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Flow problems in micropolar fluids.

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FLOW PROBLEMS

IN

MICROPOLAR FLUIDS

by

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Respectfully Dedicated to my beloved parents

ABSTRACT

This dissertation deals with various flow problems in micropolar fluids:

1. Unsteady, laminar and incompressible flows of a micropolar fluid near an accelerated infinite flat plate, with no slip or spin on the plate, are investigated. The method of perturbation expansion of functions of a similarity variable is applied to reduce the coupled partial differential equations to a set of ordinary differential equations. Series solutions in terms of parabolic cylinder functions are found for the particular case $(\mu+\kappa)j = \gamma$. Special cases of the flows near uniformly and suddenly accelerated plates are also obtained.

2. Steady porous plane Couette and Poiseuille flows of a micropolar fluid are studied. It is found, for certain ranges of the rate of suction and injection, that the velocity is composed of a linear combination of real exponential terms, whereas for other values products of exponentials and sinusoidal terms occur, provided that the material constants satisfy certain inequalities.

3. Unsteady, laminar and incompressible flows of a micropolar fluid near an accelerated infinite porous flat plate with variable suction, are investigated. Particular cases of the flows near uniformly and suddenly accelerated porous flat plates with variable suction, are studied.

4. The steady flow of a micropolar fluid through an elliptic tube is investigated. A method of successive approximations is applied to uncouple the governing partial differential equations. The equations, subject to the appropriate boundary conditions, are solved using the semi-inverse method. The graphs for the mean fluxes against eccentricity are drawn. It is observed that the steady flow of micropolar fluids in an elliptic tube is not secondary.

5. Finally, the partial differential equations governing the two dimensional unsteady flow of micropolar fluids, with no external forces and no body couples, are formulated. Kampé de Fériet's (1932) method is extended to find solutions such that the vorticity and spin are constant along a streamline at any particular time. Taylor's (1923) motions of micropolar fluids are studied. Motion due to a single vortex and/or spin filament is investigated. A semi-inverse solution for fluid velocity and spin, in the case $(\mu + \frac{1}{2}\kappa)j = \gamma$, in terms of Bessel's functions is found.

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CHAPTER I

INTRODUCTION

Section 1. Introduction to fluids with local effects.

Several authors have formulated fluid theories which take into account the micro-structure of the fluid. Einstein (1896) studied the motion of spherical particles immersed in a viscous fluid. The presence of the particles influences the properties of the fluid and in particular, its viscosity will be increased.

Jeffery (1922) extended the work of Einstein to the case of ellipsoidal particles. He calculated the fluid motion in the vicinity of a suspended ellipsoid and used the results to find the increase in viscosity due to the presence of ellipsoidal particles in a Newtonian fluid. He ignored particle interactions and observed that the increase in viscosity is greatest when the particles are spherical.

It is well known that many liquids do not move according to the Navier-Stokes' equations because of the non-spherical structure of their molecules and thus many recent fluid theories have been developed.

Oseen (1933) gave a theory of liquid crystals, in which the fluid particles are not spherical. According to him, the forces which cause molecules to combine so as to form a liquid crystal are not of electrostatic nature but are molecular forces. Newton's Law of Viscosity for the stress tensor does not hold for liquid crystals. Oseen assumed that stress is a function of the rate of strain and particle substructure.

Prager (1957) considered a suspension of non-interacting dumbbell particles and found a constitutive equation for the stress and an equation determining the 'preferred' direction adopted by the particles.

Ericksen (1960) gave a theory of anisotropic fluids. Ericksen pointed out that his equations governing the motion of a particle with a single preferred direction can be shown to be the same as the equations obtained by Jeffery (1922) for the motion of an oblate or prolate spheroid. The preferred direction coincides with the axis of revolution of the ellipsoid. However, Ericksen's equations do not account for rotation about this axis. Ericksen's equations may be classified as those governing a fluid with non-interacting substructure. The stress tensor for anisotropic fluids is symmetric because of the absence of couple stress.

Hand (1962) proposed a theory of anisotropic fluids in which the stress tensor is a function of the rate of the deformation tensor and a symmetric tensor describing the microscopic structure of a fluid. He considered the fluid to be incompressible and its properties independent of temperature. These fluids exhibit non-Newtonian behavior typical of certain higher polymer solutions. The expression for the stress tensor has been written using results from the Hamilton-Cayley theorem. This theory is shown to contain Prager's (1957) theory of dumbbell suspensions as a special case. Hand compared the results of the anisotropic theory with experiments on high polymers such as polyisobutylene.

Eringen (1966) has presented a theory of micropolar fluids. The theory of micropolar fluids has appeared under a variety of names (Cosserat fluids, generalised fluids, polar fluids, ...) during the past decade, with a corresponding multiplicity of notation. Following Eringen (1966), micropolar fluids are viscous fluids with five additional coefficients of viscosity when compared with the usual Newtonian fluids. These fluids differ from non-Newtonian fluids in that they exhibit microinertial effects and can support couple stresses and body couples. Shearing stress components in these fluids are affected by the vorticity and micro-rotation of the fluid and are no longer symmetric. The important feature of these fluids is the micro-rotation. According to Eringen (1966), micro-rotation bears a resemblance to the vorticity in as much as only those components of it are non-vanishing which correspond to the non-vanishing components of vorticity and they depend upon the same variable on which the vorticity components depend. Physically, some polymeric fluids and fluids containing small amounts of polymeric additives may be represented by the mathematical model underlying micropolar fluids. Animal blood happens to fall into this category. In a series of papers Lee and Eringen (1971) have applied the theory of micropolar fluids to the study of liquid crystals and have obtained many new interesting results. One may refer to Eringen (1966) for a detailed account of this theory and for a derivation of the basic equations governing the behaviour of such a fluid.

Section 2. Basic Equations of Micropolar Fluids.

The basic equations of the micropolar fluids are the field equations, the constitutive equations, boundary conditions and restrictions on the constitutive coefficients.

(i) Field Equations. The field equations for incompressible micropolar fluids as given by Eringen (1966) are,

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0 \tag{1.1}$$

$$(\lambda+2\mu+\kappa)\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{v}) - (\mu+\kappa)\underset{\sim}{\nabla}\times\underset{\sim}{\nabla}\times\underset{\sim}{v}+\kappa\underset{\sim}{\nabla}\times\underset{\sim}{\nu}-\underset{\sim}{\nabla}p+\rho\underset{\sim}{f} = \rho\dot{\underset{\sim}{v}} \tag{1.2}$$

$$(\alpha+\beta+\gamma)\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{\nu}) - \gamma(\underset{\sim}{\nabla}\times\underset{\sim}{\nabla}\times\underset{\sim}{\nu}) + \kappa\underset{\sim}{\nabla}\times\underset{\sim}{v} - 2\kappa\underset{\sim}{\nu} + \rho\underset{\sim}{\ell} = \rho j\ \dot{\underset{\sim}{\nu}} \tag{1.3}$$

where $\underset{\sim}{v}$ is the velocity, $\underset{\sim}{\nu}$ the micro-rotation or spin, p the thermodynamic pressure, $\underset{\sim}{f}$ and $\underset{\sim}{\ell}$ the body-force and -couple per unit mass, ρ the density and j the micro-inertia; λ, μ, κ, α, β and γ are the material constants (viscosity coefficients), where the dot signifies material differentiation with respect to time. Thermal effects have been neglected.

(ii) Constitutive Equations. The constitutive equations giving $t_{k\ell}$, the stress tensor, and $m_{k\ell}$, the couple stress tensor, are, in the Cartesian co-ordinates, as given by Eringen (1966),

$$t_{k\ell} = (-p+\lambda v_{r,r})\delta_{k\ell} + \mu(v_{k,\ell}+v_{\ell,k}) + \kappa(v_{\ell,k}-e_{k\ell r}\nu_r) \tag{1.4}$$

$$m_{k\ell} = \alpha\ \nu_{r,r}\ \delta_{k\ell} + \beta\ \nu_{k,\ell} + \gamma\ \nu_{\ell,k} \tag{1.5}$$

where $\delta_{k\ell}$ and $e_{k\ell m}$ are the Kronecker delta and the alternating symbol respectively, and the comma denotes partial differentiation with respect to a space co-ordinate. Repeated indices are to be summed.

(iii) Boundary Conditions. The boundary conditions of micropolar fluids at a rigid boundary are given by Eringen (1966),

$$\underset{\sim}{v}(\underset{\sim}{x}_B, t) = \underset{\sim}{v}_B$$
$$\underset{\sim}{\nu}(\underset{\sim}{x}_B, t) = \underset{\sim}{\nu}_B \tag{1.6}$$

where x_B is a point on a solid boundary having prescribed velocity $\underset{\sim}{v}_B$ and prescribed micro-rotation velocity $\underset{\sim}{\nu}_B$. These conditions express the assumption of adherence of the fluid to the solid boundary.

There have been other boundary conditions, particularly for the micro-rotation or spin, proposed by Aero et al (1965) and Condiff and Dahler (1964). As an alternative to $(1.6)_2$, which implies no spin at the solid boundary, these authors have suggested that the boundary condition in which the anti-symmetric part of the stress is zero at wall, may be more appropriate in some circumstances. Aero et al (1965) have put forward yet another boundary condition in which the angular velocity of a fluid particle is equal to the angular velocity of the surface.

(iv) Restrictions on Constitutive Coefficients. The following restrictions on the viscosity coefficients were obtained, provided that the Clausius-Duhem inequality is satisfied locally for all independent processes by Eringen (1966).

$$\left.\begin{array}{lll} (3\lambda+2\mu+\kappa) \geq 0 , & 2\mu+\kappa \geq 0 , & \kappa \geq 0 \\ (3\alpha+\beta+\gamma) \geq 0 , & -\gamma \leq \beta \leq \gamma & \end{array}\right\} \quad (1.7)$$

Section 3. Outline of the present work.

The problems in micropolar fluids are challenging because of the fact that one has to deal with coupled non-linear partial differential equations. It is interesting to study the behaviour of micropolar fluids to determine if they resemble Newtonian fluids or if they exhibit characteristic non-Newtonian behaviour. In the micropolar fluid model, these fluids have a micro-structure endowed with spin inertia and a capacity for sustaining stress and body moments. The behaviour of the micro-structure is coupled to the macroscopic behaviour, and vice versa. Eringen (1966) worked out in detail the steady flow along a circular channel. Willson (1969) analysed several other basic flows, such as plane shear flow, flow between rotating cylinders, surface waves and similar properties and in (1968) the same author had studied the stability of the flow of a micropolar fluid down an inclined plane. The main feature of all these investigations was the role played by a certain combination of the parameters describing the fluid. Willson (1970) investigated steady flows of the boundary layer type using the Kármán-Polhausen method and, in particular, flows in the neighbourhood of stagnation points.

The purpose of the present work is to investigate the generally accelerated flat plate problem, some steady and unsteady flows with suction and injection, steady flow in an elliptic tube, and the problem of the most general solutions possible when the vorticity and spin are constant along a streamline at any particular time.

In chapter II, unsteady, laminar and incompressible flows of a micropolar fluid near an accelerated infinite flat plate, with no slip or spin on the plate, are investigated. The method of perturbation expansion of functions of a similarity variable is applied to reduce the coupled partial differential equations to a set of ordinary differential equations. Series solutions in terms of parabolic cylinder functions are found for the particular case $(\mu+\kappa)j = \gamma$. Expressions for the fluid velocity, micro-rotation or spin velocity, stresses and couple stresses are obtained. For $\kappa = 0$ and vanishing micro-rotation, our solution reduces to the classical form. Special cases of the flows near uniformly and suddenly accelerated plates are obtained by putting $n = 1$ and $n = 0$ respectively, in the accelerated flat plate problem. In the last section of this chapter, some numerical results for different values of parameters for a suddenly accelerated flat plate problem are obtained.

In section 1 of chapter III, steady porous plane Couette and Poiseuille flows of a micropolar fluid are studied. It is found, for certain ranges of the rate of suction and injection, that the velocity is composed of a linear combination of real exponential terms, whereas for other values products of exponentials and sinusoidal terms occur, provided that the material constants satisfy certain inequalities.

In section 2 of chapter III, unsteady, laminar and incompressible flow of a micropolar fluid near an accelerated infinite porous flat plate with variable suction, are investigated. A perturbation expansion of functions of a dimensionless variable is used and in the case $(\mu+\kappa)j = \gamma$, a series solution in terms of parabolic cylinder functions is found. Expressions for fluid velocity, and micro-rotation or spin velocity are obtained. For $\kappa = 0$ and vanishing micro-rotation, our solution reduces to the classical form. In sections 3 and 4 of this chapter, particular cases of the flows near uniformly and suddenly accelerated porous flat plates with variable suction, are studied.

In chapter IV, the steady flow of a micropolar fluid through an elliptic tube is investigated. A method of successive approximations is applied to uncouple the governing partial differential equations. The equations, subject to the appropriate boundary conditions, are solved using the semi-inverse method. Expressions for fluid velocity, and two micro-rotation velocity components are obtained. The difference between the volume fluxes $\int w_0 dA$ and $\int w_1 dA$, in order to estimate the magnitude of the first approximation to the zeroth approximation, is found. The graphs for mean fluxes $\overline{F}_0$, $\overline{F}_1$ and $\overline{F}_2$ against e (eccentricity) are drawn. It is observed that the steady flow in an elliptic tube is not secondary but that curves of spin velocity ν = constant, and

ϕ = constant, are similar in pattern to the secondary flow type as observed in non-Newtonian fluids by Green and Rivlin (1956), and in visco-elastic fluids by Langlois and Rivlin (1963).

In section 1 of chapter V, the partial differential equations governing the two dimensional unsteady flow of micropolar fluids, with no external forces and no body couples, are formulated in the Cartesian co-ordinate system. In section 2 of this chapter Kampé de Fériet's (1932) method is extended to find solutions such as the vorticity and spin are constant along a streamline at any particular time. In section 3 of this chapter, following Taylor (1923), we study Taylor's motions of micropolar fluids. In section 4, motion due to a single vortex and/or spin filament is investigated. A semi-inverse solution for fluid velocity and spin, in the case $(\mu+\frac{1}{2}\kappa)j = \gamma$, in terms of Bessel's functions, is found. The problems mentioned in this chapter have no rigid boundaries, hence the solutions and the behaviour of fluids do not depend on the particular choice of boundary conditions (i.e. "hyperstick").

CHAPTER II

ACCELERATED FLAT PLATE PROBLEM

Unsteady flows of a micropolar fluid, also called a Cosserat or polar fluid, or a fluid with rigid microstructure, have been considered by several authors in the past few years. Peddieson and McNitt (1970) investigated the problem of the infinite flat plate, started impulsively from rest, and obtained a solution good for small times and distances from the plate by truncating a series expansion of the Laplace transform of the solution in powers of distance from the plate.

Willson (1969) considered the propagation of plane wave disturbances in an unbounded fluid and also the flow induced by an infinite plane boundary executing sinusoidal oscillations parallel to itself.

Allen and Kline (1970) obtained solutions for flow in the half-space above an infinite plate performing sinusoidal oscillations.

Kirwan and Newman (1972) considered flow in a channel of unit width when the boundary conditions are time-dependent, and examined the transition from arbitrary initial conditions to steady flow.

In this chapter we consider unsteady, laminar and incompressible flows of a micropolar fluid near a generally accelerated flat plate using Eringen's (1966) theory.

Section 1. Flow of micropolar fluids near an accelerated flat plate:

Here we consider one dimensional, unsteady, laminar and incompressible flow of a micropolar fluid near an infinite flat plate, choosing the x-axis along the flat plate, and the y-axis perpendicular to it. The flow is independent of z.

When $t \leq 0$, the fluid is assumed to be everywhere stationary, and when $t > 0$, the plate is assumed to have velocity $u = At^n$, where n is a positive integer (or half-integer) and A is a constant. As the plate is infinite in length, all the variables in this problem are functions of y and t only.

The material constants of the micropolar fluid namely λ, μ, κ, α, β and γ, the density ρ, and the micro-inertia j, are assumed to be independent of position. We neglect body forces and body couples. Setting,

$$\underset{\sim}{v} = (u(y,t), 0, 0), \quad \underset{\sim}{\nu} = (0, 0, \phi(y,t)) \tag{2.1}$$

and ρ = constant, the equation of continuity (1.1) is satisfied identically, and the field equations (1.2) and (1.3) reduce to

$$(\mu+\kappa)\frac{\partial^2 u}{\partial y^2} + \kappa\frac{\partial\phi}{\partial y} = \rho\frac{\partial u}{\partial t} \tag{2.2}$$

$$\gamma\frac{\partial^2\phi}{\partial y^2} - \kappa\frac{\partial u}{\partial y} - 2\kappa\phi = \rho j\frac{\partial\phi}{\partial t} \tag{2.3}$$

$$\frac{\partial p}{\partial y} = 0 \tag{2.4}$$

$$\frac{\partial p}{\partial z} = 0 . \tag{2.5}$$

The partial differential equations (2.2) and (2.3) are to be solved subject to the following initial and boundary conditions:

$$\begin{aligned} &y \geq 0 , \quad t \leq 0 ; \quad u = \phi = 0 \\ &y = 0 , \quad t > 0 ; \quad u = At^n , \ \phi = 0 \\ &y \to +\infty , \quad t > 0 ; \quad u \to 0 , \quad \phi \to 0 . \end{aligned} \tag{2.6}$$

The method of perturbation expansion of functions of a similarity variable is applied to reduce the partial differential equations (2.2) and (2.3) to two systems of ordinary differential equations. Series solutions in terms of parabolic cylinder functions are then obtained for two systems of ordinary differential equations. For small values of ε , where $\varepsilon = \frac{\kappa t}{\rho j}$, we expand u and ϕ in ascending powers of ε as follows:

$$\begin{aligned} u &= At^n[f_0(\eta) + \varepsilon f_1(\eta) + \varepsilon^2 f_2(\eta) + \ldots] , \\ \phi &= Bt^{n+\frac{1}{2}}[g_0(\eta) + \varepsilon g_1(\eta) + \varepsilon^2 g_2(\eta) + \ldots] , \end{aligned} \tag{2.7}$$

where $\eta = \frac{y}{\sqrt{2kt}}$, $k = \frac{\mu+\kappa}{\rho}$.

It is easily seen that ε and η are both dimensionless. A and B are constants having the dimensions $L^{1}T^{-n-1}$ and $T^{-n-3/2}$ respectively. Substituting (2.7) in (2.2) after a little simplification, we get

$$(f_0''+\varepsilon f_1''+\varepsilon^2 f_2''+\ldots)+[(\eta f_0'-2nf_0)+\varepsilon(\eta f_1'-2(n+1)f_1)$$

$$+\varepsilon^2(\eta f_2'-(2n+4)f_2)+\ldots] = b_1(\varepsilon g_0'+\varepsilon^2 g_1'+\ldots) \ , \qquad (2.8)$$

where

$$b_1 = -\frac{\sqrt{2}\ j\ B}{A\sqrt{k}} \ , \qquad (2.9)$$

and the prime denotes differentiation with respect to η. Identifying the coefficients of the like powers of ε in (2.8), we obtain the following system of ordinary differential equations:

$$f_0'' + \eta f_0' - 2nf_0 = 0 \ ,$$

$$f_1'' + \eta f_1' - 2(n+1)f_1 = b_1 g_0' \ , \qquad (2.10)$$

$$f_2'' + \eta f_2' - 2(n+2)f_2 = b_1 g_1' \ .$$

Substituting (2.7) into (2.3) and simplifying, we obtain

$$(g_0''+\varepsilon g_1''+\varepsilon^2 g_2''+\ldots)+b_2[\eta g_0'-(2n+1)g_0+\varepsilon\{\eta g_1'-(2n+3)g_1\}$$

$$+\varepsilon^2\{\eta g_2'-(2n+5)g_2\}+\ldots]-4b_2(\varepsilon g_0+\varepsilon^2 g_1+\ldots)$$

$$= b_3(f_0'+\varepsilon f_1'+\varepsilon^2 f_2'+\ldots) \qquad (2.11)$$

where

$$b_2 = k\,\frac{\rho j}{\gamma}$$

$$b_3 = \frac{\kappa}{\gamma}\,\sqrt{2k}\,\frac{\Lambda}{B} \tag{2.12}$$

and the prime denotes differentiation with respect to η. Comparing the coefficients of like powers of ε in (2.11), we obtain the following system of ordinary differential equations:

$$\begin{aligned} g_0'' + b_2\eta g_0' - b_2(2n+1)g_0 &= b_3f_0' , \\ g_1'' + b_2\eta g_1' - b_2(2n+3)g_1 &= 4b_2g_0 + b_3f_1' , \\ g_2'' + b_2\eta g_2' - b_2(2n+5)g_2 &= 4b_2g_1 + b_3f_2' . \end{aligned} \tag{2.13}$$

The boundary conditions (2.6) in the new variables become:

$$\text{at}\quad \eta = 0 , \qquad f_0 = 1 , \quad g_0 = 0 .$$

$$f_p = 0 , \quad g_p = 0 , \; p = 1,2,3,\ldots \tag{2.14}$$

$$\text{at}\quad \eta = \infty , \qquad f_p = 0 , \quad g_p = 0 , \; p = 0,1,2,\ldots \tag{2.15}$$

We need to solve the two systems of ordinary differential equations (2.10) and (2.13) subject to the boundary conditions (2.14) and (2.15).

If we substitute

$$f_0(\eta) = F_0(\eta)e^{-\eta^2/4} \tag{2.16}$$

in the first equation of (2.10), then F_0 satisfies the equation

$$F_0'' - (+\frac{1}{2} + 2n + \frac{\eta^2}{4})F_0 = 0 . \qquad (2.17)$$

Writing the above equation in the standard form of Weber's equation, which may be found in Whittaker and Watson (1965, pp 347), we obtain

$$F_0'' + [-(2n+1) + \frac{1}{2} - \frac{\eta^2}{4}]F_0 = 0 \qquad (2.18)$$

Now $2n+1$ is an integer, hence two linearly independent solutions of (2.18) are $D_{-2n-1}(\eta)$ and $D_{2n}(i\eta)$ [Erdélyl et al (Volume II, 1953, pp 117)], parabolic cylinder functions, where $D_\alpha(\eta)$ satisfies [Whittaker and Watson (1965, pp 347)] the equation

$$\frac{d^2 D_\alpha(\eta)}{d\eta^2} + (\alpha + \frac{1}{2} - \frac{\eta^2}{4}) D_\alpha(\eta) = 0 . \qquad (2.19)$$

The general solution of (2.18) is given by

$$F_0 = C_1 D_{-2n-1}(\eta) + C_2 D_{2n}(i\eta) , \qquad (2.20)$$

where C_1 and C_2 are arbitrary constants of integration. Using (2.20) in (2.16), the general solution of the first of (2.10) is found to be

$$f_0 = C_1 e^{-\eta^2/4} D_{-2n-1}(\eta) + C_2 e^{-\eta^2/4} D_{2n}(i\eta) . \qquad (2.21)$$

The behaviour of the parabolic cylinder function at infinity, is given by the asymptotic expansion, which may be found in Whittaker and Watson (1965, pp 347), as $\eta \to \infty$,

$$D_\alpha(\eta) \sim e^{-\eta^2/4} \eta^\alpha \{1 - \frac{\alpha(\alpha-1)}{2\eta^2} + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{2.4.\eta^4} \ldots\}$$

$$\text{when } |\arg \eta| < \frac{3\pi}{4} . \tag{2.22}$$

We need to find the arbitrary constants C_1 and C_2 in (2.21) subject to the boundary conditions (2.14) and (2.15). If the solution (2.21) is to be bounded at $\eta = \infty$, in view of (2.22), we must have $C_2 = 0$. The boundary condition at $\eta = 0$, gives

$$C_1 = \frac{1}{D_{-2n-1}(0)} = \Gamma(n+1).(\pi)^{-\frac{1}{2}}.2^{n+\frac{1}{2}} \tag{2.23}$$

where we made use of the formula for the series for $D_\alpha(z)$, which may be found in Whittaker and Watson (1965, pp 347). Using (2.23) in (2.21) and taking $C_2 = 0$, we obtain

$$f_0 = \Gamma(n+1).(\pi)^{-\frac{1}{2}}.2^{n+\frac{1}{2}}.e^{-\eta^2/4} D_{-2n-1}(\eta). \tag{2.24}$$

Differentiating (2.24), with respect to η , we get

$$f_0' = (-1)\Gamma(n+1)(\pi)^{-\frac{1}{2}} 2^{n+\frac{1}{2}} e^{-\eta^2/4} D_{-2n}(\eta) , \tag{2.25}$$

where we made use of the formula Erdelyl et al (1953, p. 119),

$$\frac{d}{d\eta}[e^{-\eta^2/4} D_{-2n-1}(\eta)] = (-1)e^{-\eta^2/4} D_{-2n}(\eta) . \quad (2.26)$$

Using (2.25) in the right hand side of the first of (2.13), we obtain

$$g_0'' + b_2 \eta g_0' - b_2(2n+1)g_0 = -b_3(\pi)^{-1/2} 2^{n+\frac{1}{2}} e^{-\eta^2/4} D_{-2n}(\eta) . \quad (2.27)$$

The homogeneous differential equation of (2.27) is

$$g_0'' + b_2 \eta g_0' - b_2(2n+1)g_0 = 0 . \quad (2.28)$$

In (2.28) let the equation be written in the form

$$\frac{1}{b_2}\frac{d^2 g_0}{d\eta^2} + \eta\frac{dg_0}{d\eta} - (2n+1)g_0 = 0 , \quad (2.29)$$

and let $\xi = \eta\sqrt{b_2}$,

then $$\frac{d^2 g_0}{d\xi^2} + \xi\frac{dg_0}{d\xi} - (2n+1)g_0 = 0 . \quad (2.30)$$

Now let $g_0 = G_0 e^{-\frac{1}{4}\xi^2}$, (2.31)

then

$$G_0'' + [-2n-\frac{3}{2} - \frac{1}{4}\xi^2]G_0 = 0 . \quad (2.32)$$

Writing (2.32) in the standard form of Weber's differential equation, we obtain

$$G_0'' + [-(2n+2) + \frac{1}{2} - \frac{\xi^2}{4}]G_0 = 0 . \quad (2.33)$$

Here $2n+2$ is an integer and so two linearly independent solutions of (2.33) are $D_{-2n-2}(\xi)$ and $D_{2n+1}(i\xi)$. The general solution of (2.33) is given by

$$G_0 = C_3 D_{-2n-2}(\xi) + C_4 D_{2n+1}(i\xi), \quad (2.34)$$

where C_3 and C_4 are arbitrary constants.

If we change the independent variable $\xi = \sqrt{b_2}\, \eta$, the equation (2.27) transforms into

$$\frac{d^2g_0}{d\xi^2} + \xi\frac{dg_0}{d\xi} - (2n+1)g_0 = \frac{b_3}{b_2} \cdot (\pi)^{-\frac{1}{2}} \cdot 2^{n+\frac{1}{2}} \cdot$$

$$e^{-\frac{\xi^2}{4b_2}} D_{-2n}\left(\frac{\xi}{\sqrt{b_2}}\right) . \quad (2.35)$$

Using (2.34) into (2.29) and changing the independent variable, the complementary function of (2.35) is given by

$$g_0 = C_3 e^{-\xi^2/4} D_{-2n-2}(\xi) + C_4 e^{-\xi^2/4} D_{2n+1}(i\xi) . \quad (2.36)$$

To find the particular solution of the differential equation (2.35), we let

$$g_0 = A e^{-\frac{1}{4}\xi^2/b_2} D_{-2n}(\xi/\sqrt{b_2}) \quad (2.37)$$

where A is a constant yet to be determined. Using (2.37) and the formula (2.26), the left-hand side of (2.35) becomes

$$\frac{d^2g_0}{d\xi^2} + \xi\frac{dg_0}{d\xi} - (2n+1)g_0$$

$$= \frac{A}{b_2} e^{-\frac{1}{4}\left(\frac{\xi}{\sqrt{b_2}}\right)^2} D_{-2n+2}\left(\frac{\xi}{\sqrt{b_2}}\right) - \frac{A}{\sqrt{b_2}} \xi e^{-\frac{1}{4}\left(\frac{\xi}{\sqrt{b_2}}\right)^2} D_{-2n+1}\left(\frac{\xi}{\sqrt{b_2}}\right)$$

$$- (2n+1).A.e^{-\frac{1}{4}\left(\frac{\xi}{\sqrt{b_2}}\right)^2} D_{-2n}\left(\frac{\xi}{\sqrt{b_2}}\right) . \qquad (2.38)$$

It is clear that the computation would become increasingly lengthy as we proceed, due to the occurrence of parabolic cylinder function with arguments η and $\xi = \eta\sqrt{b_2}$. However, considerable simplification is achieved if we consider the particular case $b_2 = 1$, that is $(\mu+\kappa)j = \gamma$, and this we proceed to do.

Our system of equations (2.13) then reduces to

$$g_0'' + \eta g_0' - (2n+1)g_0 = b_3 f_0' ,$$

$$g_1'' + \eta g_1' - (2n+3)g_1 = 4g_0 + b_3 f_1' , \qquad (2.39)$$

$$g_2'' + \eta g_2' - (2n+5)g_2 = 4g_1 + b_3 f_2' .$$

We shall make use of the formula Erdélyl et al (1953, pp 119):

$$D_{m+1}(\eta) - \eta D_m(\eta) + mD_{m-1}(\eta) = 0 . \qquad (2.40)$$

Using (2.40) and letting $b_2 = 1$ so that $\xi = \eta$, we find from (2.38) that

$$g_0'' + \eta g_0' - (2n+1)g_0 = -2A\, e^{-\eta^2/4} D_{-2n}(\eta) \; . \tag{2.41}$$

Also, from equation (2.35) when $b_2 = 1$ and so $\xi = \eta$, we have

$$g_0'' + \eta g_0' - (2n+1)g_0 = -b_3(\pi)^{-\frac{1}{2}} . \; 2^{n+\frac{1}{2}} . \; e^{-\eta^2/4} D_{-2n}(\eta) \; . \tag{2.42}$$

Comparing the coefficients of $e^{-\eta^2/4} D_{-2n}(\eta)$ in the right hand sides of (2.41) and (2.42), we find that

$$A = b_3(\pi)^{-\frac{1}{2}} . \; \Gamma(n+1) \; . \; 2^{n-\frac{1}{2}} \; . \tag{2.43}$$

Thus the particular integral of the first of (2.39) is given by

$$g_0 = b_3(\pi)^{-\frac{1}{2}} . \; \Gamma(n+1) \; . \; 2^{n-\frac{1}{2}} . \; e^{-\eta^2/4} D_{-2n}(\eta) \; . \tag{2.44}$$

The complementary function of the first of (2.39) is

$$g_0 = C_3\, e^{-\eta^2/4} D_{-2n-2}(\eta) + C_4\, e^{-\eta^2/4} D_{2n+1}(i\eta) \; . \tag{2.45}$$

Hence the general solution of the first of (2.39), which is the sum of the complementary function (2.45) and the particular integral (2.44), is given by

$$g_0 = C_3 e^{-\eta^2/4} D_{-2n-2}(\eta) + C_4 e^{-\eta^2/4} D_{2n+1}(i\eta)$$

$$+ b_3(\pi)^{-\frac{1}{2}} . \Gamma(n+1) 2^{n-\frac{1}{2}} e^{-\eta^2/4} D_{-2n}(\eta) . \quad (2.46)$$

We need to find C_3 and C_4 subject to the boundary conditions (2.14) and (2.15). If the solution (2.46) is to be bounded at $\eta = \infty$, we must have $C_4 = 0$, and the boundary condition at $\eta = 0$ gives

$$C_3 = -b_3(\pi)^{-\frac{1}{2}} . \Gamma(n+1) . 2^{n-\frac{1}{2}} . \frac{D_{-2n}(0)}{D_{-2n-2}(0)}$$

$$= -b_3(\pi)^{-\frac{1}{2}} . \Gamma(n+1) . 2^{n+\frac{1}{2}} . (n+\tfrac{1}{2}). \quad (2.47)$$

Using (2.47) and taking $C_4 = 0$ in (2.46), we obtain the general solution of the first of (2.39), subject to the appropriate boundary conditions, as follows:

$$g_0 = -b_3(\pi)^{-\frac{1}{2}} . \Gamma(n+1) . 2^{n+\frac{1}{2}} . (n+\tfrac{1}{2}) . e^{-\eta^2/4} D_{-2n-2}(\eta)$$

$$+ b_3(\pi)^{-\frac{1}{2}} . \Gamma(n+1) . 2^{n-\frac{1}{2}} . e^{-\eta^2/4} D_{-2n}(\eta) . \quad (2.48)$$

We shall use the formula Erdelyl et at (1953, pp 119) :

$$\frac{d^m}{d\eta^m}[e^{-\eta^2/4} D_p(\eta)] = (-1)^m e^{-\eta^2/4} D_{p+m}(\eta) \qquad (2.49)$$

$$m = 1,\ 2,\ 3,\ \ldots\ .$$

Differentiating (2.48) with respect to η and using the formula (2.49), we find that

$$g_0' = b_3(\pi)^{-\frac{1}{2}} . \Gamma(n+1) . 2^{n+\frac{1}{2}} . (n+\tfrac{1}{2}) . e^{-\eta^2/4} D_{-2n-1}(\eta)$$

$$- b_3(\pi)^{-\frac{1}{2}} . \Gamma(n+1) . 2^{n-\frac{1}{2}} . e^{-\eta^2/4} D_{-2n+1}(\eta) . \qquad (2.50)$$

Substituting (2.50) into the right hand side of the second of (2.10), the differential equation becomes

$$f_1'' + \eta f_1' - 2(n+1)f_1 = b_1 b_3(\pi)^{-\frac{1}{2}} . \Gamma(n+1) . 2^{n+\frac{1}{2}} .$$

$$(n+\tfrac{1}{2})e^{-\eta^2/4} D_{-2n-1}(\eta) - b_1 b_3(\pi)^{-\frac{1}{2}} . \Gamma(n+1) . 2^{n-\frac{1}{2}} .$$

$$e^{-\eta^2/4} D_{-2n+1}(\eta) . \qquad (2.51)$$

To find the complementary function of the differential equation (2.51), the corresponding homogeneous equation is

$$f_1'' + \eta f_1' - 2(n+1)f_1 = 0 . \qquad (2.52)$$

If we substitute

$$f_1 = F_1 \, e^{-\eta^2/4}$$

into the equation (2.52), then F_1 satisfies the equation

$$F_1'' + [-(2n+3) + \frac{1}{2} - \frac{\eta^2}{4}]F_1 = 0 . \tag{2.53}$$

Now $(2n+3)$ is an integer, hence two linearly independent solutions of the differential equation (2.53) are $D_{-(2n+3)}(\eta)$ and $D_{2n+2}(\eta i)$.

