -(c1) in Corollary 5.3.5 hold. Then by (a1),

$$(W_i \otimes I_p)\Sigma_Y(W_i \otimes I_p) = (W_i A W_i) \otimes \Sigma. \qquad (5.4.1)$$

By (b1), $r(W_i) = r(AW_i) = r(W_i A)$. So we have $ImW_i = Im(W_i A)$. Thus there exists a $B_i \in \mathcal{L}(E, E)$ such that

$$W_i = W_i A B_i = B_i' A W_i. \qquad (5.4.2)$$

Substituting (5.4.1) into (5.4.2), we obtain

$$(W_i \otimes I_p)\Sigma_Y(W_i \otimes I_p) = (B_i' A W_i A W_i) \otimes \Sigma,$$

and therefore (b)(i) follows from (b1) and (5.4.2). Also (b)(ii) is the same as (c1).

Now assume that (b) holds. Let $W = \sum_{i=1}^{L} W_i$. Then by (b)(i) and (b)(ii),

$$(W \otimes I_p)\Sigma_Y(W \otimes I_p) = W \otimes \Sigma. \tag{5.4.3}$$

Multiplying both sides of (5.4.3) by $W^+ \otimes I_p$, we obtain

$$(W^\circ \otimes I_p)\Sigma_Y(W^\circ \otimes I_p) = W^+ \otimes \Sigma. \tag{5.4.4}$$

Let $\{e_\alpha\}_{\alpha=1}^n$ be an orthonormal basis of E such that

$$W = \sum_{\alpha=1}^{s} w_\alpha e_\alpha \Box e_\alpha, \quad w_\alpha > 0, \quad \alpha = 1, 2, \dots, s \leq n,$$

where $s = r(W)$. Let $\{f_\beta\}_{\beta=1}^p$ be an orthonormal basis of $\Re^p$. Then $\{e_\alpha \Box e_{\alpha'}\}$, $\{f_\beta \Box f_{\beta'}\}$, and $\{(e_\alpha \Box e_{\alpha'}) \otimes (f_\beta \Box f_{\beta'})\}$ are orthonormal bases of $\mathcal{L}(E,E)$, $\mathcal{L}(V,V)$, and $\mathcal{L}(E,E) \otimes \mathcal{L}(V,V)$ respectively. So

$$W^\circ = \sum_{\alpha=1}^{s} e_\alpha \Box e_\alpha, \quad \Sigma = \sum_{\beta,\beta'=1}^{p} \sigma_{\beta\beta'} f_\beta \Box f_{\beta'},$$

and

$$\Sigma_Y = \sum_{\alpha,\alpha'=1}^{n} \sum_{\beta,\beta'=1}^{p} \sigma_{\alpha\alpha'\beta\beta'}(e_\alpha \Box e_{\alpha'}) \otimes (f_\beta \Box f_{\beta'}),$$

where $\sigma_{\beta\beta'} = \langle f_\beta, \Sigma(f_{\beta'})\rangle$ and $\sigma_{\alpha\alpha'\beta\beta'} = \langle e_\alpha \Box f_\beta, \Sigma_Y(e_{\alpha'} \Box f_{\beta'})\rangle$. Thus (5.4.4) implies that

$$\sum_{\alpha,\alpha'=1}^{s} \sum_{\beta,\beta'=1}^{p} \sigma_{\alpha\alpha'\beta\beta'}(e_\alpha \Box e_{\alpha'}) \otimes (f_\beta \Box f_{\beta'}) = (\sum_{\alpha=1}^{s} w_\alpha^{-1} e_\alpha \Box e_\alpha) \otimes (\sum_{\beta,\beta'=1}^{p} \sigma_{\beta\beta'} f_\beta \Box f_{\beta'}),$$

i.e.,

$$0 = \sum_{\alpha,\alpha'=1}^{s} \sum_{\beta,\beta'=1}^{p} (\sigma_{\alpha\alpha'\beta\beta'} - \upsilon_{\alpha\alpha'} w_\alpha^{-1} \sigma_{\beta\beta'})(e_\alpha \Box e_{\alpha'}) \otimes (f_\beta \Box f_{\beta'}), \tag{5.4.5}$$

where $\delta_{\alpha\alpha'}$'s are Kronecker symbols. By the linear independence of $\{(e_\alpha \Box e_{\alpha'}) \otimes (f_\beta \Box f_{\beta'})\}$, we obtain from (5.4.5),

$$\sigma_{\alpha\alpha'\beta\beta'} = \delta_{\alpha\alpha'} w_\alpha^{-1} \sigma_{\beta\beta'}, \quad \alpha, \alpha' = 1, \ldots, S, \quad \beta, \beta' = 1, \ldots, p.$$

Let

$$H = \sum_{\max(\alpha,\alpha')>s}^{n} \sum_{\beta,\beta'=1}^{p} \sigma_{\alpha\alpha'\beta\beta'} (e_\alpha \Box e_{\alpha'}) \otimes (f_\beta \Box f_{\beta'}).$$

Then

$$\begin{aligned}
\Sigma_Y &= \sum_{\alpha,\alpha'=1}^{s} \sum_{\beta,\beta'=1}^{p} \sigma_{\alpha\alpha'\beta\beta'} (e_\alpha \Box e_{\alpha'}) \otimes (f_\beta \Box f_{\beta'}) + H \\
&= \sum_{\alpha,\alpha'=1}^{s} \sum_{\beta,\beta'=1}^{p} \delta_{\alpha\alpha'} w_\alpha^{-1} \sigma_{\beta\beta'} (e_\alpha \Box e_{\alpha'}) \otimes (f_\beta \Box f_{\beta'}) + H \qquad (5.4.6) \\
&= W^+ \otimes \Sigma + H.
\end{aligned}$$

Choose $A = W^+$. By Lemma 5.2.5, $ImW_i \subseteq ImW = Im(W_1 + \cdots + W_L)$ and hence $W_i e_\alpha = 0$ for all $\alpha = s+1, \ldots, n$. So by (5.4.6),

$$(W_i \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W_i \otimes I_p) = (W_i \otimes I_p) H (W_i \otimes I_p) = 0,$$

proving (a1). By (b)(i) and (a1),

$$W_i \otimes \Sigma = W_i A W_i \otimes \Sigma.$$

Since $\Sigma \neq 0$, (b1) follows. (c1) is the same as (b)(ii). □

Pavur (1987) obtained the above result for the case where Σ is positive definite.

Proposition 5.4.4. *Assume that $Y \sim N_{n\times p}(\mu, \Sigma_Y)$. Let $W \in \mathcal{N}_E$ and $X = W^{\frac{1}{2}} Y = (W^{\frac{1}{2}} \otimes I_p)(Y)$. Then $X \sim N_{n\times p}(W^{\frac{1}{2}}\mu, \Sigma_X)$ with $\Sigma_X = (W^{\frac{1}{2}} \otimes I_p)\Sigma_Y(W^{\frac{1}{2}} \otimes I_p)$. Assume further that Σ_X is of the form $P \otimes \Sigma$, where P is idempotent of rank*

m. *Suppose that* $X'X \sim W_p(m, \Sigma, \mu' W \mu)$. *Then* A *in Corollary 5.3.5 can be chosen as* $W^{-\frac{1}{2}} P W^{-\frac{1}{2}}$.

Proof. Let $\{e_i\}_{i=1}^n$ and $\{f_j\}_{j=1}^p$ be respectively orthonormal bases of E and V, such that

$$W = \sum_{i=1}^{r(W)} w_i e_i \Box e_i, \qquad \Sigma = \sum_{j=1}^{r(\Sigma)} \sigma_j f_j \Box f_j,$$

where $w_i > 0$, $i = 1, \ldots, r(W)$ and $\sigma_j > 0$, $j = 1, \ldots, r(\Sigma)$. Then Σ_Y and P can be written as

$$\Sigma_Y = \sum_{i,i'=1}^{n} \sum_{j,j'=1}^{p} \sigma_{ii'jj'} \left(e_i \Box e_{i'}\right) \otimes \left(f_j \Box f_{j'}\right), \qquad P = \sum_{i,i'=1}^{n} p_{ii'} e_i \Box e_{i'}.$$

From $\Sigma_X = P \otimes \Sigma$, we obtain,

$$\sum_{i,i'=1}^{r(W)} \sum_{j,j'=1}^{p} w_i^{\frac{1}{2}} w_{i'}^{\frac{1}{2}} \sigma_{ii'jj'} \left(e_i \Box e_{i'}\right) \otimes \left(f_j \Box f_{j'}\right) = \sum_{i,i'=1}^{n} \sum_{j=1}^{r(\Sigma)} p_{ii'} \sigma_j \left(e_i \Box e_{i'}\right) \otimes \left(f_j \Box f_j\right).$$

So when $i > r(W)$ or $i' > r(W)$, $p_{ii'} = 0$, i.e. $Im\, P \subseteq Im\, W$. Moreover,

$$\sigma_{ii'jj'} = 0, \quad \text{for} \quad j > r(\Sigma) \quad \text{or} \quad j' > r(\Sigma); \quad i, i' = 1, \ldots, r(W),$$

$$\sigma_{ii'jj'} = 0, \quad \text{for} \quad j \neq j', \quad j, j' = 1, \ldots, r(\Sigma); \quad i, i' = 1, \ldots, r(W),$$

and

$$\sigma_{ii'jj'} = p_{ii'} \sigma_j / (w_i w_{i'})^{\frac{1}{2}}, \quad \text{for} \quad j = 1, \ldots, r(\Sigma); \quad i, i' = 1, \ldots, r(W).$$

For $i, i' = 1, \ldots, r(W)$, define $a_{ii'} = (w_i w_{i'})^{-\frac{1}{2}} p_{ii'}$ and $A = \sum_{i,i'=1}^{r(W)} a_{ii'} e_i \Box e_{i'}$. Then $A = W^{-\frac{1}{2}} P W^{-\frac{1}{2}}$. So

$$\begin{aligned}(W \otimes I_p) \Sigma_Y (W \otimes I_p) &= \sum_{i,i'=1}^{r(W)} \sum_{j=1}^{r(\Sigma)} p_{ii'} w_i^{\frac{1}{2}} w_{i'}^{\frac{1}{2}} \sigma_j \left(e_i \Box e_{i'}\right) \otimes \left(f_j \Box f_j\right) \\ &= (W \otimes I_p)(A \otimes \Sigma)(W \otimes I_p),\end{aligned}$$

$$AWAW = AW, \qquad r(AW) = r\left(W^{-\frac{1}{2}}PW^{\frac{1}{2}}\right) = r(P) = m,$$

and

$$\mu' W\mu = (W^{\frac{1}{2}}\mu)'P(W^{\frac{1}{2}}\mu) = \mu' W^{\frac{1}{2}}PW^{\frac{1}{2}}\mu$$
$$= \mu' W(W^{-\frac{1}{2}}PW^{-\frac{1}{2}})W\mu = \mu' WAW\mu.$$

