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**TAUB NUMBERS**

**By**

**Mark G. Naber**

**A dissertation submitted to the Faculty of Graduate Studies  
and Research through the Department of Physics in partial  
fulfillment of the requirements for the Degree of Doctor of  
Philosophy in Physics at the University of Windsor.**

**Windsor, Ontario, Canada**

**1993**



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## ABSTRACT

### TAUB NUMBERS

By

Mark G. Naber

Taub numbers are studied as a set of tensorial conservation laws derivable from curves of solutions to the vacuum Einstein equations. A formulation for Taub numbers of all orders is provided as well as a derivation of the Xanthopoulos theorem. Taub numbers are computed for the Schwarzschild and Kerr solutions viewed as perturbations of Minkowski spacetime and the Schwarzschild solution. They are found to give a measure of the mass and angular momentum and are free of the factor of 2 anomaly associated with the Komar numbers. Taub numbers are also computed for the stationary perturbations of the Schwarzschild solution.

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## 1 Introduction

There have been many attempts in general relativity to formulate expressions for mass and angular momentum. A few that come to mind are the Einstein and Landau-Lifshitz pseudo tensors and the Komar numbers. These attempts suffer from two types of difficulties. Some are non-tensorial and only hold for special coordinates, or if they are tensorial (Komar numbers) the normalizations for mass and angular momentum are different. In either case the results are unsatisfactory. Taub numbers provide a set of tensorial conservation laws, one for each Killing vector, and have no difference in normalizations for the mass and angular momentum quantities.

Taub numbers are formed by constructing a divergence-free vector (Taub vector) from the Einstein equations (and their perturbations), contraction with a Killing vector and then integrating the resulting vector over a Cauchy surface, either null or spacelike<sup>1</sup>. Explicitly, consider a curve of solutions to the vacuum field equations (this work will be exclusively restricted to exterior vacuum regions of spacetimes so as to concentrate solely on gravitational effects).

$$\hat{g}_{ab}(\lambda) = g_{ab} + \lambda h_{1ab} + \frac{\lambda^2}{2} h_{2ab} + \dots \quad (1.1)$$

The field equations for determining the coefficients of the series are found by equating powers of the curve parameter ( $\lambda$ ) in the series expansion of the Einstein equations for  $\hat{g}$ . These equations are most clearly expressed using functional derivatives of the Einstein tensor (see appendix 1.).

---

<sup>1</sup> Taub 1961, Taub 1971, Arms and Anderson 1986 and Glass 1993.

$$G_{ab}(g) = 0 \quad (1.2)$$

$$(D_g G_{ab}) \cdot \left( \begin{matrix} h \\ 1 \end{matrix} \right) = 0 \quad (1.3)$$

$$(D_g^2 G_{ab}) \cdot \left( \begin{matrix} h & h \\ 1 & 1 \end{matrix} \right) + (D_g G_{ab}) \cdot \left( \begin{matrix} h \\ 2 \end{matrix} \right) = 0 \quad (1.4)$$

$$(D_g^3 G_{ab}) \cdot \left( \begin{matrix} h & h & h \\ 1 & 1 & 1 \end{matrix} \right) + 3(D_g^2 G_{ab}) \cdot \left( \begin{matrix} h & h \\ 1 & 2 \end{matrix} \right) + (D_g G_{ab}) \cdot \left( \begin{matrix} h \\ 3 \end{matrix} \right) = 0 \quad (1.5)$$

etc.

In terms of the Ricci tensor and its functional derivatives these equations can be written as,

$$R_{ab}(g) - \frac{1}{2}g_{ab}g^{lm}R_{lm} = 0. \quad (1.6)$$

$$(D_g R_{ab}) \cdot \left( \begin{matrix} h \\ 1 \end{matrix} \right) - \frac{1}{2}g_{ab}g^{lm}(D_g R_{lm}) \cdot \left( \begin{matrix} h \\ 1 \end{matrix} \right) = 0. \quad (1.7)$$

$$(D_g^2 R_{ab}) \cdot \left( \begin{matrix} h & h \\ 1 & 1 \end{matrix} \right) + (D_g R_{ab}) \cdot \left( \begin{matrix} h \\ 2 \end{matrix} \right) - \frac{1}{2}g_{ab}g^{lm} \left[ (D_g^2 R_{lm}) \cdot \left( \begin{matrix} h & h \\ 1 & 1 \end{matrix} \right) + (D_g R_{lm}) \cdot \left( \begin{matrix} h \\ 2 \end{matrix} \right) \right] = 0. \quad (1.8)$$

$$(D_g^3 R_{ab}) \cdot \left( \begin{matrix} h & h & h \\ 1 & 1 & 1 \end{matrix} \right) + 3(D_g^2 R_{ab}) \cdot \left( \begin{matrix} h & h \\ 1 & 2 \end{matrix} \right) + (D_g R_{ab}) \cdot \left( \begin{matrix} h \\ 3 \end{matrix} \right) - \frac{g_{ab}g^{lm}}{2} \left[ (D_g^3 R_{lm}) \cdot \left( \begin{matrix} h & h & h \\ 1 & 1 & 1 \end{matrix} \right) + 3(D_g^2 R_{lm}) \cdot \left( \begin{matrix} h & h \\ 1 & 2 \end{matrix} \right) + (D_g R_{lm}) \cdot \left( \begin{matrix} h \\ 3 \end{matrix} \right) \right] = 0 \quad (1.9)$$

etc.

where the first n-1 field equations have been used to simplify the  $n^{\text{th}}$  field equation.

