1976

SOME STEADY MAGNETO-FLUID-DYNAMIC FLOWS.

MOHAN RATAN. GARG
University of Windsor

SOME STEADY MAGNETO FLUID DYNAMIC FLOWS

by

Mohan Ratan Garg

A Dissertation
Submitted to the Faculty of Graduate Studies through the
Department of Mathematics in Partial Fulfillment of
the Requirements for the Degree of
Doctor of Philosophy at
The University of Windsor

Windsor, Ontario

1976

Respectfully Dedicated to My Mother,

and to the Loving Memory of My Father.

ABSTRACT

In this dissertation, steady magnetofluid dynamic flows are considered under a variety of assumptions. The major part of this work is devoted to a consideration of viscous incompressible flows of infinitely conducting fluids.

(1) Constantly inclined plane flows of incompressible, viscous and perfectly conducting fluids:

Introducing the curvilinear coordinates (ϕ, ψ) in the physical plane, where ψ is the streamfunction, we transform the system of equations governing the flow when the magnetic field makes a constant angle with the velocity field. Using this transformed system, we determine all possible flows for which the streamlines are involutes of a curve and when streamlines and their orthogonal trajectories form an isometric net.

In the special case of orthogonal flows, we determine the geometries and solutions when the current density vanishes. We also establish that the streamlines in an orthogonal irrotational flow are either concurrent straight lines or parallel straight lines. Employing the hodograph transformation, we obtain a linear partial differential equation of second order which is used to obtain some particular solutions.

(2) Transverse flows of viscous incompressible fluids:

For such flows, the most general velocity field consistent with the transverse magnetic field is obtained. For the study of plane transverse flows, with the magnetic field vector normal to the plane of flow, we employ natural streamline coordinates and determine all possible flows for which the streamlines are (a) straight lines, (b) involutes of a plane curve and (c) isometric.

(3) Axisymmetric magnetohydro dynamic flows of perfectly conducting viscous fluids:

We obtain a non-linear partial differential equation to be satisfied by the Stokes streamfunction. A class of exact solutions of this equation is discussed by taking a particular solution. Finally, we study meridional motion of an inviscid fluid under the influence of a toroidal magnetic field and consider a particular flow.

(4) Plane flows of inviscid, compressible and perfectly conducting fluids:

We recast the governing system of equations in terms of the streamfunction and the magnetic flux function as independent variables. We use this system to find the geometries and solutions of irrotational flows and flows with zero current density.

ACKNOWLEDGEMENTS

The author wishes to express his sincere gratitude and appreciation to Dr. O. P. Chandna for his inspiring guidance throughout the course of this work. It was primarily through his patience, encouragement and stimulating advice that this dissertation was completed.

The author is very grateful to Dr. P. N. Kaloni and Dr. A. C. Smith for their untiring help and encouragement during the preparation of this dissertation.

He is thankful for the interest and encouragement accorded by various friends. He thanks Dr. G. P. Mathur, Dr. K. Sridhar, Dr. R. M. Barron and Dr. H. E. Toews for their help.

The author gratefully acknowledges the support given by the National Research Council of Canada in the form of Postgraduate Scholarships. He expresses his gratitude to Dr. F. W. Lemire, Chairman of the Mathematics Department, for teaching assistantships and his personal interest.

Finally, he expresses his sincere gratitude to his wife, Nirmal, whose constant encouragement and support was vital to the completion of this dissertation.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

(A) Historical Sketch

Magnetofluid dynamics (MFD) is the study of motion of an electrically conducting fluid in the presence of a magnetic field. The interactions between the magnetic field and the motion of the conducting fluid give rise to new phenomena and have provided challenging problems. Due to the fluid motion in the presence of magnetic field electric currents are produced which in turn modify the magnetic field. At the same time flow of currents in the magnetic field produces mechanical forces which modify the fluid motion. Astrophysicists and geophysicists have long studied magnetofluid dynamic flows in connection with problems such as sunspot theory and the origin of earth's magnetic field. Recently investigations in to the engineering aspects of magnetofluid dynamics, such as in the areas of direct conversion of energy and magnetofluiddynamic propulsion, have gained importance. MFD has also been applied in the construction of electromagnetic pumps and flow meters.

In theoretical investigations of magnetofluid dynamic problems it is necessary to consider fluid dynamic as well as electromagnetic equations, modified to take into account the interactions between the fluid motion and the magnetic field. Thus the mathematical study of magnetofluid dynamics is primarily concerned with a system of non-linear partial differential equations arising from the well known physical conservation laws applied to the continuum model of fluid. Most of the exact solutions of the MFD problems to date have been obtained for inviscid fluids of infinite electrical conductivity. The results obtained through such idealization of real fluids provide a basic understanding of the subject. Though it is desirable to relax such restrictive assumptions of ideality whenever possible, to do so usually necessitates recourse to approximate methods of solution.

A number of investigators have taken the approach of isolating special classes of flows in MFD that can be mathematically associated with some flows in ordinary fluid dynamics. The advantage of such an approach lies in the resulting applicability of existing fluid dynamical techniques to MFD. H. Grad (1960) established the reducibility of a number of magnetofluid dynamic problems to fluid dynamic flows by appropriate identification of variables. Aligned flows and transverse flows were given by Grad as examples of reducible flows. Later on, other classes of flows, in particular orthogonal, constantly inclined and axisymmetric flows were studied by using similar methods.

Some description of these classes of flows and the current status of the related research is summarized below.

Aligned Flows:

These are flows for which magnetic field vector is everywhere parallel to the velocity vector. Aligned flow are the most extensively studied of the magnetofluid dynamic flows. S. Chandrasekhar (1956) investigated the stability of an aligned flow solution of MFD equations for the case of inviscid incompressible fluids. In recent years, many authors have analysed aligned MFD flows by attempting to find suitable transformations that will yield the corresponding gas dynamic flow equations. I. Imai (1960) obtained such a correspondence with two dimensional irrotational gas flow, while R. Peyret (1962) associated aligned MFD flows to rotational gas flows. M. Vinokur (1961) obtained a kinematic formulation for three dimensional aligned flows of ideal gases. P. Smith (1963) generalized some of the results of steady rotational flows of ideal gases to aligned flows. G. Power and D. Walker (1964) established a correspondence between two dimensional aligned flows of a gas with arbitrary equation of state and a four-parameter class of rotational gas flows. O. P. Chandna and V. I. Nath (1972) developed a substitution principle, for fluids having arbitrary equation of state, that corresponds to Prim's substitution principle for classical gas flows.

Orthogonal Flows.

Flows are said to be orthogonal if the velocity and the magnetic fields are everywhere orthogonal to each other. Iu. P. Ladikov (1962) derived two Bernoulli type equations for orthogonal flows of inviscid fluids with infinite electrical conductivity. Power and Walker (1965), and Power and Talbot (1969), studied plane compressible orthogonal flows by reducing the problem to that of rotational gas dynamic flows. Power and Walker (1967) established the reducibility of certain steady plane orthogonal flows of viscous incompressible fluids to ordinary fluid flows of viscous compressible fluids. Nath and Chandna (1973) investigated steady plane viscous incompressible MFD flows, using the streamfunction and magnetic flux function as independent variables. These authors determined flow geometries for orthogonal flows when the streamlines are either straight lines or involutes of a curve.

Transverse Flows.

By transverse flows we mean flows for which the magnetic field is unidirectional. H. Grad (1960) studied transverse flows of inviscid perfectly conducting compressible fluid for the situation where the velocity vector lies in the plane perpendicular to the direction of magnetic field. He obtained two integrals, one relating the magnetic induction with the speed of sound and the other a generalized Bernoulli equation. R. M. Gunderson (1966, 1969) studied transverse flows using method of characteristics and extended the idea of simple waves to such flows. Chandna (1972) obtained a compatibility equation for steady plane transverse flow of inviscid compressible perfectly conducting fluids. Nath and Chandna (1973) developed a substitution principle for such flows and Chandna, Smith and Nath (1975) studied channel flow under a transverse magnetic field.

Constantly Inclined Flows.

These flows are defined to be flows for which the magnetic field makes a constant angle with the velocity field. Until 1973, there appears to be no mention of such flows in the literature. J. S. Waterhouse and J. G. Kingston (1973) investigated constantly inclined flows of incompressible non-viscous fluids with infinite electrical conductivity. Toews and Chandna (1974) considered constantly inclined flows of inviscid compressible fluids and generalized some of the results previously derived for orthogonal flows. In another paper Chandna, Toews and Nath (1975) studied these flows of viscous incompressible fluids.

Axisymmetric Flows.

By axisymmetric MFD flows we mean MFD flows where all the flow characteristics including the magnetic field have symmetry about an axis. Such flows of inviscid incompressible fluids have been investigated by many authors. V. C. A. Ferraro (1954) studied such flows in connection with equilibrium of magnetic stars. S. Chandrasekhar (1956) obtained solutions for a large class of force-free axisymmetric fields. R. R. Long (1960) and C. S. Yih (1965) have studied steady axisymmetric flows of perfectly conducting, inviscid incompressible fluids. Recently K. B. Ranger (1970) has given some interesting exact solutions for such flows. He dealt with an axisymmetric configuration where there is finite inviscid fluid motion inside a perfectly conducting liquid sphere in the presence of magnetic field. In another paper Ranger (1970) has studied slow motion of a viscous finitely conducting fluid past a sphere in the presence of a toroidal magnetic field.

(B) Outline of the Present Work.

In this work, steady flows of electrically conducting fluids in the presence of a magnetic field are investigated under a variety of assumptions. The major part of this investigation is devoted to a consideration of viscous incompressible plane flows of infinitely conducting fluids. First, the geometry of flows is investigated for constantly inclined flows for two situations. These correspond to the cases when the streamlines are either involutes of a curve or form an isometric net with their orthogonal trajectories. Aligned flows and orthogonal flows are treated as special cases of constantly inclined flows. For orthogonal flows, the geometrical implications of having zero current density and zero vorticity are investigated. Using hodograph transformation, a linear partial differential equation of second order is obtained for such flows and this equation is used to obtain some particular solutions.

Next, transverse flows of viscous incompressible fluids having finite electrical conductivity are investigated and most general velocity field, consistent with unidirectional magnetic field, is obtained. Some geometrical results are obtained for flows with velocity vector lying in the plane perpendicular to the direction of magnetic field.

Finally, axisymmetric flows, that have recently received much attention, are examined for incompressible fluids when viscous effects are not neglected and exact solutions are obtained.

In a somewhat different vein, some properties of plane steady compressible flows of perfectly conducting fluids are also investigated by introducing stream function and magnetic flux function as independent variables. Geometries and solutions are obtained in the case of irrotational flows and flows with zero current density.

A detailed outline of the present work follows.

Chapter II deals with constantly inclined viscous incompressible plane flows. In section 1, we give the equations of motion for magnetofluid dynamics in their most general form. In section 2, we obtain the flow equations for the steady plane flow of a viscous incompressible fluid having infinite electrical conductivity. In section 3, we transform these equations by introducing curvilinear coordinates (ϕ, Ψ) where Ψ is the streamfunction. Flows where streamlines are involutes of a curve are considered in section 4 and it is established that the streamlines must be concentric circles. In section 5, we consider flows where streamlines and their orthogonal trajectories form an isometric net and it is shown that the streamlines are restricted to parallel straight lines, concurrent lines, concentric circles or logarithmic spirals. Finally, in section 6, we find solutions to vortex and radial flow problems.

In chapter III, we consider plane viscous flows with orthogonal magnetic and velocity field distributions. In section 1, the flow equations are obtained with ϕ, ψ as independent variables where ϕ is the magnetic flux function and ψ is the streamfunction. Flows with zero current density are considered in section 2. In section 3, all possible irrotational flows are classified and corresponding solutions are obtained. Using Hodograph transformation a linear partial differential equation is obtained in section 4, known solutions of which can be used to find solutions of the flows under investigation. In the next section we consider some particular solutions.

In chapter IV, we study magnetofluid dynamic flows of viscous incompressible fluids having finite electrical conductivity, when the magnetic field is acting in a fixed direction. In section 1, we find the most general velocity field that is compatible with such a magnetic field. Plane transverse flows with magnetic field perpendicular to the plane of flow are studied in the remaining chapter. In section 2, the magnetic field is eliminated from the flow equations and the resulting equations are transformed to natural streamline coordinates. In section 3, we consider flows having straight streamlines and establish that streamlines must be either concurrent or parallel. In section 4, the flows whose streamlines are the involutes of a curve are determined. Flows with an isometric streamline pattern are investigated in section 5.

Chapter V deals with steady plane compressible magnetofluid dynamic flows. In section 1, the system of equations governing the flow of thermally non-conducting inviscid fluids with infinite electrical conductivity are formulated with ϕ, ψ as independent variables, where ϕ is the magnetic flux function and ψ is the stream function. In section 2, we consider irrotational orthogonal flows. We classify these flows and find the corresponding solutions. In the next section we study the geometry of constantly inclined flows with zero current density and find the solutions of the corresponding flows.

Chapter VI deals with steady axisymmetric magnetofluid dynamic flows of viscous incompressible fluids having infinite electrical conductivity. In section 1, we obtain a non-linear partial differential equation for the streamfunction employing the cylindrical polar coordinates with z - axis along the axis of symmetry. In section 2, a class of exact solutions for these flows is obtained and a particular solution is discussed. In section 3, we consider the flows when stream function ψ is a function of z only. Lastly, we consider meridional motion in section 4. We consider an inviscid flow in the meridional plane under the influence of toroidal magnetic field. This problem reduces to a non-linear partial differential equation of second order in ψ involving two arbitrary functions and a particular flow is investigated.

CHAPTER II

PLANE VISCOUS FLOWS with CONSTANTLY INCLINED MAGNETIC and VELOCITY FIELDS.

Many authors have considered orthogonal and aligned flows. However, the more general class of constantly inclined flows did not receive much attention until recently. Waterhouse and Kingston (1973) classified the possible flow configurations for constantly inclined flows of inviscid incompressible fluids of infinite electrical conductivity. Chandna, Toews, and Nath (1975) considered constantly inclined flows of viscous incompressible fluids and established that the only possible flows with straight streamlines are those with radial or parallel streamlines.

In the present chapter we study constantly inclined flows of a viscous incompressible fluid of infinite electrical conductivity. This study is carried out by adapting a technique developed originally by Martin (1971) to investigate steady plane flows of a non-conducting viscous incompressible fluid. We introduce curvilinear coordinates (ϕ, Ψ) in the physical plane, where the coordinate lines Ψ = constant are the streamlines, and transform the governing system of equations with ϕ, Ψ as independent variables. Using this system we determine all possible flows for which the streamlines are involutes of a curve and when streamlines and their orthogonal trajectories form an isometric net.

Section 1. General Equations of Motion of Magneto-Fluid-Dynamics

The fundamental equations of MFD governing the flow of thermally-nonconducting and electrically conducting fluids are as follows

$$\frac{\partial \rho}{\partial t} + \text{div } (\rho \vec{V}) = 0 \tag{21.01}$$

(Conservation of Mass)

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho (\vec{V} \cdot \text{grad}) \vec{V} + \text{grad } p = \text{div } \tau + \mu \vec{j} \times \vec{H} + \rho \vec{F} \tag{21.02}$$

(Conservation of Linear Momentum)

$$\rho T \frac{\partial s}{\partial t} + \rho T \vec{V} \cdot \text{grad} s = \phi + \text{div } (\kappa \text{ grad } T) + \frac{j^2}{\sigma} \tag{21.03}$$

(Conservation of Energy)

$$\vec{j} = \text{curl } \vec{H} \tag{21.04}$$

$$\text{curl } \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{21.05}$$

(Maxwell Equations)

$$\vec{j} = \sigma (\vec{E} + \mu \vec{V} \times \vec{H}) \tag{21.06}$$

(Ohm's Law)

$$p = p(\rho, s) \tag{21.07}$$

(Equation of State)

where $\vec{V}$ denotes the velocity vector, ρ the density, p the pressure, s the specific entropy, μ the magnetic permeability, $\vec{F}$ the external body forces per unit mass, $\vec{H}$ the magnetic field vector, $\vec{E}$ the electric intensity vector, $\vec{j}$ the current density vector, σ the electrical conductivity, κ the thermal conductivity, T the absolute temperature of the fluid, τ the viscous stress tensor and Φ the heat influx due to viscous dissipation.

If $\vec{V} = (v_1, v_2, v_3)$ and η is the coefficient of viscosity, then the components τ_{ij} of the viscous stress tensor τ are given by

$$\tau_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{2}{3}\eta\ \delta_{ij}\ \mathrm{div}\ \vec{V} \qquad (21.08)$$

where δ_{ij} is the Kronecker delta.

Heat influx due to viscous dissipation, Φ, is given by

$$\Phi = \sum_{j=1}^{3} \sum_{i=1}^{3} \tau_{ij} \frac{\partial v_i}{\partial x_j} \qquad (21.09)$$

Eliminating $\vec{E}$ and $\vec{j}$ between (21.04), (21.05) and (21.06), we get the equation for magnetic field as

$$\mathrm{curl}(\vec{V} \times \vec{H}) - \mathrm{curl}\left(\frac{1}{\mu\sigma}\ \mathrm{curl}\ \vec{H}\right) = \frac{\partial \vec{H}}{\partial t} \qquad (21.10)$$

Taking divergence of both sides of (21.05) gives

$$\frac{\partial}{\partial t}(\mathrm{div}\ \vec{H}) = 0 \quad .$$

This implies that div $\vec{H}$ is constant relative to time and consequently the magnetic field vector $\vec{H}$ is taken to be solenoidal,

$$\operatorname{div} \vec{H} = 0 \tag{21.11}$$

Equation (21.11) can be regarded as a constraint for the initial configuration of the magnetic field $\vec{H}$.

Section 2. Equations of Steady Plane Flow of a Viscous Incompressible Fluid Having Infinite Electrical Conductivity

In the case of incompressible fluids of infinite electrical conductivity equations of steady motion reduce to

$$\mathrm{div}(\rho\vec{V}) = 0$$

$$\rho(\vec{V}\cdot\mathrm{grad})\vec{V} = \mathrm{grad}\ \rho = \eta\ \nabla^2\vec{V} + \mu(\mathrm{curl}\ \vec{H}) \times \vec{H}$$

$$\mathrm{curl}(\vec{V}\times\vec{H}) = 0$$

$$\mathrm{div}\ \vec{H} = 0$$

When the flow is two-dimensional with magnetic field vector in the plane of flow, above equations become:

$$\frac{\partial}{\partial x}(\rho\mu) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{22.01}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial p}{\partial x} = \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \mu H_2\left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right) \tag{22.02}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \mu H_1\left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial x}\right) \tag{22.03}$$

$$uH_2 - vH_1 = K \tag{22.04}$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0 \tag{22.05}$$

where $\vec{V} = (u,v)$, $\vec{H} = (H_1,H_2)$ and K is an arbitrary constant which is zero for the aligned flows and non-zero in the case of non-aligned flows.

We define the following functions

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$j = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}$$

and (22.06)

$$h = \tfrac{1}{2}\rho V^2 + p$$

wherein $V^2 = u^2 + v^2$.

Using (22.06), we can rewrite equation (22.02) as

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial h}{\partial x} - \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right) = \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial \omega}{\partial y}\right) - \mu j H_2$$

or

$$-\rho v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \frac{\partial h}{\partial x} = -\eta\frac{\partial \omega}{\partial y} + \eta\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \mu j H_2 ,$$

and making use of (22.01), we get

$$\eta \frac{\partial \omega}{\partial y} - \rho v \omega + \mu j H_2 = - \frac{\partial h}{\partial x}$$

Similarly, (22.03) gives us

$$\eta \frac{\partial \omega}{\partial x} - \rho u \omega + \mu j H_1 = \frac{\partial h}{\partial y}$$

Therefore, the five partial differential equations (22.01) to (22.05) are replaced by a system of seven partial differential equations,

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \qquad (22.07)$$

(Equation of Continuity)

$$\eta \frac{\partial \omega}{\partial y} - \rho v \omega + \mu j H_2 = - \frac{\partial h}{\partial x} \qquad (22.08)$$

(Momentum Equation)

$$\eta \frac{\partial \omega}{\partial x} - \rho u \omega + \mu j H_1 = \frac{\partial h}{\partial y} \qquad (22.09)$$

(Momentum Equation)

$$u H_2 - v H_1 = K \qquad (22.10)$$

(Equation for Magnetic Field)

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0 \qquad (22.11)$$

(Solenoidal Condition on $\vec{H}$)

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega \qquad (22.12)$$

(Vorticity)

$$\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = j \qquad (22.13)$$

(Current Density)

The advantage of this system of equations over the original system is that the order of the partial differential equation has decreased from two to one.

Equation (22.07) implies the existence of a stream function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = -\rho v \ , \ \frac{\partial \psi}{\partial y} = \rho u \qquad (22.14)$$

and the curves $\psi(x,y)$ = constant define the family of streamlines. Let us take $\phi(x,y)$ = constant to be some arbitrary family of curves such that it generates, with the family of curves $\psi(x,y)$ = constant, a curvilinear coordinate system (ϕ,ψ) in the physical plane. In place of the rectangular coordinates (x,y), we introduce curvilinear coordinates (ϕ,ψ) as the independent variables. Then the first fundamental form for the physical plane is given by

$$ds^2 = Ed\phi^2 + 2Fd\phi d\psi + Gd\psi^2 \ , \tag{22.15}$$

where

$$E = (\frac{\partial x}{\partial \phi})^2 + (\frac{\partial y}{\partial \phi})^2 \ ,$$

$$F = \frac{\partial x}{\partial \phi} \cdot \frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \phi} \cdot \frac{\partial y}{\partial \psi} \ , \tag{22.16}$$

$$G = (\frac{\partial x}{\partial \psi})^2 + (\frac{\partial y}{\partial \psi})^2 \ .$$

We have

$$\frac{\partial x}{\partial \phi} = J\frac{\partial \psi}{\partial y} \ , \ \frac{\partial y}{\partial \phi} = -J\frac{\partial \psi}{\partial x} \ , \ \frac{\partial x}{\partial \psi} = -J\frac{\partial \phi}{\partial y} \ , \ \frac{\partial y}{\partial \psi} = J\frac{\partial \phi}{\partial x} \tag{22.17}$$

provided that $0<|J|<\infty$, where J denotes the Jacobian

$$J = \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \phi} \tag{22.18}$$

From (22.16) and (22.18), we have

$$J = \pm W \tag{22.19}$$

where $W = \sqrt{EG - F^2}$.

Let α be the angle made by the tangent to the coordinate line ψ = constant, directed in the sense of increasing ϕ, with the x-axis. Then, we have (see Appendix A)

$$\frac{\partial x}{\partial \phi} = \sqrt{E} \cos \alpha \ , \quad \frac{\partial y}{\partial \phi} = \sqrt{E} \sin \alpha \tag{22.20}$$

$$\frac{\partial x}{\partial \psi} = \frac{F}{\sqrt{E}} \cos \alpha - \frac{J}{\sqrt{E}} \sin \alpha \ , \quad \frac{\partial y}{\partial \psi} = \frac{J}{\sqrt{E}} \cos \alpha + \frac{F}{\sqrt{E}} \sin \alpha \tag{22.21}$$

and

$$\frac{\partial \alpha}{\partial \phi} = \frac{J}{E} \Gamma_{11}^{2} \ , \quad \frac{\partial \alpha}{\partial \psi} = \frac{J}{E} \Gamma_{12}^{2} \tag{22.22}$$

where

$$\Gamma_{11}^{2} = \frac{1}{2W^{2}}(-F\frac{\partial E}{\partial \phi} + 2E\frac{\partial F}{\partial \phi} - E\frac{\partial E}{\partial \psi})$$

$$\Gamma_{12}^{2} = \frac{1}{2W^{2}}(E\frac{\partial G}{\partial \phi} - F\frac{\partial E}{\partial \psi}) \tag{22.23}$$

$$\Gamma_{22}^{2} = \frac{1}{2W^{2}}(E\frac{\partial G}{\partial \psi} - 2F\frac{\partial F}{\partial \psi} + F\frac{\partial G}{\partial \phi})$$

The three functions E, F, and G of ϕ, ψ must satisfy the Gauss equation (see Appendix A):

$$\frac{\partial}{\partial \psi}(\frac{W}{E} \Gamma_{11}^{2}) - \frac{\partial}{\partial \phi}(\frac{W}{E} \Gamma_{12}^{2}) = 0 \tag{22.24}$$

Section 3. Transformation of Basic Equations

Hereafter in this chapter, we consider steady plane flows of an incompressible fluid of infinite electrical conductivity where the magnetic field vector lies in the plane of flow and makes a constant angle δ with the velocity vector throughout the flow region. Aligned flows where magnetic lines of force coincide with the streamlines are treated as special cases of such flows. In the case of non-aligned flows, (22.10) implies

$$HV \sin \delta = K \qquad (K \neq 0) \qquad (23.01)$$

where $H = \sqrt{H_1^2 + H_2^2}$

For aligned flows, we have

$$\vec{H} = f\vec{V}$$

where f is an arbitrary scalar function, and from (22.11) and (22.7) we get

$$u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0 \ ,$$

which implies that f is constant along streamlines so that $f = f(\psi)$. Consequently, in the case of aligned flows, instead of (23.01) we have

$$H = |f(\psi)|V \qquad (23.02)$$

Equation of Continuity: Using (22.17) in (22.14), we get

$$\frac{\partial\psi}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial\phi} = \rho u \quad , \quad -\frac{\partial\psi}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial\phi} = \rho v \qquad (23.03)$$

Equations (23.03) show that the fluid flows along the streamlines towards higher or lower parameter values ϕ accordingly as J is positive or negative.

If we introduce polar coordinates V,θ in the hodograph plane, so that θ is the direction of flow in the physical plane, then

$$u = V\cos\theta \; , \; v = V\sin\theta$$

and equations (23.03) become

$$\frac{\partial x}{\partial\phi} = \rho VJ\cos\theta \; , \; \frac{\partial y}{\partial\phi} = \rho VJ\sin\theta \qquad (23.04)$$

There are two possible cases:

Case 1. When $\theta = \alpha$, where α is the angle made by the tanget to the curve ψ = constant, directed in the sense of increasing ϕ, with x-axis. In this case (22.20) and (23.04) imply that

$$\rho VJ = \sqrt{E}\ ,\ J > 0 \tag{23.05}$$

Case 2. When $\theta = \pi + \alpha$.

In this case, from (23.04), we have

$$\frac{\partial x}{\partial \phi} = -\rho VJ \cos \alpha\ ,\quad \frac{\partial y}{\partial \phi} = -\rho VJ \sin \alpha .$$

and therefore, from (22.20), we get

$$\rho VJ = -\sqrt{E}\ ,\ J < 0 \tag{23.06}$$

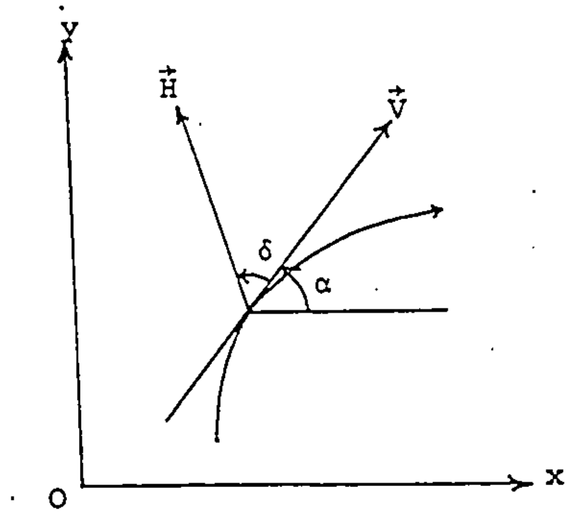
From the above two cases, we conclude that in Case 1, the fluid flows towards higher parameter values ϕ, while in Case 2, the fluid flows towards lower parameter values ϕ. In either case, from (22.19), we obtain

$$\rho VW = \sqrt{E} \tag{23.07}$$

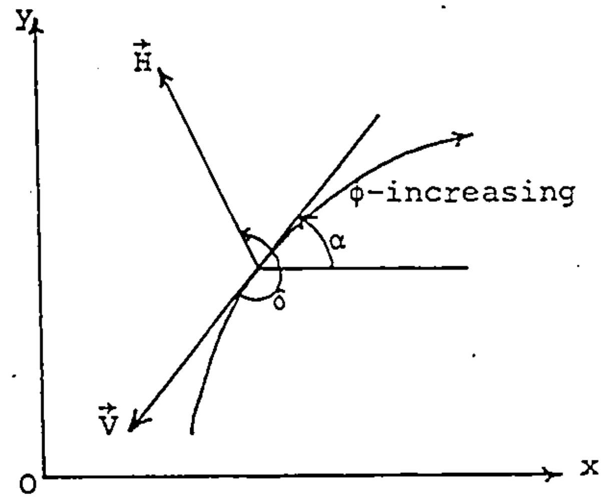
The (ϕ,ψ)-plane is mapped upon the hodograph plane by the relation

$$u + iv = \frac{\sqrt{E}}{\rho J} e^{i\alpha} \tag{23.08}$$

Solenoidal Condition on $\vec{H}$: $\vec{H}$ makes an angle $\alpha + \delta$ or $\alpha + \delta - \pi$ with the x-axis accordingly as fluid flows along the streamlines towards higher or lower parameter values ϕ .



Case 1 ($\theta = \alpha$).



Case 2 ($\theta = \pi+\alpha$).

Therefore, we have

$$H_1 = \pm H\cos(\alpha+\delta) \qquad H_2 = \pm H\sin(\alpha+\delta) \qquad (23.09)$$

wherein positive or negative sign is taken accordingly as J is positive or negative.

Using the transformation equations (22.17) in (22.05), we obtain

$$\left(\frac{\partial H_1}{\partial\phi}\cdot\frac{\partial y}{\partial\psi} - \frac{\partial H_1}{\partial\psi}\cdot\frac{\partial y}{\partial\phi}\right) - \left(\frac{\partial H_2}{\partial\phi}\cdot\frac{\partial x}{\partial\psi} - \frac{\partial H_2}{\partial\psi}\cdot\frac{\partial x}{\partial\phi}\right) = 0$$

Employing (22.20), (22.21) and (23.09) in the above equation, we get

$$\left[\frac{\partial H}{\partial \phi} \cos(\alpha+\delta) - H \sin(\alpha+\delta)\frac{\partial \alpha}{\partial \phi}\right] \left(\frac{J}{\sqrt{E}} \cos\alpha + \frac{F}{\sqrt{E}} \sin\alpha\right)$$

$$+ \left[\frac{\partial H}{\partial \psi} \sin(\alpha+\delta) + H \cos(\alpha+\delta)\frac{\partial \alpha}{\partial \psi}\right] \sqrt{E} \cos\alpha$$

$$- \left[\frac{\partial H}{\partial \psi} \cos(\alpha+\delta) - H \sin(\alpha+\delta)\frac{\partial \alpha}{\partial \psi}\right] \sqrt{E} \sin\alpha$$

$$- \left[\frac{\partial H}{\partial \phi} \sin(\alpha+\delta) + H \cos(\alpha+\delta)\frac{\partial \alpha}{\partial \phi}\right] \left(\frac{F}{\sqrt{E}} \cos\alpha - \frac{J}{\sqrt{E}} \sin\alpha\right)$$

$$= 0$$

where $\frac{\partial \alpha}{\partial \phi}$, $\frac{\partial \alpha}{\partial \psi}$ are given by (22.22). On simplification, we obtain

$$\frac{\partial H}{\partial \phi}(J \cos\delta - F \sin\delta) + \frac{\partial H}{\partial \psi} E \sin\delta - \frac{J}{E} \Gamma_{11}^{2} H(F \cos\delta +$$

$$J \sin\delta) + J \Gamma_{12}^{2} H \cos\delta = 0 \qquad (23.10)$$

The Vorticity ω: By definition,

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \left(\frac{\partial v}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial v}{\partial \psi}\frac{\partial \psi}{\partial x}\right) - \left(\frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial u}{\partial \psi}\frac{\partial \psi}{\partial y}\right)$$

Making use of transformation equations (22.17), we get

$$J\omega = \frac{\partial v}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial v}{\partial \psi}\frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial \phi}\frac{\partial x}{\partial \psi} - \frac{\partial u}{\partial \psi}\frac{\partial x}{\partial \phi}$$

On substituting

$$u = \pm V \cos \alpha \ , \quad v = \pm V \sin \alpha$$

where positive or negative sign is taken according as J is positive or negative, we find that

$$\pm J\omega = \left(\frac{\partial V}{\partial \phi} \sin \alpha + V \cos \alpha \frac{\partial \alpha}{\partial \phi}\right) \frac{\partial y}{\partial \psi} - \left(\frac{\partial V}{\partial \psi} \sin \alpha + V \cos \alpha \frac{\partial \alpha}{\partial \psi}\right) \frac{\partial y}{\partial \phi}$$

$$+ \left(\frac{\partial V}{\partial \phi} \cos \alpha - V \sin \alpha \frac{\partial \alpha}{\partial \phi}\right) \frac{\partial x}{\partial \psi} - \left(\frac{\partial V}{\partial \psi} \cos \alpha - V \sin \alpha \frac{\partial \alpha}{\partial \psi}\right) \frac{\partial x}{\partial \phi}$$

where $\frac{\partial x}{\partial \phi}, \frac{\partial y}{\partial \phi}, \frac{\partial x}{\partial \psi}, \frac{\partial y}{\partial \psi}$, are given by (22.20) and (22.21).

The above equation simplifies to

$$\sqrt{E}\, W\, \omega = F\frac{\partial V}{\partial \phi} - E\frac{\partial V}{\partial \psi} + JV\frac{\partial \alpha}{\partial \phi} \qquad (23.11)$$

where we have made use of $W = \pm J$.