The general solution of (2.53) is given by

$$F_1 = C_5 \, D_{-2n-3}(\eta) + C_6 \, D_{2n+2}(\eta i) , \tag{2.54}$$

where C_5 and C_6 are arbitrary constants of integration. Therefore, the complementary function of the differential equation (2.51) is

$$f_1 = C_5 \, e^{-\eta^2/4} D_{-2n-3}(\eta) + C_6 \, e^{-\eta^2/4} D_{2n+2}(\eta i) . \tag{2.55}$$

To find the particular integral of the differential equation (2.51), we let

$$f_1 = A_1 \, e^{-\eta^2/4} D_{-2n-1}(\eta) + A_2 \, e^{-\eta^2/4} D_{-2n+1}(\eta) , \tag{2.56}$$

where A_1 and A_2 are constants yet to be determined.

Substituting (2.56) into (2.51) and making use of the formulae (2.40) and (2.49), we find

$$\left.\begin{aligned} A_1 &= -b_1 b_3 (\pi)^{-\frac{1}{2}} \cdot (n+\tfrac{1}{2}) \cdot \Gamma(n+1) \cdot 2^{n-\frac{1}{2}} \\ A_2 &= b_1 b_3 (\pi)^{-\frac{1}{2}} \cdot \Gamma(n+1) \cdot 2^{n-\frac{5}{2}} \end{aligned}\right\} \qquad (2.57)$$

Using (2.57) in (2.56), the particular integral of the differential equation (2.51) becomes

$$f_1 = -b_1 b_3 (\pi)^{-\frac{1}{2}} \cdot (n+\tfrac{1}{2}) \cdot \Gamma(n+1) \cdot 2^{n-\frac{1}{2}} \cdot e^{-\eta^2/4} D_{-2n-1}(\eta)$$

$$+ b_1 b_3 (\pi)^{-\frac{1}{2}} \cdot \Gamma(n+1) \cdot 2^{n-\frac{5}{2}} \cdot e^{-\eta^2/4} D_{-2n+1}(\eta). \quad (2.58)$$

Thus the general solution of the differential equation (2.51), which is the sum of the complementary function (2.55) and the particular integral (2.58), is

$$f_1 = C_5 e^{-\eta^2/4} D_{-2n-3}(\eta) + C_6 e^{-\eta^2/4} D_{2n+2}(\eta i)$$

$$- b_1 b_3 (\pi)^{-\frac{1}{2}} \cdot (n+\tfrac{1}{2}) \cdot \Gamma(n+1) \cdot 2^{n-\frac{1}{2}} \cdot e^{-\eta^2/4} D_{-2n-1}(\eta)$$

$$+ b_1 b_3 (\pi)^{-\frac{1}{2}} \cdot \Gamma(n+1) \cdot 2^{n-\frac{5}{2}} \cdot e^{-\eta^2/4} D_{-2n+1}(\eta). \quad (2.59)$$

If the solution (2.59) is to be bounded at $\eta = \infty$, we must have $C_6 = 0$, and the boundary condition (2.14) at $\eta = 0$, gives

$$C_5 = b_1 b_3 (\pi)^{-\frac{1}{2}} \, . \, (n+\tfrac{1}{2}) \, . \, \Gamma(n+1) \, . \, 2^{n-\frac{1}{2}} \, . \, \frac{D_{-2n-1}(0)}{D_{-2n-3}(0)}$$

$$-b_1 b_3 (\pi)^{-\frac{1}{2}} \, . \, \Gamma(n+1) \, . \, 2^{n-\frac{5}{2}} \, . \, \frac{D_{-2n+1}(0)}{D_{-2n-3}(0)} \, ,$$

which simplifies to

$$C_5 = b_1 b_3 (n+1) \, . \, (\pi)^{-\frac{1}{2}} \, . \, 2^{n-\frac{1}{2}} \, . \, \Gamma(n+2) \, . \tag{2.60}$$

Using (2.60) in (2.59) and taking $C_6 = 0$, we obtain the general solution of (2.51), subject to the appropriate boundary conditions, as follows:

$$\begin{aligned} f_1 = {} & b_1 b_3 (n+1)(\pi)^{-\frac{1}{2}} \, . \, 2^{n-\frac{1}{2}} \, . \, \Gamma(n+2) e^{-\eta^2/4} D_{-2n-3}(\eta) \\ & -b_1 b_3 (\pi)^{-\frac{1}{2}} \, . \, (n+\tfrac{1}{2}) \, . \, \Gamma(n+1) \, 2^{n-\frac{1}{2}} \, . \, e^{-\eta^2/4} D_{-2n-1}(\eta) \\ & +b_1 b_3 (\pi)^{-\frac{1}{2}} \, . \, \Gamma(n+1) \, . \, 2^{n-\frac{5}{2}} \, . \, e^{-\eta^2/4} D_{-2n+1}(\eta) \, . \end{aligned} \tag{2.61}$$

Differentiating (2.61) with respect to η, and making use of the formula (2.49), we obtain

$$\begin{aligned} f_1' = {} & -b_1 b_3 (n+1)(\pi)^{-\frac{1}{2}} \, . \, 2^{n-\frac{1}{2}} \, . \, \Gamma(n+2) \, e^{-\eta^2/4} D_{-2n-2}(\eta) \\ & + b_1 b_3 (\pi)^{-\frac{1}{2}} \, . \, (n+\tfrac{1}{2}) \, . \, \Gamma(n+1) \, . \, 2^{n-\frac{1}{2}} \, . \, e^{-\eta^2/4} D_{-2n}(\eta) \\ & -b_1 b_3 (\pi)^{-\frac{1}{2}} \, . \, \Gamma(n+1) \, . \, 2^{n-\frac{5}{2}} \, . \, e^{-\eta^2/4} D_{-2n+2}(\eta) \, . \end{aligned} \tag{2.62}$$

Substituting (2.48) and (2.62) into the right hand side of the second of (2.39), the differential equation becomes

$$g_1'' + \eta g_1' - (2n+3)g_1$$

$$= -b_3(\pi)^{-\frac{1}{2}}.\Gamma(n+1).2^{n-\frac{1}{2}}.[4(2n+1)+b_1b_3(n+1)^2]e^{-\eta^2/4}D_{-2n-2}(\eta)$$

$$+b_3(\pi)^{-\frac{1}{2}}.\Gamma(n+1).2^{n-\frac{3}{2}}[8+b_1b_3(2n+1)]e^{-\eta^2/4}D_{-2n}(\eta)$$

$$-b_1b_3^2(\pi)^{-\frac{1}{2}}.\Gamma(n+1).2^{n-\frac{5}{2}}.e^{-\eta^2/4}D_{-2n+2}(\eta). \qquad (2.63)$$

To find the complementary function of the differential equation (2.63), the corresponding homogeneous equation is

$$g_1'' + \eta g_1' - (2n+3)g_1 = 0 . \qquad (2.64)$$

If we let

$$g_1 = G_1 e^{-\eta^2/4}$$

in (2.64), then G_1 satisfies the equation

$$G_1'' + [-(2n+4) + \frac{1}{2} - \frac{\eta^2}{4}]G_1 = 0 . \qquad (2.65)$$

The general solution of (2.65) is given by

$$G_1 = C_7 D_{-2n-4}(\eta) + C_8 D_{2n+3}(\eta i) , \qquad (2.66)$$

where C_7 and C_8 are arbitrary constants of integration. Therefore, the complementary function of the differential equation (2.63) is

$$g_1 = C_7 e^{-\eta^2/4} D_{-2n-4}(\eta) + C_8 e^{-\eta^2/4} D_{2n+3}(\eta i) \quad (2.67)$$

To find the particular integral of the differential equation (2.63), we let

$$g_1 = A_3 e^{-\eta^2/4} D_{-2n-2}(\eta) + A_4 e^{-\eta^2/4} D_{-2n}(\eta)$$

$$+ A_5 e^{-\eta^2/4} D_{-2n+2}(\eta) , \quad (2.68)$$

where A_3, A_4 and A_5 are constants yet to be determined. Substituting (2.68) into (2.63) and making use of the formulae (2.40) and (2.49), we find

$$A_3 = \frac{b_3 \Gamma(n+1) 2^{n-\frac{1}{2}}}{\sqrt{\pi}} [4(2n+1)+b_1 b_3 (n+1)^2]$$

$$A_4 = - \frac{b_3 \Gamma(n+1) 2^{n-\frac{3}{2}}}{3\sqrt{\pi}} [8+b_1 b_3 (2n+1)] \quad (2.69)$$

$$A_5 = \frac{b_1 b_3^2 \Gamma(n+1)}{5\sqrt{\pi}} 2^{n-\frac{5}{2}} .$$

Using (2.69) with (2.68), the particular integral of the differential equation (2.63) becomes

$$g_1 = b_3(\pi)^{-\frac{1}{2}}.\Gamma(n+1).2^{n-\frac{1}{2}}[4(2n+1)+b_1b_3(n+1)^2]e^{-\eta^2/4}.D_{-2n-2}(\eta)$$

$$-\frac{1}{3}b_3(\pi)^{-\frac{1}{2}}.\Gamma(n+1)2^{n-\frac{3}{2}}[8+b_1b_3(2n+1)]e^{-\eta^2/4}D_{-2n}(\eta)$$

$$+\frac{1}{5}b_1b_3^2(\pi)^{-\frac{1}{2}}.\Gamma(n+1).2^{n-\frac{5}{2}}.e^{-\eta^2/4}D_{-2n+2}(\eta) \quad (2.70)$$

The general solution of the differential equation (2.63) which is the sum of the complementary function (2.67) and the particular integral (2.70), is

$$g_1 = C_7e^{-\eta^2/4}D_{-2n-4}(\eta) + C_8e^{-\eta^2/4}D_{2n+3}(\eta i)$$

$$+b_3(\pi)^{-\frac{1}{2}}.\Gamma(n+1).2^{n-\frac{1}{2}}[4(2n+1)+b_1b_3(n+1)^2]e^{-\eta^2/4}D_{-2n-2}(\eta)$$

$$-\frac{1}{3}b_3(\pi)^{-\frac{1}{2}}.\Gamma(n+1)2^{n-\frac{3}{2}}[8+b_1b_3(2n+1)]e^{-\eta^2/4}D_{-2n}(\eta)$$

$$+\frac{1}{5}b_1b_3^2(\pi)^{-\frac{1}{2}}.\Gamma(n+1).2^{n-\frac{5}{2}}.e^{-\eta^2/4}D_{-2n+2}(\eta) . \quad (2.71)$$

If the solution (2.71) is to be bounded at $\eta = \infty$, we must have $C_8 = 0$, and the boundary condition (2.14) at $\eta = 0$, gives

$$C_7 = -b_3(\pi)^{-\frac{1}{2}}\Gamma(n+1)2^{n+\frac{1}{2}}.(n+\frac{3}{2})[4(2n+1)+b_1b_3(n+1)^2]$$

$$+\frac{1}{3}b_3(\pi)^{-\frac{1}{2}}\Gamma(n+1)2^{n+\frac{1}{2}}(n+\frac{3}{2})(n+\frac{1}{2})[8+b_1b_3(2n+1)]$$

$$-\frac{1}{5}b_1b_3^2(\pi)^{-\frac{1}{2}}\Gamma(n+1)2^{n+\frac{1}{2}}(n+\frac{3}{2})(n+\frac{1}{2})(n-\frac{1}{2}) . \quad (2.72)$$

Hence the general solution of the differential equation (2.63), subject to the appropriate boundary conditions, is

$$g_1 = C_7 e^{-\eta^2/4} D_{-2n-4}(\eta)$$

$$+b_3(\pi)^{-\frac{1}{2}}\Gamma(n+1)2^{n-\frac{1}{2}}[4(2n+1)+b_1b_3(n+1)^2]e^{-\eta^2/4} D_{-2n-2}(\eta)$$

$$-\frac{1}{3}b_3(\pi)^{-\frac{1}{2}}\Gamma(n+1)2^{n-\frac{3}{2}}[8+b_1b_3(2n+1)]e^{-\eta^2/4} D_{-2n}(\eta)$$

$$+\frac{1}{5}b_1b_3^2(\pi)^{-\frac{1}{2}}.\Gamma(n+1).2^{n-\frac{5}{2}}.e^{-\eta^2/4} D_{-2n+2}(\eta) , \qquad (2.73)$$

where C_7 is given by (2.72).

The general solutions for f_2, g_2 and higher order solutions may be found in a similar manner, although the computation becomes increasingly lengthy.

For $\kappa = 0$ and the vanishing micro-rotation, our solution (2.7) reduces to the classical form

$$u(y,t) = A\, t^n\, \Gamma(n+1)(\pi)^{-\frac{1}{2}}\, 2^{n+\frac{1}{2}}\, e^{-\eta^2/4}\, D_{-2n-1}(\eta) \qquad (2.74)$$

for a Newtonian fluid.

Using (2.1) in the constitutive equations (1.4) and (1.5), the stresses and couple stresses are found to be

$$t_{xx} = t_{yy} = t_{zz} = -p \, ,$$

$$t_{xy} = \mu \frac{\partial u}{\partial y} - \kappa\phi \, ,$$

$$t_{yx} = (\mu+\kappa)\frac{\partial u}{\partial y} + \kappa\phi \, ,$$

$$t_{xz} = t_{zx} = t_{yz} = t_{zy} = 0 \, , \tag{2.75}$$

$$m_{xx} = m_{yy} = m_{zz} = m_{xy} = m_{yx} = m_{xz} = m_{zx} = 0 \, ,$$

$$m_{zy} = \beta \frac{\partial \phi}{\partial y} \, ,$$

$$m_{yz} = \gamma \frac{\partial \phi}{\partial y} \, .$$

For $\kappa = 0$, we have the classical value

$$t_{xy} = t_{yx} = \mu \frac{\partial u}{\partial y} \, .$$

Substituting (2.7) in (2.75), we find that the non-zero components of stresses and couple stresses are given by

$$t_{xy} = \mu . \frac{At^{n-\frac{1}{2}}}{\sqrt{2k}} [f_0'(\eta)+\epsilon f_1'(\eta)+\ldots]-\kappa Bt^{n+\frac{1}{2}}[g_0(\eta)+ g_1(\eta)+\ldots] \, .$$

$$t_{yx} = (\mu+\kappa)\frac{At^{n-\frac{1}{2}}}{\sqrt{2k}}[f_0'(\eta)+\epsilon f_1'(\eta)+\ldots]+\kappa Bt^{n+\frac{1}{2}}[g_0(\eta)+\epsilon g_1(\eta)+\ldots].$$

$$m_{zy} = \beta \frac{Bt^n}{\sqrt{2k}} [g_0'(\eta)+\epsilon g_1'(\eta)+\ldots] \tag{2.76}$$

$$m_{yz} = \gamma \frac{Bt^n}{\sqrt{2k}} [g_0'(\eta)+ g_1'(\eta)+\ldots]$$

Therefore, the non-zero components of stresses and couple stresses on $y = 0$ are found as follows:

$$t_{xy} = \mu\frac{At^{n-\frac{1}{2}}}{\sqrt{2k}}\,[f_0'(0)+\epsilon f_1'(0)+\ldots]$$

$$-\kappa Bt^{n+\frac{1}{2}}[g_0(0)+\epsilon g_1(0)+\ldots] \ ,$$

$$t_{yx} = (\mu+\kappa)\,\frac{At^{n-\frac{1}{2}}}{\sqrt{2k}}\,[f_0'(0)+\epsilon f_1'(0)+\ldots] \tag{2.77}$$

$$+\kappa Bt^{n+\frac{1}{2}}[g_0(0)+\epsilon g_1(0)+\ldots] \ ,$$

$$m_{zy} = \beta\frac{\bar{B}t^{n}}{\sqrt{2k}}[g_0'+\epsilon g_1'(0)+\ldots] \ ,$$

$$m_{yz} = \gamma\frac{Bt^{n}}{\sqrt{2k}}[g_0'(0)+\epsilon g_1'(0)+\ldots] \ , \tag{2.78}$$

where $f_0'(0)$, $f_1'(0)$, $g_0(0)$, $g_1(0)$, $g_0'(0)$ and $g_1'(0)$ in (2.77) and (2.78) are given by

$$f_0'(0) = (-1)\ \Gamma(n+1)\ (\pi)^{-\frac{1}{2}}\ 2^{n+\frac{1}{2}}\ D_{-2n}(0)$$

$$= (-1)\sqrt{2}\ \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \tag{2.79}$$

$$f_1'(0) = -b_1 b_3 (n+1).(\pi)^{-\frac{1}{2}}.2^{n-\frac{1}{2}}\Gamma(n+2)\; D_{-2n-2}(0)$$

$$+b_1 b_3 (\pi)^{-\frac{1}{2}}(n+\tfrac{1}{2})\Gamma(n+1)2^{n-\frac{1}{2}} D_{-2n}(0)$$

$$-b_1 b_3 (\pi)^{-\frac{1}{2}}\Gamma(n+1)2^{n-\frac{5}{2}} D_{-2n+2}(0)$$

$$= -b_1 b_3 (n+1).2^{-\frac{3}{2}}.\frac{\Gamma(n+2)}{\Gamma(n+\frac{3}{2})} + b_1 b_3 (n+\tfrac{1}{2})2^{-\frac{1}{2}}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}$$

$$-b_1 b_3\; 2^{-\frac{3}{2}}\frac{\Gamma(n+1)}{\Gamma(n-\frac{1}{2})}. \qquad (2.80)$$

$$g_0(0) = -b_3 (\pi)^{-\frac{1}{2}}\;\Gamma(n+1).2^{n+\frac{1}{2}}(n+\tfrac{1}{2})\; D_{-2n-2}(0)$$

$$+b_3 (\pi)^{-\frac{1}{2}}\;\Gamma(n+1).2^{n-\frac{1}{2}}\; D_{-2n}(0)$$

$$= -b_3\; 2^{-\frac{1}{2}}\;(n+\tfrac{1}{2})\; . \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} + b_3\; 2^{-\frac{1}{2}}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \qquad (2.81)$$

$$g_1(0) = C_7\; D_{-2n-4}(0)+b_3(\pi)^{-\frac{1}{2}}\Gamma(n+1).2^{n-\frac{1}{2}}[4(2n+1)+b_1 b_3 (n+1)^2]$$

$$D_{-2n-2}(0)-\tfrac{1}{3}b_3(\pi)^{-\frac{1}{2}}\Gamma(n+1)2^{n-\frac{3}{2}}[8+b_1 b_3(2n+1)]D_{-2n}(0)$$

$$+\tfrac{1}{5}b_1 b_3^2(\pi)^{-\frac{1}{2}}\Gamma(n+1)2^{n-\frac{5}{2}}D_{-2n-2}(0)$$

which simplifies to

$$g_1(0) \doteq C_7 \frac{\sqrt{\pi}}{\Gamma(n+\frac{5}{2})2^{n+2}} + b_3 2^{-\frac{3}{2}} \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} [4(2n+1)+b_1 b_3 (n+1)^2]$$

$$-\frac{1}{3} b_3 \, 2^{-\frac{3}{2}} [8+b_1 b_3 (2n+1)] \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}$$

$$+\frac{1}{5} b_1 b_3^2 \, 2^{-\frac{3}{2}} \cdot \frac{\Gamma(n+1)}{\Gamma(n-\frac{1}{2})} . \tag{2.82}$$

where C_7 is given in (2.72).

$$g_0'(0) = b_3(\pi)^{-\frac{1}{2}} \Gamma(n+1) . 2^{n+\frac{1}{2}} (n+\frac{1}{2}) D_{-2n-1}(0)$$

$$-b_3(\pi)^{-\frac{1}{2}} \Gamma(n+1) 2^{n-\frac{1}{2}} D_{-2n+1}(0)$$

$$= \frac{1}{2} b_3 \tag{2.83}$$

$$g_1'(0) = -C_7 \, D_{-2n-3}(0)$$

$$-b_3(\pi)^{-\frac{1}{2}} \Gamma(n+1) 2^{n-\frac{1}{2}} [4(2n+1)+b_1 b_3 (n+1)^2] D_{-2n-1}(0)$$

$$+\frac{1}{3} b_3(\pi)^{-\frac{1}{2}} \Gamma(n+1) . 2^{n-\frac{3}{2}} [8+b_1 b_3 (2n+1)] D_{-2n+1}(0)$$

$$-\frac{1}{5} b_1 b_3^2 (\pi)^{-\frac{1}{2}} \Gamma(n+1) 2^{n-\frac{5}{2}} D_{-2n+3}(0)$$

$$= -C_7 \frac{\sqrt{\pi}}{\Gamma(n+2) . 2^{n+3/2}} - \frac{b_3}{2} [4(2n+1)+b_1 b_3 (n+1)^2]$$

$$+\frac{1}{6} b_3 n [8+b_1 b_3 (2n+1)] - \frac{1}{10} b_1 b_3^2 n(n-1) . \tag{2.84}$$

Section 2: Flow near a uniformly accelerated flat plate:

In this section, we consider the unsteady, laminar and incompressible flow of a micropolar fluid near a uniformly accelerated infinite flate plate initially at rest, under the same assumptions as used in Section 1 of this chapter. Here for $t > 0$; the plate is assumed to have velocity $u = At$.

The particular case $n = 1$ in Section 1 of this chapter, corresponds to a fluid motion caused by a uniformly accelerated flat plate.

The governing partial differential equations are the same as the equations (2.2) and (2.3) obtained in Section 1 of this chapter. The equations (2.2) and (2.3) are to be solved subject to the following initial and boundary conditions:

$$\begin{aligned} &y \geq 0 , \quad t \leq 0 ; \quad u = \phi = 0 , \\ &y = 0 , \quad t > 0 ; \quad u = At , \quad \phi = 0 , \\ &y \to \infty , \quad t > 0 ; \quad u \to 0 , \quad \phi \to 0 . \end{aligned} \tag{2.85}$$

To solve the partial differential equations (2.2) and (2.3), subject to the conditions (2.74), we let

$$\begin{aligned} u &= At[f_0(\eta) + \varepsilon f_1(\eta) + \varepsilon^2 f_2(\eta) + \ldots] \\ \phi &= Bt^{\frac{3}{2}}[g_0(\eta) + \varepsilon g_1(\eta) + \varepsilon^2 g_2(\eta) + \ldots] , \end{aligned} \tag{2.86}$$

where ε and η are the same as in Section 1 of this chapter, and in this case A and B are constants having the dimensions L^1T^{-2} and $T^{-5/2}$ respectively. Substituting (2.75) into (2.2) and (2.3), and then equating the coefficients of like powers of ε, we obtain two systems of ordinary differential equations as follows:

$$\begin{aligned} f_0'' + \eta f_0' - 2f_0 &= 0 \\ f_1'' + \eta f_1' - 4f_1 &= b_1 g_0' \\ f_2'' + \eta f_2' - 6f_2 &= b_1 g_1' \end{aligned} \tag{2.87}$$

and

$$\begin{aligned} g_0'' + \eta b_2 g_0' - 3b_2 g_0 &= b_3 f_0' \\ g_1'' + b_2 \eta g_1' - 5b_2 g_1 &= 4b_2 g_0 + b_3 f_1' \\ g_2'' + b_2 \eta g_2' - 7b_2 g_2 &= 4b_2 g_1 + b_3 f_2' \end{aligned} \tag{2.88}$$

where b_1, b_2 and b_3 are given by (2.9) and (2.12). In this case, the boundary conditions (2.85) in the new variables become the same as the conditions (2.14) and (2.15). As we have seen in Section 1 of this chapter, that the computation becomes increasingly lengthy due to the occurrence of parabolic cylinder functions of different arguments; hence we consider the particular case $b_2 = 1$, that is $(\mu+\kappa)j = \gamma$, so that the equations (2.88) reduce to

$$g_0'' + \eta g_0' - 3g_0 = b_3 f_0'$$

$$g_1'' + \eta g_1' - 5g_1 = 4g_0 + b_3 f_1' \qquad (2.89)$$

$$g_2'' + \eta g_2' - 7g_2 = 4g_1 + b_3 f_2' .$$

The general solutions of the differential equations (2.87) and (2.89) subject to the boundary conditions (2.14) and (2.15), for f_0, g_0, f_1 and g_1 are found by putting $n = 1$ in (2.24), (2.48), (2.58) and (2.73) as follows:

$$f_0 = 2^{\frac{3}{2}} (\pi)^{-\frac{1}{2}} e^{-\eta^2/4} D_{-3}(\eta) \qquad (2.90)$$

$$g_0 = -3b_3(\pi)^{-\frac{1}{2}} 2^{\frac{1}{2}} e^{-\eta^2/4} D_{-4}(\eta)$$

$$+b_3(\pi)^{-\frac{1}{2}} 2^{\frac{1}{2}} e^{-\eta^2/4} D_{-2}(\eta) \qquad (2.91)$$

$$f_1 = b_1 b_3(\pi)^{-\frac{1}{2}} 2^{\frac{5}{2}} e^{-\eta^2/4} D_{-5}(\eta) - 3b_1 b_3(\pi)^{-\frac{1}{2}} 2^{-\frac{1}{2}} e^{-\eta^2/4} D_{-3}(\eta)$$

$$+b_1 b_3(\pi)^{-\frac{1}{2}} 2^{-\frac{3}{2}} e^{-\eta^2/4} D_{-1}(\eta) \qquad (2.92)$$

and

$$G_1 = C_7\, e^{-\eta^2/4} D_{-6}(\eta) + (12+4b_1b_3)b_3(\pi)^{-\frac{1}{2}}2^{-\frac{1}{2}}e^{-\eta^2/4} D_{-4}(\eta)$$

$$-(8+3b_1b_3)b_3(\pi)^{-\frac{1}{2}}.2^{-\frac{5}{2}}e^{-\eta^2/4} D_{-2}(\eta)$$

$$+\frac{1}{3}b_1b_3^2(\pi)^{-\frac{1}{2}}2^{-\frac{5}{2}}e^{-\eta^2/4} D_0(\eta) \qquad (2.93)$$

where C_7 , in this case, is given by

$$C_7 = -5b_3(\pi)^{-\frac{1}{2}}.2^{\frac{5}{2}}(4+b_1b_3)+5b_3(\pi)^{-\frac{1}{2}}.2^{-\frac{1}{2}}(8+3b_1b_3)$$

$$-3b_1b_3^2(\pi)^{-\frac{1}{2}}2^{-\frac{3}{2}} . \qquad (2.94)$$

For $\kappa = 0$ and the vanishing of micro-rotation, our solution (2.86) reduces to

$$u(y,t) = At \;.\; 2^{\frac{3}{2}} \;.\; (\pi)^{-\frac{1}{2}} . e^{-\eta^2/4} D_{-3}(\eta) . \qquad (2.95)$$

Section 3: Flow near a suddenly accelerated flat plate.

Kline and Allen (1970) studied the flow when the flat plate is accelerated from rest, obtaining a solution in closed form when the gradient of the micro-rotation vanishes on the plate and the transform of the solution when the micro-rotation itself vanishes on the plate.

Chawla (1972) studied the boundary layer growth when the flat plate is accelerated from rest impulsively in its own plane.

In this section, our object is to find the solution of the suddenly accelerated flat plate problem as a special case of the most general problem of the accelerated flat plate studied in Section 1 of this chapter.

The particular case $n = 0$ in Section 1 of this chapter corresponds to fluid motion caused by a suddenly accelerated flat plate. The governing partial differential equations are the same as equations (2.2) and (2.3) obtained in Section 1 of this chapter.

The initial and boundary conditions are:

$$\begin{aligned} &y \geq 0 \ , \ t \leq 0 \ ; \ u = \phi = 0 \\ &y = 0 \ , \ t > 0 \ ; \ u = A \ , \ \phi = 0 \\ &y = \infty \ , \ t > 0 \ ; \ u = 0 \ , \ \phi = 0 \ . \end{aligned} \tag{2.96}$$

To solve the differential equations (2.2) and (2.3) subject to the conditions (2.96), we let

$$u = A[f_0(\eta)+\epsilon f_1(\eta)+\epsilon^2 f_2(\eta)+\dots]$$

$$\phi = Bt^{\frac{1}{2}}[g_0(\eta)+\epsilon g_1(\eta)+\epsilon^2 g_2(\eta)+\dots] , \tag{2.97}$$

where ϵ and η are the same as in Section 1 of this chapter. Substituting the relations (2.97) into equations (2.2) and (2.3) and equating the coefficients of like powers of ϵ, we obtain two systems of ordinary differential equations, whose form is simplified if we set $(\mu+\kappa)j = \gamma$ as follows:

$$\begin{aligned} f_0'' + \eta f_0' &= 0 \\ f_1'' + \eta f_1' - 2f_1 &= b_1 g_0' \\ f_2'' + \eta f_2' - 4f_2 &= b_1 g_1' \end{aligned} \tag{2.98}$$

and

$$\begin{aligned} g_0'' + \eta g_0' - g_0 &= b_3 f_0' \\ g_1'' + \eta g_1' - 3g_1 &= 4g_0 + b_3 g_1' \\ g_2'' + \eta g_2' - 5g_2 &= 4g_1 + b_3 f_2' \end{aligned} \tag{2.99}$$

where b_1 and b_3 are given by (2.9) and (2.12) respectively.

