1992

A constraint on existence of torsion free lie modules.

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University of Windsor

A CONSTRAINT ON EXISTENCE OF TORSION FREE LIE MODULES

BY

Vahid Tarokh

A Thesis
Submitted to the Faculty of Graduate Studies and Research
Through the Department of Mathematics and Statistics
in Partial Fulfillment
of the Requirements for The degree of
Master of Science
at the University of Windsor

Windsor, Ontario, Canada
1992

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ABSTRACT

For any simple Lie algebra L with any maximal toral subalgebra H, the classification of all simple H diagonalizable L modules having a finite dimensional weight space is known to depend on determining the simple torsion free L modules of finite degree. It is further known that the only simple Lie algebras which admit simple torsion free modules of finite degree are those of types A_{n-1} and C_m. For the case of A_{n-1} we show that there are no simple torsion free A_{n-1} modules of degree k for $n \geq 5$ and $2 \leq k \leq n-3$. We conclude with some examples showing that there exist simple torsion free A_{n-1} modules of degrees $1, n-2$ and $n-1$, whenever $n \geq 3$.

Dedicated to:

My Family, Mr. R. Matthewman and the U.N.H.C.R.

ACKNOWLEDGEMENTS

My thanks goes to all the nice people of the University of Windsor, particularly those at the Department of Mathematics. A special package of my thanks goes to the two giant mathematicians, Dr. D. J. Britten and Dr. F. W. Lemire, who I am truly indebted to for their help, encouragement and support.

Of course, had the people and government of Canada not sponsored our research, this work would not exist. Thank you all generous and proud Canadians. Finally, I shall thank all the authors of the books and papers used in this research work specially Dr. J. Humphreys and Dr. N. Jacobson.

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CHAPTER 0

Introduction

In this chapter we provide a brief overview of the background material relevant to the problem to be discussed in this thesis. No attempt has been made to provide precise definitions or statements of results as these will be developed in detail in subsequent chapters.

Let L denote a finite dimensional simple Lie algebra over the complex numbers $\mathbb{C}$ and let H be a fixed maximal toral subalgebra of L. A stimulating problem is to provide explicit constructions for all possible L modules i.e. realizations of L as an algebra of linear transformations on a complex vector space. This question has been successfully answered by Block[B] only for the simple Lie algebra A_1, of traceless 2×2 complex matrices. The general problem for other simple Lie algebras of higher rank is still far beyond reach. In this thesis, we restrict our attention to the category $\mathcal{W}(L,H)$ of (L,H) finitely generated weight modules having all finite dimensional weight spaces. These are the representations of L which are H diagonalizable and possess finite dimensional weight spaces.

Fernando treated this problem in [F]. He provided some global information concerning the simple L modules in the category $\mathcal{W}(L,H)$. In particular he associated with each simple module M in the category $\mathcal{W}(L,H)$ a pair of parabolic subalgebras P_M and P_M^- of L, a reductive subalgebra $l_M = P_M \cap P_M^-$ and an l_M submodule M^{u_M} of M which uniquely determines M up to equivalence. The algebra l_M further decomposes into a direct sum $L_1 \oplus L_2$ consisting of a semisimple subalgebra L_1 and a reductive subalgebra L_2 and the module M^{u_M} decomposes into a tensor product of a L_1 module M_1 which is simple and torsion free and an L_2 module M_2 which is simple and finite dimensional. It follows then that the classification of all simple modules in the category $\mathcal{W}(L,H)$ reduces to the problem of determining all simple finite dimensional modules and all simple torsion free modules with finite dimensional weight spaces for simple Lie algebras.

In light of the classical results of Cartan, Weyl, and Harish-Chandra the finite dimensional modules of complex simple Lie algebra are well understood. Since such modules find applications in numerous fields such as physics, chemistry and other areas of mathematics the amount of detailed information in this area is enormous. In particular there are numerous tabulations and formulas which provide the possible dimensions of such modules as well as the multiplicities of their weight spaces. For each of the four families of classical Lie algebras these dimensions and multiplicities can be expressed as polynomial expressions in terms of the rank of the algebra. For instance, in [BBL] the authors show that the form of these polynomials is independent of the rank of the simple algebra.

In contrast, little is known about the torsion free L modules in the category $\mathcal{W}(L, H)$. Using sophisticated techniques, Fernando first observed that the only simple Lie algebras which admit torsion free modules are the classical Lie algebras of types A or C. Using a modification of a construction familiar to physicists, Fernando provided an explicit construction of certain families of torsion free modules for the algebras of types A and C. These examples have the property that all weight spaces are one dimensional. In [BL1] it is shown that these examples in fact exhaust all simple torsion free modules having one dimensional weight spaces. For the algebra A_2, Futorny has classified all simple A_2 modules having finite dimensional weight spaces in [Fu]. For the simple algebras of higher rank there exist only sporatic examples of torsion free modules constructed either by extension of the Gel'fand basis formalism as in [LP] or by decomposition of certain tensor products of modules. In particular there is no general understanding of what weight multiplicities can occur in simple torsion free modules. An answer to this question is a first step in determining all such modules and hence in the completion of Fernando's program of classifying all simple modules in category $\mathcal{W}(L, H)$. In this work we make a contribution to the solution to this particular problem. More precisely when M is a simple torsion free A_{n-1} module, $n \geq 2$, with a weight space of dimension $k \geq 1$, it turns out that all the weight spaces of M are k-dimensional. We call this dimension the degree of M and prove the following theorem.

Main Theorem: There does not exist a simple torsion free module of degree k for the algebra A_{n-1}, whenever $n \geq 5$ and $2 \leq k \leq (n-3)$.

In the first five chapters of this thesis we review the fundamental definitions and results in the theory of Lie algebras and their representations. The main theorem then will be established in chapter 6. A more detailed outline of the contents of these chapters follows below.

In chapter 1, we introduce complex simple Lie algebras. We first prove the Ado-Iwasawa theorem for simple Lie algebras and conclude that any finite dimensional simple Lie algebra can be realized as an algebra of square matrices. We continue by defining maximal toral subalgebras for Lie algebras of square matrices. Next we give realizations for all but a finite number of simple Lie algebras in terms of $n \times n$ matrices. In these realizations, we recover maximal toral subalgebras as sets of diagonal matrices. We will pay special attention to the algebra of traceless complex $n \times n$ matrices $A_{n-1}, n \geq 2$, since our main result involves the representation theory of such algebras. In particular, due to the importance of the conjugacy theorem, in our developments, we will prove this result for the A_{n-1} case in a simple manner. We proceed by studying the root systems. We will prove that the set of roots of a Lie algebra of type A_{n-1} forms a root system that possess a base. We dedicate the rest of the chapter to studying the bases of A_{n-1}.

We cover tensor products in chapter 2. We concentrate on the general concept and therefore begin with a brief review of the associative rings module theory. This approach pays us back in chapter 5, when we will consider Fernando's work. Next we study the relations between tensor products and weak direct sums. We use this to realize tensor products of vector spaces, which we will later use in the next chapter. We will finish the chapter by studying induced modules.

We devote chapter 3 to the studies of structure of enveloping algebras. We will first define and construct the universal enveloping algebra U of a simple Lie algebra L. We will continue with the P.B.W. theorem, which we will not prove. Our major development is the definition of cycle subalgebras. Using the conjugacy theorem, we will demonstrate the independence of the structure of a cycle subalgebra of the universal enveloping algebra of a simple Lie algebra from the choice of maximal toral subalgebras. Finally we prove that cycle subalgebras are generated by certain monomials of U , which we will call the n-cycles.

In chapter 4, we will introduce and develop some basic representation theory for simple Lie algebras. We will start by defining modules and representations of Lie algebras and continue by defining internal direct sums and complete reducibility and by covering the Peter-Weyl theorem. One of our main results will relate the representations of a simple Lie algebra L with those of the universal enveloping algebra of L. We will proceed by defining weight modules and after establishing few intermediatory results, we will relate the weight modules of L to the modules of the cycle subalgebras of L.

In chapter 5, we review Fernando's results. Much of these results are proved using affine spaces techniques in algebraic geometry and by applying the Gelfand-Krillov dimension. We therefore will not prove the results and will be content with stating them.

In chapter 6, we will prove the Main Theorem by a sequence of lemmas, and will give examples showing that the bounds given in our result cannot be improved.

CHAPTER 1

Complex Simple Lie Algebras

In this chapter, we develop some of the theory of complex simple Lie algebras.

Definition 1.1: A *Lie algebra* L is a vector space over a ground field F, with an operation $[.,.] : L \times L \rightarrow L$, called the *bracket*, having the following properties:

(i) The bracket operation is bilinear.

(ii) $[x, x] = 0 \quad \forall x \in L$.

(iii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \forall x, y, z \in L$.

Condition (iii) of definition 1.1 is called the *Jacobi* identity. By applying (ii) to $[x + y, x + y]$, we obtain $[x, y] = -[y, x]$.

In this work, we assume that the ground field is the field of complex numbers $\mathbb{C}$ and all the Lie algebras are finite dimensional vector spaces over $\mathbb{C}$. In particular, all the Lie algebras are complex finite dimensional Lie algebras. We may use the term 'algebra' for 'complex finite dimensional Lie algebra' , whenever there is no ambiguity.

Example 1.2: Let R denote an associative algebra and xy denote the product of x and y in R. Define a bracket operation on R by $[x, y] = xy - yx$. Then R endowed with this bracket operation is a Lie algebra. An important special case occurs when R is the algebra $End(V)$ of all the endomorphisms of a vector space V with NM the composite of N and M in $End(V)$. Then $End(V)$ endowed with the bracket operation as defined before is a Lie algebra, which we denote by $gl(V)$.

A *subalgebra* L' of L is a subspace which is an algebra on its own with respect to the bracket operation of L. Any subalgebra of the Lie algebra $gl(V)$ given in 1.2 is called a *linear Lie algebra*.

Definition 1.3: Let L and L' be complex Lie algebras. A linear transformation $\pi : L \rightarrow L'$ is called a Lie algebra *homomorphism* if $\pi([x, y]) = [\pi(x), \pi(y)]$ for all $x, y \in L$. If π is one to one and onto, we call π an *isomorphism* of L onto L'. L and L' are isomorphic if an isomorphism of L onto L' exists.

Definition 1.4: An *ideal* I of a Lie algebra L, is a subalgebra of L such that $[x, y] \in I$ whenever $x \in I$ and $y \in L$. We write $I \trianglelefteq L$. Clearly (0) and L are ideals of L. We call these two *trivial* ideals. Any other ideal of L is called a ***proper*** ideal of L.

If $\pi : L \rightarrow L'$ is a homomorphism and $ker(\pi) = \{x \in L \mid \pi(x) = 0\}$, it is easy to verify that $ker(\pi) \trianglelefteq L$. Conversely if $I \trianglelefteq L$ define a bracket operation on the quotient vector space L/I by $[x + I, y + I] = [x, y] + I$. Then it is easy to see that L/I endowed with this bracket operation is a Lie algebra. Define $\pi : L \rightarrow L/I$ by $\pi(x) = x + I$, then π is a homomorphism with $I = ker(\pi)$. This indicates that there is an intimate relation between ideals and kernels of homomorphisms. We note that the isomorphism theorems of associative algebras and group theory are still valid in Lie algebra theory. This resemblance and the idea of simplicity (for instance in group theory) motivates the following definition.

Definition 1.5: A Lie algebra L is called *simple* if $dim_{\mathbb{C}} L > 1$ and L does not possess proper ideals.

We will later give some examples of simple Lie algebras. However, being motivated by the concept of conjugacy in group theory and its application in realization of simplicity, we give the following definition:

Definition 1.6: For $x \in L$, we define the endomorphism $ad(x) \in End(L)$ by $ad(x)y = [x, y] \quad \forall y \in L$.

In its full generality, the following theorem (due to Ado and Iwasawa) does not require simplicity and is an analogue to the Cayley's Theorem in group theory.

Theorem 1.7: Any simple Lie algebra is isomorphic to a linear Lie algebra.

Proof: We prove this theorem by showing that

(i) For any algebra L the mapping $ad : L \rightarrow gl(L)$ by $ad : x \rightarrow ad(x)$ is a Lie algebra homomorphism.

(ii) If L is simple this mapping is also injective.

(i) By the Jacobi identity $[[x,y],z] = [x,[y,z]] - [[y,x],z]$, or:

$$ad([x,y])z = ad(x)ad(y)z - ad(y)ad(x)z.$$

This being true for all $z \in L$ gives that $ad([x,y]) = ad(x)ad(y) - ad(y)ad(x)$, so that ad is a homomorphism of L and $gl(L)$.

(ii) If I denotes the kernel of this homomorphism, then $I \trianglelefteq L$. By virtue of the simplicity of L we must have $I = (0)$ or $I = L$. We claim that $I = (0)$. If not, $I = L$ and we choose $x \neq 0$ in L. Since $x \in ker(ad)$, we have $[x,y] = ad(x)y = 0, \forall y \in L$. We conclude that $(0) \neq Span_{\mathbb{C}}\{x\} \triangleleft L$. But this is impossible since L is simple. Hence $I = (0)$ and ad is one-to-one.

Now let $L' = Im(ad)$, the image of ad, then L and L' are isomorphic. ∎

In the light of Theorem 1.7, to realize finite dimensional simple Lie algebras, we can restrict our attention to simple linear Lie algebras. Hence we will pay attention to those simple Lie algebras consisting of square matrices. In this context, we present an important technical result closely related to the Jordan decomposition theorem in linear algebra.

Theorem 1.8: Let V denote a finite dimensional vector space, then

(i) Any $x \in gl(V)$ can be uniquely written as $x = x_s + x_n$, with x_s and x_n respectively diagonalizable and nilpotent elements of $gl(V)$ such that x_s and x_n commute.

(ii) There exists polynomials $p(z)$ and $q(z)$ in one indeterminate and without constant terms such that $x_s = p(x)$ and $x_n = q(x)$.

Proof: Let

$$C(z) = \prod_{1 \leq i \leq k} (z - a_i)^{m_i}$$

denote the factorization of the characteristic polynomial of x, where $a_i, 1 \leq i \leq k$ are distinct. If $a_i \neq 0$ for all $1 \leq i \leq k$, set $a_o = 0$ and $m_0 = 1$ and replace $1 \leq i \leq k$ by $0 \leq i \leq k$ everywhere in the rest of the following. Since the greatest common divisor

$$((z - a_i)^{m_i}, \prod_{1 \leq j \neq i \leq k} (z - a_j)^{m_j}) = 1$$

for all $1 \leq i \leq k$, we can use division algorithm and find polynomials $r_i(z)$ and $s_i(z)$ such that :

$$r_i(z)(z - a_i)^{m_i} + s_i(z) \prod_{1 \leq j \neq i \leq k} (z - a_j)^{m_j} = 1$$

for $1 \leq i \leq k$. Let

$$p(z) = \sum_{i=1}^{k} (a_i s_i(z) \prod_{1 \leq j \neq i \leq k} (z - a_j)^{m_j}),$$

then since $a_i = 0$ for some i, $p(z)$ does not have any constant term and it is easy to see that $p(z) = a_i \quad mod(z - a_i)^{m_i}$ for all $1 \leq i \leq k$. If V_i denotes the generalized eigenspace of x belonging to the eigenvalue a_i, then $(x - a_i)^{m_i} V_i = 0$. Since $p(z) = a_i \quad mod(z - a_i)^{m_i}$, we have $(p(x) - a_i I)V_i = 0$ and since $V = \oplus \sum_{i=1}^{k} V_i$, $p(x)$ is diagonalizable on V.

Let $q(z) = z - p(z)$, then $q(x)^{m_i} V_i = 0$. Let $m = max_i(m_i), 1 \leq i \leq k$, then $q(x)^m = 0$, and $q(x)$ is nilpotent. Hence if $x_s = p(x)$ and $x_n = q(x)$ then conditions (i) and (ii) hold. Now if $x = y + w$ with y and w satisfying conditions (i) and (ii) then $y - x_s = -w + x_n$ and in view of the part (ii) all the endomorphisms in sight commute, so that $-x_s + y$ is a diagonalizable endomorphism which is also nilpotent because it is equal to $x_n - w$. This forces $-x_s + y = 0$ or $y = x_s$ and $w = x_n$. ∎

Definition 1.9: With the notations of theorem 1.8, we call x_s and x_n respectively the *semisimple* and *nilpotent* parts of x , and $x = x_s + x_n$, the *Jordan decomposition* of x.

The next theorem relates simplicity, Jordan decomposition and homomorphisms. It is a major result whose proof is beyond the scope of this thesis.

Theorem 1.10: Let L denote a simple linear algebra and $x \in L$, with $x = x_s + x_n$ the Jordan decomposition of x. Then $x_s, x_n \in L$. Moreover if $\pi : L \to gl(W)$ is a homomorphism, where W is a complex finite dimensional vector space, then $\pi(x) = \pi(x_s) + \pi(x_n)$ is the Jordan decomposition of $\pi(x)$ in $gl(W)$.

Proof: We refer the reader to section 4.2 of [H] . ∎

Next we consider those subalgebras of a simple Lie algebra which are as close as possible to the sets of diagonal matrices, in the sense that their elements are simultaneously diagonalizable.

Definition 1.11: A *toral subalgebra* of a linear Lie algebra L is a nonzero subalgebra of L consisting of diagonalizable elements. A *maximal toral subalgebra* of L is maximal with respect to the inclusion ordering.

Theorem 1.12: A toral subalgebra T of a simple linear Lie algebra is abelian.(i.e. $[x, y] = 0$ for all $x, y \in T$.)

Proof: We must show that the restriction of $ad(x)$ (c.f. 1.6.1) to T , denoted by $ad_T(x)$ is the zero endomorphism of T, for all $x \in T$. By Theorem 1.7 $ad : L \to gl(L)$ is a homomorphism. Since x is diagonalizable, we can use Theorem 1.10 and observe that $ad(x)$ is diagonalizable. But then $ad_T(x)$ is also diagonalizable and to show that $ad_T(x) = 0$ amounts to showing that $ad_T(x)$ does not have any non-zero eigenvalues. Suppose to the contrary that $[x, y] = ay$ for some $a \neq 0$ and for some $y \in T$. Then $ad_T(y)x = -ay \neq 0$ is an eigenvector of $ad_T(y)$ related to the eigenvalue zero. On the other hand, with a similar argument to that of $ad_T(x)$, one observes that $ad_T(y)$ is diagonalizable and hence there exists a base $\{y_1, \ldots, y_m\}$ of T consisting of eigenvectors of $ad_T(y)$. Let $ad_T(y)y_i = b_i y_i$ for all $1 \leq i \leq m$ and let $x = \sum_{i=1}^{m} a_i y_i$, then by the preceding $ad_T(y)ad_T(y)x = 0$, which gives $\sum_{i=1}^{m} a_i b_i^2 y_i = 0$. We conclude that $a_i b_i^2 = 0$ for all $1 \leq i \leq m$ and hence $a_i b_i = 0$ for all $1 \leq i \leq m$. This in turn gives $-ay = ad_T(y)x = \sum_{i=1}^{m} a_i b_i y_i = 0$. This contradiction completes the proof. ∎

Definition 1.13: Let L denote a simple Lie algebra and K denote a subset of L. The *centralizer* $C_L(K)$ of K in L is the set $\{x \in L \mid [x,y] = 0 \quad \forall y \in K\}$.

Next let L be a simple linear Lie algebra. It is known that L has a toral subalgebra. Hence L, being finite dimensional, has a maximal toral subalgebra H. For any $x \in H$, since $ad : L \to gl(L)$ is a Lie algebra homomorphism and x is diagonalizable, we can use Theorem 1.10 to deduce that $ad(x)$ is a diagonalizable element of $End(L)$. We conclude that $ad(H)$ is a commutative family of diagonalizable endomorphisms of L. By a standard result in linear algebra the elements of $ad(H)$ are simultaneously diagonalizable. That is L is the vector space direct sum of

$$L_\alpha = \{x \in L \mid [h,x] = \alpha(h)x, \quad \forall h \in H\},$$

where α ranges over H^* , the dual space of H. Since H is abelian $H \subseteq L_0$ and hence $L_0 \neq (0)$. Clearly $L_0 = C_L(H)$. Denote by ϕ the set of all $0 \neq \alpha \in H^*$, for which $L_\alpha \neq (0)$. Then $L = C_L(H) \oplus \sum_{\alpha \in \phi} L_\alpha$.

If it happens that $C_L(H) = H$, then $L = H \oplus \sum_{\alpha \in \phi} L_\alpha$, which we call a *root space decomposition* of L with respect to H. We call the nonzero elements of L_α *root vectors* belonging to α and the elements of ϕ *roots* of L with respect to H.

The simple linear complex Lie algebras have been classified into four infinite families called the *classical* simple Lie algebras and five *exceptional* Lie algebras. In order to focus on simple Lie algebras, we will give a realization of classical simple Lie algebras in the following examples.

Notation: Throughout the rest of this work, we let e_{ij} denote the $n \times n$ matrix whose k,l-th element is $\delta_{ik}\delta_{jl}$ where δ is the Kronecker's delta function.