Notice that each equation is of the same form,

$$A_{ab} - \frac{1}{2}g_{ab}g^{lm}A_{lm} = 0. \quad (1.10)$$

$$\Rightarrow g^{in}A_{lm} = 0. \quad (1.11)$$

$$\Rightarrow A_{ab} = 0. \quad (1.12)$$

The sequential set of Einstein field equations can then be simplified to form,

$$R_{ab}(g) = 0, \quad (1.13)$$

$$(D_g R_{ab}) \cdot \binom{h}{1} = 0, \quad (1.14)$$

$$(D_g^2 R_{ab}) \cdot \binom{h \cdot h}{1 \quad 1} + (D_g R_{ab}) \cdot \binom{h}{2} = 0, \quad (1.15)$$

$$(D_g^3 R_{ab}) \cdot \binom{h \cdot h \cdot h}{1 \quad 1 \quad 1} + 3(D_g^2 R_{ab}) \cdot \binom{h \cdot h}{1 \quad 2} + (D_g R_{ab}) \cdot \binom{h}{3} = 0, \quad (1.16)$$

etc.

These are of course the same equations that would have been produced had we begun by expanding the Ricci tensor, i.e.  $R_{ab}(\hat{g}) = 0$ . In terms of the series for the metric and the connection (see A1.21) these equations can be written as,

$$R_{ab}(g) = 0, \quad (1.17)$$

$$(D_g R_{ab}) \cdot \binom{h}{1} = 0, \quad (1.18)$$

$$(D_g R_{ab}) \cdot \binom{h}{n} = \sum_{i=1}^{n-1} \frac{2n!}{i!(n-i)!} \left\{ \binom{h^m \Gamma^l}{i \quad n-i} \alpha^{[b]} \right\}_{:m] + \Gamma_{n-i}^m \alpha^{[b} \Gamma^l_{i \quad m]a} \right\} n \geq 2. \quad (1.19)$$

The notation is explained in appendix 1. Since the right hand side of (1.19) will be used frequently, we make use of the following definition.

$$S_{ab} \binom{h}{n-1, n-2, \dots, 1} = \sum_{i=1}^{n-1} \frac{2n!}{i!(n-i)!} \left\{ \binom{h^m \Gamma^l}{i \quad n-i} \alpha^{[b]} \right\}_{:m] + \Gamma_{n-i}^m \alpha^{[b} \Gamma^l_{i \quad m]a} \right\} \quad (1.20)$$

Thus, (1.19) and (1.3) - (1.5) may be written as,

$$(D_g R_{ab}) \cdot \binom{h}{n} = S_{ab}, \quad (1.21)$$

$$(D_g G_{ab}) \cdot \binom{h}{n} = S_{ab} - \frac{1}{2} g_{ab} g^{lm} S_{lm}. \quad (1.22)$$

We wish to keep both sets of field equations because there exists a superpotential for (1.22) when contracted with a Killing vector (for the background geometry).

However (1.21) is more natural to use if we were solving, sequentially, for each coefficient of a curve of solutions.  $D_g R_{ab}$  is a linear second order hyperbolic operator.

Thus, if we were attempting to generate a curve of vacuum solutions, each coefficient of the series for the metric would be computed by inverting  $D_g R_{ab}$  for the given source,  $S_n^{ab}$ .

Equations (1.2),(1.3) and (1.22) are divergence free (the first two are trivial).

Taub vectors are then defined:

$$t_0^a = k^b G_{ab}. \quad (1.23)$$

$$t_n^a = k^b (D_g G_{ab}) \cdot \left( \frac{h}{n} \right) \quad n \geq 1. \quad (1.24)$$

Here  $k^b$  is a Killing vector associated with  $g$ , the background metric. Equation (1.22) implies

$$t_n^a = k^b S_n^{ab} - \frac{1}{2} k_a g^{lm} S_n^{lm} \quad n \geq 2. \quad (1.25)$$

Taub numbers are defined (without normalization) as

$$\tau_n(k) = \int_{\sigma} \sqrt{-g} t_n^a ds_a. \quad (1.26)$$

where  $\sigma$  is a Cauchy surface (null or spacelike) for the background spacetime  $g$ .

Thus there is one Taub number for each Killing vector present on the background geometry. There is no applicable superpotential for (1.23)<sup>2</sup>. However if we replace the Einstein tensor by the Ricci tensor (equivalent field quantities for vacuum regions) we may then use the Komar superpotential to compute the zero order Taub numbers.  $\tau_0$  is then fundamentally different from  $\tau_1, \tau_2, \dots$ , hence we expect the normalization for the zero-order Taub numbers to be different from the normaliza-

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<sup>2</sup> This will be discussed again in a later section.

tion for the higher order Taub numbers. To use equation (1.24) to compute  $t_n$  requires knowledge of  $h_n$ , which means solving (1.21) unless a complete curve has been given, while (1.25) requires knowledge of  $h_1$  through  $h_{n-1}$ .