In this case also the boundary conditions (2.96) in the new variables become the same as conditions (2.14) and (2.15). The general solutions of the differential equations (2.98) and (2.99), subject to the boundary conditions (2.14) and (2.15), for f_0, g_0, f_1, and g_1 are found by putting $n = 0$ in (2.24), (2.48), (2.58) and (2.73) as follows:

$$\begin{aligned} f_0 &= \sqrt{\frac{2}{\pi}}\; e^{-\eta^2/4}\, D_{-1}(\eta) \\ &= 1 - \operatorname{erf} \eta \\ &= \operatorname{erfc} \eta \end{aligned} \tag{2.100}$$

where erf η is the error funotion and erfc η is the complementary error function. The error and complementary error functions are defined as

$$\begin{aligned} \operatorname{erf} \eta &= \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-t^2}\, dt \\ \operatorname{erfc} \eta &= \frac{2}{\sqrt{\pi}} \int_{\eta}^{\infty} e^{-t^2}\, dt \,. \end{aligned} \tag{2.101}$$

$$g_0 = -b_3(\pi)^{-\frac{1}{2}}2^{-\frac{1}{2}}e^{-\eta^2/4}D_{-2}(\eta)+b_3(\pi)^{-\frac{1}{2}}2^{-\frac{1}{2}}e^{-\eta^2/4}D_0(\eta) \tag{2.102}$$

$$\begin{aligned} f_1 = {} & b_1b_3(\pi)^{-\frac{1}{2}}2^{-\frac{1}{2}}e^{-\eta^2/4}D_{-3}(\eta)-b_1b_3(\pi)^{-\frac{1}{2}}2^{-\frac{3}{2}}e^{-\eta^2/4}D_{-1}(\eta) \\ & +b_1b_3(\pi)^{-\frac{1}{2}}2^{-\frac{5}{2}}e^{-\eta^2/4}D_1(\eta) \end{aligned} \tag{2.103}$$

and

$$g_1 = C_7 e^{-\eta^2/4} D_{-4}(\eta) + (4+b_1b_3)b_3(\pi)^{-\frac{1}{2}} 2^{-\frac{3}{2}} e^{-\eta^2/4} D_{-2}(\eta)$$

$$-(8+b_1b_3)b_3(\pi)^{-\frac{1}{2}} 2^{-\frac{7}{2}} e^{-\eta^2/4} D_0(\eta)$$

$$+\frac{1}{3}b_1b_3^2(\pi)^{-\frac{1}{2}} 2^{-\frac{7}{2}} e^{-\eta^2/4} D_2(\eta) , \qquad (2.104)$$

where in this case

$$C_7 = -3(4+b_1b_3)b_3(\pi)^{-\frac{1}{2}}.2^{-\frac{1}{2}} + b_3(\pi)^{-\frac{1}{2}}.2^{-\frac{3}{2}}.(8+b_1b_3)$$

$$+\frac{3}{5}b_1b_3^2(\pi)^{-\frac{1}{2}} 2^{-\frac{5}{2}} . \qquad (2.105)$$

For $\kappa = 0$ and vanishing micro-rotation or spin, our solution (2.97) reduces to the classical form

$$u = A\sqrt{\frac{2}{\pi}}\, e^{-\eta^2/4} D_{-1}(\eta)$$

$$= A(1 - \mathrm{erf}\ \eta) , \qquad (2.106)$$

$$= A\ \mathrm{erfc}\ \eta$$

for a Newtonian fluid.

Section 4. Numerical results for a suddenly accelerated flat plate problem.

In this section, we give some numerical values for the functions f_0, g_0, f_1 and g_1 which occur in the expressions for the velocity component $u(y,t)$ and the micro-rotation component $\phi(y,t)$, when the flat plate is given sudden acceleration. The general solutions for f_0, g_0, f_1 and g_1 have been obtained, subject to the appropriate boundary conditions, in the preceeding section of this chapter and are given by the equations (2.100), (2.102), (2.103), and (2.104) respectively. However, these functions f_0, g_0, f_1 and g_1 depend upon the parameters b_1 and b_3 defined in (2.9) and (2.12) respectively, and a set of values for them have been obtained by giving some suitable numerical values to these parameters.

We point out that the parabolic cylinder function defined by the equation (2.19) bears the following relationship with the parabolic cylinder function $U(a,\eta)$ as used in Abramowitz and Stegun (1965, p. 687),

$$U(a,\eta) = D_{-a-\frac{1}{2}}(\eta) .$$

We have made use of the tables given by Abramowitz and Stegun (1965, pp. 702-710) for the parabolic cylinder functions, in computing the values of f_0, g_0, f_1 and g_1 for various numerical combinations of the parameters b_1 and b_3 .

Tables 1 to 4 show the different possible variations. The following observations from these tables are in order.

Table 1 shows the variation of f_0 for different values of η. As can be seen from the table f_0 decreases significantly as the value of η increases. We note that b_1 and b_3 do not appear in the expression for f_0.

In the tables 2(a), 2(b) and 2(c), we observe the variation of g_0 for different values of η when the parameter b_3 is small, medium and large. For small values of η, g_0 increases but for large η, g_0 decreases as may be seen from the tables. We note that b_1 does not appear in the expression for g_0 and b_3 appears linearly.

Tables 3(a) through 3(f) show the variation of f_1 for various values of η. It is observed that for small values of η, f_1 decreases; for intermediate values of η, f_1 increases and for large values of η, f_1 decreases significantly. We also note that b_1 and b_3 both appear linearly in a product form in the expression for f_1. Tables 3(a) through 3(f) are computed for small, medium and large values of the parameters b_1 and b_3.

In tables 4(a) through 4(f), the variation of g_1 is observed for different values of η when the parameters b_1 and b_3 are small, medium and large. In the expression for g_1, the parameters b_1 and b_3 do not occur linearly. For small and medium values of the parameters, g_1 decreases as η increases for small values of η and for large η, g_1 increases. It is quite interesting to observe that for large values of the parameters, g_1 increases as η increases, and for large η, g_1 decreases significantly. These observations are quite apparent from the tables 4(a) through 4(f).

Table 1.

η	erf η	f_0
0.0	0.00000	1.00000
0.1	0.11246	0.88754
1.0	0.84270	0.15730
2.0	0.99532	0.00468
5.0	0.9999994267	5.7330745×10^{-7}

Table 2(a)

When $b_3 = 0.1$

η	g_0
0.0	0.00000
0.1	4.6014516×10^{-3}
1.0	1.5865409×10^{-2}
2.0	4.5500388×10^{-3}
5.0	1.4324600×10^{-7}

Table 2(b)

When $b_3 = 1.0$

η	g_0
0.0	0.00000
0.1	4.6014516×10^{-2}
1.0	1.5865409×10^{-1}
2.0	4.5500388×10^{-2}
5.0	1.4324600×10^{-6}

Table 2(c)

When $b_3 = 10.0$

η	g_0
0.0	0.00000
0.1	4.6014516×10^{-1}
1.0	1.5865409
2.0	4.5500388×10^{-1}
5.0	1.4324600×10^{-5}

Table 3(a)

When $b_1 = 0.1$, $b_3 = 0.1$

η	f_1
0.0	0.00000
0.1	$-7.6286281 \times 10^{-5}$
1.0	1.8834425×10^{-4}
2.0	2.8778397×10^{-4}
5.0	1.7251131×10^{-8}

Table 3(b)

When $b_1 = 0.1$, $b_3 = 1.0$

n	f_1
0.0	0.00000
0.1	$-7.6286281 \times 10^{-4}$
1.0	1.8834425×10^{-3}
2.0	2.8778397×10^{-3}
5.0	1.7251131×10^{-7}

Table 3(c)

When $b_1 = 0.1$, $b_3 = 10.0$

n	f_1
0.0	0.00000
0.1	$-7.6286281 \times 10^{-3}$
1.0	1.8834425×10^{-2}
2.0	2.8778397×10^{-2}
5.0	1.7251131×10^{-6}

Table 3(d)

When $b_1 = 1.0$, $b_3 = 1.0$

n	f_1
0.0	0.00000
0.1	$-7.6286281 \times 10^{-3}$
1.0	1.8834425×10^{-2}
2.0	2.8778397×10^{-2}
5.0	1.7251131×10^{-6}

Table 3(e)

When $b_1 = 1.0$, $b_3 = 10.0$

n	f_1
0.0	0.00000
0.1	$-7.6286281 \times 10^{-2}$
1.0	1.8834425×10^{-1}
2.0	2.8778397×10^{-1}
5.0	1.7251131×10^{-5}

Table 3(f)

When $b_1 = 10.0$, $b_3 = 10.0$

η	f_1
0.0	0.00000
0.1	$-7.6286281 \times 10^{-1}$
1.0	1.8834425
2.0	2.8778397
5.0	1.7251131×10^{-4}

Table 4(a)

When $b_1 = 0.1$, $b_3 = 0.1$

η	g_1
0.0	1.4672448×10^{-1}
0.1	$-5.7570650 \times 10^{-2}$
1.0	$-1.9730211 \times 10^{-2}$
2.0	$-4.4314732 \times 10^{-3}$
5.0	$-1.3809440 \times 10^{-7}$

Table 4(b)

When $b_1 = 0.1$, $b_3 = 1.0$

η	g_1
0.0	$-6.828562029 \times 10^{-1}$
0.1	$-5.890977064 \times 10^{-1}$
1.0	-0.199493037
2.0	$-4.405708849 \times 10^{-2}$
5.0	$-1.261079860 \times 10^{-6}$

Table 4(c)

When $b_1 = 0.1$, $b_3 = 10.0$

η	g_1
0.0	-8.444278267
0.1	-7.241126337
1.0	-2.214023224
2.0	$-4.215551618 \times 10^{-1}$
5.0	$-6.982956100 \times 10^{-7}$

Table 4(d)

When $b_1 = 1.0$, $b_3 = 1.0$

η	g_1
0.0	$-8.444278267 \times 10^{-1}$
0.1	-0.722993410
1.0	-0.221402322
2.0	-0.042155516
5.0	$-6.982956100 \times 10^{-8}$

Table 4(e)

When $b_1 = 1.0$, $b_3 = 10.0$

η	g_1
0.0	-24.601440620
0.1	-20.619504530
1.0	- 4.404951749
2.0	0.928208014
5.0	$1.184267343 \times 10^{-4}$

Table 4(f)

When $b_1 = 10.0$, $b_3 = 10.0$

η	ε_1
0.0	-186.173064100
0.1	-154.515209000
1.0	- 26.314237030
2.0	1.670174420
5.0	$1.309677034 \times 10^{-3}$

CHAPTER III

FLOWS WITH SUCTION AND INJECTION

Ariman, Cakmak and Hill (1967) applied Eringen's (1966) theory to the steady flow of micropolar fluids between two concentric cylinders. They investigated Couette and Poiseuille flows.

Verma and Sehgal (1968) studied Couette flow of micropolar fluids which is the flow between two coaxial right circular cylinders of different radii and rotating about the common axis with constant angular velocities. The object of the paper was to investigate the nature of the flow in the annulus. Explicit expressions for fluid velocity, micro-rotation velocity, the stresses, couple stresses, rate of shear and shearing stress differences were obtained.

Ariman and Cakmak (1968) solved problems in some basic viscous flows in micropolar fluids. These are plane Couette and Poiseuille flows between two parallel plates and the problem of a rotating fluid with a free surface.

It is observed that not much attention has been given to problems in micropolar fluids with suction and injection. In this chapter, we have investigated some steady and non-steady flows with suction and injection.

Section 1(a). Steady Flows with Suction and Injection.

Consider steady, laminar and incompressible flow of a micropolar fluid between two infinite parallel plates, a distance h apart, the lower of which is fixed and the upper of which moves with constant speed U in the x-direction. The y-direction is taken perpendicular to the plates and the flow is assumed to be independent of z.

Fluid is injected along the upper plate with constant normal velocity V and sucked out along the lower plate. The material constants of the micropolar fluid are assumed to be independent of position. The body forces and body couples are neglected.

Setting

$$\underset{\sim}{v} = (u(y), -V, 0), \quad \underset{\sim}{\nu} = (0, 0, \phi(y)) \tag{3.1}$$

and ρ = constant, the equation of continuity (1.1) is satisfied identically and the momentum equations (1.2) and (1.3) yield

$$(\mu+\kappa)u'' + \rho V u' + \kappa\phi' = \frac{\partial p}{\partial x}, \tag{3.2}$$

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0, \tag{3.3}$$

$$\gamma\phi'' + \rho j V\phi' - 2\kappa\phi - \kappa u' = 0 \tag{3.4}$$

where the prime denotes differentiation with respect to y.

It is clear from (3.2) and (3.3) that $\frac{\partial p}{\partial x}$ must be constant, hence we write

$$\frac{\partial p}{\partial x} = 2\rho VP \tag{3.5}$$

Section 1 (b). The Nature of the Solution.

If we set

$$u = Ce^{my} \quad , \quad \phi = De^{my} \tag{3.6}$$

in the homogeneous equations obtained from (3.2) and (3.4) by equating the right hand side of (3.2) to zero, we obtain a quartic in m

$$m(m^3 + am^2 + bm + c) = 0 \tag{3.7}$$

where

$$a = \rho V \frac{(\mu+\kappa)j + \gamma}{(\mu+\kappa)\gamma} ,$$

$$b = \frac{\rho^2 j\ V^2}{(\mu+\kappa)\gamma} - \frac{(2\mu+\kappa)\kappa}{(\mu+\kappa)\gamma} , \tag{3.8}$$

$$c = -\rho V \frac{2\kappa}{(\mu+\kappa)\gamma} ,$$

whence the solution is obtained in the form (where the roots are distinct)

$$u = C_0 + C_1 e^{m_1 y} + C_2 e^{m_2 y} + C_3 e^{m_3 y} ,$$

$$\phi = a_1 C_1 e^{m_1 y} + a_2 C_2 e^{m_2 y} + a_3 C_3 e^{m_3 y} \tag{3.9}$$

where $m_0 = 0$, m_1 , m_2 and m_3 are the roots of the equation (3.7) and

$$\frac{D_i}{C_i} = a_i = -\frac{(\mu+\kappa)m_i + \rho V}{\kappa}$$

$$= \frac{\kappa m_i}{\gamma m_i^2 + \rho j m_i V - 2\kappa} \qquad (3.10)$$

$$(i = 1, 2, 3)$$

It is well known from the theory of equations by USPENSKY (1948, p.84) that the non-zero roots of (3.7) are real and distinct, or include a complex pair according as the sign of Δ is negative or positive, where

$$\begin{aligned} 108\Delta &= 4p^3 + 27q^2 \\ 3p &= 3b - a^2 \\ 27q &= 2a^3 - 9ab + 27c \;. \end{aligned} \qquad (3.11)$$

Substituting (3.8) into (3.11) and setting $\overline{V}^2 = \rho^2 \frac{V^2}{\kappa}$,

we obtain

$$\overline{\Delta} = \frac{108\Delta(\mu+\kappa)^4\gamma^4}{\kappa^3} = A\overline{V}^6 + B\overline{V}^4 + C\overline{V}^2 + D \qquad (3.12)$$

where

$$A = -j^2[(\mu+\kappa)j - \gamma]^2 \,,$$

$$B = 2\{6j(\mu+\kappa)\gamma[(\mu+2\kappa)j+3\gamma] - [(2\mu+3\kappa)j+4\gamma][(\mu+\kappa)j+\gamma]^2\} \,,$$

$$C = 12(\mu+\kappa)(\mu+2\kappa)\gamma[3\gamma-j(2\mu+\kappa)] - (2\mu+\kappa)^2([\mu+\kappa]j+\gamma)^2 \,,$$

$$D = -4(2\mu+\kappa)^3(\mu+\kappa)\gamma \,. \qquad (3.13)$$

The coefficients A and D are never positive, hence if $(\mu+\kappa)j \neq \gamma$, $\Delta < 0$ when V^2 is small and when V^2 is large, that is the roots of (3.7) are real and distinct. For intermediate values of V^2, Δ is positive for some values provided that

$$B^2 - 3AC > 0 , \tag{3.14}$$

$$\left(\frac{B^2 - 3AC}{A^2}\right)^{\frac{1}{2}} - \frac{B}{A} > 0 , \tag{3.15}$$

$$\frac{2B^3 - 9ABC + 27A^2D}{A^3} - 2\left(\frac{B^2 - 3AC}{A^2}\right)^{\frac{3}{2}} > 0 , \tag{3.16}$$

inequalities which follow, respectively, from the conditions that the cubic (3.12) in $\overline{V}^2$ should have a maximum and a minimum; that the maximum should occur for a positive value of $\overline{V}^2$; and that the value of $\overline{\Delta}$ at the maximum should be positive. The positive square root of $B^2 - 3AC$ is to be taken in (3.15) and (3.16).

In the particular case $(\mu+\kappa)j = \gamma$, (3.12) reduces to

$$\overline{\Delta} = \frac{4\gamma^2}{\kappa j}\{\overline{v}^4 - (\mu^2 - 8\kappa\mu - 11\kappa^2)\overline{v}^2 - (2\mu+\kappa)^3\kappa\} \tag{3.17}$$

where $\overline{v}^2 = \kappa j \overline{V}^2$, whence it follows that the non-zero roots of (3.7) are real and distinct, or include a complex pair, according as $\overline{v}^2$ is less than, or greater than $\overline{v}_0{}^2$, the positive root of the quadratic in $\bar{v}^2$ obtained by setting (3.17) equal to zero.

Thus it appears that when $(\mu+\kappa)j \neq \gamma$, and the inequalities (3.14 - 16) are satisfied, the transverse components of velocity and spin will contain terms oscillatory in space for a certain range of injection, $V_1^{\ 2} \leq V^2 < V_2^{\ 2}$, but only real exponential terms for the remaining values $0 \leq V^2 \leq V_1^{\ 2}$, $V_2^{\ 2} < V^2$; when $(\mu+\kappa)j = \gamma$, the flow is oscillatory for all rates of injection $V^2 > V_0^{\ 2}$.

The calculation of explicit solutions in the various cases is a matter of straightforward, if lengthy, manipulation. Two solutions will be obtained in the next part of this section.

Section 1(c). Particular Solutions.

(i) Flow due to movement of the upper plate with no pressure gradient.

The boundary conditions are

$$y = 0 \; ; \quad u = 0 \, , \quad \phi = 0$$

$$y = h \, , \quad u = U \, , \quad \phi = 0 \tag{3.18}$$

and $\dfrac{\partial p}{\partial x} = 0$ in (3.2) .

Assuming that the four roots are distinct, and substituting (3.9) into (3.18), we obtain, with the aid of (3.10),

$$C_i = \frac{U a_j a_k (e^{m_k h} - e^{m_j h})}{M} \tag{3.19}$$

where

$$M = \sum_{i=1}^{3} a_j a_k (e^{m_i h} - 1)(e^{m_k h} - e^{m_j h})$$

$$C_0 = -(C_1 + C_2 + C_3) \tag{3.20}$$

and where ijk is a cyclic permutation of 123 .

The tangential stress and couple stress across a plane y = constant are given by

$$t_x = (\mu+\kappa)u' + \kappa\phi$$

$$m_z = \gamma\phi' . \tag{3.21}$$

The corresponding solution for a Newtonian fluid is

$$u = \frac{U(1 - e^{my})}{(1 - e^{mh})} \quad \text{where} \quad m = -\frac{\rho V}{\mu} .$$

(ii) Flow due to pressure gradient alone, when the plates remain stationary.

The boundary conditions are

$$\begin{aligned} y = 0 , \quad u = 0 , \quad \phi = 0 , \\ y = h , \quad u = 0 , \quad \phi = 0 . \end{aligned} \tag{3.22}$$

The complete solution in this case is

$$\begin{aligned} u &= C_0 + C_1 e^{m_1 y} + C_2 e^{m_2 y} + C_3 e^{m_3 y} + 2Py \\ \phi &= a_1 C_1 e^{m_1 y} + a_2 C_2 e^{m_2 y} + a_3 C_3 e^{m_3 y} - P \end{aligned} \tag{3.23}$$

where $2\rho VP = \frac{\partial p}{\partial x}$.

Substituting (3.23) into (3.22), and using (3.10) we find that

$$C_i = \frac{P}{M}[2ha_j a_k (e^{m_j h} - e^{m_k h}) + (a_j - a_k)(e^{hm_j} - 1)(e^{hm_k} - 1)] \tag{3.24}$$

where again ijk is a cyclic permutation of 123 , M is given by (3.20), and $C_0 = -C_1 - C_2 - C_3$. The tangential stress and couple stress are obtained from (3.21).

The corresponding solution for a Newtonian fluid is

$$u = 2P\{y - \frac{h(1 - e^{my})}{(1 - e^{mh})}\} , \tag{3.25}$$

where

$$\frac{\partial p}{\partial x} = 2P\rho V \quad \text{and} \quad m = -\rho\frac{V}{\mu} .$$

Section 2: Flow Near an Accelerated Porous Flat Plate With Variable Suction.

We consider unsteady, laminar and incompressible flow of a micropolar fluid near an infinite porous flat plate with variable suction. Let x and y be the co-ordinates along the plate and perpendicular to it respectively and u and v be the corresponding components of velocity. Micro-rotation has only one non-zero component about z-axis and let that be denoted by ϕ.

When $t \leq 0$, the fluid is assumed to be everywhere stationary, and when $t > 0$, the plate is accelerated at the velocity $u = At^n$ (where $n \geq 0$ is an integer and A is a constant). As the plate is infinite in length, all the variables in this problem are functions of y and t only. The material constants of the micropolar fluid are assumed to be independent of position. We neglect body forces and body couples.

Thus, for an infinite porous flat plate, using the above assumptions, the equation of continuity (1.1) and the momentum equations (1.2) and (1.3), provided pressure gradient in x-direction is zero, reduce to

$$\frac{\partial v}{\partial y} = 0 \tag{3.26}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{3.27}$$

$$(\mu+\kappa)\ \frac{\partial^2 u}{\partial y^2} + \kappa\ \frac{\partial \phi}{\partial y} = \rho(\frac{\partial u}{\partial t} + v\ \frac{\partial u}{\partial y}) \tag{3.28}$$

$$\gamma\ \frac{\partial^2 \phi}{\partial y^2} - \kappa\ \frac{\partial u}{\partial y} - 2\kappa\phi = \rho j(\frac{\partial \phi}{\partial t} + v\ \frac{\partial \phi}{\partial y}) . \tag{3.29}$$

Equation (3.26), upon integration, gives

$$v = -V(t) = -V_0\left(\frac{1}{t}\right)^{\frac{1}{2}}\ (\text{say}), \tag{3.30}$$

where $V_0(>0)$ represents the suction.

The reason we choose $v = -V_0\left(\frac{1}{t}\right)^{\frac{1}{2}}$ is to permit similarity solutions to be valid.

Introducing (3.30) into (3.27), (3.28) and (3.29), and then integrating (3.27), we get the following equations:

$$p = p_0 - \frac{V_0}{2}\ \rho y\ t^{-3/2}\ , \tag{3.31}$$

$$(\mu+\kappa)\ \frac{\partial^2 u}{\partial y^2} + \kappa\ \frac{\partial \phi}{\partial y} = \rho(\frac{\partial u}{\partial t} - \frac{V_0}{t^{1/2}}\ \frac{\partial u}{\partial y})\ , \tag{3.32}$$

$$\gamma\ \frac{\partial^2 \phi}{\partial y^2} - \kappa\ \frac{\partial u}{\partial y} - 2\kappa\phi = \rho j(\frac{\partial \phi}{\partial t} - \frac{V_0}{t^{1/2}}\ \frac{\partial \phi}{\partial y})\ , \tag{3.33}$$

where p_0 is the value of p at $y = 0$.

The boundary conditions are

$$\begin{aligned} u(0,\ t) &= At^n\ , \quad \phi(0,\ t) = 0\ , \\ u(\infty,\ t) &= 0\ , \quad \phi(\infty,\ t) = 0\ . \end{aligned} \tag{3.34}$$

A method of perturbation expansion of functions of a dimensionless variable is applied to solve the equations (3.32) and (3.33) and series solutions in terms of parabolic cylinder functions, in the particular case $(\mu+\kappa)j \doteq \gamma$, are found.

For small values of ε, where $\varepsilon = \frac{\kappa t}{\rho j}$, we expand u and ϕ in ascending powers of ε as follows:

$$u = At^{n}[f_0(\eta) + \varepsilon f_1(\eta) + \varepsilon^2 f_2(\eta) + \dots] ,$$

$$\phi = Bt^{n+\frac{1}{2}}[g_0(\eta) + \varepsilon g_1(\eta) + \varepsilon^2 g_2(\eta) + \dots] , \tag{3.35}$$

where $\eta = \frac{y}{\sqrt{2kt}}$, $k = \frac{\mu+\kappa}{\rho}$.

We note that ε and η are both dimensionless quantities. Substituting the relations (3.35) into the equations (3.32) and (3.33) and equating the like powers of ε, we get two systems of ordinary differential equations:

$$f_0'' + (\eta+\eta_0)f_0' - 2nf_0 = 0 \tag{3.36}$$

$$f_1'' + (\eta+\eta_0)f_1' - 2(n+1)f_1 = b_1 g_0' \tag{3.37}$$

$$f_2'' + (\eta+\eta_0)f_2' - 2(n+2)f_2 = b_1 g_1' \tag{3.38}$$

and

$$g_0'' + b_2(\eta+\eta_0)g_0' - b_2(2n+1)g_0 = b_3 f_0'$$

$$g_1'' + b_2(\eta+\eta_0)g_1' - b_2(2n+3)g_1 = b_3 f_1' + 4b_2 g_0 \qquad (3.39)$$

$$g_2'' + b_2(\eta+\eta_0)g_2' - b_2(2n+5)g_2 = b_3 f_2' + 4b_2 g_1 .$$

where the prime denotes differentiation with respect to η , and

$$b_1 = -\frac{2^{\frac{1}{2}} j\; B}{A\; k^{\frac{1}{2}}} \;, \quad b_2 = k\,\frac{\rho j}{\gamma} \;, \quad b_3 = \frac{\kappa}{\gamma}\sqrt{2k}\;\frac{A}{B} \;,$$

$$\eta_0 = V_0 \left(\frac{2}{k}\right)^{\frac{1}{2}} . \qquad (3.40)$$

The boundary conditions (3.34) in the new variables become

$$f_0(0) = 1 \;, \quad g_0(0) = 0$$

$$f_k(0) = g_k(0) = 0 \;, \quad k = 1, 2, 3, \ldots \qquad (3.41)$$

$$f_k(\infty) = g_k(\infty) = 0 \;, \quad k = 0, 1, 2, \ldots$$

We have seen that the computation would become increasingly lengthy due to the occurrence of parabolic cylinder functions with arguments $(\eta+\eta_0)$ and $(\eta+\eta_0)\sqrt{b_2}$. However, considerable simplification is introduced if we consider the particular case $b_2 = 1$, that is $(\mu+\kappa)j = \gamma$. Therefore, the system of equations (3.39) reduce to

$$g_0'' + (\eta+\eta_0)g_0' - (2n+1)g_0 = b_3 f_0' \tag{3.42}$$

$$g_1'' + (\eta+\eta_0)g_1' - (2n+3)g_1 = b_3 f_1' + 4g_0 \tag{3.43}$$

$$g_2'' + (\eta+\eta_0)g_2' - (2n+5)g_2 = b_3 f_2' + 4g_1 \ . \tag{3.44}$$

If we substitute

$$f_0(\eta) = F_0(\eta)\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \tag{3.45}$$

into equation (3.36), then F_0 satisfies the equation

$$F_0'' + [-2n - \frac{1}{2} - \frac{1}{4}(\eta+\eta_0)^2]F_0 = 0 \ . \tag{3.46}$$

Writing the above equation in the standard form of Weber's differential equation, we have

$$F_0'' + [-(2n+1) + \frac{1}{2} - \frac{1}{4}(\eta+\eta_0)^2]F_0 = 0 \ . \tag{3.47}$$

The general solution of (3.47) is given by

$$F_0 = C_1\, D_{-2n-1}(\eta+\eta_0) + C_2\, D_{2n}((\eta+\eta_0)i) \ , \tag{3.48}$$

where C_1 and C_2 are arbitrary constants, and $D_{-2n-1}(\eta+\eta_0)$ and $D_{2n}((\eta+\eta_0)i)$ are the parabolic cylinder functions. Using (3.48) with (3.45), the general solution of the differential equation (3.36) is

$$f_0 = C_1 e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-1}(\eta+\eta_0)$$

$$+ C_2 e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{2n}((\eta+\eta_0)i) \quad . \tag{3.49}$$

If the solution (3.49) is to be bounded at $\eta = \infty$, we must have $C_2 = 0$. Therefore, we have

$$f_0 = C_1 e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-1}(\eta+\eta_0) \quad . \tag{3.50}$$

The boundary condition at $\eta = 0$ gives

$$C_1 = \frac{1}{D_{-2n-1}(\eta_0)} \quad . \tag{3.51}$$

Using (3.51) in (3.50), we find that

$$f_0 = e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \cdot \frac{D_{-2n-1}(\eta+\eta_0)}{D_{-2n-1}(\eta_0)} \quad . \tag{3.52}$$

Differentiating (3.52) with respect to η, and making use of the formula (2.26), we get

$$f_0' = (-1) e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \frac{D_{-2n}(\eta+\eta_0)}{D_{-2n-1}(\eta_0)} \quad . \tag{3.53}$$

Substituting (3.53) into (3.42), the differential equation becomes

$$g_0'' + (\eta+\eta_0)g_0' - (2n+1)g_0 = (-1)b_3\, e^{-\frac{1}{4}(\eta+\eta_0)^2+\frac{\eta_0^2}{4}}\, \frac{D_{-2n}(\eta+\eta_0)}{D_{-2n-1}(\eta_0)} . \tag{3.54}$$

To find the complementary function of the differential equation (3.54), the corresponding homogeneous equation is

$$g_0'' + (\eta+\eta_0)g_0' - (2n+1)g_0 = 0 . \tag{3.55}$$

If we substitute

$$g_0 = G_0\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \tag{3.56}$$

into (3.55), then G_0 satisfies the equation

$$G_0'' + [-(2n+1) - \frac{1}{2} - \frac{1}{4}(\eta+\eta_0)^2]G_0 = 0 . \tag{3.57}$$

Writing the above equation in the standard form of Weber's differential equation, we have

$$G_0'' + [-(2n+2) + \frac{1}{2} - \frac{1}{4}(\eta+\eta_0)^2]G_0 = 0 . \tag{3.58}$$

The general solution of (3.58) is

$$G_0 = C_3\, D_{-2n-2}(\eta+\eta_0) + C_4\, D_{2n+1}((\eta+\eta_0)i) , \tag{3.59}$$

where C_3 and C_4 are arbitrary constants.