From (23.07), on using identities (A.17) and (A.18), we get

$$\frac{\partial V}{\partial \phi} = \frac{1}{2V\rho^2} \frac{\partial}{\partial \phi}\left(\frac{E}{W^2}\right) = \frac{1}{\rho \sqrt{E}W} (F\ \Gamma^2_{11} - E\ \Gamma^2_{12})$$

$$\frac{\partial V}{\partial \psi} = \frac{1}{2V\rho^2} \frac{\partial}{\partial \psi}\left(\frac{E}{W^2}\right) = \frac{1}{\rho \sqrt{E}W} (F\ \Gamma^2_{12} - E\ \Gamma^2_{22})$$

Using (23.07), (22.22) and above expressions for $\frac{\partial V}{\partial \phi}$, $\frac{\partial V}{\partial \psi}$ in (23.11), we obtain

$$\sqrt{E}\, W\, \omega = \frac{F}{\rho\sqrt{EW}}\,(F\,\Gamma_{11}^{2} - E\,\Gamma_{12}^{2}) - \frac{E}{\rho\sqrt{EW}}\,(F\,\Gamma_{12}^{2} - E\,\Gamma_{22}^{2}) + \frac{J}{\rho\sqrt{EW}}\,\Gamma_{11}^{2}$$

or

$$\rho E W^{2} \omega = F^{2}\Gamma_{11}^{2} - 2F\Gamma_{12}^{2} + E^{2}\Gamma_{22}^{2} + W^{2}\Gamma_{11}^{2}$$

$$= EG\Gamma_{11}^{2} - 2EF\Gamma_{12}^{2} + E^{2}\Gamma_{22}^{2}$$

or

$$\rho\omega = \frac{1}{W^{2}}\,(G\Gamma_{11}^{2} - 2F\Gamma_{12}^{2} + E\Gamma_{22}^{2})$$

Employing the identity (A.19), we get

$$\rho\omega = \frac{1}{W}\left[\frac{\partial}{\partial\phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial\psi}\left(\frac{E}{W}\right)\right] \tag{23.12}$$

<u>The Current Density j:</u> By the definition of j and (22.17), we have

$$j = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}$$

$$= \frac{1}{J}\left(\frac{\partial H_2}{\partial\phi}\,\frac{\partial y}{\partial\psi} - \frac{\partial H_2}{\partial\psi}\,\frac{\partial y}{\partial\phi} + \frac{\partial H_1}{\partial\phi}\,\frac{\partial x}{\partial\psi} - \frac{\partial H_1}{\partial\psi}\,\frac{\partial x}{\partial\phi}\right)$$

Using (23.09), (22.20), (22.21) and (22.22), we obtain

$$j = \frac{1}{\sqrt{EW}}\Big[\frac{\partial H}{\partial \phi}(F \cos \delta + J \sin \delta) - \frac{\partial H}{\partial \psi} E \sin \delta$$

$$+ \frac{JH}{E} \Gamma^2_{11}(J \cos \delta - F \sin \delta) + JH \Gamma^2_{12} \sin \delta\Big]. \quad (23.13)$$

The Momentum Equations: Using (22.14) and (23.09) in (22.08), we get

$$\eta(\frac{\partial \omega}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial \omega}{\partial \psi}\frac{\partial \psi}{\partial y}) + \omega\frac{\partial \psi}{\partial x} \pm \mu\, j\, H \sin(\alpha+\delta) = -(\frac{\partial h}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial h}{\partial \psi}\frac{\partial \psi}{\partial x})$$

On employing (22.17) and the fact that $W = \pm J$, above equation reduces to

$$\eta(-\frac{\partial \omega}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial \omega}{\partial \psi}\frac{\partial x}{\partial \phi}) - \omega\frac{\partial y}{\partial \phi} + \mu\, W\, H\, j \sin(\alpha+\delta) = -\frac{\partial h}{\partial \phi}\frac{\partial y}{\partial \psi} + \frac{\partial h}{\partial \psi}\frac{\partial y}{\partial \phi} \quad (23.14)$$

Likewise, (22.09) takes the form

$$\eta(\frac{\partial \omega}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial \omega}{\partial \psi}\frac{\partial y}{\partial \phi}) - \omega\frac{\partial x}{\partial \phi} + \mu\, W\, H\, j \sin(\alpha+\delta) = -\frac{\partial h}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial h}{\partial \psi}\frac{\partial x}{\partial \phi} \quad (23.15)$$

Multiplying (23.14) by $\frac{\partial y}{\partial \phi}$ and (23.15) by $\frac{\partial x}{\partial \phi}$ and adding yields

$$\eta \frac{\partial \omega}{\partial \phi}\left(\frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \phi}\right) - \omega\left[\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2\right] + \mu\ W\ H\ j\left[\frac{\partial x}{\partial \phi}\cos(\alpha+\delta)\right.$$

$$\left. + \frac{\partial y}{\partial \phi}\sin(\alpha+\delta)\right] = -\frac{\partial h}{\partial \phi}\left(\frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \psi}\right) + \frac{\partial h}{\partial \psi}\left[\left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2\right]$$

or

$$\eta\ J\ \frac{\partial \omega}{\partial \phi} - E\ \omega + \partial\ W\ H\ j\ \sqrt{E}\ \cos\ \delta = -F\ \frac{\partial h}{\partial \phi} + E\ \frac{\partial h}{\partial \psi} \tag{23.16}$$

where (22.16), (22.18) and (22.20) have been used. Likewise, multiplying (23.14) by $\frac{\partial y}{\partial \psi}$ and (23.15) by $\frac{\partial x}{\partial \psi}$ and adding results in

$$\eta\ J\ \frac{\partial \omega}{\partial \psi} - F\omega + \mu\ W\ H\ j\left(\frac{F}{\sqrt{E}}\cos\ \delta + \frac{J}{\sqrt{E}}\sin\ \delta\right)$$

$$= -G\ \frac{\partial h}{\partial \phi} + F\ \frac{\partial h}{\partial \psi} \tag{23.17}$$

Summing up, we have the following theorem:

<u>Theorem 1</u>: If the streamlines ψ = constant and an arbitrarily chosen family of curves ϕ = constant generate a curvilinear coordinate system (ϕ,ψ) in the physical plane of the fluid flow under study, the system of seven partial differential equations (22.07) to (22.13) involving seven unknowns u, v, H_1, H_2, ω, j and h are replaced by the following system

$$\eta\, J\, \frac{\partial\omega}{\partial\phi} - E\omega + \mu\, W\, H\, j\, \sqrt{E}\cos\delta = E\,\frac{\partial h}{\partial\psi} - F\,\frac{\partial h}{\partial\phi}\,,$$

$$\eta\, J\, \frac{\partial\omega}{\partial\psi} - F\omega + \mu\, W\, H\, j\, \left(\frac{F}{\sqrt{E}}\cos\delta + \frac{J}{\sqrt{E}}\sin\delta\right)$$

$$= F\,\frac{\partial h}{\partial\psi} - G\,\frac{\partial h}{\partial\phi}$$

(Momentum Equations)

$$\omega = \frac{\Upsilon}{\rho W}\left(\frac{\partial}{\partial\phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial\psi}\left(\frac{E}{W}\right)\right)$$

(Vorticity)

$$\frac{\partial H}{\partial\phi}(J\cos\delta - F\sin\delta) + \frac{\partial H}{\partial\psi}\,E\sin\delta - \frac{JH}{E}\,\Gamma^2_{11}$$

$$(F\cos\delta + J\sin\delta) + JH\,\Gamma^2_{12}\cos\delta = 0,$$

(Solenoidal Condition on $\vec{H}$) (23.18)

$$\sqrt{E}\,W\,j = \frac{\partial H}{\partial\phi}(F\cos\delta + J\sin\delta) - \frac{\partial H}{\partial\psi}\,E\cos\delta$$

$$+ \frac{JH}{E}\,\Gamma^2_{11}\,(J\cos\delta - F\sin\delta) + JH\,\Gamma^2_{12}\sin\delta,$$

(Current Density Equation)

$$\frac{\partial}{\partial\psi}\left(\frac{W}{E}\,\Gamma^2_{11}\right) - \frac{\partial}{\partial\phi}\left(\frac{W}{E}\,\Gamma^2_{12}\right) = 0\,,$$

(Gauss Equation)

of six partial differential equations in seven unknowns E, F, G, ω, j, H and h. Here E, F, G are given by

$$E = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2$$

$$F = \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \phi} + \frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \psi}$$

$$G = \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2$$

and $W = \sqrt{EF - F^2}$, $J = \pm W$,

the positive or negative sign being taken accordingly as the parameter value ϕ increases or decreases in the direction of flow.

Given a solution

$$E = E(\phi,\psi),\ F = F(\phi,\psi),\ G = G(\phi,\psi),\ H = H(\phi,\psi),$$

$$\omega = \omega(\phi,\psi),\ h = h(\phi,\psi),\ j = j(\phi,\psi)$$

of the system (23.18), we can find x, y as functions of ϕ,ψ from

$$z = x + iy = \int \frac{\exp(i\alpha)}{\sqrt{E}} \{E d\phi + (F + iJ)d\psi\} \quad (23.19)$$

where

$$\alpha = \int \frac{J}{E} \{\Gamma_{11}^{2}\, d\phi + \Gamma_{12}^{2}\, d\psi\} \qquad (23.20)$$

E, F, G, H, ω, j and h may now be obtained as functions of x, y. Then H_1, H_2, u, v and p are obtained from

$$H_1 + iH_2 = \frac{HW}{J} \exp\left[i(\alpha+\delta)\right]$$

$$u + iv = \frac{\sqrt{E}}{\rho J} \exp(i\alpha)$$

$$p = h - \frac{E}{2\rho W^2} \quad .$$

The system (23.18) is an underdetermined system as there are seven unknowns and only six equations. It can be made a determinate system in several ways. We consider two different methods of making the above system determinate.

Method 1: The orthogonal trajectories of the streamlines ψ = constant may be chosen as the coordinate lines ϕ = constant. In this case F = 0 and the momentum equations (23.16) and (23.17) become

$$\eta\, J \frac{\partial \omega}{\partial \phi} - E\omega + \mu\, E \sqrt{G}\, H\, j \cos \delta = E \frac{\partial h}{\partial \psi} \qquad (23.21)$$

$$\eta\, J \frac{\partial \omega}{\partial \psi} + \mu\, J \sqrt{G}\, H\, j \sin \delta = -G \frac{\partial h}{\partial \phi} \qquad (23.22)$$

Using the integrability condition $\frac{\partial^2 h}{\partial\phi\partial\psi} = \frac{\partial^2 h}{\partial\psi\partial\phi}$, we get

$$\eta\left[\frac{\partial}{\partial\phi}\left(\frac{J}{E}\frac{\partial\omega}{\partial\phi}\right) + \frac{\partial}{\partial\psi}\left(\frac{J}{G}\frac{\partial\omega}{\partial\psi}\right)\right] - \frac{\partial\omega}{\partial\phi} + \mu\left[\frac{\partial}{\partial\phi}(jH\sqrt{G})\cos\delta + \frac{\partial}{\partial\psi}\left(j\,H\,\frac{J}{\sqrt{G}}\right)\sin\delta\right] = 0 \ .$$

Therefore, we have the corollary,

Corollary 1: If streamlines ψ = constant and their orthogonal trajectories ϕ = constant are taken as the curvilinear coordinate system (ϕ,ψ) in the physical plane, the fluid flow under study is represented by the system:

$$\eta\left[\frac{\partial}{\partial\phi}\left(\frac{J}{E}\frac{\partial\omega}{\partial\phi}\right) + \frac{\partial}{\partial\psi}\left(\frac{J}{G}\frac{\partial\omega}{\partial\psi}\right)\right] - \frac{\partial\omega}{\partial\phi} + \mu\left[\frac{\partial}{\partial\phi}(j\,H\sqrt{G})\cos\delta + \frac{\partial}{\partial\psi}\left(j\,H\,\frac{J}{\sqrt{G}}\right)\sin\delta\right] = 0 \ . \qquad (23.23)$$

$$\omega = -\frac{1}{\rho W}\frac{\partial}{\partial\psi}\left(\frac{E}{W}\right) \qquad (23.24)$$

$$J\cos\delta\,\frac{\partial H}{\partial\phi} + E\sin\delta\,\frac{\partial H}{\partial\psi} + \tfrac{1}{2}H\sin\delta\,\frac{\partial E}{\partial\psi} + \frac{JH}{2G}\cos\delta\,\frac{\partial G}{\partial\phi} = 0 \qquad (23.25)$$

$$\sqrt{E}\,W\,j = J\sin\delta\,\frac{\partial H}{\partial\phi} - E\cos\delta\,\frac{\partial H}{\partial\psi} - \tfrac{1}{2}H\cos\delta\,\frac{\partial E}{\partial\psi} + \frac{JH}{2G}\sin\delta\,\frac{\partial G}{\partial\phi} \qquad (23.26)$$

$$\frac{\partial}{\partial\psi}\left(\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial\psi}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial\phi}\right) = 0 \qquad (23.27)$$

of five partial differential equations in five unknowns E, G, ω, H and $\dot{j}$.

Method 2. If the magnetic lines and the streamlines do not coincide anywhere in the flow region, i.e., if the flow is not an aligned flow, then magnetic lines can be taken as the coordinate lines ϕ = constant. Equation (22.11) implies the existence of a magnetic flux function ϕ such that

$$\frac{\partial\phi}{\partial x} = H_2 \quad , \quad \frac{\partial\phi}{\partial y} = -H_1 \tag{23.28}$$

and the curves ϕ = constant define the family of magnetic lines. From the transformation equations (22.17) and (23.28) we get

$$\frac{\partial x}{\partial\psi} = JH_1 \quad , \quad \frac{\partial y}{\partial\psi} = JH_2 \tag{23.29}$$

and therefore, we obtain

$$H = \sqrt{H_1^2 + H_1^2} = \frac{\sqrt{G}}{W} \tag{23.30}$$

Conversely, equation (23.30) implies the solenoidal condition on $\vec{H}$, namely, equation (22.11). Observing that, in this case, we have

$$J = \frac{\partial x}{\partial\phi}\frac{\partial y}{\partial\psi} - \frac{\partial x}{\partial\psi}\frac{\partial y}{\partial\phi} = \sqrt{EG}\sin\delta$$

and

$$F = \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \phi} + \frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \psi} = \sqrt{EG}\cos\delta \qquad (23.31)$$

the equations (22.21) yield

$$\frac{\partial x}{\partial \psi} = \frac{F}{\sqrt{E}}\cos\alpha - \frac{J}{\sqrt{E}}\sin\alpha = \sqrt{G}\cos\delta\cos\alpha - \sqrt{G}\sin\delta\sin\alpha$$

$$= H\ W\cos(\alpha+\delta)$$

and

$$\frac{\partial y}{\partial \psi} = H\ W\sin(\alpha+\delta)$$

These equations imply that

$$\frac{\partial x}{\partial \psi} = \pm WH_1$$

$$\frac{\partial y}{\partial \psi} = \pm WH_2$$

where positive sign is taken if the fluid flows along the streamlines towards higher parameter values of ϕ, i.e., if $J > 0$. Therefore, equation (23.30) is equivalent to

$$\frac{\partial x}{\partial \psi} = JH_1 \quad , \quad \frac{\partial y}{\partial \psi} = JH_2$$

or

$$-\frac{\partial \phi}{\partial y} = H_1 \quad , \quad \frac{\partial \phi}{\partial x} = H_2$$

establishing the equivalence of (23.30) and (22.11).

Using (23.31) in the Gauss equation (22.24), we get

$$\frac{\partial}{\partial\psi}\left[\frac{1}{2E\sqrt{EG}\sin\delta}\left\{-\sqrt{EG}\cos\delta\frac{\partial E}{\partial\phi}+2E\cos\delta\frac{\partial}{\partial\phi}(\sqrt{EG})-E\frac{\partial E}{\partial\psi}\right\}\right]$$

$$-\frac{\partial}{\partial\phi}\left[\frac{1}{2E\sqrt{EG}\sin\delta}\left(E\frac{\partial G}{\partial\phi}-\cos\delta\sqrt{EG}\frac{\partial E}{\partial\psi}\right)\right] = 0$$

or

$$\frac{\partial}{\partial\psi}\left[-\frac{\cos\delta}{E}\frac{\partial E}{\partial\phi}-\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial\psi}\right]-\frac{\partial}{\partial\phi}\left(\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial\phi}\right)$$

$$+\cos\delta\frac{\partial^2}{\partial\phi\partial\psi}\left[2\ln E+\ln G\right] = 0$$

or

$$\frac{\partial}{\partial\psi}\left[\frac{\cos\delta}{E}\frac{\partial E}{\partial\phi}-\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial\psi}\right]-\frac{\partial}{\partial\phi}\left[\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial\phi}-\frac{\cos\delta}{G}\frac{\partial G}{\partial\psi}\right] = 0 \tag{23.32}$$

Employing (23.30) and (23.31) in the current density equation (23.13), we have

$$\sqrt{E}\,W\,j = \frac{\partial}{\partial\phi}\left(\frac{\sqrt{G}}{W}\right)\sqrt{EG}-\frac{\partial}{\partial\psi}\left(\frac{\sqrt{G}}{W}\right)E\cos\delta$$

$$+\frac{J\sqrt{G}\sin\delta}{W}\frac{1}{2W^2}\left(E\frac{\partial G}{\partial\phi}-\cos\delta\sqrt{EG}\frac{\partial E}{\partial\psi}\right)$$

or

$$j = \frac{\partial}{\partial\phi}\left(\frac{\sqrt{G}}{W}\right)\left(\frac{\sqrt{G}}{W}\right) - \frac{\partial}{\partial\psi}\left(\frac{\sqrt{G}}{W}\right)\frac{\sqrt{E}\cos\delta}{W} + \frac{1}{2W^2E}\left[E\frac{\partial G}{\partial\phi} - \cos\delta\sqrt{EG}\frac{\partial E}{\partial\psi}\right]$$

$$= \tfrac{1}{2}\frac{\partial}{\partial\phi}\left(\frac{G}{W^2}\right) + \frac{1}{2W^2}\frac{\partial G}{\partial\phi} - \frac{\cos\delta}{W}\left[\sqrt{E}\frac{\partial}{\partial\psi}\left(\frac{\sqrt{G}}{W}\right) + \frac{\sqrt{G}}{W}\frac{1}{2\sqrt{E}}\frac{\partial E}{\partial\psi}\right]$$

$$= \frac{1}{W}\frac{\partial}{\partial\phi}\left(\frac{G}{W}\right) - \frac{\cos\delta}{W}\left[\frac{\partial}{\partial\psi}\left(\frac{\sqrt{EG}}{W}\right)\right]$$

$$= \frac{1}{W}\frac{\partial}{\partial\phi}\left(\frac{G}{W}\right) \qquad (23.33)$$

Likewise employing (23.30) and (23.31) in the momentum equations (23.16) and (23.17), we get

$$\eta\sqrt{EG}\sin\delta\frac{\partial\omega}{\partial\phi} - E\,\omega + \mu\sqrt{G}\,j\sqrt{E}\cos\delta$$

$$= E\frac{\partial h}{\partial\psi} - \sqrt{EG}\cos\delta\frac{\partial h}{\partial\phi}$$

or

$$\eta\sqrt{G}\sin\delta\frac{\partial\omega}{\partial\phi} - \sqrt{E}\,\omega + \mu\sqrt{G}\,j\cos\delta$$

$$= \sqrt{E}\frac{\partial h}{\partial\psi} - \sqrt{G}\cos\delta\frac{\partial h}{\partial\phi} \qquad (23.34)$$

and similarly

$$\eta\sqrt{E}\sin\delta\frac{\partial\omega}{\partial\psi} - \sqrt{E}\cos\delta\,\omega + \mu\sqrt{G}\,j$$

$$= \sqrt{E}\cos\delta\frac{\partial h}{\partial\psi} - \sqrt{G}\frac{\partial h}{\partial\phi} \qquad (23.35)$$

The vorticity equation (23.12) becomes

$$\omega = -\frac{1}{\rho W}\frac{\partial}{\partial\psi}\left(\frac{E}{W}\right) \tag{23.36}$$

We, therefore, have:

<u>Corollary 2</u>: When the streamlines ψ = constant and the magnetic lines ϕ = constant are taken as the curvilinear coordinate system (ϕ,ψ) in the physical plane, the flow is given by the following system of five partial differential equations:

$$\eta\sqrt{G}\sin\delta\frac{\partial\omega}{\partial\phi} - \sqrt{E}\,\omega + \mu\sqrt{G}\,j\cos\delta = \sqrt{E}\frac{\partial h}{\partial\psi} - \sqrt{G}\cos\delta\frac{\partial h}{\partial\phi},$$

$$\eta\sqrt{E}\sin\delta\frac{\partial\omega}{\partial\psi} - \sqrt{E}\cos\delta\,\omega + \mu\sqrt{G}\,j = \sqrt{E}\cos\delta\frac{\partial h}{\partial\psi} - \sqrt{G}\frac{\partial h}{\partial\phi},$$

$$\frac{\partial}{\partial\psi}\left[\frac{\cos\delta}{E}\frac{\partial E}{\partial\phi} - \frac{1}{\sqrt{EG}}\frac{\partial E}{\partial\psi}\right] - \frac{\partial}{\partial\phi}\left[\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial\phi} - \frac{\cos\delta}{G}\frac{\partial G}{\partial\psi}\right] = 0, \tag{23.37}$$

$$\omega = -\frac{1}{\rho W}\frac{\partial}{\partial\psi}\left(\frac{E}{W}\right),$$

$$j = \frac{1}{W}\frac{\partial}{\partial\phi}\left(\frac{G}{W}\right),$$

in E, G, h, ω and j.

Section 4. Flows in which Streamlines are Involutes of a Curve C

We now investigate flows in which streamlines ψ = constant are involutes of curve C. Martin (1971) has studied such flows for electrically non-conducting fluids and established that the streamlines must be concentric circles. Nath and Chandna (1973), extended this result to the magnetohydrodynamic flows having orthogonal magnetic and velocity fields.

Introducing the orthogonal coordinate system formed by the tangent lines and involutes of C, the squared element of arc length is given by (see Appendix A)

$$ds^2 = d\xi^2 + (\xi-\sigma)^2 d\beta^2 \qquad (24.01)$$

where ξ denotes the parameter constant along an involute representing the length of string used to generate the involute, β is the angle of elevation of a tangent line to C, and $\sigma = \sigma(\beta)$ is the arc length along C measured from a fixed point.

We are seeking those flows for which

$$\phi = \phi(\beta) \quad , \quad \psi = \psi(\xi) \qquad (24.02)$$

Using (24.02) in (22.15), we get

$$ds^2 = E\phi'^2 d\beta^2 + 2F\phi'\psi' \, d\beta d\xi + G\psi'^2 d\xi^2$$

and comparing it with (24.01) we find that

$$E = \left(\frac{\xi-\sigma}{\phi'}\right)^2 \quad , \quad F = 0 \quad , \quad G = \frac{1}{\psi'^2}$$

and

$$J = \frac{\xi-\sigma}{\phi'\psi'} \tag{24.03}$$

Employing (24.02) and (24.03) in (23.23), we get

$$\eta \frac{1}{\phi'} \frac{\partial}{\partial\beta}\left[\frac{1}{\psi'(\xi-\sigma)} \frac{\partial\omega}{\partial\beta}\right] + \eta \frac{1}{\psi'} \frac{\partial}{\partial\xi}\left[\frac{\xi-\sigma}{\phi'} \frac{\partial\omega}{\partial\xi}\right] - \frac{1}{\phi'} \frac{\partial\omega}{\partial\beta}$$

$$+ \mu \frac{1}{\phi'} \frac{\partial}{\partial\beta}\left[\frac{jH}{\psi'}\right] \cos\delta + \mu \frac{1}{\psi'} \frac{\partial}{\partial\xi}\left[jH\frac{(\xi-\sigma)}{\phi'}\right] \sin\delta = 0$$

or

$$\eta \frac{\partial}{\partial\beta}\left[\frac{1}{(\xi-\sigma)} \frac{\partial\omega}{\partial\beta}\right] + \eta \frac{\partial}{\partial\xi}\left[(\xi-\sigma)\frac{\partial\omega}{\partial\xi}\right] - \psi' \frac{\partial\omega}{\partial\beta} + \mu \frac{\partial}{\partial\beta}(jH) \cos\delta$$

$$+ \mu \frac{\partial}{\partial\xi}\left[jH(\xi-\sigma)\right] \sin\delta = 0 \tag{24.04}$$

Similarly, (23.24) yields

$$\omega = -\frac{\phi'\psi'}{\rho(\xi-\sigma)} \frac{1}{\psi'} \frac{\partial}{\partial\xi}\left[\frac{(\xi-\sigma)\psi'}{\phi'}\right]$$

or

$$\omega = -\frac{1}{\rho(\xi-\sigma)} \frac{\partial}{\partial\xi}\left[(\xi-\sigma)\psi'\right] \tag{24.05}$$

From (23.25), we obtain

$$\frac{(\xi-\sigma)}{\phi'\psi'} \cos\delta \frac{1}{\phi'} \frac{\partial H}{\partial\beta} + \frac{(\xi-\sigma)^2}{\phi'^2} \sin\delta \frac{1}{\psi'} \frac{\partial H}{\partial\xi} + \tfrac{1}{2} H \sin\delta \frac{1}{\psi'} \frac{2(\xi-\sigma)}{\phi'^2}$$

$$+ \frac{\psi'(\xi-\sigma)}{2\phi'^2} H \cos\delta \frac{\partial}{\partial\beta} \left(\frac{1}{\psi'^2}\right) = 0$$

or

$$\frac{\partial H}{\partial\beta} \cos\delta + \frac{\partial H}{\partial\xi} (\xi-\sigma) \sin\delta + H \sin\delta = 0 , \qquad (24.06)$$

and (23.26) reduces to

$$\frac{(\xi-\sigma)^2}{\phi'^2\psi'} \ddot{j} = \frac{(\xi-\sigma)}{\phi'^2\psi'} \sin\delta \frac{\partial H}{\partial\beta} - \frac{(\xi-\sigma)^2 \cos\delta}{\phi'^2\psi'} \frac{\partial H}{\partial\xi}$$

$$- \tfrac{1}{2} H \cos\delta \frac{2(\xi-\sigma)}{\psi'\phi'^2} + \frac{(\xi-\sigma)\psi'}{2\phi'^2} H \sin\delta \frac{\partial}{\partial\beta} \left(\frac{1}{\psi'^2}\right)$$

or

$$(\xi-\sigma)\ddot{j} = \frac{\partial H}{\partial\beta} \sin\delta - (\xi-\sigma) \frac{\partial H}{\partial\xi} \cos\delta - H \cos\delta \qquad (24.07)$$

Equation (23.27) becomes

$$\frac{1}{\psi'} \frac{\partial}{\partial\xi}\left[\frac{\phi'}{(\xi-\sigma)} \frac{2(\xi-\sigma)}{\phi'^2}\right] + \frac{1}{\phi'} \frac{\partial}{\partial\beta}\left[\frac{\phi'}{(\xi-\sigma)} \frac{1}{\phi'} \frac{\partial}{\partial\beta}\left(\frac{1}{\psi'^2}\right)\right] = 0$$

which is identically satisfied.

Using (24.05) to eliminate ω from (24.04), we get

$$-\frac{\eta}{\rho}\frac{\partial}{\partial\beta}\left[\frac{\psi'\sigma'}{(\xi-\sigma)^3}\right]+\frac{\eta}{\rho}\frac{\partial}{\partial\xi}\left[-\psi''+\frac{\psi'}{(\xi-\sigma)}-\psi'''(\xi-\sigma)\right]$$

$$+\frac{\psi'^2\sigma'}{\rho(\xi-\sigma)^2}+\mu\frac{\partial}{\partial\beta}(jH)\cos\delta+\mu\frac{\partial}{\partial\xi}\left[jH(\xi-\sigma)\right]\sin\delta=0$$

or

$$\eta\left[\frac{3\psi'\sigma'^2}{(\xi-\sigma)^4}+\frac{\psi'\sigma''}{(\xi-\sigma)^3}+\frac{\psi'}{(\xi-\sigma)^2}-\frac{\psi''}{(\xi-\sigma)}+2\,\psi'''+\psi^{(iv)}(\xi-\sigma)\right]$$

$$-\frac{\psi'^2\sigma'}{(\xi-\sigma)^2}-\mu\rho\frac{\partial}{\partial\beta}(jH)\cos\delta-\mu\rho\frac{\partial}{\partial\xi}\left[jH(\xi-\sigma)\right]\sin\delta=0$$

(24.08)

Equations (24.06) to (24.08) are valid for non-aligned as well as aligned flows.

Non-aligned Flows: For such flows, we have an additional restriction,

$$H\,V\sin\delta=K\ ,$$

which gives

$$H=\frac{\rho K}{\sin\delta}\frac{W}{\sqrt{E}}=\frac{\rho K}{\sin\delta}\frac{1}{\psi'} \qquad (24.09)$$

From (24.06) and (24.07), we get

$$(\xi-\sigma)j\sin\delta=\frac{\partial H}{\partial\beta}\ ,$$

and by virtue of (24.09), it implies that $j = 0$. Equation (24.08), therefore, reduces to

$$3\eta\psi'\sigma'^2 + \eta\psi'\sigma''(\xi-\sigma) + (\eta\psi' - \sigma'\psi'^2)(\xi-\sigma)^2 - \eta\psi''(\xi-\sigma)^3$$

$$+ 2\eta\psi'''(\xi-\sigma)^4 + \eta\psi^{(iv)}(\xi-\sigma)^5 = 0 \qquad (24.10)$$

For the relation (24.10) to hold identically, it must hold on the curve C given by $\xi = \sigma(\beta)$ and consequently

$$3\eta\psi'\sigma'^2 = 0$$

implying that $\sigma' = 0$. It means that radius of curvature of C is identically zero and the streamlines are concentric circles.

<u>Aligned Flows</u>: In this case (24.06) implies that $H = F(\xi)$, where F is an arbitrary function of ξ.

From (24.07) we have

$$j = -F'(\xi) - \frac{F(\xi)}{(\xi-\sigma)} ,$$

and (24.08) becomes

$$3\eta\psi'\sigma'^2 + \eta\psi'\sigma''(\xi-\sigma) + \left[\eta\psi' - \sigma'\psi'^2 + \rho\mu F^2(\xi)\ \sigma'\right](\xi-\sigma)^2$$

$$- \eta\psi''(\xi-\sigma)^3 + 2\eta\psi'''(\xi-\sigma)^4 + \eta\psi^{(iv)}(\xi-\sigma)^5 = 0 .$$

By the same argument as used earlier, we see that $q' = 0$ and streamlines are concentric circles. We, therefore, have,

<u>Theorem 2</u>: If the streamlines are involutes of some curve C then C must be a point and the streamlines are circles concentric at that point.

Section 5. Isometric Net

In this section we study those flows in which streamlines and their orthogonal trajectories coincide with the curves in an isometric net (ξ,σ). These flows for electrically non-conducting fluids have been investigated by Martin (1971).

Let

$$z = x + i\,y = z\,(\zeta) \tag{25.01}$$

be an analytic function of $\zeta = \xi + i\sigma$. The curves ξ = constant, and σ = constant in the z-plane (physical plane) form an isometric net. We want to determine all flows for which

$$\phi = \phi(\xi) \quad , \quad \psi = \psi(\sigma) \tag{25.02}$$

the functions $\phi(\xi)$, $\psi(\sigma)$ being at our disposal.

The squared element of arc length in the isometric net (ξ,σ) is given by

$$ds^2 = \lambda(\xi,\sigma)(d\xi^2 + d\sigma^2) \tag{25.03}$$

where $\lambda = |z'(\zeta)|^2$. In this case the coefficients E, F, and G of the first fundamental form

$$ds^2 = Ed\phi^2 + 2Fd\phi\,d\psi + Gd\psi^2$$

are given by

$$E = \frac{\lambda(\xi,\sigma)}{(\phi'(\xi))^2} \quad , \quad F = 0 \quad , \quad G = \frac{\lambda(\xi,\sigma)}{(\psi'(\sigma))^2} \qquad (25.04)$$

by using (25.02) and comparing with (25.03).

Using (25.02) to introduce ξ,σ as independent variables in the Gauss equation (22.24) and substituting for E, F, G from (25.04), we get

$$\frac{1}{\psi'} \frac{\partial}{\partial\sigma}\left[\frac{\phi'}{\lambda} \frac{\partial E}{\partial\sigma}\right] + \frac{1}{\phi'} \frac{\partial}{\partial\xi}\left[\frac{\psi'}{\lambda} \frac{\partial G}{\partial\xi}\right] = 0$$

or

$$\frac{1}{\phi'\psi'}\left[\frac{\partial^2}{\partial\sigma^2} (\ln \lambda) + \frac{\partial^2}{\partial\xi^2} (\ln \lambda)\right] = 0 \qquad (25.05)$$

which implies that $\ln \lambda$ is a harmonic function.

Since

$$\lambda = \left(\frac{\partial x}{\partial\xi}\right)^2 + \left(\frac{\partial y}{\partial\xi}\right)^2$$

$$= \frac{\partial x}{\partial\xi}\frac{\partial y}{\partial\sigma} - \frac{\partial y}{\partial\xi}\frac{\partial x}{\partial\sigma} = \frac{\partial(x,y)}{\partial(\xi,\sigma)} \quad ,$$

from (25.02), we have

$$J = \frac{\partial(x,y)}{\partial(\phi,\psi)} = \frac{1}{\phi'\psi'}\frac{\partial(x,y)}{\partial(\xi,\sigma)} = \frac{\lambda}{\phi'\psi'} \qquad (25.06)$$

Using (25.04), and (25.06) in (23.20), we get

$$\alpha = \int \frac{J}{E} \left[- \frac{E}{2W^2} \frac{\partial E}{\partial \psi} d\phi + \frac{E}{2W^2} \frac{\partial G}{\partial \phi} d\psi \right]$$

$$= \int \frac{\phi'^2}{\phi' \psi'} \frac{\psi'^2}{2\lambda} \left[- \frac{1}{\psi' \phi'^2} \frac{\partial \lambda}{\partial \sigma} \phi' \, d\xi + \frac{1}{\phi' \psi'^2} \frac{\partial \lambda}{\partial \xi} \psi' \, d\sigma \right]$$

$$= \tfrac{1}{2} \int \left[- \frac{1}{\lambda} \frac{\partial \lambda}{\partial \sigma} d\xi + \frac{1}{\lambda} \frac{\partial \lambda}{\partial \xi} d\sigma \right]$$

$$= \tfrac{1}{2} \int \left[- \frac{\partial \ell}{\partial \sigma} d\xi + \frac{\partial \ell}{\partial \xi} d\sigma \right] \qquad (25.07)$$

where $\ell = \ln \lambda$.

If m denotes a harmonic function conjugate to ℓ, then

$$\frac{\partial \ell}{\partial \xi} = \frac{\partial m}{\partial \sigma} \quad , \quad \frac{\partial \ell}{\partial \sigma} = - \frac{\partial m}{\partial \xi}$$

and (25.07) reduces to

$$\alpha = \tfrac{1}{2} \int \left[\frac{\partial m}{\partial \xi} d\xi + \frac{\partial m}{\partial \sigma} d\sigma \right]$$

$$= \alpha_o + \frac{m}{2} \qquad (25.08)$$

where α_o is an arbitrary constant.

From (23.19), z is given by

$$z = \int \frac{\exp(i\alpha)}{\sqrt{E}} \left[E\, d\phi + i\, J\, d\psi \right]$$

$$= \int \frac{\exp(i\alpha)}{\sqrt{\lambda}} \phi' \left[\frac{\lambda}{\phi'^2} \phi' d\xi + i \frac{\lambda}{\phi'\psi'} \psi' \, d\sigma \right]$$

$$= \int \exp\,(i\alpha)\, \sqrt{\lambda}\, (d\xi + i\, d\sigma)$$

$$= e^{i\alpha_0} \int \exp\left[\tfrac{1}{2}(\ell + im)\right] d\zeta \qquad (25.09)$$

Substituting for E, F, G from (25.04) in the system of equations (23.23) to (23.27), we obtain

$$\eta \left[\frac{\partial^2 \omega}{\partial \xi^2} + \frac{\partial^2 \omega}{\partial \sigma^2} \right] - \psi' \frac{\partial \omega}{\partial \xi} + \mu \left[\frac{\partial}{\partial \xi}(jH\sqrt{\lambda} \cos\delta) + \frac{\partial}{\partial \sigma}(jH\sqrt{\lambda} \sin\delta) \right] = 0, \qquad (25.10)$$

$$\omega = - \frac{\psi''}{\rho\lambda}, \qquad (25.11)$$

$$\cos\delta \frac{\partial H}{\partial \xi} + \sin\delta \frac{\partial H}{\partial \sigma} + \tfrac{1}{2} H \sin\delta \frac{\partial \ell}{\partial \sigma} + \tfrac{1}{2} H \cos\delta \frac{\partial \ell}{\partial \xi} = 0 \qquad (25.12)$$

and

$$\sqrt{\lambda}\, j = \sin\delta \frac{\partial H}{\partial \xi} - \cos\delta \frac{\partial H}{\partial \sigma} - \tfrac{1}{2}H \cos\delta \frac{\partial \ell}{\partial \sigma} + \tfrac{1}{2}H \sin\delta \frac{\partial \ell}{\partial \xi} \qquad (25.13)$$

We consider the aligned and non-aligned flows separately.

Aligned Flows: In this case $\sin\delta = 0$ and (25.12) become

$$\frac{\partial H}{\partial \xi} + \tfrac{1}{2}H\,\frac{\partial \ell}{\partial \xi} = 0$$

which implies that

$$\sqrt{\lambda}\,H = F(\sigma)$$

where $F(\sigma)$ is an arbitrary function of σ.

Equation (25.13) reduces to

$$\sqrt{\lambda}\,j = -\frac{\partial H}{\partial \sigma} - \tfrac{1}{2}H\,\frac{\partial \ell}{\partial \sigma}$$

$$= -\frac{1}{\sqrt{\lambda}}F'(\sigma) + \frac{F(\sigma)}{2\lambda\sqrt{\lambda}}\frac{\partial \lambda}{\partial \sigma} - \frac{F(\sigma)}{2\lambda\sqrt{\lambda}}\frac{\partial \lambda}{\partial \sigma}$$

or,

$$j = -\frac{F'(\sigma)}{\lambda}$$

On eliminating ω, H and j from (25.10), we get

$$-\frac{\eta}{\rho}\left[\psi''\left\{\frac{1}{\lambda}\left(\frac{\partial \ell}{\partial \xi}\right)^2 - \frac{1}{\lambda}\frac{\partial^2 \ell}{\partial \xi^2}\right\} + \frac{\psi''}{\lambda}\left(\frac{\partial \ell}{\partial \sigma}\right)^2 - \frac{\psi''}{\lambda}\frac{\partial^2 \ell}{\partial \sigma^2} - \frac{2\psi'''}{\lambda}\frac{\partial \ell}{\partial \sigma} + \frac{\psi^{(iv)}}{\lambda}\right]$$

$$-\frac{\psi'\psi''}{\rho\lambda}\frac{\partial \ell}{\partial \xi} + \mu\left[-\frac{\partial}{\partial \xi}\left(\frac{FF'}{\lambda}\right)\right] = 0$$

or

$$\eta \cdot \left[\psi'' \left\{ \left(\frac{\partial \ell}{\partial \xi}\right)^2 + \left(\frac{\partial \ell}{\partial \sigma}\right)^2 \right\} - 2\psi''' \frac{\partial \ell}{\partial \sigma} + \psi^{(iv)} \right] + \psi' \psi'' \frac{\partial \ell}{\partial \xi}$$

$$- \rho \mu F F' \frac{\partial \ell}{\partial \xi} = 0$$

or

$$\left(\frac{\partial \ell}{\partial \xi}\right)^2 + \left(\frac{\partial \ell}{\partial \sigma}\right)^2 - \frac{2\psi'''}{\psi''} \frac{\partial \ell}{\partial \sigma} + \frac{1}{\eta} \left[\psi' - \frac{\mu \rho}{\psi''} F(\sigma) F'(\sigma) \right]$$

$$\cdot \frac{\partial \ell}{\partial \xi} + \frac{\psi^{(iv)}}{\psi''} = 0 \qquad (25.14)$$

provided that $\psi'' \neq 0$, i.e. $\omega \neq 0$.