The advantage of using (1.24) is that there is a superpotential for  $t_n$  in terms of  $h_n$ , the disadvantage occurs for  $n \geq 2$  which requires inverting (1.8), unless we have been given a complete series. Equation (1.24) has a superpotential because it is a linear wave equation in one variable,  $h_n$ , while (1.25) is a nonlinear equation which depends upon several variables. Equation (1.25) has a superpotential as well, however it is not 'nice', and will be discussed later.

It will later be shown that for asymptotically simple spacetimes the zero order Taub numbers (the Komar quantities) give a measure of mass and angular momentum, and that the higher order Taub numbers give a measure of the perturbing mass, angular momentum and gravitational radiation, provided that the appropriate Killing vector is present on the background spacetime  $g$ . Thus we may use Taub numbers to study perturbations of spacetimes with Killing vectors and curves of solution of the Einstein equations which pass through a spacetime which has a Killing vector(s).

In the following section we shall briefly review linearization stability. It will be shown that for closed cosmologies the Taub numbers can be used to exclude some perturbations,  $h$ , which, while being solutions of  $D_g R_{ab} = 0$ , are not actually tangent to the solution manifold (called 'spurious solutions' in Glass 1993). Stability will also be demonstrated for asymptotically simple spacetimes and discussed where local implications are concerned.

In the third section we shall discuss the perturbation gauge invariance of the Taub numbers and vectors. Gauge invariance shall be proven in all cases with the exception of gauge transformations on the set of fields  $\{h_1, \dots, h_{n-1}\}$  for background

geometries which have non-compact Cauchy surfaces. This remains an open question. The Geroch-Xanthopoulos gauge condition will also be discussed. This condition is used as a perturbation constraint to preserve  $\mathfrak{S}^+$ .

In section 4 the Komar and Glass superpotentials will be constructed as well as an argument for the lack of a useful potential for the right hand side of (1.25). These superpotentials shall then be applied to the Schwarzschild and Kerr solutions, and again to these solutions when they are viewed as perturbations of Minkowski and Schwarzschild. These results will be used to fix an interpretation and normalization for the Taub numbers. To connect the Schwarzschild and Minkowski solutions a derivation of the Xanthopoulos theorem will also be provided. In this dissertation the word 'perturbation' shall be used to mean a symmetric 2-tensor to be added to the metric, not necessarily a solution of the linearized field equations, as is the case when viewing Kerr as a 'perturbation' of Minkowski or Schwarzschild.

In section 5 we shall compute the Taub numbers for the stationary perturbations of Schwarzschild as found by Regge and Wheeler (1957), Vishveshwara (1970) and Zerilli (1970). The results here will confirm the mass and angular momentum interpretations found in the previous section.

Section 6 is left for a discussion of some of the open questions raised and to examine some future applications of the Taub numbers and the formalisms developed. In particular a solution generation technique and a superpotential for the zero order Taub numbers will be discussed.

Appendices 1 and 2 contain all mathematical formalisms and notations. This includes recursion relations for the geometrical quantities computed from curves of solutions, functional derivatives and the Newman-Penrose (NP) formalism. appendix 3 contains the Schwarzschild and Kerr solutions, the Minkowski metric and gives the perturbation results of Vishveshwara and Zerilli.

## 2 Linearization Stability

The question of linearization stability is concerned with the existence of a curve of solutions, in the space of solutions of the Einstein equations, which passes through a particular point with a given tangent (a linearized solution). Do solving and linearizing commute? How reliable is the linear approximation<sup>3</sup>? It has been shown that asymptotically simple solutions are linearization stable<sup>4</sup> and that spacetimes which are closed and that can be foliated by compact Cauchy surfaces without boundary are linearization stable provided they have no Killing vectors<sup>5</sup>. If there are symmetries then there are restrictions placed upon the perturbations (tangents). It has also been shown that, locally, vacuum spacetimes are also linearization stable<sup>6</sup>. This may be important for astrophysical applications. We shall primarily be concerned with asymptotically simple spacetimes in this work, however we shall give a brief treatment of closed spacetimes for completeness.

Let us denote the space of Lorentzian metrics on  $V$ , a 4-manifold, as  $Lor$ .  $g \in Lor$  if  $g$  is a covariant symmetric two tensor of signature (1,-1,-1,-1). Let  $Ein$  denote the set of solutions of the Einstein equations (vacuum).  $g \in Ein$  if  $g \in Lor$  and  $R_{ab}(g) = 0$ .  $Ein$  is thus a subset of  $Lor$ . The set of points of  $Lor$  which make up curves of solutions of the field equations forms a subset of  $Ein$  which we shall denote as  $Cur$ . Denote by  $EinCC$  as the subset of  $Ein$  for which the spacetimes have compact Cauchy surfaces without boundary.

$$Cur \subset Ein \subset Lor \quad (2.1)$$

$$EinCC \subset Ein \subset Lor \quad (2.2)$$

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3 Brill 1982.