Substituting (3.59) into (3.56), we find that the complementary function of the differential equation (3.54) is

$$g_0 = C_3\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-2}(\eta+\eta_0)$$

$$+ C_4\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{2n+1}((\eta+\eta_0)i) \quad (3.60)$$

To find the particular integral of (3.54), we let

$$g_0 = A\, e^{-\frac{1}{4}(\eta+\eta_0)^2} D_{-2n}(\eta+\eta_0) , \quad (3.61)$$

where A is an unknown constant to be determined. Substituting (3.61) into the left hand side of (3.54) and making use of the formulae (2.40) and (2.49), we find that

$$g_0'' + (\eta+\eta_0)g_0' - (2n+1)g_0 = -2A\, e^{-\frac{1}{4}(\eta+\eta_0)^2} D_{-2n}(\eta+\eta_0) . \quad (3.62)$$

Comparing the coefficients in the right hand sides of the equations (3.54) and (3.62), we have

$$A = \frac{b_3}{2} \frac{e^{\eta_0^2/4}}{D_{-2n-1}(\eta_0)} \quad (3.63)$$

Hence, the particular integral becomes

$$g_0 = \frac{b_3}{2} \frac{D_{-2n}(\eta+\eta_0)}{D_{-2n-1}(\eta_0)} e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \quad (3.64)$$

Therefore, the general solution of (3.42) or (3.54), which is the sum of the complementary function (3.60) and the particular integral (3.64), is

$$g_0 = C_3\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-2}(\eta+\eta_0)$$

$$+ C_4\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{2n+1}((\eta+\eta_0)i)$$

$$+ \frac{b_3}{2}\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \frac{D_{-2n}(\eta+\eta_0)}{D_{-2n-1}(\eta_0)} . \tag{3.65}$$

If the solution (3.65) is to be bounded at $\eta = \infty$, we must have $C_4 = 0$, and thus

$$g_0 = C_3\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-2}(\eta+\eta_0)$$

$$+ \frac{1}{2} b_3 \frac{D_{-2n}(\eta+\eta_0)}{D_{-2n-1}(\eta_0)}\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} . \tag{3.66}$$

The boundary condition at $\eta = 0$, gives

$$C_3 = -\frac{b_3}{2} \frac{D_{-2n}(\eta_0)}{D_{-2n-1}(\eta_0) D_{-2n-2}(\eta_0)} . \tag{3.67}$$

Differentiating (3.66) with respect to η and using the formula (2.49), we find that

$$g_0' = -C_3\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-1}(\eta+\eta_0)$$

$$- \frac{b_3}{2}\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \frac{D_{-2n+1}(\eta+\eta_0)}{D_{-2n-1}(\eta_0)} . \tag{3.68}$$

Substituting (3.68) into (3.37), the differential equation becomes

$$f_1'' + (\eta+\eta_0)f_1' - 2(n+1)f_1 = -b_1 C_3\, e^{-\frac{1}{4}(\eta+\eta_0)^2+\frac{\eta_0^2}{4}} D_{-2n-1}(\eta+\eta_0)$$

$$- \frac{b_1}{2} b_3\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \frac{D_{-2n+1}(\eta+\eta_0)}{D_{-2n-1}(\eta_0)} \tag{3.69}$$

The homogeneous differential equation of (3.69) is

$$f_1'' + (\eta+\eta_0)f_1' - 2(n+1)f_1 = 0 \ . \tag{3.70}$$

If we substitute

$$f_1 = F_1\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} , \tag{3.71}$$

into (3.70), then F_1 satisfies the equation

$$F_1'' + [-2(n+1) - \frac{1}{2} - \frac{1}{4}(\eta+\eta_0)^2)F_1 = 0 , \tag{3.72}$$

which may be written in the form

$$F_1'' + [-(2n+3) + \frac{1}{2} - \frac{1}{4}(\eta+\eta_0)^2]F_1 = 0 \ . \tag{3.73}$$

The general solution of (3.73) is

$$F_1 = C_5\, D_{-2n-3}(\eta+\eta_0) + C_6\, D_{2n+2}((\eta+\eta_0)i) , \tag{3.74}$$

where C_5 and C_6 are arbitrary constants of integration. Substituting (3.74) into (3.71), we find that the complementary function of the differential equation (3.69) is

$$f_1 = C_5\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-3}(\eta+\eta_0)$$

$$+ C_6\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{2n+2}((\eta+\eta_0)i) . \qquad (3.75)$$

In order to find the particular integral of the differential equation (3.69), we let

$$f_1 = A_1\, e^{-\frac{1}{4}(\eta+\eta_0)^2} D_{-2n-1}(\eta+\eta_0)$$

$$+ A_2\, e^{-\frac{1}{4}(\eta+\eta_0)^2} D_{-2n+1}(\eta+\eta_0) , \qquad (3.76)$$

where A_1 and A_2 are unknown constants yet to be determined.

Substituting (3.76) into the left hand side of (3.69) and using the formula (2.40) and (2.49), we obtain

$$f_1'' + (\eta+\eta_0)f_1' - 2(n+1)f_1 = -2A_1\, e^{-\frac{1}{4}(\eta+\eta_0)^2} D_{-2n-1}(\eta+\eta_0)$$

$$-4A_2\, e^{-\frac{1}{4}(\eta+\eta_0)^2} D_{-2n+1}(\eta+\eta_0) .$$

$$(3.77)$$

Comparing the right hand sides of equations (3.69) and (3.77), we find that

$$A_1 = \frac{b_1}{2} C_3 e^{\frac{1}{4}\eta_0^2} ,$$

$$A_2 = \frac{b_1 b_3}{8} \frac{e^{\frac{1}{4}\eta_0^2}}{D_{-2n-1}(\eta_0)} . \tag{3.78}$$

Hence by (3.78), the particular integral (3.76) becomes

$$f_1 = \frac{b_1}{2} C_3 e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-1}(\eta+\eta_0)$$

$$+ \frac{b_1 b_3}{8} e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \frac{D_{-2n+1}(\eta+\eta_0)}{D_{-2n-1}(\eta_0)} . \tag{3.79}$$

The general solution of (3.77) or (3.69), which is the sum of the complementary function (3.75) and the particular integral (3.79), is

$$f_1 = C_5 e^{-\frac{1}{4}(\eta+\eta_0)^2+\frac{\eta_0^2}{4}} D_{-2n-3}(\eta+\eta_0)$$

$$+ C_6 e^{-\frac{1}{4}(\eta+\eta_0)^2+\frac{\eta_0^2}{4}} D_{2n+2}((\eta+\eta_0)i)$$

$$+ \frac{b_1}{2} C_3 e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-1}(\eta+\eta_0)$$

$$+ \frac{b_1 b_3}{8} e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \frac{D_{-2n+1}(\eta+\eta_0)}{D_{-2n-1}(\eta_0)} . \tag{3.80}$$

To find C_5 and C_6, we apply the boundary conditions (3.41), and if the solution (3.80) is to be bounded at $\eta = \infty$, we must have $C_6 = 0$ and then

$$f_1 = C_5\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-3}(\eta+\eta_0)$$

$$+ \frac{b_1}{2} C_3\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-1}(\eta+\eta_0)$$

$$+ \frac{b_1 b_3}{8}\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \frac{D_{-2n+1}(\eta+\eta_0)}{D_{-2n-1}(\eta_0)} . \qquad (3.81)$$

The boundary condition at $\eta = 0$ gives

$$C_5 = \frac{1}{D_{-2n-3}(\eta_0)} \left[- \frac{b_1 C_3}{2} D_{-2n-1}(\eta_0) - \frac{b_1 b_3}{8} \frac{D_{-2n+1}(\eta_0)}{D_{-2n-1}(\eta_0)}\right] . \qquad (3.82)$$

Using (3.67) in (3.82) we find that

$$C_5 = \frac{b_1 b_3}{8\, D_{-2n-3}(\eta_0)} \left[2 \frac{D_{-2n}(\eta_0)}{D_{-2n-2}(\eta_0)} - \frac{D_{-2n+1}(\eta_0)}{D_{-2n-1}(\eta_0)}\right] . \qquad (3.83)$$

Differentiating (3.81) with respect to η, and making use of the formulae (2.40) and (2.49), we obtain

$$f_1' = -C_5\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}}\, D_{-2n-2}(\eta+\eta_0)$$

$$- \frac{b_1}{2} C_3\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}}\, D_{-2n}(\eta+\eta_0)$$

$$- \frac{b_1}{8} b_3\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}}\, \frac{D_{-2n+2}(\eta+\eta_0)}{D_{-2n-1}(\eta_0)} \, . \qquad (3.84)$$

Substituting (3.66) and (3.84) into (3.43), the differential equation becomes

$$g_1'' + (\eta+\eta_0)g_1' - (2n+3)g_1$$

$$= (4C_3 - b_3C_5)\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}}\, D_{-2n-2}(\eta+\eta_0)$$

$$+ \left(\frac{2\, b_3}{D_{-2n-1}(\eta_0)} - \frac{b_1b_3C_3}{2}\right) e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}}\, D_{-2n}(\eta+\eta_0)$$

$$- \frac{b_1b_3^2}{8}\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}}\, \frac{D_{-2n+2}(\eta+\eta_0)}{D_{-2n-1}(\eta_0)} \, . \qquad (3.85)$$

The homogeneous differential equation of (3.85) is

$$g_1'' + (\eta+\eta_0)g_1' - (2n+3)g_1 = 0 \, . \qquad (3.86)$$

If we substitute

$$g_1(\eta) = G_1(\eta)\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \tag{3.87}$$

into (3.86), then G_1 satisfies the equation

$$G_1'' + [-(2n+3) - \frac{1}{2} - \frac{1}{4}(\eta+\eta_0)^2]G_1 = 0 . \tag{3.88}$$

The equation (3.88) may be written in the standard form of Weber's differential equation

$$G_1'' + [-(2n+4) + \frac{1}{2} - \frac{1}{4}(\eta+\eta_0)^2]G_1 = 0 . \tag{3.89}$$

The general solution of (3.89) is

$$G_1 = C_7\, D_{-2n-4}(\eta+\eta_0) + C_8\, D_{2n+3}((\eta+\eta_0)i) , \tag{3.90}$$

where C_7 and C_8 are arbitrary constants of integration. Using (3.90) in (3.87), the complementary function of the differential equation (3.85) becomes

$$g_1 = C_7\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-4}(\eta+\eta_0)$$

$$+ C_8\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{2n+3}((\eta+\eta_0)i) . \tag{3.91}$$

To find the particular integral of the differential equation (3.85), we let

$$g_1 = B_1 e^{-\frac{1}{4}(\eta+\eta_0)^2} D_{-2n-2}(\eta+\eta_0) + B_2 e^{-\frac{1}{4}(\eta+\eta_0)^2} D_{-2n}(\eta+\eta_0)$$

$$+ B_3 e^{-\frac{1}{4}(\eta+\eta_0)^2} D_{-2n+2}(\eta+\eta_0) , \qquad (3.92)$$

where B_1, B_2 and B_3 are unknown constants yet to be determined.

Substituting (3.92) into the left hand side of (3.85) and using the formulae (2.40) and (2.49), we find that

$$g_1'' + (\eta+\eta_0)g_1' - (2n+3)g_1$$

$$= -2B_1 e^{-\frac{1}{4}(\eta+\eta_0)^2} D_{-2n-2}(\eta+\eta_0)$$

$$-4B_2 e^{-\frac{1}{4}(\eta+\eta_0)^2} D_{-2n}(\eta+\eta_0)$$

$$-6B_3 e^{-\frac{1}{4}(\eta+\eta_0)^2} D_{-2n+2}(\eta+\eta_0) . \qquad (3.93)$$

Comparing the right hand sides of the equations (3.85) and (3.93), we obtain

$$B_1 = -\frac{1}{2}(4C_3 - b_3\ C_5)e^{\frac{\eta_0^2}{4}} ,$$

$$B_2 = -\frac{1}{4}\left(\frac{2b_3}{D_{-2n-1}(\eta_0)} - \frac{b_1 b_3}{2} C_3\right)e^{\frac{\eta_0^2}{4}} , \qquad (3.94)$$

$$B_3 = \frac{1}{6}\,\frac{b_1 b_3^2}{8}\, e^{\frac{\eta_0^2}{4}} \cdot \frac{1}{D_{-2n-1}(\eta_0)} .$$

Hence the particular integral (3.92), by aid of (3.94), becomes

$$\varepsilon_1 = -\frac{1}{2}(4C_3 - b_3 C_5)\ e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}}\ D_{-2n-2}(\eta+\eta_0)$$

$$+ \frac{1}{4}\left(\frac{b_1 b_3 C_3}{2} - \frac{2b_3}{D_{-2n-1}(\eta_0)}\right)e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}}\ D_{-2n}(\eta+\eta_0)$$

$$+ \frac{b_1 b_3^2}{48} \cdot \frac{1}{D_{-2n-1}(\eta_0)} \cdot e^{-\frac{1}{4}(\eta+\eta_0)^2}\ D_{-2n+2}(\eta+\eta_0) .$$

(3.95)

Therefore the general solution of the differential equation (3.43) or (3.85), which is the complementary function (3.91) plus the particular integral (3.95), is

$$g_1 = C_7 e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-4}(\eta+\eta_0)$$

$$+ C_8 e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{2n+3}((\eta+\eta_0)i)$$

$$- \frac{1}{2}(4C_3 - b_3C_5) e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-2}(\eta+\eta_0)$$

$$+ \frac{1}{4}\left(\frac{b_1b_3C_3}{2} - \frac{2b_3}{D_{-2n-1}(\eta_0)}\right)e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n}(\eta+\eta_0)$$

$$+ \frac{b_1b_3^2}{48} \cdot \frac{1}{D_{-2n-1}(\eta_0)} \cdot e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n+2}(\eta+\eta_0) . \tag{3.96}$$

We need to find C_7 and C_8, subject to the boundary conditions (3.41), and if the solution (3.96) is to be bounded at $\eta = \infty$, we must have $C_8 = 0$, and therefore from (3.96), we have

$$g_1 = C_7\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-4}(\eta+\eta_0)$$

$$- \frac{1}{2}(4C_3 - b_3C_5)\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n-2}(\eta+\eta_0)$$

$$+ \frac{1}{4}\left(\frac{b_1b_3C_3}{2} - \frac{2b_3}{D_{-2n-1}(\eta_0)}\right) . \, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n}(\eta+\eta_0)$$

$$+ \frac{b_1b_3^2}{48} \cdot \frac{1}{D_{-2n-1}(\eta_0)} \cdot e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2n+2}(\eta+\eta_0) \; . \tag{3.97}$$

The boundary condition at $\eta = 0$, gives us

$$C_7 = \frac{1}{D_{-2n-4}(\eta_0)}\Big[\frac{1}{2}(4C_3 - b_3C_5)D_{-2n-2}(\eta_0)$$

$$- \frac{1}{4}\left(\frac{b_1b_3C_3}{2} - \frac{2b_3}{D_{-2n-1}(\eta_0)}\right)D_{-2n}(\eta_0)$$

$$- \frac{b_1b_3^2}{48} \; \frac{D_{-2n+2}(\eta_0)}{D_{-2n-1}(\eta_0)} \Big] . \tag{3.98}$$

The general solutions of the differential equations (3.38) and (3.44), subject to the boundary conditions (3.41), may be found similarly, although the computation would become increasingly lengthy.

It is interesting to note that for $\kappa = 0$, and vanishing micro-rotation, our solution (3.35) reduces to the classical solution for Newtonian fluids,

$$u(y,t) = At^n f_0(\eta) , \quad \text{where}$$

$$f_0(\eta) = e^{-\frac{1}{\pi}(\eta+\eta_0)^2 + \frac{\eta_0^2}{\pi}} [D_{-2n-1}(\eta_0)]^{-1} \cdot D_{-2n-1}(\eta+\eta_0) .$$

Section 3. Flow near a uniformly accelerated porous flat plate with variable suction.

The particular case $n = 1$ in section 2 of this chapter corresponds to a fluid motion caused by uniformly accelerating a porous flat plate with variable suction. Our two systems of ordinary differential equations (3.36) through (3.38), and (3.42) through (3.44), reduce to

$$f_0'' + (\eta+\eta_0)f_0' - 2f_0 = 0 , \qquad (3.99)$$

$$f_1'' + (\eta+\eta_0)f_1' - 4f_1 = b_1 g_0' , \qquad (3.100)$$

$$f_2'' + (\eta+\eta_0)f_2' - 6f_2 = b_1 g_1' , \qquad (3.101)$$

and

$$g_0'' + (\eta+\eta_0)g_0' - 3g_0 = b_3 f_0' , \qquad (3.102)$$

$$g_1'' + (\eta+\eta_0)g_1' - 5g_1 = b_3 f_1' + 4g_0 , \qquad (3.103)$$

$$g_2'' + (\eta+\eta_0)g_2' - 7g_2 = b_3 f_2' + 4g_1 , \qquad (3.104)$$

where we have assumed that $b_2 = 1$, that is the particular case when $(\mu+\kappa)j = \gamma$.

The general solutions of the differential equations (3.99), (3.102), (3.100) and (3.103), subject to the boundary conditions (3.41), are found by putting $n = 1$ in the solutions (3.52), (3.66), (3.81) and (3.97) as follows:

$$f_0 = e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \cdot \frac{D_{-3}(\eta+\eta_0)}{D_{-3}(\eta_0)}, \tag{3.105}$$

$$f_0 = c_3\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}}\, D_{-4}(\eta+\eta_0) + \frac{b_3}{2}\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}}\, \frac{D'_{-2}(\eta+\eta_0)}{D_{-3}(\eta_0)} \tag{3.106}$$

where

$$c_3 = -\frac{b_3}{2} \cdot \frac{D_{-2}(\eta_0)}{D_{-3}(\eta_0)\, D_{-4}(\eta_0)},$$

$$f_1 = c_5\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}}\, D_{-5}(\eta+\eta_0) + \frac{b_1 c_3}{2}\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}}\, D_{-3}(\eta+\eta_0) + \frac{b_1 b_3}{8}\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}}\, \frac{D_{-1}(\eta+\eta_0)}{D_{-3}(\eta_0)}, \tag{3.107}$$

where

$$c_5 = \frac{1}{D_{-5}(\eta_0)} \left[\frac{b_1 b_3}{4} \frac{D_{-2}(\eta_0)}{D_{-4}(\eta_0)} - \frac{b_1 b_3}{8} \frac{D_{-1}(\eta_0)}{D_{-3}(\eta_0)} \right]$$

and

$$g_1 = C_7 e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-6}(\eta+\eta_0)$$

$$- \frac{1}{2}(4C_3 - b_3C_5)e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-4}(\eta+\eta_0)$$

$$+ \frac{1}{4}\left(\frac{b_1b_3C_3}{2} - \frac{2b_3}{D_{-3}(\eta_0)}\right)e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2}(\eta+\eta_0)$$

$$+ \frac{b_1b_3^2}{48} e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \frac{D_0(\eta+\eta_0)}{D_{-3}(\eta_0)} \qquad (3.108)$$

where

$$C_7 = \frac{1}{D_{-6}(\eta_0)}\left[\frac{1}{2}(4C_3 - b_3C_5)D_{-4}(\eta_0)\right.$$

$$- \frac{1}{4}\left(\frac{b_1b_3C_3}{2} - \frac{2b_3}{D_{-3}(\eta_0)}\right)D_{-2}(\eta_0)$$

$$\left. - \frac{b_1b_3^2}{48} \frac{D_0(\eta_0)}{D_{-3}(\eta_0)}\right].$$

The general solutions of the differential equations (3.101) and (3.104), subject to the boundary conditions (3.41), may be found similarly.

The velocity and micro-rotation or spin, found by putting $n = 1$ in (3.35), are

$$u = At[f_0(\eta) + \epsilon f_1(\eta) + \epsilon^2 f_2(\eta) + \ldots]$$

$$\phi = Bt^{\frac{3}{2}}[g_0(\eta) + \epsilon g_1(\eta) + \epsilon^2 g_2(\eta) + \ldots] , \qquad (3.109)$$

where A and B are constants having the dimensions L^1T^{-2} and $T^{-5/2}$ respectively. The functions f_0, g_0, f_1 and g_1 have values as given in (3.105), (3.106), (3.107) and (3.108) respectively.

$$\frac{B}{A} = -\frac{\sqrt{k}\, b_1}{\sqrt{2}\, j}, \quad k = \frac{(\mu+\kappa)}{\rho} \quad \text{and} \quad \epsilon = \frac{\kappa t}{\rho j} .$$

For $\kappa = 0$ and vanishing spin, our solution (3.109) reduces to

$u = At\, f_0(\eta)$, where

$$f_0(\eta) = e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \frac{D_{-3}(\eta+\eta_0)}{D_{-3}(\eta_0)} .$$

Section 4. Flow near a suddenly accelerated porous flat plate with variable suction.

The particular case $n = 0$ in section 2 of this chapter corresponds to a fluid motion caused by the sudden acceleration of a porous flat plate with variable suction. Therefore, the two systems of ordinary differential equations (3.36) through (3.38), and (3.42) through (3.44) reduce to

$$f_0'' + (\eta+\eta_0)f_0' = 0 \tag{3.110}$$

$$f_1'' + (\eta+\eta_0)f_1' - 2f_1 = b_1 g_0' \tag{3.111}$$

$$f_2'' + (\eta+\eta_0)f_2' - 4f_2 = b_1 g_1' \tag{3.112}$$

and

$$g_0'' + (\eta+\eta_0)g_0' - g_0 = b_3 f_0' \tag{3.113}$$

$$g_1'' + (\eta+\eta_0)g_1' - 3g_1 = b_3 f_1' + 4g_0 \tag{3.114}$$

$$g_2'' + (\eta+\eta_0)g_2' - 5g_2 = b_3 f_2' + 4g_1 , \tag{3.115}$$

where we have considered the particular case when $(\mu+\kappa)j = \gamma$.

The general solutions of the differential equations (3.110), (3.113), (3.111) and (3.114), subject to the boundary conditions (3.41), are found by putting $n = 0$ in the solutions (3.52), (3.66), (3.81) and (3.97) as follows:

$$f_0 = e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \frac{D_{-1}(\eta+\eta_0)}{D_{-1}(\eta_0)} . \tag{3.116}$$

The right hand side in (3.116) can be written in terms of the error function with the aid of the formula given in Erdélyi et al. (1953, Volume 2, p. 119),

$$D_{-1}(z) = \sqrt{2}\, e^{\frac{1}{4}z^2} \, \mathrm{Erfc}(\frac{z}{\sqrt{2}}) , \tag{3.117}$$

where Erfc is the complementary error function, which is defined in (2.101).

Therefore, we have

$$D_{-1}(\eta+\eta_0) = \sqrt{2}\, e^{\frac{1}{4}(\eta+\eta_0)^2} \, \mathrm{Erfc}\, (\frac{\eta+\eta_0}{\sqrt{2}}) \tag{3.118}$$

and

$$D_{-1}(\eta_0) = \sqrt{2}\, e^{\frac{\eta_0^2}{4}} \, \mathrm{Erfc}\, (\frac{\eta_0}{\sqrt{2}}) . \tag{3.119}$$

Using (3.118) and (3.119) in (3.116), we find that

$$f_0 = \frac{\mathrm{Erfc}(\frac{\eta+\eta_0}{\sqrt{2}})}{\mathrm{Erfc}(\frac{\eta_0}{\sqrt{2}})} \tag{3.120}$$

$$g_0 = C_3\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2}(\eta+\eta_0)$$

$$+ \frac{b_3}{2}\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \frac{D_0(\eta+\eta_0)}{D_{-1}(\eta_0)} , \tag{3.121}$$

where

$$C_3 = -\frac{b_3}{2}\,\frac{D_0(\eta_0)}{D_{-1}(\eta_0)D_{-2}(\eta_0)}$$

$$f_1 = C_5\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-3}(\eta+\eta_0)$$

$$+ \frac{b_1}{2} C_3\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-1}(\eta+\eta_0)$$

$$+ \frac{b_1 b_3}{8}\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \frac{D_1(\eta+\eta_0)}{D_{-1}(\eta_0)}\ , \qquad (3.122)$$

where

$$C_5 = \frac{1}{D_{-3}(\eta_0)}\left[\frac{b_1 b_3}{4}\,\frac{D_0(\eta_0)}{D_{-2}(\eta_0)} - \frac{b_1 b_3}{8}\,\frac{D_1(\eta_0)}{D_{-1}(\eta_0)}\right] .$$

$$g_1 = C_7\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-4}(\eta+\eta_0)$$

$$- \frac{1}{2}(4C_3 - b_3 C_5)\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_{-2}(\eta+\eta_0)$$

$$+ \frac{1}{4}\left(\frac{b_1 b_3 C_3}{2} - \frac{2b_3}{D_{-1}(\eta_0)}\right) e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} D_0(\eta+\eta_0)$$

$$+ \frac{b_1}{48} b_3^2\, e^{-\frac{1}{4}(\eta+\eta_0)^2 + \frac{\eta_0^2}{4}} \frac{D_2(\eta+\eta_0)}{D_{-1}(\eta_0)} . \qquad (3.123)$$

where

$$C_7 = \frac{1}{D_{-4}(\eta_0)}\left[\frac{1}{2}(4C_3 - b_3C_5)D_{-2}(\eta_0)\right.$$

$$\left. - \frac{1}{4}\left(\frac{b_1b_3C_3}{2} - \frac{2b_3}{D_{-1}(\eta_0)}\right)D_0(\eta_0) - \frac{b_1}{48}\, b_3^2\, \frac{D_2(\eta_0)}{D_{-1}(\eta_0)}\right] .$$

The general solutions of the differential equations (3.112) and (3.115), subject to the boundary conditions (3.41), may be found in a similar manner.

The velocity and micro-rotation or spin, found by putting $\eta = 0$ in (3.35), are

$$\begin{aligned} u &= A[f_0(\eta) + \varepsilon f_1(\eta) + \varepsilon^2 f_2(\eta) + \ldots] \\ \phi &= Bt^{\frac{1}{2}}[g_0(\eta) + \varepsilon g_1(\eta) + \varepsilon^2 g_2(\eta) + \ldots] \end{aligned} \qquad (3.124)$$

where A and B are constants having the dimensions L^1T^{-1} and $T^{-\frac{3}{2}}$ respectively. The functions f_0, g_0, f_1, and g_1 have solutions as given in (3.120), (3.121), (3.122) and (3.123) respectively.

$$\frac{B}{A} = -\frac{\sqrt{k}\; b_1}{\sqrt{2}\; j}\ , \quad k = \frac{(\mu+\kappa)}{\rho} \quad \text{and} \quad \varepsilon = \frac{\kappa t}{\rho j}\ .$$

It should be noted here that for $\kappa = 0$, and vanishing micro-rotation, our solution (3.124) reduces to

$u = A\ f_0$, where

$$f_0 = \frac{\operatorname{Erfc}\left(\frac{\eta+\eta_0}{\sqrt{2}}\right)}{\operatorname{Erfc}\left(\frac{\eta_0}{\sqrt{2}}\right)} ,$$

which is the same as the solution for Newtonian fluids.

CHAPTER IV

STEADY FLOW IN AN ELLIPTIC TUBE

Section 1. Equations of motion.

Green and Rivlin (1956) have worked out the case of non-Newtonian fluids in a tube of elliptic cross-section and observed that the flow is secondary.

Langlois and Rivlin (1963) have studied the slow steady-state flow of visco-elastic fluids through non-circular tubes and also observed that the flow is secondary.

Eringen (1966) investigated the steady flow of micropolar fluids through a circular pipe.

It is quite interesting to work out the case of micropolar fluids in an elliptic tube and to see whether the flow is secondary or not.

Here we consider steady, incompressible flow of micropolar fluids with no external forces and body couples, and take axes Ox, Oy, Oz, where Oz is parallel to the generators of the elliptic tube with axes 2a, 2b and Ox, Oy are perpendicular thereto. (Figure (4.1).

We look for a solution for which

$$\underset{\sim}{v} = (0,\ 0,\ w(x,y)) \quad \text{and} \qquad\qquad (4.1)$$

$$\underset{\sim}{\nu} = (v(x,y),\ \phi(x,y),\ 0)\ .$$

The equation of continuity (1.1) is satisfied identically,

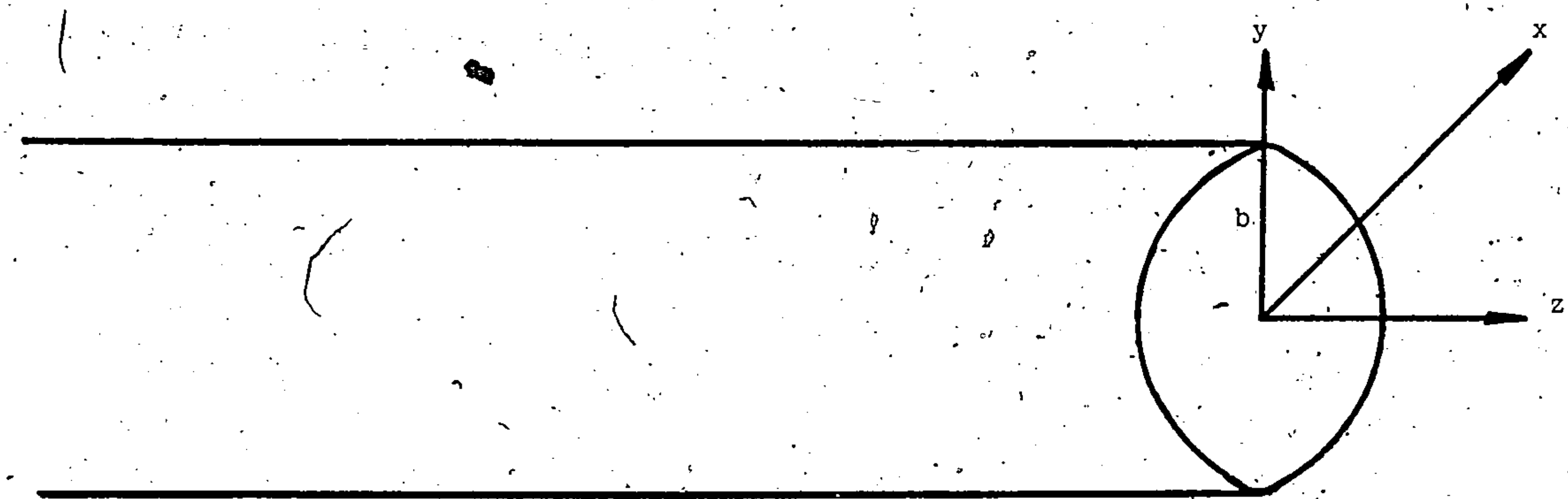


Figure 4.1 Co-ordinate system for flow through elliptic tube.

and the field equations (1.2) and (1.3), due to (4.1), reduce to

$$\frac{\partial p}{\partial x} = 0 \,, \tag{4.2}$$

$$\frac{\partial p}{\partial y} = 0 \,, \tag{4.3}$$

$$(\mu+\kappa)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \kappa\left(\frac{\partial \phi}{\partial x} - \frac{\partial \nu}{\partial y}\right) - \frac{\partial p}{\partial z} = 0 \,, \tag{4.4}$$

$$(\alpha+\beta+\gamma)\left(\frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 \phi}{\partial x \partial y}\right) - \gamma\left(\frac{\partial^2 \phi}{\partial y \partial x} - \frac{\partial^2 \nu}{\partial y^2}\right) + \kappa \frac{\partial w}{\partial y} - 2\kappa\nu = 0 \,, \tag{4.5}$$

and

$$(\alpha+\beta+\gamma)\left(\frac{\partial^2 \nu}{\partial y \partial x} + \frac{\partial^2 \phi}{\partial y^2}\right) + \gamma\left(\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \nu}{\partial x \partial y}\right) - \kappa \frac{\partial w}{\partial x} - 2\kappa\phi = 0 \,. \tag{4.6}$$

Equations (4.2) and (4.3) show that $p = p(z)$ only, so that equation (4.4) becomes

$$(\mu+\kappa)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \kappa\left(\frac{\partial \phi}{\partial x} - \frac{\partial \nu}{\partial y}\right) = \frac{dp}{dz} \,.$$

In the above equation, the left-hand side is a function of x and y only and the right-hand side is a function of z only. We deduce that both are constants, and write

$$\frac{dp}{dz} = (\mu+\kappa)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \kappa\left(\frac{\partial \phi}{\partial x} - \frac{\partial \nu}{\partial y}\right) = -P \,. \tag{4.7}$$

We need to solve the partial differential equations (4.5), (4.6) and (4.7), subject to the following boundary conditions:

$$\text{when} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \; ; \quad w(x,y) = \nu(x,y) = \phi(x,y) = 0 \; . \tag{4.8}$$

The method of successive approximations is applied to uncouple the partial differential equations (4.5), (4.6) and (4.7), which are then solved using the semi-inverse method [Langlois (1964, p. 119-120)] .