If we set

$$u = \frac{\partial \ell}{\partial \xi} \;, \quad v = - \frac{\partial \ell}{\partial \sigma} \;, \quad a = -\frac{1}{2\eta}\left[\psi' - \frac{\mu \rho}{\psi''} F(\sigma) F'(\sigma)\right]$$

$$b = - \frac{\psi'''}{\psi''} \;, \quad c = \frac{\psi^{(iv)}}{\psi''} \qquad (25.15)$$

then $u + iv$ is an analytic function of ζ and (26.14) becomes

$$u^2 + v^2 - 2au - 2bv + c = 0$$

which can be rewritten as

$$(u - a)^2 + (v - b)^2 - R^2 = 0 \qquad (25.16)$$

where $R^2 = a^2 + b^2 - c$, with a, b and c being functions of σ only. The following lemma shows that (25.16) implies the function u, y are constants.

Lamma: If u + iv is an analytic function of $\xi + i\sigma$, such that u, v satisfy the relation

$$(u-a)^2 + (v-b)^2 - R^2 = 0$$

where $R^2 = a^2 + b^2 - c$ and $a = a(\sigma)$, $b = b(\sigma)$, and $c = c(\sigma)$, then u, v must be constants.

Proof: Differentiating (26.16) with respect to ξ and σ respectively, we get

$$2(u-a)\frac{\partial u}{\partial \xi} + 2(v-b)\frac{\partial v}{\partial \xi} = 0$$

$$2(u-a)\frac{\partial u}{\partial \sigma} + 2(v-b)\frac{\partial v}{\partial \sigma} - 2(u-a)a'(\sigma) - 2(v-b)b'(\sigma) - 2RR' = 0$$

Using Cauchy-Riemann equations to eliminate $\frac{\partial v}{\partial \xi}$ and $\frac{\partial v}{\partial \sigma}$ from these equations, we obtain

$$2(u-a)\frac{\partial u}{\partial \xi} - 2(v-b)\frac{\partial u}{\partial \sigma} = 0$$

$$2(u-a)\frac{\partial u}{\partial \sigma} + 2(v-b)\frac{\partial u}{\partial \sigma} = 2(u-a)a' + 2(v-b)b' + 2RR'$$

Solving the preceding equations for $\frac{\partial u}{\partial \xi}$ and $\frac{\partial u}{\partial \sigma}$, we get

$$\frac{\partial u}{\partial \xi} = \frac{(u-a)a' + (v-b)b' + RR'}{R^2}(v-b)$$

(25.17)

$$\frac{\partial u}{\partial \sigma} = \frac{(u-a)a' + (v-b)b' + RR'}{R^2}(u-a)$$

where (25.16) has been used.

Differentiating (25.16) twice with respect to ξ, twice with respect to σ and adding, we get

$$\left(\frac{\partial u}{\partial \xi}\right)^2 + \left(\frac{\partial u}{\partial \sigma}\right)^2 + \left(\frac{\partial v}{\partial \xi}\right)^2 + \left(\frac{\partial v}{\partial \sigma}\right)^2 - 2a' \frac{\partial u}{\partial \sigma} - 2b' \frac{\partial v}{\partial \sigma}$$

$$- (u-a)a'' - (v-b)b'' + a'^2 + b'^2 - R'^2 - RR'' = 0 \tag{25.18}$$

Using Cauchy-Riemann equations to eliminate $\frac{\partial v}{\partial \xi}$, $\frac{\partial v}{\partial \sigma}$ and (25.17) to eliminate $\frac{\partial u}{\partial \xi}$, $\frac{\partial u}{\partial \sigma}$ from (25.18), we find that u, v must satisfy a second relation

$$\frac{2\left[(u-a)a'+(v-b)b'+RR'\right]^2}{R^2}$$

$$- \frac{2\left[a'(u-a)+b'(v-b)\right]\cdot\left[(u-a)a'+(v-b)b'+RR'\right]}{R^2}$$

$$- (u-a)a''-(v-b)b'' + a'^2 + b'^2 - R'^2 - RR'' = 0$$

or

$$\left(\frac{2a'R'}{R} - a''\right)(u-a) + \left(\frac{2b'R'}{R} - b''\right)(v-b) + a'^2 + b'^2 + R'^2 = 0 \tag{25.19}$$

Assuming that vorticity ω does not vanich identically, $a' \neq 0$ and (25.19) cannot vanish identically. Consequently, u, v can be expressed as functions of σ alone on solving

(25.16) and (25.19). It means that

$$v_\sigma = u_\xi = 0$$

and

$$u_\sigma = -v_\xi = 0$$

Hence u and v must be constant.

If we take u, v to be constants a_o, b_o then from (25.15) we have

$$\frac{\partial \ell}{\partial \xi} = a_o \quad , \quad \frac{\partial \ell}{\partial \sigma} = -b_o$$

and therefore,

$$\ell = a_o\xi - b_o\sigma + c_o \quad , \quad m = b_o\xi + a_o\sigma + d_o$$

where c_o, d_o are arbitrary constants. Now (25.09) yields

$$z = e^{i\alpha_o} \int \exp\left[\tfrac{1}{2}(a_o+ib_o)\xi + \frac{i}{2}(a_o+ib_o)\,\sigma + \tfrac{1}{2}(c_o+id_o)\right] d\zeta$$

$$= C \int \exp(A\zeta)\, d\zeta$$

where $A = \frac{1}{2}(a_o+ib_o)$ and $C = \exp\left[i\alpha_o+\frac{1}{2}(c_o+id_o)\right]$.

On integrating we get

$$z = \begin{cases} z_o + C\zeta & \text{if } A = 0 \\ z_o + \dfrac{C}{A}\, e^{A\zeta} & \text{if } A \neq 0 \end{cases} \qquad (25.20)$$

where z_o is an arbitrary constant.

Since the streamlines are the transforms of the lines σ = constant, they are restricted to (1) parallel lines if $A = 0$, (2) concurrent lines if $a_o \neq 0$, $b_o = 0$, (3) concentric circles if $a_o = 0$, $b_o \neq 0$, (4) logarithmic spirals if $a_o \neq 0$, $b_o \neq 0$.

Non-aligned Flows: From (23.01), (23.07) and (25.04), we have

$$H = \frac{\rho K\sqrt{\lambda}}{\psi' \sin\delta} \tag{25.21}$$

Using (25.21) in (25.12), we obtain

$$\frac{\rho K \cos\delta}{\psi' \sin\delta}\frac{1}{2\sqrt{\lambda}}\frac{\partial\lambda}{\partial\xi} + \frac{\rho K}{\psi'}\frac{1}{2\sqrt{\lambda}}\frac{\partial\lambda}{\partial\sigma} - \frac{\rho K\sqrt{\lambda}}{\psi'^2}\psi'' + \frac{\rho K \cos\delta}{2\psi' \sin\delta}\sqrt{\lambda}\frac{\partial\ell}{\partial\xi}$$

$$+ \frac{\rho K}{2\psi'}\sqrt{\lambda}\frac{\partial\ell}{\partial\sigma} = 0$$

or

$$\cos\delta\frac{\partial\ell}{\partial\xi} + \sin\delta\frac{\partial\ell}{\partial\sigma} = \frac{\psi''}{\psi'}\sin\delta \tag{25.22}$$

Differentiating (25.22) with respect to ξ, we have

$$\cos\delta\frac{\partial^2\ell}{\partial\xi^2} + \sin\delta\frac{\partial^2\ell}{\partial\xi\partial\sigma} = 0$$

Noting that ℓ is a harmonic function, we get

$$\frac{\partial}{\partial \sigma}\left[\sin \delta \frac{\partial \ell}{\partial \xi} - \cos \delta \frac{\partial \ell}{\partial \sigma}\right] = 0$$

or,

$$\sin \delta \frac{\partial \ell}{\partial \xi} - \cos \delta \frac{\partial \ell}{\partial \sigma} = f(\xi) \qquad (25.23)$$

Equations (25.22) and (25.23) imply that

$$\frac{\partial \ell}{\partial \xi} = f(\xi)\sin \delta + \frac{\psi''}{\psi'} \sin \delta \cos \delta$$

$$\frac{\partial \ell}{\partial \sigma} = \frac{\psi''}{\psi'} \sin^2 \delta - f(\xi) \cos \delta \qquad (25.24)$$

where $f(\xi)$ is an arbitrary function of ξ.
Using the fact that $\frac{\partial^2 \ell}{\partial \xi^2} + \frac{\partial^2 \ell}{\partial \sigma^2} = 0$, we obtain

$$-f'(\xi) = \frac{d}{d\sigma}\left(\frac{\psi''}{\psi'}\right)\sin \delta = \text{constant, } A$$

so that

$$\psi'(\sigma) = a \exp\left[(\tfrac{1}{2}A\sigma^2 + B\sigma) \operatorname{cosec} \delta\right], \quad a \neq 0$$

$$f(\xi) = -A\xi + C \qquad (25.25)$$

where a, A, B, C are arbitrary constants.

Eliminating ω from (25.10) using (25.11), we get

$$\eta\left[\frac{\psi''}{\rho\lambda}\left(\frac{\partial^2 \ell}{\partial \xi^2}+\frac{\partial^2 \ell}{\partial \sigma^2}\right)-\frac{\psi''}{\rho\lambda}\left\{\left(\frac{\partial \ell}{\partial \xi}\right)^2+\left(\frac{\partial \ell}{\partial \sigma}\right)^2\right\}-\frac{\psi^{(iv)}}{\rho\lambda}+\frac{2\psi'''}{\rho\lambda}\frac{\partial \ell}{\partial \sigma}\right]$$

$$-\frac{\psi'\psi''}{\rho\lambda}\frac{\partial \ell}{\partial \xi}+\mu\left[\frac{\partial}{\partial \xi}(jH\sqrt{\lambda})\cos\delta+\frac{\partial}{\partial \sigma}(jH\sqrt{\lambda})\sin\delta\right]=0 \tag{25.26}$$

Using (25.21), (25.22) in (25.13), we obtain

$$\sqrt{\lambda}j=\frac{\rho K}{2\psi'\sqrt{\lambda}}\frac{\partial \lambda}{\partial \xi}-\cot\delta\,\frac{\rho K}{2\psi'\sqrt{\lambda}}\frac{\partial \lambda}{\partial \sigma}+\cot\delta\,\frac{\rho K\sqrt{\lambda}}{\psi'^2}\psi''$$

$$-\frac{\rho K\sqrt{\lambda}}{2\psi'}\cot\delta\,\frac{\partial \ell}{\partial \sigma}+\frac{\rho K\sqrt{\lambda}}{2\psi'}\frac{\partial \ell}{\partial \xi}$$

or,

$$j=\frac{\rho K}{\psi'}\left[\frac{\partial \ell}{\partial \xi}-\cot\delta\,\frac{\partial \ell}{\partial \sigma}+\cot\delta\,\frac{\psi''}{\psi'}\right]$$

$$=\frac{\rho K}{\psi'\sin^2\delta}\frac{\partial \ell}{\partial \xi} \tag{25.27}$$

Equations (25.22), (25.26) and (25.27) yield

$$-\frac{\eta\psi''}{\rho\lambda}\left[\left(\frac{\partial \ell}{\partial \xi}\right)^2+\left(\frac{\partial \ell}{\partial \sigma}\right)^2\right]-\frac{\eta}{\rho\lambda}\psi^{(iv)}+\frac{2\eta\psi'''}{\rho\lambda}\frac{\partial \ell}{\partial \sigma}-\frac{\psi'\psi''}{\rho\lambda}\frac{\partial \ell}{\partial \xi}$$

$$+\frac{\mu\rho^2K^2\lambda}{\sin^2\delta\,\psi'^2}\left[\frac{\partial^2 \ell}{\partial \xi^2}\cot\delta+\left(\frac{\partial \ell}{\partial \xi}\right)^2\cot\delta+\frac{\partial^2 \ell}{\partial \xi\partial \sigma}\right.$$

$$\left.+\frac{\partial \ell}{\partial \xi}\frac{\partial \ell}{\partial \sigma}-2\frac{\partial \ell}{\partial \xi}\frac{\partial \ell}{\partial \sigma}\frac{\psi''}{\psi'}\right]=0$$

or

or

$$(\frac{\partial \ell}{\partial \xi})^2 + (\frac{\partial \ell}{\partial \sigma})^2 + \frac{\psi^{(iv)}}{\psi''} - \frac{2\psi'''}{\psi''} \frac{\partial \ell}{\partial \sigma} + \frac{\psi'}{\eta} + \frac{\partial \ell}{\partial \xi}$$

$$+ \frac{\mu \rho^3 K^2 \bar{\lambda}^2}{\eta \sin^2\delta \ \psi'^3} \frac{\partial \ell}{\partial \xi} = 0 \qquad (25.28)$$

As a result of (25·24) and (25.25), we obtain

$$\ell = \tfrac{1}{2}A\sin\delta\ \sigma^2 + A\cos\delta .\xi\sigma - \tfrac{1}{2}A\sin\delta\,\xi^2 + (B\cos\delta + C\sin\delta)\xi$$

$$+ (B\sin\delta - C\cos\delta)\sigma + D_1 \qquad (25.29)$$

and

$$m = \tfrac{1}{2}A\cos\delta.\sigma^2 - A\sin\delta.\xi\sigma - \tfrac{1}{2}A\cos\delta\ \xi^2 - (B\sin\delta - C\cos\delta)\xi$$

$$+ (B\cos\delta + C\sin\delta)\sigma + D_2 \qquad (25.30)$$

where D_1, D_2 are arbitrary constants.

From (25.25), (25.28) and (25.29), we get

$$(A\sigma + B)^3 + (-A\xi + C)^2(A\sigma + B) - 2\left[(A\sigma + B)^2\text{cosec}\,\delta + A\right]$$

$$\cdot (A\sigma\sin\delta + A\xi\cos\delta + B\sin\delta - C\cos\delta)$$

$$+ (A\sigma + B)\left[(A\sigma + B)^2\text{cosec}\,\delta + 3A\right]\text{cosec}\,\delta \qquad \text{(cont'd)}$$

$$+ \frac{a}{\eta}(A\sigma + B)(A\sigma\cos\delta - A\xi\sin\delta + B\cos\delta + C\sin\delta)$$

$$\cdot \exp\left[(\tfrac{1}{2}A\sigma^2 + B\sigma)\operatorname{cosec}\delta\right] + \frac{\mu\rho^3 K^2}{a^3\eta\sin^2\delta}(A\sigma + B)$$

$$\cdot (A\sigma\cos\delta - A\xi\sin\delta + B\cos\delta + C\sin\delta)$$

$$\cdot \exp\left[A\sigma^2(\sin\delta - \tfrac{3}{2}\operatorname{cosec}\delta) - A\xi^2\sin\delta + 2A\xi\sigma\cos\delta\right.$$

$$+ 2(B\cos\delta + C\sin\delta)\xi + (2B\sin\delta - 2C\cos\delta - 3B\operatorname{cosec}\delta)\sigma$$

$$\left. + 2D_1\right] = 0 \qquad (25.31)$$

For the relation (25.31) to hold identically, $A = 0$ and either $B = 0$ or $B\cos\delta + C\sin\delta = 0$. From (25.09), we have

$$z = \begin{cases} z_o + D\zeta & \text{if } B = 0,\ C = 0 \\ z_o + \dfrac{2Di}{(B\sin\delta - C\cos\delta)}\exp\left[-\tfrac{i}{2}(B\sin\delta - C\cos\delta)\zeta\right] & \text{if } B\cos\delta + C\sin\delta = 0,\ B\sin\delta - C\cos\delta \neq 0. \\ z_o + \dfrac{2D(\sin\delta - i\cos\delta)}{C}\exp\left[\tfrac{C}{2}(\sin\delta + i\cos\delta)\zeta\right] & \text{if } B = 0,\ C \neq 0. \end{cases}$$

where z_o is an arbitrary constant and $D = \frac{1}{2}(D_1 + iD_2)$.

As the streamlines are the transforms of the lines φ = constant, they are restricted to (1) parallel lines if $B = C = 0$, (2) concurrent lines if $B = 0$, $C \neq 0$ and $\delta = \pm \frac{\pi}{2}$, (3) concentric circles if $B \sin \delta - C \cos \delta \neq 0$ and $B \cos \delta + C \sin \delta = 0$, (4) logarithmic spirals if $B = 0$, $C \neq 0$ and $\delta \neq \pm \frac{\pi}{2}$. Summing up, we have,

<u>Theorem 3</u>: If the streamlines and their orthogonal trajectories coincide with the curves in an isometric net then the streamlines are restricted to (1) parallel straight lines (2) concurrent lines, (3) concentric circles and (4) logarithmic spirals..

Section 6. Solutions for Some Particular Flows

Here we study vortex and radial flows whentthe magnetic field vector $\vec{H}$ makes a constant non-zero angle δ with the velocity vector.

(1) Vortex Flows: Using the polar coordinates (r,θ), the square of the element of arc length is given by

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{26.01}$$

Since the streamlines ψ = constant are concentric circles, taking the origin at the common centre, we have

$$\psi = \psi(r) \quad , \quad \phi = \phi(\theta) \tag{26.02}$$

where ϕ = constant are the orthogonal trajectories of the family of streamlines.

From (22.15) and (26.02), we get

$$ds^2 = E\phi'^2 d\theta^2 + 2F\phi'\psi' d\theta dr + G\psi'^2 dr^2 \tag{26.03}$$

Comparison of (26.01) and (26.03) yields

$$E = \frac{r^2}{\phi'^2} \quad , \quad F = 0 \quad , \quad G = \frac{1}{\psi'^2} \tag{26.04}$$

and

$$J = \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \phi}$$

$$= \frac{1}{\phi'\psi'}\left[\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} - \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta}\right]$$

$$= -\frac{r}{\phi'\psi'} \qquad (26.05)$$

Without any loss of generality, we can assume $\phi'(\theta) > 0$ and that fluid is flowing in the clockwise direction so that $J < 0$ and $\psi'(r) > 0$.
From (26.04 (23.01) and (23.07), we get

$$V = \frac{\psi'}{\rho} \quad , \quad H = \frac{\rho K}{\sin\delta\,\psi'} \qquad (26.06)$$

Using (26.04) in the vorticity equation (23.12), we obtain

$$\omega = -\frac{1}{\rho W}.\frac{\partial}{\partial \psi}\left(\frac{E}{W}\right)$$

or

$$\omega = -\frac{\phi'}{\rho r}\frac{\partial}{\partial r}\left(\frac{r\psi'}{\phi'}\right) = -\frac{1}{\rho r}\left[\psi' + r\psi''\right]. \qquad (26.07)$$

Employing (26.04) and (26.05) in (23.25), we get

$$-\frac{r\cos\delta}{\psi'\phi'^2}\frac{\partial H}{\partial \theta} + \frac{r^2\sin\delta}{\psi'\phi'^2}\frac{\partial H}{\partial r} + \tfrac{1}{2} H \sin\delta\,\frac{2r}{\psi'\phi'^2} = 0$$

$$(26.08)$$

Using (26.06) in (26.08), we find that

$$-\frac{r^2\sin\delta}{\psi'\phi'^2}\,\frac{\rho K}{\sin\delta}\,\frac{\psi''}{\psi'^2}+\frac{\rho K}{\psi'}\,\frac{r}{\phi'^2\psi'}=0$$

or

$$\frac{\psi''}{\psi'}=\frac{1}{r}$$

which on integration gives

$$\psi' = Ar \qquad (26.09)$$

where A is an arbitrary non-zero constant.

From the current density equation (23.26) and using (26.04) to (26.06), we see that

$$\frac{r^2}{\phi'^2\psi'}\,j=\frac{r^2\cos\delta}{\phi'^2\psi'}\,\frac{\rho K}{\sin\delta}\,\frac{\psi''}{\psi'^2}-\tfrac{1}{2}\,\frac{\rho K\cos\delta}{\sin\delta}\,\frac{2r}{\phi'^2\psi'^2}$$

or

$$j=\rho K\cot\delta\left[\frac{\psi''}{\psi'^2}-\frac{1}{r\psi'}\right] \qquad (26.10)$$

Using (26.09) in (26.10), we get

$$j = 0 \qquad (26.11)$$

Similarly, from (26.09) and (26.07), we have

$$\omega = -\frac{2A}{\rho} \qquad (26.12)$$

Momentum equations (23.21) and (23.22), with the help of (26.11) and (26.12), simplify to

$$\frac{\partial h}{\partial \psi} = \frac{2A}{\rho}$$

and $$\frac{\partial h}{\partial \phi} = 0 \qquad (26.13)$$

Equations (26.13) imply that

$$\frac{dh}{dr} = \frac{2A}{\rho}\psi'$$
$$= \frac{2A^2}{\rho} r$$

which on integration yields

$$h = \frac{A^2 r^2}{\rho} + B \qquad (26.14)$$

where B is an arbitrary constant.

From (26.06) and (26.09) we have

$$V = \frac{Ar}{\rho}, \quad H = \frac{\rho K}{A \sin \delta \, r} \qquad (26.15)$$

As $h = \frac{1}{2}\rho V^2 + p$, the pressure is given by

$$p = \frac{A^2 r^2}{\rho} - \frac{1}{2}\frac{A^2 r^2}{\rho} + B$$
$$= \frac{A^2 r^2}{2\rho} + B .$$

(2) Radial Flows:

In this case the streamlines are concurrent straight lines emerging from the origin and their orthogonal trajectories ϕ = constant are concentric circles. Therefore, we have

$$\phi = \phi(r) \quad , \quad \psi = \psi(\theta) \tag{26.16}$$

From (22.15) and (26.16), we obtain

$$ds^2 = E\phi'^2 dr^2 + 2F\phi'\psi' dr\, d\theta + G\psi'^2 d\theta^2 \tag{26.17}$$

Comparing (26.17) with (26.01), we have

$$E = \frac{1}{\phi'^2} \quad , \quad F = 0 \quad , \quad G = \frac{r^2}{\psi'^2}$$

and (26.18)

$$J = \frac{r}{\phi'\psi'}$$

For the outward flow $J > 0$ (taking $\phi'(r) > 0$), we have $\psi'(\theta) > 0$. From (26.18), (23.01) and (23.07), we obtain

$$V = \frac{\sqrt{E}}{\rho W} = \frac{\psi'}{\rho r} \quad ,$$

(26.19)

$$H = \frac{\rho K}{\sin \delta} \frac{r}{\psi'}$$

Employing (26.18) in the vorticity equation (23.24), we obtain

$$\omega = -\frac{\phi'\psi'}{\rho r}\frac{1}{\psi'}\frac{\partial}{\partial\theta}\left(\frac{\psi'}{r\phi'}\right)$$

$$= -\frac{\psi''}{\rho r^2} \tag{26.20}$$

From (26.18) and (23.25), we get

$$\frac{r}{\phi'\psi'}\cos\delta\frac{1}{\phi'}\frac{\partial}{\partial r}\left(\frac{\rho K r}{\sin\delta\,\psi'}\right) + \frac{\sin\delta}{\phi'^2\psi'}\frac{\partial}{\partial\theta}\left(\frac{\rho K r}{\sin\delta\,\psi'}\right)$$

$$+ \frac{\rho K r}{2\psi'}\frac{1}{\psi'}\frac{\partial}{\partial\theta}\left(\frac{1}{\phi'^2}\right) + \frac{r^2\rho K}{\phi'\psi'^2\sin\delta}\frac{\psi'^2\cos\delta}{2r^2}\frac{1}{\phi'}\frac{\partial}{\partial r}\left(\frac{r^2}{\psi'^2}\right) = 0$$

or

$$r\cot\delta - \frac{r\psi''}{\psi'} + r\cot\delta = 0$$

or

$$\frac{\psi''}{\psi'} = 2\cot\delta \tag{26.21}$$

which on integration yields

$$\psi' = A\,e^{2\theta\cot\delta} \tag{26.22}$$

Using (26.18) and (26.19) in the current density equation (23.26), we find that

$$\frac{r}{\phi'^2\psi'} j = \frac{r \sin \delta}{\phi'\psi'} \frac{1}{\phi'} \frac{\partial}{\partial r}\left(\frac{\rho K r}{\sin \psi'}\right) - \frac{\cos \delta}{\phi'^2\psi'} \frac{\partial}{\partial \theta} \left(\frac{\rho K r}{\sin \psi'}\right)$$

$$- \frac{\rho K r \cos \delta}{\psi'^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\phi'^2}\right) + \frac{r^2 K}{\phi'\psi'^2} \frac{\psi'^2}{2r^2\phi'} \frac{\partial}{\partial r} \left(\frac{r^2}{\psi'^2}\right)$$

or

$$j = \frac{\rho K}{\psi'} + K \cot \delta \frac{\psi''}{\psi'^2} + \frac{\rho K}{\psi'}$$

$$= \frac{\rho K}{\psi'} \left[2 + \frac{\psi''}{\psi'} \cot \delta \right] \qquad (26.23)$$

Using (26.22) in (26.23), we obtain

$$j = \frac{\rho K}{Ac^{2\theta} \cot \delta} \left[2 + 2\cot^2\delta \right]$$

$$= \frac{2\rho K}{A\sin^2\delta \, e^{2\theta} \cot \delta} \qquad (26.24)$$

From (23.23) with ω and j given by (26.20) and (26.24), we get

$$\frac{\eta}{\phi'} \frac{\partial}{\partial r} \left(\frac{r\phi'\psi''}{\phi'\psi'} \frac{2}{\rho r^3}\right) + \frac{\eta}{\psi'} \frac{\partial}{\partial \theta} \left(- \frac{\psi'}{r\phi'\psi'} \frac{\psi'''}{\rho r^2}\right) + \frac{\psi''}{\rho\phi'} \left(- \frac{2}{r^3}\right)$$

$$+ \frac{\mu \cos \delta}{\phi'} \frac{\partial}{\partial r} \left(\frac{2\rho^2K^2r^2}{\sin^3\delta \, \psi'^3}\right) + \frac{\mu \sin \delta}{\psi'} \frac{\partial}{\partial \theta} \left(\frac{2\rho^2K^2r}{\sin^3\delta\psi'^2\phi'}\right) = 0$$

or

$$4\eta\psi'' + \eta\psi^{(iv)} + 2\psi'\psi'' - \frac{4\mu \cot\delta\, \rho^3 K^2}{\psi'^2 \sin^2\delta} r^4 + \frac{4\mu \rho^3 K^2 \psi''}{\psi'^3 \sin^2\delta} r^4 = 0 \qquad (26.25)$$

Equation (26.25) must hold for all values of (r,θ). Since the left hand side of (26.25) is a polynomial of degree four in r, all its coefficients must be zero. In particular, equating the coefficient of r to zero, we obtain

$$\frac{4\mu \rho^3 K^2}{\psi'^2 \sin^2\delta} \left(\frac{\psi''}{\psi'} - \cot\delta\right) = 0$$

or

$$\frac{\psi''}{\psi'} = \cot\delta \qquad (26.26)$$

Equation (26.26) together with (26.21) implies that $\cot\delta = 0$, $\psi'' = 0$ and (26.25) is identically satisfied. Thus we conclude that radial flow is possible only when the magnetic field is orthogonal to velocity field. In this case, from (26.20) and (26.24), we get

$$\left.\begin{aligned} \omega &= 0 \\ j &= \frac{2\rho K}{A} \end{aligned}\right\} \qquad (26.27)$$

where $\psi' = A$.

From momentum equations (23.21) and (23.22) with ω, j given by (26.27), we obtain

$$\frac{\partial h}{\partial \psi} = 0$$

and

$$\frac{\partial h}{\partial \phi} = -\mu \frac{r}{\phi' \psi'} \frac{\psi'}{r} \frac{\rho K r}{\psi'} \frac{2\rho K}{A}$$

which imply that

$$\frac{dh}{dr} = -\frac{2\mu\rho^2 K^2 r}{A^2}$$

On integration we get

$$h = -\frac{\mu\rho^2 K^2 r^2}{A^2} + B$$

and therefore, pressure is given by

$$p = -\frac{\mu\rho^2 K^2 r^2}{A^2} - \frac{A^2}{2\rho r^2} + B$$

CHAPTER III

VISCOUS ORTHOGONAL FLOWS

In this chapter steady orthogonal flows of a viscous incompressible fluid having infinite electrical conductivity are investigated. Power and Walker (1967) established the reducibility of certain viscous orthogonal flows of incompressible fluids to plane flows of non-conducting fluids. In the case of inviscid incompressible fluids, Kingston and Talbot (1969) classified completely the possible flow configurations and in doing so obtained some interesting spiral flows. Nath and Chandna (1973) established that the only viscous incompressible orthogonal flows with straight streamlines are parallel or radial flows.

First we determine the geometry of the orthogonal flows with zero current density. We also establish that the streamlines in an irrotational orthogonal flow are either concurrent straight lines or parallel straight lines. Finally, using the hodograph transformation the flow problem is reduced to the solution of a linear partial differential equation of second order. The usefulness of this approach is illustrated by considering some particular examples.

Section 1. Flow Equations

The steady flow of a viscous incompressible fluid of infinite electrical conductivity is governed by the following system of equations:

$$\operatorname{div}(\rho\vec{V}) = 0 \qquad (31.01)$$

$$\rho(\vec{V}\cdot\operatorname{grad})\vec{V} + \operatorname{grad} p = \mu \operatorname{curl} \vec{H} \times \vec{H} + \eta\nabla^2\vec{V} \qquad (31.02)$$

$$\operatorname{curl} (\vec{V} \times \vec{H}) = 0 \qquad (31.03)$$

$$\operatorname{div} \vec{H} = 0 \qquad (31.04)$$

In the case of two dimensional flows, with $\vec{H}$ in the plane of flow and orthogonal to velocity vector $\vec{V}$, we have

$$\vec{V} = (u,v) \text{ and } \vec{H} = (H_1,H_2)$$

with

$$H_1 = -\lambda v \ , \ H_2 = \lambda u \qquad (31.05)$$

where λ is a scaler function.

From (31.03), we get

$$uH_2 - vH_1 = K \qquad (31.06)$$

where K is an arbitrary non-zero constant.

Equations (31.05) and (31.06) yield

$$\vec{H} = \left(-\frac{Kv}{V^2}, \frac{Ku}{V^2}\right) \quad \text{and} \quad H^2V^2 = K^2 \qquad (31.07)$$

where $V = |\vec{V}|$ and $H = |\vec{H}|$.

Equations (31.01) and (31.04) imply the existence of streamfunction $\psi(x,y)$ and magnetic flux function $\phi(x,y)$ such that

$$\frac{\partial\psi}{\partial x} = -\rho v \quad , \quad \frac{\partial\psi}{\partial y} = \rho u$$

and (31.08)

$$\frac{\partial\phi}{\partial x} = H_2 \quad , \quad \frac{\partial\phi}{\partial y} = -H_1$$

The curves ψ = constant and the curves ϕ = constant are the streamlines and the magnetic field lines respectively. These curves form an orthogonal curvilinear coordinate system in the physical (x,y)-plane. Using (31.08) in (31.06), we obtain

$$\rho K = \rho u H_2 - \rho v H_1$$

$$= \frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y} = \frac{\partial(\phi,\psi)}{\partial(x,y)}$$

or

$$J = \frac{\partial(x,y)}{\partial(\phi,\psi)} = \frac{1}{\rho K} \qquad (31.09)$$

Introducing the functions

$$h = \tfrac{1}{2}\rho V^2 + p$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$j = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}$$

and with ϕ, ψ as independent variables the system equations (23.37), when $\delta = \pi/2$, reduces to

$$\eta\sqrt{G}\,\frac{\partial \omega}{\partial \phi} - \sqrt{E}\,\omega = \sqrt{E}\,\frac{\partial h}{\partial \psi} \qquad (31.10)$$

$$\eta\sqrt{E}\,\frac{\partial \omega}{\partial \psi} + \mu\sqrt{G}\,j = -\sqrt{G}\,\frac{\partial h}{\partial \phi} \qquad (31.11)$$

$$\frac{\partial}{\partial \psi}\left(\frac{1}{\sqrt{EG}}\,\frac{\partial E}{\partial \psi}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sqrt{EG}}\,\frac{\partial G}{\partial \phi}\right) = 0 \qquad (31.12)$$

$$\omega = -\frac{1}{\rho J}\,\frac{\partial}{\partial \psi}\left(\frac{E}{J}\right) \qquad (31.13)$$

$$j = \frac{1}{J}\,\frac{\partial}{\partial \phi}\left(\frac{G}{J}\right) \qquad (31.14)$$

where E, G are given by (22.16).

Section 2. Zero Current Density

In this section we consider the special class of plane orthogonal flows for which the magnetic field term vanishes from the momentum equations. This will be the case if curl $\vec{H} \times \vec{H} = 0$ and since curl $\vec{H}$ is orthogonal to the plane of flow, we must have curl $\vec{H} = 0$. Using (31.07), we see that a necessary and sufficient condition for the above possibility is

$$\frac{\partial}{\partial x}\left(\frac{Ku}{V^2}\right) + \frac{\partial}{\partial y}\left(\frac{Kv}{V^2}\right) = 0$$

or

$$\frac{1}{V^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial}{\partial x}\left(\frac{1}{V^2}\right) + v\frac{\partial}{\partial y}\left(\frac{1}{V^2}\right) = 0$$

Using (31.01), we get

$$\vec{V}\cdot\text{grad}\left(\frac{1}{V^2}\right) = 0 \qquad (32.01)$$

which implies that velocity magnitude is constant along streamlines. Thus, we have

Theorem 1: For a steady incompressible viscous, plane flow with orthogonal magnetic and velocity field distributions, the magnetic force on a fluid element is zero if and only if velocity magnitude V is constant along streamlines.

We investigate the geometric implication of the above possibility. Taking $j = \sigma$ in (31.14), we get

$$\frac{\partial}{\partial\phi}\left(\frac{G}{J}\right) = 0 \qquad (32.02)$$

Using (31.09) in (32.02), we have

$$\frac{\partial}{\partial\phi}(\rho K G) = 0$$

or

$$G = \frac{1}{\rho K} f(\psi) \qquad (32.03)$$

where $f(\psi)$ is an arbitrary function of ψ.

From (31.09), since $F = 0$, we have

$$J = \pm\sqrt{EG} = \frac{1}{\rho K} \qquad (32.04)$$

From (32.03) and (32.04), we find that

$$E = \frac{1}{\rho K f(\psi)} \qquad (32.05)$$

Substituting the expressions for E,G given by (32.03), (32.05) in (31.12), we obtain

$$\frac{\partial^2}{\partial\psi^2}\left(\frac{1}{f(\psi)}\right) + \frac{\partial^2}{\partial\phi^2}(f(\psi)) = 0$$

or

$$\frac{1}{f(\psi)} = A\psi + B \qquad (32.06)$$

where A, B are arbitrary constants.

Using (32.06) in (32.05) and (32.03) respectively, we get

$$E = \frac{A\psi + B}{\rho K} \quad , \quad G = \frac{1}{\rho K(A\psi + B)} \tag{32.07}$$

Let α be the angle made by the tangent to the curve ψ = constant, directed in the sense of increasing , with x-axis. From (23.20), with F = 0 and using (32.07), we get

$$\alpha = \int \frac{J}{E}\left[-\frac{E}{2J^2}\frac{\partial E}{\partial \psi}\,d\phi + \frac{E}{2J^2}\frac{\partial G}{\partial \phi}\,d\psi\right]$$

$$= -\frac{A}{2}\int d\phi \tag{32.08}$$

Integration of (32.08) yields

$$\alpha = \alpha_o - \tfrac{1}{2}A\phi \tag{32.09}$$

where α_o is an arbitrary constant.

Introducing the complex variable $z = x + i\,y$, from (23.19) we have

$$z = \int e^{i\alpha}\left(\sqrt{E}\,d\phi + i\,\frac{J}{\sqrt{E}}\,d\psi\right)$$

$$= \int \exp\left[i(\alpha_o - \tfrac{1}{2}A\phi)\right]\cdot\left[\sqrt{\frac{A\psi+B}{\rho K}}\,d\phi + i\,\frac{1}{\sqrt{\rho K(A\psi+B)}}\,d\psi\right]$$

or

$$z = \begin{cases} z_o + \frac{2i}{A} \exp\left[i(\alpha_o - \frac{1}{2}A\phi)\right] \cdot \sqrt{\frac{A\psi + B}{K}} & \text{if } A \neq 0 \\ z_o + \exp(i\alpha_o) \cdot \left[\sqrt{\frac{B}{\rho K}}\,\phi + i\sqrt{\frac{1}{B\rho K}}\,\psi\right] & \text{if } A = 0 \end{cases} \qquad (32.10)$$

From (32.10) we see that the streamlines ψ = constant are either concentric circles or a family of parallel straight lines. Hence, we have

<u>Theorem 2</u>: If in a steady incompressible viscous plane orthogonal flow of infinitely conducting fluid the current density is zero, then the streamlines must be either concentric circles or parallel straight lines.