Example 1.14: Let $A_{n-1}, n \geq 2$ denote the set of all $n \times n$ complex matrices of trace 0. Then A_{n-1}, as the kernel of the trace functional, is a $n^2 - 1$ dimensional. Since $tr(AB) = tr(BA)$, we observe that A_{n-1} endowed with the bracket operation $[A, B] = AB - BA$ is a Lie algebra.

Let H denote the subspace of diagonal matrices in A_{n-1}. The matrices $h_i = e_{i,i} - e_{i+1,i+1}, 1 \leq i \leq n$ form a basis for H. These matrices together with $e_{pq}, 1 \leq p \neq q \leq n$ form a basis for A_{n-1}.

We prove that A_{n-1} is simple by showing that:

(i) all nonzero ideals of A_{n-1} have non-trivial intersections with H, and

(ii) the ideal generated by any nonzero element of H is A_{n-1}.

To prove (i) let $I \neq (0)$ denote an ideal of A_{n-1}. Choose $0 \neq x \in I$. Then $x = \sum_{l=1}^{n-1} d_l h_l + \sum_{1 \leq p \neq q \leq n} c_{pq} e_{pq}$, for d_l and c_{pq} 's complex numbers. Set $x_0 = x$ and for $1 \leq i \leq (n-1)$, let $x_i = x_{i-1} - [h_i, [h_i, x_{i-1}]]$. Clearly $x_i \in I$. Let $c = Card(\{i, i+1\} \cap \{p, q\})$. Then

$$[h_i, [h_i, e_{pq}]] = \begin{cases} 0, & \text{if } c = 0, \\ e_{pq}, & \text{if } c = 1, \\ 4e_{pq}, & \text{if } c = 2. \end{cases}$$

The coefficients of $e_{ir}, e_{ri}, 1 \leq r \neq i+1 \leq n$ and $e_{i+1,s}, e_{s,i+1}, 1 \leq s \neq i \leq n$ is zero in x_i. If $n > 2$, we have $x_{n-1} = \sum_{l=1}^{n-1} d_l h_l$ and if $n = 2$, then $d_1 h_1 = (4x - [h_1, [h_1, x]])/4 \in I$. So that in any case $\sum_{l=1}^{n-1} d_l h_l \in I \cap H$. If $d_i \neq 0$ for some $1 \leq i \leq n-1$, then the assertion (i) follows. Otherwise, since $x \neq 0$ we have $c_{j,k} \neq 0$ for some $1 \leq k \neq j \leq n$. Without loss of generality we can assume that $k < j$. Then by considering $x' = [x, e_{k,j}] \in I$ and noticing that the coefficient of h_i is nonzero for any $k \leq i \leq j-1$, we revert to the case above.

To prove (ii), let $h = \sum_{l=1}^{n-1} d_l h_l \neq 0$. Then $d_k \neq 0$ for some $1 \leq k \leq n-1$. Let J denotes the ideal generated by h. Assume to the contrary that $J \neq A_{n-1}$.

We claim that $e_{k,k+1} \notin J$ for all $1 \leq k \leq n-1$. To prove the claim assume that $e_{k,k+1} \in J$, then $e_{k+1,k+2} = [e_{k+1,k}, [e_{k,k+1}, e_{k+1,k+2}]] \in J$ whenever $k+1 \leq n-1$. Similarly $e_{k-1,k} \in J$, whenever $2 \leq k \leq n$. We conclude that $e_{i,i+1} \in J$ for all $1 \leq i \leq n-1$. It follows from the relation

$$h_i = [e_{i,i+1}, e_{i+1,i}]$$

that $h_i, 1 \leq i \leq n-1$ are in J and hence $H \subseteq J$. But then from the relation

$$[h_p + \ldots + h_{q-1}, e_{pq}] = 2e_{pq}, \text{ whenever } p < q$$

and a similar identity whenever $p > q$ that all the basis elements of A_{n-1} are in J and hence $A_{n-1} = J$. A contradiction!. Hence the claim is true.

It follows from the relation

$$[h, e_{i,i+1}] = \begin{cases} (2d_1 - d_2)e_{1,2}, & \text{if } i = 1, \\ (-d_{i-1} + 2d_i - d_{i+1})e_{i,i+1}, & \text{if } 1 < i < n-1, \\ (-d_{n-2} + 2d_{n-1})e_{n-1,n}, & \text{if } i = n-1. \end{cases}$$

and the claim that $2d_1 - d_2 = 0$, $-d_{i-1} + 2d_i - d_{i+1} = 0$ for all $1 < i < n-1$ and $-d_{n-2} + 2d_{n-1} = 0$. But these give that $d_i = 0, 1 \leq i \leq n-1$ and $h = 0$. A contradiction that proves (ii), and proves that A_{n-1} is simple.

The subalgebra H is by definition a toral subalgebra of A_{n-1}. We prove that H is a maximal toral subalgebra of A_{n-1} by proving that $H = C_L(H)$, where $C_L(H)$ was defined in 1.13. Clearly $H \subseteq C_L(H)$. If $x \in C_L(H)$ then $[x, e_{ii} - e_{jj}] = 0, 1 \leq i \neq j \leq n$. This gives that x is diagonal and hence in H. Hence $H = C_L(H)$ follows and H is a maximal toral subalgebra of L.

Since $C_L(H) = H$, we know that L has a root space decomposition with respect to H. We recover this root space decomposition by defining $\omega_i \in H^*$ by $\omega_i(e_{jj} - e_{j+1,j+1}) = \delta_{i,j} - \delta_{i,j+1}$ for all $1 \leq j < n$ and $1 \leq i \leq n$. Then e_{pq} belongs to the root $\omega_p - \omega_q$ of H in L, and $L = H \oplus \sum_{1 \leq p \neq q \leq n} span_{\mathbb{C}}\{e_{pq}\}$ gives a root space decomposition of L with respect to H.

Example 1.15: We consider the algebra $C_m, m \geq 2$ of $2m \times 2m$ complex matrices, skew symmetric with respect to non-degenerate skew bilinear form $x_1y_{2m} + \ldots + x_my_{m+1} - x_{m+1}y_m - \ldots - x_{2m}y_1$. Then C_m consists of all matrices $[a_{i,j}]$ such that $a_{2m+1-i,2m+1-j} = -a_{ji}\epsilon(i)\epsilon(j)$, where

$$\epsilon(k) = \begin{cases} 1, & \text{if} \quad 1 \leq k \leq m, \\ -1, & \text{if} \quad m+1 \leq k \leq 2m, \\ 0, & \text{otherwise.} \end{cases}$$

It can be proved, as in 1.14, that C_m is simple and the set H of all diagonal matrices of C_m is a maximal toral subalgebra of C_m satisfying: $C_{C_m}(H) = H$.

The matrices $e_{i,i} - e_{2m+1-i,2m+1-i}$ form a basis for H. These matrices together with:

(1.15.1) $\quad e_{i,j} - e_{2m+1-j,2m+1-i}, \quad i \neq j, \quad 1 \leq i,j \leq m$

(1.15.2) $\quad e_{i,j} + e_{2m+1-j,2m+1-i}, \quad m+1 \leq i \leq 2m, \quad j \leq 2m+1-i$

(1.15.3) $\quad e_{i,j} + e_{2m+1-j,2m+1-i}, \quad m+1 \leq j \leq 2m, \quad i \leq 2m+1-j$

form a basis for C_m.

Define linear forms $\omega_i \in H^*$ by $\omega_i(e_{j,j} - e_{2m+1-j,2m+1-j}) = \delta_{i,j}$ for all $1 \leq i,j \leq m$. Then the matrices of type 1.15.1 belong to roots $\omega_i - \omega_j$, those of type 1.15.2 belong to the roots $-(\omega_{2m+1-i} + \omega_j)$ and those of type 1.15.3 belong to the roots $(\omega_{2m+1-j} + \omega_i)$. This characterizes the root space decomposition of C_m with respect to H.

Example 1.16: We consider the algebra L of $n \times n$ complex matrices (B_m if $n = 2m+1$, D_m if $n = 2m$) skew symmetric with respect to the non-degenerate symmetric bilinear form $x_1 y_n + x_2 y_{n-1} + \ldots + x_n y_1$. Then L consists of all matrices $[a_{i,j}]$ satisfying $a_{n+1-i,n+1-j} = -a_{j,i}, 1 \leq i,j \leq n$. As in 1.14 it can be proved that L is simple with H, the set of diagonal matrices in L, a maximal toral subalgebra satisfying $H = C_L(H)$.

The matrices $e_{i,i} - e_{n+1-i,n+1-i}, 1 \leq i \leq m$ form a basis for H. These matrices together with

$$(1.16.1) \qquad e_{i,j} - e_{n+1-j,n+1-i}, \quad 1 \leq i \neq j \leq m$$

$$(1.16.2) \qquad e_{i,j} - e_{n+1-j,n+1-i}, \quad \frac{n+1}{2} < i \leq n, \quad j < n+1-i$$

$$(1.16.3) \qquad e_{i,j} - e_{n+1-j,n+1-i}, \quad \frac{n+1}{2} < j \leq n, \quad i < n+1-j$$

$$(1.16.4) \qquad e_{m+1,j} - e_{n+1-j,m+1}, \quad 1 \leq j < m+1$$

$$(1.16.5) \qquad e_{j,m+1} - e_{m+1,n+1-j}, \quad 1 \leq j < m+1$$

if $n = 2m+1$ and with only 1.16.1, 1.16.2 and 1.16.3 if $n = 2m$ form a basis for L.

Define linear forms $\omega_i \in H^*$ by $\omega_i(e_{j,j} - e_{n+1-j,n+1-j}) = \delta_{i,j}$ for $1 \leq i,j \leq m$. Then the matrices of types 1.16.1, 1.16.2 and 1.16.3 respectively belong to the roots $(\omega_i - \omega_j), -(\omega_{n+1-i} + \omega_j)$ and $(\omega_{n+1-j} + \omega_i)$ of H in L. This characterizes the root space decomposition of L with respect to H , if $n = 2m$(type D_m). If $n = 2m+1$ (type B_m), the matrices of type 1.16.4 and 1.16.5 respectively belong to the roots $-\omega_j$ and ω_j of H in L. This finishes the characterization of root space decomposition of B_m with respect to H.

To complete the classification of simple linear Lie algebras, we refer the reader to [H] for the proof of the following *classification theorem.* For a realization of the exceptional Lie algebras, we refer the reader to [J].

Theorem 1.17: Any simple linear complex Lie algebra is either isomorphic to one of the algebras $A_m(m \geq 1), B_m(m > 2), C_m(m \geq 2), D_m(m \geq 4)$, called the classical Lie algebras, or is isomorphic to one of the algebras $E_m(6 \leq m \leq 8), F_4, G_2$ called the exceptional Lie algebras.

We next study the class of maximal toral subalgebras of a simple Lie algebra L. This will be important when we will treat the cycle subalgebras of the universal enveloping algebra of L.

Theorem 1.18: If L is a simple linear Lie algebra, then there exists a maximal toral subalgebra H of L satisfying $C_L(H) = H$.

Proof: If L is a classical Lie algebra, then H was constructed for an isomorphic copy of L in 1.14, 1.15 and 1.16. If L is an exceptional Lie algebra, we refer the reader to [J]. ∎

Part of the following result is known as the *conjugacy* theorem. We give a proof of it for the A_{n-1} case and for the general proof, we refer the reader to [H].

Theorem 1.19: If L is a simple linear Lie algebra , then for any two maximal toral subalgebra H and H' of L , there exists an automorphism σ of L (an isomorphism of L with itself) such that $\sigma(H) = H'$. In particular $H' = C_L(H')$ and L has a root space decomposition

$$L = H' \oplus \sum_{\alpha' \in \phi'} L_{\alpha'}$$

with respect to H' where ϕ' is the set of roots of L with respect to H' and $L_{\alpha'}$ is the root space of L belonging to $\alpha' \in \phi'$. Moreover for $\alpha' \in \phi'$ the root space $L_{\alpha'}$ is one dimensional. If α', β' and $(\alpha' + \beta') \in \phi'$ and $X_{\alpha'}$ and $X_{\beta'}$ respectively denote arbitrary root vectors belonging to α' and β', then $0 \neq [X_{\alpha'}, X_{\beta'}] \in L_{\alpha'+\beta'}$.

Proof: Assume that L is of type A_{n-1}. Since a composition of automorphisms of L is again an automorphism of L, we may assume that H is as in 1.14. Let H' denote another maximal toral subalgebra of L.

The subalgebra H' is a commutative family of diagonalizable matrices , and is therefore simultaneously diagonalizable. Hence an $n \times n$ matrix S exists such that $S^{-1}H'S$ is a family of diagonal matrices. Since conjugacy preserves trace, we conclude that $S^{-1}H'S \subset H$. Hence $H' \subseteq SHS^{-1}$. Since conjugacy preserves diagonalizability and trace, SHS^{-1} is a toral subalgebra of L containing H'. The latter being maximal, we conclude that $SHS^{-1} = H'$. Hence $\sigma : X \to SXS^{-1}$ for $X \in L$, is an automorphism of L sending H to H'. It is easy to see that $\sigma(C_L(H)) = C_L(H')$ and hence $H' = \sigma(H) = \sigma(C_L(H)) = C_L(H')$ and L has a root space decomposition with respect to H'. The final assertion of the thorem follows from the existence of σ and the identity $[e_{rs}, e_{st}] = e_{rt}$ for r, s and t distinct elements of $\{1, \ldots n\}$. ∎

Next we study the root systems.

Definition 1.20: Let E denote a Euclidean space and $(.,.)$ be the inner product of E. A subset ϕ of E, is called a *root system* in E if:

(i) ϕ is finite, spans E, and does not contain 0,

(ii) If $\alpha \in \phi$, the only multiples of α in ϕ are $\pm\alpha$,

(iii) If $\alpha \in \phi$, the endomorphism σ_α of E given by $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ leaves ϕ invariant and,

(iv) If α, β are in ϕ then $< \alpha, \beta > \in Z$ where $< \alpha, \beta >= 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$.

Definition 1.21: Let ϕ denote a root system of E. A subset $\Delta \subseteq \phi$ is called a *base*, if it is a base of E and any element α of ϕ can be written as a linear combination of elements of Δ with all the coefficients non-negative or non-positive integers. In the non-negative case we refer to α as a positive element of ϕ,(with respect to Δ), and in the non-positive case we call α a negative element of ϕ.

In the next Theorem , we freely borrow the notations of example 1.14.

Theorem 1.22: Let H denote the maximal toral subalgebra of simple linear Lie algebra A_{n-1} constructed in Example 1.14.

(i) The set ϕ (of the roots of A_{n-1} with respect to H) forms a root system, which has a base $\Delta = \{\omega_i - \omega_{i+1} \mid 1 \leq i \leq n-1\}$,

(ii) If S_n denotes the symmetric group on n objects and $\sigma \in S_n$,then the set $\Delta_\sigma = \{\omega_{\sigma(i)} - \omega_{\sigma(i+1)} \mid 1 \leq i \leq n-1\}$ is a base of ϕ.

(iii) If Δ' is a base of ϕ, there exists $\sigma' \in S_n$ such that $\Delta' = \{\omega_{\sigma'(i)} - \omega_{\sigma'(i+1)} \mid 1 \leq i \leq n-1\}$.

Proof: Let $\{\omega_i \mid 1 \leq i \leq n\}$ form an orthonormal basis for R^n and $(.,.)$ denote the related inner product. Let E denote the vector subspace spanned by

$$\Delta = \{\omega_i - \omega_{i+1} \mid 1 \leq i \leq n-1\}.$$

Then E endowed with the inherited inner product $(.,.)$ is a Euclidean space.

Let ϕ denote the set of roots of L with respect to H. By direct computation the conditions (i), (ii) , (iii) and (iv) of the definition of the root system are readily seen to be true. Furthermore, the set $\Delta = \{\omega_i - \omega_{i+1} \mid 1 \leq i \leq n-1\}$ is easily seen to be a base of ϕ. This proves (i).

(ii) Let $\sigma \in S_n$. Define linear transformation $L_\sigma \in End(E)$ by $L_\sigma(\omega_i) = \omega_{\sigma(i)}$ and extend linearly. Then $L_\sigma \in End(E)$ has $L_{\sigma^{-1}}$ as its inverse, and maps ϕ into ϕ. Let $\gamma \in \phi$, then $L_{\sigma^{-1}}(\gamma) \in \phi$ and hence

$$L_{\sigma^{-1}}(\gamma) = \sum_{\beta \in \Delta} k_\beta \beta$$

with k_β all non-positive or all non-negative integers. By applying L_σ to both sides of this identity we have:

$$\gamma = \sum_{\beta \in \Delta} k_\beta L_\sigma(\beta),$$

which is an expression of γ in terms of the elements of Δ_σ with all non-positive or all non-negative integers. Since γ was arbitrary, we see that the elements of Δ_σ spans E and since there are $n-1$ elements in Δ_σ , they must be linearly independent. Finally by definition of base we are done.

(iii) We know that each element α of ϕ is of the form $\omega_i - \omega_j$ where $1 \leq i \neq j \leq n$. Let Δ' be a base of ϕ.

First, we claim that each ω_i ($1 \leq i \leq n$) occurs in at most two elements of Δ' and whenever ω_i occurs in two elements of Δ' these occurences have opposite signs. If the claim is not true for say ω_1, we could assume that there exist two elements α and β of ϕ such that ω_1 appears in both of these elements with positive signs. But then $\alpha - \beta \in \phi$ is a root with coefficients of α and β respectively positive and negative. This violates the assumption that Δ' is a base and proves the claim.

Now, since Δ' is a base of ϕ it must consist of $n-1$ elements. Also since each $\omega_r - \omega_s$, $1 \leq r \neq s \leq n$ is expressible in terms of elements of Δ' each ω_i for $1 \leq i \leq n$ must occur in some element of Δ'. The pigeon hole principle and the first step imply that two elements of the set $\Omega = \{\omega_i \mid 1 \leq i \leq n\}$ say ω_{i_1} and ω_{i_n} occur exactly once in distinct roots of ϕ with distinct signs and each of the rest of the elements of Ω occurs in exactly two elements of Δ' with distinct signs. We conclude that there exists $i_1, \ldots, i_n$ distinct elements of the set $\{1, \ldots, n\}$ such that: $\Delta' = \{\omega_{i_j} - \omega_{i_{j+1}} \mid 1 \leq j \leq n-1\}$. Define σ' by $\sigma'(j) = i_j$ for $1 \leq j \leq n-1$, then (iii) immediately follows. ∎

Definition 1.23: An ordered base $\Delta' = \{\alpha_1, \ldots, \alpha_{n-1}\}$ of ϕ ,the root system of (A_{n-1}, H), is *standard* if there exists a $\sigma \in S_n$ such that $\alpha_i = \omega_{\sigma(i)} - \omega_{\sigma(i+1)}$ for all $1 \leq i \leq n-1$.

By part (iii) of 1.22, every ordered base of (A_{n-1}, H) can be re-ordered to be standard. Also by part (ii) and by definition an element of S_n with the action defined in 1.22, takes standard bases of (A_{n-1}, H) into standard bases.

The following theorem partially generalizes Theorem 1.22.

Theorem 1.24: Let L denote a simple linear Lie algebra and H denote a maximal toral subalgebra of L. The set of roots of L with respect to H forms a root system ϕ which possess a base Δ.

Proof: We refer the reader to [H]. ∎

The following theorem is an easy but crucial observation.

Theorem 1.25: Let L denote a simple linear Lie algebra and H be a maximal toral subalgebra of L. Let ϕ denote the root system of roots of L with respect to H and Δ be a base of ϕ, then

(i) If ϕ^+ and ϕ^- denote the sets of positive and negative elements of ϕ with respect to Δ, then

$$\phi^+ = -\phi^- = \{-\alpha \mid \alpha \in \phi^-\}.$$

(ii) Let $\{h_i, 1 \leq i \leq k\}$ denote a base of H. For $\alpha \in \phi$, let L_α denote the root space of L belonging to α and choose $0 \neq X_\alpha \in L_\alpha$ and $0 \neq Y_\alpha \in L_{-\alpha}$ for $\alpha \in \phi^+$. Order the finite set ϕ^+ in an arbitrary manner, then $\{Y_\alpha, \alpha \in \phi^+\} \cup \{h_i, 1 \leq i \leq k\} \cup \{X_\alpha, \alpha \in \phi^+\}$ is an ordered basis of L.

Proof: (i) It follows easily from part (ii) of Definition 1.20 and the Definition 1.21.

(ii) By theorem 1.19, for any $\alpha \in \phi$, L_α is one dimensional. Since $L = H \oplus \sum_{\alpha \in \phi} L_\alpha$ and $\phi = \phi^+ \cup \phi^-$, the assertion of the theorem is obvious. ∎

We continue with some standard definitions.

Definition 1.26: Let L denote a complex linear Lie algebra, then

(i) L is an *internal algebra direct sum* of subalgebras $L_i, 1 \leq i \leq m$ of L if L is a vector space internal direct sum of $L_i, 1 \leq i \leq m$ and $[x, y] = 0$ whenever $x \in L_i, y \in L_j$ and $1 \leq i \neq j \leq m$. We write $L = \oplus \sum_{i=1}^m L_i$.