□

Now consider the multivariate components of variance model, see e.g. Anderson et al. (1986) and Mathew (1989):

$$Y = XB + \sum_{s=1}^{k} C_s'\mathcal{E}_s, \tag{5.4.7}$$

where $X \in M_{n\times q}$ is known, $B \in M_{q\times p}$ is unknown, $C_s \in M_{r_s} \times n$ is known, $\mathcal{E}_s \sim N(0, I_{r_s} \otimes \Sigma_s)$ and $\mathcal{E}_s$'s are independent. Thus

$$E(Y) = XB, \qquad \Sigma_Y = \sum_{s=1}^{k} V_s \otimes \Sigma_s, \tag{5.4.8}$$

where $V_s = C_s'C_s$, $s = 1, \cdots, k$. Note that Σ_Y in (5.4.8) is not of the form $A \otimes \Sigma$. For (5.4.7), A in Theorem 5.3.4 can be so chosen that it no longer depends on $\{W_i\}$:

Proposition 5.4.5. *For Y in (5.4.7), let $i \in \{1, \cdots, \ell\}$, $W_i \in \mathcal{N}_n$, $m_i > 0$, $\Sigma \neq 0$, and $Q_i(Y) = Y'W_iY$. Then $\{Q_i(Y)\}$ is a family of independent $W_p(m_i, \Sigma, \lambda_i)$ random matrices if and only if for any distinct $i, j = 1, \cdots, \ell$,*

(a) $(W_i \otimes I_p)(\Sigma_Y - P \otimes \Sigma)(W_i \otimes I_p) = 0$,

(b) $PW_iPW_i = PW_i, \quad r(PW_i) = m_i$,

(c) $\lambda_i = B'X'W_iXB = B'X'W_iPW_iXB$,

and

(d) $(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) = 0$,

where $P = \sum_{s=1}^{k} \alpha_s V_s$ and $\alpha_s = tr(\Sigma_s)/tr(\Sigma)$. Moreover, if (a) and (b) hold, then

$$\Sigma = \frac{1}{m}\sum_{s=1}^{k} tr(V_sW_i)\Sigma_s, \quad i = 1, \cdots, \ell.$$

The proof of Proposition 5.4.5 is similar to the proof of Theorem 5.3.1 with $W_iPW_i = W_iAW_i$ for all $i \in \{1, \dots, \ell\}$.

Two remarks are in order:

(a) The operator A in Theorem 5.3.4 is by no means unique. Indeed, the A constructed in our proof satisfies $ImA \subseteq Im(\sum_{i=1}^{\ell} W_i)(= \sum_{i=1}^{\ell}(ImW_i))$, a property that is not required. Now, for any nonzero $B \in \mathcal{N}_E$ with $ImB \subsetneq ker(\sum_{i=1}^{\ell} W_i)$ $(= \cap_{i=1}^{\ell}(kerW_i))$, $A + B$ is another A that satisfies the conditions in Theorem 5.3.4.

(b) In practice, mostly, it is the sufficient conditions (a) - (c) of Theorem 5.3.4 that are used in statistical inference. For this, we note that in (a) - (c), even if A is replaced by $A_i \in \mathcal{N}_E$, $\{Y'W_iY\}$ is still an independent family of $W_p(m_i, \Sigma, \lambda_i)$ random operators; when these A_i's are all equal to A with $\Sigma_Y = A \otimes \Sigma$, we obtain the corresponding standard result in, e.g., Khatri (1980) and DeGunst (1987).

CHAPTER SIX

COCHRAN THEOREMS FOR AN MULTIVARIATE ELLIPTICALLY CONTOURED MODEL

6.1. Introduction

So far, the Cochran theorems we obtained are for the normal setting. As mentioned in Chapter 1, many properties of MEC distributions are very similar to those of multivariate normal distributions. In this chapter we shall extend the Cochran Theorems in Chapter 5, from the normal setting to the MEC setting. The results given in this chapter are extensions of the corresponding results of Anderson and Fang (1982a, 1982b) and Pavur (1987).

In section 6.2 we shall state some basic properties of the MEC distributions and define noncentral generalized Wishart distributions. Two versions of Cochran's theorem will be discussed in Section 6.3. In the last section, we shall give some examples, which are extensions of the results given in Section 5.4. In the appendix, we give an alternative proof for the 'Only if part' of Theorem 6.3.1 by using the formulae for the first and second order moments of quadratic forms of Y given in Chapter 3.

6.2. Basic properties of multivariate elliptically contoured distributions

We shall use $u^{(np)}$ to denote a uniformly distributed random vector on the unit sphere in $\Re^{np}$. Suppose that $X \sim MEC_{n\times p}(0, I_n \otimes I_p, \phi)$. Then the stochastic spherical representation of X is given by

$$X \overset{d}{=} RU, \qquad \text{vec}\, U \overset{d}{=} u^{(np)}, \tag{6.2.1}$$

where (a) $X \stackrel{d}{=} Y$ means that X, Y have the same distribution, (b) U is a random operator in $\mathcal{L}(V, E)$ and R is a nonnegative random variable that is independent of U, and (c) the distribution function, F, of R is related to ϕ by

$$\phi(\langle T, T \rangle) = \int_0^\infty \psi(r^2 \langle T, T \rangle) dF(r), \qquad (6.2.2)$$

where

$$\psi(\langle S, S \rangle) = E\left[e^{i\langle S, U \rangle}\right],$$

see Fang, Kotz and Ng (1990), and Fang and Zhang (1990).

Lemma 6.2.1. *Suppose that X and Y are two random operators of a probabilty space into $\mathcal{L}(V, E)$ such that $X \stackrel{d}{=} Y$. Let g_j, $j = 1, 2, \cdots, \ell$, be Borel functions. Then*

$$(g_1(X), g_2(X), \cdots, g_\ell(X)) \stackrel{d}{=} (g_1(Y), g_2(Y), \cdots, g_\ell(Y)). \qquad (6.2.3)$$

In particular,

$$(X'W_1X, X'W_2X, \cdots, X'W_\ell X,) \stackrel{d}{=} (Y'W_1Y, Y'W_2Y, \cdots, Y'W_\ell Y),$$

where $W_1, W_2, \cdots, W_\ell$ are constant operators in $\mathcal{L}(E, E)$.

Lemma 6.2.2. *Assume that X is a random operator of a probabilty space into $\mathcal{L}(V, E)$. Then the following statements are equivalent:*

(i) $X \sim MEC_{n\times p}(0, I_n \otimes I_p, \phi)$;

(ii) the c.f. of X has the form $\phi(\langle T, T \rangle)$ given in (6.2.2);

(iii) $X \stackrel{d}{=} RU$, where R and U are given in (6.2.1);

(iv) $X \stackrel{d}{=} \Gamma X$ for every $\Gamma \in \mathcal{O}_E$, where $\mathcal{O}_E$ is the set of all orthogonal operators in $\mathcal{L}(E, E)$.

For proofs of the above two lemmas, see Fang and Zhang (1990).

Now suppose that Y is a random operator of a probability space $(\Omega, \mathcal{A}, P)$ into $\mathcal{L}(V, E)$ such that $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$. Then

$$Y \stackrel{d}{=} \mu + R\Sigma_Y^{\frac{1}{2}}(U), \tag{6.2.4}$$

where R and U are given in (6.2.1) and $\Sigma_Y \in \mathcal{L}(E, E) \otimes \mathcal{L}(V, V)$. Note that here $\Sigma_Y^{\frac{1}{2}}(U)$ is the random operator

$$\omega \to \Sigma_Y^{\frac{1}{2}}(U(\omega)) \in \mathcal{L}(V, E), \quad \omega \in \Omega.$$

Lemma 6.2.3. *Assume that $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$. Let $K \in \mathcal{L}(E, E)\otimes\mathcal{L}(V, V)$. Then*

$$K(Y) \sim MEC_{n\times p}(\, K(\mu),\; K\Sigma_Y K',\; \phi\,). \tag{6.2.5}$$

In particular, if $K = B \otimes C$ with $B \in \mathcal{L}(E, E)$ and $C \in \mathcal{L}(V, V)$, then

$$(B \otimes C)(Y) = BYC' \sim MEC_{n\times p}(\, B\mu C', (B \otimes C)\Sigma_Y(B \otimes C)', \phi\,). \tag{6.2.6}$$

Definition 6.2.1. *Let $E = E_1 \oplus E_2 \oplus \cdots \oplus E_\ell \oplus E_{\ell+1}$, the direct sum of E_1, $E_2, \cdots, E_\ell$ and $E_{\ell+1}$, where E_1, $E_2, \cdots$, E_ℓ, $E_{\ell+1}$ are, respectively, $m_1, m_2, \cdots$, m_ℓ, $m_{\ell+1}$-dimensional subspaces of E and $m_{\ell+1} = n - \sum_{j=1}^{\ell} m_j$. Suppose that $X \sim MEC_{n\times p}(\,\nu, I_n \otimes \Sigma, \phi\,)$. Partition X, $\nu \in \mathcal{L}(V, E)$ into $\ell + 1$ parts, i.e.,*

$$X = \{X_1, X_2, \cdots, X_\ell, X_{\ell+1}\} \tag{6.2.7}$$

and

$$\nu = \{\nu_1, \nu_2, \ldots, \nu_\ell, \nu_{\ell+1}\}, \tag{6.2.8}$$

where $X_j, \nu_j \in \mathcal{L}(V, E_j)$, j =1, 2, $\cdots$, ℓ, $\ell + 1$. Then the joint distribution of $(X_1'X_1, X_2'X_2, \cdots, X_\ell'X_\ell)$, denoted by $GW_p(m_1, m_2, \cdots, m_\ell;\, m_{\ell+1}; \Sigma; \lambda_1, \lambda_2, \ldots, \lambda_\ell;$

ϕ), *is called the generalized Wishart distribution with parameters* $m_1, m_2, \cdots, m_\ell$, $m_{\ell+1}$, Σ, $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ *and* ϕ, *and* $\lambda_j = \nu_j'\nu_j$, $j = 1, 2, \ldots, \ell$. *In particular, if* $\ell = 1$, $m = m_1$, *then*

$$X_1'X_1 \sim GW_p(m; n-m; \Sigma; \lambda_1; \phi). \tag{6.2.9}$$

We shall use $GW_p(m_1, m_2, \cdots, m_\ell; m_{\ell+1}; \Sigma; \phi)$ to denote $GW_p(m_1, m_2, \cdots, m_\ell; m_{\ell+1}; \Sigma; 0, 0, \ldots, 0; \phi)$ and use $GW_p(m; n-m; \Sigma; \phi)$ to denote $GW_p(m; n-m; \Sigma; 0; \phi)$. More specifically, let $\{e_k\}_{k=1}^n$ be an orthonormal basis of E and

$$E_j = \langle\{e_{m_{j-1}+1}, \cdots, e_{m_j}\}\rangle, \quad j = 1, 2, \ldots, \ell, \ell+1.$$

Then the above X_j's are given by

$$X_j = \sum_{k=m_{j-1}+1}^{m_j} (e_k \Box e_k) X \sim MEC_{n\times p}(\nu_j, \sum_{k=m_{j-1}+1}^{m_j} (e_k \Box e_k) \otimes \Sigma, \phi). \tag{6.2.10}$$

Note that for the case where $X \sim N_{n\times p}(\nu, I_n \otimes \Sigma)$, $GW_p(m_1, m_2, \cdots, m_\ell; m_{\ell+1}; \Sigma; \lambda_1, \lambda_2, \ldots, \lambda_\ell; \phi)$ is nothing but a joint distribution of independent Wishart $W_p(m_1, \Sigma, \lambda_1)$, $W_p(m_2, \Sigma, \lambda_2) \cdots$, $W_p(m_\ell, \Sigma, \lambda_\ell)$ random operators and no longer depends on $m_{\ell+1}$. For the case where $\nu = 0$ and $\Sigma > 0$, the joint probability density function of $(X_1'X_1, X_2'X_2, \cdots, X_\ell'X_\ell)$ and its properties were obtained by Andersion and Fang (1982a).