Section 2. Solution by successive approximation and semi-inverse methods.

We seek a solution of the partial differential equations (4.5), (4.6), and (4.7) in powers of ε, assuming that ε is a parameter which is sufficiently small, where

$$\varepsilon = \frac{\kappa L}{\rho_0 j_0 v_0} .$$

Thus we expand w, ν, and ϕ in ascending powers of ε as follows:

$$\begin{aligned}
w(x,y) &= w_0(x,y) + \varepsilon w_1(x,y) + \varepsilon^2 w_2(x,y) + \varepsilon^3 w_3(x,y) + \dots \\
\nu(x,y) &= \nu_0(x,y) + \varepsilon \nu_1(x,y) + \varepsilon^2 \nu_2(x,y) + \varepsilon^3 \nu_3(x,y) + \dots \qquad (4.9)\\
\phi(x,y) &= \phi_0(x,y) + \varepsilon \phi_1(x,y) + \varepsilon^2 \phi_2(x,y) + \varepsilon^3 \phi_3(x,y) + \dots .
\end{aligned}$$

Substituting the relations (4.9) into equations (4.7), (4.5) and (4.6), we find that

$$\begin{aligned}
&\mu[(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}) + \varepsilon(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}) + \varepsilon^2(\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2}) \\
&+ \varepsilon^3(\frac{\partial^2 w_3}{\partial x^2} + \frac{\partial^2 w_3}{\partial y^2}) + \dots] + \varepsilon b_1[(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}) \\
&+ \varepsilon(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}) + \varepsilon^2(\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2}) + \dots] + \varepsilon b_1[(\frac{\partial \phi_0}{\partial x} - \frac{\partial \nu_0}{\partial y}) \\
&+ \varepsilon(\frac{\partial \phi_1}{\partial x} - \frac{\partial \nu_1}{\partial y}) + \varepsilon^2(\frac{\partial \phi_2}{\partial x} - \frac{\partial \nu_2}{\partial y}) + \dots] = -P . \qquad (4.10)
\end{aligned}$$

$$(\alpha+\beta+\gamma)[(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial x \partial y}) + \varepsilon (\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial x \partial y})$$

$$+ \varepsilon^2(\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial x \partial y}) + \varepsilon^3(\frac{\partial^2 v_3}{\partial x^2} + \frac{\partial^2 \phi_3}{\partial x \partial y}) + \ldots]$$

$$- \gamma[(\frac{\partial^2 \phi_0}{\partial y \partial x} - \frac{\partial^2 v_0}{\partial y^2}) + \varepsilon(\frac{\partial^2 \phi_1}{\partial y \partial x} - \frac{\partial^2 v_1}{\partial y^2}) + \varepsilon^2(\frac{\partial^2 \phi_2}{\partial y \partial x} - \frac{\partial^2 v_2}{\partial y^2})$$

$$+ \varepsilon^3(\frac{\partial^2 \phi_3}{\partial y \partial x} - \frac{\partial^2 v_3}{\partial y^2}) + \ldots] + b_1\varepsilon(\frac{\partial w_0}{\partial y} + \varepsilon\frac{\partial w_1}{\partial y} + \varepsilon^2 \frac{\partial w_2}{\partial y} + \ldots)$$

$$= 2b_1\varepsilon(v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots) \qquad (4.11)$$

and

$$(\alpha+\beta+\gamma)[(\frac{\partial^2 v_0}{\partial y \partial x} + \frac{\partial^2 \phi_0}{\partial y^2}) + \varepsilon(\frac{\partial^2 v_1}{\partial y \partial x} + \frac{\partial^2 \phi_1}{\partial y^2}) + \varepsilon^2(\frac{\partial^2 v_2}{\partial y \partial x} + \frac{\partial^2 \phi_2}{\partial y^2})$$

$$+ \varepsilon^3(\frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial^2 \phi_3}{\partial y^2}) + \ldots] + \gamma[(\frac{\partial^2 \phi_0}{\partial x^2} - \frac{\partial^2 v_0}{\partial x \partial y}) + \varepsilon(\frac{\partial^2 \phi_1}{\partial x^2} - \frac{\partial^2 v_1}{\partial x \partial y})$$

$$+ \varepsilon^2(\frac{\partial^2 \phi_2}{\partial x^2} - \frac{\partial^2 v_2}{\partial x \partial y}) + \varepsilon^3(\frac{\partial^2 \phi_3}{\partial x^2} - \frac{\partial^2 v_3}{\partial x \partial y}) + \ldots]$$

$$- b_1\varepsilon (\frac{\partial w_0}{\partial x} + \varepsilon \frac{\partial w_1}{\partial x} + \varepsilon^2 \frac{\partial w_2}{\partial x} + \ldots)$$

$$= 2b_1\varepsilon(\phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \ldots), \qquad (4.12)$$

where

$$b_1 = \frac{\rho_0 j_0 v_0}{L} .$$

Identifying like powers of ε in equations (4.10), (4.11) and (4.12), we find the following:

Zeroth order approximation:

$$\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2} = -\frac{P}{\mu} ,$$

$$\left\{\begin{aligned} &(\alpha+\beta+\gamma)\left(\frac{\partial^2 \nu_0}{\partial x^2} + \frac{\partial^2 \phi_0}{\partial x \partial y}\right) - \gamma\left(\frac{\partial^2 \phi_0}{\partial y \partial x} - \frac{\partial^2 \nu_0}{\partial y^2}\right) = 0 , \\ &(\alpha+\beta+\gamma)\left(\frac{\partial^2 \nu_0}{\partial y \partial x} + \frac{\partial^2 \phi_0}{\partial y^2}\right) + \gamma\left(\frac{\partial^2 \phi_0}{\partial x^2} - \frac{\partial^2 \nu_0}{\partial x \partial y}\right) = 0 . \end{aligned}\right. \tag{4.13}$$

First order approximation:

$$\mu\left(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}\right) + b_1\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right) + b_1\left(\frac{\partial \phi_0}{\partial x} - \frac{\partial \nu_0}{\partial y}\right) = 0 ,$$

$$(\alpha+\beta+\gamma)\left(\frac{\partial^2 \nu_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial x \partial y}\right) - \gamma\left(\frac{\partial^2 \phi_1}{\partial y \partial x} - \frac{\partial^2 \nu_1}{\partial y^2}\right) + b_1 \frac{\partial w_0}{\partial y} = 2b_1 \nu_0 , \tag{4.14}$$

$$(\alpha+\beta+\gamma)\left(\frac{\partial^2 \nu_1}{\partial y \partial x} + \frac{\partial^2 \phi_1}{\partial y^2}\right) + \gamma\left(\frac{\partial^2 \phi_1}{\partial x^2} - \frac{\partial^2 \nu_1}{\partial x \partial y}\right) - b_1 \frac{\partial w_0}{\partial x} = 2b_1 \phi_0 .$$

Second order approximation:

$$\mu(\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2}) + b_1(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}) + b_1(\frac{\partial \phi_1}{\partial x} - \frac{\partial \nu_1}{\partial y}) = 0 ,$$

$$(\alpha+\beta+\gamma)(\frac{\partial^2 \nu_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial x \partial y}) - \gamma(\frac{\partial^2 \phi_2}{\partial y \partial x} - \frac{\partial^2 \nu_2}{\partial y^2}) + b_1\frac{\partial w_1}{\partial y} = 2b_1\nu_1, \quad (4.15)$$

$$(\alpha+\beta+\gamma)(\frac{\partial^2 \nu_2}{\partial y \partial x} + \frac{\partial^2 \phi_2}{\partial y^2}) + \gamma(\frac{\partial^2 \phi_2}{\partial x^2} - \frac{\partial^2 \nu_2}{\partial x \partial y}) - b_1\frac{\partial w_1}{\partial x} = 2b_1\phi_1 .$$

Third order approximation:

$$\mu(\frac{\partial^2 w_3}{\partial x^2} + \frac{\partial^2 w_3}{\partial y^2}) + b_1(\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2}) + b_1(\frac{\partial \phi_2}{\partial x} - \frac{\partial \nu_2}{\partial y}) = 0 ,$$

$$(\alpha+\beta+\gamma)(\frac{\partial^2 \nu_3}{\partial x^2} + \frac{\partial^2 \phi_3}{\partial x \partial y}) - \gamma(\frac{\partial^2 \phi_3}{\partial y \partial x} - \frac{\partial^2 \nu_3}{\partial y^2}) + b_1\frac{\partial w_2}{\partial y} = 2b_1\nu_2 , \quad (4.16)$$

$$(\alpha+\beta+\gamma)(\frac{\partial^2 \nu_3}{\partial y \partial x} + \frac{\partial^2 \phi_3}{\partial y^2}) + \gamma(\frac{\partial^2 \phi_3}{\partial x^2} - \frac{\partial^2 \nu_3}{\partial x \partial y}) - b_1\frac{\partial w_2}{\partial x} = 2b_1\phi_2 .$$

The boundary conditions (4.8), in view of (4.9), become:

$$\text{when} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ; \quad w_k(x,y) = \nu_k(x,y) = \phi_k(x,y) = 0 . \quad (4.17)$$

$$k = 0, 1, 2, 3, \ldots .$$

The partial differential equations (4.13) through (4.16), subject to the boundary conditions (4.17), are solved using the semi-inverse method [Langlois (1964, p. 119-120)]. The solution of the first of (4.13), subject to the boundary condition that $w_0(x,y) = 0$ when

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is easily found as

$$w_0(x,y) = C_1(b^2x^2 + a^2y^2 - a^2b^2) , \tag{4.18}$$

where

$$C_1 = - \frac{P}{2\mu(a^2+b^2)} .$$

Therefore, we have

$$w_0(x,y) = - \frac{P}{2\mu(a^2+b^2)}(b^2x^2 + a^2y^2 - a^2b^2) . \tag{4.19}$$

The pair of equations $(4.13)_{2,3}$ is elliptic and the boundary conditions are $\nu_0 = \phi_0 = 0$, when $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. A solution satisfying the equations and boundary conditions is

$$\nu_0 \equiv 0 , \quad \phi_0 \equiv 0 . \tag{4.20}$$

Hence this is the unique solution.

Using (4.20) in equations (4.14), the equations become

$$\mu(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}) + b_1(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}) = 0 ,$$

$$(\alpha+\beta+\gamma)(\frac{\partial^2 \nu_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial x \partial y}) - \gamma(\frac{\partial^2 \phi_1}{\partial y \partial x} - \frac{\partial^2 \nu_1}{\partial y^2}) + b_1\frac{\partial w_0}{\partial y} = 0 , \tag{4.21}$$

$$(\alpha+\beta+\gamma)(\frac{\partial^2 \nu_1}{\partial y \partial x} + \frac{\partial^2 \phi_1}{\partial y^2}) + \gamma(\frac{\partial^2 \phi_1}{\partial x^2} - \frac{\partial^2 \nu_1}{\partial x \partial y}) - b_1\frac{\partial w_0}{\partial x} = 0 .$$

Using the first of (4.13) in the first of (4.21), we have

$$\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2} = \frac{b_1 P}{\mu^2} . \qquad (4.22)$$

The solution of the partial differential equation (4.22), subject to the boundary condition that $w_1(x,y) = 0$ when $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is

$$w_1(x,y) = C_2(b^2x^2 + a^2y^2 - a^2b^2) \qquad (4.23)$$

where

$$C_2 = \frac{b_1 P}{2\mu^2(a^2 + b^2)} .$$

Therefore we have

$$w_1(x,y) = \frac{b_1 P}{2\mu^2(a^2 + b^2)}(b^2x^2 + a^2y^2 - a^2b^2) . \qquad (4.24)$$

Using (4.19) in the second and third of (4.21) and after a little simplification, the equations become

$$(\alpha+\beta+\gamma)\left(\frac{\partial^2 \nu_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial x \partial y}\right) - \gamma\left(\frac{\partial^2 \phi_1}{\partial y \partial x} - \frac{\partial^2 \nu_1}{\partial y^2}\right) = \frac{b_1 P a^2 y}{\mu(a^2+b^2)}$$

and

$$(4.25)$$

$$(\alpha+\beta+\gamma)\left(\frac{\partial^2 \nu_1}{\partial y \partial x} + \frac{\partial^2 \phi_1}{\partial y^2}\right) + \gamma\left(\frac{\partial^2 \phi_1}{\partial x^2} - \frac{\partial^2 \nu_1}{\partial x \partial y}\right) = - \frac{b_1 P b^2 x}{\mu(a^2 + b^2)} .$$

$$(4.26)$$

The solutions of the partial differential equations (4.25) and (4.26), subject to the boundary conditions that $\nu_1(x,y) = \phi_1(x,y) = 0$ when $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, using the semi-inverse method, are found to be

$$\nu_1(x,y) = C_3 y(b^2x^2 + a^2y^2 - a^2b^2) , \tag{4.27}$$

$$\phi_1(x,y) = C_4 x(b^2x^2 + a^2y^2 - a^2b^2) , \tag{4.28}$$

provided that $x \neq 0$, $y \neq 0$, and where C_3 and C_4 are constants, which are found to be

$$C_3 = \frac{b_1 \; Pa^2[(\alpha+\beta)(a^2+b^2) + \gamma(3b^2+a^2)]}{2\mu(a^2+b^2)\gamma[(\alpha+\beta)a^2b^2+3b^4(\alpha+\beta)+a^2(3a^2+b^2)(\alpha+\beta+\gamma) + 3b^2\gamma(3a^2+b^2)]} \tag{4.29}$$

and

$$C_4 = -\frac{b_1 \; Pb^2[(\alpha+\beta)(a^2+b^2) + \gamma(3a^2+b^2)]}{2\mu(a^2+b^2)\gamma[(\alpha+\beta)a^2b^2+3b^4(\alpha+\beta)+a^2(3a^2+b^2)(\alpha+\beta+\gamma) + 3b^2\gamma(3a^2+b^2)]} \tag{4.30}$$

We would like to point out here that if w_0 is a polynomial of degree 2, and $\phi_0 = \nu_0 = 0$, then w_1 could be a polynomial of degree 2, and ϕ_1, ν_1 could be polynomials of degree 3.

Similarly w_2 could be a polynomial of degree 4, and

ν_2, ϕ_2 could be polynomials of degree 5.

We make ν odd in y and even in x, ϕ odd in x and even in y, w even in both, because the partial differential equations are not satisfied otherwise.

We find the difference between the volume fluxes $\int w_0 \, dA$ and $\int w_1 \, dA$ in order to estimate the magnitude of first approximation to the zeroth approximation.

The volume flux for the zeroth order approximation is F_0 given by

$$F_0 = \iint w_0(x,y)\,dx\,dy$$

$$= 4\int_{y=0}^{b}\int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} \left\{-\frac{P}{2\mu(a^2+b^2)}\right\}(b^2x^2+a^2y^2-a^2b^2)\,dx\,dy \; ; \quad (4.31)$$

which on integration gives us

$$F_0 = \frac{\pi P a^3 b^3}{4\mu(a^2+b^2)} \qquad (4.32)$$

The volume flux for the first order approximation is

$$F_1 = \iint w_1(x,y)\,dx\,dy$$

$$= \frac{4b_1 P}{2\mu^2(a^2+b^2)}\int_{y=0}^{b}\int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} (b^2x^2+a^2y^2-a^2b^2)\,dx\,dy\,,$$

$$= -\frac{\pi b_1 P a^3 b^3}{4\mu^2(a^2+b^2)} \qquad (4.33)$$

Using (4.24), (4.27) and (4.28) in the partial differential equations (4.15) and after a little simplification, the equations become

$$\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2} = \frac{b_1 b^2}{\mu}(C_3-3C_4)x^2 + \frac{b_1 a^2}{\mu}(3C_3-C_4)y^2 - \frac{b_1^2 P}{\mu^3} + \frac{b_1 a^2 b^2}{\mu}(C_4-C_3)\ , \tag{4.34}$$

$$(\alpha+\beta+\gamma)\left(\frac{\partial^2 \nu_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial x \partial y}\right) - \gamma\left(\frac{\partial^2 \phi_2}{\partial y \partial x} - \frac{\partial^2 \nu_2}{\partial y^2}\right)$$
$$= b_1 y\left[2C_3 b^2 x^2 + 2C_3 a^2 y^2 - 2C_3 a^2 b^2 - \frac{b_1 P a^2}{\mu^2(a^2+b^2)}\right] \tag{4.35}$$

and

$$(\alpha+\beta+\gamma)\left(\frac{\partial^2 \nu_2}{\partial y \partial x} + \frac{\partial^2 \phi_2}{\partial y^2}\right) + \gamma\left(\frac{\partial^2 \phi_2}{\partial x^2} - \frac{\partial^2 \nu_2}{\partial x \partial y}\right)$$
$$= b_1 x\left[2C_4 b^2 x^2 + 2C_4 a^2 y^2 - 2C_4 a^2 b^2 + \frac{b_1 P b^2}{\mu^2(a^2+b^2)}\right] . \tag{4.36}$$

The solution of the partial differential equation (4.34), subject to the boundary condition that $w_2(x,y) = 0$ when $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is found to be

$$w_2(x,y) = (b^2x^2+a^2y^2-a^2b^2)(A_1x^2+A_2y^2+A_3), \qquad (4.37)$$

where

$$A_1 = \frac{b_1b^2(b^2+6a^2)(C_3-3C_4)-b_1a^2b^2(3C_3-C_4)}{12\mu(a^4+b^4+6a^2b^2)}, \qquad (4.38)$$

$$A_2 = \frac{b_1a^2(a^2+6b^2)(3C_3-C_4)-b_1a^2b^2(C_3-3C_4)}{12\mu(a^4+b^4+6a^2b^2)} \qquad (4.39)$$

and

$$A_3 = \frac{1}{2(a^2+b^2)}\left[2a^2b^2(A_1+A_2) - \frac{b_1^2\,P}{\mu^3} - \frac{b_1a^2b^2}{\mu}\,(C_3-C_4)\right]. \qquad (4.40)$$

The solutions of the partial differential equations (4.35), and (4.36), subject to the boundary conditions that $\nu_2(x,y) = \phi_2(x,y) = 0$ when $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, are found using the semi-inverse method as follows:

$$\nu_2(x,y) = y(b^2x^2+a^2y^2-a^2b^2)(A_4x^2+A_5y^2+A_6), \qquad (4.41)$$

$$\phi_2(x,y) = x(b^2x^2+a^2y^2-a^2b^2)(A_7x^2+A_8y^2+A_9), \qquad (4.42)$$

provided that $x \neq 0$, $y \neq 0$, and where A_4, A_5, A_6, A_7, A_8 and A_9 are constants given by

$$A_4 = \frac{\Delta_1}{\Delta}, \quad A_5 = \frac{\Delta_2}{\Delta}, \quad A_6 = \frac{\Delta_3}{\Delta},$$

$$A_7 = \frac{\Delta_4}{\Delta}, \quad A_8 = \frac{\Delta_5}{\Delta}, \quad A_9 = \frac{\Delta_6}{\Delta}. \tag{4.43}$$

and where Δ , Δ_1 , Δ_2 , Δ_3 , Δ_4 , Δ_5 and Δ_6 are

$$\Delta = [(\alpha+\beta+\gamma)a^2+3\gamma b^2][(\alpha+\beta+\gamma)b^2+3\gamma a^2]\Delta_0 - a^2b^2(\alpha+\beta)^2\Delta_0 , \tag{4.44}$$

$$\Delta_0 = \begin{vmatrix} (\alpha+\beta+\gamma)a^2 & (\alpha+\beta+\gamma)b^2+10a^2\gamma & 0 & 2(\alpha+\beta)a^2 \\ 6(\alpha+\beta+\gamma)b^2+3a^2\gamma & 3b^2\gamma & 3a^2(\alpha+\beta) & 3b^2(\alpha+\beta) \\ 2(\alpha+\beta)b^2 & 0 & (\alpha+\beta+\gamma)a^2+10b^2\gamma & (\alpha+\beta+\gamma)b^2 \\ 3(\alpha+\beta)a^2 & 3(\alpha+\beta)b^2 & 3\gamma a^2 & 6a^2(\alpha+\beta+\gamma)+3\gamma b^2 \end{vmatrix} .$$

$$\Delta_1 = [(\alpha+\beta+\gamma)a^2+3\gamma b^2][(\alpha+\beta+\gamma)b^2+3\gamma a^2]\Delta' - a^2b^2(\alpha+\beta)^2\Delta' , \tag{4.45}$$

$$\Delta' = \begin{vmatrix} b_1a^2C_3 & (\alpha+\beta+\gamma)b^2+10a^2\gamma & 0 & 2(\alpha+\beta)a^2 \\ b_1b^2C_3 & 3b^2\gamma & 3a^2(\alpha+\beta) & \cancel{2(\alpha+\beta)b^2} \\ b_1b^2C_4 & 0 & (\alpha+\beta+\gamma)a^2+10b^2\gamma & (\alpha+\beta+\gamma)b^2 \\ b_1a^2C_4 & 3(\alpha+\beta)b^2 & 3\gamma a^2 & 6a^2(\alpha+\beta+\gamma)+3\gamma b^2 \end{vmatrix}$$

and

$$\Delta_2 = [(\alpha+\beta+\gamma)a^2+3\gamma b^2][(\alpha+\beta+\gamma)b^2+3\gamma a^2]\Delta'' - a^2b^2(\alpha+\beta)^2\Delta'' , \tag{4.46}$$

where

$$\Delta'' = \begin{vmatrix} (\alpha+\beta+\gamma)a^2 & b_1a^2C_3 & 0 & 2(\alpha+\beta)a^2 \\ 6(\alpha+\beta+\gamma)b^2+3a^2\gamma & b_1b^2C_3 & 3a^2(\alpha+\beta) & 3(\alpha+\beta)b^2 \\ 2(\alpha+\beta)b^2 & b_1b^2C_4 & (\alpha+\beta+\gamma)a^2+10b^2\gamma & \gamma(\alpha+\beta+\gamma)b^2 \\ 3(\alpha+\beta)a^2 & b_1a^2C_4 & 3\gamma a^2 & 6a^2(\alpha+\beta+\gamma)+3\gamma b^2 \end{vmatrix} .$$

$$\Delta_3 = [(\alpha+\beta+\gamma)a^2+3\gamma b^2]\Delta_{30}-a^2(\alpha+\beta)\Delta_{31} , \tag{4.47}$$

$$\Delta_{30} = \begin{vmatrix} (\alpha+\beta+\gamma)a^2 & (\alpha+\beta+\gamma)b^2+10a^2\gamma & b_1a^2C_3 & 0 & 2(\alpha+\beta)a^2 \\ 6(\alpha+\beta+\gamma)b^2+3a^2\gamma & 3b^2\gamma & b_1b^2C_3 & 3a^2(\alpha+\beta) & 3b^2(\alpha+\beta) \\ 2(\alpha+\beta)b^2 & 0 & b_1b^2C_4 & (\alpha+\beta+\gamma)a^2+10b^2\gamma & (\alpha+\beta+\gamma)b^2 \\ 3(\alpha+\beta)a^2 & 3(\alpha+\beta)b^2 & b_1a^2C_4 & 3\gamma a^2 & 6a^2(\alpha+\beta+\gamma)+3\gamma b^2 \\ -(\alpha+\beta+\gamma)a^2b^2 & -9\gamma a^2b^2 & a_{11} & 0 & -(\alpha+\beta)a^2b^2 \end{vmatrix} ,$$

$$a_{11} = -b_1C_3a^2b^2 + \frac{b_1^2\,P\,a^2}{2\mu^2(a^2+b^2)} ,$$

$$\Delta_{31} = \begin{vmatrix} (\alpha+\beta+\gamma)a^2 & (\alpha+\beta+\gamma)b^2+10a^2\gamma & b_1a^2C_3 & 0 & 2(\alpha+\beta)a^2 \\ 6(\alpha+\beta+\gamma)a^2+3a^2\gamma & 3b^2\gamma & b_1b^2C_3 & 3a^2(\alpha+\beta) & 3b^2(\alpha+\beta) \\ 2(\alpha+\beta)b^2 & 0 & b_1b^2C_4 & \begin{matrix}(\alpha+\beta+\gamma)a^2 \\ +10b^2\gamma\end{matrix} & (\alpha+\beta+\gamma)b^2 \\ 3(\alpha+\beta)a^2 & 3(\alpha+\beta)b^2 & b_1a^2C_4 & 3\gamma a^2 & 6a^2(\alpha+\beta+\gamma)+3\gamma b^2 \\ -(\alpha+\beta)a^2b^2 & 0 & a_{12} & -3\gamma a^2b^2 & -(\alpha+\beta+\gamma)a^2b^2 \end{vmatrix}$$

$$a_{12} = -b_1\, C_4\, a^2b^2 + \frac{b_1^2\, P\, b^2}{2\mu^2(a^2+b^2)} .$$

$$\Delta_4 = [(\alpha+\beta+\gamma)a^2+3\gamma b^2][(\alpha+\beta+\gamma)b^2+3\gamma a^2]\Delta''' - a^2b^2(\alpha+\beta)^2\Delta''' , \tag{4.48}$$

where

$$\Delta''' = \begin{vmatrix} (\alpha+\beta+\gamma)a^2 & (\alpha+\beta+\gamma)b^2+10a^2\gamma & b_1a^2C_3 & 2(\alpha+\beta)a^2 \\ 6(\alpha+\beta+\gamma)b^2+3\gamma a^2 & 3b^2\gamma & b_1b^2C_3 & 3b^2(\alpha+\beta) \\ 2(\alpha+\beta)b^2 & 0 & b_1b^2C_4 & (\alpha+\beta+\gamma)b^2 \\ 3(\alpha+\beta)a^2 & 3(\alpha+\beta)b^2 & b_1a^2C_4 & 6a^2(\alpha+\beta+\gamma)+3\gamma b^2 \end{vmatrix} .$$

$$\Delta_5 = [(\alpha+\beta+\gamma)a^2+3\gamma b^2][(\alpha+\beta+\gamma)b^2+3\gamma a^2]\Delta^{IV} - a^2b^2(\alpha+\beta)^2\Delta^{IV} , \tag{4.49}$$

$$\Delta^{IV} = \begin{vmatrix} (\alpha+\beta+\gamma)a^2 & (\alpha+\beta+\gamma)b^2+10a^2\gamma & 0 & b_1a^2C_3 \\ 6(\alpha+\beta+\gamma)b^2+3a^2\gamma & 3b^2\gamma & 3a^2(\alpha+\beta) & b_1b^2C_3 \\ 2(\alpha+\beta)b^2 & 0 & (\alpha+\beta+\gamma)a^2+10b^2\gamma & b_1b^2C_4 \\ 3(\alpha+\beta)a^2 & 3(\alpha+\beta)b^2 & 3\gamma a^3 & b_1a^2C_4 \end{vmatrix} .$$

$$\Delta_6 = [(\alpha+\beta+\gamma)b^2+3\gamma a^2]\Delta_{60}-(\alpha+\beta)b^2\Delta_{61} , \qquad (4.50)$$

where

$$\Delta_{60} = \begin{vmatrix} (\alpha+\beta+\gamma)a^2 & (\alpha+\beta+\gamma)b^2+10a^2\gamma & 0 & 2(\alpha+\beta)a^2 & b_1a^2C_3 \\ 6(\alpha+\beta+\gamma)b^2+3a^2\gamma & 3b^2\gamma & 3a^2(\alpha+\beta) & 3(\alpha+\beta)b^2 & b_1b^2C_3 \\ 2(\alpha+\beta)b^2 & 0 & \begin{matrix}(\alpha+\beta+\gamma)a^2 \\ +10b^2\gamma\end{matrix} & (\alpha+\beta+\gamma)b^2 & b_1b^2C_4 \\ 3(\alpha+\beta)a^2 & 3(\alpha+\beta)b^2 & 3\gamma a^2 & \begin{matrix}6a^2(\alpha+\beta+\gamma) \\ +3\gamma b^2\end{matrix} & b_1a^2C_4 \\ -(\alpha+\beta)a^2b^2 & 0 & -3\gamma a^2b^2 & -(\alpha+\beta+\gamma)a^2b^2 & a_{12} \end{vmatrix}$$

and

$$\Delta_{61}=\begin{vmatrix} (\alpha+\beta+\gamma)a^2 & (\alpha+\beta+\gamma)b^2+10a^2\gamma & 0 & 2(\alpha+\beta)a^2 & b_1a^2C_3 \\ 6(\alpha+\beta+\gamma)b^2+3a^2\gamma & 3b^2\gamma & 3a^2(\alpha+\beta) & 3(\alpha+\beta)b^2 & b_1b^2C_3 \\ 2(\alpha+\beta)b^2 & 0 & (\alpha+\beta+\gamma)a^2+10b^2\gamma & (\alpha+\beta+\gamma)b^2 & b_1b^2C_4 \\ 3(\alpha+\beta)a^2 & 3(\alpha+\beta)b^2 & 3\gamma a^2 & 6a^2(\alpha+\beta+\gamma)+3\gamma b^2 & b_1a^2C_4 \\ -(\alpha+\beta+\gamma)a^2b^2 & -9\gamma a^2b^2 & 0 & -(\alpha+\beta)a^2b^2 & a_{11} \end{vmatrix}$$

The volume flux for the second order approximation is

$$F_2 = \iint w_2(x,y)\,dx\,dy$$

$$= 4\int_{y=0}^{b}\int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} (b^2x^2+a^2y^2-a^2b^2)(A_1x^2+A_2y^2+A_3)\,dxdy$$

$$= 4\int_{y=0}^{b}\int_{x=0}^{\frac{a}{b}\sqrt{b^2-y^2}} \{A_1b^2x^4+(A_1a^2+A_2b^2)x^2y^2 +x^2(A_3-A_1a^2)b^2+A_2a^2y^4 +(A_3-A_2b^2)a^2y^2-A_3a^2b^2\}\,dxdy\,, \qquad (4.51)$$

which on integration gives

$$F_2 = -\frac{\pi}{12}\,[A_1 a^5 b^3 + A_2 a^3 b^5 + 6A_3 a^3 b^3] \tag{4.52}$$

where

$$A_1 = \frac{b_1^2 b^2 P[3s(a^2+b^2)(a^4+b^4+6a^2b^2)+3a^6+3b^6+29a^2b^4+61a^4b^2]}{24\mu^2\gamma(a^2+b^2)(a^4+b^4+6a^2b^2)[s(3a^4+3b^4+2a^2b^2)+3a^4+3b^4+10a^2b^2]} \tag{4.53}$$,

$$A_2 = \frac{b_1^2 a^2 P[3s(a^2+b^2)(a^4+b^4+6a^2b^2)+3a^6+3b^6+29a^4b^2+61a^2b^4]}{24\mu^2\gamma(a^2+b^2)(a^4+b^4+6a^2b^2)[s(3a^4+3b^4+2a^2b^2)+3a^4+3b^4+10a^2b^2]} \tag{4.54}$$

and

$$A_3 = \frac{1}{2(a^2+b^2)}\,[2a^2b^2(A_1+A_2) - \frac{b_1^2 P}{\mu^3}]$$

$$+ \frac{b_1^2 a^2 b^2 P[s(a^2+b^2)^2+a^4+b^4+6a^2b^2]}{4\mu^2\gamma(a^2+b^2)^2[s(3a^4+3b^4+2a^2b^2)+3a^4+3b^4+10a^2b^2]}\,, \tag{4.55}$$

$$s = \frac{\alpha+\beta}{\gamma}\,.$$

Clearly, $F_1 = -\left(\frac{b_1}{4\mu}\right)F_0$ (4.56)

Area of an ellipse with semi-axes a and b

$$= \pi ab\,, \tag{4.57}$$

hence mean fluxes:

$$\overline{F}_0 = \frac{P\ a^2b^2}{4\mu(a^2+b^2)} ,$$

$$\overline{F}_1 = -\frac{b_1Pa^2b^2}{16\mu^2(a^2+b^2)} , \tag{4.58}$$

and

$$\overline{F}_2 = -\frac{1}{12}/[A_1a^4b^2 + A_2a^2b^4 + 6A_3a^2b^2] .$$

The effect of spin does not enter F_1 (since $\nu_0 = \phi_0 = 0$), but F_2 is definitely effected.