(ii) L is *semisimple* if L is an internal algebra direct sum of simple subalgebras of L.

(iii) If L_1 and L_2 are ideals of L the *commutator* algebra $[L_1, L_2]$ of L_1 and L_2 is defined by

$$[L_1, L_2] = \{\sum_{i=1}^{k}[x_i, y_i] \mid x_i \in L_1, y_i \in L_2 \text{ and } k \in \mathbb{N}\}.$$

One can easily check that $[L_1, L_2]$ is infact an ideal of L.

(iv) The *derived algebra* of L is the subalgebra (in fact the ideal) $[L, L]$ of L. The *lower central series* of L is the chain of subalgebras of L given by $L^0 = L$, $L^1 = [L, L]$, $L^2 = [L, L^1]$, $\ldots, L^i = [L, L^{i-1}]$. L is *nilpotent* if $L^m = 0$ for some $m \in \mathbb{N}$. An ideal I of L is called *a nilpotent ideal* of L if it is nilpotent as a Lie algebra.

(v) L is *reductive* if L is an internal algebra direct sum

$$L = Z(L) \oplus [L, L],$$

with $[L, L]$ semisimple. Note that $Z(L)$ and $[L, L]$ are respectively the centralizer $C_L(L)$ and the derived subalgebra of L.

Definition 1.27: Let L denote a simple linear Lie algebra and H denote a maximal toral subalgebra of L. Let ϕ denote the root system of (L, H) and Δ denote a base of ϕ. Let ϕ^+ and ϕ^- respectively denote the sets of respectively positive and negative elements of ϕ with respect to Δ. For $\alpha \in \phi$, let L_α denote the corresponding root space of L with respect to H, then

(i) The algebras $L^+ = H \oplus \sum_{\alpha\in\phi^+} L_\alpha$ and $L^- = H \oplus \sum_{\alpha\in\phi^-} L_\alpha$ are referred to as the H diagonalizable *Borel subalgebras* of L.

(ii) An H diagonalizable *parabolic subalgebra* of L is a subalgebra of L containing an H diagonalizable Borel subalgebra of L.

Definition 1.28: Let ϕ denote a root system. a *root subsystem* ϕ' of ϕ is a subset of ϕ, which is a root system on its own. A *closed subset* ϕ^* of ϕ is a subset of ϕ such that $\alpha, \beta \in \phi^*$ and $\alpha + \beta \in \phi$ imply that $\alpha + \beta \in \phi^*$.

Theorem 1.29: Let L denote a simple linear Lie algebra and H denote a maximal toral subalgebra of L. Let ϕ denote the root system of (L, H) and P denote a H diagonalizable parabolic subalgebra of L. For $\alpha \in \phi$, let L_α denote the corresponding root space of L with respect to H. Let Δ denote a base of ϕ and ϕ^+ and ϕ^- respectively denote the sets of respectively positive and negative elements of ϕ with respect to Δ. There exists a closed subset ϕ' of ϕ such that $\phi^+ \subseteq \phi'$ or $\phi^- \subseteq \phi'$ and $P = H \oplus \sum_{\alpha \in \phi'} L_\alpha$.

Proof: We freely borrow the notations of Definition 1.27. Without loss of generality we may assume that $L^+ \subseteq P$.

For $\alpha \in \phi$, by virtue of Theorem 1.19, L_α is one dimensional and hence either $L_\alpha \subseteq P$ or $L_\alpha \cap P = (0)$. We conclude that if $\phi' = \{\alpha \in \phi \mid L_\alpha \cap P \neq (0)\}$ then we have $P = H \oplus \sum_{\alpha \in \phi'} L_\alpha$ and since $L^+ \subseteq P$ we have $\phi^+ \subseteq \phi'$. It remains to prove that ϕ' is a closed subsystem of ϕ. To this end take $\alpha, \beta \in \phi'$ with $\alpha + \beta \in \phi$ and choose $0 \neq X_\alpha \in L_\alpha$ and $0 \neq X_\beta \in L_\beta$. Then by Theorem 1.19 $0 \neq [X_\alpha, X_\beta] \in L_{\alpha+\beta} \cap P$. We conclude that $L_{\alpha+\beta} \subseteq P$ and $\alpha + \beta \in \phi'$. ∎

The preceding theorem motivates the following definition.

Definition 1.30: Let L denote a simple linear Lie algebra and H denote a maximal toral subalgebra of L. Let ϕ be the root system of (L, H) and Δ be a base of ϕ. Let ϕ^+ and ϕ^- respectively denote the sets of positive and negative elements of ϕ with respect to Δ. A *parabolic subset* ϕ' of ϕ is a closed subset of ϕ containing ϕ^+ or ϕ^-.

We continue with the following theorem which we will only prove for the A_{n-1} case.

Theorem 1.31: Let L denote a simple linear Lie algebra and H denote a maximal toral subalgebra of L. Let ϕ denote the root system of (L, H) and

$$L = H \oplus \sum_{\alpha \in \phi} L_\alpha$$

the corresponding root space decomposition. There exists an automorphism σ_{op} of L such that $\sigma_{op}(H) = H$ and $\sigma_{op}(L_\alpha) = L_{-\alpha}$ for all $\alpha \in \phi$.

proof: We freely borrow the notations of Example 1.14. By virtue of Theorem 1.19, we could assume that L and H are as in Example 1.14. Define σ_{op} by $\sigma_{op}(e_{rs}) = e_{sr}$ and $\sigma_{op}(h_i) = -(h_i)$ for $1 \leq r \neq s \leq n$ and $1 \leq i \leq n-1$. It is readily seen that σ_{op} is the desired automorphism. ∎

The following definition relates the preceding theorems and definitions.

Definition 1.32: Let L denote a simple linear Lie algebra and H denote a maximal toral subalgebra of L. Let P be a H diagonalizable parabolic subalgebra of L. Let σ_{op} be as in Theorem 1.30. The subalgebra $P^* = \sigma_{op}(P)$ of L is easily seen to be a H diagonalizable parabolic subalgebra of L and is called the *opposite* subalgebra for P. P and P^* together are refered to as a *pair of opposite* H diagonalizable parabolic subalgebras of L.

We close this chapter with the following theorem.

Theorem 1.33: Let L denote a finite dimensional complex linear Lie algebra. Then L possesses a unique maximal nilpotent ideal $nil(L)$, called the *nilradical* of L that contains all the nilpotent ideals of L.

Proof: We prove the theorem by showing that

(i) The sum (vector space sum) of any two nilpotent ideals of L is a nilpotent ideal of L.

(ii) The sum of all nilpotent ideals of L is the desired nilpotent ideal of L.

To prove (i), let I and J be nilpotent ideals of L. It is easy to see that $(I + J) \trianglelefteq L$. By definition there exists m_1 and m_2 natural numbers such that $I^{m_1} = (0)$ and $J^{m_2} = (0)$. By induction and by using Jacobi identity, it can be easily shown that

$$(I + J)^m = I^m + [I^{m-1}, J^1] + \ldots + [I, J^{m-1}] + J^m$$

for all $m \in \mathbb{N} \cup \{0\}$. Hence if $m > (m_1 + m_2)$ we have $(I + J)^m = (0)$ and $(I + J)$ is nilpotent.

To prove (ii), we observe that by induction and by part (i), the sum of a finite number of nilpotent ideals of L is nilpotent. Let O denote the collection of all nilpotent ideals of L. Let $nil(L) = \sum_{I \in O} I$. Since L is finite dimensional, then $nil(L)$ is the sum of a finite number of elements of O. By the preceding remarks, $nil(L)$ is a nilpotent ideal of L. If I' is another nilpotent ideal of L, then $I \subseteq nil(L)$ and hence $nil(L)$ has the desired property. ∎

CHAPTER 2

Tensor Products

In this chapter, we review tensor products of modules of associative rings. In doing so, we follow the terminologies and proofs of [P]. It is assumed that the reader is familiar with the definition of a ring. We also assume that all the rings contain identity.

Definition 2.1: Let R denote a ring. An additive abelian group V is said to be a *left R module* if there exists a map $R \times V \to V$, written by $(r, v) \to rv$ satisfying:

(i) $r(v_1 + v_2) = rv_1 + rv_2$

(ii) $(r_1 + r_2)v = r_1v + r_2v$

(c) $(r_2r_1)v = r_2(r_1v)$

(iv) $1v = v$

for all $v, v_1, v_2 \in V$ and $r, r_1, r_2 \in R$. Similarly, V is a right R-module if and only if there exists a multiplicative map $V \times R \to V$ satisfying analogous conditions.

We note that if R is the complex field $\mathbb{C}$, then a left (resp. right) module of R is precisely a left (resp. right) complex vector space.

Definition 2.2: Let R be a ring M, N, M' and $\{M_i\}_{i \in I}$ denote left R modules.

(i) An additive group homomorphism $\phi : M \to M'$ is called an *R module homomorphism* if $\phi(rm) = r\phi(m)$ for all $r \in R$ and $m \in M$. The R module homomorphism ϕ is called an *R module isomorphism*, if it is a group isomorphism between M and M'.

(ii) An *R submodule* N of M is a subgroup of M which is an R module with respect to the module multiplication of M.

(iii) If M is the group generated by R submodules $\{M_i\}_{i \in I}$ of M, then we call M the *R module sum* of $\{M_i\}_{i \in I}$ and write $M = \sum_{i \in I} M_i$. If, in addition M is the internal group sum of $\{M_i\}_{i\in I}$ then we call M the *internal weak direct sum* of $\{M_i\}_{i\in I}$ and we write $M = \oplus \sum_{i\in I} M_i$.

(iv) For $a_i \in M$, let $Ra_i = \{ra_i \mid r \in R\}$ for $1 \leq i \leq n$, then Ra_i is an R submodules of M. We say that M is *finitely generated by* $a_i, 1 \leq i \leq n$ as an R module, if $M = \sum_{i=1}^{n} Ra_i$.

(v) Assume that N is an R submodule of M. The operation given by

$$r.(m+N) = r.m + N \quad \text{for all} \quad r \in R \quad \text{and} \quad m \in M$$

defined on the quotient group M/N is easily seen to be well defined and M/N endowed with this operation is a R module, called the *quotient module* M divided by N.

(vi) M is a *simple* R module, if it does not possess any R submodule distinct from (0) and M.

Theorem 2.3: Let R be a ring and M denote an R module with $N \neq M$ a submodule of M. Assume that there is no R submodule P of M satisfying $N \subset P \subset M$. (We may say N is a *maximal* R submodule of M.) Then M/N is a simple R module.

Proof: Assume to the contrary that M/N is not simple and hence has a R submodule $P' \neq (0)$ such that $P' \neq M/N$. Define $P = \{x \in M \mid x + N \in P'\}$, then it is easy to see that P is an R submodule of M containing N. Since $P' \neq \{0+N\}$ we have $P \neq N$ and since $P' \neq M/N$ we have $P \neq M$. This contradiction gives the result. ▮

We continue by another definition.

Definition 2.4: Let R be a ring and A and B denote right and left R-modules respectively. If U is an additive abelian group, then $\Phi : A \times B \to U$ is said to be a *balanced map* provided that:

(i) $\Phi(a_1 + a_2, b) = \Phi(a_1, b) + \Phi(a_2, b)$,

(ii) $\Phi(a, b_1 + b_2) = \Phi(a, b_1) + \Phi(a, b_2)$, and

(iii) $\Phi(ar, b) = \Phi(a, rb)$,

for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$ and $r \in R$. Notice that if $\epsilon : U \to U'$ is a group homomorphism, then $\epsilon \circ \Phi$ is also a balanced map.

Definition 2.5: The *tensor product* of A and B over R, is an ordered pair (X, Θ) of an abelian group X and a balanced map $\Theta : A \times B \to X$ such that for any other balanced map $\Phi : A \times B \to U$, there exists a unique group homomorphism $\alpha : X \to U$ such that $\Phi = \alpha \circ \Theta$.

An explicit construction of the tensor product of A and B can be given as follows. Define $S(A, B)$ to be the set of finite sums of the form

$$\sum_{(a,b)\in A\times B} z_{a,b}(a, b)$$

where $z_{a,b} \in \mathbb{Z}$. Then define an addition operation on $S(A, B)$ by

$$\sum_{(a,b)\in A\times B} z_{a,b}(a, b) + \sum_{(a,b)\in A\times B} z'_{a,b}(a, b) = \sum_{(a,b)\in A\times B} (z_{a,b} + z'_{a,b})(a, b).$$

It is then clear that $S(A, B)$ is an additive group. Let $S_0(A, B)$ be the subgroup of $S(A, B)$ generated by the elements of the form

$$(a_1 + a_2, b) - (a_1, b) - (a_2, b)$$

$$(a, b_1 + b_2) - (a, b_1) - (a, b_2)$$

$$(ar, b) - (a, rb)$$

with $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$ and $r \in R$. Now define $A \otimes B = A \otimes_R B$ to be the additive abelian group $S(A, B)/S_0(A, B)$. Define $\Theta : A \times B \to A \otimes B$ by

$\Theta(a, b) = (a, b) + S_0(A, B)$ for $a \in A$ and $b \in B$. We denote the image of (a, b) under Θ by $a \otimes b$. Then by definition of $S_0(A, B)$ we have:

$$(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$$

$$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$$

$$ar \otimes b = a \otimes rb$$

for appropriate elements of A, B and R. In particular Θ is a balanced map.

Theorem 2.6: Let A and B respectively denotes right and left modules of R. Then $(A \otimes_R B, \Theta)$ is the tensor product of A and B over R. Furthermore if (X, Θ') denote another tensor product, then there exists an abelian group isomorphism $\sigma : A \otimes_R B \to X$ such that $\Theta' = \sigma \circ \Theta$.

Proof: As we observed, $A \otimes B$ is an abelian group and $\Theta : A \times B \to A \otimes B$ is a balanced map. Now let $\Phi : A \times B \to U$ be any balanced map. We define an abelian group homomorphism $\epsilon : S \to U$ by:

$$\epsilon : \sum_{(a,b) \in A \times B} z_{(a,b)}(a, b) \to \sum_{(a,b) \in A \times B} z_{(a,b)} \Phi(a, b).$$

Furthermore since Φ is balanced, it follows easily that ϵ maps the generators of S_0 and hence S_0 to 0. Hence if we define, σ by $\sigma(a \otimes b) = \epsilon(a, b)$ for all $a \otimes b \in A \otimes B$ and extend additively, then σ is a group homomorphism satisfying $\sigma \circ \Theta = \Theta'$. The homomorphism σ is unique, since σ is determined uniquely by $\sigma(a \otimes b) = \Phi(a, b)$.

Let (X, Θ') be any other such tensor product. Since the map $\Theta' : A \times B \to X$ is balanced, there exists a homomorphism $\beta : A \otimes B \to X$ satisfying $\Theta' = \beta \circ \Theta$. Similarly since $\Theta : A \times B \to A \otimes B$ is balanced, there exists a homomorphism γ with $\Theta = \gamma \circ \Theta'$. Thus $\Theta' = \beta \circ \gamma \circ \Theta'$ and $\Theta = \gamma \circ \beta \circ \Theta$. Consider the definition of tensor product and observe that there exists a unique map $\delta : A \otimes B \to A \otimes B$ such that $\Theta = \delta \circ \Theta$, but then $\delta = \gamma \circ \beta$ and the identity mapping of $A \otimes B$, both satisfy this, so that $\gamma \circ \beta$ is the identity mapping on $A \otimes B$. Similarly $\beta \circ \gamma$ is the identity mapping on X. We conclude that β is the desired isomorphism. ∎

The following is one of the key properties of tensor products.

Lemma 2.7: Let A, A', A'' be right R modules and B, B', B'' be left R modules. If $\alpha : A \to A'$ and $\beta : B \to B'$ are R module homomorphisms, then there exists a natural abelian group homomorphism $\alpha \otimes \beta : A \otimes B \to A' \otimes B'$ given by:

$$\alpha \otimes \beta(a \otimes b) = \alpha(a) \otimes \beta(b),$$

for all $a \in A$ and $b \in B$. Further if $\alpha' : A' \to A''$ and $\beta' : B' \to B''$ are also homomorphisms, then

$$(\alpha' \otimes \beta') \circ (\alpha \otimes \beta) = (\alpha' \circ \alpha) \otimes (\beta' \circ \beta).$$

In particular, if α and β are isomorphisms, then so is $\alpha \otimes \beta$.

Proof: The map $A \times B \to A' \otimes B'$ given by $(a, b) \to \alpha(a) \otimes \beta(b)$ is easily checked to be balanced. The definition of tensor products implies that this gives rise to the abelian group homomorphism $\alpha \otimes \beta : A \otimes B \to A' \otimes B'$ with $\alpha \otimes \beta : a \otimes b \to \alpha(a) \otimes \beta(b)$. The remaining parts follow by evaluating the appropriate maps on the set $\{a \otimes b \mid a \in A, b \in B\}$ which generates $A \otimes B$ and by noticing that the inverse of an R module isomorphism is an R module isomorphism. ∎

It often happens in algebra that an abelian group is a right module for some ring and a left module for another ring and these two structures are compatible. This situation suggests the following definition.

Definition 2.8: Let R and S denote rings. Assume that M is a left R module and also a right S module. If we also have:

$$r(ms) = (rm)s$$

for all $r \in R, m \in M$ and $s \in S$, then we call M an R and S *bimodule.*

For instance if R is a ring, then R is an R and R bimodule with the natural action of R. The following theorem relates definitions 2.4 and 2.8.

Theorem 2.9: Let A be an R and S bimodule and B a left S module and respectively. Then $A \otimes_S B$ is a left R module via the action: $r(a \otimes b) = (ra) \otimes b$.

Proof: Let $r \in R$ and let 1_B denote the identity mapping of B. Then by lemma 2.7, since the mapping given by $a \to rb$ for all $a \in A$ is an S module homomorphism, $r \otimes 1_B : A \otimes B \to A \otimes B$ given by $r \otimes 1_B : a \otimes b \to (ra) \otimes b$ is an endomorphism of the abelian group $A \otimes_S B$. Define $r.x = r \otimes 1_B(x)$ for all $x \in A \otimes B$. By lemma 2.7, we observe that $(rr').x = (rr' \otimes 1_B)(x) = (r \otimes 1_B) \circ (r' \otimes 1_B)(x) = r \otimes 1_B(r'.x) = r.(r'.x)$, for all r and $r' \in R$. The other conditions of 2.1 are also easily verified. ∎

The next theorem relates internal weak direct sums and tensor products.

Theorem 2.10: Let R be a ring and B a left R module. Assume that $A = \oplus \sum_{i \in I} A_i$, where $A_i, i \in I$ are R submodules of right R module A, then

(i)

$$A \otimes B = \oplus \sum_{i \in I} A_i \otimes B,$$

(ii) The R module $A \otimes R$ is R module isomorphic to A via the mapping $a \otimes r \to ar$ for $a \in A$ and $r \in R$.

Proof: We refer the reader to [P]. ∎

We next study the tensor product of vector spaces. It is easy to see that a (complex) vector space can be considered as a $\mathbb{C}$ bimodule. We conclude that if V and W denote complex vector spaces, it makes sense to talk about $V \otimes_{\mathbb{C}} W$. In fact by theorem 2.9 (and an analogous result) , it is clear that $V \otimes_{\mathbb{C}} W$ is a complex bimodule and hence a complex vector space. We close this chapter with the following theorem which provides a summary of the basic properties of this vector space.

Theorem 2.11: Let V and W denote complex vector spaces. Let $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ respectively denote arbitrary bases of V and W, then:

(i) If $0 \neq v \in V$ and $0 \neq w \in W$ then $v \otimes_{\mathbf{C}} w \neq 0$.

(ii) If $0 \neq u_j \in V$ for $j \in J$, the elements $\{u_j \otimes_{\mathbf{C}} w_j\}_{j\in J}$ are linearly independent elements of $V \otimes_{\mathbf{C}} W$.

(iii) The elements of the set $\{v_i \otimes_{\mathbf{C}} w_j \mid i \in I, j \in J\}$ form a base of $V \otimes_{\mathbf{C}} W$.

(iv) If V_1, V_2 denote subspaces of V and W_1 and W_2 denote subspaces of W then $(V_1 \otimes_{\mathbf{C}} W_1) \cap (V_2 \otimes_{\mathbf{C}} W_2) = (0)$ provided that $V_1 \cap V_2 = (0)$ or $W_1 \cap W_2 = (0)$.

Proof: Let $W_j = span_{\mathbf{C}}(w_j)$ and $V_i = span_{\mathbf{C}}(v_i)$ Then W_j 's and V_i 's are respectively $\mathbf{C}$ submodules of W and V.

(i) Let $V' = span_{\mathbf{C}}(v)$ and $W' = span_{\mathbf{C}}(w)$ then V', W' are $\mathbf{C}$ modules isomorphic to $\mathbf{C}$ via the isomorphisms determined by $v \to 1$ and $w \to 1$. It follows from lemma 2.7 that $V' \otimes W'$ is $\mathbf{C}$ module isomorphic to $\mathbf{C} \otimes_{\mathbf{C}} \mathbf{C}$, which by theorem 2.10 part (ii), is in turn a $\mathbf{C}$ module isomorphic to $\mathbf{C}$. So that $V' \otimes_{\mathbf{C}} W'$ and $\mathbf{C}$ are isomorphic complex vector spaces and have the same dimension. Moreover $V' \otimes_{\mathbf{C}} W'$ is easily seen to be spanned by $v \otimes_{\mathbf{C}} w$. Since $dim(V' \otimes_{\mathbf{C}} W') = 1$, we have $v \otimes_{\mathbf{C}} w \neq 0$.