6.3. Cochran theorems for an multivariate elliptically contoured model

Theorem 6.3.1. *Assume that* Y *is a random operator of a probabilty space* $(\Omega, \mathcal{A}, P)$ *such that* $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$ *with* $P(Y = \mu) = 1$. *Let* $W \in \mathcal{N}_E$ *and* $Q(Y) = (Y-\mu)'W(Y-\mu)$. *Then*

$$Q(Y) \sim GW_p(m; n-m; \Sigma; \phi)$$

if and only if there exists an $A \in \mathcal{N}_E$ such that

(a) $(W \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W \otimes I_p) = 0$

and

(b) $AWAW = AW, \qquad r(AW) = m.$

Proof. Suppose that $Q(Y) \sim GW_p(m; n-m; \Sigma; \phi)$. Let $E = E_1 \oplus E_2$, where E_1 and E_2 are, respectively, m- and $(n-m)$-dimensional subspaces of E. Let $X = \{X_1, X_2\} \sim MEC_{n\times p}(0, I_n \otimes \Sigma, \phi)$ with $X_1 \in (V, E_1)$ and $X_2 \in (V, E_2)$. Then by the definition of $GW_p(m; n-m; \Sigma; \phi)$,

$$Q(Y) = (Y-\mu)'W(Y-\mu) \stackrel{d}{=} X_1'X_1. \tag{6.3.1}$$

Let $X_* = RU \sim MEC_{n\times p}(0, I_n \otimes I_p, \phi)$, where R and U are given by (6.2.1). By (6.2.4) and (6.3.1),

$$R^2(\Sigma_Y^{\frac{1}{2}}(U))'W(\Sigma_Y^{\frac{1}{2}}(U)) \stackrel{d}{=} R^2\Sigma^{\frac{1}{2}}U_1'U_1\Sigma^{\frac{1}{2}}, \tag{6.3.2}$$

where $X_i \stackrel{d}{=} RU_i\Sigma^{\frac{1}{2}}$, $i = 1, 2$, and $U = \{U_1, U_2\}$ with $U_1 \in (V, E_1)$ and $U_2 \in (V, E_2)$. Since $P(R=0) = P(Y=\mu) = 1$, there exists $K \in (0, \infty)$ such that $P(0 < R < K) > 0$. Multiplying (6.3.2) by the indicator $I_{(0,K)}$, we obtain

$$R_*^2(\Sigma_Y^{\frac{1}{2}}(U))'W(\Sigma_Y^{\frac{1}{2}}(U)) \stackrel{d}{=} R_*^2\Sigma^{\frac{1}{2}}U_1'U_1\Sigma^{\frac{1}{2}},$$

where $R_*^2 = I_{(0,K)}R^2$. Since $P(R_* > 0) > 0$, we may, without loss of generality, assume that R is less that K. This also implies that $0 < E(R^{2\ell}) < \infty$ for all $\ell = 0, 1, 2, \ldots$. Choose a chi random variable χ_{np} with np degrees of freedom that is independent of R and U. Let $Z = \chi_{np}U$, $Z_i = \chi_{np}U_i$, and $Z_* = \Sigma_Y^{\frac{1}{2}}(Z)$. Multiplying (6.3.2) by χ_{np}^2, we obtain

$$R^2Z_*'WZ_* \stackrel{d}{=} R^2\Sigma^{\frac{1}{2}}Z_1'Z_1\Sigma^{\frac{1}{2}}.$$

Thus

$$E(e^{\langle T, R^2 Z_*' W Z_* \rangle}) = E(e^{\langle T, R^2 \Sigma^{\frac{1}{2}} Z_1' Z_1 \Sigma^{\frac{1}{2}} \rangle}), \qquad T \in \mathcal{S}_E. \tag{6.3.3}$$

Since $Z \sim N_{n\times p}(0, I_n \otimes I_p)$, $Z_* \sim N_{n\times p}(0, \Sigma_Y)$ and $Z_1\Sigma^{\frac{1}{2}} \sim N_{n\times p}(0, \mathrm{diag}(I_m, 0) \otimes \Sigma)$. So by Corollary 4.2.2 and 4.2.3,

$$E\left(e^{\langle T, R^2 Z_*' W Z_* \rangle} | R^2\right) = |I_{np} - 2R^2 \Sigma_Y^{\frac{1}{2}} (W \otimes T) \Sigma_Y^{\frac{1}{2}}|^{-\frac{1}{2}}, \quad T \in N_0 \tag{6.3.4}$$

and

$$E\left(e^{\langle T, R^2 \Sigma^{\frac{1}{2}} Z_1' Z_1 \Sigma^{\frac{1}{2}} \rangle} | R^2\right) = |I_p - 2R^2 \Sigma^{\frac{1}{2}} T \Sigma^{\frac{1}{2}}|^{-\frac{m}{2}}, T \in N_0, \tag{6.3.5}$$

where N_0 denotes a neighborhood of $0 \in \mathcal{S}_V$. By (6.3.4) and (6.3.5), (6.3.3) becomes

$$E\left(|I_{np} - 2R^2 \Sigma_Y^{\frac{1}{2}} (W \otimes T) \Sigma_Y^{\frac{1}{2}}|^{-\frac{1}{2}}\right) = E\left(|I_p - 2R^2 \Sigma^{\frac{1}{2}} T \Sigma^{\frac{1}{2}}|^{-\frac{m}{2}}\right), \quad T \in N_0. \tag{6.3.6}$$

Let $c \in \Re$ and $P \in \mathcal{L}(E, E)$ be an idempotent operator of rank m. Then, replacing $2T$ in (6.3.6) by cT, we obtain

$$E\left(|I_{np} - cR^2 \Sigma_Y^{\frac{1}{2}} (W \otimes T) \Sigma_Y^{\frac{1}{2}}|^{-\frac{1}{2}}\right) = E\left(|I_{mp} - cR^2 P \otimes (\Sigma^{\frac{1}{2}} T \Sigma^{\frac{1}{2}})|^{-\frac{1}{2}}\right). \tag{6.3.7}$$

Let v_j and τ_j $(j = 1, 2, \ldots, np)$ be the eigenvalues of $\Sigma_Y^{\frac{1}{2}}(W \otimes T)\Sigma_Y^{\frac{1}{2}}$ and $P \otimes (\Sigma^{\frac{1}{2}} T \Sigma^{\frac{1}{2}})$ respectively. Then (6.3.7) is reduced to

$$E(f(cR^2)) = E(g(cR^2)), \tag{6.3.8}$$

where

$$f(x) = \prod_{j=1}^{np} (1 - xv_j)^{-\frac{1}{2}}, \quad g(x) = \prod_{j=1}^{np} (1 - x\tau_j)^{-\frac{1}{2}}. \tag{6.3.9}$$

Now there exist $r > 0$ and sequences $\{a_\ell\}$, $\{b_\ell\}$ such that

$$f(x) = \sum_{\ell=0}^{\infty} a_\ell x^\ell, \quad g(x) = \sum_{\ell=0}^{\infty} b_\ell x^\ell, \quad |x| < r, \tag{6.3.10}$$

e.g., we may choose r such that

$$r \max(\{|v_j| : j \geq 1\} \cup \{|\tau_j| : j \geq 1\}) < 1.$$

By (6.3.8) and (6.3.10),

$$\sum_{\ell=0}^{\infty} a_\ell c^\ell E(R^{2\ell}) = \sum_{\ell=0}^{\infty} b_\ell c^\ell E(R^{2\ell}), \quad |c| < r/K^2. \tag{6.3.11}$$

Since $E(R^{2\ell}) > 0$ for $\ell = 0, 1, 2, \ldots$, by comparing the coefficients of $c^0, c^1, c^2, \cdots$ on both sides of (6.3.11), we obtain

$$a_\ell = b_\ell, \quad \text{for} \quad \ell = 0, 1, 2, \ldots .$$

So by (6.3.10),

$$f(x) = g(x) \quad \text{for} \quad |x| < r,$$

whence, by (6.3.9) - (6.3.11),

$$|I_{np} - c\Sigma_Y^{\frac{1}{2}}(W \otimes T)\Sigma_Y^{\frac{1}{2}}|^{-\frac{1}{2}} = |I_{np} - cP \otimes (\Sigma^{\frac{1}{2}} T \Sigma^{\frac{1}{2}})|^{-\frac{1}{2}}.$$

So by the theorem of identity or analytic continuation,

$$|I_{np} - \Sigma_Y^{\frac{1}{2}}(W \otimes T)\Sigma_Y^{\frac{1}{2}}| = |I_{np} - P \otimes (\Sigma^{\frac{1}{2}} T \Sigma^{\frac{1}{2}})|. \tag{6.3.12}$$

By Corollary 4.4.4, (6.3.12) is equivalent to

$$Z_*' W Z_* \sim W_p(m, \Sigma). \tag{6.3.13}$$

Thus by Theorem 5.3.1, there exists an $A \in \mathcal{N}_E$ such that (a) and (b) hold.