We may find the effect of eccentricity e of the ellipse on flux: $b^2 = a^2(1-e^2)$. We take $a = 1$, then

$$\overline{F}_0 = \frac{P(1-e^2)}{4\mu(2-e^2)} ,$$

$$\overline{F}_1 = -\frac{b_1P(1-e^2)}{16\mu^2(2-e^2)} , \tag{4.59}$$

and $\overline{F}_2 = -\frac{1}{12}[A_1(1-e^2) + A_2(1-e^2)^2 + 6A_3(1-e^2)]$.

We plot $\overline{F}_0$, $\overline{F}_1$ and $\overline{F}_2$ [Figures 4.2 through 4.7] against e (eccentricity). F_2 shows the effect of the constants α , β and γ . These in turn might be calculated by running the fluid through elliptic pipes of different eccentricity.

If we let $P = 5$, $b_1 = 0.1$, $\mu = 0.5$, then we have the tables and graphs for various s, γ as follows:

Table 1

e	$\overline{F}_0$
0.00	1.250000
0.25	1.209625
0.50	1.071375
0.75	0.761250

Table 2

e	$\overline{F}_1$
0.00	-0.062500
0.25	-0.060481
0.50	-0.053569
0.75	-0.038063

Table 3(a)

When $s = 3$, $\gamma = 3$

e	$\overline{F}_2$
0.00	0.022743
0.25	0.047048
0.50	0.041609
0.75	0.029587

Table 3(b)

When $s = 3$, $\gamma = 1$

e	$\overline{F}_2$
0.00	0.044792
0.25	0.043516
0.50	0.039062
0.75	0.028620

Table 3(c)

When $s = 3, \gamma = \frac{1}{3}$

e	$\overline{F}_2$
0.00	0.034375
0.25	0.033775
0.50	0.031471
0.75	0.024990

Table 3(d)

When $s = 1, \gamma = 1$

e	$\overline{F}_2$
0.00	0.044792
0.25	0.041574
0.50	0.039057
0.75	0.029041

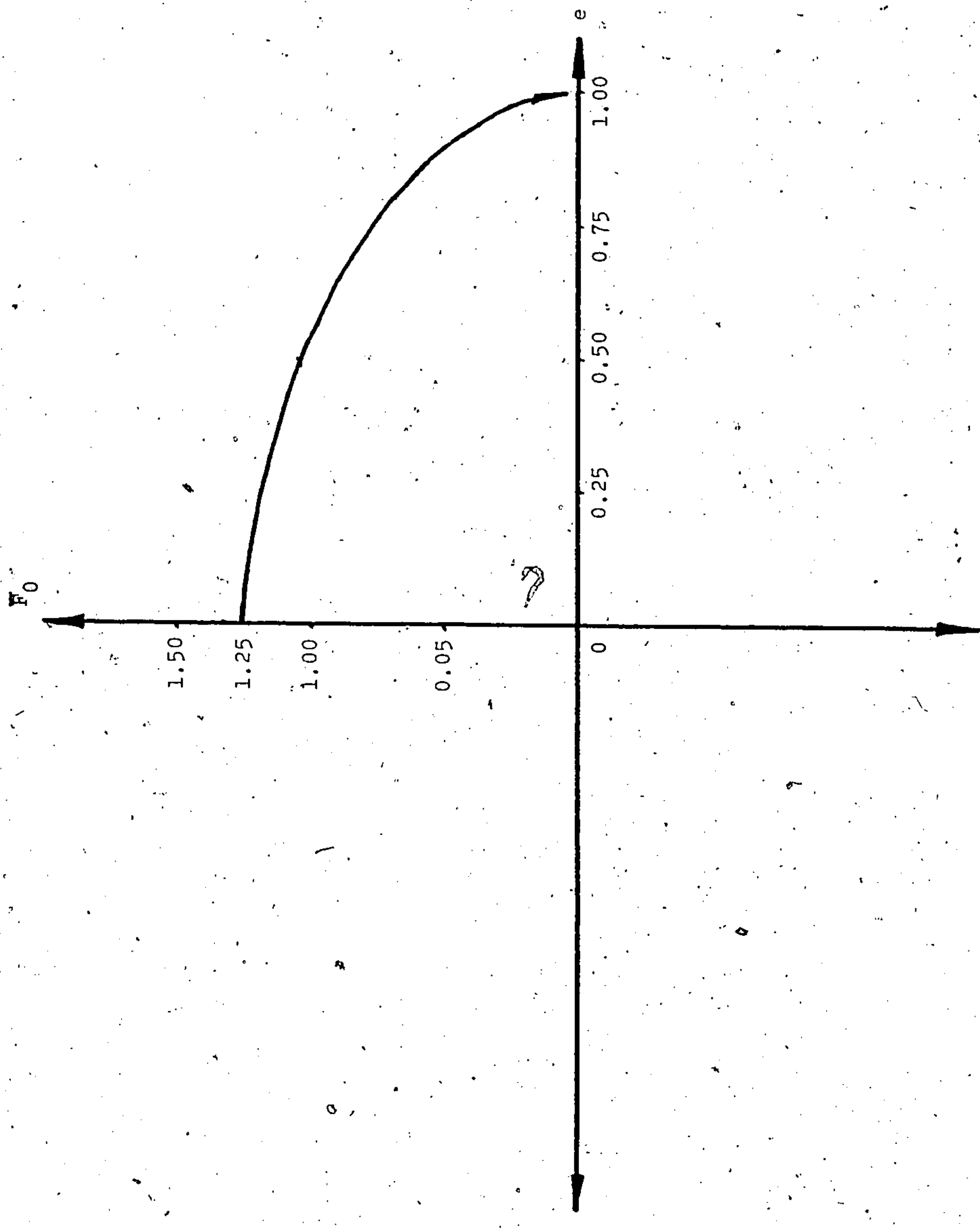


Figure 4.2

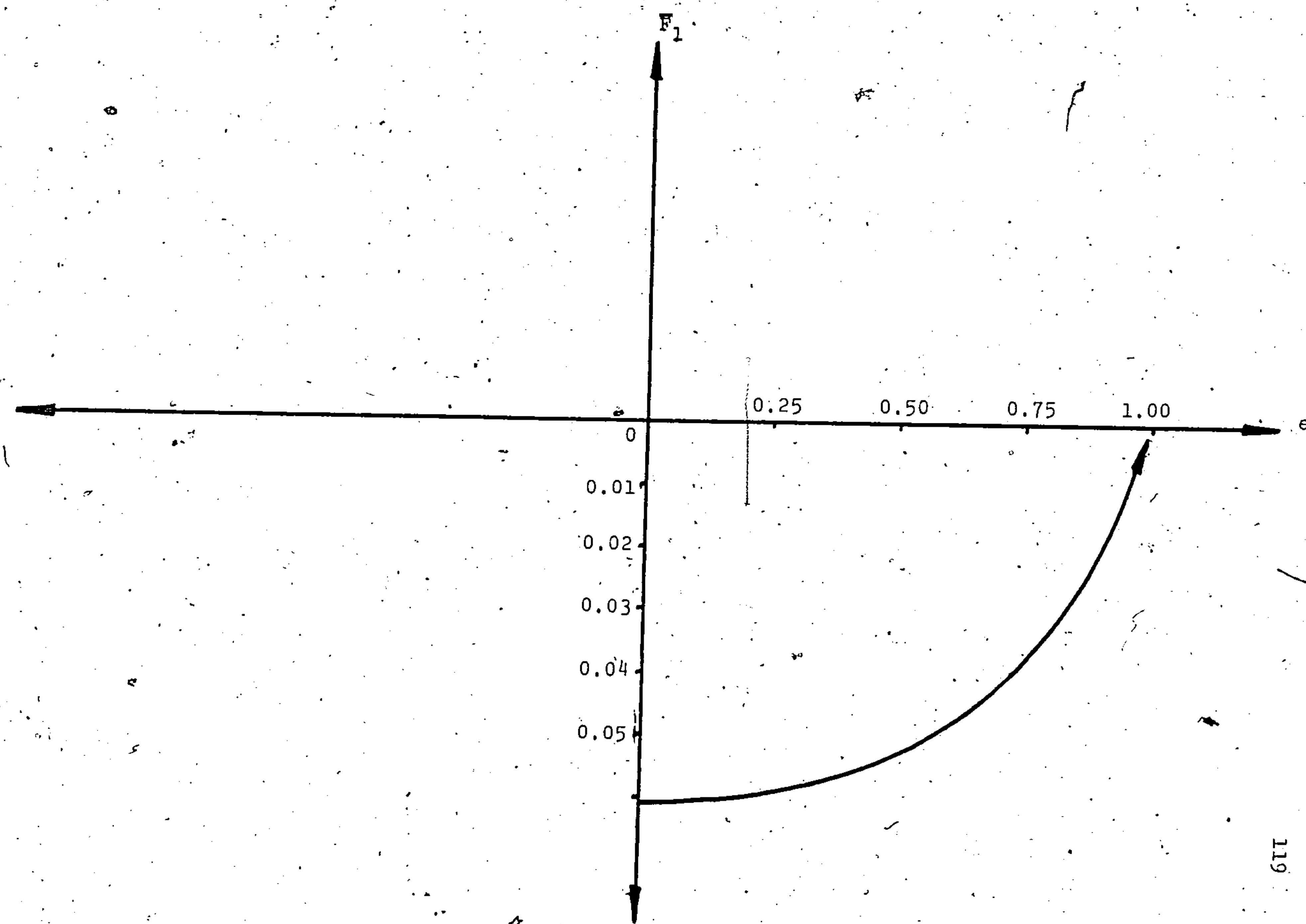


Figure 4.3

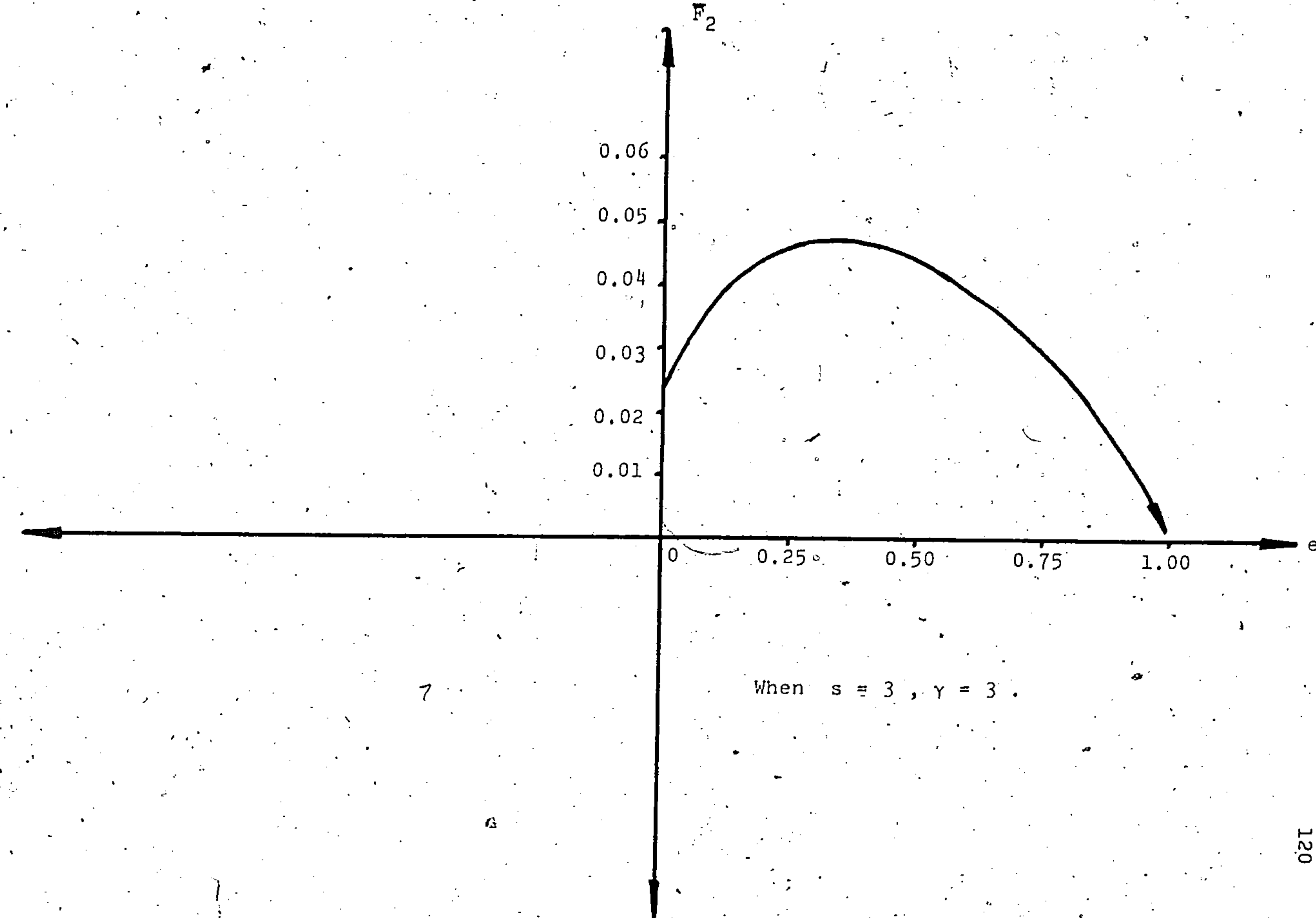


Figure 4.4

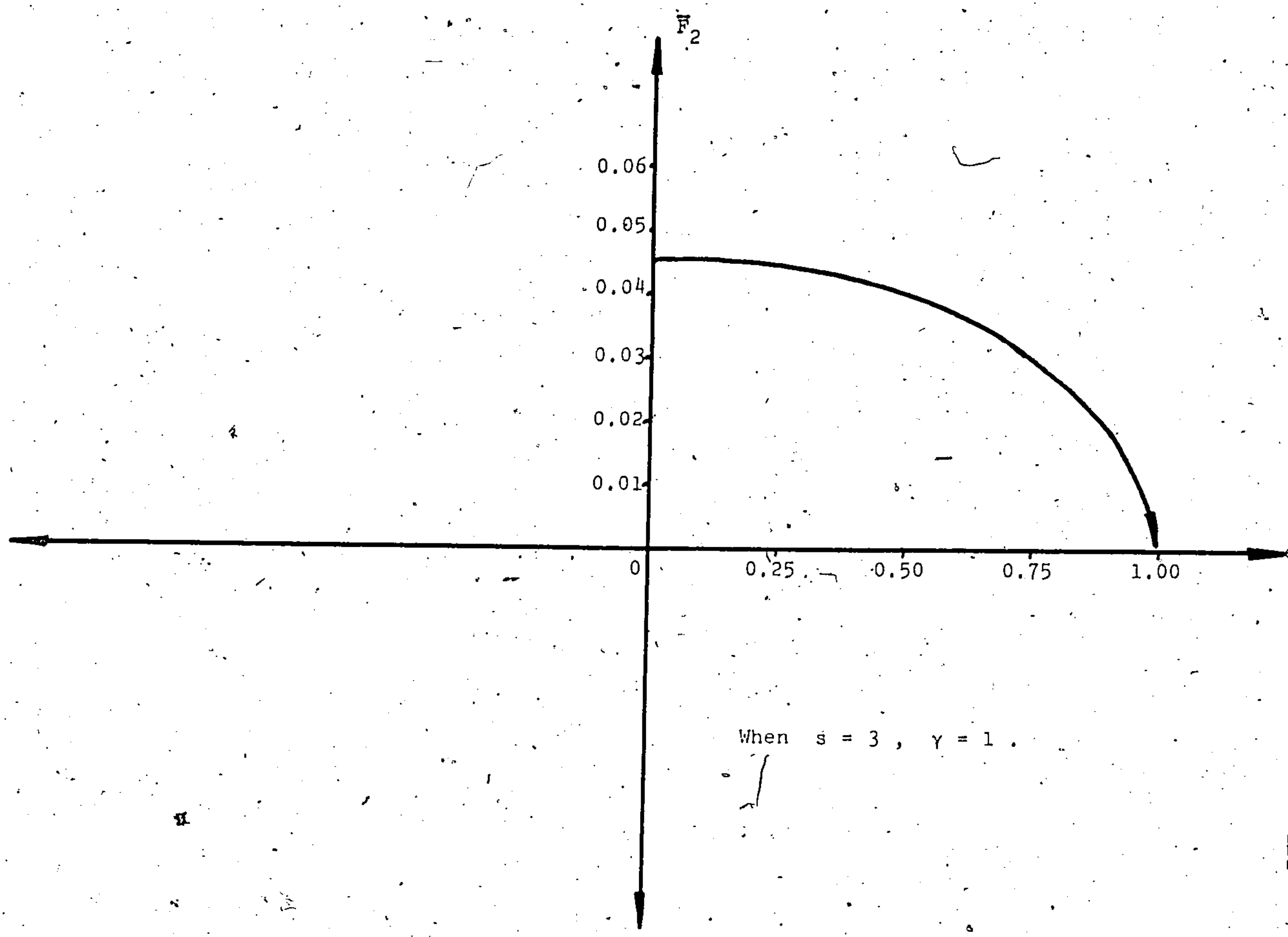


Figure 4.5

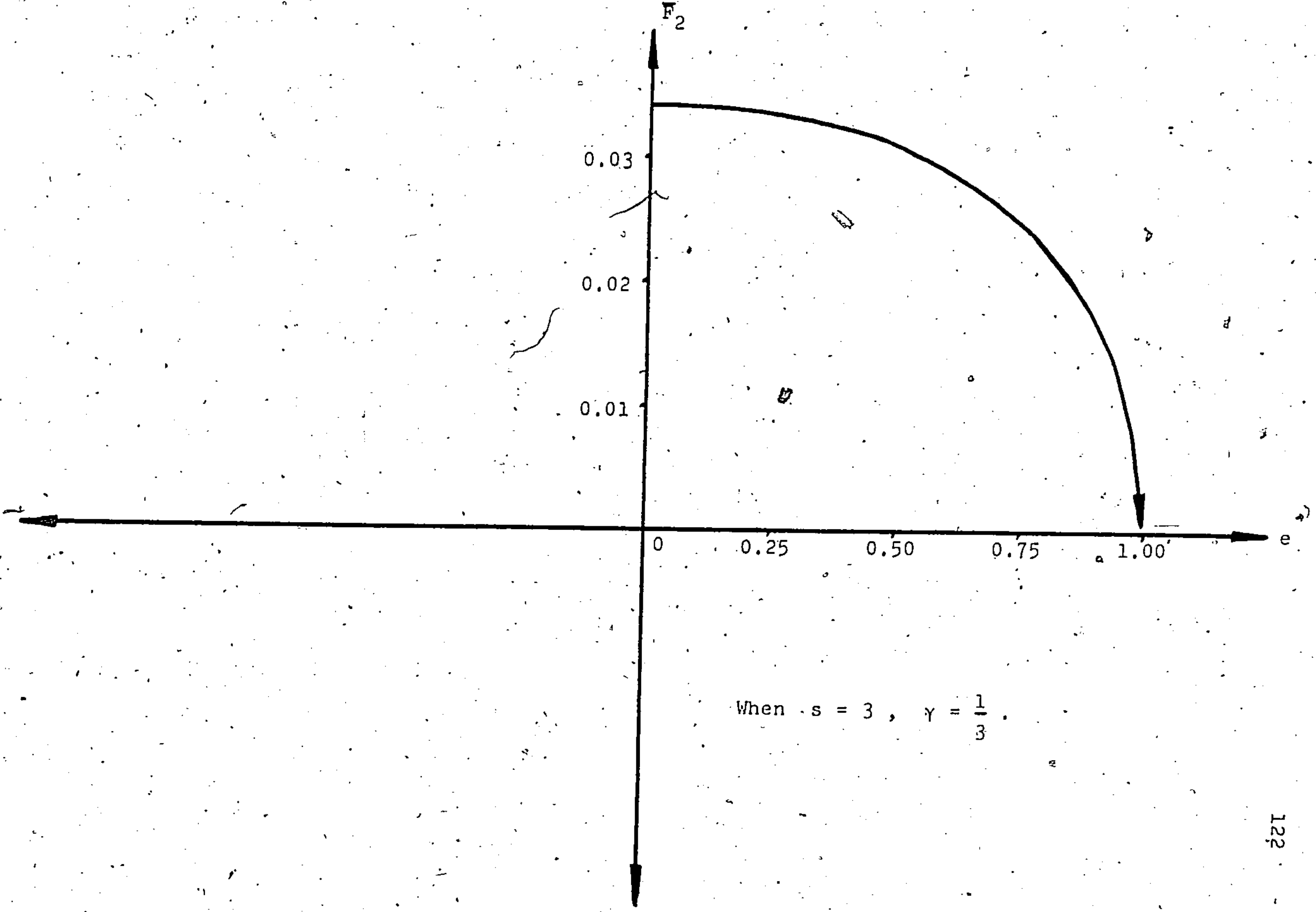


Figure 4.6

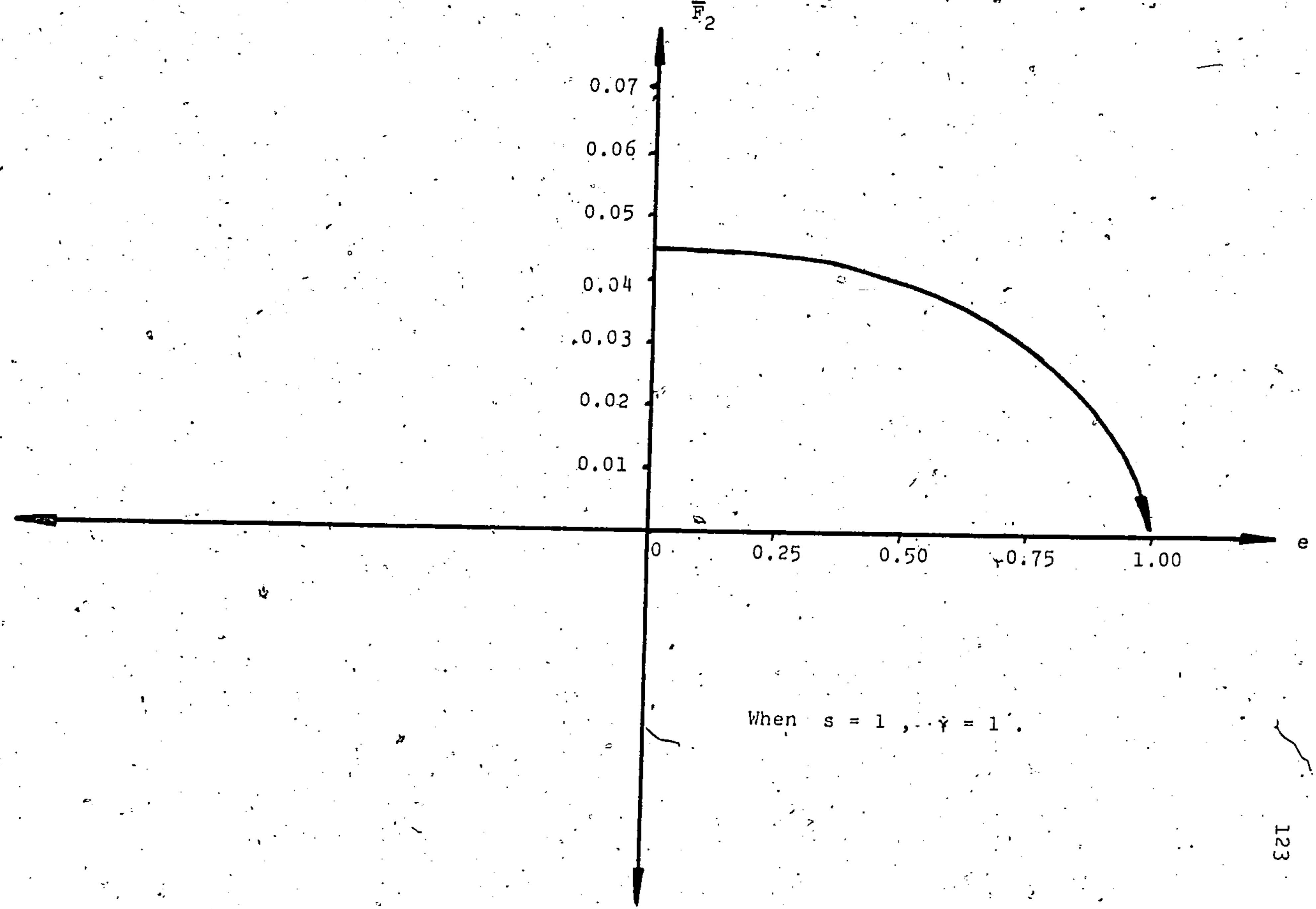


Figure 4.7

The solutions of the partial differential equations (4.16), subject to the boundary conditions that $w_3(x,y) = \nu_3(x,y) = \phi_3(x,y) = 0$ when $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, are of the form

$$w_3(x,y) = (b^2x^2+a^2y^2-a^2b^2)(B_1x^4+B_2x^2y^2+B_3y^4+B_4x^2+B_5y^2+B_6), \tag{4.60}$$

$$\nu_3(x,y) = y(b^2x^2+a^2y^2-a^2b^2)(B_7x^4+B_8x^2y^2+B_9y^4+B_{10}x^2+B_{11}y^2+B_{12}), \tag{4.61}$$

and

$$\phi_3(x,y) = x(b^2x^2+a^2y^2-a^2b^2)(B_{13}x^4+B_{14}x^2y^2+B_{15}y^4+B_{16}x^2+B_{17}y^2+B_{18}) \tag{4.62}$$

provided that $x \neq 0$, $y \neq 0$, and where B's are constants which may be found in a similar manner as the constants in the zeroth, first and second order approximations.

From the constitutive equations of micropolar fluids, namely (1.4) and (1.5), we obtain the stresses and couple stresses as follows:

$$
\begin{aligned}
t_{xx} &= t_{yy} = t_{zz} = -p \,, \\
t_{xy} &= t_{yx} = 0 \,, \\
t_{yz} &= (\mu+\kappa)\frac{\partial w}{\partial y} - \kappa\nu \,, \\
t_{zy} &= \mu \frac{\partial w}{\partial y} + \kappa\nu \,, \\
t_{xz} &= (\mu+\kappa)\frac{\partial w}{\partial x} + \kappa\phi \,, \\
t_{zx} &= \mu \frac{\partial w}{\partial x} - \kappa\phi \,,
\end{aligned}
\qquad (4.63)
$$

and

$$
\begin{aligned}
m_{xx} &= (\alpha+\beta+\gamma)\frac{\partial \nu}{\partial x} + \alpha \frac{\partial \phi}{\partial y} \,, \\
m_{yy} &= (\alpha+\beta+\gamma)\frac{\partial \phi}{\partial y} + \alpha \frac{\partial \nu}{\partial x} \,, \\
m_{zz} &= \alpha\left(\frac{\partial \nu}{\partial x} + \frac{\partial \phi}{\partial y}\right) \,, \\
m_{xy} &= \beta \frac{\partial \nu}{\partial y} + \gamma \frac{\partial \phi}{\partial x} \,, \\
m_{yx} &= \beta \frac{\partial \phi}{\partial x} + \gamma \frac{\partial \nu}{\partial y} \,, \\
m_{xz} &= m_{zx} = m_{yz} = m_{zy} = 0 \,.
\end{aligned}
\qquad (4.64)
$$

Section 3. Remarks

1. For $\kappa = 0$ and the vanishing micro-rotation or spin, our solution (4.9) reduces to the classical solution for Newtonian fluids,

$$w(x,y) = w_0(x,y) = -\frac{P}{2\mu(a^2+b^2)}\,(b^2x^2+a^2y^2-a^2b^2) \,. \tag{4.65}$$

2. The flow is not secondary here, whereas in the case of non-Newtonian fluids and Visco-elastic fluids, the flow is observed to be secondary by Green and Rivlin (1956) and by Langlois and Rivlin (1963) respectively.

3. The curves of spin velocity, ν=constant and ϕ = constant [Figure 4.8], are similar in pattern to the secondary flow type as observed in non-Newtonian fluids by Green and Rivlin (1956), and in visco-elastic fluids by Langlois and Rivlin (1963).

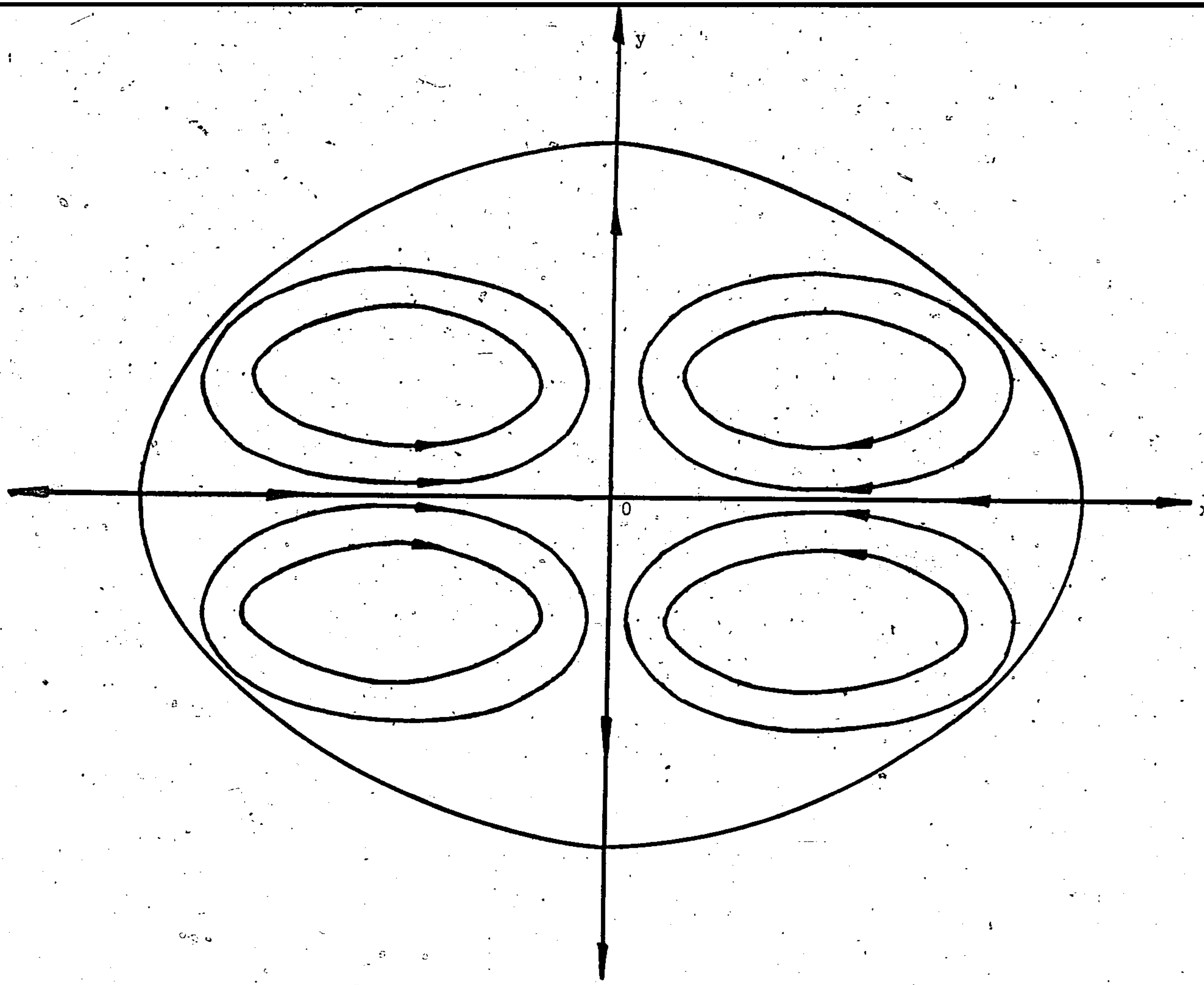


Figure 4.8 Graph of curves, ν=constant, and ϕ=constant.