(ii) By a result analogous to 2.10, we have:

$$V \otimes W = \oplus \sum_{j\in J} V \otimes_{\mathbf{C}} W_j. \tag{2.11.1}$$

Since by part (i) $0 \neq u_j \otimes_{\mathbf{C}} w_j \in V \otimes_{\mathbf{C}} W_j$ and by directness of vector space sum (ii) follows.

(iii) By a similar argument to that of part (ii), we have:

$$V \otimes_{\mathbf{C}} W_j = \oplus \sum_{i\in I} V_i \otimes_{\mathbf{C}} W_j.$$

Combining this with 2.11.1 gives:

$$V \otimes_{\mathbf{C}} W = \oplus \sum_{(i,j) \in I \times J} V_i \otimes_{\mathbf{C}} W_j.$$

we conclude as in (ii) that the elements of $\{v_i \otimes_{\mathbf{C}} w_j \mid i \in I, j \in J\}$ are linearly independent. Moreover since $V_i \otimes_{\mathbf{C}} W_j$ is spanned by $v_i \otimes_{\mathbf{C}} w_j$, the elements of the mentioned set span $V \otimes_{\mathbf{C}} W$. This proves (iii).

(iv) Without loss of generality, assume that $V_1 \cap V_2 = (0)$ and let V_3 denote a complement to $V_1 \oplus V_2$ in V. Then by theorem 2.10 we have:

$$V \otimes_{\mathbf{C}} W = (V_1 \otimes_{\mathbf{C}} W) \oplus (V_2 \otimes_{\mathbf{C}} W) \oplus (V_3 \otimes_{\mathbf{C}} W)$$

and hence $(V_1 \otimes_{\mathbf{C}} W) \cap (V_2 \otimes_{\mathbf{C}} W) = (0)$. Since subspaces of separated spaces are separated (iv) follows. ∎

CHAPTER 3

The Universal Enveloping Algebras

In example 1.2, we associated a Lie algebra with an associative algebra in a natural manner. Next, we study the reverse idea. It turns out that one can embed a Lie algebra L in many non-isomorphic associative algebras. Among these the universal enveloping algebra U of L is the most important, since any representation of L naturally extends to a unique representation of U.

Definition 3.1: Let V be a finite dimensional complex vector space. Then V is a $\mathbb{C}$ bimodule. Define $T^0V = \mathbb{C}, T^1V = V, T^2V = V \otimes_{\mathbb{C}} V, \ldots, T^nV = V \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V \ldots \otimes_{\mathbb{C}} V$ (n copies). Let

$$I(V) = \oplus \sum_{i=o}^{\infty} T^iV$$

and define an associative product on $I(V)$ by

$$(v_1 \otimes v_2 \ldots \otimes v_k)(w_1 \otimes w_2 \otimes \ldots \otimes w_m) = (v_1 \otimes \ldots \otimes v_k \otimes w_1 \otimes w_2 \ldots \otimes w_m)$$

for $k, m \geq 0$ and extend linearly. This multiplication turns $I(V)$ into an algebra with identity , called the *tensor algebra* on V.

Definition 3.2: Let L be a complex Lie algebra, a *universal enveloping algebra (U.E.A.)* of L is an ordered pair (U, i), where U is an associative complex algebra with identity, $i : L \to U$ a linear map satisfying

$$i([x, y]) = i(x)i(y) - i(y)i(x)$$

for all $x, y \in L$, such that for any associative complex algebra U' and any linear map $j : L \to U'$ satisfying

$$j([x, y]) = j(x)j(y) - j(y)j(x)$$

for all $x, y \in L$, there exists a unique associative algebra homomorphism $\pi : U \to U'$ with $\pi(1) = 1$ such that $\pi \circ i = j$.

The existence and uniqueness of the U.E.A. of a complex Lie algebra is established in the next theorem. The proof uses the same idea as in 2.4 and 2.5.

Theorem 3.3: Let L denote a Lie algebra. Then the U.E.A. of L exists and is unique up to an isomorphism.

Proof: (uniqueness) Assume that (U, i) and (B, i') are both U.E.A. 's of L. By definition, there exist algebra homomorphisms $\Theta : U \to B$ and $\psi : B \to U$ such that $i' = \Theta \circ i$ and $i = \psi \circ i'$. So that $\psi \circ \Theta \circ i = i$ and $\Theta \circ \psi \circ i' = i'$. But by uniqueness part in 3.2 and by considering $U' = U$ and $j = i$ in the definition there exists a unique homomorphism $\pi : U \to U$ such that $\pi \circ i = i$. But then $\psi \circ \Theta$ and 1_U the identity map of U satisfy the requirement for π. We conclude that $\psi \circ \Theta = 1_U$. Similarly $\Theta \circ \psi = 1_B$. So that Θ is an isomorphism of U and U'. This proves the uniqueness of U.E.A. of L up to an isomorphism.

(existence) Let $I(L)$ be as in 3.1 and let J be the two sided ideal in $I(L)$ generated by

$$x \otimes_{\mathbf{C}} y - y \otimes_{\mathbf{C}} x - [x, y]$$

for all $x, y \in L$.

Define $U = I(L)/J$ and let $i : L \to U$ by $i(x) = x + J$. Clearly $i([x, y]) = i(x)i(y) - i(y)i(x)$. Let U' denote another associative algebra with identity with $j : L \to U'$ satisfying $j([x, y]) = j(x)j(y) - j(y)j(x)$. Then consider $I(L)$ and define $\pi' : I(L) \to U'$ by $\pi'(1) = 1$ and $\pi'(x_1 \otimes_{\mathbf{C}} \ldots \otimes_{\mathbf{C}} x_n) = j(x_1) \ldots j(x_n)$ and extend linearly. Then π' is an algebra homomorphism $\pi' : L \to U'$ extending j. Since $j([x, y]) = j(x)j(y) - j(y)j(x)$ we have $x \otimes_{\mathbf{C}} y - y \otimes_{\mathbf{C}} x - [x, y] \in ker(\pi')$ for all $x, y \in L$. We conclude that $J \triangleleft ker(\pi')$ and π' factors through U. Define

$\pi : U = I(L)/J \to U'$ by $\pi(x + J) = \pi'(J)$, then π is an algebra homomorphism satisfying $\pi \circ i = j$. ∎

The next theorem provides a basis of the U.E.A. of L, whenever a basis of L is given. It is due to Poincare, Birkhoff, and Witt.

Theorem 3.4 (The P.B.W. Theorem): Let $(x_1, \ldots, x_n)$ denote an ordered basis of L. Then 1 along with the elements

$$x_{i(1)}x_{i(2)}\ldots x_{i(m)} = x_{i(1)} \otimes_{\mathbf{C}} x_{i(2)} \ldots \otimes_{\mathbf{C}} x_{i(m)} + J, \quad i(1) \leq i(2) \leq \ldots \leq i(m)$$

(with the notations of Theorem 3.3) form a basis for the U.E.A. of L, in particular i is injective.

Proof: We refer the reader to [H]. ∎

We next introduce subalgebras of the U.E.A. of a simple Lie algebra which will be usefull in studying Lie modules. Since by the P.B.W. Theorem i is injective, we may assume that i is the identity mapping.

Definition 3.5: Let L be any simple linear Lie algebra, $U(L)$ denote the universal enveloping algebra of L and H denote an arbitrary maximal toral subalgebra of L. The *cycle subalgebra* $CY(L, H)$ is the set: $\{x \in U(L) \mid xy - yx = 0, \forall y \in H\}$. In other words $CY(L, H)$ is the centralizer of H in $U(L)$.

The next theorem is a key result in studying the universal enveloping algebra of a simple Lie algebra.

Theorem 3.6: Let L denote a simple linear Lie algebra and H be a maximal toral subalgebra of L. Let U denote the U.E.A. of L, and ϕ denote the root system of roots of L in H . Let Δ be a base of ϕ. Let S denote the set $\{\sum_{\alpha\in\Delta} k_\alpha \alpha \mid k_\alpha \in \mathbf{Z}\} \subset H^*$, then

(i) U has a vector space decomposition $U = \oplus \sum_{\gamma\in S} U_\gamma$ such that $[h, u_\gamma] = \gamma(h)u_\gamma$ for all $h \in H$ and $u \in U_\gamma$,

(ii) $U_0 = CY(L, H)$,

(iii) $U_\gamma U_\epsilon \subseteq U_{\gamma+\epsilon}$ for all $\epsilon, \gamma \in S$ where

$$U_\gamma U_\epsilon = \{\sum_{i=1}^{n} u_i v_i \mid u_i \in U_\gamma, v_i \in U_\epsilon, n \in \mathbb{N}\}.$$

Proof: We freely borrow the notations of Theorem 1 25. We order ϕ^+ in an arbitrary manner and observe that by part (ii) of Theorem 1.25 the set

$$\{Y_\alpha, \alpha \in \phi^+\} \cup \{h_i, 1 \leq i \leq k\} \cup \{X_\alpha, \alpha \in \phi^+\}$$

is an ordered basis for L. By Theorem 3.4 the monomials

$$\prod_{\alpha \in \phi^+} Y_\alpha^{m(\alpha)} \prod_{1 \leq i \leq k} h_i^{l(i)} \prod_{\alpha \in \phi^+} X_\alpha^{n(\alpha)}, \tag{3.6.1}$$

where $n(\alpha), m(\alpha)$ and $l(i)$ are non-negative integers, for $\alpha \in \phi^+$ and $1 \leq i \leq k$, form a basis for U. Note that any element raised to the power zero is defined to be 1.

For $\omega \in S$, we define B_ω to be the set of all the monomials 3.6.1 such that

$$\sum_{\alpha \in \phi^+} (n(\alpha) - m(\alpha))\alpha = \omega.$$

Then define $U_\omega = Span_{\mathbb{C}}\{B_\omega\}$. Clearly $U_\omega \cap U_{\omega'} = (0)$ whenever $\omega \neq \omega'$ and $U = \oplus \sum_{\omega \in S} U_\omega$. If $h \in H$ by direct computation, we have

$$[h, \prod_{\alpha \in \phi^+} Y_\alpha^{m(\alpha)} \prod_{1 \leq i \leq k} h_i^{l(i)} \prod_{\alpha \in \phi^+} X_\alpha^{n(\alpha)}] =$$

$$= \sum_{\alpha \in \phi^+} (n(\alpha) - m(\alpha))\, \alpha(h)\, (\prod_{\alpha \in \phi^+} Y_\alpha^{m(\alpha)} \prod_{1 \leq i \leq k} h_i^{l(i)} \prod_{\alpha \in \phi^+} X_\alpha^{n(\alpha)}).$$

We conclude that if $z \in B_\omega$, then $[h, z] = \omega(h)z$. Since the elements of B_ω span U_ω, part (i) follows. Now part (ii) follows easily from part (i).

To prove (iii), let $u \in U_\gamma$, $v \in U_\epsilon$ and $h \in H$, then

$$[h, uv] = [h, u]v + u[h, v] = (\gamma + \epsilon)(h)uv.$$

This and part (i) prove that uv is an element of $U_{\gamma+\epsilon}$. By considering this and the definition of $U_\gamma U_\epsilon$, we have the desired result. ∎

Corollary 3.7: With the notations of Theorem 3.6, the cycle subalgebra $CY(L,H)$ is spanned by all the monomials 3.6.1, for which $\sum_{\alpha\in\phi^+}(n(\alpha)-m(\alpha))\alpha = 0$.

Proof: It follows immediately from the definition of U_0 and from set equality: $CY(L,H) = U_0$. ∎

Definition 3.8: We refer to the monomials of U_0 as *cycles*. A cycle is *basic* if, when considered as a commutative monomial, it can not be written as the product of two other cycles, both distinct from 1. An basic cycle is called an *n-cycle*, if, when considered as a commutative monomial, it is of degree n.

The following theorem reveals the importance of basic cycles.

Theorem 3.9: The n-cycles, $n \geq 0$, generate the algebra $CY(L,H)$.

Proof: For $z \in B_0$, we define, the degree of z denoted by $deg(z)$ to be the degree of z when it is considered as a commutative monomial. We prove by induction on $deg(z)$ that z is an element of G, the subalgebra of $CY(L,H)$ generated by all the basic cycles. Since by Corollary 3.7, B_0 spans $CY(L,H)$, this will give the result.

If $deg(z) = 0$ or $deg(z) = 1$, then z is a basic cycle, and is in G trivially. Assume that $z \in G$ for all z such that $deg(z) \leq k$ where $k \geq 1$. If w is a cycle with $deg(w) = k+1$, then if w is basic, clearly it is in G. If w is not basic, by definition there exist cycles x and y such that w is the product of x and y, when x, y and w are considered as commutative monomials. We claim that in $CY(L,H)$ we have

$$xy = w + \text{ a linear combination of cycles of lower degree.}$$

To observe this, let $Y_\alpha = X_{-\alpha}$ whenever α is negative with respect to Δ. It follows from Theorem 1.19 that, $[X_\alpha, X_\beta] = c_{\alpha+\beta}X_{\alpha+\beta}$, for some $c_{\alpha+\beta} \neq 0$, whenever α, β, and $\alpha+\beta \in \phi$.

On the other hand, assume that $\alpha, \beta \in \phi$ and $(\alpha+\beta) \notin \phi$. Since

$$[h,[X_\alpha,X_\beta]] = [[h,X_\alpha],X_\beta] + [X_\alpha,[h,X_\beta]] = (\alpha+\beta)(h)[X_\alpha,X_\beta],$$

for all $h \in H$, we must have $[X_\alpha, X_\beta] = 0$. Using the relation $[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha$, we see that we can replace $X_\alpha X_\beta$ by $X_\beta X_\alpha$ in xy at the cost of a constant multiple of a monomial (if any) of lower degree than w. By repeated application of this argument we could rearrange the elements of xy to recover w at a cost of a linear combination of cycles of lower degree. Hence the claim is true.

Now, induction hypothesis will give the result. ∎

We next study the basic cycles of the algebra A_{n-1}. To this end, we let H, $\{h_i, \quad 1 \leq i \leq n-1\}$, $\omega_i, \quad 1 \leq i \leq n$ and ϕ be as in Example 1.14. Then by theorem 1.22 part (i), the set $\Delta = \{\omega_i - \omega_{i+1}, \quad 1 \leq i \leq n-1\}$ is a base of ϕ. Let ϕ^+ denote the sets of positive roots of ϕ with respect to this base. For $\alpha = \omega_i - \omega_j$, $1 \leq i \neq j \leq n$, let X_α denote e_{ij}. Define an arbitrary ordering on ϕ^+ , then by the preceding theorems the set of basic cycles of the type

$$\prod_{\alpha\in\phi^+} X_{-\alpha}^{m(\alpha)} \prod_{1\leq i\leq n-1} h_i^{l(i)} \prod_{\alpha\in\phi^+} X_\alpha^{n(\alpha)} \tag{3.10}$$

generate $CY(A_{n-1}, H)$. For a basic cycle c as in 3.10 but with all $l(i) = 0, 1 \leq i \leq n-1$, we let $r(c)$ denote the collection of all roots appearing as subscript in c, where any root α is repeated as many times as the exponent of X_α in c. Then by definition of a basic cycle the elements of $r(c)$ add to 0 but no proper subcollection of $r(c)$ has this property. If we define $\mu(i,j) = \omega_i - \omega_j$ for $i,j \in \{1,\ldots,n\}$, then the collection $r(c)$ can be described using the following Theorem of [BL2].

Theorem 3.11: For each sequence $a_1, ..., a_k$ of $k \geq 2$ distinct elements in $\{1,\ldots,n\}$ the set of roots $\{\mu(a_1,a_2),\ldots,\mu(a_{k-1},a_k),\mu(a_k,a_1)\}$ equals $r(c)$ for some basic cycle in $CY(A_{n-1}, H)$. Moreover if c denotes a basic cycle of $CY(A_{n-1}, H)$, then either

$c = h_i$ for some $1 \le i \le n-1$ or there exists a sequence $a_1, \ldots, a_k$ of $k \ge 2$ distinct elements of $\{1, \ldots, n\}$ such that $r(c) = \{\mu(a_1, a_2), \ldots, \mu(a_{k-1}, a_k), \mu(a_k, a_1)\}$.

Proof: For simplicity we let

$$c\{a_1, \ldots, a_k\}$$

denote the collection $\{\mu(a_1, a_2), \ldots, \mu(a_{k-1}, a_k)\}$ everywhere in this proof. Clearly

$$\mu(a_1, a_2) + \ldots + \mu(a_{k-1}, a_k) + \mu(a_k, a_1) = 0.$$

We must prove that

$$\mu(a_{i_1}, a_{i_1+1}) + \mu(a_{i_2}, a_{i_2+1}) + \ldots + \mu(a_{i_m}, a_{i_m+1}) \neq 0. \qquad (3.11.1)$$

for any nonempty proper subcollection

$$\{\mu(a_{i_1}, a_{i_1+1}), \ldots, \mu(a_{i_m}, a_{i_m+1})\} \subsetneq \{\mu(a_1, a_2), \ldots, \mu(a_{k-1}, a_k), \mu(a_k, a_1)\}.$$

To prove this, let j be the minimal element in $S = \{a_{i_1}, \ldots, a_{i_m}\}$ such that $j+1$ read $mod(n)$ is not in S. Since S has less than n elements, such a j always exists. Then the left hand side of of 3.11.1, when expanded using the basis Δ, has a (-1) appearing of the coefficient of ω_{j+1}. This proves the first part of the assertion of the theorem.

Next let c be a basic cycle of $CY(A_{n-1}, H)$. Notice that $h_i, 1 \le i \le n-1$ are trivially basic cycles of $CY(A_{n-1}, H)$. Since c, when considered as a commutative monomial can not be written as a product of two elements of $CY(A_{n-1}, H)$, we observe that if $l(i) \neq 0$ for some $1 \le i \le n-1$, we must have $c = h_i$. Otherwise we could assume that $l(i) = 0$ for all $1 \le i \le n-1$.

To prove the remainder of the theorem, we do an inductive proof on the number k of elements of the collection $r(c)$. Then $k \ge 2$. If $k = 2$ then we have $r(c) = \{\mu, -\mu\}$ with $\mu = \omega_i - \omega_j$ for some $1 \le i \neq j \le n$. We take a_1, a_k to be i and j respectively and our assertion is true trivially.

Assume now that the result is true up to but not including the case $k = K$. Let c be a basic cycle with

$$r(c) = \{\alpha_1, \ldots, \alpha_K\}$$

with $\alpha_i \in \phi$ (not necessarily distinct elements) for $1 \leq i \leq K$. Then $\alpha_1 = \omega_p - \omega_q$ for some $1 \leq p \neq q \leq n$. Since the sum of elements of $r(c)$ is 0, we must have $\alpha_r = \omega_q - \omega_s$ for some $1 \leq r \leq K$ and $1 \leq s \leq n$. Without loss of generality we could assume that $r = 2$. Furthermore, since $\alpha_1 + \alpha_2 \neq 0$ we have $s \neq p$. We conclude that $(\alpha_1 + \alpha_2) \in \phi$ and hence there exist a basic cycle c' such that:

$$r(c') = \{\alpha_1 + \alpha_2, \alpha_3, \ldots, \alpha_k\}.$$

The induction hypothesis now implies that

$$\{\alpha_1 + \alpha_2, \alpha_3, \ldots, \alpha_k\} = c\{a_1, \ldots, a_{K-1}\}$$

with the a_i, $1 \leq i \leq K - 1$ distinct elements in $\{1, \ldots, n\}$. Since $\alpha_1 + \alpha_2 = \omega_p - \omega_s \in r(c')$, there exists $1 \leq i \leq K - 1$ such that $a_i = p$ and $a_{i+1} = s$ but then:

$$\{\alpha_1, \ldots, \alpha_K\} = c\{a_1, \ldots, a_i, q, a_{i+1}, \ldots, a_{K-1}\}.$$

If $q = a_l$ for some $1 \leq l \leq K - 1$, then c is not basic. By induction we are done. ∎

CHAPTER 4

Representation Theory Of Simple Lie Algebras

In this chapter we study some basic facts about the representations of simple Lie algebras. we start by stating some standard definitions.

Definition 4.1: Let L denote a complex Lie algebra. A complex vector space M is called an *L-module*, if there is an operation $L \times M \to M$ denoted by: $(x, v) \to xv$; and satisfying

(i) $x(c_1 v_1 + c_2 v_2) = c_1(xv_1) + c_2(xv_2)$,

(ii) $(x_1 + x_2)v = x_1 v + x_2 v$, and

(iii) $[x_1, x_2]v = x_1(x_2 v) - x_2(x_1 v)$,

for all $x, x_1, x_2 \in L$ and $v, v_1, v_2 \in V$ and $c_1, c_2 \in \mathbb{C}$.