4 Choquet-Bruhat, Fischer and Marsden 1976 and Choquet-Bruhat and York 1980.

5 Choquet-Bruhat and York 1980, O'Murchadha and York 1974, Fischer and Marsden 1973, Moncrief 1976 and Brill and Desser 1973.

6 Brill, Reula and Schmidt 1987.

$$Cur \cap EinCC \neq \emptyset \quad (2.3)$$

If we linearize the field equations about a point  $g \in Ein$ , and the solutions of the linearized field equations,  $(D_g R_{ab}) \cdot (h) = 0$ , coincide with the tangent space of  $Ein$  at  $g$  then the field equations are said to be linearization stable<sup>7</sup> at  $g$  and  $g \in Cur$ . We shall denote this subset of  $Cur$  as  $LS$ .

$$LS \subset Cur \quad (2.4)$$

$$LS \cap EinCC \neq \emptyset \quad (2.5)$$

If this is not the case then we say that the field equations are linearization unstable at  $g$ . We shall denote this subset of  $Ein$  as  $LU$ .

$$LU \cap LS = \emptyset \quad (2.6)$$

The following theorem, due to Moncrief, will demonstrate that there is always a neighborhood within which a Lorentzian metric plus a given symmetric two tensor will remain Lorentzian.

**Theorem (2.1)<sup>8</sup>:** Let  $g \in Lor$  and let  $h$  be a symmetric covariant two tensor on the 4-manifold  $V$ . Then for any point  $p \in V$  there exists a neighborhood  $N_p \subset V$  of  $p$  and a constant  $\alpha > 0$  such that  $g + \lambda h \in Lor$  on  $N_p$  for all  $\lambda \in (-\alpha, \alpha)$ .

**Proof:** Let  $M_p$  be a neighborhood of  $p$ ; choose an orthonormal tetrad  $x_{(a)}^b$  such that  $g_{ab} x_{(a)}^c x_{(b)}^d = \eta_{(a)(b)}$ , where (a) signifies a tetrad index and  $\eta_{(a)(b)} = diag(1, -1, -1, -1)$  is the Minkowski metric. Let  $K_p$  be a compact region about  $p$  such that  $K_p \subset M_p$ . Given  $h$  let us define the following on  $M_p$ .

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<sup>7</sup> Fischer, Marsden and Moncrief 1980.

<sup>8</sup> Moncrief 1976.



$$Y_{(a)(b)} = x_{(a)}^a x_{(b)}^b h_{ab} \quad (2.7)$$

$$\Gamma = \max_{((a),(b))} \sup_{K_p} |Y_{(a)(b)}| \quad (2.8)$$

$\Gamma$  will be finite due to continuity of the tetrad and  $h$ , and the compactness of  $K_p$ .

Let  $N_p$  be a neighborhood of  $p$  such that  $N_p \subset K_p$ . If  $\Gamma > 0$  let

$\alpha = \Gamma/4$ ; if  $\Gamma = 0$  let  $\alpha = 1$ . We may now show that

$g_{ab}(\lambda) = g_{ab} + \lambda h_{ab} \in Lor$  on  $N_p$  for all  $\lambda \in (-\alpha, \alpha)$ . Consider the vector fields of the form  $v^a = v^{(i)} x_{(i)}^a$  ( $i = 1, 2, 3$ ) on  $N_p$ . Given the above choice of  $\alpha$  we have,

$$g_{ab}(\lambda) v^a v^b \leq 0 \quad (2.9)$$

with equality for  $v = 0$ . Thus the three space spanned by  $x_{(i)}^a$  is spacelike for

$g_{ab}(\lambda)$  on  $N_p$ . Similarly we may show that  $x_{(0)}^a$  is timelike for  $g_{ab}(\lambda)$  on  $N_p$ .

Thus  $g_{ab}(\lambda) \in Lor$ , and we may conclude that for any  $g \in Ein$  and any symmetric two tensor  $h$ ,  $g + \lambda h \in Lor$  for sufficiently small  $\lambda > 0$ .

An  $h$  which is a solution of

$$(D_g R_{ab}) \cdot (h) = 0 \quad (2.10)$$

is said to be integrable<sup>9</sup> if on some compact set  $D \subset V$  there is a curve  $g(\lambda)$  of exact solutions

$$R_{ab}(g(\lambda)) = 0 \quad (2.11)$$

on  $D$  such that  $g(0) = g \in Ein$ , and  $h = \frac{dg(\lambda)}{d\lambda} \Big|_{\lambda=0}$ .  $g \in LS$  if all solu-

tions of  $(D_g R_{ab}) \cdot (h) = 0$  are integrable. Conversely if at  $g$  there is a solution

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<sup>9</sup> Fischer, Marsden and Moncrief 1980.

of the linearized field equations which is not integrable then the field equations are linearization unstable<sup>10</sup> at  $g$  (i.e.  $g \in LU$ ). That is to say, there are solutions of the linearized field equations which are not tangent to the solution manifold.

Consider the following two theorems due to Taub<sup>11</sup>.