Now suppose that (a) and (b) hold. Let $Y_* = W^{\frac{1}{2}}(Y - \mu) = (W^{\frac{1}{2}} \otimes I_p)(Y - \mu)$. Then by (6.2.6) and (a),

$$Y_* \sim MEC_{n\times p}(0, (W^{\frac{1}{2}} \otimes I_p)\Sigma_Y(W^{\frac{1}{2}} \otimes I_p), \phi) = MEC_{n\times p}(0, \tilde{W} \otimes \Sigma, \phi),$$

where $\tilde{W} = W^{\frac{1}{2}}AW^{\frac{1}{2}}$. Let $X \sim MEC_{n\times p}(0, I_n \otimes \Sigma, \phi)$. Then

$$Y_{*} \stackrel{d}{=} W^{\frac{1}{2}}A^{\frac{1}{2}}X. \tag{6.3.14}$$

Thus

$$Q(Y) = Y_{*}'Y_{*} \stackrel{d}{=} (W^{\frac{1}{2}}A^{\frac{1}{2}}X)'W^{\frac{1}{2}}A^{\frac{1}{2}}X = X'A^{\frac{1}{2}}WA^{\frac{1}{2}}X. \tag{6.3.15}$$

Let $\{e_k\}_{k=1}^{n}$ be an orthonormal basis of E. From (b), we know that $A^{\frac{1}{2}}WA^{\frac{1}{2}}$ is idempotent of rank m. So there exists an orthogonal $\Gamma \in \mathcal{L}(E,E)$ such that $\Gamma'A^{\frac{1}{2}}WA^{\frac{1}{2}}\Gamma = \sum_{k=1}^{m} e_k \Box e_k$. Let $X_{*} = \Gamma'X$. Then $X_{*} \sim MEC_{n\times p}(0, I_n \otimes \Sigma, \phi)$. So with $X_{*} = \{X_{*1}, X_{*2}\}$, $E_1 = \langle\{e_1, e_2, \ldots, e_m\}\rangle$ and $X_{*1} = \sum_{k=1}^{m}(e_k \Box e_k)X_{*} \in \mathcal{L}(V, E_1)$, $X_{*1}'X_{*1} \sim GW_p(m; n-m; \Sigma; \phi)$. Since

$$X'A^{\frac{1}{2}}WA^{\frac{1}{2}}X \stackrel{d}{=} X_{*}'\Gamma'A^{\frac{1}{2}}WA^{\frac{1}{2}}\Gamma X_{*} \stackrel{d}{=} X_{*}'(\sum_{k=1}^{m}(e_k \Box e_k)X_{*} \stackrel{d}{=} X_{*1}'X_{*1},$$

we obtain from (6.3.15), $Q(Y) \stackrel{d}{=} X_{*1}'X_{*1}$, and therefore $Q(Y) \sim GW_p(m; n-m; \Sigma; \phi)$. □

Note that Theorem 6.3.1 still holds if the assumption $P(Y = \mu) = 0$ is replaced by a weaker condition $P(Y = \mu) < 1$, i.e. $P(R > 0) > 0$.

Corollary 6.3.2. *(Anderson and Fang (1982a)). Assume that $Y \sim MEC_{n\times p}(0, I_n \otimes \Sigma, \phi)$ with Σ being positive definite and $P(Y = 0) = 0$. Let $W \in \mathcal{S}_n$. Then $Y'WY \sim GW_p(m; n-m; \Sigma; \phi)$ if and only if*

$$W^2 = W, \qquad r(W) = m.$$

Theorem 6.3.3. *Suppose that Y is a random operator of a probability space $(\Omega, \mathcal{A}, P)$ such that $Y \sim MEC_{n\times p}(\mu, \Sigma_Y\, \phi)$ and $P(Y = \mu) < 1$. Let $i \in$*

$\{1,2,\cdots,\ell\}$, $W_i \in \mathcal{N}_E$, $m_i \in \{1,2,\cdots,\}$, $\Sigma(\neq 0) \in \mathcal{N}_V$ and $Q_i(Y) = (Y-\mu)'W_i(Y-\mu)$. *Then (a) and (b) are equivalent:*

$$(a) \quad (Q_1(Y), Q_2(Y), \cdots, Q_\ell(Y)) \sim GW_p(m_1, m_2, \cdots, m_\ell; n - \sum_{i=1}^{\ell} m_i; \Sigma; \phi).$$

(b) For some $A \in \mathcal{N}_E$ and for any distinct $i, j \in \{1,2,\ldots,\ell\}$,

(i) $(W_i \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W_i \otimes I_p) = 0,$

(ii) $AW_iAW_i = AW_i, \qquad r(AW_i) = m_i,$

(iii) $W_iAW_j = 0,$

and

(iv) $(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) = 0.$

Proof. Suppose that (a) holds. Then by the definition of $GW_p(m_1,\ldots,m_\ell; n - \sum_{i=1}^{\ell} m_i; \Sigma; \phi)$, there exists a random matrix $X \sim MEC_{n\times p}(0, I_n \otimes \Sigma, \phi)$ such that

$$(Q_1(Y),\ldots,Q_\ell(Y)) \stackrel{d}{=} (X_1'X_1,\ldots,X_\ell'X_\ell), \tag{6.3.16}$$

where $X = \{X_1,\ldots,X_\ell,X_{\ell+1}\}$, $X_i \in \mathcal{L}(V,E_i)$, $i = 1,\ldots,\ell,\ell+1$, and $m_{\ell+1} = n - \sum_{i=1}^{\ell} m_i$. Thus for each $i = 1,2,\ldots,\ell$, by definition 6.2.1, the distribution of $Q_i(Y)$ is $GW_p(m_i; n - m_i; \Sigma; \phi)$. So by Theorem 6.3.1, there exists an $A_i \in \mathcal{N}_E$ such that

(d1) $\quad (W_i \otimes I_p)(\Sigma_Y - A_i \otimes \Sigma)(W_i \otimes I_p) = 0$

and

(d2) $\quad A_iW_iA_iW_i = A_iW_i, \qquad r(A_iW_i) = m_i.$

Now let $W = \sum_{i=1}^{\ell} W_i$ and $m = \sum_{i=1}^{\ell} m_i$. Then by (6.3.16),

$$(Y-\mu)'W(Y-\mu) = \sum_{i=1}^{\ell} Q_i(Y) \stackrel{d}{=} \sum_{i=1}^{\ell} X_i'X_i = X_{\cdot 1}'X_{\cdot 1}, \tag{6.3.17}$$

where $X_{\bullet 1} = \{X_1, X_2, \dots X_\ell\} \in \mathcal{L}(V, E_1 \oplus E_2 \oplus \cdots \oplus E_\ell)$ and $X = \{X_{\bullet 1}, X_{\ell+1}\} \sim MEC_{n\times p}(0, I_n \otimes \Sigma; \phi)$. Thus

$$(Y-\mu)'W(Y-\mu) \stackrel{d}{=} X_{\bullet 1}'X_{\bullet 1} \sim GW_p(m; n-m; \Sigma; \phi).$$

By Theorem 6.3.1 again, we obtain that for some $A \in \mathcal{N}_n$,

$$(W \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W \otimes I_p) = 0,$$

which is equivalent to

$$(W^\circ \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W^\circ \otimes I_p) = 0, \tag{6.3.18}$$

where $W^\circ = WW^+ = W^+W$. Multiplying both sides of (6.3.18) by $(W_i \otimes I_p)$, we have

$$(W_iW^\circ \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W^\circ W_i \otimes I_p) = 0. \tag{6.3.19}$$

By Lemma 5.2.5, we obtain $ImW_i \subseteq ImW$ and therefore $W_iW^\circ = W^\circ W_i = W_i$. So by (d1) and (6.3.19),

$$W_iA_iW_i \otimes \Sigma = W_iAW_i \otimes \Sigma.$$

Since $\Sigma \neq 0$,

$$W_iA_iW_i = W_iAW_i. \tag{6.3.20}$$

So (b)(i) follows from (d1) and (6.3.20). Also by (6.3.20) and (d2),

$$W_iAW_i = W_iA_iW_i = W_iA_iW_iA_iW_i = W_iAW_iAW_i$$

and $m_i = r(A_iW_i) = r(AW_i)$, proving (b)(ii).

For (b)(iii) and (b)(iv), let $W_\bullet = W_i + W_j$. Then by (6.3.16), $(Y-\mu)'W_\bullet(Y-\mu) \sim GW_p(m_i, m_j; n-m_i-m_j; \Sigma; \phi)$. By Theorem 6.3.1 again, there exists an $A_\bullet \in \mathcal{N}_n$ such that

$$(W_\bullet \otimes I_p)(\Sigma_Y - A_\bullet \otimes \Sigma)(W_\bullet \otimes I_p) = 0 \tag{6.3.21}$$

and

$$W_* A_* W_* A_* W_* = W_* A_* W_*. \tag{6.3.22}$$

Since $ImW_* \subseteq ImW$, by the above argument, we obtain

$$W_* A_* W_* = W_* A W_*. \tag{6.3.23}$$

So by (6.3.22), (6.3.23) and (b)(ii),

$$A(W_i + W_j)A(W_i + W_j)A = A(W_i + W_j)A,$$

i.e.,

$$AW_iAW_jA + AW_jAW_iA = 0. \tag{6.3.24}$$

By Section 5.22 of Wong (1986), (6.3.24) implies that

$$AW_iAW_jA = 0,$$

and therefore (b)(iii) follows. Again by (6.3.23), (b)(i), (b)(ii), and (b)(iii), (6.3.21) is reduced to

$$(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) + (W_j \otimes I_p)\Sigma_Y(W_i \otimes I_p) = 0.$$

So by Section 5.22 of Wong (1986) again, (b)(iv) follows.