Definition 4.2: A *representation* of a Lie algebra L is a Lie algebra homomorphism $\rho : L \to gl(V)$ for some complex vector space V.

Definition 4.3: Let M be an L module, then an L *submodule* N of M is a subspace N of M such that $xv \in N$ whenever $x \in L$ and $v \in N$.

If M is an L module, clearly (0) and M are L submodules of M. We call these *trivial* L submodules of M. If N is a non-trivial L submodule of M, we call N a *proper* L submodule of M. We may write submodule for L submodule when there is no ambiguity.

If N is an L submodule of M, the quotient vector space M/N endowed with the operation

$$x.(m + N) = x.m + N \quad \text{for all} \quad x \in L \quad \text{and} \quad m \in M$$

is easily seen to be an L module called the *quotient module* of M divided by N.

Definition 4.4: An L module M is *simple* if it does not possess proper submodules.

Definition 4.5: If $M_i, i \in I$ denote L submodules of a module M, their vector space sum is called the *sum* of $M_i, i \in I$ and is denoted by $\sum_{i \in I} M_i$. If it happens that this sum is a direct vector space sum, we refer to it as *internal direct sum* of $M_i, i \in I$ and we denote it by $\oplus \sum_{i \in I} M_i$.

Next let L denote a Lie algebra. Let M and N denote L modules. Then M and N are $\mathbb{C}$ bimodules. By Theorem 2.8, we observe that $M \otimes_{\mathbb{C}} N$ is a $\mathbb{C}$ bimodule and hence a complex vector space. We could turn this vector space into an L module as the next definition suggests.

Definition 4.6: The *tensor product* of L modules M and N is the complex vector space $M \otimes_{\mathbb{C}} N$ endowed with the module operation determined by

$$x.(m \otimes_{\mathbb{C}} n) = (xm) \otimes_{\mathbb{C}} n + m \otimes_{\mathbb{C}} (xn).$$

It is easy to check that this operation in fact turns $M \otimes_{\mathbb{C}} N$ into an L module. For notational simplicity, we drop the subscript $\mathbb{C}$ everywhere in the rest of this work, whenever this does not cause any ambiguity.

Definition 4.7: Let L be a Lie algebra. An L module M is

(i) *completely reducible*, if for any L submodule N of M, there exists a submodule N' of M such that $M = N \oplus N'$.

(ii) *indecomposable*, if M is not the internal direct sum of two proper submodules of M.

We now prove some results relating these definitions.

Theorem 4.8: Any L module M gives rise to a representation $\rho : L \to gl(M)$ of Lie algebra L. Conversely if such a representation exists, M is naturally an L module.

Proof: If M is an L module, then define $\rho : L \to gl(M)$ by $\rho(x).v = xv$ for all $x \in L$ and $v \in M$. It is easy to check that ρ is a representation of Lie algebra L.

Conversely, if $\rho : L \to gl(M)$ is a representation of L, define $xv = \rho(x).v$, for $x \in L$ and $v \in V$. It is again easy to see that M is an L module with this operation. ∎

Lemma 4.9: Let M be an L module and let $M_i, i \in I$ denote a family of simple submodules of M satisfying $\sum_{i \in I} M_i = M$. If N is a submodule of M, then there exists a subset J of I such that: $(\oplus \sum_{j \in J} M_j) \oplus N = M$.

Proof: Let S be the set of all subsets K of I such that $\oplus \sum_{k \in K} M_k \oplus N$ is an internal direct sum. We note that S is nonempty, since $\emptyset \in S$. Furthermore, the property of being a direct sum is finitary, so Zorn's lemma implies that there exists a maximal element $T \in S$. By definition, we know that $N' = N \oplus \sum_{t \in T} M_t$ is direct.

Suppose by way of contradiction that $N' \neq M$. We observe from the hypothesis that there exists M_k such that $M_k \cap N' \neq M_k$. But M_k is simple. This forces $M_k \cap N' = (0)$. Hence $N' \oplus M_k$ is a direct sum. We conclude that $T \cup \{k\} \in S$. This violates the assumption that T is maximal. ∎

Theorem 4.10: Let L be a Lie algebra, and M an L module. Then the following are equivalent.

(i) M is a sum of simple L submodules of M.

(ii) M is an internal direct sum of simple L submodules of M.

(iii) M is completely reducible.

Proof: The implication (i) $\Rightarrow$ (ii) follows from the lemma with $N = (0)$ and the implication (ii)$\Rightarrow$ (iii) follows from the general case of the lemma.

Thus we need only to prove that (iii) $\Rightarrow$ (i). To this end, let M be completely reducible and let S denote the sum of all simple submodules of M. The goal is to show that $S = M$. If this is not the case fix $v \in M - S$.

By Zorn's lemma we can choose a submodule V of M maximal with the properties $S \subset V$ and $v \notin V$. Now M is completely reducible, so $M = V \oplus U$ for some nonzero submodule $U \subset M$. Furthermore since $S \subset V$ we know that U is not simple. In particular there exists $A \neq (0)$ a submodule of U. Since M is completely reducible we let $M = A \oplus A'$ for A' a submodule of M. Then $U = A \oplus (A' \cap U)$. We let $B = A' \cap U$, then A and B are submodules of M. By maximality of V, we have $v \in V \oplus A$ and $v \in V \oplus B$. We conclude that $v \in (V \oplus A) \cap (V \oplus B) = V$. A contradiction. So that $M = S$. ∎.

In light of the preceding result, if M is a completely reducible L module, then the structure of simple L modules completely determine the structure of M. This reveals the importance of the following theorem.

Theorem 4.11 (Peter-Weyl): Any finite dimenional L module of a linear simple Lie algebra is completely reducible.

Proof: We refer the reader to [J]. ∎

It must be mentioned that there exists indecomposable, infinite dimensional modules of simple linear Lie algebras. An example of these will be given in chapter 6.

We next relate the modules of a Lie algebra L with those of the U.E.A. of L.

Theorem 4.12: Let L be a Lie algebra and $U(L)$ denote the U.E.A. of L. Let M denote an (a simple) L module, then

(i) The action of the elements of L on M can be uniquely extended to a representation of $U(L)$. Therefore M is naturally a (simple) $U(L)$ module. Conversely any $U(L)$ (simple) module is an (a simple) L module by restriction.

(ii) If M and $N \subset M$ are L modules such that there is no L module P satisfying $N \subset P \subset M$ (We may say N is a *maximal* submodule of M), then M/N is a simple L module.

Proof: If $0 \neq x \in L$, extend x to a base $x_1, x_2, \ldots, x_n$ of L with $x_1 = x$ and observe that if the pair $(U(L), i)$ denote the U.E.A. of L, by the P.B.W. Theorem , $i(x) \neq 0$. So that $i : L \rightarrow U(L)$ is an injection. We hence can identify L with $i(L)$ and assume that $L \subset U(L)$. With this observation the second part of (i) follows immediately.

Next Let M denote an L module and let $\rho : L \rightarrow gl(V)$ be the associated representation of L constructed in 4.8. We can apply definition 3.2 with i the identity mapping, to obtain a homomorphism $\pi : U(L) \rightarrow gl(V)$ satisfying $\pi \circ i = \rho$. Hence π when restricted to L is ρ. By theorem 4.8 we see that M is a $U(L)$ module. The fact that π is unique follows from 3.2.

If M is a simple L (respectively $U(L)$ module), the statement of the theorem follows easily from the definition of simplicity and the preceding conclusions. This proves (i) and part (ii) follows immediately from part (i) and Theorem 2.3. ∎

Next we introduce weight modules. The study of these is the main goal of this work.

Definition 4.13: Let L denote a simple linear Lie algebra and H be a maximal toral subalgebra of L. A *weight module* M of (L, H) is one with a vector space decomposition $M = \oplus \sum_{\theta \in H^*} M_\theta$ such that if $h \in H$ and $v_\omega \in M_\omega$ for $\omega \in H^*$ then $hv_\omega = \omega(h)v_\omega$. If $M_\omega \neq 0$ for $\omega \in H^*$, then M_ω is the *weight space* of M belonging to ω and ω is a *weight* of M.

Notation: Throughout the rest of this chapter, we let L, U, H, ϕ, Δ and $W(L, H)$ respectively denote a simple linear Lie algebra, the U.E.A. of L, a maximal toral subalgebra of L, the root system of (L, H), a base of ϕ and the category of (L, H) weight modules. For $\alpha \in \phi$, we let X_α denote a root vector belonging to α. We also let M denote an L module and whenever $M \in W(L, H)$, we let $wt(M)$ denote the set of weights of M.

Theorem 4.14: If $dim_{\mathbb{C}} M < \infty$, then $M \in W(L, H)$.

Proof: Let $\rho : L \to gl(M)$ denote the associated representation constructed in 4.8. By theorem 1.10 for $h \in H$, the endomorphism $\rho(h)$ of $gl(M)$ is diagonalizable. Hence the algebra $\rho(H)$ being the image of a commutative algebra H is a commutative algebra of diagonalizable endomorphisms of M. By a standard result in linear algebra, the elements of $\rho(H)$ are simultaneously diagonalizable. We conclude that there exists a natural number k and $\theta_i \in H^*, 1 \leq i \leq k$ such that $M = \oplus \sum_{i=1}^{k} M_{\theta_i}$, and $hv_{\theta_i} = \rho(h)v_{\theta_i} = \theta_i(h)v_{\theta_i}$ for $h \in H$ and $v_{\theta_i} \in M_{\theta_i}$, and $1 \leq i \leq k$. ∎

We should mention that there exists $M \in W(L, H)$ such that $M \notin W(L, H')$ for any other maximal toral subalgebra H' of L. In light of preceding theorem such an M cannot be finite dimensional. An example of such modules is given in Chapter 6.

On the other hand, U is an infinite dimensional module of L such that $U \in W(L, H)$ for any maximal toral subalgebra H of L (see Theorem 3.6).

Lemire constructed a module M such that $M \notin W(L, H)$ for any maximal toral subalgebra H of L. We refer the interested reader to [L].

We next study some properties of weight modules.

Theorem 4.15: Let $M \in W(L, H)$ and $M = \oplus \sum_{\theta \in wt(M)} M_\theta$ denote the weight space decomposition of M with respect to (L, H). If $\theta \in wt(M)$, then

(i) $X_\alpha v_\theta \in M_{\theta+\alpha}$ for $\alpha \in \phi$

(ii) If N is an L submodule of M, then $N \in W(L, H)$. Moreover,

$$N = \oplus \sum_{\theta \in wt(M)} (N \cap M_\theta)$$

gives a decomposition of N with respect to (L, H).

Proof: (i) If $h \in H$ we have $h(X_\alpha v_\theta) = X_\alpha(hv_\theta) + ([h, X_\alpha])v_\theta = (\theta + \alpha)(h)v_\theta$. This proves (i).

(ii) Assume that $v \neq 0$ and $v \in N$, then $v \in M$, hence $v = \sum_{i=1}^{k} v_i$ where $v_i \in M_{\theta_i}$ and $\theta_i \in H^*$ are distinct for $1 \leq i \leq k$. We wish to prove that $v_i \in N$ for $1 \leq i \leq k$. Assume on the contrary that this is not the case for some $v \in N$. Among all such v's , we choose one such that k is minimal.

Choose $h \in H$ such that $\theta_1(h) \neq \theta_2(h)$. (This is possible, since θ_1 and θ_2 are distinct functionals). Then $(h - \theta_1(h).1)v \in N$. This gives that $\sum_{i=2}^{k}[\theta_i - \theta_1](h)v_i \in N$. By minimality of k , we have $(\theta_2 - \theta_1)(h)v_2 \in N$. since $\theta_1(h) - \theta_2(h) \neq 0$, we have $v_2 \in N$. Therefore $\sum_{i=1, i\neq 2}^{k} v_i \in N$. By minimality of k again $v_i \in N$ for all $1 \leq i \leq k, i \neq 2$. But $v_2 \in N$ was established before and all $v_i \in N$ for all $1 \leq i \leq k$. A contradiction. We hence have $v \in \oplus \sum_{\theta \in H^*}(M_\theta \cap N)$ for any $v \in N$. This gives the result immediately. ∎

Theorem 4.16: Let $CY(L, H)$ denote the cycle subalgebra of U with respect to H, and $M \in W(L, H)$ with a weight space decomposition $M = \oplus \sum_{\theta \in wt(M)} M_\theta$. Fix $\omega \in wt(M)$, then

(i) M_ω is a $CY(L, H)$ module.

(ii) If M is a simple (L, H) module, then M_ω is a simple $CY(L, H)$ module. In this case

$$wt(M) \subseteq \{\omega + \sum_{\alpha \in \Delta} k_\alpha \alpha \mid k_\alpha \in \mathbb{Z}\}.$$

Proof: We freely borrow the notations of theorem 3.6. Fix M_ω a weight space of M. Let $z \in B_0$ and $v \in M_\omega$. Then by repeated applications of part (i) of Theorem 4.15, we have: $zv \in M_\omega$. Since B_0 spans U_0, it follows immediately that M_ω is a $CY(L, H)$ module. This proves (i).

Next, let M be simple. Assume to the contrary that M_ω is not a simple U_0 module. Then M_ω has a proper U_0-module N_ω. Let $N = UN_\omega$. Then N is a U

and hence an L submodule of M. By part (ii) of 4.15, $N \in W(L, H)$. By Theorem 3.6, we have $U = \oplus \sum_{\alpha \in S} U_\alpha$. So that $N = \sum_{\alpha \in S} U_\alpha N_\omega$ as a vector space sum.

By repeated application of part (i) of 4.15, we have

(4.16.1) $$B_\alpha M_\omega \subseteq M_{\omega+\alpha}.$$

This proves that the mentioned vector space sum is direct and

(4.16.2) $$N = \oplus \sum_{\alpha \in S} U_\alpha N_\omega$$

is a decomposition of N with respect to (L, H).

By virtue of part (ii) of Theorem 4.15, we have $N \cap M_\omega = N_\omega$. Since $N_\omega = U_0 N_\omega = N_\omega \neq M_\omega$ we observe that $N \neq M$. Hence N is a proper submodule of M. This violates the simplicity of M. We conclude that M_ω is a simple $CY(L, H)$ module. Next replace N_ω by M_ω in (4.16.2) to have $M = \oplus \sum_{\alpha \in S} U_\alpha M_\omega$. We conclude that the weight spaces of M are of the form $U_\alpha M_\omega$ for $\alpha \in S$. But by (4.16.1) we have:

$$U_\alpha M_\omega \subseteq M_{\alpha+\omega}.$$

By directness of sum the preceding inclusion is in fact set equality. This and definition of S gives the result. ∎

We next consider the tensor product of two L modules.

Theorem 4.17: Assume that $M, N \in W(L, H)$ with weight space decompositions

$$M = \oplus \sum_{\alpha \in wt(M)} M_\alpha$$

and

$$N = \oplus \sum_{\beta \in wt(N)} N_\beta$$

For $\alpha \in wt(M)$ and $\beta \in wt(N)$, let $\{v_{\alpha,i}\}_{i\in I_\alpha}$ and $\{w_{\beta,j}\}_{j\in J_\beta}$ denote vector space bases of M_α and N_β respectively, then $M \otimes N \in W(L,H)$ with a weight space decomposition

$$M \otimes N = \oplus \sum_{\omega \in wt(M)+wt(N)} (M \otimes N)_\omega,$$

where $wt(M) + wt(N) = \{\alpha + \beta \mid \alpha \in wt(M), \beta \in wt(N)\}$. Moreover for $\omega \in wt(M \otimes N)$ we have

$$(M \otimes N)_\omega = span_{\mathbf{C}} \bigcup_{\alpha+\beta=\omega} \{v_{\alpha,i} \otimes w_{\beta,j} \mid i \in I_\alpha, j \in J_\beta\},$$

and

$$(M \otimes N)_\omega = \oplus \sum_{\alpha+\beta=\omega} M_\alpha \otimes N_\beta$$

as a vector space direct sum.

Proof: It is clear that $\bigcup_{\alpha\in wt(M)} \bigcup_{i\in I_\alpha} v_{\alpha,i}$ and $\bigcup_{\beta\in wt(N)} \bigcup_{j\in J_\beta} w_{\beta,j}$ are respectively bases for the vector spaces M and N. It follows from part (iii) of theorem 2.11 that the set

$$\bigcup_{\alpha\in wt(M),\beta\in wt(N)} \bigcup_{(i,j)\in I_\alpha\times J_\beta} \{v_{\alpha,i} \otimes w_{\beta,j}\}$$

forms a basis for $M \otimes N$.

We next observe that if $h \in H$, then

$$h.(v_{\alpha,i} \otimes w_{\beta,j}) = (hv_{\alpha,i}) \otimes w_{\beta,j} + v_{\alpha,i} \otimes (hw_{\beta,j}).$$

Since $hv_{\alpha,i} = \alpha(h)v_{\alpha,i}$ and $hw_{\beta,j} = \beta(h)w_{\beta,j}$ this amounts to

(4.17.1) $$h.(v_{\alpha,i} \otimes w_{\beta,j}) = (\alpha + \beta)(h)(v_{\alpha,i} \otimes w_{\beta,j}).$$

Define the vector spaces $(M \otimes N)_\omega$ by:

$$(M \otimes N)_\omega = span_{\mathbf{C}} \bigcup_{\alpha+\beta=\omega} \{v_{\alpha,i} \otimes w_{\beta,j} \mid i \in I_\alpha, j \in J_\beta\}.$$

Then it is clear that

$$M \otimes N = \oplus \sum_{\omega \in wt(M)+wt(N)} (M \otimes N)_\omega$$

as a vector space direct sum. Using the identity 4.17.1 we see that for the generators $v_{\alpha,i} \otimes w_{\beta,j}$ of $(M \otimes N)_\omega$ we have: $h.(v_{\alpha,i} \otimes w_{\beta,j}) = \omega(h)(v_{\alpha,i} \otimes w_{\beta,j})$ whenever $h \in H$. This and linearity prove that if $h \in H$ and $z \in (M \otimes N)_\omega$ we have: $h.z = \omega(h)z$. The preceding and the identity

$$M_\alpha \otimes N_\beta = Span_{\mathbf{C}}\{v_{\alpha,i} \otimes w_{\beta,j} \mid i \in I_\alpha, j \in J_\beta\}$$

(given by part(iii) of 2.11) prove that $M \otimes N$ is in $W(L, H)$ with the desired description. ∎

We next define the torsion free modules.

Definition 4.18: Let $M \in W(L, H)$ is called an (L, H) *torsion free* module provided that for all $\alpha \in \phi$, X_α acts in an injective manner on the weight spaces of M.

The following is a simple but crucial observation.

Theorem 4.19: Assume that M is a simple torsion free (L, H) module with a weight space decomposition $M = \sum_{\theta \in wt(M)} M_\theta$. If $dim_{\mathbf{C}}(M_\omega) = k < \infty$, then

(i) all the weight spaces of M are of dimension k and the set $wt(M) = \{\omega + \sum_{\alpha \in \Delta} k_\alpha \alpha \mid k_\alpha \in \mathbf{Z}\}$, and

(ii) the action of any elementary cycle of $CY(L, H)$ on an arbitrary weight space M_θ of M is a linear automorphism of M_θ.

Proof: If $\alpha \in \Delta$, then for any $m \geq 0$, by torsion free assumption and by repeated applications of Theorem 4.15 part (i), X_α^m and $X_{-\alpha}^m$ are respectively injective linear transformations of $M_\omega \to M_{\omega+m\alpha}$ and $M_{\omega+m\alpha} \to M_\omega$. So that $dim_{\mathbf{C}}(M_\omega) \leq dim_{\mathbf{C}}(M_{\omega+m\alpha})$ and $dim_{\mathbf{C}}(M_{\omega+m\alpha}) \leq dim_{\mathbf{C}}(M_\omega)$. We conclude that

$dim_{\mathbb{C}}(M_{\omega+m\alpha}) = dim_{\mathbb{C}}(M_\omega) = k$ for all $m \geq 0$. We can replace α by $-\alpha$ everywhere in the mentioned argument and conclude that $dim_{\mathbb{C}} M_{\omega+m\alpha} = k$ for all $m \in \mathbb{Z}$. Now by induction and part (ii) of 4.16 part (i) follows.

Since a composition of injective linear transformations is an injective linear transformation and since the action of root vectors maps a weight space of M into another injectively (part i), it follows that the action of any arbitrary elementary cycle of $CY(L, H)$ is an injective linear transformation of M_ω. Since an injective endomorphism of any finite dimensional vector space is also surjective (ii) follows. ∎

Definition 4.20: Let M denote an (L, H) torsion free module having all k dimensional weight spaces, for a cardinal number k, then we say that M is of *degree* k and write $deg(M) = k$. If $k = 1$, we say that M is *pointed*.

In view of Theorem 4.19, if M is a simple torsion free L module having a $k < \infty$ dimensional weight space then $deg(M) = k$.

The following is a key result in studying torsion free modules.

Theorem 4.21: Let M denote a simple pointed torsion free (L, H) module and N denote a simple L module with $dim(N) < \infty$. Then $M \otimes N$ is a torsion free (L, H) module with $deg(M \otimes N) = dim(N)$.