Without any loss of generality the origin can be taken at z_o and x-axis in the direction making $\alpha_o = 0$. In the case of streamlines being a family of concentric circles ($A \neq 0$), we have

$$z = \frac{2i}{A} \exp(-\tfrac{1}{2}iA\phi) \cdot \sqrt{\frac{A\psi + B}{\rho K}}$$

or

$$x = \frac{2}{A} \sin \frac{A\phi}{2} \cdot \sqrt{\frac{A\psi + B}{\rho K}}, \quad y = \frac{2}{A} \cos \frac{A\phi}{2} \cdot \sqrt{\frac{A\psi + B}{\rho K}} \qquad (32.11)$$

From (32.11), we get

$$A\psi + B = \tfrac{1}{4}A^2 \rho K r^2 \qquad (32.12)$$

and

$$\frac{A\phi}{2} = \tan^{-1}\left(\frac{x}{y}\right) = \pi/2 - \theta \qquad (32.13)$$

where θ represents the vectorial angle in the (x,y)-plane. From (32.12) and (31.08), we find that

$$u = \tfrac{1}{2}AKy \quad , \quad v = -\tfrac{1}{2}AKx \qquad (32.14)$$

Similarly, from (31.08) and (32.12), we get

$$H_1 = -\frac{\partial\phi}{\partial y} = \frac{2x}{Ar^2} \quad , \quad H_2 = \frac{\partial\phi}{\partial x} = \frac{2y}{Ar^2} \qquad (32.15)$$

Also from (31.13), we have

$$\omega = -KA \qquad (32.16)$$

Equations (31.10) and (31.11) now yield

$$\frac{\partial h}{\partial\psi} = KA \quad , \quad \frac{\partial h}{\partial\phi} = 0$$

implying that

$$h = KA\psi + \text{constant}$$

$$= \tfrac{1}{4}K^2A^2\rho r^2 + p_o \qquad (32.17)$$

where p_o is an arbitrary constant.

Since $p = h - \frac{1}{2}\rho V^2$, pressure is given by

$$p = p_o + \frac{1}{4} K^2 A^2 \rho r^2 - \frac{1}{8}\rho A^2 K^2 r^2$$

$$= p_o + \frac{1}{8}\rho A^2 K^2 r^2 .$$

Section 3. Irrotational Flows

Since $\omega = 0$, (31.13) yields

$$\frac{\partial}{\partial\psi}\left(\frac{E}{J}\right) = 0 \qquad (33.01)$$

Using (31.09) in (33.01), we get

$$\frac{\partial}{\partial\psi}(\rho KE) = 0$$

or

$$E = \frac{g(\phi)}{\rho K}\ , \qquad (33.02)$$

where $g(\phi)$ is an arbitrary function[9] of ϕ. As $EG = J^2$, from (31.09) and (33.02), we obtain

$$G = \frac{1}{\rho K g(\phi)} \qquad (33.03)$$

Substituting E, G from (33.02), (33.03) in (31.12) gives

$$\frac{\partial^2}{\partial\psi^2}(g(\phi)) + \frac{\partial^2}{\partial\phi^2}\left(\frac{1}{g(\phi)}\right) = 0$$

or

$$\frac{1}{g(\phi)} = a\phi + b \qquad (33.04)$$

where a, b are arbitrary constants.

Using (33.04) in (33.02) and (33.03), we have

$$E = \frac{1}{\rho K(a\phi + b)} \quad , \quad G = \frac{a\phi + b}{\rho K} \tag{33.05}$$

As in the previous section, these forms of E, G imply that

$$\alpha = \frac{a}{2}\psi + \alpha_o \tag{33.06}$$

where α_o is an arbitrary constant, and

$$z = \int e^{i\alpha}\left[\frac{d\phi}{\sqrt{\rho K(a\phi + b)}} + i\sqrt{\frac{a\phi + b}{\rho K}}\, d\psi\right]$$

$$= \begin{cases} z_o + \frac{2}{a}\sqrt{\frac{a\phi+b}{\rho K}}\exp\left[i(\alpha_o + \frac{a}{2}\psi)\right] & \text{if } a \neq 0 \\ z_o + \left[\sqrt{\frac{1}{\rho K b}}\,\phi + i\sqrt{\frac{b}{\rho K}}\,\psi\right]e^{i\alpha_o} & \text{if } a = 0 \end{cases}$$

(33.07)

where z_o is an arbitrary complex number.
Therefore, the streamlines ψ = constant are either concurrent straight lines or parallel straight lines and we have:

Theorem 3: If a steady incompressible viscous plane orthogonal flow of infinitely conducting fluid is irrotational then the streamlines must be either concurrent straight lines or parallel straight lines.

As before z_o and α_o can be taken to be zero. If $a \neq 0$, then from (33.07), we have

$$x = \frac{2}{a}\sqrt{\frac{a\phi+b}{\rho K}}\cos\frac{a\psi}{2} \quad , \quad y = \frac{2}{a}\sqrt{\frac{a\phi+b}{\rho K}}\sin\frac{a\psi}{2} \tag{33.08}$$

The functions ϕ and ψ are given by

$$a\phi+b = \tfrac{1}{4}\rho K a^2 r^2$$

and (33.09)

$$\psi = \frac{2}{a}\tan^{-1}(y/x) = \frac{2}{a}\theta$$

From (31.08) and (33.09), we find that

$$u = \frac{2x}{a\rho r^2} \quad , \quad v = \frac{2y}{a\rho r^2}$$

and (33.10)

$$H_1 = -\tfrac{1}{2}a\rho K y \quad , \quad H_2 = \tfrac{1}{2}a\rho K x$$

From (31.14), we have

$$j = a\rho K \tag{33.11}$$

With the help of (33.05) and (33.11), the equations (31.10) and (31.11) reduce to

$$\frac{\partial h}{\partial \phi} = -\mu a\rho K \quad , \quad \frac{\partial h}{\partial \psi} = 0$$

giving

$$h = -\mu a \rho K \phi + \text{constant}$$

$$= -\tfrac{1}{4}\mu\rho^2 K^2 a^2 r^2 + c \qquad (33.12)$$

where c is an arbitrary constant. Pressure is given by

$$p = h - \tfrac{1}{2}\rho V^2$$

$$= -\tfrac{1}{4}\mu\rho^2 K^2 a^2 r^2 - \frac{2}{\rho a^2 r^2} + c \ .$$

Section 4. Hodograph Transformation

Using (31.07) in (31.04), we get

$$\frac{\partial}{\partial x}\left(-\frac{Kv}{V^2}\right) + \frac{\partial}{\partial y}\left(\frac{Ku}{V^2}\right) = 0$$

or

$$-\frac{K}{V^2}\frac{\partial v}{\partial x} + \frac{K}{V^2}\frac{\partial u}{\partial y} + \frac{Kv}{V^4}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right) - \frac{Ku}{V^4}\left(2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}\right) = 0,$$

which simplifies to

$$(v^2-u^2)\frac{\partial u}{\partial y} + 2uv\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + (v^2-u^2)\frac{\partial v}{\partial x} = 0 \qquad (34.01)$$

Equation (31.01) can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (34.02)$$

(34.01) and (34.02) are two partial differential equations in two dependent variables u, v. This non-linear system of equations can be made linear by the hodograph transformation. We introduce u and v as independent variables and regard x, y as functions of u and v, assuming that the Jacobian

$$J = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \neq 0$$

By means of the transformation relations

$$\frac{\partial u}{\partial x} = J\frac{\partial y}{\partial v} \quad , \quad \frac{\partial u}{\partial y} = -J\frac{\partial x}{\partial v} \quad , \quad \frac{\partial v}{\partial x} = -J\frac{\partial y}{\partial u} \quad , \quad \frac{\partial v}{\partial y} = J\frac{\partial x}{\partial u}$$

equations (34.01) and (34.02) are transformed to

$$-(v^2-u^2)\frac{\partial x}{\partial v} + 2uv\left(\frac{\partial y}{\partial v} - \frac{\partial x}{\partial u}\right) - (v^2-u^2)\frac{\partial y}{\partial u} = 0 , \tag{34.03}$$

$$\frac{\partial y}{\partial v} + \frac{\partial x}{\partial u} = 0 . \tag{34.04}$$

Equation (34.04) implies that there is a function $\Psi(u,v)$ such that

$$\frac{\partial \Psi}{\partial u} = -y \quad , \quad \frac{\partial \Psi}{\partial v} = x . \tag{34.05}$$

Employing (34.05) in (34.03), we obtain

$$(v^2-u^2)\frac{\partial^2 \Psi}{\partial v^2} + 4uv\frac{\partial^2 \Psi}{\partial u \partial v} - (v^2-u^2)\frac{\partial^2 \Psi}{\partial u^2} = 0 \tag{34.06}$$

Introducing polar coordinates V,θ in the (u,v)-plane through the relations

$$u = V\cos\theta \quad , \quad v = V\sin\theta \tag{34.07}$$

we get

$$\frac{\partial V}{\partial u} = \cos\theta \quad , \quad \frac{\partial V}{\partial v} = \sin\theta$$

$$(34.08)$$

$$\frac{\partial \theta}{\partial u} = -\frac{\sin\theta}{V} \quad , \quad \frac{\partial \theta}{\partial v} = \frac{\cos\theta}{V}$$

and therefore

$$\frac{\partial \Psi}{\partial u} = \frac{\partial \Psi}{\partial V}\frac{\partial V}{\partial u} + \frac{\partial \Psi}{\partial \theta}\frac{\partial \theta}{\partial u}$$

$$= \frac{\partial \Psi}{\partial V}\cos\theta - \frac{\partial \Psi}{\partial \theta}\frac{\sin\theta}{V}$$

$$\frac{\partial \Psi}{\partial v} = \frac{\partial \Psi}{\partial V}\frac{\partial V}{\partial v} + \frac{\partial \Psi}{\partial \theta}\frac{\partial \theta}{\partial v}$$

$$= \frac{\partial \Psi}{\partial V}\sin\theta + \frac{\partial \Psi}{\partial \theta}\frac{\cos\theta}{V}$$

$$\frac{\partial^2 \Psi}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial \Psi}{\partial V}\right)\cos\theta + \frac{\partial \Psi}{\partial V}\frac{\sin^2\theta}{V} - \frac{\partial}{\partial u}\left(\frac{\partial \Psi}{\partial \theta}\right)\frac{\sin\theta}{V}$$

$$+ \frac{\partial \Psi}{\partial \theta}\left(\frac{\cos\theta\,\sin\theta}{V^2} + \frac{\sin\theta}{V^2}\cos\theta\right)$$

$$= \frac{\partial^2 \Psi}{\partial V^2}\cos^2\theta - \frac{\partial^2 \Psi}{\partial V \partial \theta}\frac{2\sin\theta\,\cos\theta}{V} + \frac{\partial^2 \Psi}{\partial \theta^2}\frac{\sin^2\theta}{V^2}$$

$$+ \frac{\partial \Psi}{\partial V}\frac{\sin^2\theta}{V} + \frac{\partial \Psi}{\partial \theta}\frac{2\sin\theta\,\cos\theta}{V^2}$$

$$\frac{\partial^2 \Psi}{\partial u \partial v} = \frac{\partial^2 \Psi}{\partial V^2}\sin\theta\,\cos\theta + \frac{\partial^2 \Psi}{\partial V \partial \theta}\left(\frac{\cos^2\theta - \sin^2\theta}{V}\right) - \frac{\partial^2 \Psi}{\partial \theta^2}\frac{\sin\theta\,\cos\theta}{V^2}$$

$$- \frac{\partial \Psi}{\partial V}\frac{\sin\theta\,\cos\theta}{V} - \frac{\partial \Psi}{\partial \theta}\left(\frac{\cos^2\theta - \sin^2\theta}{V^2}\right)$$

$$\frac{\partial^2 \Psi}{\partial v^2} = \frac{\partial^2 \Psi}{\partial V^2}\sin^2\theta + \frac{\partial^2 \Psi}{\partial V \partial \theta}\frac{2\sin\theta\,\cos\theta}{V} + \frac{\partial^2 \Psi}{\partial \theta^2}\frac{\cos^2\theta}{V^2}$$

$$+ \frac{\partial \Psi}{\partial V}\frac{\cos^2\theta}{V} - \frac{\partial \Psi}{\partial \theta}\frac{2\sin\theta\,\cos\theta}{V^2}$$

In terms of V, θ as independent variables (34.06) becomes

$$V^2\cos 2\theta \left\{\frac{\partial^2\Psi}{\partial V^2}\cos^2\theta - \frac{\partial^2\Psi}{\partial V\partial\theta}\frac{\sin 2\theta}{V} + \frac{\partial^2\Psi}{\partial\theta^2}\frac{\sin^2\theta}{V^2} + \frac{\partial\Psi}{\partial V}\frac{\sin^2\theta}{V}\right.$$

$$\left. + \frac{\partial\Psi}{\partial\theta}\frac{\sin 2\theta}{V^2}\right\} + V^2\sin 2\theta \left\{\frac{\partial^2\Psi}{\partial V^2}\sin 2\theta + 2\frac{\partial^2\Psi}{\partial V\partial\theta}\frac{\cos 2\theta}{V}\right.$$

$$\left. - \frac{\partial^2\Psi}{\partial\theta^2}\frac{\sin 2\theta}{V^2} - \frac{\partial\Psi}{\partial V}\frac{\sin 2\theta}{V} - 2\frac{\partial\Psi}{\partial\theta}\frac{\cos 2\theta}{V^2}\right\} - V^2\cos 2\theta \left\{\frac{\partial^2\Psi}{\partial V^2}\sin^2\theta\right.$$

$$\left. + \frac{\partial^2\Psi}{\partial V\partial\theta}\frac{\sin 2\theta}{V} + \frac{\partial^2\Psi}{\partial\theta^2}\frac{\cos^2\theta}{V^2} + \frac{\partial\Psi}{\partial V}\frac{\cos^2\theta}{V} - \frac{\partial\Psi}{\partial\theta}\frac{\sin 2\theta}{V^2}\right\} = 0$$

which simplifies to

$$\frac{\partial^2\Psi}{\partial V^2} - \frac{1}{V^2}\frac{\partial^2\Psi}{\partial\theta^2} - \frac{1}{V}\frac{\partial\Psi}{\partial V} = 0 \qquad (34.09)$$

Knowing a solution $\Psi(V, \theta)$ of (34.09), from (34.05) we have

$$x = \frac{\partial\Psi}{\partial v}, \quad y = -\frac{\partial\Psi}{\partial u} \qquad (34.10)$$

where $u = V\cos\theta$, $v = V\sin\theta$

From these equations we can find u,v as functions of x,y provided that

$$\frac{\partial(x, y)}{\partial(u, v)} \neq 0$$

However, the velocity field thus obtained must satisfy the integrability condition for pressure, in which case p can be determined from momentum equations. The magnetic field vector $\vec{H}$ is given by (31.07).

In the next section we consider some examples.

Section 5. Some Particular Solutions

(1) Radial Flow: A simple solution of (34.09) is given by

$$\Psi = k_o \theta = k_o \tan^{-1} \left(\frac{v}{u} \right) \tag{35.01}$$

where k_o is a positive constant.

From (34.10), we get

$$x = \frac{\partial \Psi}{\partial v} = \frac{k_o u}{V^2}$$

(35.02)

$$y = - \frac{\partial \Psi}{\partial u} = \frac{k_o v}{V^2}$$

Equations (35.02) yield

$$x^2 + y^2 = \frac{k_o^2}{V^2}$$

or

$$V = \frac{k_o}{r}$$

where $r^2 = x^2 + y^2$

Therefore

$$u = \frac{V^2}{k_o} x = \frac{k_o x}{r^2} ,$$

(35.03)

$$v = \frac{V^2 y}{k_o} = \frac{k_o y}{r^2}$$

Equations (35.03) represent a purely radial flow. The magnetic field is given by

$$\vec{H} = (-\frac{Kv}{V^2} , \frac{Ku}{V^2})$$

$$= (- \frac{Ky}{k_o} , \frac{Kx}{k_o}) \qquad (35.04)$$

The expressions for $\vec{V}$ and $\vec{H}$ thus obtained must satisfy the momentum equations (31.02) which may be written as

$$\eta \frac{\partial \omega}{\partial y} - \rho\omega v + \mu j H_2 = - \frac{\partial h}{\partial x} \qquad (35.05)$$

and

$$\eta \frac{\partial \omega}{\partial x} - \rho\omega u + \mu j H_1 = \frac{\partial h}{\partial y} \qquad (35.06)$$

From definitions of ω and j, we find

$$\omega = - \frac{k_o y}{r^4} 2 x + \frac{k_o x}{r^4} 2 y = 0 \qquad (35.07)$$

$$j = \frac{2K}{k_o} \qquad (35.08)$$

Using (35.03), (35.04), (35.07) and (35.08) in (35.05) and (35.06) respectively, we obtain

$$\frac{\partial h}{\partial x} = - \frac{2\mu K^2 x}{k_o^{\,2}} , \quad \frac{\partial h}{\partial y} = - \frac{2\mu K^2 y}{k_o^{\,2}} \qquad (35.09)$$

From (35.09), on integration, we get

$$h = \overset{*}{C} - \frac{\mu K^2 r^2}{k_o^{\,2}} \quad ,$$

and the pressure is given by

$$p = C - \frac{\rho k_o^{\,2}}{2r^2} - \frac{\mu K^2 r^2}{k_o^{\,2}} \quad .$$

(2) Vortex Flow: If we assume that $\Psi = \Psi$ (V), then (34.09) gives

$$\frac{\Psi''}{\Psi'} = \frac{1}{V}$$

which yields

$$\Psi' = k_1 V \qquad (35.10)$$

where k_1 is an arbitrary positive constant.

On integrating (35.10), we get

$$\Psi = \frac{1}{2} k_1 \ (u^2 + v^2) + k_2 \qquad (35.11)$$

From (34.10), we have

$$x = \frac{\partial \Psi}{\partial v} = k_1 v$$
$$y = - \frac{\partial \Psi}{\partial u} = - k_1 u \qquad (35.12)$$

The velocity field is given by

$$u = -\frac{y}{k_1} \quad , \qquad v = \frac{x}{k_1} \tag{35.13}$$

These relations represent a circulatory flow with constant angular velocity $1/k_1$.

From (31.07) magnetic field vector $\vec{H}$ is given by

$$H_1 = -\frac{Kk_1 x}{r^2} \quad , \qquad H_2 = -\frac{Kk_1 y}{r^2} \tag{35.14}$$

Vorticity ω and current density j are given by

$$\omega = \frac{2}{K_1}$$

$$j = -\frac{2Kk_1 yx}{r^4} + \frac{2Kk_1 xy}{r^4} = 0 \tag{35.15}$$

Using (35.15) in the momentum equations (35.05) and (35.06), we obtain

$$\frac{\partial h}{\partial x} = \frac{2\rho x}{k_1^{\,2}}$$

$$\frac{\partial h}{\partial y} = \frac{2\rho y}{k_1^{\,2}}.$$

which imply that

$$h = \frac{\rho r^2}{k_1^2} + D$$

where D is an arbitrary constant. The pressure is given by

$$p = h - \frac{1}{2} \rho V^2 = \frac{\rho r^2}{2k_1^2} + D . \qquad (35.16)$$

(3) Spiral Flows: Superimposing the above two solutions of (34.09) we consider

$$\Psi = k_o \theta + \frac{1}{2} k_1 V^2 + k_2 \qquad (35.17)$$

From (34.10), we have

$$x = k_o \frac{u}{u^2 + v^2} + k_1 v$$

$$(35.18)$$

$$y = k_o \frac{v}{u^2 + v^2} - k_1 u$$

The Jacobian

$$\frac{\partial (x,y)}{\partial (u,v)} = \{\frac{k_o}{u^2+v^2} - \frac{2k_o u^2}{(u^2+v^2)^2}\} \{\frac{k_o}{u^2+v^2} - \frac{2k_o v^2}{(u^2+v^2)^2}\}$$

$$- \{\frac{2k_o uv}{(u^2+v^2)^2} + k_1\} \{\frac{2k_o uv}{(u^2+v^2)^2} - k_1\} = k_1^2 - \frac{k_o^2}{(u^2+v^2)^2} ,$$

vanishes when $V^2 = \frac{k_o}{k_1}$.

From (35.18), we obtain

$$r^2 = \frac{k_o^2}{v^2} + k_1^2 v^2$$

which gives

$$v^2 = \frac{r^2 \pm \sqrt{r^4 - 4k_o^2 k_1^2}}{2k_1^2} \qquad (35.19)$$

in the region $x^2 + y^2 > 2k_o k_1$.

From (35.19) we see that two different flows are possible, one with

$$v^2 = \frac{r^2 + \sqrt{r^4 - 4k_o^2 k_1^2}}{2k_1^2}$$

and the other with

$$v^2 = \frac{r^2 - \sqrt{r^4 - 4k_o^2 k_1^2}}{2k_1^2}$$

Equations (35.18) can be rewritten as

$$\frac{k_o}{v^2} u + k_1 v = x$$

$$- k_1 u + \frac{k_o}{v^2} v = y$$

On solving for u, we get

$$(k_1^{\,2} + \frac{k_o^{\,2}}{V^4}) \; u = \frac{k_o x}{V^2} - k_1 \, y$$

or

$$u = k_o \frac{x}{r^2} - k_1 \frac{V^2 y}{r^2} \qquad (35.20a)$$

and similarly,

$$v = k_o \frac{y}{r^2} + k_1 \frac{V^2 x}{r^2} \, , \qquad (35.20b)$$

where V is given by (35.19).

If (r, β) denote the polar co-ordinates in the (x,y)-plane and v_r, v_β represent the velocity components in the direction of increasing r and β respectively, at a point (r, β) then from (35.20), we have

$$v_r = \frac{k_o}{r} \; , \qquad v_\beta = k_1 \frac{V^2}{r} \qquad (35.21)$$

and the magnetic field vector $\vec{H}$, being orthogonal to the velocity vector, is given by

$$H_r = - \frac{K v_\beta}{V^2} = - \frac{K k_1}{r} \; ,$$

$$H_\beta = \frac{K v_r}{V^2} = \frac{K k_o}{r V^2} \; . \qquad (35.22)$$

Vorticity ω is given by

$$\omega = \frac{1}{r} \{ \frac{\partial}{\partial \beta} (r \, v_\beta) - \frac{\partial}{\partial \beta} (v_r) \}$$

$$= \frac{1}{r} \{ k_1 \frac{\partial}{\partial r} (V^2) \}$$

$$= \frac{1}{k_1} \{ 1 \pm \frac{r^2}{\sqrt{r^4 - 4 \, k_o^2 \, k_1^2}} \} \qquad (35.23)$$

Similarly, current density j is given by

$$j = \frac{1}{r} \{ \frac{\partial}{\partial r} (r \, H_\beta) - \frac{\partial}{\partial \beta} (H_r) \}$$

$$= \frac{K k_o}{r} \frac{\partial}{\partial r} (\frac{1}{V^2})$$

$$= \frac{K}{k_o} \{ 1 \mp \frac{r^2}{\sqrt{r^4 - 4 \, k_o^2 \, k_1^2}} \} \qquad (35.24)$$

In terms of polar co-ordinates, momentum equations (35.05) and (35.06) become

$$\frac{\partial h}{\partial r} = -\eta \frac{1}{r} \frac{\partial \omega}{\partial \beta} + \rho \omega v_\beta - \mu K j \frac{v_r}{V^2} , \qquad (35.25)$$

$$\frac{1}{r} \frac{\partial h}{\partial \beta} = \eta \frac{\partial \omega}{\partial r} - \rho \omega v_r - \mu K j \frac{v_\beta}{V^2} \qquad (35.26)$$

Using (35.21), (35.23) and (35.24) in (35.25) and (35.26) respectively, we get

$$\frac{\partial h}{\partial r} = \frac{\rho V^2}{r}\{1 \pm \frac{r^2}{(r^4-4k_o{}^2k_1{}^2)^{\frac{1}{2}}}\} - \frac{\mu K^2}{rV^2}\{1 \mp \frac{r^2}{(r^4-4k_o{}^2k_1{}^2)^{\frac{1}{2}}}\} \quad (35.27)$$

and

$$\frac{\partial h}{\partial \beta} = \mp 8\eta k_o{}^2 k_1 \frac{r^2}{(r^4-4K_o{}^2k_1{}^2)^{3/2}} - \frac{\rho k_o}{k_1}\{1 \pm \frac{r^2}{(r^4-4k_o{}^2k_1{}^2)^{\frac{1}{2}}}\}$$

$$- \frac{\mu K^2 k_1}{k_o}\{1 \mp \frac{r^2}{(r^4-4k_o{}^2k_1{}^2)^{\frac{1}{2}}}\} \quad (35.28)$$

From the integrability condition, $\frac{\partial^2 h}{\partial r \partial \beta} = \frac{\partial^2 h}{\partial \beta \partial r}$, for h, we get

$$\frac{\partial}{\partial r}\left[- \frac{\rho k_o}{k_1} - \frac{\mu K^2 k_1}{k_o} \mp \left(\frac{\rho k_o}{k_1} - \frac{\mu K^2 k_1}{k_o}\right) \frac{r^2}{(r^4-4K_o{}^2k_1{}^2)^{\frac{1}{2}}} \mp 8\eta\, k_o{}^2 k_1 \frac{r^2}{(r^4-4k_o{}^2k_1{}^2)^{3/2}}\right] = 0$$

or

$$\left(\frac{\rho k_o}{k_1} - \frac{\mu K^2 k_1}{k_o}\right)(r^4-4k_o{}^2k_1{}^2) + \frac{4\eta}{k_1}(r^4+2k_o{}^2k_1{}^2) = 0 \quad (35.29)$$

As (35.29) must hold for all values of r, we have

$$\frac{4\eta}{k_1} + \frac{\rho k_o{}^2}{k_1} - \frac{\mu K^2 k_1}{k_o} = 0 \quad ,$$

$$\frac{2\eta}{k_1} - \frac{\rho k_o{}^2}{k_1} + \frac{\mu K^2 k_1}{k_o} = 0$$

which require that

$$\eta = 0, \qquad \frac{k_o{}^2}{k_1{}^2} = \frac{\mu K^2}{\rho} \quad (35.30)$$

Using (35.30) in (35.27) and (35.28), we get

$$\frac{\partial h}{\partial r} = \pm \frac{\rho}{2k_1^{\,2}} \left\{ \frac{2r^3}{(r^4-4k_o^{\,2}k_1^{\,2})^{\frac{1}{2}}} + \frac{(r^4-4k_o^{\,2}k_1^{\,2})^{\frac{1}{2}}}{r^2} \cdot 2r \right\}$$

and

$$\frac{\partial h}{\partial \beta} = - \frac{2\rho k_o}{k_1}$$

which imply that

$$h = \pm \frac{\rho}{2k_1^{\,2}} \left\{ 2\sqrt{r^4-4k_o^{\,2}k_1^{\,2}} - 2k_o k_1 \cos^{-1} \frac{2k_o k_1}{r^2} \right\} - \frac{2\rho k_o}{k_1} \beta + C \qquad (35.31)$$

where C is an arbitrary constant.

Pressure is given by

$$\begin{aligned} p &= h - \frac{1}{2}\rho V^2 \\ &= \pm \frac{\rho}{4k_1^{\,2}} \left\{ 3\sqrt{r^4-4k_o^{\,2}k_1^{\,2}} - 4k_o k_1 \cos^{-1} \frac{2k_o k_1}{r^2} \right\} - \frac{\rho r^2}{4k_1^{\,2}} \\ &\quad - \frac{2\rho k_o}{k_1} \beta + C . \end{aligned}$$

It is interesting to note from (35.30) that in order to have a sprial flow, the fluid must be inviscid. Kingston and Talbot (1969) obtained such spiral flows while classifying the possible plane orthogonal flows for an inviscid incompressible fluid.

CHAPTER IV

TRANSVERSE VISCOUS FLOWS

Magnetofluid dynamic flows with magnetic field acting in a fixed direction, referred as transverse flows, have been studied by several authors in the past few years. H. Grad (1960) considered transverse flows of inviscid compressible fluids and derived two integrals. R. M. Gunderson (1966,1969) studied these flows using the method of charecteristics. O. P. Chandna (1972) obtained a compatibility equation for such flows and used it to obtain particular solutions. Nath and Chandna (1973) developed a substitution principle for these flows and in another paper Chandna, Smith and Nath (1975) considered transverse flow through a logarithmic channel.

In this chapter, we consider the transverse flows of viscous incompressible fluids having finite electrical conductivity and obtain the most general velocity field consistent with the transverse magnetic field. For the study of plane transverse flows, with the magnetic field perpendicular to the plane of flow, we employ natural streamline co-ordinates and recast the governing system of equations in terms of these co-ordinates. Using the transformed system, we determine all possible flows for which the streamlines are (a) straight lines, (b) involutes of a plane curve and (c) isometric.

Section 1. Flow Equations.

The steady flow of a viscous incompressible fluid of electrical conductivity σ in the presence of magnetic field is governed by the following system of equations

$$\operatorname{div} \vec{V} = 0 . \qquad (41.01)$$

$$\rho(\vec{V}.\ \operatorname{grad})\ \vec{V} + \operatorname{grad} p = \eta\nabla^2\vec{V} + \mu(\operatorname{curl}\ \vec{H})\times\vec{H} \qquad (41.02)$$

$$\operatorname{curl}\ (\vec{V} \times \vec{H}) + \frac{1}{\mu\sigma}\nabla^2\ \vec{H} = \vec{0} \qquad (41.03)$$

where the magnetic field vector is solenoidal , i.e.

$$\operatorname{div} \vec{H} = 0 \qquad (41.04)$$

Throughout this chapter, we consider the magnetic field to be acting in a constant direction and without loss of generality we may take $\vec{H}$ parallel to z - axis.

If x, y, z are the spatial coordinates and $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit vectors along x, y and z - axis respectively then we may write

$$\vec{H} = H\,\vec{k}, \qquad \vec{V} = (u,\ v,\ w) \qquad (41.05)$$

and from (41.04), we have

$$\frac{\partial H}{\partial z} = 0 \quad \text{or} \quad H = H(x,\ y) \qquad (41.06)$$

Substitution of (41.05) in (41.03) yields

$$\frac{\partial}{\partial z}(u\ H) = \frac{\partial}{\partial z}(v\ H) = 0 \qquad (41.07)$$

and

$$\frac{\partial}{\partial x}(u\ H) + \frac{\partial}{\partial y}(v\ H) = \frac{1}{\mu\sigma}\ \nabla^2 H \qquad (41.08)$$

Using (41.06) in (41.07) and (41.08), we get

$$\frac{\partial u}{\partial z} = 0 , \qquad \frac{\partial v}{\partial z} = 0 . \qquad (41.09)$$

and

$$\frac{\partial}{\partial x}(uH) + \frac{\partial}{\partial y}(vH) = \frac{1}{\mu\sigma}\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right). \qquad (41.10)$$

From the continuity equation (41.01), we have

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \qquad (41.11)$$

Differentiating (41.11) with respect to z and using (41.09) yields

$$\frac{\partial^2 w}{\partial z^2} = 0 \qquad (41.12)$$

Equations (41.11) and (41.12) imply that

$$w = f(x, y) - z\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

where f(x, y) is an arbitrary function of x, y.

Thus the most general velocity field which is consistent with the transverse magnetic field is given by

$$u = u(x, y), \qquad v = v(x, y)$$
$$w = f(x, y) - z\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (41.13)$$

Using (41.01) and (41.06) in (41.02), we obtain

$$\rho(\text{curl } \vec{V}) \times \vec{V} + \frac{1}{2}\rho \text{ grad } V^2 + \text{grad } p$$

$$= -\eta \text{ curl}(\text{curl } \vec{V}) - \frac{1}{2}\mu \text{ grad } H^2,$$

and introducing vorticity vector $\vec{\omega} = \text{curl } \vec{V}$, we get

$$\rho\vec{\omega} \times \vec{V} + \text{grad}(p + \frac{1}{2}\mu H^2 + \frac{1}{2}\rho V^2) + \eta \text{ curl } \vec{\omega} = \vec{0} \qquad (41.14)$$

The velocity field (41.13) must satisfy the dynamical equations (41.14) which can be interpreted as integrability conditions for $(p+\frac{1}{2}\mu H^2+\frac{1}{2}\rho V^2)$.

For example, let us consider a plane flow in the (y, z)-plane with all the variables independent of x. In this case

$$u = 0, \quad v = v(y), \quad w = f(y)-z\,v'(y)$$

and

$$H = H(y) \tag{41.15}$$

Substituting (41.15) in (41.14), we get

$$\rho(-f\,f' + zf'v' + zfv'' - z^2v'v'') + \frac{\partial}{\partial y}(p + \frac{1}{2}\mu H^2 + \frac{1}{2}\rho V^2) - \eta v'' = 0 \tag{41.16a}$$

and

$$\rho(vf' - zvv'') + \frac{\partial}{\partial z}(p + \frac{1}{2}\mu H^2 + \frac{1}{2}\rho V^2) + \eta(zv''' - f'') = 0 \tag{41.16b}$$

Integrability condition for $(p+\frac{1}{2}\mu H^2+\frac{1}{2}\rho V^2)$ is

$$\rho(f'v' + fv'' - 2zv'v'' - v'f' - vf'' + zvv''' + zv'v'') = \eta(zv^{(iv)} - f''')$$

which requires that

$$\rho(vv''' - v'v'') = \eta v^{(iv)}, \quad f''' = 0$$

or

$$\rho\left[vv'' - (v')^2\right] - \eta v''' = c_1,$$
$$f'' = c_2 \tag{41.17}$$

where c_1 and c_2 are arbitrary constants.

Having determined the velocity field from (41.17), we obtain H from (41.10),

$$\frac{d}{dy}(vH) = \frac{1}{\mu\sigma}\frac{d^2H}{dy^2}$$

which on integration gives

$$\mu\sigma\, vH + k_1 = \frac{dH}{dy} \quad .$$

Integrating once more, we get

$$H = \exp(\mu\sigma\int v\, dy)\cdot\left[k_1\int \exp(-\mu\sigma\int v\, dy)\, dy + k_2\right] \qquad (41.18)$$

where k_1 and k_2 are arbitrary constants.

Section 2. Plane Flows with $\vec{H}$ Perpendicular to the Plane of Flow.

In this section and the subsequent sections, we consider plane flows in the (x,y) -plane. Let the family of curves $\psi(x,y)$ = constant represent the streamlines and $\phi(x,y)$ = constant be their orthogonal trajectories. In the orthogonal curvilinear co-ordinate system (ϕ,ψ), the squared element of arc length for the (x,y)-plane is given by

$$ds^2 = E(\phi,\psi)\ d\phi^2 + G(\phi,\psi)\ d\psi^2 \qquad (42.01)$$

where the coefficients E, G must satisfy the Gauss equation

$$\frac{\partial}{\partial\phi}\left(\frac{1}{\sqrt{E}}\frac{\partial}{\partial\phi}\sqrt{G}\right) + \frac{\partial}{\partial\psi}\left(\frac{1}{\sqrt{G}}\frac{\partial}{\partial\psi}\sqrt{E}\right) = 0\ . \qquad (41.02)$$

In terms of the streamline co-ordinates (ϕ,ψ), the system of equations (41.01), (41.10) and (41.14) representing the transverse flow is replaced by

$$\frac{\partial}{\partial\phi}(V\sqrt{G}) = 0\ , \qquad (42.03)$$

$$\sqrt{G}\ V\frac{\partial H}{\partial\phi} - \frac{1}{\mu\sigma}\left[\frac{\partial}{\partial\phi}\left(\sqrt{\frac{G}{E}}\frac{\partial H}{\partial\phi}\right) + \frac{\partial}{\partial\psi}\left(\sqrt{\frac{E}{G}}\frac{\partial H}{\partial\psi}\right)\right] = 0 \qquad (42.04)$$

$$\omega = -\frac{1}{\sqrt{EG}}\frac{\partial}{\partial\psi}(\sqrt{E}\ V) \qquad (42.05)$$

$$\rho V\frac{\partial V}{\partial\phi} + \frac{\partial}{\partial\phi}\left(p + \frac{1}{2}\mu H^2\right) + \eta\sqrt{\frac{E}{G}}\frac{\partial\omega}{\partial\psi} = 0 \qquad (42.06)$$

$$\rho\frac{V^2}{\sqrt{E}}\ \frac{\partial}{\partial\psi}\sqrt{E} - \frac{\partial}{\partial\psi}\left(p + \frac{1}{2}\mu H^2\right) + \eta\sqrt{\frac{G}{E}}\frac{\partial\omega}{\partial\phi} = 0 \qquad (42.07)$$

where $\vec{\omega} = \omega\vec{k}$.