Now assume that (b)(i) - (b)(iv) hold. Let $X = \{X_1, \ldots, X_\ell, X_{\ell+1}\} \sim MEC_{n\times p}$ $(0,\, I_n \otimes \Sigma,\, \phi)$, $Z_i = W_i^{\frac{1}{2}}(Y-\mu)$ and $Z_{*i} = W_i^{\frac{1}{2}} A^{\frac{1}{2}} X$, $i = 1, \ldots, \ell$. Then

$$(Z_1', \ldots, Z_\ell')' \sim MEC_{n\ell\times p}(0, \Sigma_Z, \phi) \tag{6.3.25}$$

and

$$(Z_{*1}', \ldots, Z_{*\ell}')' \sim MEC_{n\ell\times p}(0, \Sigma_{Z_*}, \phi), \tag{6.3.26}$$

where

$$\Sigma_Z = \begin{pmatrix} W_1^{\frac{1}{2}} \otimes I_p \\ \vdots \\ W_\ell^{\frac{1}{2}} \otimes I_p \end{pmatrix} \Sigma_Y \left(W_1^{\frac{1}{2}} \otimes I_p, \ldots, W_\ell^{\frac{1}{2}} \otimes I_p \right)$$

$$= \begin{pmatrix} (W_1^{\frac{1}{2}} \otimes I_p)\Sigma_Y(W_1^{\frac{1}{2}} \otimes I_p) & \ldots & (W_1^{\frac{1}{2}} \otimes I_p)\Sigma_Y(W_\ell^{\frac{1}{2}} \otimes I_p) \\ \vdots & & \vdots \\ (W_\ell^{\frac{1}{2}} \otimes I_p)\Sigma_Y(W_1^{\frac{1}{2}} \otimes I_p) & \ldots & (W_\ell^{\frac{1}{2}} \otimes I_p)\Sigma_Y(W_\ell^{\frac{1}{2}} \otimes I_p) \end{pmatrix}$$

$$= \text{diag}\,(W_1^{\frac{1}{2}} A W_1^{\frac{1}{2}} \otimes \Sigma, \ldots, W_\ell^{\frac{1}{2}} A W_\ell^{\frac{1}{2}} \otimes \Sigma) \quad \text{(by (b)(i) and (b)(iv))}$$

and

$$\Sigma_{Z_*} = \begin{pmatrix} W_1^{\frac{1}{2}} A^{\frac{1}{2}} \otimes I_p \\ W_\ell^{\frac{1}{2}} A^{\frac{1}{2}} \otimes I_p \end{pmatrix} (I_n \otimes \Sigma)(A^{\frac{1}{2}} W_1^{\frac{1}{2}} \otimes I_p, \ldots, A^{\frac{1}{2}} W_\ell^{\frac{1}{2}} \otimes I_p)$$

$$= \text{diag}\,(W_1^{\frac{1}{2}} A W_1^{\frac{1}{2}} \otimes \Sigma, \ldots, W_\ell^{\frac{1}{2}} A W_\ell^{\frac{1}{2}} \otimes \Sigma) \quad \text{(by (b)(iii))}.$$

Thus by definition (2.2),

$$(Z_1', \ldots, Z_\ell')' \stackrel{d}{=} (Z_{*1}', \ldots, Z_{*\ell}')'. \tag{6.3.27}$$

Let $g(z_1, \ldots, z_\ell) = (z_1' z_1, \ldots, z_\ell' z_\ell)$, $z_i \in \mathcal{L}(V, E_i)$. Then by Lemma 6.2.1,

$$(Z_1' Z_1, \ldots, Z_\ell' Z_\ell) \stackrel{d}{=} (Z_{*1}' Z_{*1}, \cdots, Z_{*\ell}' Z_{*\ell}),$$

i.e.,

$$(Q_1(Y), \ldots, Q_\ell(Y)) \stackrel{d}{=} (X_1' \tilde{W}_1 X, \ldots, X' \tilde{W}_\ell X), \tag{6.3.28}$$

where $\tilde{W}_i = A^{\frac{1}{2}} W_i A^{\frac{1}{2}}$, $i = 1, \ldots, \ell$. Let $\{e_k\}_{k=1}^n$ be an orthonormal basis of E. Note that (b)(ii) and (b)(iii) imply that each $\tilde{W}_i$ is an idempotent operator of rank m_i and $\tilde{W}_i \tilde{W}_j = 0$ for all distinct i, j. So there exists an orthogonal operator $\Gamma \in \mathcal{L}(E, E)$ such that

$$\Gamma' \tilde{W}_i \Gamma = \sum_{k=m_{i-1}+1}^{m_i} e_k \Box e_k, \qquad i = 1, \ldots, \ell.$$

Let $X_* = \Gamma' X$. Then $X_* \sim MEC_{n\times p}(0, I_n \otimes \Sigma, \phi)$. Thus with E_i being given in (6.2.10) with $\nu = 0$, $X_* = \{X_{*1}, \ldots, X_{*\ell}, X_{*\ell+1}\}$ and $X_{*i} = (\sum_{k=m_{i-1}+1}^{m_i} e_k \Box e_k) X_* \in \mathcal{L}(V, E_i)$, $i = 1, \ldots, \ell$,

$$\begin{aligned}(X'\bar{W}_1 X, \ldots, X'\bar{W}_\ell X) &\stackrel{d}{=} (X_*'\Gamma'\bar{W}_1\Gamma X_*, \ldots, X_*'\Gamma'\bar{W}_\ell\Gamma X_*) \\ &\stackrel{d}{=} (X_{*1}'X_{*1}, \ldots X_{*\ell}'X_{*\ell}).\end{aligned} \tag{6.3.29}$$

By (6.3.28) and (6.3.29),

$$(Q_1(Y), \ldots Q_\ell(Y)) \stackrel{d}{=} (X_{*1}'X_{*1}, \ldots X_{*\ell}'X_{*\ell}),$$

proving (a). □

Corollary 6.3.4. *In Theorem 6.3.3, if each $m_i = r(W_i)$, then (a′) and (b′) are equivalent:*

(a′) $(Q_1(Y), Q_2(Y), \ldots, Q_\ell(Y)) \sim GW_p(m_1, m_2, \ldots, m_\ell; n - \sum_{i=1}^{\ell} r(W_i); \Sigma; \phi)$.

(b′) For any distinct $i, j = 1, 2, \ldots, \ell$,

(i) $(W_i \otimes I_p)\Sigma_Y(W_i \otimes I_p) = W_i \otimes \Sigma$,

and

(ii) $(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) = 0$.

Proof. . The proof for $(a') \Rightarrow (b')$ is the same as that of Proposition 5.4.3. For the proof of $(b') \Rightarrow (a')$, by Proposition 5.4.3, it sufficies to show that (b)(iii) holds for any distinct $i, j \in \{1, 2, \ldots, \ell\}$. By Corollary 5.2.5 again, $Im W_i$, $Im W_j \subseteq Im W$, where $W = \sum_{i=1}^{\ell} W_i$. Since $W_i e_\alpha = 0$ for all $\alpha = s+1, \ldots, n$, we obtain from (5.4.6),

$$(W_i \otimes I_p) H (W_j \otimes I_p) = \sum_{\max(\alpha,\alpha')>s}^{n} \sum_{\beta,\beta'=1}^{p} \sigma_{\alpha\alpha'\beta\beta'}(W_i e_\alpha \Box W_j e_{\alpha'}) \otimes (f_\beta \Box f_{\beta'}) = 0.$$

Thus by (b′)(ii) and (5.4.7),

$$\begin{aligned}0 = (W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) &= (W_i \otimes I_p)(A \otimes \Sigma + H)(W_j \otimes I_p) \\ &= W_i A W_j \otimes \Sigma.\end{aligned}$$

Hence (b)(iii) follows as $\Sigma \neq 0$. □

4. Further applications

We shall now extend the following three results given in Section 5.4 from the normal settings to the MEC settings.

Proposition 6.4.1. . *Assume that $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$. Let $W \in \mathcal{N}_E$ and $X = W^{\frac{1}{2}}(Y-\mu)$. Then $X \sim MEC_{n\times p}(0, \Sigma_X, \phi)$ with $\Sigma_X = (W^{\frac{1}{2}} \otimes I_p)\Sigma_Y(W^{\frac{1}{2}} \otimes I_p)$. Assume further that Σ_X is of the form $P \otimes \Sigma$, where P is self-adjoint and idempotent of rank m. Suppose that $P(Y = \mu) < 1$ and $X'X \sim GW_p(m; n-m; \Sigma; \phi)$. Then A in Theorem 6.3.3 can be chosen as $W^{-\frac{1}{2}}PW^{-\frac{1}{2}}$.*

In MANOVA models with balanced subsample sizes, the following properties are given for matrices $W_1, \ldots, W_k$ in $\mathcal{N}_n$,

$$W_iW_j = \delta_{ij}W_i, \qquad \sum_{i=1}^{k} W_i = I_n - J_n,$$

where J_n is an $n \times n$ matrix and each component of J_n is equal to $1/n$, see, e.g., Pavur (1987). Suppose that $Y \sim MEC_{n\times p}(0, \Sigma_Y, \phi)$ and Σ_Y is of the form $\sum_{i=1}^{k} W_i \otimes \Sigma + H$, where H is an $np \times np$ matrix such that $ImH \subseteq Im(J_n \otimes I_p)$. Let $Q_i(Y) = Y'W_iY$, $i = 1, \ldots, k$. Then from Theorem 6.3.3, we obtain that

$$(Q_1(Y), \ldots, Q_k(Y)) \sim GW_p(r(W_1), \ldots, r(W_k); n - \sum_{i=1}^{k} r(W_i); \Sigma; \phi).$$

More generally, we can obtain:

Proposition 6.4.2. *Assume that $Y \sim MEC_{n\times p}(0, \Sigma_Y, \phi)$ with $\Sigma_Y = A \otimes \Sigma + H$ for some $A \in \mathcal{N}_n$ and H, such that $ImH \subseteq Im(M \otimes I_p)$ for some M in $\mathcal{N}_n$. Let $i \in \{1, \ldots, \ell\}$, $W_i \in \mathcal{N}_n$ and $Q_i(Y) = Y'W_iY$. Suppose that for all distinct $i, j \in \{1, \ldots, \ell\}$,*

(a) *$AW_iAW_i = AW_i, \qquad r(AW_i) = m,$*

(b) $W_i M = 0$,

and

(c) $W_i A W_j = 0$.

Then $(Q_1(Y), \ldots Q_\ell(Y)) \sim GW_p(m_1, \ldots, m_\ell; n - \sum_{i=1}^{\ell} m_i; \Sigma; \phi)$.

Proposition 6.4.3. *Consider the multivariate components of variance model:* $Y \sim MEC_{n\times p}(XB, \Sigma_Y, \phi)$ *with* $\Sigma_Y = \sum_{j=1}^{k} A_j \otimes \Sigma_j$. *Note that* Σ_Y *here is not of the form* $A \otimes \Sigma$. *Let* $i \in \{1, \ldots, \ell\}$, $W_i \in \mathcal{N}_n$, $m_i > 0$, $\Sigma \neq 0$, *and* $Q_i(Y) = (Y - XB)'W_i(Y - XB)$. *Then*

$$(Q_1(Y), \ldots, Q_\ell(Y)) \sim GW_p(m_1, \ldots, m_\ell; n - \sum_{i=1}^{\ell} m_i; \Sigma; \phi)$$

if and only if for any distinct $i, j = 1, 2, \ldots, \ell$,

(a) $(W_i \otimes I_p)(\Sigma_Y - P \otimes \Sigma)(W_i \otimes I_p) = 0$,

(b) $PW_iPW_i = PW_i, \quad r(PW_i) = m_i$ *and*

(c) $W_iPW_j = 0$, $i, j \in \{1, \ldots, \ell\}$,

and

(d) $(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) = 0$,

where $P = \sum_{s=1}^{k} \alpha_s A_s, \quad \alpha_s = tr(\Sigma_s)/tr(\Sigma)$.

Now for $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$ with $\mu \neq 0$, we shall give sufficient conditions under which

$$Y'WY \sim GW_p(m; n - m; \Sigma; \lambda, \phi), \qquad (6.4.1)$$

where $W \in \mathcal{N}_E$.