Theorem (2.2): If  $g \in Ein$  and  $h$  is any symmetric covariant two tensor. Then,

$$\nabla^a(g)(D_g G_{ab}) \cdot (h) = 0. \quad (2.12)$$

Proof: For  $g \in Ein$  we have (trivially),

$$\nabla^a(g)G_{ab}(g) = 0. \quad (2.13)$$

Consider the functional derivative, with respect to  $g$ , in the direction of  $h$ . Then,

$$(D_g \nabla^a) \cdot (h)G_{ab}(g) + \nabla^a(g)(D_g G_{ab}) \cdot (h) = 0. \quad (2.14)$$

Since  $g \in Ein$  the first term is zero. This leaves the desired result.

Theorem (2.3): If  $g \in Ein$  and  $(D_g R_{ab}) \cdot (h) = 0$ . Then,

$$\nabla^a(D_g^2 G_{ab}) \cdot (h, h) = 0. \quad (2.15)$$

Proof: Consider the functional derivative, with respect to the metric, of the result of the previous theorem in the direction of  $h$ .

$$(D_g \nabla^a) \cdot (h)(D_g G_{ab}) \cdot (h) + \nabla^a(g)(D_g^2 G_{ab}) \cdot (h, h) = 0 \quad (2.16)$$

The first term is zero because  $h$  is a solution of the linearized field equations. This leaves the desired result.

If  $k^b$  is a Killing vector on  $g$  then we can construct two conserved quantities.

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<sup>10</sup> Fischer, Marsden and Moncrief 1980.

<sup>11</sup> Taub 1961 and 1971.

$$t_{1a} = k^b (D_g G_{ab}) \cdot (h), \quad (2.17)$$

$$\nabla^b t_{1b} = 0, \quad (2.18)$$

$$t_{2a} = k^b (D_g^2 \dot{G}_{ab}) \cdot (h, h), \quad (2.19)$$

$$\nabla^b t_{2b} = 0, \quad (2.20)$$

where  $h$  need only be symmetric in the first, and  $h$  must be a solution of the linearized field equations in the second.

For a spacetime  $g \in \text{EinCC}$  the integrals of  $t_1$  and  $t_2$  over compact spacelike hypersurfaces are hypersurface independent. We then have the following results.

**Theorem (2.4)<sup>12</sup>:** Let  $g \in \text{EinCC}$ ,  $k$  be a Killing vector of  $g$ ,  $h$  a symmetric covariant two tensor and  $\Sigma$  a compact spacelike hypersurface, then the first Taub number vanishes:

$$\int_{\Sigma} k^a (D_g G_{ab}) \cdot (h) z^b \sqrt{-g} d^3x = 0, \quad (2.21)$$

where  $z^a$  is a unit forward pointing normal to the hypersurface.

**Proof:** Since the integral is hypersurface independent we have the following, for  $\Sigma_1$  and  $\Sigma_2$  being disjoint compact spacelike hypersurfaces.

$$\int_{\Sigma_1} t_{1a} z^a \sqrt{-g} d^3x = \int_{\Sigma_2} t_{1a} z^a \sqrt{-g} d^3x \quad (2.22)$$

Let us replace  $h$  by  $h'$  which equals  $h$  on  $\Sigma_1$  and goes smoothly to zero on  $\Sigma_2$ .

Then the right hand side of (2.14) is zero and we have the desired result.

We will now see that the second Taub number vanishes if the field equations are linearization stable at  $g$ .

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<sup>12</sup> Fischer, Marsden and Moncrief 1980.

Theorem (2.5)<sup>13</sup>: Let  $g_0 \in EinCC$ ,  $k$  be a Killing vector with respect to  $g_0$  and  $h_1$

any integrable solution of the linearized field equations, then the second Taub number vanishes when computed over any compact spacelike hypersurface  $\Sigma$ .

Proof: Let  $g(\lambda)$  be a curve of solutions in  $EinCC$  which passes through  $g_0$  with tangent  $h_1$ . Then,

$$R_{ab}(g_0) = 0. \quad (2.23)$$

$$(D_g G_{ab}) \cdot (h_1) = 0. \quad (2.24)$$

$$(D_g^2 G_{ab}) \cdot (h_1, h_1) + (D_g G_{ab}) \cdot (h_2) = 0, \quad (2.25)$$

where  $h_2 = \frac{d^2}{d\lambda^2} g(\lambda)|_{\lambda=0}$ . Then,

$$\int_{\Sigma} k^a \left[ (D_g^2 G_{ab}) \cdot (h_1, h_1) + (D_g G_{ab}) \cdot (h_2) \right] \sqrt{-g} d^3x = 0. \quad (2.26)$$

The second integral vanishes by the previous theorem, this leaves the desired result.

The converse of this theorem is also true.

Theorem (2.6)<sup>14</sup>: If  $g_0 \in EinCC$ ,  $h$  satisfies the linearized field equations and

has a zero second-order Taub number for all Killing vectors, then  $h$  is integrable at  $g_0$ . Curves tangent to the above  $h$ 's at  $g_0$  generate all the solutions to  $R_{ab}(g(\lambda))$  near  $g_0$ .