Equations (42.02) to (42.07) constitute a system of six partial differential equations with six dependent variables, namely, V, E, G, p, H and ω.

Eliminating $(p+\frac{1}{2}\mu H^2)$ from (42.06) and (42.07), we get

$$\eta\left[\frac{\partial}{\partial\psi}\left(\sqrt{\frac{E}{G}}\frac{\partial\omega}{\partial\psi}\right)+\frac{\partial}{\partial\phi}\left(\sqrt{\frac{G}{E}}\frac{\partial\omega}{\partial\phi}\right)\right]+\rho\left[\frac{\partial}{\partial\psi}\left(V\frac{\partial V}{\partial\phi}\right)+\frac{\partial}{\partial\phi}\left(\frac{V^2}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial\psi}\right)\right]=0$$

or

$$\eta\left[\frac{\partial}{\partial\psi}\left(\sqrt{\frac{E}{G}}\frac{\partial\omega}{\partial\psi}\right)+\frac{\partial}{\partial\phi}\left(\sqrt{\frac{G}{E}}\frac{\partial\omega}{\partial\phi}\right)\right]+\rho\frac{\partial}{\partial\phi}\left[\frac{V}{\sqrt{E}}\frac{\partial}{\partial\psi}(\sqrt{E}\,V)\right]=0$$

or

$$\eta\left[\frac{\partial}{\partial\psi}\left(\sqrt{\frac{E}{G}}\frac{\partial\omega}{\partial\psi}\right)+\frac{\partial}{\partial\phi}\left(\sqrt{\frac{G}{E}}\frac{\partial\omega}{\partial\phi}\right)\right]-\rho\frac{\partial}{\partial\phi}(\sqrt{G}\,V\omega)=0,$$

using (42.03), we get

$$\eta\left[\frac{\partial}{\partial\psi}\left(\sqrt{\frac{E}{G}}\frac{\partial\omega}{\partial\psi}\right)+\frac{\partial}{\partial\phi}\left(\sqrt{\frac{G}{E}}\frac{\partial\omega}{\partial\phi}\right)\right]-\rho\sqrt{G}\,V\frac{\partial\omega}{\partial\phi}=0. \qquad (42.08)$$

Equations (42.02), (42.03) and (42.08) with ω given by (42.05) form a system of three non-linear partial differential equations in three unknowns V, E and G. Knowing a solution $V = V(\phi,\psi)$, $E = E(\phi,\psi)$, $G = G(\phi,\psi)$ of this system of equations, we can find H from (42.04) and then pressure can be determined from (42.06) and (42.07).

Section 3. Straight Streamlines

Among the familiar flow patterns of a plane flow having straight streamlines are the parallel flows and radial flows. In this section we investigate all possible flows with straight streamlines. We assume that streamlines are not parallel and envelope a curve C. The tangent lines to C and the involutes of C form an orthogonal curvilinear net for which the squared element of arc length is given by (see Appendix A)

$$ds^2 = d\phi^2 + \left[\phi - \sigma(\psi)\right]^2 d\psi^2 \qquad (43.01)$$

where ϕ is the parameter constant along each individual involute, σ denotes the arc length along C from some fixed point, and ψ denotes the angle of inclination of the tangent line to C with the x-axis. The function $\sigma = \sigma(\psi)$ depends on the curve C. Clearly the curves ϕ = constant are the involutes of C and the curves ψ = constant are its tangent lines.

From (43.01) and (42.01), we get

$$E = 1, \qquad G = \left[\phi - \sigma(\psi)\right]^2 \qquad (43.02)$$

Substituting for E, G from (43.02) in the Gauss equation (42.02) we obtain

$$\frac{\partial^2}{\partial\phi^2}\left[\phi - \sigma(\psi)\right] = 0,$$

which is identically satisfied. From (42.03) and (43.02), we have

$$V = g(\psi) \; / \; \left[\phi - \sigma(\psi)\right] \qquad (43.03)$$

where $g(\psi)$ is an arbitrary function of ψ.

Using (43.02), (43.03) in vorticity equation (42.05), we get

$$\omega = -\frac{1}{\phi - \sigma(\psi)} \frac{\partial}{\partial \psi}\left[g(\psi)/\{\phi - \sigma(\psi)\}\right]$$

$$= -\frac{1}{\{\phi - \sigma(\psi)\}^2}\left[g'(\psi) + \frac{g(\psi) . \sigma'(\psi)}{\phi - \sigma(\psi)}\right] \qquad (43.04)$$

Eq. (42.08), with E and G given by (43.02), yields

$$\eta\frac{\partial}{\partial\psi}\left[\frac{1}{\phi-\sigma(\psi)} \frac{\partial\omega}{\partial\psi}\right] + \eta\frac{\partial}{\partial\phi}\left[(\phi-\sigma(\psi)) \frac{\partial\omega}{\partial\phi}\right] - \rho(\phi-\sigma(\psi))\cdot V\frac{\partial\omega}{\partial\phi} = 0. \qquad (43.05)$$

Eliminating V and ω between (43.03), (43.04) and (43.05), we obtain

$$- \eta\frac{\partial}{\partial\psi}\left[\frac{1}{(\phi-\sigma)^5}\left\{g(\psi)\sigma''(\psi)(\phi-\sigma) + g''(\psi)(\phi-\sigma)^2 + 3g'(\psi)\sigma'(\psi)\cdot(\phi-\sigma) + 3g(\psi)\sigma'^2(\psi)\right\}\right] + \eta\frac{\partial}{\partial\phi}\left[\frac{1}{(\phi-\sigma)^3}\left\{2g'(\psi)(\phi-\sigma) + 3\, g(\psi)\sigma'(\psi)\right\}\right] - \frac{\rho g(\psi)}{(\phi-\sigma)^4}\left[2\, g'(\psi)\;(\phi-\sigma) + 3g(\psi)\sigma'(\psi)\right] = 0,$$

which simplifies to

$$- 15\eta\, g\, \sigma'^3 - (\phi-\sigma)(10\eta\, g\, \sigma'\, \sigma'' + 15\eta\, g'\sigma'^2)$$
$$- (\phi-\sigma)^2(6\eta\, g''\sigma' + 4\eta\, g'\sigma'' + \eta\, g\, \sigma''' + 9\eta\, g\, \sigma' + 3\rho g^2\sigma')$$
$$- (\phi-\sigma)^3(\eta\, g''' + 4\eta\, g' + 2\rho\, g\, g') = 0. \qquad (43.06)$$

Since ϕ,ψ are independent variables, the identity (43.06) can hold only if all coefficients vanish identically. In particular this requires that

$$15\eta\, g(\psi)\ \sigma'^3(\psi) = 0.$$

As $g(\psi)$ cannot vanish identically, we see that

$$\sigma'(\psi) = 0.$$

But $\sigma'(\psi)$ represents the radius of curvature of C and therefore C must reduce to a point and streamlines are concurrent straight lines.

We therefore have:

<u>Theorem 1</u>. If the streamlines in a steady plane transverse flow of a viscous electrically conducting fluid are straight lines they must be either concurrent or parallel.

Section 4. Streamlines are involutes of a curve C

In this section we investigate the geometric implication of prescribing the streamlines to be involutes of a curve. We consider the orthogonal net formed by the streamlines, the involutes of C, and their orthogonal trajectories, the tangent lines to C. Let ψ denotes the parameter which is constant along each individual involute, σ the arc length along C from a fixed point on C, and ϕ the angle of inclination of the tangent lines to C with the x-axis. The streamlines are represented by ψ = constant and their orthogonal trajectories, the tangent lines to C, by ϕ = constant. The squared element of arc length in this (ϕ,ψ) net is given by

$$ds^2 = d\psi^2 + \{\psi-\sigma(\phi)\}^2\, d\phi^2 \tag{44.01}$$

On comparing (44.01) with (42.01), we obtain

$$E = \{\psi-\sigma(\phi)\}^2, \qquad G = 1. \tag{44.02}$$

From these expressions for E and G, we see that the Gauss equation is identically satisfied.
Substituting for E, G from (44.02) into (42.03) and (42.05), we get

$$V = V(\psi) \tag{44.03}$$

and

$$\begin{aligned}\omega &= -\frac{1}{\{\psi-\sigma(\phi)\}}\frac{\partial}{\partial\psi}\Big[\{\psi-\sigma(\phi)\}V(\psi)\Big] \\ &= -\Big[V'(\psi) + V(\psi)/\{\psi-\sigma(\phi)\}\Big]\end{aligned} \tag{44.04}$$

From (44.02) and (42.08), we obtain

$$\eta \frac{\partial}{\partial\psi}\left[\{\psi-\sigma(\phi)\}\frac{\partial\omega}{\partial\psi}\right] + \eta\frac{\partial}{\partial\phi}\left[\frac{1}{\psi-\sigma(\phi)}\frac{\partial\omega}{\partial\phi}\right] - \rho\, V(\psi)\frac{\partial\omega}{\partial\phi} = 0.$$

Using (44.04) to eliminate ω, we get

$$\eta\frac{\partial}{\partial\psi}\left[\{\psi-\sigma(\phi)\}\, V''(\psi) + V'(\psi) - V(\psi)/\{\psi-\sigma(\phi)\}\right]$$

$$- \eta V(\psi)\frac{\partial}{\partial\phi}\left[\sigma'(\phi)/\{\psi-\sigma(\phi)\}^3\right] + \rho V^2(\psi)\sigma'(\phi)/\{\psi-\sigma(\phi)\}^2 = 0,$$

or

$$3\eta\, V(\psi)\sigma'^2(\phi) + 3\eta V(\psi)\sigma''(\phi)\{\psi-\sigma(\phi)\}$$
$$+ \{\eta V(\psi) - \rho V^2(\psi)\sigma'(\phi)\}\,\{\psi-\sigma(\phi)\}^2 - \eta V'(\psi)\{\psi-\sigma(\phi)\}^3$$
$$+ 2\eta\, V''(\psi)\{\psi-\sigma(\phi)\}^4 + \eta V'''(\psi)\{\psi-\sigma(\phi)\}^5 = 0. \quad (44.05)$$

For the relation (44.05) to hold identically, it must hold on the curve C whose equation is $\psi = \sigma(\phi)$ and therefore

$$3\eta\, V(\psi)\,\sigma'^2(\phi) = 0.$$

As $V(\psi)$ is not identically zero, the radius of curvature $\sigma'(\phi)$ of C must be zero and the curve C reduces to a point. Therefore the stream-lines must be concentric circles with this point as the common centre.

Thus, we have:

Theorem 2. If the stream-lines in a steady plane transverse flow of a viscous ellectrically conducting fluid are involutes of a curve then the stream-lines must be concentric circles.

Section 5. Flows with Isometric Geometry

In this section we study the implication of requiring that the streamlines ψ = constant and their orthogonal trajectories ϕ = constant form an isometric net, so that E and G are everywhere equal. Let

$$E = G = g^2(\phi,\psi) \tag{45.01}$$

The Gauss equation (42.02), in this case, reduces to

$$\frac{\partial^2}{\partial\phi^2}\ln g + \frac{\partial^2}{\partial\psi^2}\ln g = 0. \tag{45.02}$$

From (42.03), we get

$$g\,V = f(\psi) \tag{45.03}$$

where f is an arbitrary function of ψ. Using (45.03) in (42.05), yields

$$\omega = -\frac{1}{g^2}\,f'(\psi) \tag{45.04}$$

From (45.01) and (42.08), we have

$$\eta\left(\frac{\partial^2\omega}{\partial\psi^2} + \frac{\partial^2\omega}{\partial\phi^2}\right) - \rho g V\,\frac{\partial\omega}{\partial\phi} = 0. \tag{45.05}$$

Using (45.03) and (45.04) in (45.05), we obtain

$$\begin{aligned} &-\frac{4\eta}{g^2}\,f'(\psi)\left[\left(\frac{\partial}{\partial\phi}\ln g\right)^2 + \left(\frac{\partial}{\partial\psi}\ln g\right)^2\right] \\ &+\frac{2\eta}{g^2}\,f'(\psi)\left[\frac{\partial^2}{\partial\phi^2}\ln g + \frac{\partial^2}{\partial\psi^2}\ln g\right] + \frac{4\eta}{g^2}\,f''(\psi)\,\frac{\partial}{\partial\psi}\ln g \\ &-\frac{\eta}{g^2}\,f'''(\psi) - \frac{2\rho}{g^2}\,f(\psi)f'(\psi)\,\frac{\partial}{\partial\phi}\ln g = 0\,. \end{aligned}$$

Making use of (45.02), we get

$$- \frac{4\eta}{g^2} f'(\psi) \left[(\frac{\partial}{\partial\phi} \ln g)^2 + (\frac{\partial}{\partial\psi} \ln g)^2 \right] + \frac{4\eta}{g^2} f''(\psi) \frac{\partial}{\partial\psi} \ln g$$

$$- \frac{\eta}{g^2} f'''(\psi) - \frac{2\rho}{g^2} f(\psi)\ f'(\psi)\ \frac{\partial}{\partial\phi} \ln g = 0$$

Assuming that the vorticity is not identically zero, we have $f'(\psi) \neq 0$ and on dividing throughout by $-\frac{4\eta f'(\psi)}{g^2}$, we get

$$(\frac{\partial}{\partial\phi}\ln g)^2 + (\frac{\partial}{\partial\psi}\ln g)^2 + \frac{\rho}{2\eta} f(\psi)\ \frac{\partial}{\partial\phi}\ln g - \frac{f''(\psi)}{f'(\psi)} \frac{\partial}{\partial\psi}\ln g + \frac{f'''(\psi)}{4f'(\psi)} = 0. \quad (45.06)$$

If we set

$$\xi = \frac{\partial}{\partial\phi}\ln g, \qquad \zeta = -\frac{\partial}{\partial\psi}\ln g, \qquad a = -\frac{\rho}{4\eta} f(\psi),$$

$$b = -\frac{f''(\psi)}{2f'(\psi)}, \qquad \text{and} \qquad c = \frac{f'''(\psi)}{4f'(\psi)}, \quad (45.07)$$

then $\xi + i\zeta$ is an analytic function of $\phi + i\psi$, and (45.07) becomes

$$\xi^2 + \zeta^2 - 2a\xi - 2b\zeta + c = 0,$$

which can be rewritten as

$$(\xi - a)^2 + (\zeta - b)^2 - R^2 = 0 \quad (45.08)$$

where $R^2 = a^2 + b^2 - c$ with a, b, c being functions of ψ only. As a consequence of the Lemma in section 5 of Chapter II we see that the functions ξ and ζ must be constant. If we take these constants to be a_0, b_0 respectively, from (45.07), we get

$$\xi = \frac{\partial}{\partial\phi}\ln g = a_0$$

$$\zeta = -\frac{\partial}{\partial\psi}\ln g = b_0$$

and therefore

$$\ln g = a_0\phi - b_0\psi + c_0 \qquad (45.09)$$

If α denotes the local angle of inclination of the tangent to a streamline, then

$$\frac{\partial\alpha}{\partial\phi} = -\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial\psi}, \qquad \frac{\partial\alpha}{\partial\psi} = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial\phi} \qquad (45.10)$$

Using (45.01) and (45.09) in (45.10), we obtain

$$\frac{\partial\alpha}{\partial\phi} = -\frac{1}{g}\frac{\partial g}{\partial\psi} = -\frac{\partial}{\partial\psi}\ln g = b_0$$

$$\frac{\partial\alpha}{\partial\psi} = \frac{1}{g}\frac{\partial g}{\partial\phi} = \frac{\partial}{\partial\phi}\ln g = a_0$$

which imply that

$$\alpha = b_0\phi + a_0\psi + d_0 \qquad (45.11)$$

where d_0 is an arbitrary constant.

Introducing the complex variable $z = x+iy$, we have

$$\frac{\partial z}{\partial\phi} = \sqrt{E}\exp(\,i\alpha\,)$$

$$= \exp(\,a_0\phi - b_0\psi + c_0).\ \exp i(b_0\phi + a_0\psi + d_0)$$

$$= \exp\{(a_0 + ib_0)\phi + i(a_0 + ib_0)\psi + (c_0 + id_0)\}$$

and

$$\frac{\partial z}{\partial\psi} = i\sqrt{G}\exp(i\alpha)$$

$$= i\exp\{(a_0 + ib_0)\phi + i(a_0 + ib_0)\psi + (c_0 + id_0)\}$$

giving

$$z = z_0 + (B/A)\exp\{A(\phi+i\psi)\} \qquad \text{if } A\neq 0,$$
$$\quad = z_0 + B\,(\phi+i\psi) \qquad \text{if } A=0, \qquad (45.12)$$

where z_0 is an arbitrary constant, $A = a_0+ib_0$, $B = \exp(c_0+id_0)$.

Since the streamlines are given by ψ = constant, as a consequence of (45.12), we see that the streamlines are restricted to

(a) parallel straight lines if $A = 0$ i.e. $a_0=b_0=0$,

(b) concurrent straight lines if $a_0\neq 0$, $b_0=0$,

(c) concentric circles, if $a_0=0$, $b_0\neq 0$,

(d) logarithmic spirals if $a_0\neq 0$, $b_0\neq 0$.

Summing up, we have:

Theorem 3. If the streamlines in a plane transverse flow of a viscous electrically conducting fluid and their orthogonal trajectories form an isometric net then the streamlines are restricted to the parallel straight lines, concurrent lines, concentric circles or logarithmic spirals.

CHAPTER V

PLANE COMPRESSIBLE MFD FLOWS

We investigate steady plane flows of perfectly conducting compressible fluids in this chapter. Iu. P. Ladikov (1962) derived two important Bernoulli type equations for these flows under certain assumptions and used these equations to study the flows having orthogonal magnetic and velocity fields. He also studied homentropic radial and vortex flows of polytropic gases. Power and Walker (1965), and Power and Talbot (1969) studied plane compressible orthogonal flows by reducing the problem to that of rotational gasdynamic flows. Chandna and Nath (1973) generalised some properties of gasdynamics, originally investigated by Chandna and Smith (1971), to these flows. Recently Toews and Chandna (1974) considered plane compressible flows when the magnetic and velocity fields are constantly inclined to each other, and extended some of the results previously derived for orthogonal flows.

First we reformulate the system of equations governing the flow with ϕ, ψ as independent variables where ϕ is the magnetic flux function and ψ is the stream function. We use this system to find the geometry of the irrotational compressible flows with orthogonal magnetic and velocity fields and obtain corresponding solutions. We also determine the geometry of constantly inclined flows with zero current density and find the corresponding solutions.

Section 1. Flow Equations

The steady flow of an inviscid, thermally non-conducting compressible fluid having infinite electrical conductivity, in the absence of external forces, is governed by the following system of equations:

$$\operatorname{div}(\rho\vec{V}) = 0 \tag{51.01}$$

$$\rho(\vec{V}.\operatorname{grad})\vec{V} + \operatorname{grad} p = \mu\vec{j} \times \vec{H} \tag{51.02}$$

$$\operatorname{curl}(\vec{V} \times \vec{H}) = \vec{0} \tag{51.03}$$

$$\operatorname{div}\vec{H} = 0 \tag{51.04}$$

$$\vec{j} = \operatorname{curl}\vec{H} \tag{51.05}$$

$$\vec{V}.\operatorname{grad} s = 0 \tag{51.06}$$

together with an appropriate equation of state $\rho = \rho(p,s)$. Using the identity

$$(\vec{V}.\operatorname{grad})\vec{V} = \tfrac{1}{2}\operatorname{grad} V^2 - \vec{V} \times \vec{\omega}$$

where the vorticity vector $\vec{\omega}$ is defined by

$$\vec{\omega} = \operatorname{curl}\vec{V} \quad .$$

and $V = |\vec{V}|$, the momentum equation (51.02) can be rewritten as

$$\operatorname{grad} p + \tfrac{1}{2}\rho\operatorname{grad} V^2 - \rho\vec{V} \times \vec{\omega} = \mu\vec{j} \times \vec{H} \tag{51.07}$$

In the case of plane flows with $\vec{H}$ in the plane of flow and using cartesian coordinate (x,y) with

$$\vec{V} = (u,v), \qquad \vec{H} = (H_1, H_2)$$

we have

$$\vec{\omega} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} \equiv \omega\vec{k},$$

$$\tag{51.08}$$

$$\vec{j} = \left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right)\vec{k} \equiv j\vec{k}$$

where $\vec{k}$ is the unit vector perpendicular to the (x,y)-plane. Equation (51.03) implies that

$$u\,H_2 - v\,H_1 = K$$

where K is an arbitrary constant which is zero in the case of aligned flows and non-zero for non-aligned flows.

We assume that velocity and magnetic field vectors are nowhere parallel and therefore K is non-zero. The governing equations (51.01) to (51.06) for plane flow take the form

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \qquad (51.09)$$

$$\frac{\partial p}{\partial x} + \tfrac{1}{2}\,\rho\frac{\partial V^2}{\partial x} - \rho\omega v = -\mu\, jH_2 \qquad (51.10)$$

$$\frac{\partial p}{\partial y} + \tfrac{1}{2}\,\rho\frac{\partial V^2}{\partial y} + \rho\omega u = \mu\, jH_1 \qquad (51.11)$$

$$uH_2 - vH_1 = K \qquad (51.12)$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0 \qquad (51.13)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega \qquad (51.14)$$

$$\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = j \qquad (51.15)$$

$$u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = 0 \qquad (51.16)$$

Equations (51.09) and (51.13) imply the existence of a streamfunction $\psi(x,y)$ and a magnetic flux function $\phi(x,y)$

such that

$$\frac{\partial\psi}{\partial x} = -\rho v, \qquad \frac{\partial\psi}{\partial y} = \rho u \tag{51.17}$$

and

$$\frac{\partial\phi}{\partial x} = H_2, \qquad \frac{\partial\phi}{\partial y} = -H_1 \tag{51.18}$$

We introduce ϕ and ψ as independent variables with

$$x = x(\phi,\psi), \qquad y = y(\phi,\psi)$$

defining a system of curvilinear coordinates in the (x,y)-plane. The coordinate curves $\psi(x,y)=$ constant represent streamlines while the curves $\phi(x,y)=$ constant represent magnetic field lines.

Using (51.17) and (51.18) in (51.12), we find that

$$\rho K = \frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y} = \frac{\partial(\phi,\psi)}{\partial(x,y)} = \frac{1}{J} \tag{51.19}$$

where Jacobian J is defined by (22.18).

Equation of Continuity. As in section 3 of Chapter II, we find that the fluid flows along the streamlines towards higher or lower parameter values ϕ accordingly as J is positive or negative and

$$\rho W V = \sqrt{E} \tag{51.20}$$

$$u + iv = \frac{\sqrt{E}}{\rho J} e^{i\alpha} \tag{51.21}$$

Solenoidal condition on $\vec{H}$: From (51.18) and (A.4), we get

$$\frac{\partial x}{\partial\psi} = JH_1, \qquad \frac{\partial y}{\partial\psi} = JH_2 .$$

Proceeding exactly as in the section 3 of Chapter II in the case of "Equation of Continuity", we find that

$$W\,H = \sqrt{G}\ , \tag{51.22}$$

$$H_1 + iH_2 = \frac{\sqrt{G}}{J}\, e^{i\beta} \tag{51.23}$$

where β is the angle between the tangent to the co-ordinate line ϕ = constant, directed in the sense of increasing ψ, with the x-axis.

<u>The vorticity ω</u> : From (51.14) and (A.4), we get

$$\omega = \left(\frac{\partial v}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial v}{\partial \psi}\frac{\partial \psi}{\partial x}\right) - \left(\frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial u}{\partial \psi}\frac{\partial \psi}{\partial y}\right)$$

$$= \frac{1}{J}\left(\frac{\partial v}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial v}{\partial \psi}\frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial \phi}\frac{\partial x}{\partial \psi} - \frac{\partial u}{\partial \psi}\frac{\partial x}{\partial \phi}\right)$$

Making use of

$$u = \pm\, V\cos\alpha \qquad\qquad v = \pm\, V\sin\alpha$$

where positive or negative sign is taken according as J is positive or negative, and using (A.7) and (A.10), we obtain

$$\sqrt{E}\,W\,\omega = F\frac{\partial V}{\partial \phi} - E\frac{\partial V}{\partial \psi} + JV\frac{\partial \alpha}{\partial \phi} \tag{51.24}$$

From (51.20), we have

$$\frac{\partial V}{\partial \phi} = \frac{1}{2V\rho^2}\frac{\partial}{\partial \phi}\left(\frac{E}{W^2}\right) - \frac{E}{VW^2\rho^3}\frac{\partial \rho}{\partial \phi}\ ,$$

$$\frac{\partial V}{\partial \psi} = \frac{1}{2V\rho^2}\frac{\partial}{\partial \psi}\left(\frac{E}{W^2}\right) - \frac{E}{VW^2\rho^3}\frac{\partial \rho}{\partial \psi}\ .$$

Employing the identities (A.17) and (A.18), we get

$$\frac{\partial V}{\partial \phi} = \frac{1}{\rho\sqrt{EW}}\{F\,\Gamma^2_{11} - E\,\Gamma^2_{12}\} - \frac{E}{\sqrt{E}\,W\,\rho^2}\frac{\partial \rho}{\partial \phi}\ ,$$

$$\frac{\partial V}{\partial \psi} = \frac{1}{\rho\sqrt{EW}}\{F\,\Gamma^2_{12} - E\,\Gamma^2_{22}\} - \frac{E}{\sqrt{E}\,W\,\rho^2}\frac{\partial \rho}{\partial \psi}\ .$$

Eliminating $\frac{\partial V}{\partial \phi}$, $\frac{\partial V}{\partial \psi}$ from (51.24) and using (A.11), we obtain

$$\sqrt{E}\, W\omega = \frac{F}{\rho\sqrt{E}\, W} (F\, \Gamma_{11}^{2} - E\, \Gamma_{12}^{2}) - \frac{E}{\rho\sqrt{E}\, W} (F\, \Gamma_{12}^{2} - E\, \Gamma_{22}^{2})$$
$$- \frac{EF}{\rho^{2} W\sqrt{E}} \frac{\partial \rho}{\partial \phi} + \frac{E^{2}}{\rho^{2} W\sqrt{E}} \frac{\partial \rho}{\partial \psi} + \frac{\sqrt{E} W^{2}}{\rho E W} \Gamma_{11}^{2}$$

or

$$\rho E W^{2}\, \omega = EG\, \Gamma_{11}^{2} - 2EF\, \Gamma_{12}^{2} + E^{2}\Gamma_{22}^{2} - \frac{E}{\rho} (F\frac{\partial \rho}{\partial \phi} - E\frac{\partial \rho}{\partial \psi})$$

or

$$\omega = \frac{1}{\rho W}\{ \frac{\partial}{\partial \phi}(\frac{F}{W}) - \frac{\partial}{\partial \psi}(\frac{E}{W})\} - \frac{1}{W\rho^{2}}(\frac{F}{W} \frac{\partial \rho}{\partial \phi} - \frac{E}{W} \frac{\partial \rho}{\partial \psi})$$

$$= \frac{1}{W}\{ \frac{\partial}{\partial \phi}(\frac{F}{\rho W}) - \frac{\partial}{\partial \psi}(\frac{E}{\rho W})\} \qquad (51.25)$$

where identity (A.19) has been used.

<u>The Current density j:</u> From (51.15) and (A.4), we have

$$Jj = \frac{\partial H_2}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial H_2}{\partial \psi} \frac{\partial y}{\partial \phi} + \frac{\partial H_1}{\partial \phi} \frac{\partial x}{\partial \psi} - \frac{\partial H_1}{\partial \psi} \frac{\partial x}{\partial \phi}$$

On substituting

$$H_1 = \pm H \cos \beta, \qquad H_2 = \pm H \sin \beta,$$

we find that

$$\pm Jj = (\frac{\partial H}{\partial \phi} \sin \beta + H \cos\beta\frac{\partial \beta}{\partial \phi}) \frac{\partial y}{\partial \psi} - (\frac{\partial H}{\partial \psi} \sin \beta + H \cos\beta\frac{\partial \beta}{\partial \psi})\frac{\partial y}{\partial \phi}$$

$$+ (\frac{\partial H}{\partial \phi} \cos \beta - H \sin\beta\frac{\partial \beta}{\partial \phi}) \frac{\partial x}{\partial \psi} - (\frac{\partial H}{\partial \psi} \cos \beta - H \sin\beta\frac{\partial \beta}{\partial \psi})\frac{\partial x}{\partial \phi} .$$

(51.26)

Making use of the relations

$$\frac{\partial x}{\partial \psi} = \sqrt{G} \cos \beta, \qquad \frac{\partial y}{\partial \psi} = \sqrt{G} \sin \beta$$

$$\frac{\partial x}{\partial \phi} = \frac{F}{\sqrt{G}} \cos \beta + \frac{J}{\sqrt{G}} \sin \beta \;, \qquad \frac{\partial y}{\partial \phi} = \frac{F}{\sqrt{G}} \sin \beta - \frac{J}{\sqrt{G}} \cos \beta \;,$$

in (51.26), after some simplification, we get

$$\sqrt{G}Wj = G\frac{\partial H}{\partial \phi} - F\frac{\partial H}{\partial \psi} + HJ\frac{\partial \beta}{\partial \psi} \tag{51.27}$$

Eliminating H from (51.27) with the help of (51.22) and using

$$\frac{\partial \beta}{\partial \psi} = \frac{J}{G} \Gamma^2_{11} \;, \qquad \frac{\partial \beta}{\partial \phi} = \frac{J}{G} \Gamma^2_{12}$$

(51.27) takes the form

$$\sqrt{G}Wj = \frac{GW}{\sqrt{G}} \frac{\partial}{\partial \phi}\left(\frac{G}{2W^2}\right) - \frac{FW}{\sqrt{G}} \frac{\partial}{\partial \psi}\left(\frac{G}{2W^2}\right) + \frac{\sqrt{G}}{W} \frac{J^2}{G} \Gamma^2_{11}$$

$$= \frac{GW}{\sqrt{G}} \frac{1}{W^2}(G \, \Gamma^2_{22} - F \, \Gamma^2_{12}) - \frac{FW}{\sqrt{G}} \frac{1}{W^2}(G \, \Gamma^2_{12} - F \, \Gamma^2_{11})$$

$$+ \frac{\sqrt{G}}{W} \frac{J^2}{G} \Gamma^2_{11}$$

or

$$Wj = \frac{1}{W}\{ G \, \Gamma^2_{22} - 2F \, \Gamma^2_{12} + E \, \Gamma^2_{11} \}$$

or

$$j = \frac{1}{W} \left\{ \frac{\partial}{\partial \phi}\left(\frac{G}{W}\right) - \frac{\partial}{\partial \psi}\left(\frac{F}{W}\right) \right\} \tag{51.28}$$

Momentum Equations. Equation (51.10) can be written as

$$\frac{\partial p}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial p}{\partial \psi} \frac{\partial \psi}{\partial x} + \tfrac{1}{2}\rho\left(\frac{\partial V^2}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial V^2}{\partial \psi} \frac{\partial \psi}{\partial x}\right) - \rho \omega v = -\mu j H_2$$

or

$$\frac{\partial p}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial p}{\partial \psi} \frac{\partial y}{\partial \phi} + \tfrac{1}{2}\rho \frac{\partial V^2}{\partial \phi} \frac{\partial y}{\partial \psi} - \tfrac{1}{2}\rho \frac{\partial V^2}{\partial \psi} \frac{\partial y}{\partial \phi} - \omega \frac{\partial y}{\partial \phi} = -\mu j \frac{\partial y}{\partial \psi} \tag{51.29}$$

where (51.17), (51.18) and (A.4) have been used.

Similarly, from (51.11), we get

$$- \frac{\partial p}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial p}{\partial \psi}\frac{\partial x}{\partial \phi} - \tfrac{1}{2}\rho\frac{\partial V^2}{\partial \phi}\frac{\partial x}{\partial \psi} + \tfrac{1}{2}\rho\frac{\partial V^2}{\partial \psi}\frac{\partial x}{\partial \phi} + \omega\frac{\partial x}{\partial \phi} = \mu\, j\frac{\partial x}{\partial \psi} \tag{51.30}$$

Multiplying (51.29) by $\frac{\partial x}{\partial \phi}$, (51.30) by $\frac{\partial y}{\partial \phi}$ and adding, yields

$$J\left(\frac{\partial p}{\partial \phi} + \tfrac{1}{2}\rho\,\frac{\partial V^2}{\partial \phi}\right) = -\mu J j \quad ,$$

and using (51.20), we obtain

$$\frac{\partial p}{\partial \phi} + \tfrac{1}{2}\rho\,\frac{\partial}{\partial \phi}\left(\frac{E}{\rho^2 W^2}\right) + \mu\, j = 0 \tag{51.31}$$

Again, multiplying (51.29) by $\frac{\partial x}{\partial \psi}$, (51.30) by $\frac{\partial y}{\partial \psi}$ and adding gives

$$- J\left(\frac{\partial p}{\partial \psi} + \tfrac{1}{2}\rho\,\frac{\partial V^2}{\partial \psi}\right) - J\omega = 0$$

and using (51.20) to eliminate V, we get

$$\frac{\partial p}{\partial \psi} + \tfrac{1}{2}\rho\,\frac{\partial}{\partial \psi}\left(\frac{E}{\rho^2 W^2}\right) + \omega = 0 \tag{51.32}$$

<u>Energy equation</u>. Equation (51.16) transforms to

$$\left(\frac{\partial s}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial s}{\partial \psi}\frac{\partial \psi}{\partial x}\right)\frac{\partial \psi}{\partial y} - \left(\frac{\partial s}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial s}{\partial \psi}\frac{\partial \psi}{\partial y}\right)\frac{\partial \psi}{\partial x} = 0$$

or

$$\frac{\partial s}{\partial \phi}\left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial x}\right) = 0$$

which implies that

$$\frac{\partial s}{\partial \phi} = 0, \quad \text{or} \quad s = s(\psi). \tag{51.33}$$

In view of preceding derivations we see that the system of equations (51.09) to (51.16) is replaced by

$$\frac{\partial p}{\partial \phi} + \tfrac{1}{2}\rho \frac{\partial}{\partial \phi}\left(\frac{E}{\rho^2 W^2}\right) + \mu j = 0 \qquad (51.34)$$

(momentum)

$$\frac{\partial p}{\partial \psi} + \tfrac{1}{2}\rho \frac{\partial}{\partial \psi}\left(\frac{E}{\rho^2 W^2}\right) + \omega = 0 \qquad (51.35)$$

$$\frac{\partial}{\partial \psi}\left(\frac{W}{E}\,\Gamma^2_{11}\right) - \frac{\partial}{\partial \phi}\left(\frac{W}{E}\,\Gamma^2_{12}\right) = 0 \qquad (51.36)$$

(Gauss)

$$\omega = \frac{1}{W}\left\{ \frac{\partial}{\partial \phi}\left(\frac{F}{\rho W}\right) - \frac{\partial}{\partial \psi}\left(\frac{E}{\rho W}\right) \right\} \qquad (51.37)$$

(Vorticity)

$$j = \frac{1}{W}\left\{ \frac{\partial}{\partial \phi}\left(\frac{G}{W}\right) - \frac{\partial}{\partial \psi}\left(\frac{F}{W}\right) \right\} \qquad (51.38)$$

(Current density)

$$\frac{\partial s}{\partial \phi} = 0 \qquad \text{(Energy)} \qquad (51.39)$$

with

$$J = \frac{1}{K\rho} \qquad (51.40)$$

and an equation of state

$$\rho = \rho\,(p,\ s) \qquad (51.41)$$

We can find x, y as functions of ϕ, ψ from

$$z = x + iy = \int \frac{\exp(i\alpha)}{\sqrt{E}} \{E\,d\phi + (F + iJ)\,d\psi\} \qquad (51.42)$$

where

$$\alpha = \int \frac{J}{E}(\Gamma^2_{11}\,d\phi + \Gamma^2_{12}\,d\psi)\,. \qquad (51.43)$$

Section 2: Irrotational Flows.

In this section we consider irrotational plane flows of compressible fluids when the magnetic and velocity fields are everywhere orthogonal. We investigate the geometries of these flows and find the corresponding solutions.