Proposition 6.4.4. *Suppose that* $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$. *Let* $W \in \mathcal{N}_E$. *Then (6.4.1) holds if there exists an* $A \in \mathcal{N}_E$ *such that*

(i) $(W \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W \otimes I_p) = 0$,

(ii) $AWAW = AW, \qquad r(AW) = m$,

and

(iii) $\lambda = \mu' W \mu = \mu' W A W \mu$.

Proof. Let $Y_* = W^{\frac{1}{2}} Y = (W^{\frac{1}{2}} \otimes I_p)(Y)$. Then by Lemma 6.2.3 and (i),

$$Y_* \sim MEC_{n\times p}(\,W^{\frac{1}{2}}\mu,\ W^{\frac{1}{2}}AW^{\frac{1}{2}} \otimes \Sigma,\ \phi\,). \tag{6.4.2}$$

By (ii), $W^{\frac{1}{2}}AW^{\frac{1}{2}}$ is an idempotent of rank m and hence $I_n - W^{\frac{1}{2}}AW^{\frac{1}{2}}$ is also idempotent. So there exists an orthogonal $\Gamma \in \mathcal{L}(E, E)$ such that

$$W^{\frac{1}{2}}AW^{\frac{1}{2}} = \Gamma\left(\sum_{k=1}^{m} c_k \Box c_k\right)\Gamma', \tag{6.4.3}$$

where $\{c_k\}_{k=1}^{n}$ is an orthonormal basis of E. By (iii),

$$0 = \mu' W \mu - \mu' W A W \mu = \mu' W^{\frac{1}{2}}(I_n - W^{\frac{1}{2}}AW^{\frac{1}{2}})W^{\frac{1}{2}}\mu,$$

which by (ii) implies that

$$(I_p - W^{\frac{1}{2}}AW^{\frac{1}{2}})W^{\frac{1}{2}}\mu = 0,$$

i.e.,

$$W^{\frac{1}{2}}\mu = W^{\frac{1}{2}}AW\mu. \tag{6.4.4}$$

Now let $X \sim MEC_{n\times p}(\,W^{\frac{1}{2}}\mu, I_n \otimes \Sigma,\ \phi\,)$. Then by (6.4.2) - (6.4.4),

$$Y_* \stackrel{d}{=} W^{\frac{1}{2}}AW^{\frac{1}{2}}X = \Gamma\left(\sum_{k=1}^{m} e_k \Box e_k\right)\Gamma' X. \tag{6.4.5}$$

Let $X_* = \Gamma' X$ and $X_{*1} = \sum_{k=1}^{m}(e_k \Box e_k)X_*$. Then $X_* \sim MEC_{n\times p}(\,\Gamma' W^{\frac{1}{2}}\mu,\ I_n \otimes \Sigma,\ \phi\,)$ and by definition 6.2.1,

$$X_{*1}' X_{*1} \sim GW_p(\,m;\ n-m;\ \Sigma;\ \lambda_*;\ \phi\,), \tag{6.4.6}$$

where

$$\lambda_* = \left(\sum_{k=1}^{m}(e_k\Box c_k)\Gamma' W^{\frac{1}{2}}\mu\right)' \left(\sum_{k=1}^{m}(e_k\Box c_k)\Gamma' W^{\frac{1}{2}}\mu\right)$$
$$= \mu' W^{\frac{1}{2}}\Gamma \sum_{k=1}^{m}(e_k\Box c_k)\Gamma' W^{\frac{1}{2}}\mu = \mu' W A W \mu = \lambda.$$

Since

$$Y'WY = Y_*'Y_* \overset{d}{=} X'W^{\frac{1}{2}}AW^{\frac{1}{2}}X \overset{d}{=} X_*' \sum_{k=1}^{m}(e_k\Box c_k)X_*$$
$$= \left(\sum_{k=1}^{m}(e_k\Box e_k)X_*\right)' \left(\sum_{k=1}^{m}(e_k\Box c_k)X_*\right) \overset{d}{=} X_{*1}'X_{*1}.$$

the desired result follows from (6.4.6). □

The following extension of Proposition 6.4.4 can be proved by Proposition 6.4.4 and Theorem 6.3.3.

Proposition 6.4.5. *Suppose that $Y \sim MEC_{n\times p}(\mu, \Sigma_Y, \phi)$. Let $i \in \{1, 2 \ldots, \ell\}$, $W_i \in \mathcal{N}_E$. Then*

$$(Y'W_1Y, \ldots, Y'W_\ell Y) \sim GW_p(m_1, \ldots, m_\ell; n - \sum_{i=1}^{\ell} m_i; \Sigma; \lambda_1, \ldots, \lambda_\ell; \phi)$$

if there exists some $A \in \mathcal{N}_E$ such that for any distinct i, $j \in \{1, 2, \ldots, \ell\}$,

(a) $(W_i \otimes I_p)(\Sigma_Y - A \otimes \Sigma)(W_i \otimes I_p) = 0,$

(b) $AW_iAW_i = AW_i, \qquad r(AW_i) = m_i,$

(c) $\lambda_i = \mu' W_i \mu = \mu' W_i A W_i \mu,$

(d) $W_i A W_j = 0,$

and

(e) $(W_i \otimes I_p)\Sigma_Y(W_j \otimes I_p) = 0.$

Appendix

Let $\{e_i\}_{i=1}^n$ and $\{f_j\}_{j=1}^p$ be orthonormal bases of E and V respectively such that

$$W = \sum_{i=1}^{s} w_i e_i \Box e_i, \quad w_i > 0, \quad i = 1, 2, \ldots, s \tag{A1}$$

and

$$\Sigma = \sum_{j=1}^{r} \sigma_j f_j \Box f_j, \quad \sigma_j > 0, \quad j = 1, 2, \ldots, r, \tag{A2}$$

where $s = r(W)$ and $r = r(\Sigma)$. Then $\Sigma_Y \in \mathcal{L}(E, E) \otimes \mathcal{L}(V, V)$ can be written as

$$\Sigma_Y = \sum_{i,i'=1}^{n} (e_i \Box e_{i'}) \otimes \Sigma_{ii'}, \quad \Sigma_{ii'} = \sum_{jj'=1}^{p} \sigma_{ii'jj'} f_j \Box f_{j'}. \tag{A3}$$

Now suppose that $Y'WY \sim W_p(m, \Sigma, \phi)$. Let $E = E_1 \oplus E_2$, where E_1 and E_2 are, respectively, m- and $(n-m)$-dimensional subspaces of E. Let $X = \{X_1, X_2\} \sim MEC_{n \times p}(0, I_n \otimes \Sigma, \phi)$ with $X_1 \in (V, E_1)$ and $X_2 \in (V, E_2)$. Then by the definition of $GW_p(m; n-m; \Sigma; \phi)$,

$$Q(Y) = (Y - \mu)'W(Y - \mu) \overset{d}{=} X_1' X_1. \tag{A4}$$

Then

$$E(Y'WY) = E(X_1'X_1) \quad \text{and} \quad \mathrm{Cov}(Y'WY) = \mathrm{Cov}(X_1'X_1). \tag{A5}$$

By (A1), Corollary 3.2.4 and Example 3.2.1, (A5), upon simplification, is reduced to

$$\sum_{ii'=1}^{n} (e_i' W e_{i'}) \Sigma_{ii'} = \sum_{i=1}^{s} w_i \Sigma_{ii} = m\Sigma \tag{A6}$$

and

$$\begin{aligned} &\sum_{ii'=1}^{s} w_i w_{i'} \{4\phi''(0) [\Sigma_{ii'} \otimes \Sigma_{ii'} + K_{p,p}(\Sigma_{i'i} \otimes \Sigma_{i'i})] \\ &\qquad +4 \left[\phi''(0) - (\phi'(0))^2\right] \mathrm{vec}\, \Sigma_{ii} (\mathrm{vec}\, \Sigma_{i'i'})' \} \\ &= m(I_{p^2} + K_{p,p})(\Sigma \otimes \Sigma) + 4\left[\phi''(0) - (\phi'(0))^2\right] m^2 \mathrm{vec}\, \Sigma (\mathrm{vec}\, \Sigma)'. \end{aligned} \tag{A7}$$

By (A6), (A7) becomes

$$\sum_{ii'=1}^{s} w_i w_{i'} \{\Sigma_{ii'} \otimes \Sigma_{ii'} + K_{p,p}(\Sigma_{i'i} \otimes \Sigma_{i'i})\} \\ = m(I_{p^2} + K_{p,p})(\Sigma \otimes \Sigma). \tag{A8}$$

Now choose $t = \text{vec}(f_j \Box f_j) = f_j \otimes f_j,\ \ j = 1, 2, \ldots, p$. Then

$$t'(\Sigma_{ii'} \otimes \Sigma_{ii'} + K_{p,p}(\Sigma_{i'i} \otimes \Sigma_{i'i}))\,t$$
$$= (f_j \otimes f_j)(\Sigma_{ii'} \otimes \Sigma_{ii'} + K_{p,p}(\Sigma_{i'i} \otimes \Sigma_{i'i}))(f_j \otimes f_j)$$
$$= \sigma^2_{ii'jj} + \sigma^2_{i'ijj} = 2\sigma^2_{i'ijj}$$

and

$$t'(I_{p^2} \otimes K_{p,p})(\Sigma \otimes \Sigma)t = 2\sigma_j^2.$$

Thus by (A8), we obtain

$$\sum_{ii'=1}^{s} w_i w_{i'} \sigma^2_{ii'jj} = m\sigma_j^2, \quad j = 1, 2, \ldots, p. \tag{A9}$$

Let $t = f_j \otimes f_j + f_{j'} \otimes f_{j'}$. Then

$$t'(\Sigma_{ii'} \otimes \Sigma_{ii'} + K_{p,p}(\Sigma_{i'i} \otimes \Sigma_{i'i}))\,t$$
$$= \sigma^2_{ii'jj} + \sigma^2_{ii'jj'} + \sigma^2_{ii'j'j} + \sigma^2_{ii'j'j'} + \sigma^2_{i'ijj} + \sigma^2_{i'ijj'} + \sigma^2_{i'ij'j} + \sigma^2_{i'ij'j'}$$
$$= 2(\sigma^2_{i'ijj} + \sigma^2_{i'ijj'} + \sigma^2_{i'ij'j} + \sigma^2_{i'ij'j'})$$

and

$$t'(I_{p^2} \otimes K_{p,p})(\Sigma \otimes \Sigma)t = 2(\sigma_j^2 + \sigma_{j'}^2).$$

Thus by (A8), we obtain

$$\sum_{ii'=1}^{s} w_i w_{i'} (\sigma^2_{i'ijj} + \sigma^2_{i'ijj'} + \sigma^2_{i'ij'j} + \sigma^2_{i'ij'j'}) = m(\sigma_j^2 + \sigma_{j'}^2). \tag{A10}$$