Proof: We freely borrow the notation of Theorem 3.6. By Theorem 4.14, it is clear that $N \in W(L, H)$. For $\omega \in wt(N)$ (resp. in $wt(M)$), let N_ω (resp. M_ω) denote the weight space of N (resp. M) belonging to ω. By part (ii) of Theorem 4.16, for $\omega, \omega' \in wt(N)$ (resp. in M) then $(\omega - \omega') \in S$. Say $\omega' \leq^* \omega$ whenever $\omega - \omega' = \sum_{\alpha\in\Delta} k_\alpha \alpha$ with $k_\alpha \geq 0$ for all $\alpha \in \Delta$, then it is easy to see that $\leq^*$ is a partial order on $wt(N)$ (resp. $wt(M)$). For $\omega \in wt(M)$, let $0 \neq v_\theta \in M_\omega$, then by pointed assumption we have $M_\omega = \mathbb{C}v_\omega$ It follows immediately from Theorem 4.17 that $M \otimes N$ is an (L, H) weight module with $wt(M \otimes N) = wt(M) + wt(N)$ with

$wt(M) + wt(N)$ as defined in 4.17. Moreover if $\omega \in wt(M \otimes N)$, then

$$(M \otimes N)_\omega = \oplus \sum_{\beta+\gamma=\omega} M_\beta \otimes N_\gamma \tag{4.21.1}$$

as a vector space direct sum. Since $dim(M_\beta) = 1$, it follows from part (i) of Theorem 4.19 and the preceding that

$$dim((M \otimes N)_\omega) = \sum_{\gamma \in wt(N)} dim(N_\gamma) = dim(N).$$

We would like to show that for $\mu \in \phi$, X_μ acts injectively on the weight spaces of $M \otimes N$. To this end, let $z \in (M \otimes N)_\gamma$,then

$$0 \neq z = \sum_{\beta+\gamma=\omega} c_{\beta,\gamma} v_\beta \otimes u_\gamma$$

for $u_\gamma \in N_\gamma$ and $0 \neq c_{\beta,\gamma} \in \mathbb{C}$. Assume on the contrary that $X_\mu.z = 0$. Without loss of generality, we could assume that μ is positive with respect to Δ. (Otherwise, change minimal to maximal and vice versa everywhere in the remainder of this proof.)

Let I and J respectively denote the sets of subscripts of elements of M and N appearing in the expansion of z. Since $z \neq 0$, I and J are nonempty. Since $wt(N)$ is a finite set so is J, hence any $\leq^*$ chain of elements of J is finitary, and J possess a $\leq^*$ minimal element θ. We conclude that $\omega - \theta$ is $\leq^*$ maximal in I. But

$$X_\mu.(v_{\omega-\theta} \otimes u_\theta) = (X_\mu.v_{\omega-\theta}) \otimes u_\theta + v_{\omega-\theta} \otimes (X_\mu.u_\theta),$$

where $0 \neq X_\mu.v_{\omega-\theta} \in M_{\mu+\omega-\theta}$ by the torsion free assumption. Since $\omega - \theta$ is a maximal element of I, and since the subscripts of elements of M appearing in the expansion $X_\mu.z$ are contained in $I \cup \{\alpha + \mu \mid \alpha \in I\}$ no other subscript of the elements of M appearing in the expansion of $X_\mu.z$ can equal to $\omega - \theta + \mu$. So that by Theorem 2.11 part (ii), $(X_\mu.v_{\omega-\theta}) \otimes u_\theta$ and the rest of elements appearing in the expansion of $X_\mu.z$ are linearly independent and since $c_{\omega-\theta,\theta}$ was assumed to be nonzero, $X_\mu.z \neq 0$. A contradiction!. We conclude that $M \otimes N$ is torsion free with the required property. ▮

Theorem 4.22: Let M be an (L, H) torsion free module of degree $k < \infty$ with a weight space decomposition

$$M = \oplus \sum_{\theta \in wt(M)} M_\theta$$

with respect to (L, H). Let U_0 denote the cycle algebra $CY(L, H)$, then

(i) any submodule N of M is torsion free,

(ii) if $\alpha \in wt(M)$ and P_α is a U_0-submodule of M_α and $P = UP_\alpha$, then P is a torsion free module , having P_α as a weight space and satisfying $deg(P) = dim(P_\alpha)$, and

(iii) M possess a simple nonzero submodule.

Proof: (i) By part (ii) of Theorem 4.15, N is an (L, H) weight module with a decomposition

$$N = \oplus \sum_{\theta \in wt(M)} (N \cap M_\theta)$$

with respect to (L, H). Since the restriction of any injective map to a subset of its domain is injective, the root vectors of (L, H) act injectively on $N \cap M_\theta$, for any $\theta \in wt(M)$. This proves (i).

(ii) It is clear that P is a U submodule and hence an L submodule of M. By part (i), P is a torsion free (L, H) module. We freely borrow the notations of Theorem 3.6. By part (i) of Theorem 3.6, U has a decomposition

$$U = \oplus \sum_{\gamma \in S} U_\gamma$$

with respect to (L, H), where $U_\gamma = span_{\mathbf{C}} B_\gamma$. We hence have

$$P = \sum_{\gamma \in S} U_\gamma P_\alpha.$$

With an argument similar to that of 4.16.2, this vector space sum is direct and hence

$$P = \oplus \sum_{\gamma \in S} U_\gamma P_\alpha,$$

is a decomposition of P with respect to (L, H). It is clear that $P_\alpha = 1.P_\alpha \subseteq U_0 P_\alpha \subseteq P_\alpha$, hence $P_\alpha = U_0 P_\alpha$ is a weight space of P. Next let $z \in B_\omega$. Since z acts injectively on P_θ we have $dim(z.P_\alpha) = dim(P_\alpha)$ and hence $dim(U_\omega P_\alpha) \geq dim(P_\alpha)$. Similarly $dim(U_{-\omega} U_\omega P_\alpha) \geq dim(U_\omega P_\alpha)$. But by Theorem 3.6 part (iii), we have $U_{-\omega} U_\omega \subseteq U_0$ and hence $dim(P_\alpha) \geq dim(U_{-\omega} U_\omega P_\alpha) \geq dim(U_\omega P_\alpha) \geq dim(P_\alpha)$. We conclude that $dim(U_\omega P_\alpha) = dim(P_\alpha)$ and hence all the weight spaces of P are of dimension $dim(P_\alpha)$. This proves (ii).

(iii) We induct on the $deg(M) = k$.

if $k = 1$, let $N \neq (0)$ be a submodule of M. Then as in the preceding

$$N = \oplus \sum_{\theta \in wt(M)} (N \cap M_\theta).$$

For $\theta \in wt(M)$, let $N_\theta = N \cap M_\theta$. Since $N \neq (0)$, we must have $N_\mu = N \cap M_\mu \neq (0)$ for some $\mu \in wt(M)$, hence $N_\mu = M_\mu$. By considering $M' = UM_\mu$ a submodule of M which is torsion free and of degree 1, and by the previous parts, we observe that there is no loss of generality in assuming $M = UM_\mu$. But then $M = UM_\mu = UN_\mu \subseteq N \subseteq M$. We conclude that $M = N$, and by definition of simplicity M is simple and (iii) is trivially true.

Assume that (iii) is true whenever $k < m$ and $m \geq 2$. Let M be torsion free of degree m. If M is simple we are done, otherwise it possess a proper submodule N' which is torsion free by the previous part and has a decomposition

$$N' = \oplus \sum_{\theta \in wt(M)} (N' \cap M_\theta)$$

with respect to (L, H). Since $N \neq (0)$, we must have $N'_\mu = N' \cap M_\mu \neq (0)$ for some $\mu \in wt(M)$ and as in the preceding we could assume $M = UM_\mu$. We claim that $N'_\mu \neq M_\mu$. Otherwise $M = UM_\mu = UN'_\mu \subseteq N' \subseteq M$ and $N' = M$, which violates the assumption that N' is a proper submodule of M. Hence the claim is true and by part (ii), we have $deg(N') = dim(N'_\mu) < m$. By applying the induction hypothesis to N' the result follows immediately. ∎

We close this chapter by the following theorem.

Theorem 4.23: Let M denote a torsion free (L, H) module of degree $k < \infty$ with a weight space decomposition $M = \oplus \sum_{\theta \in wt(M)} M_\theta$. Let N be a submodule of M such that $dim(N \cap M_\theta) = k'$ for all $\theta \in wt(M)$. Then the quotient module M/N is a simple (L, H) torsion free module of degree $k - k'$.

Proof: For $\alpha \in wt(M)$, define

$$((M_\alpha + N)/N) = \{m + N \mid m \in M_\alpha\},$$

then it is easy to see that $((M_\alpha + N)/N)$ is a complex vector space of dimension $k - k'$. It is clear that $M/N = \sum_{\theta \in wt(M)}((M_\theta + N)/N)$ as a vector space sum. Also for $h \in H$ and $x = m + N \in ((M_\alpha + N)/N)$, we have

$$h.x = h.(m + N) = h.m + N = \alpha(h).m + N = \alpha(h)(m + N) = \alpha(h)x.$$

We claim that the vector space sum

$$M/N = \sum_{\theta \in wt(M)} ((M_\theta + N)/N)$$

is direct. If not we could find $0 = \sum_{i=1}^{n} x_n$ such that $N = \bar{0} \neq x_i \in ((M_{\theta_i} + N)/N)$ For $\theta_i \in wt(M)$ and $1 \leq i \leq n$ with n minimal with this property. Since $\theta_1 \neq \theta_2$ there exists $h \in H$ with $\theta_1(h) \neq \theta_2(h)$. but then $0 = \alpha_1(h).0 - h.0 = \sum_{i=2}^{n}(\theta_1(h) - \theta_n(h))x_n$ and minimality of n forces $(\theta_1(h) - \theta_i(h))x_i = 0$ for $2 \leq i \leq n$. In particular $(\theta_1(h) - \theta_2(h))x_2 = 0$ and hence $x_2 = 0$. A contradiction!. We conclude that the mentioned sum is direct and M/N is an (L, H) weight module with the given weight space decomposition with all weight spaces of dimension $k - k'$.

Finally we claim that M/N is torsion free. To this end, let $\mu \in \phi$ and $\bar{0} = N \neq m + N \in ((M_\beta + N)/N)$ for $\beta \in wt(M)$. We claim that $X_\mu.(m + N) \neq \bar{0} = N$. If not $X_\mu.(m + N) = N$ and hence $X_\mu.m \in N$. By part (i) of Theorem 4.22, N is torsion free with a weight space decomposition

$$N = \oplus \sum_{\theta \in wt(M)} N_\theta,$$

where $N_\theta = N \cap M_\theta$ for all $\theta \in wt(M)$. Then by part (i) of Theorem 4.15 $X_\mu.m \in M_{\beta+\mu} \cap N = N_{\mu+\beta}$. Since N is torsion free the action of X_μ on N_β denoted by $X_\mu : N_\beta \rightarrow N_{\mu+\beta}$ is injective and since $dim(N_\beta) = dim(N_{\mu+\beta})$ it is also surjective. Hence there exists $n \in N_\beta \subseteq M_\beta$ with $X_\mu.n = X_\mu.m$. Since M is torsion free $m = n$. But we assumed that $m + N \neq N$ and hence $m \notin N$. A contradiction!. We conclude that the action of X_μ on $((M_\theta + N)/N)$ is injective and M/N is torsion free. ∎

CHAPTER 5

A Review Of Fernando's Results

In this chapter, we provide an outline of Fernando's program. This program highlights the role of simple torsion free modules in classifying the general elements of the category of all simple Lie modules having finite dimensional weight spaces. Since the machinery required for the proofs is sophisticated, we will be content with stating the results and providing a general overview.

Let L denote a finite dimensional complex simple linear Lie algebra, H be a maximal toral subalgebra of L and ϕ be the root system of (L, H). Let $U(L)$ denote the U.E.A. of L. For $\alpha \in \phi$, let L_α denote the corresponding root space. Let $\mathcal{W}(L, H)$ be the category of all (L, H) weight modules that are finitely generated $U(L)$ modules and possess all finite dimensional weight spaces.

For s in L, let $< s >$ denote the smallest associative subalgebra of $U(L)$ containing s. Let $M \in \mathcal{W}(L, H)$, then we have the following notations and definitions.

Notation: Let $S \subseteq U(L)$, we let $M^{[S]}$, $M^{(S)}$ and M^S respectively denote the subsets

$$M^{[S]} = \{m \in M \mid dim(< s > m) < \infty, \quad \forall s \in S\},$$

$$M^{(S)} = \{m \in M \mid \text{if } s \in S \text{ there exists } r = r(s, m) \in \mathbb{N} \text{ such that } s^r m = 0\}$$

and

$$M^S = \{m \in M \mid s.m = 0, \quad \forall s \in S\}$$

of M. It can be shown that $M^{[S]}$ and $M^{(S)}$ are L submodules of M.

Definition 5.1: (i) If $M^{[S]} = M$, we say that S is *locally finite* on M and if $M^{(S)} = M$, we say that S is *locally nilpotent* on M. If $\alpha \in \phi$ and X_α denote the root vector of L belonging to α, we denote $M^{[\{X_\alpha\}]}$ by $M^{[\alpha]}$ and $M^{(\{X_\alpha\})}$ by $M^{(\alpha)}$. We say M is *α-finite* (respectively *α-free*) if $M^{[\alpha]} = M$ (respectively $M^{(\alpha)} = (0)$).

(ii) If $g = H' \oplus \sum_{\alpha \in \phi_g} L_\alpha$ is a subalgebra of L such that $H' \subseteq H$ and ϕ_g a closed subsystem of ϕ, we say g is *torsion free* on M, if M is α-free for all $\alpha \in \phi_g$.

The first important result of Fernando is

Lemma 5.2: Let $M \in \mathcal{W}(L, H)$ be simple then, for $\alpha \in \phi$, M is either α-free or α-finite. Further if

$$F = \{\alpha \in \phi \mid M \text{ is } \alpha\text{-finite}\}$$

and

$$T = \{\alpha \in \phi \mid M \text{ is } \alpha\text{-free }\},$$

then F and T are disjoint closed subsets of ϕ satisfying $F \cup T = \phi$.

Notation: For any $R \subseteq \phi$, we let

$$R_s = \phi \cap span_{\mathbb{Z}}(R), \qquad R^s = R \cap -R,$$

$$-R = \{-\alpha \mid \alpha \in R\}, \qquad R^a = R \backslash (-R).$$

Theorem 5.3: Let $M \in \mathcal{W}(L, H)$ be simple and F and T be as in Lemma 5.2. Let

$$P_M = H \oplus \sum_{\alpha \in F \cup T^a} L_\alpha,$$

$$P_M^- = H \oplus \sum_{\alpha \in F^a \cup T} L_\alpha,$$

$$u_M = \oplus \sum_{\alpha \in F^a} L_\alpha,$$

and $l_M = P_M \cap P_M^-$. Then P_M and P_M^- are opposite parabolic subalgebras of L having u_M as nilradical. Moreover P_M is the algebra direct sum $l_M \oplus u_M$ and M^{u_M} is a simple module for algebras P_M and l_M. Let $U(P_M)$ denote the U.E.A. of P_M, then $U(P_M)$ is naturally a subalgebra of $U(L)$ and hence $U(L)$ is a $U(L)$

and $U(P_M)$ bimodule. The $U(L)$ module $M^* = U(L) \otimes_{U(P_M)} M^{u_M}$ has a unique maximal submodule N^* and M^*/N^* (unique simple quotient of M^*) is a $U(L)$ module isomorphic to M.

We fix the previous notations. The following is a major result of Fernando's paper.

Theorem 5.4: (i) The algebra l_M decomposes into the algebra direct sum $l_M = L_1 \oplus L_2$ where

$$L_1 = H_1 \oplus \sum_{\alpha \in T^s} L_\alpha,$$

$$L_2 = H_2 \oplus Z(l_M) \oplus \sum_{\alpha \in F^s} L_\alpha,$$

where $H_1 = \sum_{\alpha \in T^s} [L_\alpha, L_{-\alpha}]$, $Z(l_M)$ denotes the centralizer of l_M, and $H_2 = \sum_{\alpha \in F^s} [L_\alpha, L_{-\alpha}]$. Moreover L_1 and L_2 are ideals of l_M which are respectively semisimple and reductive Lie algebras on their own.

(ii) M^{u_M} is isomorphic to the tensor product $M_1 \otimes_{\mathbf{C}} M_2$, where M_1 is a simple torsion free L_1 module and M_2 is a simple finite dimensional L_2 module.

(iii) If $L_1 = \oplus \sum_{i=1}^m I_i$ denote the decomposition of L_1 into simple ideals, then M_1 decomposes into $M_1^1 \otimes_{\mathbf{C}} \ldots \otimes_{\mathbf{C}} M_1^m$, where $M_1^i, 1 \leq i \leq m$ is a simple torsion free module in $\mathcal{W}(I_i, I_i \cap H)$.

Conversely if a pair of opposite H diagonalizable parabolic subalgebras P_M and P_M^- of L, the algebra decompositions $l_M = L_1 \oplus L_2$, $L_1 = \oplus \sum_{i=1}^m I_i$ and an L_2 finite dimensional module M_2 and I_i torsion free modules $M_1^i, 1 \leq i \leq m$ are given, then M can be recovered by considering L_1 module $M_1 = M_1^1 \otimes_{\mathbf{C}} \ldots \otimes_{\mathbf{C}} M_1^m$ and by considering the unique simple quotient of $U(L)$ module $U(L) \otimes_{U(P_M)} (M_1 \otimes_{\mathbf{C}} M_2)$.

Next Fernando considers the problem of existence of torsion free modules (of finite degree) for simple Lie algebras. In particular, he proves that if L is a simple Lie algebra that possess a torsion free module then L is either of type $A_{n-1}, n \geq 2$ or of type $C_m, m \geq 2$. Following his predecessors, Fernando provides examples of simple torsion free modules of degree one for algebras of type A and C. This gives a necessary and sufficient condition for possessing a simple torsion free module of finite degree for a simple Lie algebra.

In summary Fernando's results imply that for any simple module $M \in \mathcal{W}(L, H)$ we can select a base Δ and two orthogonal subsets Δ_1 and Δ_2 (–i.e. if $\alpha \in \Delta_1$, and $\beta \in \Delta_2$ then $(\alpha + \beta) \notin \phi$) such that

(i) M is Δ_1 free and Δ_2 finite and,

(ii) if ϕ_1 is the root subsystem generated by Δ_1, and

$$H_1 = \sum_{\alpha \in \phi_1} [L_\alpha, L_{-\alpha}],$$

then the semisimple Lie algebra $H_1 \oplus \sum_{\alpha \in \phi_1} L_\alpha$ has only simple ideals of types A_{n-1} or C_m. The module M is uniquely determined from this data by reversing the decomposition as follows.

First step: Select a base Δ of ϕ and consider two subsets Δ_1 and Δ_2 of Δ satisfying the conditions (i) and (ii) of the preceding paragraph. Then the semisimple Lie algebra L_1 has a decomposition $L_1 = \oplus \sum_{i=1}^{m} I_i$ into simple ideals.

Second step: Let ϕ_2 denote the root subsystem generated by Δ_2 and

$$L_2 = H_2 \oplus Z \oplus \sum_{\alpha \in \phi_2} L_\alpha,$$

where

$$H_2 = \sum_{\alpha \in \phi_2} [L_\alpha, L_{-\alpha}]$$

and

$$Z = \{x \in H \mid [x, X_\alpha] = 0, \quad \forall \alpha \in \phi_1 \cup \phi_2\}.$$

Then L_2 is a reductive algebra.

Third step: Select a simple finite dimensional L_2 module M_2 and a simple torsion free module M_1^i in $\mathcal{W}(I_i, I_i \cap H)$. Let $M_1 = M_1^1 \otimes_{\mathbf{C}} \dots \otimes_{\mathbf{C}} M_1^m$ and $Y = M_1 \otimes_{\mathbf{C}} M_2$, then Y is a simple $l_M = L_1 \oplus L_2$ module.

Let S^+ (respectively S^-) denote the set of positive (respectively negative) elements of $S = \phi \backslash (\phi_1 \cup \phi_2)$. Let $T = \phi_1 \cup S^-$ and $F = \phi_2 \cup S^+$. Define

$$P_M = H \oplus \sum_{\alpha \in F \cup T^\bullet} L_\alpha,$$

$$P_M^- = H \oplus \sum_{\alpha \in F^\bullet \cup T} L_\alpha,$$

and

$$u_M = \oplus \sum_{\alpha \in S^+} L_\alpha.$$

Then P_M and P_M^- are a pair of H diagonalizable opposite parabolic subalgebras satisfying $P_M \cap P_M^- = l_M$. Moreover, u_M is the nilradical of P_M and $P_M = u_M \oplus l_M$. Let u_M act trivially on Y, then Y is a simple P_M module.

Fourth step: The L module $U(L) \otimes_{U(P_M)} Y$ has a unique simple homomorphic image which is simple and in $\mathcal{W}(L, H)$. This construction yields every possible simple module in $\mathcal{W}(L, H)$.