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<sup>13</sup> Moncrief 1976, Fischer and Marsden 1979 and Fischer et. al. 1980.

<sup>14</sup> Fischer, Marsden and Moncrief 1980, and Arms, Marsden and Moncrief 1982.

The proof of this theorem is somewhat involved since one must consider the problem within the context of the initial value formulation of the Einstein equations and show that there is a conical singularity on the solution manifold. This requires the Liapunov-Schmidt procedure and an application of the Morse lemma<sup>15</sup>.

This leaves the final theorem for spacetimes  $g \in EinCC$ .

**Theorem (2.7)<sup>16</sup>:** Let  $g \in EinCC$  then  $g \in LS$  if and only if  $g$  has no

**Killing vectors.**

The proof of this theorem uses the same techniques as the previous proof and will not be presented.

Asymptotically simple spacetimes with non-compact Cauchy surfaces have also been studied with regard to linearization stability. A spacetime  $(M, g)$  is asymptotically simple<sup>17</sup> if there exists a spacetime  $(\tilde{M}, \tilde{g})$  and an imbedding  $\Theta: M \rightarrow \tilde{M}$  which imbeds  $M$  with a smooth boundary  $\partial M$  in  $\tilde{M}$  such that,

(1) there is a smooth function  $\Omega$  on  $\tilde{M}$  such that on  $\Theta(M)$   $\Omega > 0$  and

$$\tilde{g} = \Omega^2 g;$$

(2) on  $\partial M$ ,  $\Omega = 0$  and  $d\Omega \neq 0$ ;

(3) every null geodesic in  $M$  has two end points on  $\partial M$ .

A spacetime is asymptotically flat<sup>18</sup> if it is asymptotically simple and  $R_{ab} = 0$  in an open neighborhood of  $\partial M$ . We shall usually be considering spacetimes which

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<sup>15</sup> Fischer, Marsden and Moncrief 1980 and Arms, Marsden and Moncrief 1982.

<sup>16</sup> Fischer and Marsden 1973, Moncrief 1975 and 1976 and Fischer, Marsden and Moncrief 1980.

<sup>17</sup> Page 222 Hawking and Ellis 1973.

<sup>18</sup> Page 222 Hawking and Ellis 1973.

are weakly simple. That is, we relax condition (3) to allow for null orbits and black holes. We shall denote the set of these spacetimes as  $Ein.AF$ . Given the above definitions we have the following result due to Choquet-Bruhat, Fischer and Marsden. Theorem (2.8)<sup>19</sup>: Let  $g \in Ein.AF$ . Then,  $g \in LS$ . i.e.  $Ein.AF \subset LS$ .

As in the previous cases, this proof is most easily produced using the initial value techniques and will not be presented here. The implication of this theorem is that all solutions of  $D_g R_{ab} = 0$  will be tangent to the solution manifold, and the Taub numbers are then free to be non-zero.

Note that the above results on linearization stability are global results. The question of linearization stability may also be cast in a local form. Consider the following theorem.

Theorem (2.9)<sup>20</sup>: Let  $g \in Ein$  and  $\Omega$  be a region of spacetime which is compact, has a smooth boundary, globally hyperbolic and with  $g$  square integrable together with its derivatives to order 6. Then for any  $h$  which is square integrable together with its derivatives to order 6 and which is a solution of the linearized equations and any open region  $U \subset \Omega$  there exists a curve of solutions on  $U$  which passes through  $g$  with tangent  $h$ .

Thus we may be able to use Taub numbers for a local characterization of solutions and perturbations, provided the integral in the definition is suitably changed.

In this section we have seen that Taub numbers are useful for constructing a constraint by which spurious linearized solutions may be rejected, for spacetimes with compact Cauchy surfaces and Killing vectors. For spacetimes which are linearization stable all perturbations may be considered and curves of solution may be constructed. For this circumstance the resulting Taub numbers have a physical significance which we shall discuss later.

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<sup>19</sup> Choquet-Bruhat, Fischer and Marsden 1979.

<sup>20</sup> Brill, Reula and Schmidt 1987.

### 3 Gauge Invariance

In this section we shall examine the gauge invariance of both the sequential set of field equations (1.18) and (1.21) (or (1.22)) and the Taub numbers. For example: if we use (1.24) to compute a Taub number what are the permissible gauge transformations of  $h_i$ ? If we use (1.25) and we wish to gauge transform  $h_i$  ( $i \leq n-1$ ) what are the transformations induced on the higher order fields  $h_{i-1}, h_{i-2}, \dots, h_{n-1}$ ?

Let  $F$  denote a diffeomorphism of the 4-manifold  $V$ .

$$F: V \rightarrow V \quad (3.1)$$

Let  $F^*$  denote the pullback corresponding to  $F$ . Then the globalized version of general covariance<sup>21</sup> of the Einstein tensor can be written as,

$$G_{ab}(F^*g) = F^*G_{ab}(g). \quad (3.2)$$

Let  $x$  be an arbitrary vector on  $V$ . Denote by  $F_\lambda$  the flow<sup>22</sup> of  $x$  parameterized by  $\lambda$  such that at  $\lambda = 0$ ,  $F_0$  is the identity diffeomorphism.

$$F_\lambda^* = F_0^* + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \mathcal{L}_x^n \quad (3.3)$$

$\mathcal{L}_x$  is the Lie derivative. Consider the following theorem.