Geometry. Since F=0, and from (51.37), we find that

$$\omega = -\frac{\partial}{\partial\psi}\left(\frac{E}{\rho W}\right) = 0 \ ,$$

and using (51.40), we have

$$\frac{\partial E}{\partial\psi} = 0 \ , \qquad \text{or} \qquad E = E(\phi) \tag{52.01}$$

Since $J^2 = EG = \frac{1}{\rho^2 K^2}$, G is given by

$$G = \frac{1}{K^2\rho^2 E} \tag{52.02}$$

From (51.38) and (51.40), we get

$$j = \frac{1}{W}\frac{\partial}{\partial\phi}\left(\frac{G}{W}\right) = K^2\rho\frac{\partial}{\partial\phi}(\rho G)$$

$$= \rho\frac{\partial}{\partial\phi}(1/\rho E) \tag{52.03}$$

Substitution of (52.03) and (51.40) in (51.34) gives

$$\frac{\partial p}{\partial\phi} + \tfrac{1}{2}\rho K^2 \ \frac{\partial E}{\partial\phi} + \mu\rho\frac{\partial}{\partial\phi}(1/\rho E) = 0. \tag{52.04}$$

Employing (51.40) and (52.01) in (51.35), we find that

$$\frac{\partial p}{\partial\psi} = 0 \qquad \text{or} \qquad p = p(\phi) \tag{52.05}$$

Assuming that the equation of state is of the product form

$$\rho = P_1(p)\ S_1(s)$$

by virtue of (51.39) and (52.05), we can write

$$\rho = P(\phi)\ S(\psi) \tag{52.06}$$

where $P(\phi) = P_1(p)$, $S(\psi) = S_1(s)$.

Employing (52.05), (52.06) and (52.01) in (52.04), we obtain

$$p'(\phi) + \tfrac{1}{2}K^2 S(\psi)P(\phi)\ E'(\phi) - \mu\{P'(\phi)E(\phi) + P(\phi)E'(\phi)\}/\{P(\phi)E^2(\phi)\} = 0. \tag{52.07}$$

Using (52.01) and (52.02) in the Gauss equation, we get

$$\frac{\partial}{\partial\psi}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial\psi}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial\phi}\right) = 0,$$

or

$$\frac{\partial}{\partial\phi}\left\{\frac{1}{\rho^2 E}\left(\frac{\partial\rho}{\partial\phi} + \frac{\rho}{2E}\frac{\partial E}{\partial\phi}\right)\right\} = 0,$$

or

$$E\frac{\partial\rho}{\partial\phi} + \tfrac{1}{2}\rho\, E'(\phi) = \rho^2 E^2\ f(\psi), \tag{52.08}$$

where $f(\psi)$ is an arbitrary function of ψ.

Substituting $\rho = P(\phi)\ S(\psi)$ in (52.08), we find that

$$E\ P'(\phi)S(\psi) + \tfrac{1}{2}E'(\phi)P(\phi)S(\psi) = E^2\ P^2(\phi)\ S^2(\psi)f(\psi),$$

or

$$\frac{1}{E^2P^2}\{E\ P'(\phi) + \tfrac{1}{2}E'(\phi)P\} = S(\psi)f(\psi).$$

Since the left hand side is a function of ϕ alone, we must have

$$\frac{1}{E^2P^2}\{ E\, P'(\phi) + \tfrac{1}{2}P\, E'(\phi)\} = S(\psi)f(\psi) \qquad (52.09)$$
$$= \text{constant, } A.$$

From (52.07) and (52.09), we obtain

$$p'(\phi) + \tfrac{1}{2}K^2\, S(\psi)P(\phi)E'(\phi) - \frac{\mu}{E^2P}\{ A\, E^2P^2 + \tfrac{1}{2}P\, E'(\phi)\} = 0,$$

or

$$\frac{1}{2}K^2S(\psi) = \frac{1}{PE'(\phi)}\left[\frac{\mu}{E^2}\{ AE^2P + \tfrac{1}{2}E'(\phi)\} - p'(\phi)\right] \qquad (52.10)$$

As ϕ,ψ are independent variables, (52.10) requires that

$$S(\psi) = \text{constant},$$

putting a restriction on the equation of state, which must be of the form

$$\rho = \rho(p) \qquad (52.11)$$

Equations (52.02) and (52.11) give us $\rho = \rho(\phi)$,

and therefore $G = G(\phi)$

Again, from Gauss equation, we have

$$\frac{\partial}{\partial\phi}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial\phi}\right) = 0,$$

therefore writing $\sqrt{E} = g'(\phi)$, we have

$$\sqrt{G} = A\, g(\phi) + B \qquad (52.12)$$

where A, B are arbitrary constants.

From (51.43) and (52.12) we find that

$$\frac{\partial\alpha}{\partial\phi} = 0,$$

$$\frac{\partial\alpha}{\partial\psi} = \frac{1}{2J}\frac{\partial G}{\partial\phi} = \pm\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial\phi} = \pm A$$

giving us

$$\alpha = \alpha_0 \pm A\psi \tag{52.13}$$

where α_0 is an arbitrary constant.

Equations (51.42), (52.12) and (52.13) yield

$$\frac{\partial z}{\partial \phi} = \exp(i\alpha)\cdot g'(\phi),$$

$$\frac{\partial z}{\partial \psi} = \pm\, i \exp(i\alpha)\cdot\{ A\, g(\phi) + B\}.$$

On integration we get

$$z = \begin{cases} z_0 + \frac{1}{A}\exp(i\alpha_0 \pm iA\psi)\cdot\{ A\, g(\phi) + B\}, \text{ if } A \neq 0; \\ z_0 + \exp(i\alpha_0)\cdot\{ g(\phi) + iB\psi\}, \text{ if } A = 0. \end{cases} \tag{52.14}$$

As a consequence of (53.14) we see that the curves ψ = constant are concurrent lines if $A \neq 0$ and parallel straight lines if $A = 0$. By suitably choosing co-ordinates axis, we can take $z_0 = 0$ and $\alpha_0 = 0$. We also assume that $J > 0$.

Solutions

(a) If $A \neq 0$, without loss of generality we can take $B = 0$. From (52.14), we get

$$x = g(\phi)\cos(A\psi), \quad y = g(\phi)\sin(A\psi) \tag{52.15}$$

Solving (52.15) for ϕ and ψ, we find that

$$g(\phi) = \sqrt{x^2 + y^2} = r, \quad \psi = (1/A)\tan^{-1}(y/x) = \theta/A. \tag{52.16}$$

where (r,θ) are the polar coordinates.

From (52.12) and (52.16), we have

$$\sqrt{E} = g'(\phi) = 1/(\frac{d\phi}{dr})$$

$$\sqrt{G} = Ar$$

and (51.40) yields

$$A\, r/(\frac{d\phi}{dr}) = 1/(\rho K)$$

or

$$\frac{d\phi}{dr} = \rho KAr \qquad (52.17)$$

Velocity magnitude is given by

$$V = \frac{\sqrt{E}}{\rho W} = K\sqrt{E} = 1/(A\rho r). \qquad (52.18)$$

From (51.22), we see that

$$H = \frac{\sqrt{G}}{W} = \rho KAr. \qquad (52.19)$$

Current density is given by

$$j = \frac{1}{W}\frac{\partial}{\partial\phi}(\frac{G}{W}) = \rho K^2 \frac{\partial}{\partial\phi}(\rho G)$$

$$= \frac{AK}{r}\frac{d}{dr}(\rho r^2) \qquad (52.20)$$

where $\rho = \rho(r)$.

From momentum equations (51.34) and (51.35), we obtain

$$\frac{dp}{d\phi} + \tfrac{1}{2}K^2\rho\frac{dE}{d\phi} + \mu j = 0,$$

or

$$\frac{dp}{dr} + \frac{\rho}{2A^2}\frac{d}{dr}(\frac{1}{\rho^2 r^2}) + \mu K^2A^2\rho\frac{d}{dr}(\rho r^2) = 0. \qquad (52.21)$$

Using the equation of state, which is of the form $\rho = \rho(p)$, we can determine ρ and p as functions of r from (52.21).

(b). If $A = 0$, then the stream-lines and the magnetic field lines are orthogonal families of parallel straight lines and we have

$$x = g(\phi), \qquad y = B\psi . \qquad (52.22)$$

From (52.12) we get

$$\sqrt{E} = 1/(\frac{d\phi}{dx}), \qquad \sqrt{G} = B,$$

and (51.40) yields

$$B/(\frac{d\phi}{dx}) = \frac{1}{\rho K}, \qquad \text{i.e.} \quad \frac{d\phi}{dx} = \rho KB. \qquad (52.23)$$

V, H and j are now given by

$$V = K\sqrt{E} = \frac{1}{\rho B}, \qquad H = K\sqrt{G} = \rho KB ,$$
$$j = \rho K^2 \frac{\partial}{\partial\phi}(\rho G) = KB \frac{d\rho}{dx} . \qquad (52.24)$$

From momentum equations, we obtain

$$\frac{1}{\rho KB}\{\frac{dp}{dx} + \tfrac{1}{2}\rho K^2 \frac{d}{dx}(\frac{1}{\rho^2K^2B^2}) + \mu\rho K^2B^2 \frac{d\rho}{dx}\} = 0,$$

which integrates to yield

$$p = p_0 - \tfrac{1}{2}\mu K^2B^2\rho^2 + \frac{1}{\rho B^2} . \qquad (52.25)$$

Summing up, we have

<u>Theorem 1</u>. If a steady plane orthogonal flow, of an inviscid compressible fluid, is irrotational then it must be either radial or parallel flow. For the radial flows

$$\vec{V} = \frac{1}{A\rho r}\vec{e}_r , \quad \vec{H} = KA\rho r\, \vec{e}_\theta, \quad \vec{j} = \frac{KA}{r}\frac{d}{dr}(\rho r^2)\, \vec{k}$$

and $$p = p_0 - \int\{\frac{\rho}{2A^2}\frac{d}{dr}(\frac{1}{\rho^2r^2}) + \mu K^2A^2\rho\frac{d}{dr}(\rho r^2)\}dr$$

In the case of parallel flows

$$\vec{V} = \frac{1}{\rho B}\vec{e}_1, \quad \vec{H} = \rho KB\, \vec{e}_2, \quad \vec{j} = KB\, \rho'(x)\, \vec{k}$$

and $$p = p_0 - \tfrac{1}{2}\mu K^2B^2\rho^2 + \frac{1}{B^2\rho} .$$

Section 3. Flows with zero Current density

We now consider plane flows when the magnetic field vector $\vec{H}$ makes a constant non-zero angle δ with the velocity vector and investigate the implication of zero current density j.

Geometry:

In this case, we have

$$J = \sqrt{EG}\,\sin\delta = \frac{1}{\rho K}, \qquad (53.01)$$

$$F = \sqrt{EG}\,\cos\delta = \frac{1}{\rho K}\cot\delta. \qquad (53.02)$$

From (51.38), (53.01) and (53.02), we get

$$j = \frac{1}{W}\frac{\partial}{\partial\phi}\left(\frac{G}{W}\right) = \rho K^2 \frac{\partial}{\partial\phi}(\rho G).$$

Therefore, current density $j = 0$ is equivalent to

$$\frac{\partial}{\partial\phi}(\rho G) = 0, \quad \text{i.e.} \quad \rho G = f(\psi), \qquad (53.03)$$

where $f(\psi)$ is an arbitrary function of ψ. From (53.01) and (53.03), we find that

$$\rho E = \frac{1}{\rho G K^2 \sin^2\delta} = \frac{1}{K^2 \sin^2\delta\, f(\psi)}. \qquad (53.04)$$

Since $j = 0$, (51.34) and (51.40) yield

$$\frac{\partial p}{\partial\phi} + \tfrac{1}{2} K^2 \rho \frac{\partial E}{\partial\phi} = 0,$$

or

$$c^2 \frac{\partial\rho}{\partial\phi} + \tfrac{1}{2} K^2 \left\{ \frac{\partial}{\partial\phi}(\rho E) - E\frac{\partial\rho}{\partial\phi} \right\} = 0,$$

where $c^2 = \frac{\partial p}{\partial\rho}$, From (53.04) we see that $\frac{\partial}{\partial\phi}(\rho E) = 0$, therefore

$$(c^2 - \tfrac{1}{2}K^2 E)\frac{\partial\rho}{\partial\phi} = 0. \qquad (53.05)$$

If $(c^2 - \tfrac{1}{2}K^2 E) = c^2 - \tfrac{1}{2}V^2 \neq 0$, i.e. the Mach number is not equal to $\sqrt{2}$, then $\frac{\partial \rho}{\partial \phi} = 0$ meaning that the density remains constant along the stream-lines and $\rho = \rho(\psi)$. From (53.03) and (53.04) we conclude that E and G are functions of ψ alone. Gauss equation (51.36), reduces to

$$\frac{\partial}{\partial \psi}\left(- \frac{1}{2EW} E \frac{\partial E}{\partial \psi}\right) = 0,$$

or

$$\frac{1}{2\sqrt{EG}} \frac{\partial E}{\partial \psi} = a, \tag{53.06}$$

where a is an arbitrary constant. Writing $\sqrt{G} = g'(\psi)$, eq. (53.06) gives

$$\frac{\partial \sqrt{E}}{\partial \psi} = a\ g'(\psi),$$

or

$$\sqrt{E} = a\ g(\psi) + b, \tag{53.07}$$

where b is an arbitrary constant. Using (53.07) in (51.43), we get

$$\frac{\partial \alpha}{\partial \phi} = \frac{1}{2EJ}\left(- E \frac{\partial E}{\partial \psi}\right) = - \frac{1}{2\sqrt{EG}\ \sin\delta} \frac{\partial E}{\partial \psi}$$

$$= - \frac{1}{\sqrt{G}\ \sin\delta} \frac{\partial \sqrt{E}}{\partial \psi} = - \frac{a}{\sin\delta} \tag{53.08}$$

and

$$\frac{\partial \alpha}{\partial \psi} = \frac{1}{2EJ}\left(- F \frac{\partial E}{\partial \psi}\right) = - \frac{\cot\delta}{2E} \frac{\partial E}{\partial \psi}. \tag{53.09}$$

Equations (53.08) and (53.09) yield

$$\alpha = \alpha_0 - \frac{a}{\sin\delta} \phi + \cot\delta\ \ln\{a\ g(\psi)+b\} \tag{53.10}$$

where α_0 is an arbitrary constant.

From (51.42), we obtain

$$\frac{\partial z}{\partial \phi} = \sqrt{E} \exp(i\alpha) = \{a\ g(\psi)+b\} \exp(i\alpha),$$

$$\frac{\partial z}{\partial \psi} = \left(\frac{F}{\sqrt{E}} + i \frac{J}{\sqrt{E}}\right) \exp(i\alpha) = g'(\psi) \exp\{i(\alpha+\delta)\}$$

and z is given by

$$z = \begin{cases} z_0 + i \exp(i\alpha)\dfrac{\{a\, g(\psi) + b\}}{a} \sin\delta, & \text{if } a \neq 0; \\ z_0 + \exp(i\alpha_0)\{g(\psi) \exp(i\delta) + b\phi\}, & \text{if } a=0. \end{cases} \tag{53.11}$$

From (52.11) we conclude that the streamlines ψ = constant are either concentric circles (when $a \neq 0$) or parallel straight lines (when $a = 0$).

Solutions.

By suitably choosing co-ordinate axes, we can take $z_0 = 0$. We also assume that $J>0$ i.e. $\sin\delta>0$.

(a) When $a \neq 0$, without loss of generality we can take $b = 0$. From (53.11), we get

$$x = -\sin\delta \sin\alpha\, g(\psi), \qquad y = \sin\delta \cos\alpha\, g(\psi) \tag{53.12}$$

or

$$g(\psi) = \frac{r}{\sin\delta}$$

and

$$\phi = \frac{\sin\delta}{a}(\alpha_1 + \cot\delta \ln r - \theta), \tag{53.13}$$

where $\alpha_1 = \frac{1}{2}\pi + \alpha_0 + \cot\delta \ln(a/\sin\delta)$.

From (53.07) and (53.13), we get .

$$\sqrt{E} = ar/\sin\delta, \qquad \sqrt{G} = 1/\left(\frac{d\psi}{dr}\sin\delta\right)$$

and (51.40), we find that

$$\frac{d\psi}{dr} = \frac{aK\rho r}{\sin\delta} \tag{53.14}$$

V and H are now given by

$$V = \sqrt{E/\rho W} = aKr/\sin\delta ,$$
$$H = \sqrt{G/W} = 1/ar . \tag{53.15}$$

Vorticity is given by

$$\omega = - K^2 \rho \frac{dE}{d\psi} = - 2aK/\sin\delta \tag{53.16}$$

From momentum equations (51.34) and (51.35), we have

$$\frac{\partial p}{\partial \phi} = 0 ,$$

$$\frac{\partial p}{\partial \psi} + \tfrac{1}{2} K^2 \rho \frac{\partial E}{\partial \psi} - K^2 \rho \frac{\partial E}{\partial \psi} = 0$$

implying that

$$\frac{dp}{dr} = \tfrac{1}{2} K^2 \rho \frac{dE}{dr} = \frac{a^2 K^2 \rho r}{\sin^2\delta}$$

or

$$p = p_0 + \frac{a^2 K^2}{\sin^2\delta} \int \rho \, r dr \tag{53.17}$$

where $\rho = \rho(r)$ which can be determined by (53.17) and the equation of state.

(b) If $a = 0$, then streamlines and magnetic field lines are two families of parallel straight lines making an angle δ with each other. In this case, (53.11) yields

$$x = b \cos\alpha_0 \cdot \phi + \cos(\alpha_0 + \delta) g(\psi) ,$$
$$y = b \sin\alpha_0 \cdot \phi + \sin(\alpha_0 + \delta) g(\psi) \tag{53.18}$$

Choosing x-axis along streamlines we have $\alpha_0 = 0$, and

$$\phi = \frac{1}{b}(x - y \cot \delta), \; g(\psi) = y/\sin \delta. \tag{53.19}$$

From (53.07), (53.19) and (51.40), we get

$$\frac{d\psi}{dy} = b\,K\,\rho \tag{53.20}$$

V, H and ω are given by

$$V = K\sqrt{E} = b\,K\,,$$

$$H = \rho K\sqrt{G} = \frac{1}{b \sin \delta}$$

and $$\omega = -K^2 \rho \frac{\partial E}{\partial \psi} = 0$$

Momentum equations imply that p = constant, p_0.

Summing up, we have

<u>Theorem 2</u>: If the current density is zero throughout a constantly inclined compressible plane flow and Mach number is not equal to $\sqrt{2}$, then the streamlines are either concentric circles or parallel straight lines.

For the circular streamlines, solution is given by

$$V = \frac{aKr}{\sin \delta}, \quad H = \frac{1}{ar}\,, \quad \rho = \rho(r)\,, \quad s = s(r)\,,$$

$$p = p_0 + \int \frac{a^2 K^2}{\sin^2 \delta} \rho r \, dr\,;$$

while in the case of parallel flows

$$V = bK, \quad H = \frac{1}{b \sin \delta}, \quad p = p_0, \quad \rho = \rho(y), \; s = s(y).$$

CHAPTER VI

AXISYMMETRIC MAGNETOHYDRODYNAMIC FLOWS

In recent years many authors have studied axisymmetric magnetohydrodynamic flows of an infinitely conducting inviscid fluid. V. C. A. Ferraro (1954) found a general condition to be satisfied by any poloidal magnetic field in equilibrium with an incompressible fluid and gave a particular solution. S. Chandrasekhar (1956) gave solutions to a large class of force-free fields. R. R. Long (1960) and C. S. Yih (1965) have also considered steady axisymmetric flows of perfectly conducting, inviscid imcompressible fluids. Recently K. B. Ranger (1970) has given some interesting exact solutions of steady MHD equations under above assumptions. He considered finite fluid motion inside a liquid sphere. C. Sozou (1972) extended some of the solutions given by Ranger by taking into account the gravitational potential of the fluid.

We consider steady axisymmetric MHD flows of incompressible infinitely conducting fluid when viscosity is also taken into account and obtain a non-linear partial differential equation for the streamfunction ψ. We then give a class of exact solutions and also discuss a particular solution. Finally, we study meridional motion of an inviscid fluid under the influence of toroidal magnetic field and consider a particular flow.

Section 1. Flow Equations.

The steady-state equations of motion for a viscous infinitely conducting incompressible fluid are as follows

$$\text{div}\ \vec{V} = 0 \tag{61.01}$$

$$\rho(\vec{V}.\text{grad})\ \vec{V} + \text{grad}\, p = \eta\nabla^2\vec{V} + \mu(\text{curl}\ \vec{H})\times\vec{H} \tag{61.02}$$

$$\text{curl}(\vec{V}\times\vec{H}) = \vec{0} \tag{61.03}$$

$$\text{div}\ \vec{H} = 0 \tag{61.04}$$

We make use of cylindrical polar coordinates (r,ϕ,z) and consider a flow in which all the dependent variables are function of r and z only. By virtue of (61.01) and (61.04) velocity and magnetic fields may be expressed by (Ranger,1970)

$$\begin{aligned} \vec{V} &= -\frac{1}{r}\frac{\partial\psi}{\partial r}\vec{e}_z + \frac{1}{r}\frac{\partial\psi}{\partial z}\vec{e}_r + \frac{U}{r}\vec{e}_\phi \\ \vec{H} &= -\frac{1}{r}\frac{\partial\chi}{\partial r}\vec{e}_z + \frac{1}{r}\frac{\partial\chi}{\partial z}\vec{e}_r + \frac{T}{r}\vec{e}_\phi \end{aligned} \tag{61.05}$$

where ψ is the Stoke's streamfunction, χ is the flux function for the poloidal magnetic field, $\frac{U}{r}$ is the rotational or swirl component of the fluid velocity field, $\frac{T}{r}$ is the toroidal component of the magnetic field and $\vec{e}_z$, $\vec{e}_r$, $\vec{e}_\phi$, are the unit vectors at a point (r,ϕ,z) in the directions of increasing z, r, ϕ, respectively.

Equation (61.02) can be rewritten as

$$\frac{\mu}{\rho}\vec{H}\times\text{curl}\,\vec{H} - \vec{V}\times\text{curl}\,\vec{V} = -\,\text{grad}\,P - \frac{\eta}{\rho}\,\text{curl}(\text{curl}\ \vec{V}) \tag{61.06}$$

where $P = \frac{p}{\rho} + \frac{1}{2}V^2$.

Using the expressions (61.05) for $\vec{V}$ and $\vec{H}$ in (61.06) and resolving along $\vec{e}_z$, $\vec{e}_r$ and $\vec{e}_\phi$, we get three scalar equations

$$\frac{\mu}{\rho}\frac{1}{r^2}\frac{\partial\chi}{\partial z}L(\chi) - \frac{1}{r^2}\frac{\partial\psi}{\partial z}L(\psi) + \frac{\mu}{\rho}\frac{T}{r^2}\frac{\partial T}{\partial z} - \frac{U}{r^2}\frac{\partial U}{\partial z}$$

$$= -\frac{\partial P}{\partial z} - \frac{\eta}{\rho r}\frac{\partial}{\partial r}L(\psi) \qquad (61.07)$$

$$\frac{\mu}{\rho}\frac{1}{r^2}\frac{\partial\chi}{\partial r}L(\chi) - \frac{1}{r^2}\frac{\partial\psi}{\partial r}L(\psi) + \frac{\mu}{\rho}\frac{T}{r^2}\frac{\partial T}{\partial r} - \frac{U}{r^2}\frac{\partial U}{\partial r}$$

$$= -\frac{\partial P}{\partial r} + \frac{\eta}{\rho r}\frac{\partial}{\partial z}L(\psi) \qquad (61.08)$$

$$\frac{\mu}{\rho}\left(\frac{1}{r^2}\frac{\partial\chi}{\partial r}\frac{\partial T}{\partial z} - \frac{1}{r^2}\frac{\partial\chi}{\partial z}\frac{\partial T}{\partial r}\right) - \left(\frac{1}{r^2}\frac{\partial\psi}{\partial r}\frac{\partial U}{\partial z} - \frac{1}{r^2}\frac{\partial\psi}{\partial z}\frac{\partial U}{\partial r}\right)$$

$$= \frac{\eta}{\rho r}L(U) \qquad (61.09)$$

where the Stokes operator L is defined by

$$L \equiv \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}.$$

Employing the integrability condition $\frac{\partial^2 P}{\partial z\partial r} = \frac{\partial^2 P}{\partial r\partial z}$, from (61.07) and (61.08), we obtain

$$\frac{\mu}{\rho}\left[\frac{\partial}{\partial z}\left\{\frac{\partial\chi}{\partial r}\frac{L(\chi)}{r^2}\right\} - \frac{\partial}{\partial r}\left\{\frac{\partial\chi}{\partial z}\frac{L(\chi)}{r^2}\right\}\right] - \left[\frac{\partial}{\partial z}\left\{\frac{\partial\psi}{\partial r}\frac{L(\psi)}{r^2}\right\}\right.$$

$$\left. - \frac{\partial}{\partial r}\left\{\frac{\partial\psi}{\partial z}\frac{L(\psi)}{r^2}\right\}\right] + \frac{\mu}{\rho}\left[\frac{\partial}{\partial z}\left(\frac{T}{r^2}\frac{\partial T}{\partial r}\right) - \frac{\partial}{\partial r}\left(\frac{T}{r^2}\frac{\partial T}{\partial z}\right)\right] - \left[\frac{\partial}{\partial z}\left(\frac{U}{r^2}\frac{\partial U}{\partial r}\right)\right.$$

$$\left. - \frac{\partial}{\partial r}\left(\frac{U}{r^2}\frac{\partial U}{\partial z}\right)\right] = \frac{\eta}{\rho}\left[\frac{1}{r}\frac{\partial^2}{\partial z^2}L(\psi) + \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}L(\psi)\right\}\right]$$

or

$$\frac{2\mu T}{\rho r^3}\frac{\partial T}{\partial z} - \frac{2U}{r^3}\frac{\partial U}{\partial z} + \frac{\partial(\psi, L(\psi)/r^2)}{\partial(z, r)} - \frac{\mu}{\rho}\frac{\partial(\chi, L(\chi)/r^2)}{\partial(z, r)}$$

$$= \frac{\eta}{\rho r}L\left[L(\psi)\right]. \qquad (61.10)$$

Equation (61.09) can be rewritten as

$$\frac{\partial(\psi,U)}{\partial(z,r)} - \frac{\mu}{\rho}\frac{\partial(\chi,T)}{\partial(z,r)} = \frac{\eta}{\rho} rL(U) \tag{61.11}$$

Similarly, using (61.05) in (61.03), we get three scaler equations

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\left(\frac{\partial\psi}{\partial z}\frac{\partial\chi}{\partial r} - \frac{\partial\chi}{\partial z}\frac{\partial\psi}{\partial r}\right)\right] = 0 \tag{61.12}$$

$$\frac{\partial}{\partial z}\left[\frac{1}{r}\left(\frac{\partial\psi}{\partial z}\frac{\partial\chi}{\partial r} - \frac{\partial\chi}{\partial z}\frac{\partial\psi}{\partial r}\right)\right] = 0 \tag{61.13}$$

$$\frac{\partial}{\partial z}\left(\frac{T}{r^2}\frac{\partial\psi}{\partial r} - \frac{U}{r^2}\frac{\partial\chi}{\partial r}\right) - \frac{\partial}{\partial r}\left(\frac{T}{r^2}\frac{\partial\psi}{\partial z} - \frac{U}{r^2}\frac{\partial\chi}{\partial z}\right) = 0 \tag{61.14}$$

Equations (61.12) and (61.13) imply that

$$\frac{1}{r}\left(\frac{\partial\psi}{\partial z}\frac{\partial\chi}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial\chi}{\partial z}\right) = \text{constant, } C$$

or

$$\frac{1}{r}\frac{\partial(\psi,\chi)}{\partial(z,r)} = C\ .$$

We consider the case when $C = 0$. This will be the case, for instance, if there is a stagnation point in the meridian plane. Then we have

$$\frac{\partial(\psi,\chi)}{\partial(z,r)} = 0 \tag{61.15}$$

Equation (61.14) simplifies to

$$\frac{\partial(\psi,T/r^2)}{\partial(z,r)} - \frac{\partial(\chi,U/r^2)}{\partial(z,r)} = 0 \tag{61.16}$$

The governing equations of motion (61.01) to (61.04) are, therefore, replaced by (61.10), (61.11), (61.15) and (61.16) in terms of the functions ψ, χ, U and T.

Equation (61.15) implies, in general, that

$$\chi = f(\psi) \tag{61.17}$$

where f is an arbitrary function of ψ.

Using (61.17) in (61.16), we get

$$\frac{\partial(\psi, T/r^2)}{\partial(z,r)} - \frac{\partial(\psi, f'(\psi)U/r^2)}{\partial(z,r)} = 0$$

Since

$$\frac{\partial(\psi, f'(\psi)Ur^2)}{\partial(z,r)} = \frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}(f'(\psi)U/r^2) - \frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}(f'(\psi)U/r^2)$$

$$= f'(\psi)\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}(\frac{U}{r^2}) - f'(\psi)\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}(\frac{U}{r^2})$$

$$= \frac{\partial(f(\psi), U/r^2)}{\partial(z,r)} .$$

Hence (61.16) reduces to

$$\frac{\partial(\psi, (T-f'(\psi)U)/r^2)}{\partial(z,r)} = 0,$$

which implies that

$$T - f'(\psi)\,U = r^2 g(\psi), \tag{61,18}$$

where $g(\psi)$ is an arbitrary function of ψ.

If we impose the restriction

$$L(U) = 0 \tag{61.19}$$

on U, then (61.11) reduces to

$$\frac{\partial(\psi,U)}{\partial(z,r)} - \frac{\mu}{\rho}\frac{\partial(\chi,T)}{\partial(z,r)} = 0 \qquad (61.20)$$

Note that in the case of inviscid fluids (61.20) holds without the restriction imposed by (61.19). Using (61.17) in (61.20), we have

$$\frac{\partial(\psi,\ U-(\mu/\rho)f'(\psi)T\ )}{\partial(z,r)} = 0,$$

which means that

$$U - \frac{\mu}{\rho} f'(\psi)\ T = h(\psi) \qquad (61.22)$$

where $h(\psi)$ is an arbitrary function of ψ.

Solving (61.18) and (61.22) for U and T, we get

$$U = \frac{h(\psi) + (\mu/\rho)\ r^2 f'(\psi)\ g(\psi)}{1 - \frac{\mu}{\rho}\left[f'(\psi)\right]^2}$$

$$T = \frac{h(\psi)\ f'(\psi) + r^2 g(\psi)}{1 - (\mu/\rho)\left[f'(\psi)\right]^2} \qquad (61.22)$$

assuming that $\quad 1 - \frac{\mu}{\rho}\left[f'(\psi)\right]^2 \neq 0.$

Using (61.17) and (61.22) in (61.10), we find that the stream function ψ must satisfy the equation

$$\frac{2\mu}{\rho r^3}\frac{r^2 g(\psi) + h(\psi)f'(\psi)}{\{1 - (\mu/\rho)f'^2(\psi)\}^2}\frac{\partial\psi}{\partial z}\Big[\{r^2 g'(\psi) + h(\psi)f''(\psi) + f'(\psi)h'(\psi)\}$$

$$+ \frac{2\mu}{\rho} f'(\psi)f''(\psi)\ \frac{r^2 g + hf'}{1 - \frac{\mu}{\rho}f'^2}\Big] - \frac{2}{r^3}\frac{h + (\mu/\rho)r^2 f' g}{(1 - \frac{\mu}{\rho}f'^2)^2}\frac{\partial\psi}{\partial z}$$

$$\cdot\{(h' + \frac{\mu}{\rho}r^2 f'\ g' + \frac{\mu}{\rho}r^2 f'' g) + \frac{h + (\mu/\rho)r^2 f' g}{1 - (\mu/\rho)\ f'^2}\frac{2\mu}{\rho}f'f''\ \}$$

$$+ \frac{\partial(\psi, L(\psi)/r^2)}{\partial(z,r)} - \frac{\mu f'}{\rho}\frac{\partial(\psi,\{f'L(\psi) + f''[(\frac{\partial\psi}{\partial z})^2 + (\frac{\partial\psi}{\partial r})^2]\}/r^2)}{\partial(z,r)}$$

(cont'd)

$$= \frac{\eta}{\rho r} L\{L(\psi)\}, \qquad (61.23)$$

since

$$L(\chi) = f' \frac{\partial^2\psi}{\partial z^2} + f'' \left(\frac{\partial\psi}{\partial z}\right)^2 + f' \frac{\partial^2\psi}{\partial r^2} + f'' \left(\frac{\partial\psi}{\partial r}\right)^2 - \frac{1}{r} f' \frac{\partial\psi}{\partial r}$$

$$= f' L(\psi) + f'' \left\{\left(\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right\} .$$

Equation (61.23) can be simplified further to give

$$\frac{2}{r^3(1-\frac{\mu}{\rho} f'^2)^2} \left[\frac{\mu}{\rho} (1-\frac{\mu}{\rho} f'^2) r^4 g g' + \frac{\mu^2}{\rho^2} r^4 f' f'' g^2 - (1-\frac{\mu}{\rho} f'^2) h h' - \frac{\mu}{\rho} f' f'' h^2\right] \frac{\partial\psi}{\partial z} + \frac{\partial(\psi, L(\psi)/r^2)}{\partial(z,r)}$$

$$- \frac{\partial\left(\psi, \frac{\mu}{\rho} f'\left[f' L(\psi) + f''\{(\frac{\partial\psi}{\partial z})^2 + (\frac{\partial\psi}{\partial r})^2\}\right]/r^2\right)}{\partial(z,r)} = \frac{\eta}{\rho r} L\{L(\psi)\},$$

or

$$\frac{\partial(\psi, G/r^2)}{\partial(z,r)} = \frac{\eta}{\rho r} L\{L(\psi)\} , \qquad (61.24)$$

where

$$G = (1-\frac{\mu}{\rho} f'^2) L(\psi) - \frac{\mu}{\rho} f' f''\{(\frac{\partial\psi}{\partial z})^2 + (\frac{\partial\psi}{\partial r})^2\}$$

$$+ \frac{\mu}{\rho} \frac{r^4 g g'}{(1-\frac{\mu}{\rho} f'^2)} + \frac{h h'}{(1-\frac{\mu}{\rho} f'^2)} + \frac{\mu}{\rho} \frac{f' f''(\frac{\mu}{\rho} r^4 g^2 + h^2)}{(1-\frac{\mu}{\rho} f'^2)}$$

$$= (1-\frac{\mu}{\rho} f'^2) L(\psi) - \frac{\mu}{\rho} f' f''\{(\frac{\partial\psi}{\partial z})^2 + (\frac{\partial\psi}{\partial r})^2\} + r^4 g_1' + h_1' , \qquad (61.25)$$

functions g_1 and h_1 are defined as

$$g_1 = \frac{\mu g^2}{2\rho(1-\frac{\mu}{\rho} f'^2)} , \qquad h_1 = \frac{h^2}{2(1-\frac{\mu}{\rho} f'^2)}$$

Equation (61.24) is a nonlinear partial differential equation for the stream function ψ.

Section 2. Some Exact solutions:

As the governing equations are non-linear, analytic solutions for the viscous flows of conducting fluids are scarce. K. B. Ranger (1970) has given some interesting examples of exact solutions for axisymmetric flows of inviscid infinitely conducting fluids. When viscosity is neglected (61.24) reduces to

$$\frac{\partial(\psi,\ G/r^2)}{\partial(z,r)} = 0,$$

which implies, in general, that

$$G \equiv (1-\frac{\mu}{\rho} f'^2)L(\psi) - \frac{\mu}{\rho} f'f''\{(\frac{\partial\psi}{\partial z})^2 + (\frac{\partial\psi}{\partial r})^2\} + r^4 g_1' + h_1'$$

$$= r^2 F(\psi) \qquad (62.01)$$

where $F(\psi)$ is an arbitrary function of ψ.

Any solution of (62.01) which simultaneously satisfies the equation

$$L\{L(\psi)\} = 0, \qquad (62.02)$$

represents a solution of the equation (61.24) and gives an analytic solution for a viscous flow problem.

The fourth-order partial differential equation (62.02) can be decomposed to a pair of second-order equations:

$$\begin{aligned} L(\psi_0) &= 0, \\ L(\psi_1) &= \psi_0 \end{aligned} \qquad (62.03)$$

and the general solution (62.02) is given by $\psi = \psi_1 + \psi_0$.

If we take $f'(\psi)$ to be constant, $g'(\psi)=0$, $h'(\psi)=0$

and $F(\psi) = -kA$ where $(1-\frac{\mu}{\rho}f'^2) = k$, then (62.01) becomes

$$L(\psi) = -Ar^2 \qquad (62.04)$$

whose general solution is

$$\psi = \frac{1}{8}Ar^4 + \psi_0$$

From (62.04), we see that

$$L\{L(\psi)\} = -L(Ar^2) = 0$$

Therefore any solution of (62.04) will represent a solution of a viscous flow problem.

Using spherical polar coordinates (R,θ,ϕ), equation (62.04) may be wtitten as

$$\frac{\partial^2\psi}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{\cos\theta}{R^2\sin\theta}\frac{\partial\psi}{\partial\theta} = -AR^2\sin^2\theta$$

or

$$\frac{\partial^2\psi}{\partial R^2} + \frac{1-\beta^2}{R^2}\frac{\partial^2\psi}{\partial\beta^2} = -AR^2(1-\beta^2) \qquad (62.05)$$

where $\beta = \cos\theta$.