CHAPTER V

TWO DIMENSIONAL UNSTEADY MICROPOLAR FLOWS

In this chapter we have extended Kampe de Feriet's (1932) method to find solutions such that the vorticity ω and the spin ϕ are constant along streamlines, that is, $\omega = \omega(\psi)$ and $\phi = \phi(\psi)$ at any time. We also study Taylor (1923) Motions and motion due to a single vortex and/or spin. One may refer to Eringen (1967, p. 253-256) for the classical treatment of these problems.

Section 1: Equations of motion.

We consider two-dimensional, unsteady, incompressible flow of micropolar fluids with equations of motion in Cartesian co-ordinates

$$(\mu+\kappa)\nabla^2 u + \kappa\frac{\partial\phi}{\partial y} - \frac{\partial p}{\partial x} = \rho(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t}) \tag{5.1}$$

$$(\mu+\kappa)\nabla^2 v - \kappa\frac{\partial\phi}{\partial x} - \frac{\partial p}{\partial y} = \rho(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t}) \tag{5.2}$$

$$\gamma\nabla^2\phi + \kappa(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) - 2\kappa\phi = \rho j(u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial t}) \tag{5.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.4}$$

where

$$\underset{\sim}{v} = (u(x,y,t),\ v(x,y,t),\ 0)$$

$$\underset{\sim}{\nu} = (0,\ 0,\ \phi(x,y,t)). \tag{5.5}$$

To eliminate pressure p between (5.1) and (5.2), we differentiate (5.1) with respect to y and (5.2) with respect to x and then subtracting, we get

$$(\mu+\kappa)\nabla^2\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \kappa\nabla^2\phi$$

$$= \rho\left[\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right], \qquad (5.6)$$

where we have made use of the equation of continuity (5.4), and the assumption that $\frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial y \partial x}$.

We introduce a stream function $\psi(x,y,t)$ such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \qquad (5.7)$$

Using (5.7) in the equations (5.6) and (5.3), after a little simplification, we find that

$$(\mu+\kappa)\nabla^4\psi + \kappa\nabla^2\phi = \rho\left|\frac{\partial(\nabla^2\psi,\psi)}{\partial(x,y)}\right| + \rho\frac{\partial}{\partial t}(\nabla^2\psi); \qquad (5.8)$$

$$\gamma\nabla^2\phi - \kappa\nabla^2\psi - 2\kappa\phi = \rho j\left|\frac{\partial(\phi,\psi)}{\partial(x,y)}\right| + \rho j\frac{\partial \phi}{\partial t}. \qquad (5.9)$$

Let $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2\psi = \omega$ (vorticity), then the equations (5.8) and (5.9) become

$$(\mu+\kappa)\nabla^2\omega - \kappa\nabla^2\phi = \rho\left|\frac{\partial(\omega,\psi)}{\partial(x,y)}\right| - \rho\frac{\partial \omega}{\partial t}, \qquad (5.10)$$

$$\gamma\nabla^2\phi + \kappa\omega - 2\kappa\phi = \rho j\left|\frac{\partial(\phi,\psi)}{\partial(x,y)}\right| + \rho j\frac{\partial \phi}{\partial t}. \qquad (5.11)$$

Section 2: Kampe de Feriet solutions in which vorticity and spin are constant along streamlines.

Following Kampe de Feriet's (1932) method, the non-linear terms in the equations (5.10) and (5.11) vanish when

$$\omega = \omega(\psi) \;, \quad \phi = \phi(\psi) \;,$$

that is, the vorticity and spin are constant along a streamline at a particular time. Then the equations (5.10) and (5.11) reduce to

$$(\mu+\kappa)\nabla^2\omega - \kappa\nabla^2\phi = \rho\,\frac{\partial\omega}{\partial t} \tag{5.12}$$

and

$$\gamma\nabla^2\phi + \kappa\omega - 2\kappa\phi = \rho j\frac{\partial\phi}{\partial t} \;. \tag{5.13}$$

Now we introduce streamline co-ordinates with metric (orthogonal)

$$ds^2 = h_1^2\,dq_1^2 + h_2^2\,dq_2^2$$

such that $\psi = q_1$.

We must have the compatibility condition (since the region is a plane)

$$\frac{\partial}{\partial q_1}\left[\frac{1}{h_1}\,\frac{\partial h_2}{\partial q_1}\right] + \frac{\partial}{\partial q_2}\left[\frac{1}{h_2}\,\frac{\partial h_1}{\partial q_2}\right] = 0 \;. \tag{5.14}$$

Now

$$\nabla^2 W = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2}{h_1}\,\frac{\partial W}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1}{h_2}\,\frac{\partial W}{\partial q_2}\right)\right] \tag{5.15}$$

hence

$$\omega = -\nabla^2\psi = -\frac{1}{h_1 h_2}\frac{\partial}{\partial q_1}\left(\frac{h_2}{h_1}\right) \tag{5.16}$$

since $\psi = q_1$

and

$$\nabla^2\omega = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2}{h_1}\frac{\partial\omega}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1}{h_2}\frac{\partial\omega}{\partial q_2}\right)\right]$$

$$= \frac{1}{h_1^2}\frac{\partial^2\omega}{\partial q_1^2} - \omega\frac{\partial\omega}{\partial q_1} \tag{5.17}$$

when $\omega = \omega(\psi) = \omega(q_1)$.

Using (5.14) through (5.17) in the equations (5.12) and (5.13), we find that

$$\frac{1}{h_1^2}\left\{(\mu+\kappa)\frac{\partial^2\omega}{\partial q_1^2} - \kappa\frac{\partial^2\phi}{\partial q_1^2}\right\} - \omega\left\{(\mu+\kappa)\frac{\partial\omega}{\partial q_1} - \kappa\frac{\partial\phi}{\partial q_1}\right\} = \rho\frac{\partial\omega}{\partial t}, \tag{5.18}$$

$$\frac{1}{h_1^2}\left\{\gamma\frac{\partial^2\phi}{\partial q_1^2} - \gamma h_1^2\,\omega\frac{\partial\phi}{\partial q_1}\right\} + \kappa\omega - 2\kappa\phi = \rho j\frac{\partial\phi}{\partial t} . \tag{5.19}$$

Now $\omega = \omega(q_1)$, $\phi = \phi(q_1)$, hence it follows from (5.18) and (5.19) that

$h_1 = h_1(q_1)$ only, in which case the streamlines are concentric circles or parallel straight lines.

<u>OR</u>

$$\therefore \quad \frac{\partial^2 \omega}{\partial q_1^2} = \frac{\partial^2 \phi}{\partial q_1^2} = 0 \ .$$

Let

$$\frac{\partial \omega}{\partial q_1} = k(t) \ ,$$

$$\frac{\partial \phi}{\partial q_1} = m(t) \ ,$$

then equations (5.18) and (5.19) reduce to

$$\frac{\partial \omega}{\partial t} + A(t)\ \omega = 0 \ , \tag{5.20}$$

$$\frac{\partial \phi}{\partial t} + \alpha\phi + B(t)\ \omega = 0 \ , \tag{5.21}$$

where

$$A(t) = \frac{(\mu+\kappa)\ k(t) - \kappa m(t)}{\rho}$$

$$B(t) = \frac{\gamma\ m(t) - \kappa}{\rho j} \tag{5.22}$$

$$\alpha = \frac{2\kappa}{\rho j} \ .$$

From (5.20) we obtain

$$\omega = e^{-C(t)}\ \tilde{W}(x,y) \tag{5.23}$$

where $C'(t) = A(t)$, and the prime denotes differentiation with respect to time t.

But $\nabla^2\omega = -\omega \dfrac{\partial\omega}{\partial q_1} = -k(t)\omega$ (5.24)

and substituting (5.23) into (5.24) we find that

$$\nabla^2 W + k(t)\,W = 0 \tag{5.25}$$

Hence $k(t) = k =$ constant.

Again $\nabla^2\phi = -\omega \dfrac{\partial\phi}{\partial q_1} = -m(t)\omega$. (5.26)

Substituting (5.23) into (5.21), we find that

$$\phi = -e^{-\alpha t}\,[V + I(t)\,W] \tag{5.27}$$

where

$$I(t) = \int^{t} B(u)\, e^{\alpha u - C(u)}\,du \tag{5.28}$$

and $V = V(x,y)$.

Substituting (5.27) and (5.23) into (5.26) and using (5.25), we find that

$$\nabla^2 V = \ell W \tag{5.29}$$

and

$$k\,I(t) + m(t)e^{\alpha t - C(t)} = \ell\ , \tag{5.30}$$

where ℓ is a constant.

Differentiating (5.30) with respect to t , we obtain the differential equation

$$\frac{dm}{dt} + \frac{\kappa}{\rho}\left[m^2(t) + m(t)\left\{\frac{\gamma-(\mu+\kappa)\rho}{\kappa j}k + \frac{2}{j}\right\} - \frac{k}{j}\right] = 0\ . \tag{5.31}$$

The nature of the solution of (5.31) depends on the roots of the quadratic

$$x^2 + x\left[\frac{\gamma-(\mu+\kappa)j}{\kappa j}k + \frac{2}{j}\right] - \frac{k}{j} = 0 \qquad (5.32)$$

which are

$$\begin{matrix} m_1 \\ m_2 \end{matrix} = \frac{1}{2}\left\{\frac{(\mu+\kappa)j - \gamma}{\kappa j} k - \frac{2}{j} \pm \Delta_1\right\}, \qquad (5.33)$$

where

$$\kappa^2 j^2 \Delta_1^2 = [\gamma-(\mu+\kappa)j]^2 k^2 + 4\kappa k[\gamma-\mu j] + 4\kappa^2 . \qquad (5.34)$$

The discriminant of (5.34) is

$$\Delta_2^2 = 32\kappa^3 j\{\gamma-[\mu+\tfrac{1}{2}\kappa]j\}. \qquad (5.35)$$

Clearly, if $\gamma < (\mu+\frac{1}{2}\kappa)j$, $\Delta_2^2 < 0$, and $\Delta_1^2 > 0$ for all values of k.

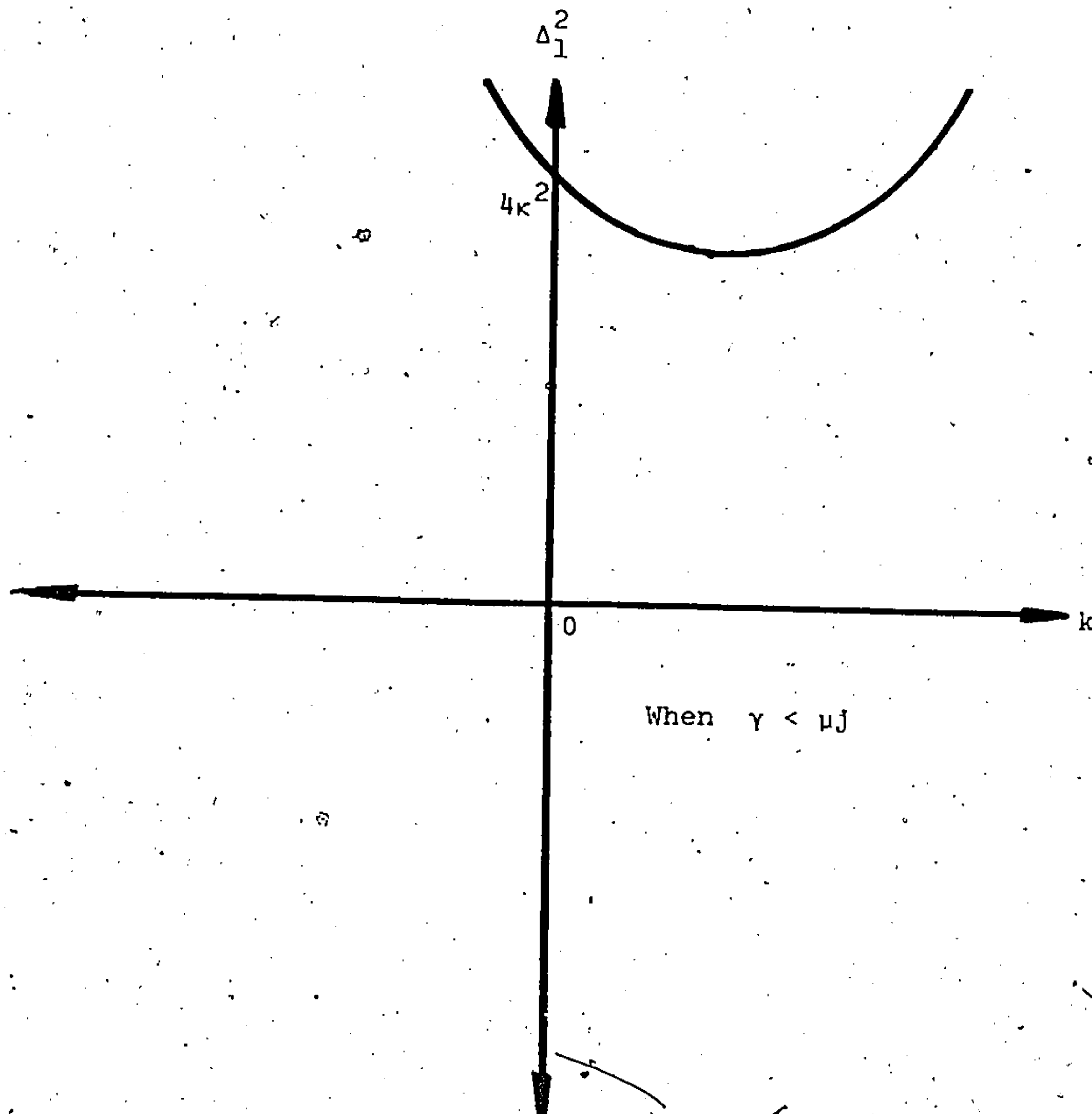
But if $\gamma > (\mu+\frac{1}{2}\kappa)j$, $\Delta_2^2 > 0$, then $\Delta_1^2 = 0$ has two real roots

$$\begin{matrix} k_1 \\ k_2 \end{matrix} = \frac{4(\mu j - \gamma)\kappa \pm \Delta_2}{2[\gamma - (\mu+\kappa)j]^2} . \qquad (5.36)$$

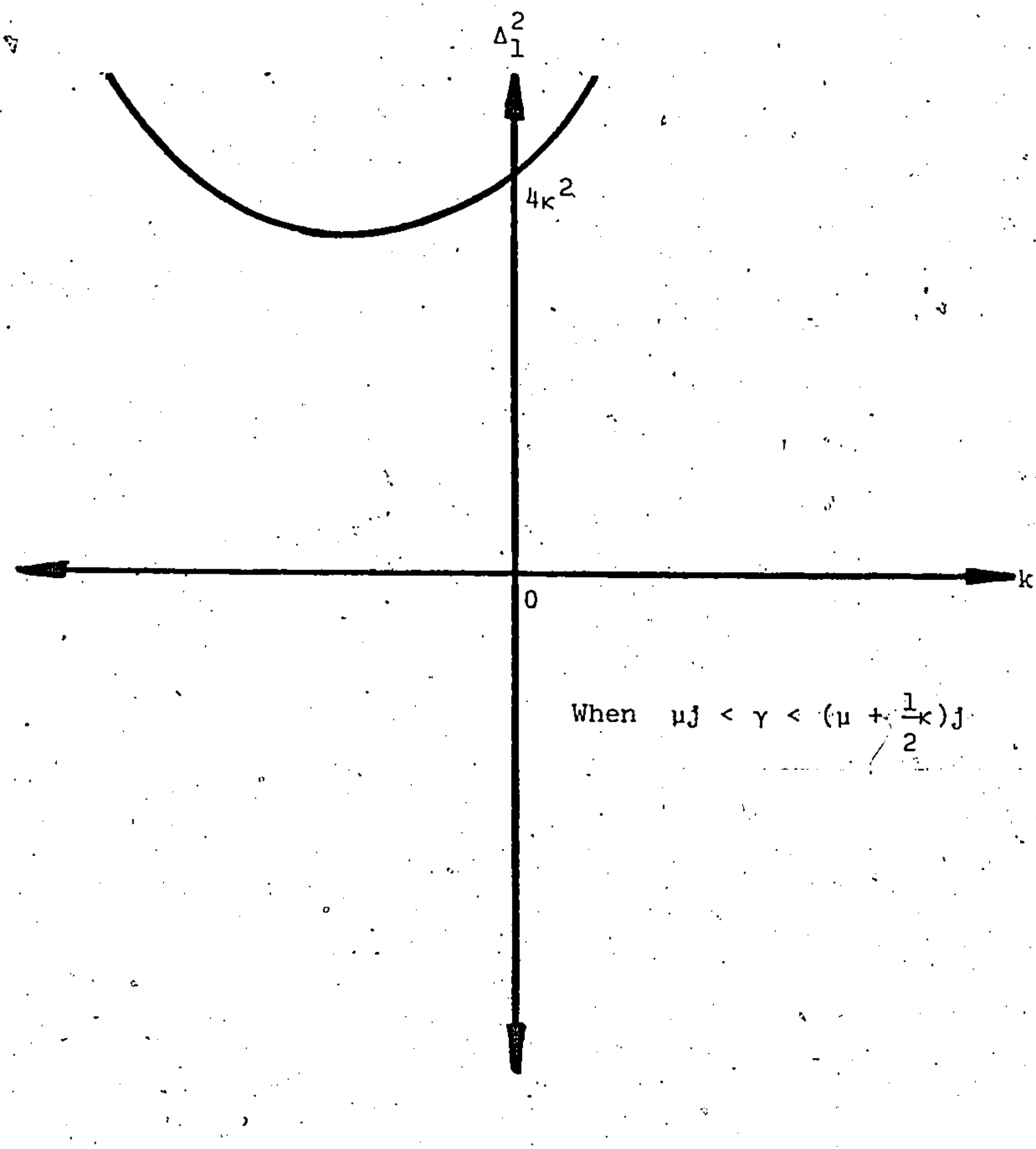
Now

$$\frac{d}{dk}[\Delta_1^2]_{k=0} = 4\kappa[\gamma - \mu j] ,$$

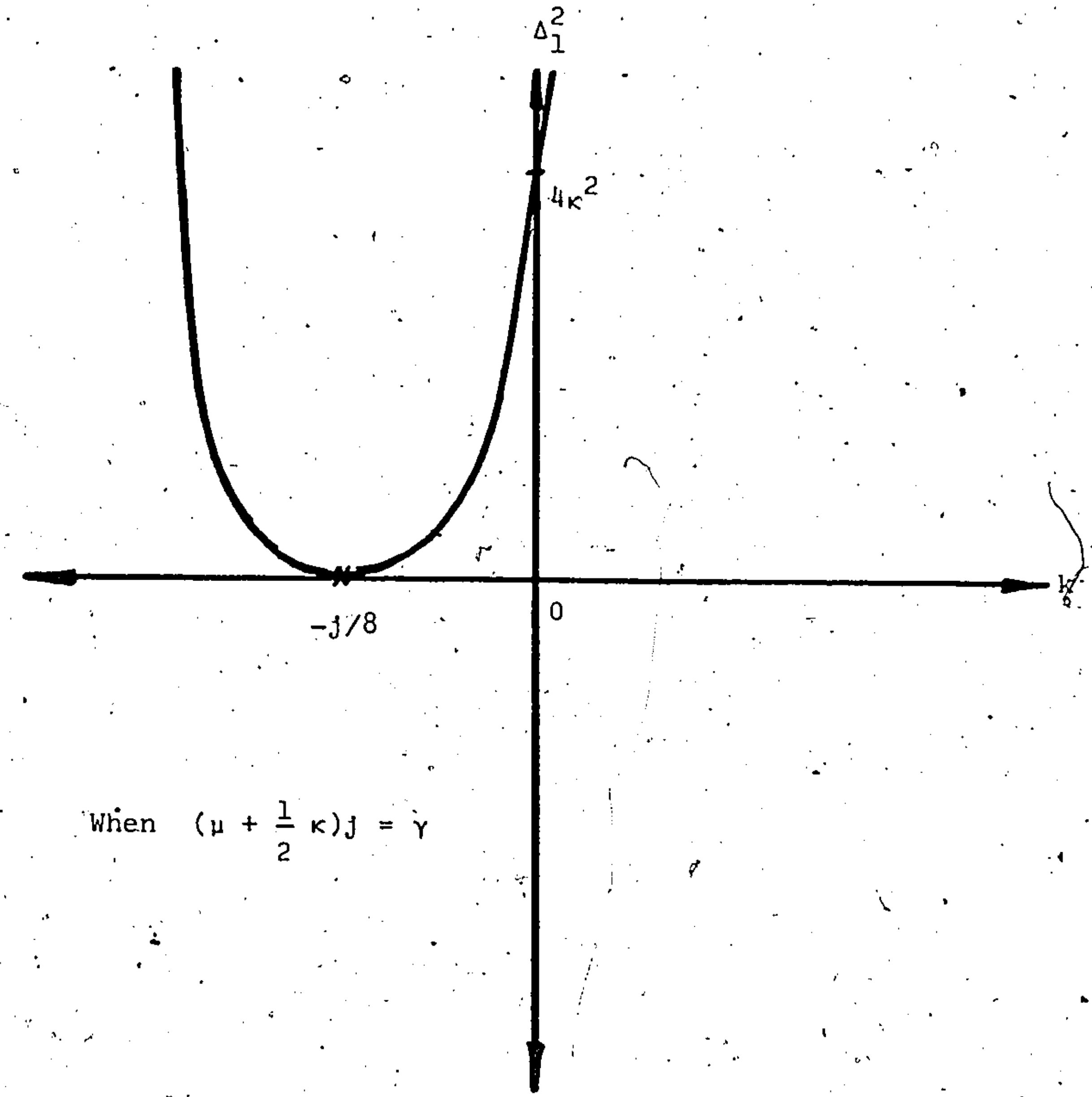
hence we can draw the graphs of Δ_1^2 as a function of k as follows:



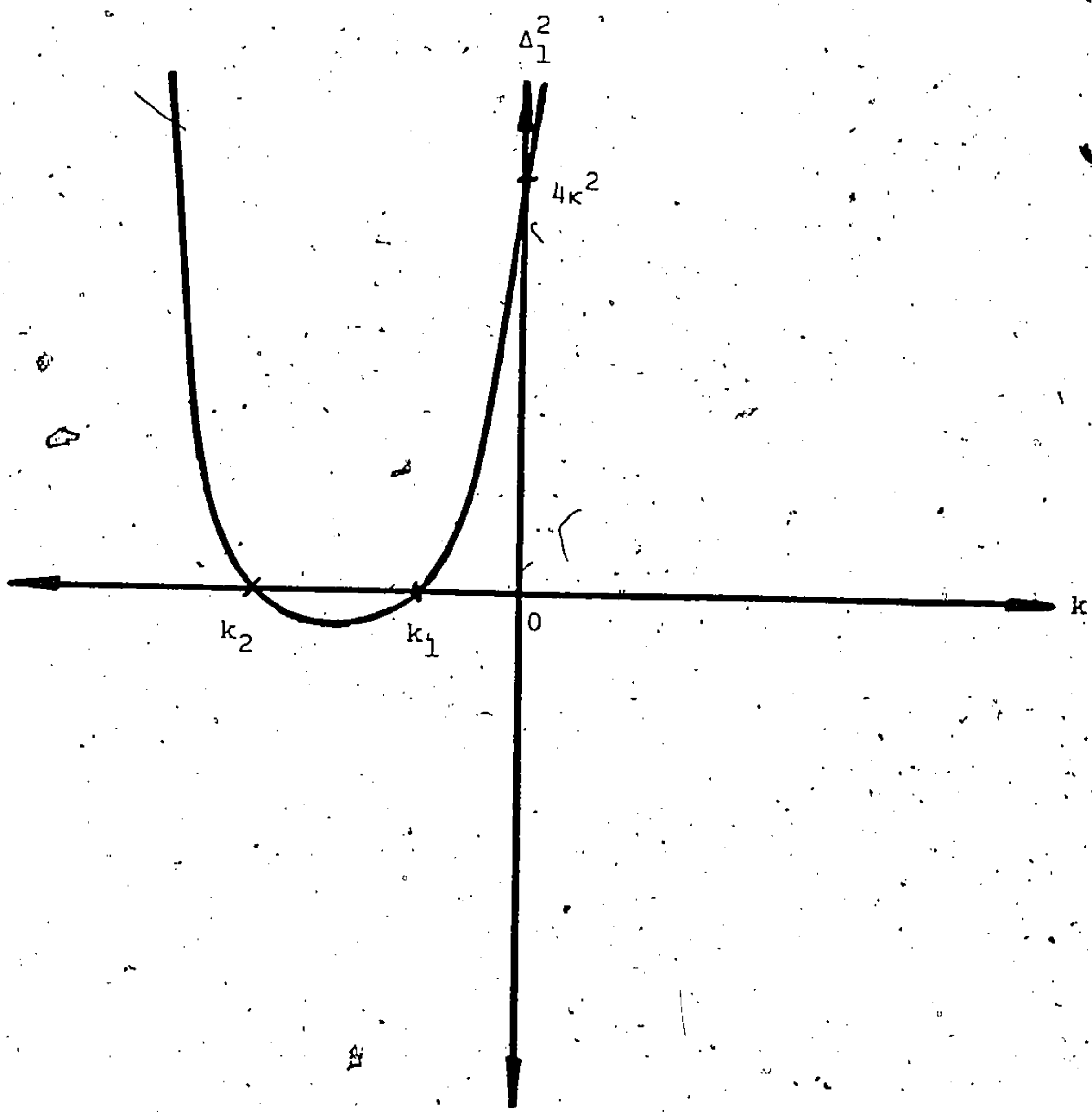
When $\gamma < \mu j$



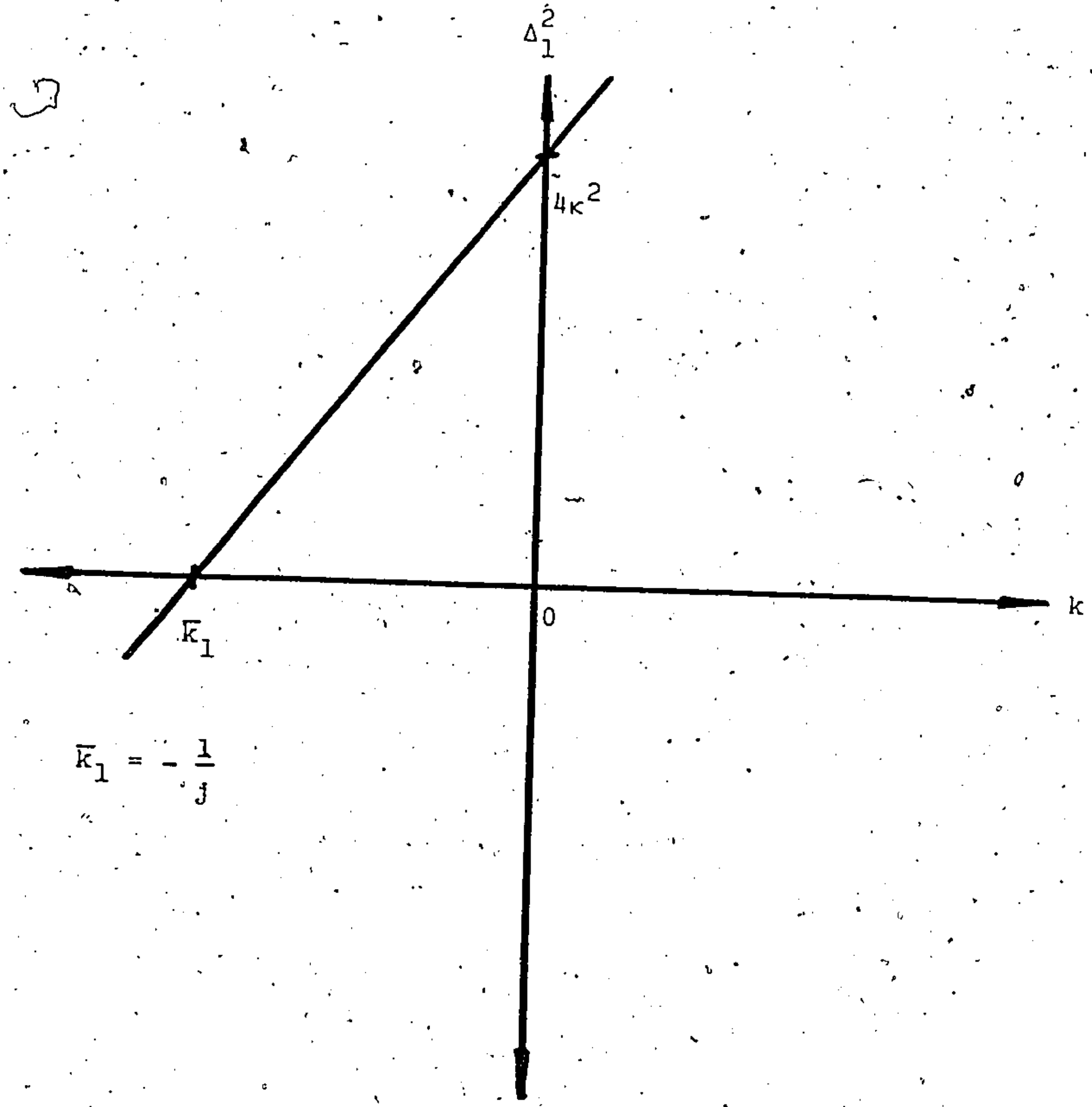
When $\mu j < \gamma < (\mu + \frac{1}{2}\kappa)j$



When $(\mu + \frac{1}{2}\kappa)j = \gamma$

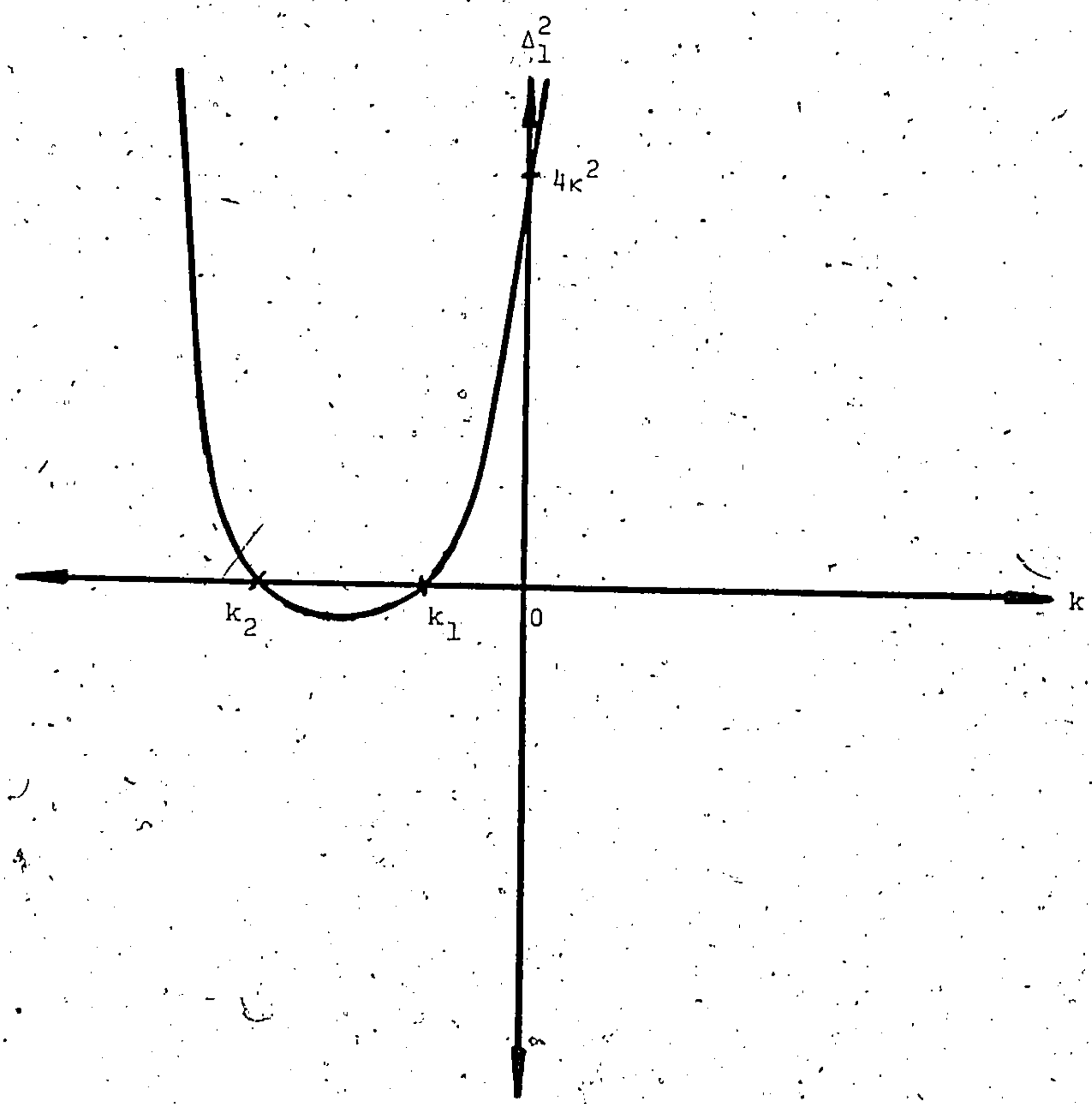


When $(\mu + \frac{1}{2}\kappa)j < \gamma < (\mu + \kappa)j$



$$\bar{k}_1 = -\frac{1}{j}$$

When $(\mu + \kappa)j = \gamma$.