In particular, this demonstrates the importance of being able to construct all the simple torsion free modules of the algebras $A_{n-1}, n \geq 2$ and $C_m, m \geq 2$. Next we illustrate this process by an example.

Example 5.5: We consider the algebra A_3 and H as in Example 1.14. Let ϕ and Δ be as in Theorem 1.22 part (i). Let $\alpha = \omega_1 - \omega_2$, $\beta = \omega_2 - \omega_3$ and $\gamma = \omega_3 - \omega_4$, where ω_i is as in Example 1.14. Let $\Delta_1 = \{\alpha\}$, and $\Delta_2 = \{\gamma\}$. It is easy to see that $\phi_1 = \{\alpha, -\alpha\}$, $\phi_2 = \{\gamma, -\gamma\}$,

$$L_1 = \mathbf{C}h_1 + L_\alpha + L_{-\alpha},$$

and

$$L_2 = \mathbb{C}(h_1 + 2h_2 + h_3) + \mathbb{C}h_3 + L_\gamma + L_{-\gamma}.$$

We will prove in chapter 6 that L_2 possess a module M_2 of dimension 2 and L_1 possess a pointed torsion free module M_1. We consider $Y = M_1 \otimes_{\mathbb{C}} M_2$ and

$$l_M = H \oplus \sum_{\mu \in \phi_1 \cup \phi_2} L_\mu,$$

then Y is an l_M module. It is easy to see that

$$S^+ = \{\beta, (\alpha+\beta), (\alpha+\beta+\gamma), (\beta+\gamma)\},$$

$$S^- = \{-\beta, -(\alpha+\beta), -(\alpha+\beta+\gamma), -(\beta+\gamma)\},$$

$$T = \{\pm\alpha, -\beta, -(\alpha+\beta), -(\alpha+\beta+\gamma), -(\beta+\gamma)\},$$

and

$$F = \{\pm\gamma, \beta, (\alpha+\beta), (\alpha+\beta+\gamma), (\beta+\gamma)\}.$$

Define P_M, P_M^- and u_M as in step 3 of the preceding construction, then it is easy to see that P_M and P_M^- are a pair of opposite H diagonalizable parabolic subalgebras of L satisfying $P_M \cap P_M^- = l_M$. Moreover if u_M is the nilradical of P_M, then we have $P_M = u_M \oplus l_M$. Let u_M act trivially on Y, then Y is a P_M module. As mentioned before $U(L) \otimes_{U(P_M)} Y$ is an L module having a unique simple homomorphic image $M \in \mathcal{W}(L, H)$. Moreover M is respectively μ-finite or μ-free, depending on whether $\mu \in F$ or $\mu \in T$.

In view of the illustrated method, the problem of classification of all simple modules in $\mathcal{W}(L, H)$ is equivalent to those of all simple finite dimensional L_2 modules and simple torsion free modules in $\mathcal{W}(I_i, I_i \cap H), 1 \le i \le m$.

However, since L_2 is reductive, we have $L_2 = Z(L_2) \oplus [L_2, L_2]$ where $[L_2, L_2]$ is semisimple and $Z(L_2)$ is the center of L_2. By a standard result in representation theory the problem of classifying simple L_2 modules is equivalent to that of simple ideals of the semisimple algebra $[L_2, L_2]$. This reduces the problem of classifying all the simple modules in $\mathcal{W}(L, H)$ to those of classifying all the simple finite dimensional and simple torsion free modules (of finite degree) of certain simple subalgebras of L.

In light of the classical results of Cartan, Weyl, Verma, Harish-Chandra and others this reduces the problem of classifying all the simple modules in $\mathcal{W}(L, H)$ to that of classifying torsion free modules of simple Lie algebras.

As mentioned before, the question of classifying all the simple torsion free modules of finite degree of the simple algebras of types A_{n-1} and C_m is still open. This motivates our work in the next chapter.

CHAPTER 6

On The Existence Of Simple Torsion Free Lie Modules

In this chapter, we consider the existence of torsion free A_{n-1} modules of finite degree. We will prove:

MAIN THEOREM: There are no simple torsion free (A_{n-1}, H) modules of degree k for $n \geq 5$ and $2 \leq k \leq n-3$, where H is any maximal toral subalgebra of A_{n-1}.

We prove our main result by a sequence of lemmas.

Lemma 6.1: Let H and H' denote maximal toral subalgebras of the simple linear algebra A_{n-1}. If (A_{n-1}, H) possess a (simple) torsion free module M of degree k, the same is true for (A_{n-1}, H').

Proof: Let M be a torsion free (A_{n-1}, H) module of degree k. We prove the lemma by constructing such a module for (A_{n-1}, H'). To this end, observe that by Theorem 1.19, there exists an automorphism σ of A_{n-1} such that $\sigma(H) = H'$. Define a new module multiplication on M by $x \circ m = \sigma^{-1}(x).m$. It is easy to see that the (A_{n-1}, H) weight spaces of M turn into (A_{n-1}, H') weight spaces. Also the actions of root vectors of (A_{n-1}, H') with respect to the operation '$\circ$' coincide with the actions of root vectors of (A_{n-1}, H) with respect to the original operation. We conclude that M endowed with '$\circ$' is an (A_{n-1}, H') torsion free module of degree k. If $N \neq M$ is an A_{n-1} submodule of M with respect to the operation $\circ$, it is routine to see that same is true with respect to the original module operation and vice versa. This observation and the definition of simplicity conclude the proof. ∎

In light of the preceding lemma, we may fix the maximal toral subalgebra H of A_{n-1} as in example 1.14 and investigate the existence of simple torsion free (A_{n-1}, H) modules.

Let $\{e_{ij} \mid 1 \leq i,j \leq n\}$, $\{\omega_i \mid 1 \leq i \leq n\}$ be as in 1.14, and ϕ denote the root system of (L,H). By theorem 1.22 $\Delta = \{\omega_i - \omega_{i+1} \mid 1 \leq i \leq n-1\}$ is a base of $\phi = \{\pm(\omega_i - \omega_j) \mid 1 \leq i < j \leq n\}$. If $i < j$ and $\alpha = \omega_i - \omega_j$, then the root vectors X_α, $X_{-\alpha}$ respectively belonging to the roots α and $-\alpha$ can respectively be chosen to be $e_{i,j}$ and $e_{j,i}$. Let $h_\alpha = e_{ii} - e_{jj}$. Then $\{h_\alpha \mid \alpha \in \Delta\}$ is a base of H. Let $U = U(A_{n-1})$ be the universal enveloping subalgebra of A_{n-1} and $U_0(A_{n-1}) = CY(A_{n-1}, H) = \{u \in U \mid [u,h] = 0, \forall h \in H\}$.

Let M denote a torsion free (A_{n-1}, H) module of degree k.

Lemma 6.2: There exists a weight θ of H in M such that $\theta(h_\alpha) \neq 0$ for all $\alpha \in \phi$.

Proof: Choose ω an arbitrary weight of H in M and let

$$m = max_{\alpha \in \phi} \|\omega(h_\alpha)\|.$$

Choose a natural number l such that $3^{l+1} > m$ and let

$$\theta = \omega + \sum_{s=1}^{n-1} 3^{(l+s)}(\omega_s - \omega_{s+1}).$$

Then by Theorem 4.19 part (i), θ is a weight of M. Let $\alpha = \omega_r - \omega_m \in \phi$. Then $h_\alpha = e_{rr} - e_{mm}$ and

$$\|\theta(h_\alpha)\| \geq \|\sum_{s=1}^{n-1} 3^{(l+s)}(\omega_s(h_\alpha) - \omega_{s+1}(h_\alpha))\| - \|\omega(h_\alpha)\| \geq$$

$$\geq \|\sum_{s=1}^{n-1} 3^{(l+s)}(\omega_s(h_\alpha) - \omega_{s+1}(h_\alpha))\| - m$$

by triangle equality. By direct computation we have:

$$\sum_{s=1}^{n-1} 3^{(l+s)}(\omega_s(h_\alpha) - \omega_{s+1}(h_\alpha)) = \sum_{s=1}^{n-1} 3^{(l+s)} b_s$$

where $b_s = \delta_{s,r} - \delta_{s,m} - \delta_{s+1,r} + \delta_{s+1,m}$. It is easy to see that $\|b_s\|$ is either $0, 1$ or 2 , $1 \leq s \leq n-1$ and hence if q is maximum with the property that $b_q \neq 0$, we have:

$$\|\sum_{s=1}^{n-1} 3^{(l+s)} b_s\| \geq \|3^{(l+q)} b_q\| - \sum_{s=1}^{q-1} \|3^{(l+s)} b_s\| \geq 3^{(l+q)} - 2\sum_{s=1}^{q-1} 3^{(l+s)}.$$

But $2\sum_{s=1}^{q-1} 3^{(l+s)} = 3^{(l+q)} - 3^{l+1}$, hence:

$$\|3^{(l+q)}b_q\| - \sum_{s=1}^{q-1} \|3^{(l+s)}b_s\| \geq 3^{(l+q)} - 2\sum_{s=1}^{q-1} 3^{(l+s)} = 3^{l+1} > m.$$

We conclude that

$$\|\theta(h_\alpha)\| \geq \| \sum_{s=1}^{n-1} 3^{(l+s)}(\omega_s(h_\alpha) - \omega_{s+1}(h_\alpha))\| - m > 0$$

and hence $\theta(h_\alpha) \neq 0$. ∎

We apply Lemma 6.2 and fix a weight space M_θ of M such that $\theta(h_\alpha) \neq 0$. Then by part (i) of Theorem 4.16, M_θ is a $U_0(A_{n-1})$ module. We denote the restriction of $X_{-\alpha}X_\alpha$ to M_θ by A_α.

Let i, j, m be any three distinct values in $\{1, \ldots, n\}$. Set

$$c_1(i,j,m) = e_{ji}e_{ij},\ c_2(i,j,m) = e_{mj}e_{jm},\ c_3(i,j,m) = e_{mi}e_{im},$$

$$c_4(i,j,m) = e_{mi}e_{ij}e_{jm},\ c_5(i,j,m) = e_{mj}e_{ji}e_{im},$$

$$h_1(i,j,m) = e_{ii} - e_{jj},\ \text{and}\ h_2(i,j,m) = e_{jj} - e_{mm}.$$

Drop (i,j,m) to simplify notation. Notice that in U we have

$$\begin{aligned}
e_{ij}e_{mj} - e_{mj}e_{ij} &= [e_{ij}, e_{mj}] = 0 \\
e_{jm}e_{ji} - e_{ji}e_{jm} &= [e_{jm}, e_{ji}] = 0 \\
e_{ji}e_{mj} - e_{mj}e_{ji} &= [e_{ji}, e_{mj}] = -e_{mi} \\
e_{jm}e_{ij} - e_{ij}e_{jm} &= [e_{jm}, e_{ij}] = -e_{im} \\
[c_1, c_2] &= c_1c_2 - c_2c_1.
\end{aligned}$$

We hence have

$$\begin{aligned}
c_1c_2 &= e_{ji}e_{ij}e_{mj}e_{jm} = e_{ji}e_{mj}e_{ij}e_{jm} \\
&= e_{mj}e_{ji}e_{ij}e_{jm} - e_{mi}e_{ij}e_{jm} \\
&= e_{mj}e_{ji}e_{jm}e_{ij} + e_{mj}e_{ji}e_{im} - e_{mi}e_{ij}e_{jm} \\
&= e_{mj}e_{jm}e_{ji}e_{ij} + e_{mj}e_{ji}e_{im} - e_{mi}e_{ij}e_{jm} \\
&= c_2c_1 + c_5 - c_4.
\end{aligned}$$

Hence we have

$$[c_1, c_2] = c_5 - c_4.$$

Similarly one can show that the following identities hold in U_0.

$$\begin{aligned}
[c_1, c_2] =&[c_2, c_3] = -[c_1, c_3] = c_5 - c_4 \\
[c_1, c_4] =& - c_2c_1 + c_3c_1 - c_4h_1 + c_3h_1 - c_5 + c_4 \\
[c_2, c_4] =&c_2c_1 - c_3c_2 - c_4h_2 + c_5 - c_4 \\
[c_1, [c_1, c_2]] =&2c_2c_1 - 2c_3c_1 + (c_5 + c_4)h_1 - 2c_3h_1 + 2[c_1, c_2].
\end{aligned}$$

Let $\tilde{c}_i$ denote the action of c_i on M_θ and $\tilde{h}_i = \theta(h_i)$. We use $W \leq M_\theta$ to indicate that W is a subspace of M_θ.

Lemma 6.3: If $W \leq M_\theta$ is invariant under A_α for all $\alpha \in \phi$, then W is a U_0–submodule of M_θ.

Proof: By Theorem 3.11 , an element $u \in U_0$ is a linear combination of products of h_α, $\alpha \in \Delta$ and elements of the form

$$(6.3.1) \qquad u = e_{i_1, i_2} \cdots e_{i_{l-1}, i_l} e_{i_l, i_1}$$

where $i_1, \ldots, i_l$ are distinct in $\{1, \ldots, n\}$. We conclude that it suffices to prove that W is invariant under the actions of elements of the form 6.3.1. We induct on l and observe that by hypothesis, this is true if $l = 2$. For $l = 3$ an element u of the form 6.3.1 is either $c_4(i, j, m) = e_{mi}e_{ij}e_{jm}$ or $c_5(i, j, m) = e_{mj}e_{ji}e_{im}$ for appropriate choices of i, j and m. Since $\theta(h_1) \neq 0$, we can use the equations

$$(6.3.2) \qquad [c_1, c_2] =[c_2, c_3] = -[c_1, c_3] = c_5 - c_4$$

$$(6.3.3) \qquad [c_1, [c_1, c_2]] =2c_2c_1 - 2c_3c_1 + (c_5 + c_4)h_1 - 2c_3h_1 + 2[c_1, c_2].$$

to solve for $\tilde{c}_4$ and $\tilde{c}_5$ and see that $\tilde{c}_4$ and $\tilde{c}_5$ leave W invariant. Assume that W is invariant under all elements of form (6.3.1) with $3 < l < m$. Since e_{i_{m-1}, i_1}

commutes with e_{i_m,i_1} and e_{i_{m-1},i_m}. The following identity is valid in U_0.

$$(e_{i_1,i_2}\cdots e_{i_{m-1},i_m}e_{i_m,i_1})(e_{i_{m-1},i_1}e_{i_1,i_{m-1}}) =$$
$$(e_{i_1,i_2}\cdots e_{i_{m-2},i_{m-1}}e_{i_{m-1},i_1})(e_{i_{m-1},i_m}e_{i_m,i_1}e_{i_1,i_{m-1}}).$$

Since the two expressions on the right hand side of the preceding equation leave W invariant as does the invertible operator $e_{i_{m-1},i_1}e_{i_1,i_{m-1}}$, the result follows by induction. ∎

Notation: Let $\Delta_* = \{\alpha_1, \dots \alpha_{n-1}\}$ denote a standard base of ϕ. Let $\phi^m_{\Delta_*}$ be the root subsystem generated by $\Delta^m_* = \{\alpha_m, \dots, \alpha_{n-1}\} \subseteq \phi$.(i.e. $\phi^m_{\Delta_*} = span_{\mathbb{Z}}(\Delta^m_*) \cap \phi$.)

If A_α acts like a complex multiple of the identity map on $W \leq M_\theta$ then we say that A_α is a *scalar* on W. By direct computation $A_\alpha - A_{-\alpha} = h_\alpha$, so that if A_α acts as a scalar on W, so does $A_{-\alpha}$ and vice versa.

Lemma 6.4: Let $\alpha, \beta, \alpha+\beta \in \phi$ and $W \leq M_\theta$. Then

(i) if A_α is a scalar on W and W is invariant under A_β, then W is invariant under $A_{(\alpha+\beta)}$.

(ii) if A_α and A_β are both scalars on W then $A_{(\alpha+\beta)}$ is a scalar on W.

(iii) if A_α is a scalar on W and A_β is a scalar on W for all $\beta \in \phi$ such that $\alpha+\beta \in \phi$, then A_γ is a scalar on W for all $\gamma \in \phi$.

Proof: Let $w \in W$ and A_α be a scalar on W. Without loss of generality, we may assume that $X_\alpha = e_{ij}$ and $X_\beta = e_{jm}$. if we let $\tilde{c}_1 = A_\alpha$ and $\tilde{c}_2 = A_\beta$ by 6.3.2 we have: $0 = [\tilde{c}_1, \tilde{c}_2]w = (\tilde{c}_5 - \tilde{c}_4)w$. and 6.3.2 gives that: $[\tilde{c}_r, \tilde{c}_s]w = 0$ for $1 \leq r, s \leq 3$. By (6.3.2) and (6.3.3), $\tilde{c}_4$ and $\tilde{c}_5$ are expressible in terms of $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$. Therefore,

(6.4.1) $$0 = [\tilde{c}_1, \tilde{c}_4]w = (-\tilde{c}_2\tilde{c}_1 + \tilde{c}_3\tilde{c}_1 - \tilde{c}_4\tilde{h}_1 + \tilde{c}_3\tilde{h}_1)w,$$

(6.4.2) $$0 = [\tilde{c}_2, \tilde{c}_4]w = (\tilde{c}_2\tilde{c}_1 - \tilde{c}_3\tilde{c}_2 - \tilde{c}_4\tilde{h}_2)w.$$

and we have

$$(6.4.3)\qquad \tilde{c}_2\tilde{c}_1(\tilde{h}_1 + \tilde{h}_2)w = \tilde{c}_3(\tilde{c}_1\tilde{h}_2 + \tilde{c}_2\tilde{h}_1 + \tilde{h}_1\tilde{h}_2)w.$$

Since $\tilde{c}_2\tilde{c}_1(\tilde{h}_1 + \tilde{h}_2)$ is invertible, $(\tilde{c}_1\tilde{h}_2 + \tilde{c}_2\tilde{h}_1 + \tilde{h}_1\tilde{h}_2)$ is an invertible operator on W and we see that $\tilde{c}_3$ leaves W invariant. This proves (i).

Also, if A_β is a scalar on W then (6.4.3) implies $\tilde{c}_3$ is a a scalar on W and (ii) follows.

For (iii), assume that $\alpha = \omega_i - \omega_r$, $1 \le i \ne j \le n$. We know that A_α and $A_{-\alpha}$ are scalars on W, hence it suffices to prove that if $1 \le p \ne q \le n-1$ then $A_{\omega_p-\omega_q}$ is a scalar on W whenever $p \ne r$ or $q \ne i$.

To this end, without loss of generality assume that $p \ne r$. By assumption $A_{\omega_p-\omega_i}$ (whenever $p \ne i$) and hence by part (ii) (and by assumption if $p = i$), $A_{\omega_p-\omega_r}$ is a scalar on W. Similarly whenever $q \ne r$, $A_{\omega_r-\omega_q}$ and hence By part (ii) $A_{\omega_p-\omega_q}$ is a scalar on W. This is also immediately true whenever $q = r$. ∎

Lemma 6.5: Let Δ_* denote a standard base of ϕ. Fix an eigenvalue λ_i of A_{α_i} for each $\alpha_i \in \Delta_*$ and let W_i be the corresponding eigenspace. For each $2 \le j \le n-1$ $(n \ge 3)$, define $W^{(j-1)} = W_1 \cap \cdots \cap W_{j-1}$. If A_{α_j} is a scalar on $W^{(j-1)} \ne 0$ and $k > 1$ then M is not simple.

Proof: By definition of standard bases, we have

$$\Delta_* = \{\alpha_1 = \omega_{i_1} - \omega_{i_2}, \ldots, \alpha_{n-1} = \omega_{i_{n-1}} - \omega_{i_n}\},$$

for $i_1, \ldots, i_n$ an appropriate permutation of $\{1, \ldots, n\}$. Assume on the contrary that M is simple of degree $k \ge 2$. By direct computation $A_\alpha A_{\alpha_i} = A_{\alpha_i} A_\alpha$, whenever $\alpha \in \phi^{j+1}_{\Delta_*}$ and $1 \le i \le j-1$. We conclude that $W^{(j-1)} \ne 0$ is invariant under A_α for each $\alpha \in \phi^{j+1}_{\Delta_*}$. Lemma 6.4 (i) implies $W^{(j-1)}$ is invariant under A_α for $\alpha \in \phi^j_{\Delta_*}$ and in turn for $\alpha \in \phi^{j-1}_{\Delta_*}$, $\alpha \in \phi^{j-2}_{\Delta_*}$,..., $\alpha \in \phi^1_{\Delta_*} = \phi$. If $W^{(j-1)} \ne M_\theta$, then by Lemma 6, $W^{(j-1)}$ is a proper U_0-submodule of M_θ which implies that M_θ

is not a simple U_0 module contradicting the simplicity of M. We hence must have $W^{(j-1)} = M_\theta$.