We try a solution of (62.05) of the form

$$\psi = H(R)(1-\beta^2) \qquad (62.06)$$

Substituting (62.06) in (62.05), we obtain

$$R^2H'' - 2H = -AR^4$$

whose general solution is

$$H(R) = C_1R^2 + C_2R^{-1} - \frac{1}{10}AR^4 \qquad (62.07)$$

The requirement of finite velocity as $R \to 0$ implies that $C_2=0$. In the case of a spherical vortex inside $R = a$, we should have $\psi = 0$ at $R = a$ and therefore

$$C_1 = \frac{1}{10} Aa^2$$

Therefore the streamfunction is given by

$$\psi = \frac{A}{10} R^2(a^2 - R^2)(1 - \beta^2) \qquad (62.08)$$

where $R \leq a$.

If g=0 and h=0, from (61.22) we see that

$$U = 0, \quad T = 0$$

and we have meridional motion. The streamfunction

$$\psi = \frac{A}{10} R^2(a^2 - R^2)(1 - \beta^2)$$

$$= \frac{A}{10} r^2(a^2 - r^2 - z^2) \quad \text{with} \quad r^2 + z^2 \leq a^2,$$

gives "Hill's Spherical Vortex" inside the sphere $R = a$ (Lamb, 1932)

However, if h=0 but $g \neq o$ then from (61.22), we get

$$\frac{U}{r} = \frac{\mu}{\rho} \frac{g f'}{(1 - \frac{\mu}{\rho} f'^2)} r, \quad \frac{T}{r} = \frac{g}{(1 - \frac{\mu}{\rho} f'^2)} r \qquad (62.09)$$

representing a rotation with constant angular velocity $\frac{\mu}{\rho} g f'/(1 - \frac{\mu}{\rho} f'^2)$ added to the above case of Hill's spherical vortex.

If we take $h \neq 0$ then the swirl component of velocity

$$\frac{U}{r} = \frac{h/r + (\mu/\rho) g f' r}{(1 - (\mu/\rho) f'^2)}$$

becomes infinite on the axis. Hence we must take h = 0. From (61.07) and (61.08) we find that

$$\frac{\partial P}{\partial z} = \frac{2\eta}{\rho} A - A\frac{\partial \psi}{\partial z} + \frac{\mu}{\rho} f'^2 A \frac{\partial \psi}{\partial z}$$

$$= \frac{2\eta}{\rho} A - kA\frac{\partial \psi}{\partial z}, \qquad k = 1 - \frac{\mu}{\rho} f'^2$$

and

$$\frac{\partial P}{\partial r} = \frac{\mu}{\rho} f'^2 A \frac{\partial \psi}{\partial r} - A\frac{\partial \psi}{\partial r} - \frac{\mu}{\rho} \frac{2rg^2}{k^2} + \left(\frac{\mu/\rho\ gf'}{k}\right)^2 2r$$

$$= -k A\frac{\partial \psi}{\partial r} - \frac{\mu}{\rho k} g^2 2r$$

which on integration give us

$$P = \frac{1}{\rho} p + \tfrac{1}{2} V^2$$

$$= \frac{2\eta}{\rho} Az - \frac{1}{10} kA^2 r^2 (a^2 - r^2 - z^2) - \frac{\mu}{k\rho} g^2 r^2 + P_0$$

for $R \leq a$, that is $r^2 + z^2 \leq a^2$. (62.10)

We now consider the stream function in the region outside the sphere R = a. From (62.07), we observe that

$$\psi^* = (C_1 R^2 + C_2 R^{-1})(1 - \beta^2) \qquad (62.11)$$

is a solution of (62.05) with A = 0.

Since the two expressions for the stream function, namely (62.08) and (62.11), must be such that the component of velocity normal to the surface of the sphere R = a is zero on both sides of surface, we require that the radial velocity given by (62.11) be zero,

$$-\frac{1}{R^2 \sin\theta}\frac{\partial\psi^*}{\partial\theta} = \frac{1}{R^2}\frac{\partial\psi^*}{\partial\beta} = 0 \qquad \text{when } R = a,$$

or

$$C_2 = -C_1 a^3$$

Therefore (62.11) becomes

$$\psi^* = C_1\left(R^2 - \frac{a^3}{R}\right)(1 - \beta^2) \tag{62.12}$$

Further, the continuity of tangential velocities on both sides of the surface $R = a$, requires that we have

$$\frac{1}{R^2\sin\theta}\frac{\partial\psi}{\partial R} = \frac{1}{R^2\sin\theta}\frac{\partial\psi^*}{\partial R} \qquad \text{when } R = a.$$

From (62.08) and (62.12), we get

$$C_1 = -\frac{1}{15} A\, a^2 \tag{62.13}$$

and therefore stream function outside the region $R \leq a$ is given by

$$\psi^* = -\frac{A}{15} a^2\left(R^2 - \frac{a^3}{R}\right)(1 - \beta^2)$$

$$= -\frac{A}{15} a^2 r^2\left\{1 - \left(\frac{a^2}{r^2+z^2}\right)^{3/2}\right\} \tag{62.14}$$

The stream function given by (62.14) determines an irrotational flow outside the sphere $R = a$ which is parallel to z – axis with velocity $-\frac{2}{15} Aa^2$ at infinity.

Section 3. Flows with $\frac{\partial\psi}{\partial r} = 0$

In this case components of velocity and magnetic field along the z - axis are zero, and the motion is in planes perpendicular to z - axis.

When $\frac{\partial\psi}{\partial r} = 0$, (61.24) can be written as

$$\frac{d\psi}{dz}\frac{\partial}{\partial r}\left(\frac{G}{r^2}\right) = \frac{\eta}{\rho r}\frac{d^4\psi}{dz^4}$$

or

$$r\left[(1 - \frac{\mu}{\rho} f'^2)\frac{d^2\psi}{dz^2}\left(-\frac{2}{r^3}\right) - \frac{\mu}{\rho} f'f''\left(\frac{d\psi}{dz}\right)^2\left(-\frac{2}{r^3}\right)\right.$$
$$+ \frac{2\mu}{\rho}\frac{rgg'}{(1-\frac{\mu}{\rho}f'^2)} + \frac{hh'}{(1-\frac{\mu}{\rho}f'^2)}\left(-\frac{2}{r^3}\right)$$
$$\left. + \frac{\mu}{\rho}\frac{f'f''}{(1-\frac{\mu}{\rho}f'^2)^2}\left(\frac{2\mu}{\rho} rg^2 - \frac{2h^2}{r^3}\right)\right] - \frac{\eta}{\rho}\frac{d^4\psi}{dz^4}\Big/\frac{d\psi}{dz} = 0 \qquad (63.01)$$

where f, g, h are functions of ψ , which is a function of z only. Equation (63.01) can be satisfied only if coefficient of different powers of r are zero. Therefore

$$(1-\frac{\mu}{\rho} f'^2)\frac{d^2\psi}{dz^2} - \frac{\mu}{\rho} f'f''\left(\frac{d\psi}{dz}\right)^2 + \frac{h\,h'}{(1-\frac{\mu}{\rho} f'^2)}$$
$$+ \frac{\mu}{\rho}\frac{f'f''h^2}{(1-\frac{\mu}{\rho} f'^2)^2} = 0 \qquad (63.02)$$

$$\frac{\mu}{\rho}\frac{gg'}{(1-\frac{\mu}{\rho} f'^2)} + \frac{\mu^2}{\rho^2}\frac{f'f''g^2}{(1-\frac{\mu}{\rho} f'^2)} = 0 \qquad (63.03)$$

$$\frac{d^4\psi}{dz^4} = 0 \qquad (63.04)$$

From (63.03) we get

$$g^2 = A(1 - \frac{\mu}{\rho} f'^2) \qquad (63.05)$$

where A is an arbitrary constant.

From (63.02), we have

$$\sqrt{(1- \frac{\mu}{\rho} f'^2)} \frac{d}{dz}\left[\sqrt{(1- \frac{\mu}{\rho} f'^2)} \frac{d\psi}{dz}\right] + \tfrac{1}{2} \frac{d}{d\psi}\left[\frac{h^2}{1-\frac{\mu}{\rho} f'^2}\right] = 0$$

or

$$(1- \frac{\mu}{\rho} f'^2)\left(\frac{d\psi}{dz}\right)^2 + \frac{h^2}{(1-\frac{\mu}{\rho} f'^2)} = B \qquad (63.06)$$

where B is an arbitrary constant.

Equation (63.06) gives

$$\frac{d\psi}{dz} = \pm \left[B(1- \frac{\mu}{\rho} f'^2) - h^2\right]^{\frac{1}{2}} \Big/ (1- \frac{\mu}{\rho} f'^2)$$

By choosing f' and h, we can determine $\psi(z)$ which must also satisfy (63.04) for viscous fluids.

Velocity field is given by

$$\vec{V} = - \frac{1}{r}\frac{\partial\psi}{\partial r}\vec{e}_z + \frac{1}{r}\frac{\partial\psi}{\partial z}\vec{e}_r + \frac{U}{r}\vec{e}_\phi$$

$$= \frac{1}{r}\left[\left\{\pm\sqrt{B(1- \frac{\mu}{\rho} f'^2)- h^2}\Big/(1- \frac{\mu}{\rho} f'^2)\right\}\vec{e}_r + \frac{h' + \frac{\mu}{\rho} r^2 f' g}{(1- \frac{\mu}{\rho} f'^2)}\vec{e}_\phi\right]$$

and magnetic field $\vec{H}$ is given by

$$\vec{H} = \frac{1}{r}\left[\left\{\pm f'\sqrt{B(1- \frac{\mu}{\rho}f'^2)- h^2}\Big/(1- \frac{\mu}{\rho} f'^2)\right\}\vec{e}_r + \frac{hf' + r^2 g}{(1- \frac{\mu}{\rho}f'^2)}\vec{e}_\phi\right]$$

where g is given by (63.05). Flow between two concentric rotating cylinders belongs to this class of solutions.

Section 4. An Inviscid Flow Problem

In this section we consider the flow of an inviscid incompressible fluid in the presence of magnetic field when the swirl or rotational component of velocity is zero, i.e.

$$U = \frac{h(\psi) + (\mu/\rho)r^2 f'(\psi)g(\psi)}{1 - \frac{\mu}{\rho} f'^2} = 0, \qquad (64.01)$$

which implies that if ψ is not a fuction of r alone then either

(i) $h(\psi) = 0, \qquad f'(\psi) = 0;$

or (ii) $h(\psi) = 0, \qquad g(\psi) = 0$

Since magnetic field $\vec{H}$ is given by

$$\vec{H} = -\frac{1}{r} f'(\psi) \frac{\partial\psi}{\partial r} \vec{e}_z + \frac{1}{r} f'(\psi) \frac{\partial\psi}{\partial z} \vec{e}_r + \frac{T}{r} \vec{e}_\phi ,$$

we see that in the first case $\vec{H} = \frac{T}{r} \vec{e}_\phi = rg(\psi)\, \vec{e}_\phi$, i.e., the magnetic field is toroidal; while in the second case T = 0 and

$$\vec{H} = f'(\psi)\{-\frac{1}{r}\frac{\partial\psi}{\partial r}\vec{e}_z + \frac{1}{r}\frac{\partial\psi}{\partial z}\vec{e}_r\}$$

implying that the magnetic field is poloidal. We thus have

<u>Theorem</u>: If the fluid flow is meridional then the magnetic field is either toroidal or poloidal unless the flow is along the axis alone.

In the case of meridional flow under the influence of a toroidal magnetic field, we have f' = 0, h = 0 and so the equation (61.24) reduces to

$$L(\psi) + \frac{\mu}{\rho} r^4 g(\psi)g'(\psi) = r^2 F(\psi), \qquad (64.02)$$

where $F(\psi)$ is an arbitrary function of ψ.

K. B. Ranger (1970) considered the cases when

(i) $g(\psi)g'(\psi) = K$, K being a positive constant, $F(\psi) = 0$

(ii) $g(\psi)g'(\psi) = K$, $F(\psi) = -\alpha$, $(\alpha > 0)$.

In these cases (64.02) reduces to a linear partial differential equation of second order in ψ.

We consider the case when $g(\psi)g'(\psi) = \frac{\mu}{\rho} A\psi^2$, $F(\psi) = 0$. In this case (64.02) becomes

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + A\,\psi^2 r^4 = 0 \qquad (64.03)$$

We look for "invariant solutions" of (64.03) employing the group theory method developed by A. J. A. Morgan(1952). Definitions pertinent to this method and main results of Morgan's theory are given in Appendix B. Our first step is to find a possible one parameter transformation group such that the differential form

$$\phi = \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + A\psi^2 r^4 \qquad (64.04)$$

is "Conformally invariant" under the second enlargements of that group. We try a transformation group of the form

$$\begin{aligned} \bar{z} &= a^m z \\ \bar{r} &= a^n r \\ \bar{\psi} &= a^p \psi \end{aligned} \qquad (64.05)$$

where m, n, p are the real numbers, a is the parameter of the group; and find m, n, p such that the differential form ϕ is conformally invariant under the transformations (64.05).

Employing (64.05) in (64.04), we get

$$\frac{\partial^2\bar{\psi}}{\partial\bar{z}^2} + \frac{\partial^2\bar{\psi}}{\partial\bar{r}^2} - \frac{1}{\bar{r}}\frac{\partial\bar{\psi}}{\partial\bar{r}} + A\bar{\psi}^2\bar{r}^4$$
$$= a^{p-2m}\frac{\partial^2\psi}{\partial z^2} + a^{p-2n}\frac{\partial^2\psi}{\partial r^2} - a^{p-2n}\frac{1}{r}\frac{\partial\psi}{\partial r} + Aa^{2p+4n}\psi^2 r^4$$

which shows that the differential form ϕ given by (64.04) is an "absolute invariant" under the continuous one parameter group of transformations defined by (64.05) if

$$p - 2m = p - 2n = 2p + 4n = 0$$

or

$$m = n, \qquad p = -6n \quad .$$

It follows that ϕ is an absolute invariant under the transformation group

$$\begin{aligned} \bar{z} &= a^m z = bz \\ \bar{r} &= a^m r = br \\ \bar{\psi} &= a^{-6m}\psi = b^{-6}\psi \end{aligned} \qquad (64.06)$$

As a consequence of Theorem 2, Appendix B, we see that the invariant solutions of the partial differential equation (64.03) can be expressed in terms of the solutions of a new eqation with the number of independent variables reduced by one. As there are only two independent variables in the original equation, the problem reduces to the solution of an ordinary differential equation.

We can express the differential form ϕ in terms of new variables η and G and the derivatives of G with respect to η. The variable η is to be an absolute invariant of the subgroup of the transformations of the independent variables z, r. By definition of absolute invariant of a group, η is a function such that

$$\eta(\bar{z}, \bar{r}) = \eta(z, r)$$

where

$$\bar{z} = b\,z, \qquad \bar{r} = b\,r \tag{64.07}$$

There is no well defined manner for finding an absolute invariant, but recognising that the transformation involves powers of b, we might try

$$\eta = z\,r^s$$

and seek a value of s such that zr^s would be invariant under (64.07). This means that

$$\bar{z}\,\bar{r}^s = z\,r^s$$

or

$$b\,z\,b^s\,r^s = z\,r^s$$

Therefore, we must have

$$s = -1$$

This choice of s means that

$$\eta = z/r \tag{64.08}$$

is an absolute invariant of the subgroup of transformations defined by (64.07).

The function G is defined by

$$G(\eta) = g(z,r,\psi) \tag{64.09}$$

where g is an absolute invariant of the group of transformations (64.06) for both dependent and independent variables.

As there is only one dependent variable ψ, we have to find just one absolute invariant of (64.06) which is functionally independent of η. We try g of the form

$$g = \psi \, r^t$$

and look for a value of t such that ψr^t is invariant under (64.06). This implies that

$$b^{-6+t} \psi r^t = \psi \, r^t$$

which requires that

$$t = 6$$

Therefore

$$g = \psi \, r^6 \qquad (64.10)$$

According to Theorem 2 (Appendix B), the invariant solutions of equation (64.03) can now be expressed in terms of η and the function $G(\eta)$ defined by (64.09).

From (64.09) and (64.10), we have

$$g = \psi \, r^6 = G(\eta)$$

so that,

$$\psi = r^{-6} \, G(\eta) \qquad (64.11)$$

where $\eta = z/r$.

Substituting $\psi = r^{-6} \, G(\eta)$ in (64.03) and employing $\eta = z/r$, we get

$$r^{-6} \frac{\partial}{\partial z}\left[\frac{1}{r} G'(\eta)\right] + \frac{\partial}{\partial r}\left[- 6r^{-7} \, G(\eta) - r^{-6} \frac{z}{r^2} G'(\eta)\right]$$

$$- \frac{1}{r}\left[-6r^{-7} \, G(\eta) - r^{-6} \, \frac{z}{r^2} \, G'(\eta)\right] + Ar^{-12} \, G^2(\eta) \, r^4 = 0$$

or

$$(1+\eta^2)\ G''(\eta) + 15\eta\ G'(\eta) + 48\ G(\eta) + A\ G^2(\eta) = 0 \tag{64.12}$$

We have thus reduced the problem of solving the partial differential equation (64.03) to that of finding the solution of a non-linear ordinary differential equation in $G(\eta)$.

Let us consider meridional flow inside an infinite cone $z = r$. On the boundary of cone, $\eta = 1$ and since the flow must be tangential to it, we must have

$$G(1) = 0 \tag{64.13}$$

We specify the other boundary condition of the form

$$G'(1) = K$$

where K is a constant.

We solve (64.12) numerically subject to the boundary conditions (64.13) and (64.14) by applying Runge-Kutta fourth-order method to the system of equations

$$G' = u$$

$$u' = -(15\eta\ u + 48G + AG^2)/\ (1 + \eta^2),$$

for specific values of the constants A and K; and plot some streamlines for the resulting flows.

Taking K = 40, for different values of A, the solutions are given by the following tables:

A = 4.0

η	G(η)	G'(η)
1.0	0.0	40.0
1.1	2.696	15.755
1.2	3.484	1.499
1.3	3.253	-5.071
1.4	2.628	-6.872
1.5	1.954	-6.378
1.6	1.377	-5.105
1.7	0.934	-3.781
1.8	0.614	-2.674
1.9	0.390	-1.834
2.0	0.239	-1.229
2.1	0.138	-0.808
2.2	0.073	-0.521
2.3	0.031	-0.328
2.4	0.005	-0.200

A = 2.0

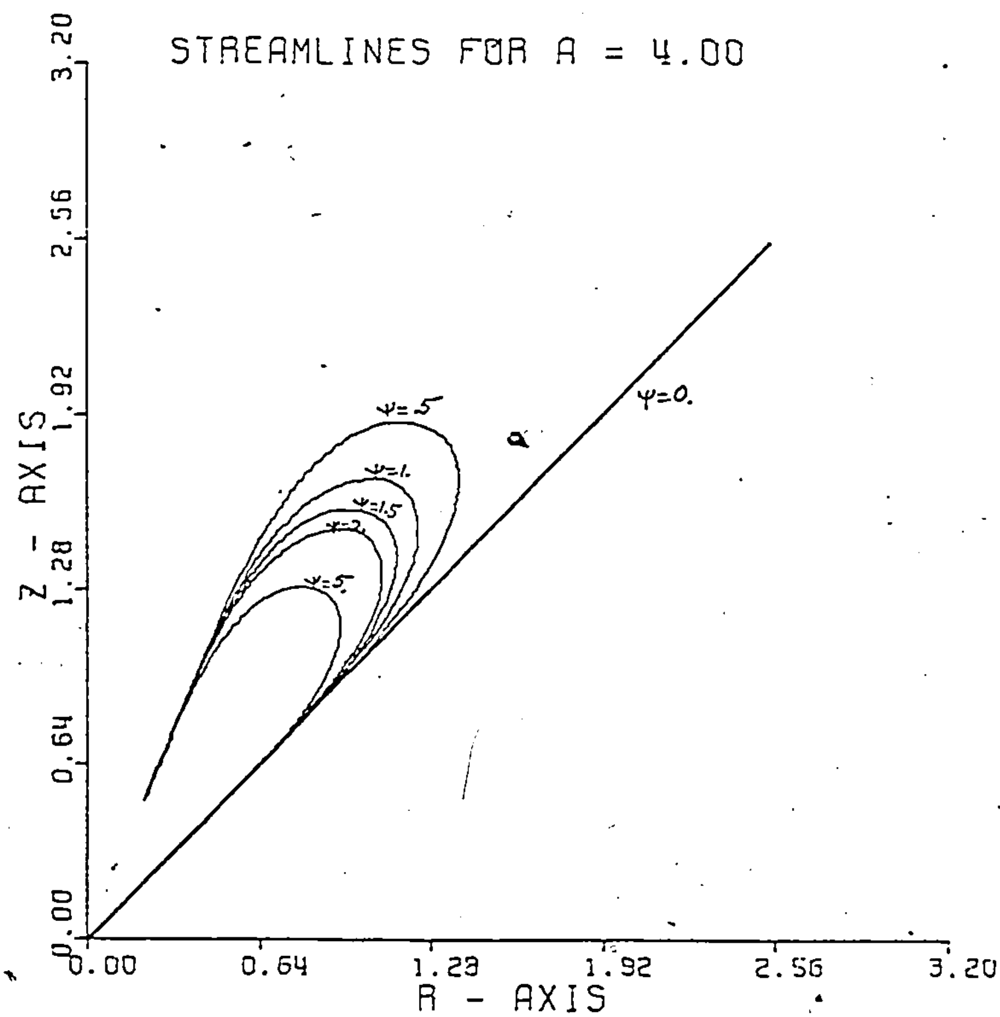
η	G(η)	G'(η)
1.0	0.0	40.0
1.1	2.703	15.98
1.2	3.537	2.19
1.3	3.386	-4.26
1.4	2.831	-6.31
1.5	2.196	-6.16
1.6	1.626	-5.17
1.7	1.167	-4.02
1.8	0.819	-2.98
1.9	0.564	-2.15
2.0	0.382	-1.52
2.1	0.255	-1.06
2.2	0.166	-0.73
2.3	0.105	-0.50
2.4	0.064	-0.34
2.5	0.036	-0.22
2.6	0.018	-0.15
2.7	0.006	-0.09

A = -2.0

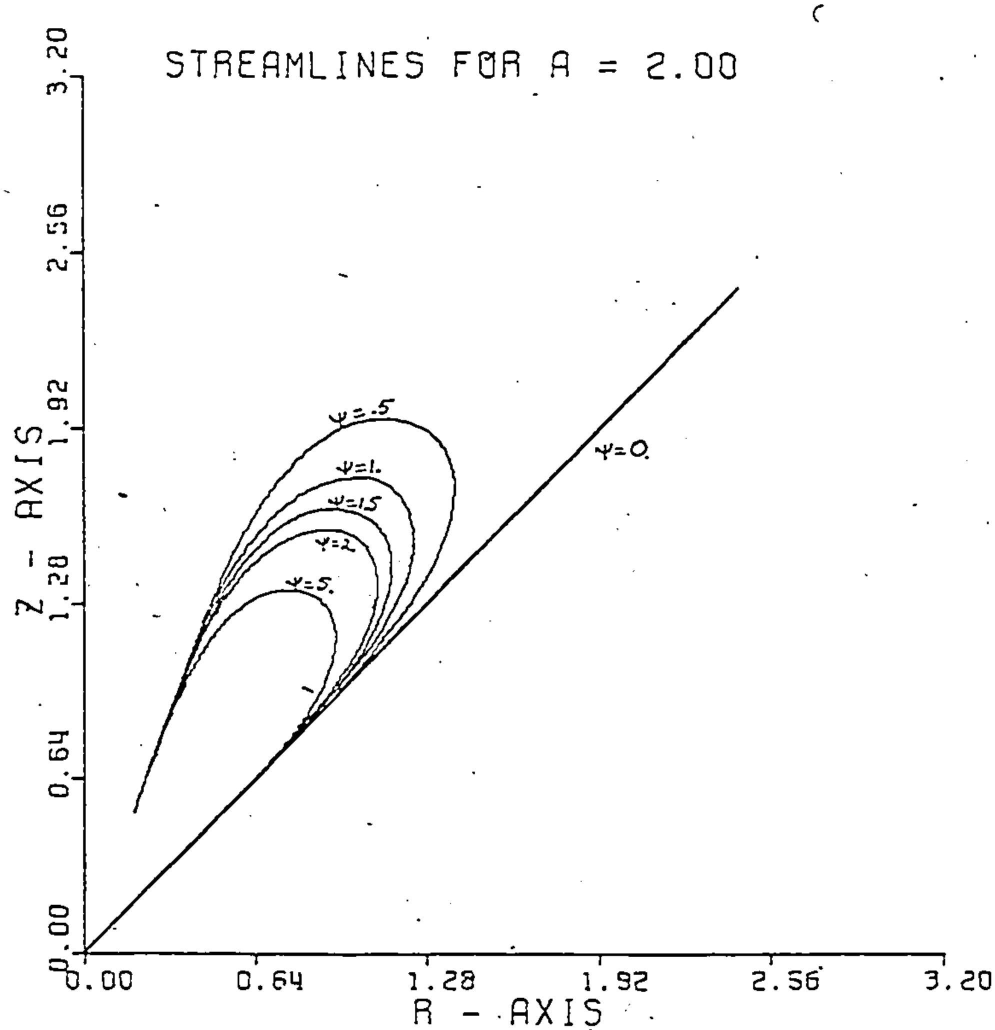
η	G(η)	G'(η)
1.0	0.0	40.0
1.1	2.716	16.433
1.2	3.645	3.655
1.3	3.670	-2.390
1.4	3.291	-4.763
1.5	2.779	-5.282
1.6	2.226	-4.942
1.7	1.801	-4.266
1.8	1.412	-3.518
1.9	1.096	-2.819
2.0	0.845	-2.217
2.1	0.648	-1.723
2.2	0.497	-1.327
2.3	0.380	-1.017
2.4	0.291	-0.777
2.5	0.223	-0.593
2.6	0.171	-0.452
2.7	0.131	-0.345
2.8	0.101	-0.264
2.9	0.078	-0.202
3.0	0.060	-0.655
3.1	0.046	-0.119
3.2	0.036	-0.092
3.3	0.028	-0.071
3.4	0.022	-0.055
3.5	0.017	-0.042
3.6	0.013	-0.033
3.7	0.010	-0.026
3.8	0.008	-0.020
3.9	0.006	-0.016
4.0	0.005	-0.012
4.1	0.004	-0.010
4.2	0.003	-0.008
4.3	0.002	-0.006
4.5	0.001	-0.004

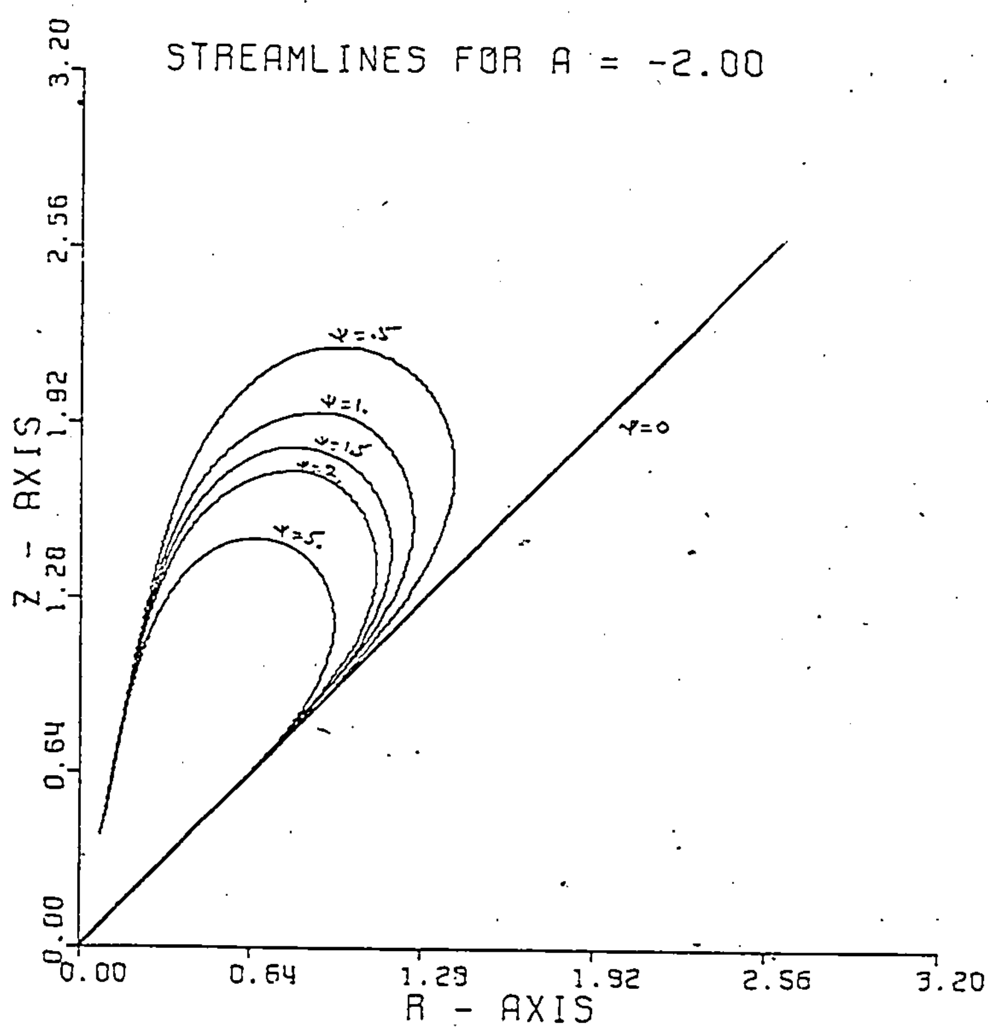
A = −4.0

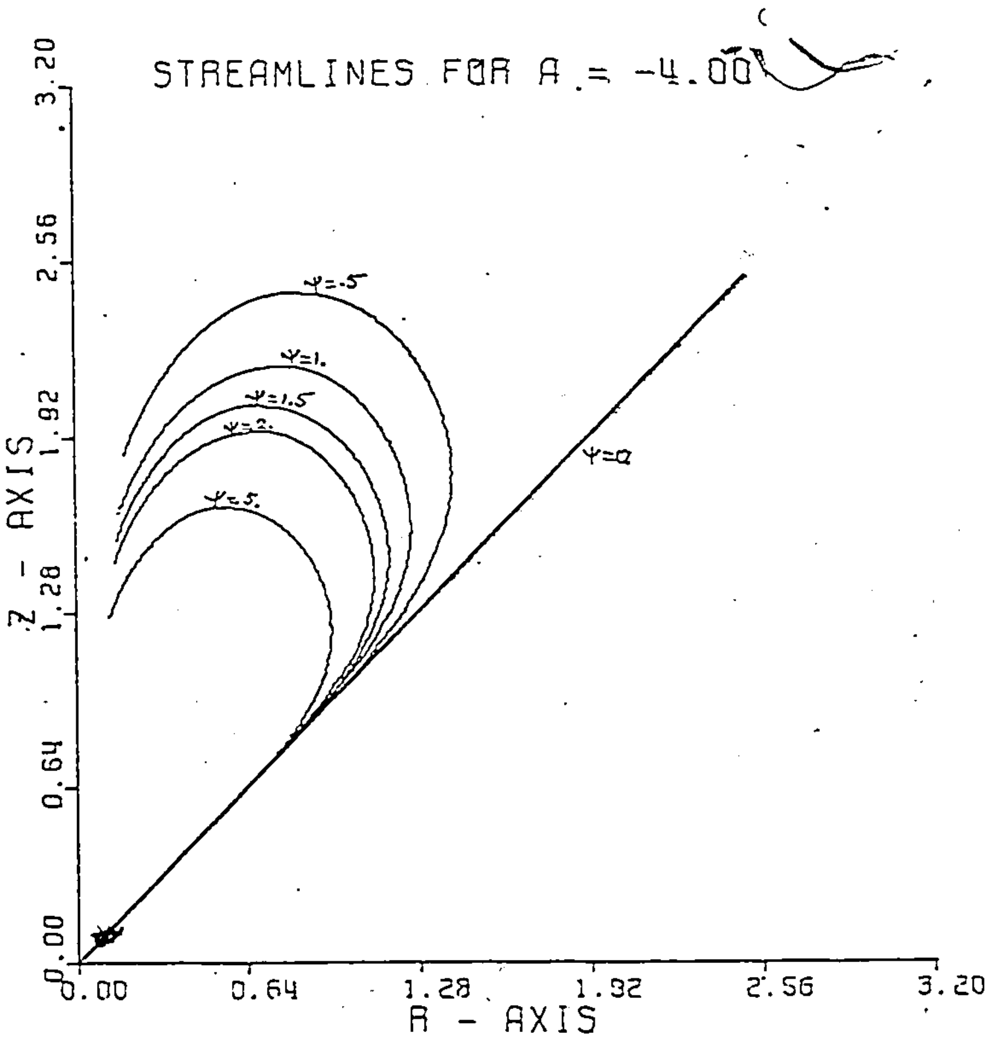
η	G(η)	G'(η)
1.0	0.0	40.0
1.1	2.723	16.662
1.2	3.705	4.422
1.3	3.822	−1.315
1.4	3.551	−3.731
1.5	3.130	−4.514
1.6	2.674	−4.508
1.7	2.240	−4.139
1.8	1.851	−3.628
1.9	1.515	−3.089
2.0	1.232	−2.579
2.1	0.998	−2.123
2.2	0.805	−1.731
2.3	0.649	−1.401
2.4	0.523	−1.129
2.5	0.422	−0.907
2.6	0.340	−0.727
2.7	0.275	−0.583
2.8	0.223	−0.467
2.9	0.181	−0.374
3.0	0.147	−0.301
3.1	0.120	−0.242
3.2	0.098	−0.195
3.3	0.081	−0.158
3.4	0.067	−0.128
3.5	0.055	−0.104
3.6	0.046	−0.085
3.8	0.032	−0.057
4.0	0.022	−0.039
4.2	0.016	−0.026
4.4	0.011	−0.018
4.6	0.008	−0.013
4.8	0.006	−0.011
5.0	0.004	−0.007
5.2	0.003	−0.005
5.5	0.002	−0.003
6.0	0.001	−0.001
7.0	0.0003	−0.0004
8.0	0.0001	−0.0001



STREAMLINES FOR A = 4.00
Z - AXIS
R - AXIS
0.00
0.64
1.28
1.92
2.56
3.20
ψ=.5
ψ=1.
ψ=1.5
ψ=2.
ψ=5.
ψ=0.


STREAMLINES FOR A = 2.00
Z - AXIS
R - AXIS
0.00
0.64
1.28
1.92
2.56
3.20
ψ=.5
ψ=1.
ψ=1.5
ψ=2.
ψ=5.
ψ=0.