**Theorem (3.1)<sup>23</sup>:** Let  $x$  be any vector on  $V$  and  $h$  a symmetric 2-tensor. Then,

$$(D_g G_{ab}) \cdot \left( \mathcal{L}_x g \right) = \mathcal{L}_x [G_{ab}(g)] \quad (3.4)$$

$$\text{and } (D_g^2 G_{ab}) \cdot (h, \mathcal{L}_x g) + (D_g G_{ab}) \cdot \left( \mathcal{L}_x h \right) = \mathcal{L}_x [(D_g G_{ab}) \cdot (h)]. \quad (3.5)$$

**Proof:** Consider the requirement of general covariance as imposed along the flow of  $x$ .

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<sup>21</sup> Page 57 Wald 1984 and Fischer, Marsden and Moncrief 1980.

<sup>22</sup> Fischer, Marsden and Moncrief 1980 and page 196 Abraham, Marsden and Ratiu 1983.

<sup>23</sup> Fischer, Marsden and Moncrief 1980.

$$G_{ab}(F_\lambda^* g) = F_\lambda^* G_{ab}(g) \quad (3.6)$$

Consider the derivative of this with respect to  $\lambda$

$$(D_g G_{ab})(F_\lambda^* g) \cdot \left( F_\lambda^* \left( \frac{\mathcal{L}}{x} g \right) \right) = F_\lambda^* \left( \frac{\mathcal{L}}{x} G_{ab}(g) \right), \quad (3.7)$$

$$\text{at } \lambda = 0, (D_g G_{ab}) \cdot \left( \frac{\mathcal{L}}{x} g \right) = \frac{\mathcal{L}}{x} (G_{ab}(g)). \quad (3.8)$$

This yields the first result. Now consider the functional derivative of (3.7) with respect to the metric in the direction of  $h$ , and the proof is complete.

$$(D_g^2 G_{ab}) \cdot \left( h, \frac{\mathcal{L}}{x} g \right) + (D_g G_{ab}) \cdot \left( \frac{\mathcal{L}}{x} h \right) = \frac{\mathcal{L}}{x} [(D_g G_{ab}) \cdot (h)] \quad (3.9)$$

$D_g$  passes through  $\frac{\mathcal{L}}{x}$  since the Lie derivative is independent of the metric. Alternatively (3.4) and (3.5) can be derived by an application of the chain rule, since the derivative with respect to the metric and the Lie derivative are both derivations and the Einstein tensor is a function of the metric. The notion of flow will be useful later when we need see how gauge transformations on one field affect higher order fields.

By considering the special cases for  $h$  and  $x$  we can readily generate the following results. If  $x$  is a Killing vector and  $h$  is an arbitrary symmetric two-tensor then,

$$(D_g G_{ab}) \cdot \left( \frac{\mathcal{L}}{x} h \right) = \frac{\mathcal{L}}{x} [(D_g G_{ab}) \cdot (h)], \quad (3.10)$$

$$\frac{\mathcal{L}}{x} [G_{ab}(g)] = 0, \quad (3.11)$$

$$\text{or, for } g \in \text{Ein}, (D_g R_{ab}) \cdot \left( \frac{\mathcal{L}}{x} h \right) = \frac{\mathcal{L}}{x} [(D_g R_{ab}) \cdot (h)]. \quad (3.12)$$

The Lie derivative with respect to a symmetry passes through the functional derivative with respect to the metric, as we would expect. If  $h$  obeys the linearized field equations and  $x$  is arbitrary then,

$$(D_g^2 G_{ab}) \cdot \left( h, \frac{\mathcal{L}}{x} g \right) = -(D_g G_{ab}) \cdot \left( \frac{\mathcal{L}}{x} h \right), \quad (3.13)$$



$$\text{or, for } g \in \text{Ein}, (D_g^2 R_{ab}) \cdot \left( h \cdot \frac{\mathcal{L}g}{x} \right) = -(D_g R_{ab}) \cdot \left( \frac{\mathcal{L}h}{x} \right). \quad (3.14)$$

If  $x$  is a Killing vector and  $h$  obeys the linearized equations then,

$$(D_g G_{ab}) \cdot \left( \frac{\mathcal{L}h}{x} \right) = 0. \quad (3.15)$$

From (3.8) we may write out the gauge transformation. Let  $g \in \text{Ein}$  and  $h$  and  $x$  be arbitrary then,

$$(D_g G_{ab}) \cdot \left( \frac{\mathcal{L}g}{x} \right) = 0, \quad (3.16)$$

$$\Rightarrow (D_g G_{ab}) \cdot \left( h + \frac{\mathcal{L}g}{x} \right) = (D_g G_{ab}) \cdot (h). \quad (3.17)$$