When $(\mu + \kappa)j < \gamma$

The roots m_1 and m_2 are real and distinct if $\Delta_1^2 > 0$; real and equal if $\Delta_1^2 = 0$ and complex if $\Delta_1^2 < 0$.

We obtain the solutions of (5.31) for the three cases.

<u>Case (1).</u> <u>m_1 and m_2 real and distinct.</u>

Then (i), if $(\mu+\frac{1}{2}\kappa)j > \gamma$, k can assume any value.

(ii) if $(\mu+\frac{1}{2}\kappa)j < \gamma$, $k < k_2$, $k > k_1$.

(iii) if $(\mu+\kappa)j = \gamma$, $k > \bar{k}_1$.

The integral of (5.31) is then

$$m(t) = \frac{m_1 - m_2 c \exp[-\delta_1 t]}{1-c \exp[-\delta_1 t]}$$

where $\delta_1 = \frac{\kappa}{\rho} \Delta_1$

and c is an arbitrary constant depending on the initial value of $m(t)$,

$$c = \frac{m(0) - m_1}{m(0) - m_2} .$$

If $m(0) < m_2$, $c > 1$; if $m_2 < m(0) < m_1$ then $c < 0$; and if $m_1 < m(0)$, $0 < c < 1$.

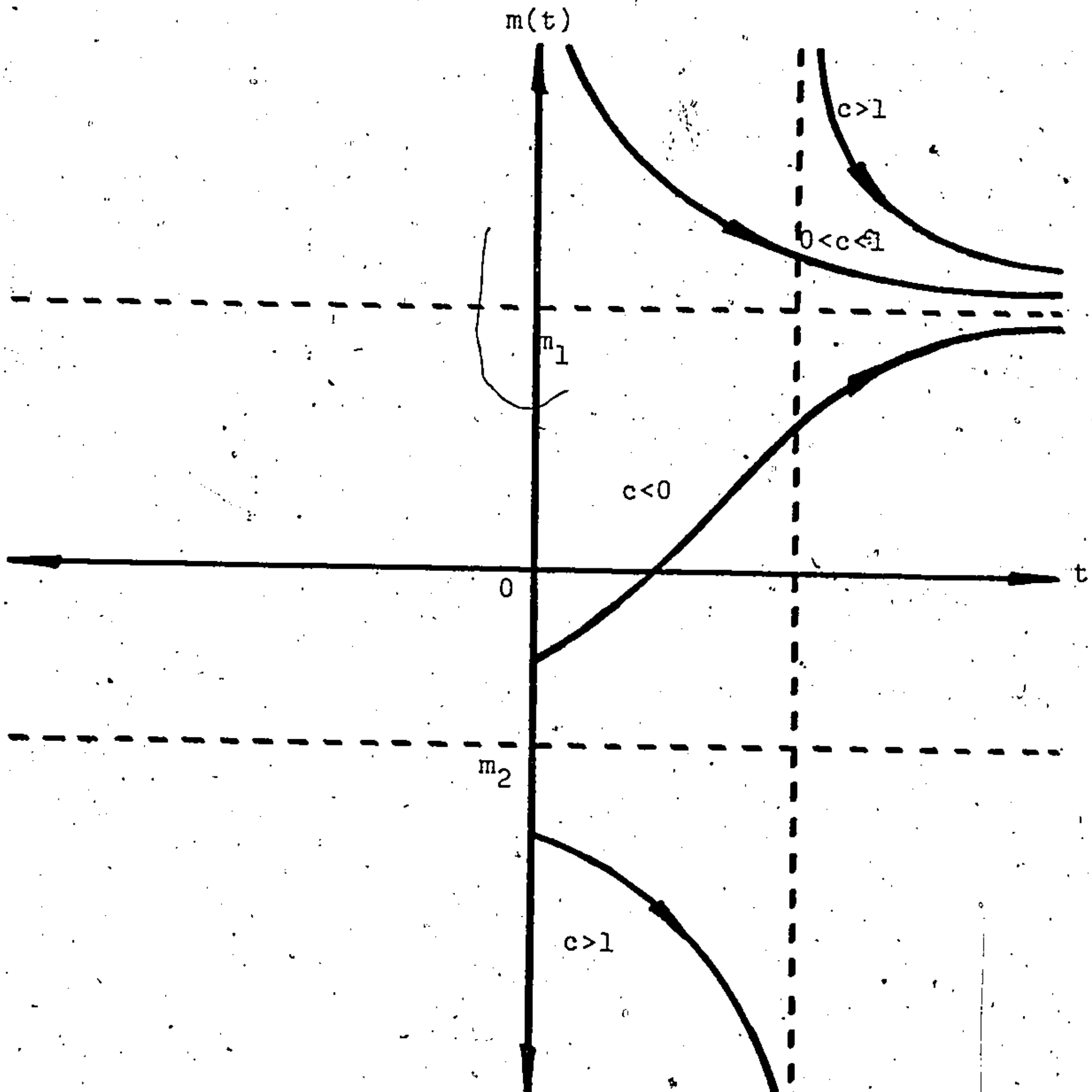


Figure 5.1 Case (1). m_1 and m_2 real and distinct.

<u>Case (2).</u> <u>m_1 and m_2 real and equal, $m_1 = m_2 = m_0$</u> .

Then $\Delta_1 = 0$, and hence

(i) if $(\mu+\frac{1}{2}\kappa)j \leq \gamma < (\mu+\kappa)j$, $k = k_1$ or k_2 .

(ii) if $(\mu+\kappa)j < \gamma$, $k = k_1$ or k_2 .

(iii) if $(\mu+\kappa)j = \gamma$, $k = \bar{k}_1$.

Then (5.31) integrates to give

$$m(t) = m_0 + \frac{\rho}{\kappa(t-t_0)} ,$$

where $t_0 = 0$ without loss of generality,

and

$$m_0 = \frac{(\mu+\kappa)j - \gamma}{2\kappa j} k .$$

<u>Case (3).</u> <u>m_1 and m_2 complex.</u>

Then $\Delta_1^2 < 0$, hence we have

(i) $(\mu+\frac{1}{2}\kappa)j < \gamma < (\mu+\kappa)j$, $k_2 < k < k_1$.

(ii) $(\mu+\kappa)j < \gamma$, $\cancel{k_2 < k < k_1}$.

Then (5.31) integrates to give

$$m(t) = h - p \tan\left[\frac{\kappa}{\rho} p(t - t_0)\right] ,$$

where

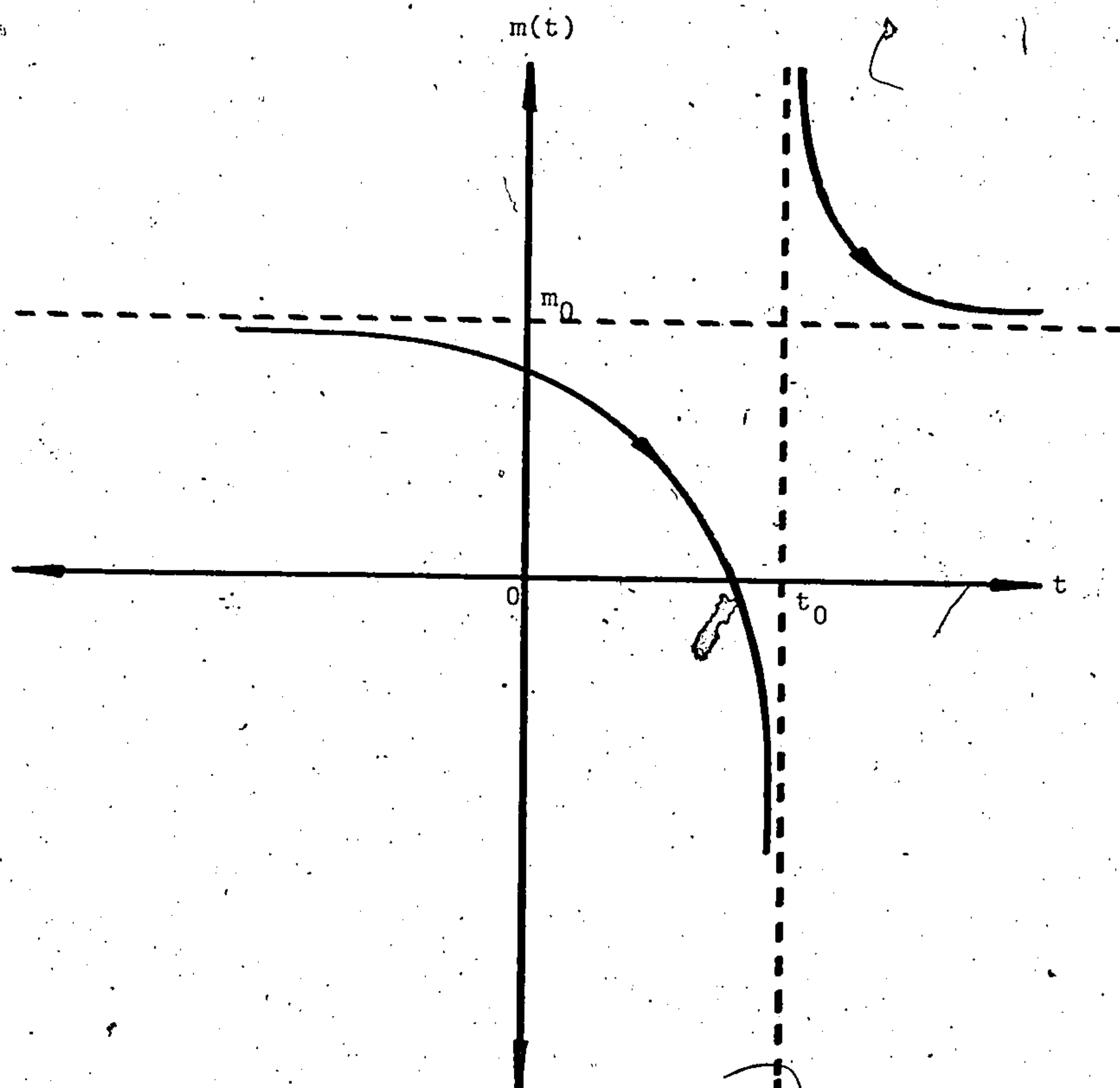


Figure 5.2 Case (2). m_1 and m_2 real and equal,
$m_1 = m_2 = m_0$.

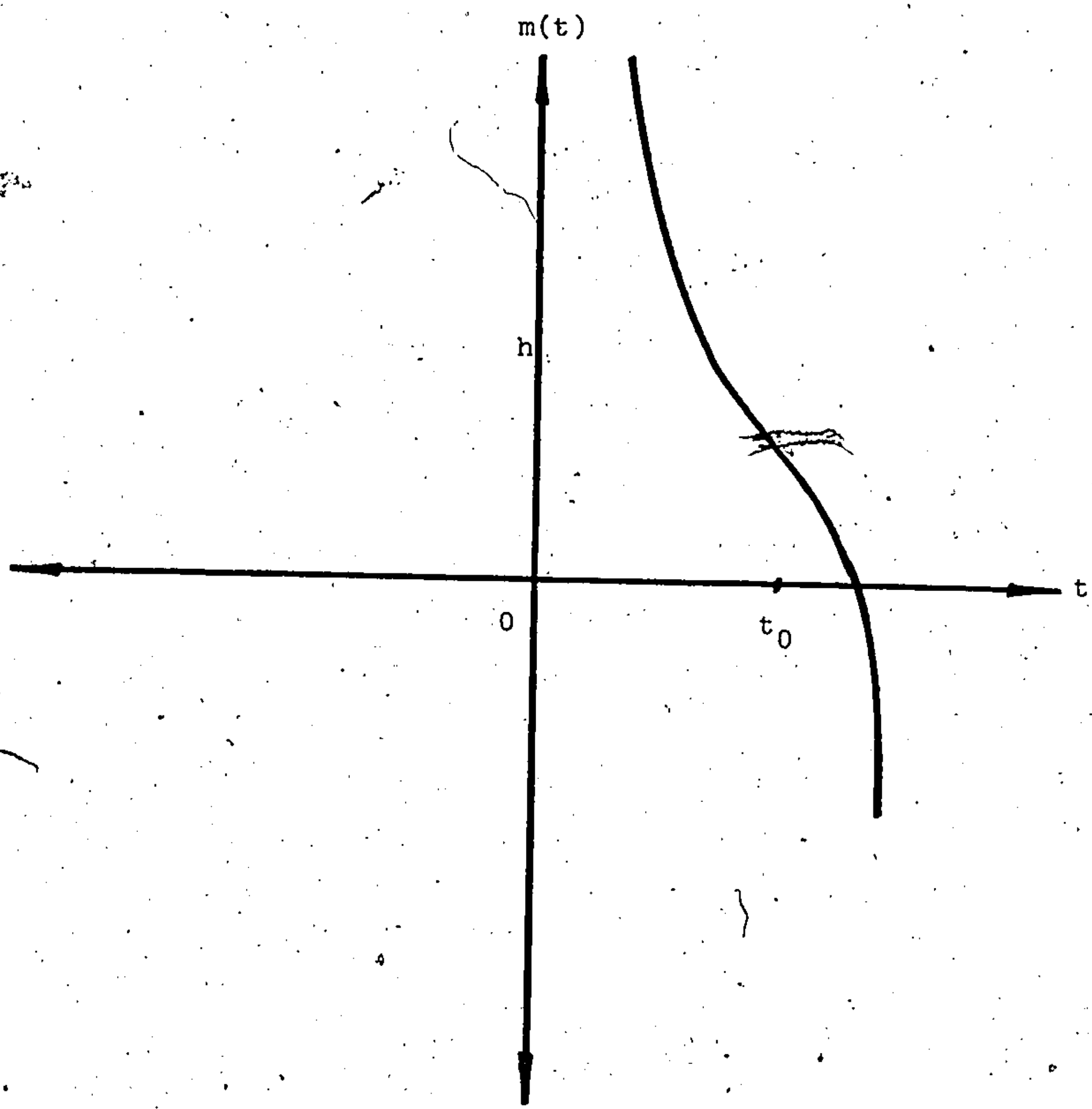


Figure 5.3 Case (3). m_1 and m_2 complex.

$$h = \frac{[(\mu+\kappa)j - \gamma]k - 2\kappa}{2\kappa j}$$

and $p^2 = -\frac{1}{4}\Delta_1^2$.

Now we had found that

$$\omega = W \exp[-C(t)]$$

$$\phi = -e^{-\alpha t}[V + I(t)W]$$

$$\nabla^2 W + kW = 0 \ , \ \nabla^2 V = \ell W \qquad (5.37)$$

$$C'(t) = A(t) = \frac{(\mu+\kappa)k - \kappa m(t)}{\rho}$$

$$I(t) = \int^t B(u)e^{\alpha u - C(u)}du \ .$$

Now $\frac{\partial\omega}{\partial q_1} = k$, $\frac{\partial\phi}{\partial q_1} = m(t)$,

$$\therefore \ \omega = kq_1 = \exp[-C(t)]W \ .$$

$$\therefore \ W = kq_1 \cdot \exp[C(t)] \ , \qquad (5.38)$$

$$\phi = -\exp[C(t) - \alpha t]\{e^{-C(t)}V + kI(t)q_1\} \ .$$

But $\frac{\partial\phi}{\partial q_1} = m(t)$

Therefore,

$$m(t)\exp[\alpha t-C(t)] + kI(t) = -\exp[-C(t)]\frac{\partial V}{\partial q_1} .$$

Using (5.30) in the above equation, we have

$$\frac{\partial V}{\partial q_1} = -\ell \exp[C(t)] .$$

Therefore $V = -\ell q_1 \exp[C(t)]$

$$= -\ell \frac{W}{k}$$

where we have made use of the first of (5.38).

Thus $\phi = -e^{-\alpha t}[-\ell\frac{W}{k} + I(t)W]$

$$= \frac{\exp[-C(t)]m(t)W}{k}$$

since $kI(t) + m(t)\exp[\alpha t-C(t)] = \ell$ ..

Then we can write (in general)

$$\begin{aligned} &\psi = W\exp[-C(t)] , \quad \nabla^2 W + kW = 0 \\ &\omega = k W \exp[-C(t)] , \quad C'(t) = A(t) \qquad (5.39) \\ &\phi = m(t) W \exp[-C(t)] \qquad = \frac{\mu+\kappa}{\rho} k - \frac{\kappa}{\rho} m(t) . \end{aligned}$$

It remains to find the functions $\exp[-C(t)]$ and $m(t) \exp[-C(t)]$ in the three cases.

Case (1):

$$\exp[-C(t)] = |1-c\ \exp(-\delta_1 t)|\ \exp[-(\frac{\mu+\kappa}{\rho}k - \frac{\kappa}{\rho} m_1)t] ,$$

$$m(t)\exp[-C(t)] = [m_1 - m_2 c\ \exp(-\delta_1 t)]\exp\{-(\frac{\mu+\kappa}{\rho}k - \frac{\kappa}{\rho} m_1)t\}$$
$$\times\ \mathrm{sgn}\ (1-\exp(-\delta_1 t)) ,$$

where $\delta_1 = \frac{\kappa}{\rho}\Delta_1$, $m_1 - m_2 = \Delta_1$ and $\frac{m_1 - m_2}{\delta_1} = \frac{\rho}{\kappa}$.

Case (2):

$$\exp(-C(t)) = t\ \exp[-\{\frac{(\mu+\kappa)k - \kappa m_0}{\rho}\}t] ,$$

$$m(t)\exp[-C(t)] = (m_0 t + \frac{\rho}{\kappa})\exp[-\{\frac{(\mu+\kappa)k - \kappa m_0}{\rho}\}t] . \qquad (5.40)$$

Case (3):

$$\exp[-C(t)] = \cos\frac{\kappa pt}{\rho}\ \exp[-\ \frac{(\mu+\kappa)k - \kappa q}{\rho}\ t] ,$$

$$m(t)\exp[-C(t)] = (q\ \cos\frac{\kappa pt}{\rho} - p\ \sin\frac{\kappa pt}{\rho})\times\ \exp[-\ \frac{(\mu+\kappa)k - \kappa q}{\rho}\ t] .$$

$$(5.41)$$

The above solves the problem of the most general solutions possible when the vorticity and spin are constant along a streamline at any particular time.

Section 3: Taylor Motions of micropolar fluids

We suppose, in the manner of G. I. Taylor (1923), that

$$\nabla^2\psi = \lambda_1\psi \ , \quad \phi = \lambda_2\psi \tag{5.42}$$

where λ_1, and λ_2 are constants.

Using (5.42) in equations (5.10) and (5.11), the non-linear terms vanish and the equations reduce to

$$(\mu+\kappa)\nabla^2\omega - \kappa\nabla^2\phi = \rho\,\frac{\partial\omega}{\partial t} \tag{5.43}$$

$$\gamma\nabla^2\phi + \kappa\omega - 2\kappa\phi = \rho j\,\frac{\partial\phi}{\partial t} \tag{5.44}$$

$$\text{where} \quad \omega = -\nabla^2\psi \ . \tag{5.45}$$

Let $\psi(x,y,t) = e^{-kt}\,F(x,y)$. (5.46)

Substituting (5.46) into equations (5.43), (5.44) and (5.45), we find that

$$\nabla^2 F = \lambda_1 F \ , \tag{5.47}$$

$$(\mu+\kappa)\lambda_1 + \kappa\lambda_2 = -k\rho \ , \tag{5.48}$$

$$\gamma\lambda_1\lambda_2 - \kappa\lambda_1 - 2\kappa\lambda_2 = -\rho j\, k\, \lambda_2 \ ; \tag{5.49}$$

whence we find

$$\lambda_1 = -\lambda_2\left[\frac{\kappa(j\lambda_2 + 2)}{\lambda_2\{(\mu+\kappa)j - \gamma\} + \kappa}\right] , \tag{5.50}$$

$$k = \frac{\kappa\lambda_2}{\rho}\,\frac{(\gamma\lambda_2 + 2\mu + \kappa)}{\lambda_2[(\mu+\kappa)j - \gamma] + \kappa} \tag{5.51}$$

and

$$\frac{\lambda_2}{\lambda_1} = -\frac{\lambda_2[(\mu+\kappa)j - \gamma] + \kappa}{\kappa(j\lambda_2 + 2)} . \tag{5.52}$$

If $F = \cos \mu x \cos \mu y$, then a solution of (5.47) is obtained when $\lambda_1 = -2\mu^2$.

Let $\mu = \frac{\pi}{d}$, then a solution is

$$\psi = -e^{-kt} \cos \frac{\pi x}{d} \cos \frac{\pi y}{d} . \tag{5.53}$$

Thus

$$u = -\frac{\pi}{d} e^{-kt} \cos \frac{\pi x}{d} \sin \frac{\pi y}{d} ,$$

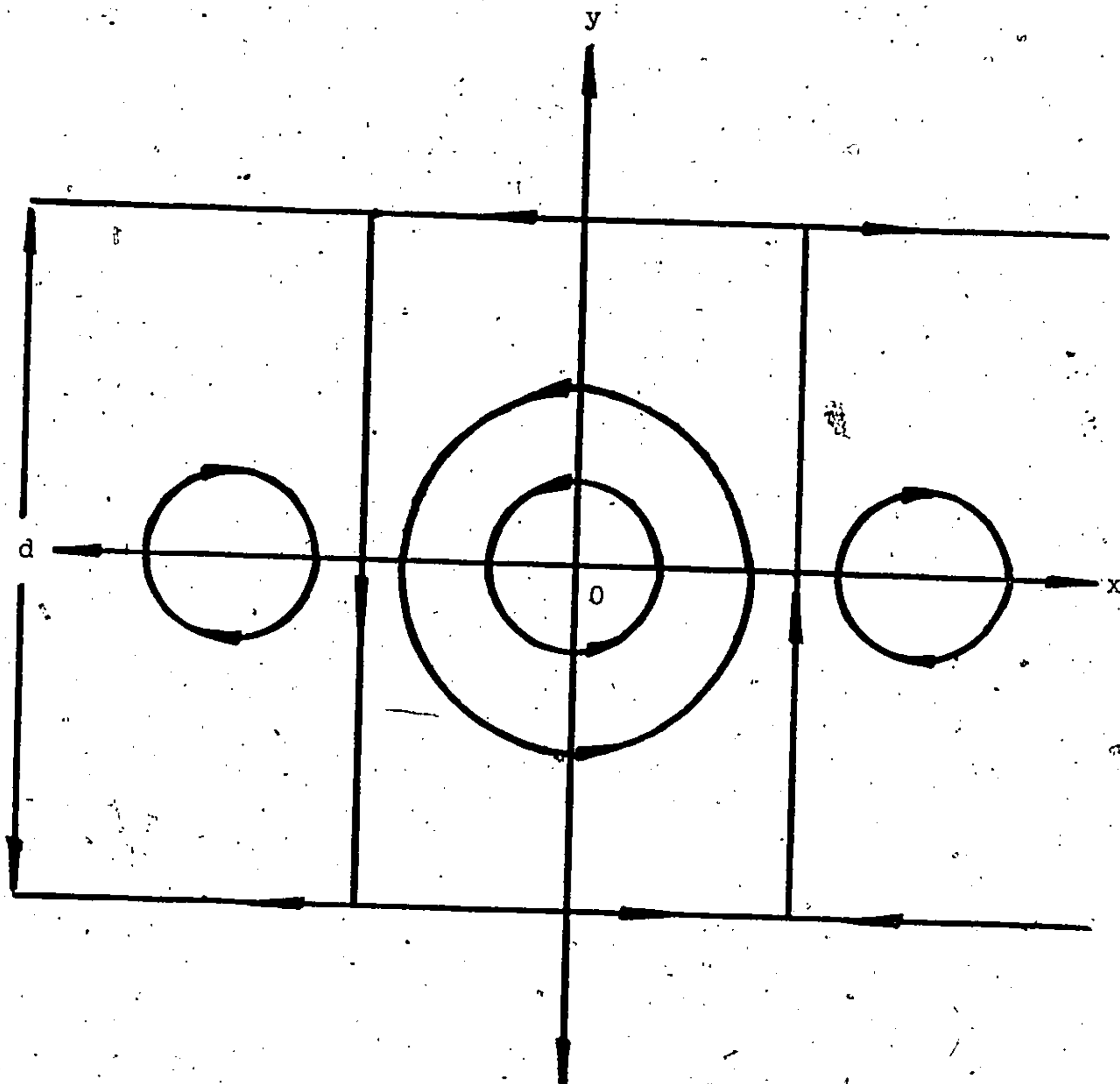
$$v = \frac{\pi}{d} e^{-kt} \sin \frac{\pi x}{d} \cos \frac{\pi y}{d} \tag{5.54}$$

and $\phi = \lambda_2 e^{-kt} \cos \frac{\pi x}{d} \cos \frac{\pi y}{d}$, (5.55)

where k is given in terms of λ_2 in (5.51) , and

$$d^2 = \frac{2\pi^2[\lambda_2\{(\mu+\kappa)j - \gamma\} + \kappa]}{\kappa\lambda_2(j\lambda_2 + 2)} . \tag{5.56}$$

The only values of λ_2 that yield "real" solutions (of interest) are those that make $d^2 > 0$ and $k > 0$.



Flow pattern of Taylor motions.

Section 4: Motion due to a single vortex and/or spin filament.

We consider

$$\underset{\sim}{v} = (0,\ v_\theta = v(r,t),\ 0)$$
$$\underset{\sim}{\nu} = (0,\ 0,\ \nu_z = \phi(r,t)) \qquad (5.57)$$

in polar co-ordinates, where $r^2 = x^2 + y^2$.

The equation of continuity (1.1) is satisfied identically for ρ = constant, and the field equations (1.2) and (1.3) reduce to

$$(\mu+\kappa)\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right] - \kappa\frac{\partial\phi}{\partial r} = \rho\frac{\partial v}{\partial t} \qquad (5.58)$$

$$\gamma\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right) + \kappa\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) - 2\kappa\phi = \rho j\frac{\partial\phi}{\partial t} \qquad (5.59)$$

We try solutions of the form

$$v(r,t) = AJ_1(kr)e^{-mt}$$

$$\phi(r,t) = BJ_0(kr)e^{-mt} \qquad (5.60)$$

where J_0 and J_1 are Bessel's functions of the first kind, and of order zero and one respectively.

Substituting (5.60) into (5.58) and (5.59), we obtain

$$(\mu+\kappa)A\left[\frac{d^2}{dr^2}J_1(kr) + \frac{1}{r}\frac{d}{dr}J_1(kr) - \frac{1}{r^2}J_1(kr)\right]$$

$$-\kappa B\frac{d}{dr}J_0(kr) = -\rho AmJ_1(kr) \qquad (5.61)$$

and

$$\gamma B\left[\frac{d^2}{dr^2} J_0(kr) + \frac{1}{r}\frac{d}{dr} J_0(kr)\right] + \kappa A\left[\frac{d}{dr} J_1(kr) + \frac{1}{r} J_1(kr)\right]$$

$$- 2\kappa B J_0(kr) = -\rho j\ BmJ_0(kr). \qquad (5.62)$$

We shall make use of the following differentiation formulae [Churchill (1963, p. 170-171)] for Bessel's functions:

$$\frac{d^2}{dr^2} J_1(kr) = -k^2 J_1(kr) + \frac{k}{r} J_2(kr) \qquad (5.63)$$

$$\frac{1}{r}\frac{d}{dr} J_1(kr) = \frac{1}{r^2} J_1(kr) - \frac{k}{r} J_2(kr) \qquad (5.64)$$

$$\frac{d}{dr} J_0(kr) = -k J_1(kr) \qquad (5.65)$$

$$\frac{d}{dr} J_1(kr) + \frac{1}{r} J_1(kr) = k J_0(kr) \qquad (5.66)$$

$$\frac{d^2}{dr^2} J_0(kr) = -k^2 J_0(kr) + \frac{k}{r} J_1(kr). \qquad (5.67)$$

Substituting (5.63), (5.64) and (5.65) into (5.61), and after a little simplification, we get

$$A[(\mu+\kappa)k^2 - \rho m] - \kappa k B = 0. \qquad (5.68)$$

Substituting (5.65), (5.66) and (5.67) into (5.62), we find that

$$-A\ k\ \kappa + B[k^2\gamma + 2\kappa - \rho j m] = 0\ . \tag{5.69}$$

If we eliminate A and B between (5.68) and (5.69), it is found that m must satisfy the quadratic

$$\rho^2\ j\ m^2 - \rho m[2\kappa + k^2\{j(\mu+\kappa) + \gamma\}] + k^2(2\mu+\kappa)\kappa + k^4(\mu+\kappa)\gamma = 0\ . \tag{5.70}$$

We consider here the particular case in which $(\mu+\frac{1}{2}\kappa)j = \gamma$, when the discriminant of the quadratic equation (5.70) is a perfect square. Then

$$m_1 = \frac{2\kappa}{\rho j} + \frac{\mu+\kappa}{\rho} k^2\ ,\quad m_2 = \frac{\gamma}{\rho j} k^2\ , \tag{5.71}$$

$$A_1 = -\frac{1}{2} j\ k\ B_1\ ,\quad A_2 = \frac{2}{k} B_2 \tag{5.72}$$

and the two solutions, each pair of which may be multiplied by an arbitrary constant, are

$$\begin{aligned} v_1 &= k\ j\ J_1(kr)\ \exp[-(\frac{2\kappa}{\rho j} + \frac{\mu+\kappa}{\rho} k^2)t]\ , \\ \phi_1 &= -2\ J_0(kr)\ \exp[-(\frac{2\kappa}{\rho j} + \frac{\mu+\kappa}{\rho} k^2)t] \end{aligned} \tag{5.73}$$

and

$$\begin{aligned} v_2 &= J_1(kr)\ \exp[-\frac{\gamma}{\rho j} k^2 t] \\ \phi_2 &= \frac{1}{2} k\ J_0(kr)\ \exp[-\frac{\gamma}{\rho j} k^2 t]\ . \end{aligned} \tag{5.74}$$

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