We claim that there exist $1 \leq p \leq n-1$ such that A_{α_p} is not a scalar on M_θ. Otherwise all A_{α_i}, $1 \leq i \leq n-1$ are scalars on M_θ and by repeated application of Lemma 6.4 part i, all A_α for $\alpha \in \phi$ are scalars on M_θ. Hence if $0 \neq v \in M_\theta$, then $span_{\mathbf{C}}(v)$ is a proper submodule of M_θ (Lemma 6.3). This again contradicts the irreducibility of M, so that such a p exists. Let l be the smallest subscript such that A_{α_l} is not a scalar on M_θ. Consider the element $\pi \in S_n$ defined by $\pi(i_j) = i_{(l+2-j)}$ for $1 \leq j \leq (l+1)$ and $\pi(i_j) = i_j$ for $l+1 < j \leq n$. Since π maps standard bases into standard bases , we conclude that $\Delta' = \pi(\Delta_*) = \{\alpha'_1 = -\alpha_l, \alpha'_2 = -\alpha_{l-1}, \ldots, \alpha'_l = -\alpha_1, \alpha'_{l+1} = (\alpha_1 + \ldots + \alpha_{l+1}), \alpha'_{l+2} = \alpha_{l+2}, \ldots, \alpha'_{n-1} = \alpha_{n-1}\}$ is a standard base of ϕ. If we let $W'_1 = W_l$ be the eigenspace corresponding to $A_{\alpha'_1}$, by the comment mentioned before the Lemma 6.4, $A_{\alpha'_2}$ is a scalar on $W^{(1)'} = W'_1 \neq (0)$. We are now back to the case handled above. ∎

Lemma 6.6: If for some $\beta \in \phi$, A_β is a scalar on M_θ and $k \geq 2$, then M is not simple.

Proof: Assume on the contrary that M is simple. We claim that there exist $\alpha \in \phi$ such that $(\alpha + \beta) \in \phi$ and A_α is not a scalar on M_θ. If not then for all those $\gamma \in \phi$ satisfying $(\beta + \gamma) \in \phi$, A_γ is a scalar on M_θ. By part (iii) of Lemma 6.4, all A_μ ($\mu \in \phi$) are scalars. Hence if $0 \neq v \in M_\theta$, then $span_{\mathbf{C}}(v)$ is a proper submodule of M_θ (Lemma 6.3). This contradicts the irreducibility of M. We conclude that such an α exist. Let $\alpha = \omega_p - \omega_q$. Without loss of generality we could assume $\beta = \omega_q - \omega_r$.(otherwise consider $-\alpha$ and $-\beta$.) Choose a permutation σ of S_n satisfying $\sigma(1) = p$, $\sigma(2) = q$ and $\sigma(3) = r$. Then σ takes Δ to a base $\Delta_* = \{\alpha = \alpha_1, \beta = \alpha_2, \ldots, \alpha_{n-1}\}$ of ϕ such that A_{α_2} acts as a scalar on any eigenspace W_1 of A_{α_1}. Lemma 6.5 gives that M is not simple. ∎

Lemma 6.7. Let $n \geq 5$ and $\Delta_* = \{\alpha_1, \ldots, \alpha_{n-1}\}$ be a standard base of ϕ. If the degree k of M satisfies $2 \leq k \leq n-3$ and v is an eigenvector of A_α for each $\alpha \in \Delta_*^3$, then M is not simple.

Proof: Suppose that M is simple and for $3 \leq i \leq n-1$ let λ_i be the eigenvalue of A_{α_i} belonging to v. Let W_i be the eigenspace of A_{α_i} belonging to λ_i. By Lemma 6.6, $\dim W_i < k \leq n-3$.

By definition of standard bases, we have

$$\Delta_* = \{\alpha_1 = \omega_{i_1} - \omega_{i_2}, \ldots, \alpha_{n-1} = \omega_{i_{n-1}} - \omega_{i_n}\},$$

for $i_1, \ldots, i_n$ an appropriate permutation of $\{1, \ldots, n\}$. Consider the permutation $\sigma \in S_n$ given by $\sigma(i_j) = i_{j+2}$ for $1 \leq j \leq n-2$, $\sigma(i_{n-1}) = i_1$ and $\sigma(i_n) = i_2$. Then σ takes Δ_* to the standard base

$$\Delta' = \{\alpha_3, \alpha_4, \ldots, \alpha_{n-3}, \alpha'_{n-2}, \alpha'_{n-1}\},$$

where $\alpha'_{n-2}, \alpha'_{n-1}$ are appropriate elements of ϕ. We now apply Lemma 6.5 to Δ' to see that $W_3 \not\subseteq W_4$ hence $dim(W_3 \cap W_4) < dim(W_3)$, similarly $W_3 \cap W_4 \not\subseteq W_5$ hence $dim(W_3 \cap W_4 \cap W_5) < dim(W_3 \cap W_4)$. By continuing in this manner we have $dim(W_3 \cap W_4) < n-4$, $dim(W_3 \cap W_4 \cap W_5) < n-5$, $\ldots$, $dim(W_3 \cap \cdots \cap W_{n-1}) < n-(n-1) = 1$. By assumption, $v \in W_3 \cap \cdots \cap W_{n-1}$. This contradiction gives the result. ∎

Corollary 6.8: If $n \geq 5$ and M is simple, then for $\alpha \in \phi$ the eigenspaces of A_α are all of dimension greater than 1.

Proof: Assume that $\alpha = \omega_p - \omega_q$. Consider any permutation σ of S_n satisfying $\sigma(1) = p$ and $\sigma(2) = q$. Then σ takes Δ to a standard base $\Delta' = \{\alpha_1, \ldots, \alpha_{n-1}\}$ of ϕ with $\alpha_1 = \alpha$. If $W = \mathbf{C}v$ is a 1-dimensional eigenspace of $A_\alpha = A_{\alpha_1}$, then by direct computation A_{α_1} and $A_{\alpha_j}, 3 \leq j \leq (n-1)$ commute, hence $A_{\alpha_3}, \ldots, A_{\alpha_{n-1}}$ all have v as an eigenvector. Lemma 6.7 implies that M is not simple. ∎

Proof of Main Theorem: Assume that M is a simple torsion free (A_{n-1}, H) module of degree k and $n \geq 5$. By lemma 6.1, we could assume that H is as in 1.14. If $k = 2$, then we have the contradiction concerning the dimension of an eigenspace of A_α given to us by Lemma 6.6 and Corollary 6.8. Hence, $k \geq 3$.

This proves that the result is true for $n = 5$ and begins our inductive proof. Now, assume that the result is true for all m, $5 \leq m < n$ and $n \geq 6$.

As our first step, we establish the claim that, when $n \geq 6$, k must be $n-3$ and each operator A_α has a unique eigenvalue whose eigenspace has dimension $d = n-4$. For $n = 6$, Lemma 6.6 and Corollary 6.8 imply that the claim is true. Continuing, we assume that $n \geq 7$. Let V be either an eigenspace or a generalized eigenspace of A_α. By Corollary 6.8, $dim(V) \geq 2$. Suppose that $2 \leq \dim V \leq n-5$. As in the proof of corollary 6.8, one can find a standard base $\Delta_* = \{\alpha_1, \ldots, \alpha_{n-1}\}$ such that $\alpha = \alpha_1$. By the definition of standard base, we have

$$\Delta_* = \{\alpha_1 = \omega_{i_1} - \omega_{i_2}, \ldots, \alpha_{n-1} = \omega_{i_{n-1}} - \omega_{i_n}\},$$

for $i_1, \ldots, i_n$ appropriate choices of distinct elements of $\{1, \ldots, n\}$.

Consider the vector space L generated by $\{h_\beta, X_\beta \mid \beta \in \phi^3_{\Delta_*}\}$. Let $H_L = span_{\mathbb{C}}\{h_\beta \mid \beta \in \phi^3_{\Delta_*}\}$. It is easy to see that L is a subalgebra of A_{n-1} which is isomorphic to an algebra of type A_{n-3} and $H_L \subset H$ is a maximal toral subalgebra of L. Also $\phi^3_{\Delta_*}$ is the root system of (L, H_L) having Δ^3_* as a standard base. By the P.B.W. Theorem, $U(L)$ is a subalgebra of $U(A_{n-1})$. So that, M is naturally an (L, H_L) torsion free module.

By direct computation, A_{α_1} commutes with A_β whenever $\beta \in \phi^3_{\Delta_*}$. We conclude that V is invariant under A_β whenever $\beta \in \phi^3_{\Delta_*}$. We conclude that if $U_0(L)$ denotes the centralizer algebra $CY(L, H_L)$, then by Lemma 6.3, V is a $U_0(L)$ submodule of M_θ. Let $N = U(L).V$, then N is a $U(L)$ submodule of M. By part (i) of Theorem 4.22, N is a torsion free (L, H_L) module and by part (ii) of the same theorem, we have $deg(N) = dim(V) \leq n-5$ and V is a weight space of N.

Since $n-3 \geq 4$, the induction assumption says that N is not a simple L module and hence by part (iii) of Theorem 4.22 has a simple submodule $0 \neq N' \neq N$. By part (i) of Theorem 4.22 N' is a torsion free (L, H_L) module. Since $N' \neq N$, we must have $deg(N') < deg(N) \leq n-5$. Induction hypothesis gives $deg(N') = 1$ and hence the (L, H_L) weight space $N'_\theta = N' \cap V$ of N' is one dimensional. Choose $0 \neq v \in N'_\theta$. Since N'_θ is a weight space of N', by Theorem 4.17 it is a $U_0(L)$ module. We conclude that for all $\beta \in \phi^3_{\Delta_*}$, A_β has v as an eigenvector. By Lemma 6.7 we have a contradiction.

If A_α has two distinct eigenvalues with corresponding dimensions d_1 and d_2, then $k \geq d_1 + d_2 > 2n - 10$ and so we have arrived at the contradiction that $2n - 10 < k \leq n - 3$ or $n < 7$. Hence, the claim is true.

Let W_α denote the unique eigenspace of A_α for $\alpha \in \phi$ and let $n \geq 6$. Our second claim is that $\dim(W_{\beta_1} \cap \ldots \cap W_{\beta_p}) \geq n - 3 - p$. where $\beta_1, \ldots, \beta_p$ are not neccessarily distinct elements of ϕ. This is known for $p = 1$. Inductively assume that $\dim(W_{\beta_1} \cap \ldots \cap W_{\beta_{p-1}}) \geq n - 3 - (p-1) = n - 2 - j$. Then using a standard result of linear algebra we have:

$$\begin{aligned} n-3 &\geq \dim(W_{\beta_1} \cap \ldots \cap W_{\beta_{p-1}} + W_{\beta_p}) \\ &= \dim(W_{\beta_1} \cap \ldots \cap W_{\beta-p-1}) + \dim(W_{\beta_p}) - \dim(W_{\beta_1} \cap \ldots \cap W_{\beta_p}) \\ &\geq n - 2 - p + n - 4 - \dim(W_{\beta_1} \cap \ldots \cap W_{\beta_p}) \end{aligned}$$

Thus $\dim(W_{\beta_1} \cap \ldots \cap W_{\beta_p}) \geq n - 3 - p$.

Let $p = n - 4$, and take $\beta_j = \alpha_j, 1 \leq j \leq p$ then $\dim(W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-4}}) \geq 1$. If

$$W_{\alpha_1} \cap \ldots \cap W_{\alpha_{j-1}} = W_{\alpha_1} \cap \ldots \cap W_{\alpha_j}$$

for some $2 \leq j \leq n-4$, then A_{α_j} is a scalar on $W = W_{\alpha_1} \cap \ldots \cap W_{\alpha_{j-1}}$ and M is not simple by Lemma 6.7. Therefore,

$$W_{\alpha_1} \supset W_{\alpha_1} \cap W_{\alpha_2} \supset \ldots \supset W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-4}} \neq \{0\}$$

and so $\dim(W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-5}}) = 2$ and $\dim(W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-4}}) = 1$.

Our third claim is that $W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-5}} \subset W_{\alpha_{n-1}}$. Otherwise, by a standard dimension argument we must have $\dim(W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-5}} \cap W_{\alpha_{n-1}}) = 1$. On the other hand, by direct computation $A_{\alpha_{n-1}}$ commutes with all A_{α_q} for all $1 \leq q \leq n-4$, hence W_{α_q} is invariant under $A_{\alpha_{n-1}}$ for all $1 \leq q \leq$ -- 4. Hence $A_{\alpha_{n-1}}$ leaves $W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-4}}$ invariant and since $\dim(W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-4}}) = 1$, we have $W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-4}} \subset W_{\alpha_{n-1}}$ and so $W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-4}} = W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-5}} \cap W_{\alpha_{n-1}}$. By a similar argument $W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-5}} \cap W_{\alpha_{n-1}} \subset W_{\alpha_{n-3}}$.

This implies $W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-4}} \subset W_{\alpha_{n-3}}$, and $A_{\alpha_{n-3}}$ is a scalar on $W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-4}}$, contrary to Lemma 6.7. Hence, our third claim is proved.

Consider the standard bases

$$\Delta_1 = \{\alpha_1, \ldots, \alpha_{n-4}, \alpha_{n-3} + \alpha_{n-2}, -\alpha_{n-2}, \alpha_{n-2} + \alpha_{n-1}\} \text{ and}$$

$$\Delta_2 = \{\alpha_1, \ldots, \alpha_{n-5}, \alpha_{n-4} + \alpha_{n-3} + \alpha_{n-2}, -\alpha_{n-2}, -\alpha_{n-3}, \alpha_{n-3} + \alpha_{n-2} + \alpha_{n-1}\}.$$

Since by the third step $A_{\alpha_{n-3}+\alpha_{n-2}+\alpha_{n-1}}$ and $A_{-(\alpha_{n-2}+\alpha_{n-1})}$ are scalars on $W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-5}}$, Lemma 6.4 says that $A_{\alpha_{n-3}}$ is a scalar on $W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-4}} \subset W_{\alpha_1} \cap \ldots \cap W_{\alpha_{n-5}}$ and we again have a contradiction to Lemma 6.7. ∎

Example: We provide examples of simple torsion free A_{n-1} modules of degrees 1, $n-2$ and $n-1$. To this end, let y_i denote the column vector whose j-th element is δ_{ij}. Then, if we define $x.v$ to be the regular matrix multiplication of $x \in A_{n-1}$ and $v \in V$, then $V = \mathbb{C}^n$ is an A_{n-1} module and hence by Theorem 4.14 is a (A_{n-1}, H) weight module. Assume that $(0) \neq V'$ is a submodule of V and let $0 \neq z = \sum_{i=1}^{n} d_i y_i \in V'$. Since $0 \neq z$, we have $d_j \neq 0$ for some $1 \leq j \leq n$, then $y_i = d_j^{-1} e_{ij}.z \in V'$ for all $i \neq j$. It follows that $y_j \in V'$ and hence $V' = V$. Hence V is simple.

Let $\{x_1, \ldots, x_n\}$ denote a set of commuting variables.

Fix $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$, where none of the components are integers and consider the vector space

$$\mathcal{N}(\vec{a}) = \text{span}_{\mathbf{C}}\{x^{\vec{b}} = x_1^{b_1}\dots x_n^{b_n} \mid b_i - a_i \in \mathbb{Z}, \text{ for all } i \text{ and } \sum_{i=1}^{n}(b_i - a_i) = 0\}.$$

If we define an operation of $\circ : A_{n-1} \times \mathcal{N}(\vec{a}) \to \mathcal{N}(\vec{a})$ where the actions of e_{ij} for $1 \leq i \neq j \leq n$ and $h_i = e_{ii} - e_{i+1,i+1}$, on $\mathcal{N}(\vec{a})$ are completely determined by

$$e_{ij} \circ (x_1^{b_1}\dots x_n^{b_n}) = b_j x_1^{b_1}\dots x_i^{b_i+1}\dots x_j^{b_j-1}\dots x_n^{b_n},$$

and

$$h_i \circ (x_1^{b_1}\dots x_n^{b_n}) = (b_i - b_{i+1})x_1^{b_1}\dots x_n^{b_n},$$

then $\mathcal{N}(\vec{a})$ endowed with $\circ$ is a (A_{n-1}, H) weight module having $span_{\mathbf{C}}(x_1^{b_1}\dots x_n^{b_n})$ as weight spaces. Since all of the a_i and hence all of the b_i are not integers, the actions of e_{ij} for all $1 \leq i \neq j \leq n$ is injective and $\mathcal{N}(\vec{a})$ is a pointed torsion free (A_{n-1}, H) module. It is easy to see that $\mathcal{N}(\vec{a})$ is simple. In fact in [BL1], it is shown that every simple torsion free A_{n-1} module of degree 1 is equivalent to such a module.

We now consider the tensor product module M given by $M = \mathcal{N}(\vec{a} - \epsilon_1) \odot V$, where $\vec{a} - \epsilon_1 = (a_1 - 1, a_2, \dots, a_n)$. By theorem 4.21, M is torsion free of degree n.

For each $i = 1, \dots, n$, we define

$$v_i = x_1^{a_1}\dots x_i^{a_i-1}\dots x_n^{a_n} \otimes y_i$$

It is easily verified that the elements v_i span the θ weight space of M where $\theta(h_i) = a_i - a_{i+1}$. By direct computation we observe that subspaces

$$W_1 = \text{span}_{\mathbf{C}}\{\sum_{i=1}^{n} a_i v_i\},\ W_0 = \text{span}_{\mathbf{C}}\{\sum_{i=1}^{n} l_i v_i | \sum_{i=1}^{n} l_i = 0\}$$

of M_θ are invariant under A_α for all $\alpha \in \phi$ and hence, by Lemma 6, they are U_0 submodules. This means that $M_0 = UW_0$ and $M_1 = UW_1$ are submodules of M

with $(UW_i)_\theta = W_i$ $(i = 1, 2)$, which are torsion free of degrees of $n-1$ and 1 respectively, (c.f. Theorem 4.22) and M_1 is isomorphic to $N(\vec{a})$.

Suppose that M_0 contains the proper nonzero submodule Y. By part (i) of Theorem 4.22, Y is a torsion free (L, H) module. If $wt(M_0)$ and $wt(Y)$ respectively denote the set of weights of M_0 and Y, we claim that $wt(Y) = wt(M_0)$ and all the weight spaces of Y have the same dimension.

To this end, Let $\omega \in wt(Y)$ be such that Y_ω the corresponding weight space is of maximal dimension. If $\gamma \in wt(M_0)$ then we could do the same argument as part (ii) of Theorem 4.19 and see the claim is true. Hence Y has a degree k.

Suppose that $k = n-2$, then no vector of the form $v_i - v_j$ with $i \neq j$ is in Y because such vectors generate W_0 and hence generates M_0 as we see from

$$e_{ik}e_{ki}(v_i - v_j) = a_i(a_k+1)(v_i - v_j) - a_k(v_i - v_k) \qquad \text{for all } k \neq i, j.$$

By a dimension argument, we know that $(v_1 - v_2) - b(v_1 - v_3) \in V$ for some $b \in \mathbb{C}$ and since $-v_2 + v_3 \notin Y$, we know that $b \neq 1$. Computing we have

$$\begin{aligned} e_{14}e_{41}((v_1 - v_2) - b(v_1 - v_3)) =& [a_1(a_4+1)(v_1 - v_2) - a_4(v_1 - v_4)] \\ & - b[a_1(a_4+1)(v_1 - v_3) - a_4(v_1 - v_4)] \end{aligned}$$

from which it follows that $v_1 - v_4 \in V$. Hence, $k \neq n-2$.

Next assume that Z is a simple submodule of M_0 of degree k'. By our Main theorem if $Z \neq M_0$, we have $k' = 1$ or $k' = n-2$. By the preceding $k' \neq n-2$ and so $k' = 1$. This means that there is some $v = \sum_{i=1}^n d_i v_i \in Z \cap M_\theta$ with not all $d_i = 0$ for $i \geq 3$, which is an eigenvector for every $e_{ik}e_{ki}$.

$$\begin{aligned} e_{12}e_{21}v =& (d_1(a_1-1)(a_2+1) + d_1 + d_2 a_1)v_1 + (d_1 a_2 + d_2 a_1 a_2)v_2 \\ & + d_3 a_1(a_2+1)v_3 + \ldots + d_n a_1(a_2+1)v_n \end{aligned}$$

tells us that the eigenvalue for this operator is $a_1(a_2+1)$ and it forces $d_1 = s a_1$ and $d_2 = s a_2$ for some $s \in \mathbb{C}$. Similar calculations show that $d_i = s a_i$. Therefore, if $\sum_{i=1}^{n} a_i \neq 0$ M_0 is simple torsion free A_{n-1} module of degree $n-1$ and $M = M_1 \oplus M_0$. Moreover it is easy to see that in this case M_0 is not an (L, H') weight module, for any other maximal toral subalgebra H' of L.

If $\sum_{i=1}^{n} a_i = 0$, then we have $\{0\} \subset M_1 \subset M_0 \subset M$. We claim that M_1 is a maximal proper submodule of M_0. If not say $M_1 \subset M_2 \subset M_0$ for some proper submodule of M_0, then M_2 is not of degrees $1, n-1$ and $n-2$. Then by Theorem 4.12 and Theorem 4.23, M_0/M_2 is a simple torsion free module of degree k_1 and $2 \leq k_1 \leq n-3$. A contradiction!. Hence M_1 is a maximal submodule of M_0 and by part (ii) of Theorem 4.12, M_0/M_1 is a simple L module and by Theorem 4.23, it is torsion free of degree $n-2$. Moreover, since any proper submodule of M_0 contains the simple submodule M_1 of M_0, we conclude that in this case M_0 is indecomposable.

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