So by (A9), (A10) is reduced to

$$\sum_{ii'=1}^{s} w_i w_{i'} (\sigma^2_{ii'jj'} + \sigma^2_{ii'j'j}) = 0, \quad j \neq j',\ j, j' = 1, 2 \ldots, p. \tag{A11}$$

Let $t = f_j \otimes f_{j'} + f_{j'} \otimes f_j$, $j \neq j', j, j' = 1, 2, \ldots, p$. Then we have

$$t' (\Sigma_{ii'} \otimes \Sigma_{ii'} + K_{p,p}(\Sigma_{i'i} \otimes \Sigma_{i'i}))\, t$$

$$= \sigma_{ii'jj}\sigma_{ii'j'j'} + \sigma_{ii'jj'}\sigma_{ii'j'j} + \sigma_{ii'j'j}\sigma_{ii'jj'} + \sigma_{ii'j'j'}\sigma_{ii'jj}$$

$$\sigma_{i'ijj}\sigma_{i'ij'j'} + \sigma_{i'ijj'}\sigma_{i'ij'j} + \sigma_{i'ij'j}\sigma_{i'ijj'} + \sigma_{i'ij'j'}\sigma_{i'ijj}$$

$$= 4(\sigma_{ii'jj}\sigma_{ii'j'j'} + \sigma_{ii'jj'}\sigma_{ii'j'j})$$

and

$$t'(I_{p^2} \otimes K_{p,p})(\Sigma \otimes \Sigma)t = 2(\sigma_j\sigma_{j'} + \sigma_{j'}\sigma_j) = 4\sigma_j\sigma_{j'}.$$

Thus by (A8), we obtain

$$\sum_{ii'=1}^{s} w_i w_{i'} (\sigma_{ii'jj}\sigma_{ii'j'j'} + \sigma_{ii'jj'}\sigma_{ii'j'j}) = m\sigma_j\sigma_{j'}, \quad j \neq j',\ j, j' = 1, 2, \ldots, p. \tag{A12}$$

Note that $w_i > 0,\ i = 1, 2, \ldots, s$. By (A2), (A9) and (A11), we obtain

$$\sigma_{ii'jj} = 0, \quad i, i' = 1, 2, \ldots, s, \quad j = r + 1, \ldots, p, \tag{A13}$$

and

$$\sigma_{ii'jj'} = \sigma_{ii'j'j} = 0, \quad j \neq j', \quad j, j' = 1, 2, \ldots, p, \quad i, i' = 1, 2, \ldots, s. \tag{A14}$$

By (A14), (A12) implies that

$$\sum_{ii'=1}^{s} w_i w_{i'} \sigma_{ii'jj}\sigma_{ii'j'j'} = m\sigma_j\sigma_{j'}, \quad j \neq j', \quad j, j' = 1, 2, \ldots, r. \tag{A15}$$

So combining (A10) and (A15), we obtain

$$\sum_{ii'=1}^{s} w_i w_{i'} \sigma_{ii'jj}\sigma_{ii'j'j'} = m\sigma_j\sigma_{j'}, \quad j, j' = 1, 2, \ldots, r. \tag{A16}$$

Therefore

$$\begin{aligned} \Sigma_Y &= \sum_{i,i'=1}^{n} \sum_{j,j'=1}^{p} \sigma_{ii'jj'}(e_i \Box e_{i'}) \otimes (f_j \Box f_{j'}) \\ &= \sum_{i,i'=1}^{s} \sum_{j,j'=1}^{r} \sigma_{ii'jj}(e_i \Box e_{i'}) \otimes (f_j \Box f_j) + H, \end{aligned} \tag{A17}$$

where

$$H = \sum_{j,j'=1}^{p} \sum_{\max(i,i')>s}^{n} \sigma_{ii'jj'}(e_i \square e_{i'}) \otimes (f_j \square f_j). \tag{A18}$$

Now let $a_{ii'j} = \sigma_{ii'jj}/\sigma_j$, $j = 1, 2, \dots, r$, and $i, i' = 1, 2, \dots, s$. Then for $j \neq j'$, $j, j' = 1, 2, \dots, r$, (A16) implies that

$$\sum_{i,i'=1}^{s} w_i w_{i'}(a_{ii'j}^2 - 2a_{ii'j}a_{ii'j'} + a_{ii'j'}^2) = \sum_{i,i'=1}^{s} w_i w_{i'}(a_{ii'j} - a_{ii'j'})^2 = 0,$$

i.e.,

$$a_{ii'j} = a_{ii'j'}, \quad i, i' = 1, 2, \dots, s, \quad j \neq j', \ j, j' = 1, 2, \dots, r. \tag{A19}$$

Let $A = \sum_{i,i'=1}^{s} a_{ii'1} e_i \square e_{i'}$. Then by a similar argument as in the proof of Corollary 5.2.3, we can prove that $A \in \mathcal{N}_E$. Thus by (A19), (A17) is reduced to

$$\Sigma_Y = \sum_{j,j'=1}^{r} \left(\sum_{i,i'=1}^{s} a_{ii'j}(e_i \square e_{i'}) \right) \otimes \sigma_j(f_j \square f_j) + H = A \otimes \Sigma + H. \tag{A20}$$

Since $We_i = 0$ for all e_i, $i = s + 1, \dots, n$,

$$(W \otimes I_p) H (W \otimes I_p) = 0.$$

Thus by (A20),

$$(W \otimes I_p)(\Sigma_Y - A \otimes \Sigma + H)(W \otimes I_p) = 0,$$

and hence (a) follows.

Now let $Y_* = W^{\frac{1}{2}}(Y - \mu)$. Then by Lemma 6.2.3 and (a),

$$Y_* \sim MEC_{n \times p}(0\, W^{\frac{1}{2}} A W^{\frac{1}{2}} \otimes \Sigma, \phi)$$

and hence

$$Y_* \stackrel{d}{=} W^{\frac{1}{2}} A^{\frac{1}{2}} X.$$

Thus by (A4),

$$(Y-\mu)'W(Y-\mu) = Y_*'Y_* \stackrel{d}{=} X'A^{\frac{1}{2}}WA^{\frac{1}{2}}X \stackrel{d}{=} X_1'X_1. \qquad \text{(A21)}$$

Now it sufficies to show that (A21) implies -hat

$$A^{\frac{1}{2}}WAWA^{\frac{1}{2}} = A^{\frac{1}{2}}WA^{\frac{1}{2}}, \qquad r(A^{\frac{1}{2}}WA^{\frac{1}{2}}) = m.$$

Let $W_* = A^{\frac{1}{2}}WA^{\frac{1}{2}}$. By (6.2.4),

$$X \stackrel{d}{=} RU\Sigma^{\frac{1}{2}} \quad \text{and} \quad X_1 \stackrel{d}{=} RU_1\Sigma^{\frac{1}{2}},$$

where $U = \{U_1, U_2\}$ with $U_1 \in \mathcal{L}(E_1, V)$ and R, U are given in (6.2.1). Thus by (A21),

$$R^2\Sigma^{\frac{1}{2}}U'W_*U\Sigma^{\frac{1}{2}} \stackrel{d}{=} R^2\Sigma^{\frac{1}{2}}U_1'U_1\Sigma^{\frac{1}{2}}. \qquad \text{(A22)}$$

Choose a chi random variable χ_{np} with np degrees of freedom that is independent of R and U and let a be a constant vector in V such that $a'\Sigma a = 1$. Then

$$z \equiv \chi_{np}U\Sigma^{\frac{1}{2}}a \sim N_n(0, I_n \otimes a'\Sigma a) = N_n(0, I_n)$$

and

$$z_1 \equiv \chi_{np}U_1\Sigma^{\frac{1}{2}}a \sim N_n(0, \sum_{k=1}^{m} e_k\Box e_k).$$

Thus by (A22), we obtain

$$\chi_{np}^2R^2a'\Sigma^{\frac{1}{2}}U'W_*U\Sigma^{\frac{1}{2}}a \stackrel{d}{=} \chi_{np}^2R^2a'\Sigma^{\frac{1}{2}}U_1'U_1\Sigma^{\frac{1}{2}}a,$$

i.e.,

$$R^2z'W_*z \stackrel{d}{=} R^2z_1'z_1 \stackrel{d}{=} R^2\chi_m^2. \qquad \text{(A23)}$$

By Lemma 1 of Anderson and Fang (1982b) or by (c) of Section 2.1.7 of Fang and Zhang (1990), (A23) implies that $z'W_*z \stackrel{d}{=} \chi_m^2$, which, by the standard univariate Cochran theorem, imples that $W_*^2 = W_*$ and $r(W_*) = m$. So the desired result follows. □

REFERENCES

1. Anderson, B.M., Anderson, T.W. and Olkin, I. (1986), *Maximum likelihood estimators and likelihood ratio criterion in multivariate components of variance*, Ann. Statist. 14, 405-417.

2. Anderson, T.W. (1984), An Introduction to Multivariate Analysis, 2nd Edition, *Wiley, New York.*

3. Anderson, T.W. (1987), *The asmyptotic distribution of likelihood ratio criterion for testing rank in multivariate components of variance*, J. Multivariate Anal. 30 72-79.

4. Anderson, T.W. and Fang, K.T (1982a), *On the theory of multivariate elliptically contoured distributions and their applications*, Technical report No. 54, Contract N00014-75-C-0442, Department of Statistics, Stanford University, California (Edited version in "Statistical Inference in Elliptically Contoured and Related Distribution" (Anderson, T.W. and Fang, K.T. eds. (1990)).

5. Anderson, T.W. and Fang, K.T. (1982b), *Distributions of quadratic forms and Cochran's theorem for elliptically contoured distributions and their applications*, Technical report No. 53, Contract N00014-75-C-0442, Department of Statistics, Stanford University, California. A condensed version "Cochran's theorem for elliptically contoured distributions" was published in Sankyha, Series A, 48 305-315 (1987).

6. Anderson, T.W. and Fang, K.T. (1990), *Preface: An introduction to elliptically contoured distributions*, In "Statistical Inference in Elliptically Contoured and Related Distributions, (K.T. Fang and T.W. Anderson, Eds), pp (i)-(vii), Allerton Press Inc., New York.

7. Anderson, T.W. and Styan, G.P.H. (1982), *Cochran's theorem, rank additivity,*

and tripotent matrices, In "Statistics and Probability: Essays in Honor of C.R. Rao" (G. Kallianpur, P.R. Krishnaiah, and J.K. Ghosh, Eds.), pp 1-23, North-Holland, Amsterdam.

8. Browne, M.W. and Neudecker, H. (1988), *The covariance matrix of a general symmetric second degree matrix polynomial under normality assumptions*, Linear Algebra Appl. **103**, 13-21.

9. Cambanis, S., Huang, S. and Simons, G. (1981),*On the theory of elliptically contoured distributions*, J. Multivariate Anal. **11**, 368-385.