STREAMLINES FOR A = -2.00
Z - AXIS
R - AXIS
0.00
0.64
1.28
1.92
2.56
3.20
ψ=.5
ψ=1.
ψ=1.5
ψ=2.
ψ=5.
ψ=0


STREAMLINES FOR A = -4.00
Ψ=.5
Ψ=1.
Ψ=1.5
Ψ=2.
Ψ=5.
Ψ=0
Z - AXIS
R - AXIS
0.00
0.64
1.28
1.92
2.56
3.20

APPENDIX A

Section 1. Some Results from Differential Geometry

Let

$$x = x(\phi,\psi), \qquad y = y(\phi,\psi) \tag{A.1}$$

define a system of curvilinear coordinates in the (x,y)-plane. With (ϕ,ψ) as curvilinear coordinates, the squared element of arc length along any curve is given by

$$ds^2 = E(\phi,\psi)\, d\phi^2 + 2F(\phi,\psi)\, d\phi\, d\psi + G(\phi,\psi) d\psi^2 \tag{A.2}$$

where,

$$\begin{aligned} E &= \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 \\ F &= \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \psi} \\ G &= \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 \end{aligned} \tag{A.3}$$

Equations (A.1) can be solved to obtain

$$\phi = \phi(x,y), \qquad \psi = \psi(x,y)$$

such that

$$\begin{aligned} \frac{\partial x}{\partial \phi} &= J\frac{\partial \psi}{\partial y}, & \frac{\partial y}{\partial \phi} &= -J\frac{\partial \psi}{\partial x}, \\ \frac{\partial x}{\partial \psi} &= -J\frac{\partial \phi}{\partial y}, & \frac{\partial y}{\partial \psi} &= J\frac{\partial \phi}{\partial x} \end{aligned} \tag{A.4}$$

provided that $0<|J|<\infty$, where J denotes the Jacobian given by

$$J = \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \phi}. \tag{A.5}$$

From (A.3) and (A.4), we find that

$$J = \pm W, \qquad \text{where } W = \sqrt{E\,G - F^2} \tag{A.6}$$

Let α be the angle made by the tangent to the coordinate line ψ = constant, directed in the sense of increasing ϕ, with x-axis. From the first equation of (A.3), we get

$$\frac{\partial x}{\partial \phi} = \sqrt{E} \cos \alpha , \qquad \frac{\partial y}{\partial \phi} = \sqrt{E} \sin \alpha . \tag{A.7}$$

The first two equations in (A.3) can be rewritten in the form

$$\frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \phi} = E$$

$$\frac{\partial x}{\partial \psi}\frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \psi}\frac{\partial y}{\partial \phi} = F$$

On solving these equations for $\frac{\partial x}{\partial \phi}$, we obtain

$$\left(\frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial y}{\partial \phi}\frac{\partial x}{\partial \psi}\right)\frac{\partial x}{\partial \phi} = E\frac{\partial y}{\partial \psi} - F\frac{\partial y}{\partial \phi}$$

or

$$J\frac{\partial x}{\partial \phi} = E\frac{\partial y}{\partial \psi} - F\frac{\partial y}{\partial \phi}$$

or

$$E\frac{\partial y}{\partial \psi} = J\frac{\partial x}{\partial \phi} + F\frac{\partial y}{\partial \phi} \tag{A.8}$$

Similarly, we find that

$$E\frac{\partial x}{\partial \psi} = F\frac{\partial x}{\partial \phi} - J\frac{\partial y}{\partial \phi} \tag{A.9}$$

Using (A.7) in (A.8) and (A.9), we get

$$\frac{\partial x}{\partial \psi} = \frac{F}{\sqrt{E}} \cos \alpha - \frac{J}{\sqrt{E}} \sin \alpha,$$
$$\frac{\partial y}{\partial \psi} = \frac{J}{\sqrt{E}} \cos \alpha + \frac{F}{\sqrt{E}} \sin \alpha. \qquad \text{(A.10)}$$

On computing the integrability conditions

$$\frac{\partial^2 x}{\partial \phi\, \partial \psi} = \frac{\partial^2 x}{\partial \psi\, \partial \phi}, \qquad \frac{\partial^2 y}{\partial \phi\, \partial \psi} = \frac{\partial^2 y}{\partial \psi\, \partial \phi}$$

from (A.7) and (A.10), we obtain

$$- \sqrt{E} \sin \alpha \frac{\partial \alpha}{\partial \psi} + \left(\frac{F}{\sqrt{E}} \sin \alpha + \frac{J}{\sqrt{E}} \cos \alpha \right) \frac{\partial \alpha}{\partial \phi}$$
$$= \left(- \frac{1}{2\sqrt{E}} \frac{\partial E}{\partial \psi} + \frac{1}{\sqrt{E}} \frac{\partial F}{\partial \phi} - \frac{F}{2E\sqrt{E}} \frac{\partial E}{\partial \phi}\right) \cos \alpha$$
$$+ \left(\frac{J}{2E\sqrt{E}} \frac{\partial E}{\partial \phi} - \frac{1}{\sqrt{E}} \frac{\partial J}{\partial \phi}\right) \sin \alpha$$

and

$$\sqrt{E} \cos \alpha \frac{\partial \alpha}{\partial \psi} - \left(\frac{F}{\sqrt{E}} \cos \alpha - \frac{J}{\sqrt{E}} \sin \alpha \right) \frac{\partial \alpha}{\partial \phi} = \left(- \frac{1}{2\sqrt{E}} \frac{\partial E}{\partial \psi}\right.$$
$$\left. + \frac{1}{\sqrt{E}} \frac{\partial F}{\partial \phi} - \frac{F}{2E\sqrt{E}} \frac{\partial E}{\partial \phi}\right) \sin \alpha - \left(\frac{J}{2E\sqrt{E}} \frac{\partial E}{\partial \phi} - \frac{1}{\sqrt{E}} \frac{\partial J}{\partial \phi}\right) \cos \alpha .$$

Solving these equations for $\frac{\partial \alpha}{\partial \phi}$ and $\frac{\partial \alpha}{\partial \psi}$, we find that

$$\frac{\partial \alpha}{\partial \phi} = \frac{1}{2EJ} \left(- F \frac{\partial E}{\partial \phi} + 2E \frac{\partial F}{\partial \phi} - E \frac{\partial E}{\partial \psi} \right),$$
$$\frac{\partial \alpha}{\partial \psi} = \frac{1}{2EJ} \left(- F \frac{\partial E}{\partial \psi} + E \frac{\partial G}{\partial \phi} \right)$$

which can be written as

$$\frac{\partial \alpha}{\partial \phi} = \frac{J}{E} \Gamma^2_{11}, \qquad \frac{\partial \alpha}{\partial \psi} = \frac{J}{E} \Gamma^2_{12} \qquad \text{(A.11)}$$

where

$$\Gamma^2_{11} = \frac{1}{2W^2}\{-F\frac{\partial E}{\partial \phi} + 2E\frac{\partial F}{\partial \phi} - E\frac{\partial E}{\partial \psi}\},$$

$$\Gamma^2_{12} = \frac{1}{2W^2}\{E\frac{\partial G}{\partial \phi} - F\frac{\partial E}{\partial \psi}\}. \qquad (A.12)$$

From (A.11), we see that the integrability condition

$$\frac{\partial^2 \alpha}{\partial \phi\,\partial \psi} = \frac{\partial^2 \alpha}{\partial \psi\,\partial \phi}$$

implies that

$$\frac{\partial}{\partial \psi}(\frac{J}{E}\Gamma^2_{11}) - \frac{\partial}{\partial \phi}(\frac{J}{E}\Gamma^2_{12}) = 0. \qquad (A.13)$$

Equation (A.13) simply means that the Gaussian curvature

$$K = \frac{1}{W}\{\frac{\partial}{\partial \psi}(\frac{W}{E}\Gamma^2_{11}) - \frac{\partial}{\partial \phi}(\frac{W}{E}\Gamma^2_{12})\}$$

of a plane equals zero, and is referred to as Gauss equation.

Conversely, if E,F,G are given as functions of ϕ,ψ such that Gauss equation (A.13) is satisfied then we show that the functions $x(\phi,\psi)$ and $y(\phi,\psi)$ can be obtained in terms of E, F and G where E,F,G satisfy (A.2).

Equation (A.13) implies the existence of $\alpha = \alpha(\phi,\psi)$ such that

$$\frac{\partial \alpha}{\partial \phi} = \frac{J}{E}\Gamma^2_{11}, \qquad \frac{\partial \alpha}{\partial \psi} = \frac{J}{E}\Gamma^2_{12}.$$

Therefore α can be obtained from

$$\alpha = \int \left(\frac{\partial \alpha}{\partial \phi} d\phi + \frac{\partial \alpha}{\partial \psi} d\psi \right)$$

$$= \int \frac{J}{E} \left(\Gamma^2_{11} \, d\phi + \Gamma^2_{12} \, d\psi \right). \qquad \text{(A.14)}$$

The functions $x(\phi,\psi)$ and $y(\phi,\psi)$ are then given by

$$x = \int \{ (\sqrt{E} \cos \alpha) d\phi + \left(\frac{F}{\sqrt{E}} \cos \alpha - \frac{J}{\sqrt{E}} \sin \alpha \right) d\psi \},$$

$$y = \int \{ (\sqrt{E} \sin \alpha) d\phi + \left(\frac{F}{\sqrt{E}} \sin \alpha + \frac{J}{\sqrt{E}} \cos \alpha \right) d\psi \}. \qquad \text{(A.15)}$$

Introducing the complex variable $z = x + iy$, equations (A.15) can be written in a concise form as

$$z = \int \frac{1}{\sqrt{E}} \exp(i\alpha) \; \{E \, d\phi + (F + iJ) \, d\psi\} \qquad \text{(A.16)}$$

where α is given by (A.14).

We sum up the above results in the form of following theorem.

Theorem: Three functions E, F, G of ϕ,ψ serve as coefficients in the first fundamental form

$$ds^2 = E d\phi^2 + 2F d\phi d\psi + G d\psi^2 \quad .$$

for a plane with a curvilinear coordinate system

$$x = x(\phi,\psi), \qquad y = y(\phi,\psi) .$$

if and only if they satisfy the Gauss equation

$$\frac{\partial}{\partial \psi}\left(\frac{J}{E} \Gamma^2_{11} \right) - \frac{\partial}{\partial \phi}\left(\frac{J}{E} \Gamma^2_{12} \right) = 0.$$

If this condition is satisfied then the functions $x(\phi,\psi)$ and $y(\phi,\psi)$ defining the curvilinear coordinate system, are given in terms of E, F, G by .

$$z = \int \frac{1}{\sqrt{E}} \exp(i\alpha) \cdot \{E \, d\phi + (F + iJ) \, d\psi\}$$

where

$$\alpha = \int \frac{J}{E} \left(\Gamma_{11}^{2} \, d\phi + \Gamma_{12}^{2} \, d\psi \right) .$$

From the relation

$$W = \sqrt{EG - F^2} ,$$

we find that

$$\frac{\partial}{\partial\phi}\left(\frac{E}{2W^2}\right) = \frac{1}{2W^2} \left[\frac{\partial E}{\partial\phi} - \frac{E}{W^2} \left(E \frac{\partial G}{\partial\phi} + G \frac{\partial E}{\partial\phi} - 2F \frac{\partial F}{\partial\phi} \right)\right]$$

$$= \frac{1}{W^2} \left(F \, \Gamma_{11}^{2} - E \, \Gamma_{12}^{2} \right) \qquad \text{(A.17)}$$

Similarly, the following identities can be established:

$$\frac{\partial}{\partial\psi}\left(\frac{E}{2W^2}\right) = \frac{1}{W^2} \left(F \, \Gamma_{12}^{2} - E \, \Gamma_{22}^{2} \right) \qquad \text{(A.18)}$$

$$\frac{\partial}{\partial\phi}\left(\frac{F}{W} \right) - \frac{\partial}{\partial\psi}\left(\frac{E}{W}\right) = \frac{1}{W} \left(G \, \Gamma_{11}^{2} - 2F \, \Gamma_{12}^{2} + E \, \Gamma_{22}^{2} \right) \qquad \text{(A.19)}$$

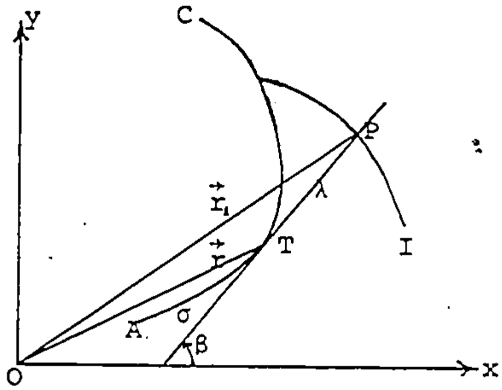
where Γ_{22}^{2} is given by

$$\Gamma_{22}^{2} = \frac{1}{2W^2} \left(E \frac{\partial G}{\partial\psi} - 2F \frac{\partial F}{\partial\psi} + F \frac{\partial G}{\partial\phi} \right). \qquad \text{(A.20)}$$

Section 2. Orthogonal curvilinear coordinate system formed by tangent lines and involutes of a curve C.

Here we consider the system of orthogonal curvilinear coordinates formed by tangents to a curve C and their orthogonal trajectories, the involutes I of C.

Let σ denotes the arc length AT along C measured from some fixed point A, then equation of an involute I of C is of the form



$$\vec{r}_1 = \vec{r}(\sigma) + \lambda(\sigma)\vec{t} \qquad \text{(A.21)}$$

where $\vec{t}$ is the unit tangent vector to C at T, $\vec{r}_1$ is the position vector of P, $\vec{r}$ the position vector of T and λ = TP.

As $\frac{d\vec{r}_1}{d\sigma}$ is a tangent vector to the involute, we have

$$\vec{t}\cdot\frac{d\vec{r}_1}{d\sigma} = 0$$

or

$$\vec{t}\cdot(\vec{t} + \lambda\kappa\vec{n} + \vec{t}\,\frac{d\lambda}{d\sigma}) = 0 \qquad \text{(A.22)}$$

where κ is the curvature and $\vec{n}$ the unit normal vector to C at T. Hence,

$$1 + \frac{d\lambda}{d\sigma} = 0$$

or

$$\lambda = \text{constant} - \sigma$$
$$= \xi - \sigma \qquad (A.23)$$

where ξ is a constant. For each value of ξ there is an involute. The equation of the involutes is therefore

$$\vec{r}_1 = \vec{r} + (\xi - \sigma)\vec{t} \qquad (A.24)$$

and they can be obtained by unwinding a string originally stretched along the curve, keeping the string taut all the time. Here $\xi = \sigma + TP$ denotes the length of string used to construct the involute.

Unit tangent vector to the involute I at P is

$$\vec{t}_1 = \frac{d\vec{r}_1}{ds_1} = \frac{d\vec{r}_1}{d\sigma}\frac{d\sigma}{ds_1} = (\xi - \sigma)\,\kappa\,\frac{d\sigma}{ds_1}\,\vec{n}$$

where s_1 represents arc length along I. Taking positive direction of $\vec{t}_1$ to be that of $\vec{n}$, we get

$$(\xi - \sigma)\,\kappa\,\frac{d\sigma}{ds_1} = 1$$

or

$$\frac{ds_1}{d\sigma} = (\xi - \sigma)\,\kappa \qquad (A.25)$$

The square of the element of arc length ds in the orthogonal curvilinear coordinate system formed by tangent lines to C and their orthogonal trajectories, the involutes I of C, is given by

$$ds^2 = ds_1^{\,2} + ds_2^{\,2}$$

where ds_1 and ds_2 are the elements of arc length along the involutes and the tangents respectively.

From (A.25), we have

$$ds_1 = (\xi - \sigma)\kappa \, d\sigma$$

Hence

$$ds^2 = d\xi^2 + (\xi - \sigma)^2 \kappa^2 \, d\sigma^2 \qquad (A.26)$$

If β is the angle which the tangent line to C at T makes with the x-axis, we have

$$\frac{d\sigma}{d\beta} = \frac{1}{\kappa} \qquad (A.27)$$

From (A.26) and (A.27), we get

$$ds^2 = d\xi^2 + (\xi - \sigma)^2 \, d\beta^2 \qquad (A.28)$$

where $\sigma = \sigma(\beta)$. In this coordinate system, the coordinate curves ξ = constant are the involutes of the curve C and the curves β = constant, its tangent lines.

APPENDIX B

I. Transformation Groups.

Let $f_i(x_1, \ldots, x_m; a)$ $(i = 1, \ldots, m)$ be a set of functions continuous in both the variables $\underset{\sim}{x} = (x_1, \ldots, x_m)$ and a. The variable a will be referred as parameter of the functions.

Given a specific value of parameter a, the values of the function are found by assigning values to the variables x_i. We regard the functions $f_i(x; a)$ as transforming the variables $(x_1, x_2, \ldots, x_m)$ into a set of variables $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)$ such that

$$\bar{x}_i = f_i(x_1, \ldots, x_m; a) \; .$$

For a particular value of the parameter a, say, a_1, we write the transformation of $\underset{\sim}{x} = (x_1, \ldots, x_m)$ into $\underset{\sim}{\bar{x}} = (\bar{x}_1, \ldots, \bar{x}_m)$ as

$$T_{a_1} \underset{\sim}{x} = \underset{\sim}{\bar{x}} \; .$$

If the set of functions f_i is "functionally independent" that is, if the Jacobian of the set of functions f_i,

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \end{vmatrix}$$

does not vanish in a region R, then we can express x_i as functions $\bar{x}_1, \bar{x}_2 \ldots\ldots \bar{x}_m$ such that

$$x_i = f_i^{-1} (\bar{x}_1, \ldots, \bar{x}_m; a_1) \quad (i = 1, \ldots, m)$$

and the transformation carrying $(\bar{x}_1, \ldots, \bar{x}_m)$ back into $(x_1, \ldots, x_m)$ can be defined as

$$T_{a_1}^{-1} \bar{\underset{\sim}{x}} = \underset{\sim}{x} \quad .$$

$T_{a_1}^{-1}$ is called the inverse transformation of T_{a_1}.

Two different transformations are defined by different values of the parameter a. Thus if a_1 and a_2 are two distinct values of a, we consider T_{a_1} and T_{a_2} to be different transformations. We consider set of all transformations T_a obtained by assigning different values to a, and form a transformation group.

By the product of two transformations T_{a_1} and T_{a_2} we mean application of the transformations successively; that is, a point $\underset{\sim}{x}$ is taken into a point $\bar{\bar{\underset{\sim}{x}}}$ as follows

$$T_{a_2} T_{a_1} \underset{\sim}{x} = T_{a_2} (T_{a_1} \underset{\sim}{x}) = T_{a_2} \bar{\underset{\sim}{x}} = \bar{\bar{\underset{\sim}{x}}}$$

A set of transformations is said to be closed under the product definition if, given any set of parametric values a_1 and a_2, a parametric value a_3 can always be found such that T_{a_3} is a unique member of the set of given transformations and

$$T_{a_1} T_{a_2} = T_{a_3}$$

A transformation which leaves each point unaltered is called an "identity transformation". Thus T_{a_0} is an identity transformation if

$$T_{a_0} \underset{\sim}{x} = \underset{\sim}{x} \quad .$$

By our previous definition of $T_{a_1}^{-1}$ we see that

$$T_{a_1}^{-1} T_{a_1} \underset{\sim}{x} = T_{a_1}^{-1} \bar{\underset{\sim}{x}} = \underset{\sim}{x} \quad ,$$

and similarly

$$T_{a_1} T_{a_1}^{-1} \bar{\underset{\sim}{x}} = \bar{\underset{\sim}{x}}$$

Thus we see that $T_{a_1}^{-1} T_{a_1} = T_{a_1} T_{a_1}^{-1} = I$, the identity transformation. We say that a set of transformations $\mathfrak{J} = \{T_a\}$ constitutes a group if

(1) The set is closed.

(2) There exists a transformation $I \varepsilon \mathfrak{J}$, such that

$$I \, T_a \underset{\sim}{x} = T_a \, I \underset{\sim}{x} = T_a \underset{\sim}{x}$$

for every transformation T_a of the set. I is called the identity transformation.

(3) The product is associative,

$$T_{a_1} (T_{a_2} T_{a_3}) \underset{\sim}{x} = (T_{a_1} T_{a_2}) T_{a_3} \underset{\sim}{x} \qquad \text{(for all } \underset{\sim}{x}\text{)}$$

(4) Given any transformation T_{a_1} an inverse transformation $T_{a_1}^{-1}$ belonging to the set, exists such that

$$T_{a_1}^{-1} T_{a_1} \underset{\sim}{x} = T_{a_1} T_{a_1}^{-1} \underset{\sim}{x} = \underset{\sim}{x} \qquad \text{(for all } \underset{\sim}{x}\text{)}$$

A subgroup of a given group of transformations is a set of elements contained in the given group and is such that these elements by themselves constitute a group.

2. Absolute Invariants

Let $\mathfrak{J}$ be a continuous transformation group with an individual member represented by T_a, a being parameter of the group, and let $\zeta(\underset{\sim}{x})$ be a function of $\underset{\sim}{x} = (x_1, \ldots, x_m)$. If

$$\bar{\underset{\sim}{x}} = T_a \underset{\sim}{x}$$

and if

$$\zeta(\bar{\underset{\sim}{x}}) = \zeta(\underset{\sim}{x})$$

for every transformation T_a and for all $\underset{\sim}{x}$ then $\zeta(\underset{\sim}{x})$ is said to be an absolute invariant of the group. If a transformation group is defined by

$$T_a \underset{\sim}{x} = \bar{\underset{\sim}{x}}$$

with

$$\bar{x}_i = f_i(x_1, \ldots, x_m; a)$$

then it is proved in general group theory that the group has (m - 1) functionally independent absolute invariants $\zeta_j(x_1, \ldots, x_m)$ (j = 1, ..., m-1); that is, functions ζ_j such that

$$\zeta_j(x_1, \ldots, x_m) = \zeta_j(\bar{x}_1, \ldots, \bar{x}_m) \quad (j = 1, \ldots, m-1)$$

3. Continuous Transformation Groups and Partial Differential Equations.

We now consider an arbitrary one-parameter continuous group of transformations defined by

$$T_a : \begin{cases} \bar{x}_i = f_i(x_1,\ldots,x_m; a) & (i=1,\ldots,m; m\geq 2) \\ \bar{y}_i = h_i(y_1,\ldots,y_n; a) & (j=;,\ldots,n; n\geq 1) \end{cases} \qquad (B.01)$$

The transformations defined by

$$\bar{x}_i = f_i(x_1,\ldots, x_m; a) \qquad (B.02)$$

are assumed to define a subgroup of the given group of transformations.

When considering a system of partial differential equations we identify x_i with the independent variables and y_j with the dependent variables of the system of partial differential equations under study. We assume that the y_j are differentiable functions of the x_i upto any required order.

If the transformations of the partial derivatives of the y_j with respect to the x_i are appended to the transformations defined by (B.01), then the resulting set of transformations is also a continuous one-parameter group. The new groups constructed in this way are called "enlargements" of the group $\mathfrak{J}$ and denoted by $\mathfrak{J}^1, \mathfrak{J}^2, \ldots, \mathfrak{J}^k$ accordingly as the transformations of the partial derivatives of the y_j upto order 1, 2, ..., k are added successively to those of $\mathfrak{J}, \mathfrak{J}^1, \ldots, \mathfrak{J}^{k-1}$.

The set of transformations defined by

$$\bar{x}_i = f_i(x_1, \dots, x_m; a)$$

form a subgroup of the group of transformations (B.01) and hence form a group having (m - 1) functionally independent "absolute invariants"

$$\eta_1(x_1, \dots, x_m), \dots\dots, \eta_{m-1}(x_1, \dots, x_m) .$$

Considering the group of transformations (B.01) as a whole, when considered as transformations of (m + n) variables namely, $x_1, \dots, x_m; y_1 \dots, y_n$, there are (m + n - 1) functionally independent absolute invariants. We therefore append the invariants

$$g_1(x_1, \dots, x_m; y_1, \dots, y_n), \dots\dots, g_n(x_1, \dots, x_m; y_1, \dots, y_n)$$

to the invariants $\eta_1, \dots, \eta_{m-1}$. We choose these later set of invariants such that the Jacobian

$$\begin{vmatrix} \frac{\partial g_1}{\partial y_1} & \cdots\cdots & \frac{\partial g_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial y_1} & \cdots\cdots & \frac{\partial g_n}{\partial y_1} \end{vmatrix} \neq 0 \qquad \text{(B.03)}$$

<u>Theorem 1</u>. If the variables $y_j, \bar{y}_j$ are implicitly defined as functions of the x_i and $\bar{x}_i$ by the equations

$$g_j(x_1, \dots, x_m; y_1, \dots, y_n) = z_j(x_1, \dots, x_m) \qquad \text{(B.04)}$$

$$g_j(\bar{x}_1, \dots, \bar{x}_m; \bar{y}_1, \dots, \bar{y}_n) = z_j(\bar{x}_1, \dots, \bar{x}_m) \qquad \text{(B.05)}$$

Then a necessary and sufficient condition for the y_j (defined implicitly as functions of the x_i by equations (B.04)) to be exactly the same functions of x_i as the $\bar{y}_j$ are of the $\bar{x}_i$ (as defined by (B.05)) is that

$$z_j(x_1,\ldots,x_m) = z_j(\bar{x}_1,\ldots,\bar{x}_m) = z_j(\bar{x}_1,\ldots,\bar{x}_m) \qquad \text{(B.06)}$$

The condition (B.06) is equivalent to

$$z_j(x_1,\ldots,x_m) = F_j(\eta_1,\ldots,\eta_{m-1}) \quad (j = 1,\ldots,n) \qquad \text{(B.07)}$$

The $\eta_1,\ldots, \eta_{m-1}$ are the functionally independent absolute invariants of the subgroup of transformations relating the variables x_i to $\bar{x}_i$ given by (B.02).

Proof of the above theorem is given in Morgan (1952). However, the equivalence of (B.06) and (B.07) can be seen easily. Relation (B.06) implies that the z_j are absolute invariants of transformations on the x_i defined by

$$\bar{x}_i = f_i(x_1,\ldots,x_m;\ a)$$

As the $\eta_1,\ldots, \eta_{m-1}$ are (m - 1) functionally independent absolute invariants of the above group, any absolute invariant of the above group is expressible as a function of these (m - 1) functionally independent invariants which form a maximal set. Therefore z_j can be expressed as a function of $\eta_1,\ldots, \eta_{m-1}$; that is,

$$z_j(x_1,\ldots, x_m) = F_j(\eta_1,\ldots, \eta_{m-1})$$

<u>Definition 1:</u> By a "differential form of the kth order in m independent variables" we mean a function, of the form

$$\phi = \phi(x_1,\ldots.x_m;\ y_1,\ldots.,y_n,\ldots,\ \frac{\partial^k y_1}{\partial x_1^k},\ldots.,\ \frac{\partial^k y_n}{\partial x_m^k}) \qquad (B.08)$$

whose arguments are the variables x_i, the functions y_j of the x_i, and the partial derivatives of the y_j with respect to the x_i upto the order k.

The differential form ϕ will be assumed to be of class C^1. For convenience consider the arguments in a given differential form ϕ , such as defined by (B.08), to be p in number, and designate them by $z_1,\ldots,z_p$, for example,

$$z_1=x_1,\ z_2=x_2,\ldots,\ z_{m+1}=y_1,\ldots,\ z_{p-1}=\frac{\partial^k y_n}{\partial x_{m-1}^k},\ z_p=\frac{\partial^k y_n}{\partial x_m^k}\ .$$

Suppose that the arguments z_i transform under the transformation laws of a continuous one-parameter transformation group

$$G_a: (z_1,\ldots,z_p) \longrightarrow (\bar{z}_1\ldots,\bar{z}_p),\quad \text{i.e.}\quad G_a\underset{\sim}{z} = \bar{\underset{\sim}{z}}.$$

<u>Definition 2</u> A differential form $\phi(z_1,\ldots,z_p)$ is said to be "conformally invariant" under a one-parameter transformation group $G_a\underset{\sim}{z} = \bar{\underset{\sim}{z}}$, if under the group transformations, it satisfies the relation

$$\phi(\bar{z}_1,\ldots,\bar{z}_p) = F(z_1,\ldots,z_p;\ a)\ .\ \phi(z_1,\ldots,z_p) \qquad (B.09)$$

where $F(z_1,\ldots,z_p;\ a)$ is some function of the z_i and the group parameter a.

If ϕ satisfies a relation

$$\phi(\bar{z}_1,\ldots,\bar{z}_p) = F(a) \cdot \phi(z_1,\ldots,z_p) \tag{B.10}$$

then the differential form ϕ is said to be "constant conformally invariant" under the group transformation.

Furthermore if $F(a) \equiv 1$, so that

$$\phi(\bar{z}_1,\ldots,\bar{z}_p) = \phi(z_1,\ldots,z_p) \tag{B.11}$$

then ϕ is said to be "absolutely invariant" under the transformation group.

Assuming again that y_j are dependent variables and x_i are independent variables, and the expressions

$$\bar{x}_i = f_i(x_1,\ldots,x_m;\ a)$$
$$\bar{y}_j = h_j(y_1,\ldots,y_m;\ a)$$

define a one- parameter group of transformations $\mathfrak{J}$ of these variables, we define invariance of a system of differential equations as follows:

<u>Definition 3:</u> A system of kth- order partial differential equations

$$\phi_j(x_1,\ldots,x_m;y_1,\ldots,y_n,\ldots,\frac{\partial^k y_1}{\partial x_1{}^k},\ldots,\frac{\partial^k y_n}{\partial x_m{}^k}) = 0 \tag{B.12}$$

is said to be "invariant under a continuous one-parameter group of transformations $\mathfrak{J}$" if each of the kth- order differential forms ϕ_j is conformally invariant under the kth-enlargement $\mathfrak{J}^k$ of $\mathfrak{J}$. It means that, given the transformation group $\mathfrak{J}^k$, the differential forms ϕ_j satisfy the relations

$$\phi_j(\bar{x}_1,\ldots,\bar{x}_m;\bar{y}_1,\ldots,\bar{y}_n;\ldots,\frac{\partial^k\bar{y}_1}{\partial\bar{x}_1{}^k},\ldots,\frac{\partial^k\bar{y}_n}{\partial x_m{}^k})$$

$$=F(x_1,\ldots,x_m;y_1,\ldots,y_n,\ldots,\frac{\partial^k y_n}{\partial x_m{}^k};\ a)\phi_j(x_1,\ldots,x_m,y_1,\ldots,\frac{\partial^k y_n}{\partial x_m{}^k})$$

(B.13)

<u>Definition 4:</u> By "invariant solutions" of a system of partial differential equations is meant that class of solutions of that system, which have the property that y_j are exactly the same function of the x_i as the $\bar{y}_j$ are of the $\bar{x}_i$.

In view of the theorem 1, the condition that invariant solutions exist can be simplified to the establishment of the relation

$$g_j(x_1,\ldots,x_m;y_1,\ldots,y_n) = F_j(\eta_1,\ldots,\eta_{m-1}) \quad \text{(B.14)}$$

where $\eta_1,\ldots,\eta_{m-1}$ are functionally independent absolute invariants of the set of transformations

$$\bar{x}_i = f_i(x_1,\ldots,x_m;\ a)$$

and $g_1, g_2,\ldots,g_n$ are n absolute invariants of the group of transformations (B.01) which togather with $\eta_1, \eta_2,\ldots,\eta_{m-1}$ form a set of $(m + n - 1)$ functionally independent absolute invariants of group (B.01) such that (B.03) is satisfied.

Assuming that $F_j \varepsilon C^1$, by Implicit Function Theorem, y_j can be expressed as

$$y_j=y_j(x_1,\ldots,x_m;\ g_1,\ldots,g_n) = y_j(x_1,\ldots,x_m;\ F_1,\ldots,F_n)$$

in some neighbourhood of the point $(x_1,\ldots,x_m)$.

We now state the principal theorem of Morgan's (1952) approach. The significance of this theorem is that it gives conditions under which the number of independent variables in a partial differential equation can be reduced by one in the process of obtaining invariant solutions.

Theorem 2: If each of the differential forms ϕ_j in the system (B.12) of partial differential equations is conformally invariant under the kth-enlargement of the group $\mathfrak{J}$ given by (B.01), then the invariant solutions of (B.12) can be expressed in terms of the solutions of a new system of partial differential equations

$$\phi_j(\eta_1,\ldots,\eta_{m-1};\ F_1,\ldots,F_n,\ldots,\frac{\partial^k F_1}{\partial \eta_1^k},\ldots,\frac{\partial^k F_n}{\partial \eta_{m-1}^k}) = 0 \qquad \text{(B.15)}$$

in (m - 1) independent variables $\eta_1,\ldots,\eta_{m-1}$.

REFERENCES

O. P. Chandna (1972) — "Steady transverse magnetohydrodynamic flows." Can. J. Phys., 50, pp. 2565-2567.

O. P. Chandna and M. R. Garg (1975) — "The flow of a viscous MHD fluid." Quart. Appl. Math., Accepted for publication.

(1976) — "Steady transverse MHD viscous flows." Can. J. Phys., 54, pp. 262-267.

O. P. Chandna and V. I. Nath (1972) — "On the uniqueness of MHD aligned flows with given streamlines." Can. J. Phys., 50, pp. 661-665.

(1972) — "Some properties of aligned MHD flows." Jap. J. Appl. Phys., 11, pp. 889-892.

(1973) — "Two dimensional steady magnetofluid dynamic flows with orthogonal magnetic and velocity field distributions." Can. J. Phys., 51, pp. 772-778

O. P. Chandna and A. C. Smith (1971) — "Some steady plane rotational flows of gases with arbitrary equation of state." J. de Mecanique, 10, pp. 315-322.

O. P. Chandna, A. C. Smith and V. I. Nath (1975) — "Steady transverse magnetogasdynamics channel flow." Prog. Math., 9, pp. 29-41.

O. P. Chandna, H. Toews and V. I. Nath (1975) — "Plane MHD steady flows with constantly inclined magnetic and velocity fields." Can. J. Phys., 53, pp. 2613-2616.

S. Chandrasekhar (1956) — "On the stability of the simplest solution of the equation of Hydromagnetics." Proc. Nat. Acad. Sci., U.S.A., 42, pp.273-276

(1956) — "On force-free magnetic fields." Proc. Nat. Acad. Sci., U.S.A., 42, pp.1-5

V. C. A. Ferraro (1954) — "On the equilibrium of magnetic stars." Astrophys. J., 119, pp. 407-412.

M. R. Garg and O. P. Chandna (1976) "Viscous orthogonal MHD flows." SIAM J. Appl. Math., 30, pp. 577-585.

H. Grad (1960) "Reducible problems in Magnetofluid dynamic steady flows." Rev. Mod. Phys., 32, pp. 830-847.

R. M. Gunderson (1966) "Steady two dimensional magnetohydrodynamic flows." Z. Angew. Math. Phys., 17, pp. 755-765

(1969) "Steady plane magnetohydrodynamics flow." J. Math. Mech., 19, pp. 357-370.

I. Imai (1960) "On flows of Conducting fluids past bodies." Rev. Mod. Phys., 32, pp. 992-999.

J. G. Kingston and R. F. Talbot (1969) "The solutions to a class of magnetohydrodynamic flows with orthogonal magnetic and velocity field distributions." Z. Angew. Math. Phys., 20, pp. 956-965.

Iu. P. Ladikov (1962) "Properties of plane and axisymmetrical stationary flows in magnetohydrodynamics." J. Appl. Math. Mech., 26, pp. 1646-1652.

H. Lamb (1932) "Hydrodynamics" (Cambridge) p 245.

R. R. Long (1960) "Steady finite motions of a conducting fluid." J. Fluid Mech., 7, pp. 108-114.

M. H. Martin (1950) "A new approach to problems in two-dimensional flow." Quart. Appl. Math., 8, pp. 137-150.

(1971) "The flow of a viscous fluid." Arch. Rat. Mech Anal., 41, pp. 266-286.

A. J. A. Morgan (1952) "The reduction by one of the number of independent variables in some system of partial differential equations." Quart. J. Math., 2, pp. 250-259.

V. I. Nath and O. P. Chandna (1973) "On plane viscous magnetohydrodynamic flows." Quart. Appl. Math., 31, pp. 351-362.

(1973) "On plane transverse MFD flows." Tensor, 27, pp. 28-32.

R. Peyret (1962) "Sur certains ecoulements homologues en magnetodynamique des fluids et en dynamique des gaz." J. de Mech., 1, pp. 31-47.

G. Power and R. Talbot (1969) "Magnetogasdynamic flows in two-dimensions with orthogonal magnetic and velocity field distributions." Z. Angew. Math. Phys., 20, pp. 358-369.

G. Power and D. Walker (1964) "Some reciprocal relations in rotational magnetogasdynamic flow." Z. Angew. Math. Phys., 15, pp. 144-154.

(1965) "Plane gasdynamic flows with orthogonal magnetic and velocity field distribution." Z. Angew. Math. Phys., 16, pp. 803-817.

(1967) "Reduction of viscous flows having orthogonal magnetic and velocity field distribution." Appl. Sc. Res., 17, pp. 223-232.

R. C. Prim (1952) "Steady rotational flow of ideal gases." J. Rat. Mech. Anal., 1, pp. 425-497.

K. B. Ranger (1970) "Spherical vortex motions of a conducting fluid." J. Fluid Mech., 44, pp. 481-492.

(1970) "Slow motion of a viscous conducting fluid past a sphere in the presence of a toroidal magnetic field." Quart. Appl. Math., 28, pp. 237-244.

P. Smith (1963) "Substitution principle for MHD flows." J. Math. Mech., 13, pp. 505-520.

C. Sozou (1972) "On some exact solution in magneto-hydrodynamics with astrophysical applications." J. Fluid Mech., 51, pp. 33-38.

H. Toews and O. P. Chandna (1974) "Steady transverse plane magnetogas-dynamic flows." Tensor, 28, pp. 184-188.

(1974) "Plane magnetofluiddynamic flows with constantly inclined magnetic and vector fields." Can. J. Phys., 52, pp. 753-758.

M. Vinokur (1961)	"Kinematic formulation of rotational flow in magnetogasdynamic." Lockheed Aircraft Corp., Tech. Report, 6-90-61-10.
J. S. Waterhouse and J. G. Kingston (1973)	"Plane magnetohydrodynamic flows with constantly inclined magnetic and velocity fields." Z. Angew. Math. Phys., 24, pp. 653-658.
C. S. Yih (1965)	"On large-amplitude magnetohydrodynamics." J. Fluid. Mech., 23, pp. 261-271.

VITA AUCTORIS

The author was born in Dehra Dun (India) on 21st June, 1942. He obtained the M. Stat. degree from the Indian Statistical Institute, Calcutta (India) in 1965 and taught at the Birla Institute of Technology and Science, Pilani (India) till 1968.

He received his M.S. in Mathematics from the Northeastern University, Boston (U.S.A.) in 1970.