10. Chmielewski, M.A. (1981), *Elliptically symmetric distributions: A review and bibliography*, Inter. Statist. Review **49**, 67-74.

11. Chipman, J. S. and Rao, M. M. (1964), *Projections, generalized inverses, and quadratic forms*, J. Math. Anal. Appl. **9**, 1-11.

12. Cochran, W. G. (1934), *The distribution of quadratic forms in a normal system, with applications to the analysis of covariance*, Proc. Cambridge Philos. Soc. **30**, 178-191.

13. Craig, A.T. (1943), *Note on the independence of certain quadratic forms in normal vectors*, Ann. Math. Statist. **14**, 195-197.

14. Dawid, A.P. (1978), *Extendability of spherical matrix distributions*, J. Multivariate Anal. **17**, 304-315.

15. DeGunst, M. C. M. (1987), *On the distribution of general quadratic functions in normal vectors*, Statistica Neerlandica **41**, 245-251.

16. Dik, J.J. and DeGunst, M.C.M. (1985), *The distribution of general quadratic forms in normal variables*, Statistica Neerlandica **39**, 14-26.

17. Dricoll, M.F. and Gundberg, W.R. (1986), *A history of the development of Craig's theorem*, The American Statistician **40**, 65-70.

18. Drygas, H. (1984), *On the Kleffe-approach for computing covariance-operators*

of tensor products, Math Operationsforsch. Statist., Ser. Statist. 15, 487-492.

19. Drygas, H. (1985), *Linear sufficiency and some applications in multilinear estimation*, J. Multivariate Anal. **16**, 71-84.

20. Eaton, M.L. (1983), Multivariate Statistics, *Wiley, New York.*

21. Fan, J. (1986), *Distributions of quadratic forms and non-central Cochran's theorem*, Acta Math. Sinica (New Ser.) **2**, 185-198.

22. Fang, K.T. and Anderson, T.W. (Eds.) (1990). Statistical Inference in Elliptically Contoured and Related Distributions, *Allerton Press Inc., New York.*

23. Fang, K.T., Kotz, S. and Ng, K.W. (1990), Symmetric Multivariate and Related Distributions, *Chapman and Hall, New York.*

24. Fang, K.T. and Wu, Y.H. (1984), *Distributions of quadratic forms and generalized Cochran's theorem*, Math. in Economics, **1**, 29-48 (in Chinese), English version in "Statistical Inference in Elliptically Contoured and Related Distributions" (K.T. Fang and T.W. Anderson Eds. (1990)).

25. Fang, K.T. and Xu, J.L. (1986), *The direct operations of symmetric and lower triangular matrices with their applications*, Northeastern Math. J. 2, 4-16.

26. Fang, K.T. and Zhang, Y. (1990), Generalized Multivariate Analysis, *Springer-Verlag, New York.*

27. Good, I. J. (1963), *On the independence of quadratic expressions*, (with an appendix by L.R. Welch) J. Roy. Statist. Soc. B **25**, 377-382 [Corrigenda 28, 584.]

28. Good, I. J. (1969), *Conditions for a quadratic form to have a chi-squared distribution*, Biometrika **56**, 215-216 [Correction 57, 225.]

29. Graybill, F.J. (1969), Introduction to Matrices with Applications in Statistics, *Wadsworth, Belmont, CA.*

30. Henderson, H.V. and Searle, S.R. (1981), *The vec-permutation matrix, the vec-*

operator and Kronecker products: a review, Linear and Multilinear Algebra **9**, 271-288.

31. Hogg, R.V. and Craig, A.T. (1958), *On the decomposition of certain chi-square variables*. Ann. Math. Statist. **29**, 608-610.

32. Jensen, D.R. and Good, I.J. (1981), *Invariant distributions associate with matrix laws under stuctural symmetry*, J. Royal Statist. Soc., B. **43**, 327-332.

33. Jinadasa, K.G. (1986), *Matrix derivatives with aplication to estimation problems in multivariate statistics*, Ph.D. thesis, Department of Mathematics and Statistics, University of Windsor, Windsor, Ontario, Canada N9B 3P4.

34. Kariya, T. and Eaton, M.L. (1977), *Robust tests for spherical symmetry*, Ann. Statist. **5**, 206-215.

35. Kelker, D. (1970), *Distribution theory of spherical distributions and a location-scale parameter generalization*, Sankyha, Ser. A **32**, 419-430.

36. Khatri, C. G. (1962), *Conditions for Wishartness and independence of second degree polynomials in normal vecters*, Ann. Math. Statist. **33**, 1002-1007.

37. Khatri, C. G. (1963), *Further contributions to Wishartness and independence of second degree polynomials in normal vecters*, J. Indian Statist. Assoc. **1**, 61-70.

38. Khatri, C. G. (1968), *Some results for the singular normal multivariate regression models*, Sankhya, Ser. A **30**, 267-280.

39. Khatri, C. G. (1977), *Quadratic forms and extension of Cochran's theorem to normal vector variables*, in "Multivariate Analysis-IV" (P. R. Krishnaiah ed.), pp 79-94, North-Holland Publishing Company, New York.

40. Khatri, C. G. (1980), *Quadratic forms in normal variables*, in "Handbook of Statistics" (P. R. Krishnaiah ed.), pp 443-469, North-Holland Publishing Company, Amsterdam.

41. Khatri, C. G. (1982), *A theorem on quadratic forms for normal variables*, in

"Statistics and Probability: Essays in Honor of C. R. Rao" (G. Kallianpur, P. R. Krishnaiah, J. K. Ghosh, eds.), pp 411-417, North-Holland Publishing Company, Amsterdam.

42. Kleffe, J. (1978), *Simultaneous estimation of expectation and covariance matrix in linear models*, Math. Operationsforsch. Statist., Ser. Statist. 9, 443-478.

43. Laha , R.G. (1956), *On the stochastic independence of two second degree polynomial statistics in normally distributed variates*, Ann. Math. Statist. 27, 790-796.

44. Li, Gang (1987), *Moments of a random vector and its quadratic forms*, J. Statist. Appl. Prob. 2, 219-229.

45. Magnus, J.R. and Neudecker, H. (1979), *The commutation matrix: some properties and applications*, Ann. Statist. 7, 381-394.

46. Mathew, T. (1989), *MANOVA in multivariate components of variance model*, J. Multivariate Anal. 29, 30-38.

47. Muirhead, R.J. (1980), *The effects of elliptical distributions on some standard procedures involving correlation coefficients*, in " Multivariate Statistical Analysis" (R.P.Gupta Ed.), pp 143-159, Norht-Holland, New York.

48. Muirhead, R.J. (1982), Aspects of Multivariate Statistical Theory. *Wiley, New York.*

49. Nagase, G. and Banerjee, K. S. (1976), *On the distribution of a second degree polynomial statistics in singular normal variates*, Comm. Statist. A-Theory Methods 5, 407-412.

50. Neudecker, H. (1985), *On the dispersion matrix of a matrix quadratic form connected with the noncentral Wishart distribution*, Linear Algebra Appl. 70, 257-262.

51. Neudecker, H. (1990), *The variance matrix of a matrix quadratic form under*

normality assumptions, Statistics **21**, 455-459.

52. Neudecker, H. and Wansbeek, T.J. (1983), *Some results on commutation matrices with statistical applications*, Canad. J. Statist. **11**, 222-231.

53. Neudecker, H. and Wansbeek, T.J. (1987), *Fourth-order properties of normally distributed random matrices*, Linear Algebra Appl. **97**, 13-21.

54. Ogasawra, T. and Takahashi, M. (1951), *Independence of quadratic quantities in a normal system*, J. Sc. Hiroshima Univ. **A 15**, 1-9.

55. Pavur, R.J. (1987), *Multivariate quadratic forms*, Canad. J. Statist. **15**, 169-176.

56. Rao, C. R. (1962), *A note on a generalized inverse of a matrix with applications to problems in mathematical statistics*, J. Roy. Statist. Soc. **B 24**, 1152-1158.

57. Rao, C. R. (1973), Linear Statistical Inference and Its Applications, 2nd ed. *John Wiley, New York.*

58. Rao, C.R. and Kleffe, J. (1988), Estimation of Variance Components and its Applications, *North-Holland, New York.*

59. Rayner, A. A. and Livingstone, D. (1965), *On the distribution of quadratic forms in singular normal variates*, S. African J. Agric. Sci. **8**, 357-370.

60. von Rosen, D. (1988), *Moments for matrix normal variables*, Statistics **19**, 575-583.

61. Schoenberg, I.J. (1938), *Metric spaces and completely monotone functions*, Ann. Math. **39**, 811-841.

62. Searle, S.R. (1971), Linear Models, *Wiley, New York.*

63. Shanbhag, D. N. (1966), *On the independence of quadratic forms*, J. Roy. Statist. Soc. **B 28**, 582-583.

64. Shanbhag, D. N. (1969), *Some remarks concerning Khatri's result on quadratic forms*, Biometrika **55**, 593-595.

65. Styan, G. P. H. (1970), *Notes on the distribution of quadratic forms in singular*

normal variables, Biometrika **56**, 567-572.

66. Tan, W. Y. (1975), *Some matrices results and extensions of Cochran's theorem*, SIAM J. Appl. Math. **28**, 547-554 [corrections (1976) **30**, 608-610].

67. Tan, W.Y. (1977), *On the distribution of quadratic forms in normal random variables*, Canad. J. Statist. **5**, 241-250.

68. Wang, Tonghui and Wong, Chi Song (1991), *Cochran theorem for a multivariate elliptically contoured model*, submitted.

69. Wong, Chi Song, (1982), *Characterization of products of symmetric matrices*, Linear Algebra Appl. **42**, 243-251.

70. Wong, Chi Song, (1985), *On the use of differentials in statistics*, Linear Algebra Appl. **70**, 282-299.

71. Wong, Chi Song, (1986), Modern Analysis and Algebra. *Northwest Telecommunication Engineering Institute (Xidian University) Publisher, Xian (in Chinese).*

72. Wong, Chi Song and Liu, Dongsheng, (1992), *Moments for left elliptically contoured random matrices*, submitted.

73. Wong, Chi Song, Masaro, J. and Wang, Tonghui, (1991), *Multivariate versions of Cochran's theorems*, J. Multivariate Anal. **39**, 154-174.

74. Wong, Chi Song and Wang, Tonghui, (1991), *Moments for elliptically contoured random matrices*, to appear in Sankhya Ser. B.

75. Wong, Chi Song and Wang, Tonghui, (1992), *Multivariate versions of Cochran's theorems II*, to appear in J. Multivariate Anal.

76. Wong, Chi Song, (1992), *Multivariate versions of Cochran's theorems III*, submitted.

INDEX OF MAIN NOTATIONS AND DEFINITIONS

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