Thus we may add the symmetrized derivative of a vector to the field  $h_n$  on the left

hand side of the field equation (1.21) or to the field  $h_n$  on the right hand side of (1.24)

$$(D_g R_{ab}) \cdot \left( h_n + \frac{\mathcal{L}g}{x} \right) = S_{ab}, \quad (3.18)$$

$$t_a = k^b (D_g G_{ab}) \cdot \left( h_n + \frac{\mathcal{L}g}{x} \right) = k^b (D_g G_{ab}) \cdot \left( h_n \right) \quad n \geq 1. \quad (3.19)$$

Thus Taub numbers computed with (1.24) are gauge invariant. The easy gauge question has been answered. We must now ask what happens to the fields

$h_1, h_2, \dots, h_{n-1}$  which comprise  $S_n$  when we make a gauge transformation on  $h_i$ ,  $i \in (1, \dots, n-1)$ . This question arises when we wish to compute Taub numbers with (1.25) rather than (1.24). Formally what we wish to construct is a gauge transformation for the set of fields  $\left( h_1, \dots, h_{n-1} \right)$ . Following Arms and Anderson 1986 we may derive a formula which will produce the induced gauge transformations. To facilitate the derivation let us introduce some notation. Let  $g(\lambda)$  be a curve of solutions which may be written as a power series in  $\lambda$ .

$$g(\lambda) = g + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} h_n \quad (3.20)$$

Let  $j^m(g(\lambda))$  denote the first  $m$  coefficients of the polynomial for  $g(\lambda)$ <sup>24</sup>. For example let  $P$  and  $\tilde{P}$  be two sets of the first  $r$  fields about  $g \in \text{Ein}$ .

$$P = \left( g_0, h_1, \dots, h_r \right) \quad (3.21)$$

$$\tilde{P} = \left( \tilde{g}_0, \tilde{h}_1, \dots, \tilde{h}_r \right) \quad (3.22)$$

Let  $g(\lambda)$  and  $\tilde{g}(\lambda)$  be curves of solutions representing  $P$  and  $\tilde{P}$ .

$$j^r(g) = P \quad (3.23)$$

$$j^r(\tilde{g}) = \tilde{P} \quad (3.24)$$

As before let  $F_\lambda$  denote a 1-parameter family of diffeomorphisms. Let us also suppose that this diffeomorphism will be the identity transformation on the first  $m$  fields,  $m \leq r$ . Then<sup>25</sup>,

$$F_\lambda^* = \sum_{k=0}^{\infty} \frac{\lambda^{mk}}{(m!)^k k!} \mathcal{L}_x^k \quad (2.25)$$

where  $x$  is the vector which generates  $F_\lambda$ . Then if we apply  $F_\lambda$  to the curve  $g$  we have,

$$F_\lambda^* g = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\lambda^{mk+n}}{(m!)^k n! k!} \mathcal{L}_x^k h_n \quad (3.26)$$

Following Arms and Anderson we may rearrange the sum. Let  $i = mk + n$ ; this will then range from zero to infinity. Then for a given value of  $i$ ,  $k$  will range from zero to  $[i/m]$ . This leaves,

$$F_\lambda^* g = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \left[ \sum_{k=0}^{[i/m]} \frac{i!}{(m!)^k (i-mk)! k!} \mathcal{L}_x^k h_{i-mk} \right] \quad (3.27)$$

Let us suppose that  $P$  and  $\tilde{P}$  are related by  $F_\lambda$ .

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<sup>24</sup> This is similar to the language of Arms and Anderson but we do not enter a discussion of jets and bundles of jets.

<sup>25</sup> Arms and Anderson 1986.

$$\tilde{P} = j^r(F_\lambda^* g) \quad (3.28)$$

$$\Rightarrow \tilde{h}_i = \sum_{k=0}^{[i/m]} \frac{i!}{(m!)^k (i-mk)! k!} \mathcal{L}_x^k h_{i-mk} \quad (3.29)$$

The last equation yields the induced gauge transformations. For example suppose we gauge transform  $h_1$  what are the induced transformations on  $h_2, \dots, h_{n-1}$  that we need for evaluation of  $S_n$ ?

$$h_1 \rightarrow h_1 + \mathcal{L}_x g \quad (3.30)$$

$$h_2 \rightarrow h_2 + 2\mathcal{L}_x h_1 + \mathcal{L}_x^2 g \quad (3.31)$$

$$h_{n-1} \rightarrow h_{n-1} + \sum_{k=1}^{n-1} \frac{(n-1)!}{(n-1-k)! k!} \mathcal{L}_x^k h_{n-1-k} \quad (3.32)$$

In the literature gauge transformations of this form are denoted as,

$$\Gamma_x^k(P) \text{ for } P = j^r(g(\lambda)) \quad k \leq r. \quad (3.33)$$

$k$  is the number of the lowest order of the field for which the transformation is not an identity. If we consider the set of  $P = j^r(g(\lambda))$  for all curves, the set of applicable gauge transformations forms a group<sup>26</sup>. Any gauge transformation,  $\Gamma$ , can be decomposed as,

$$\Gamma = \Gamma_{x^r}^r \Gamma_{x^{r-1}}^{r-1} \dots \Gamma_{x^1}^1. \quad (3.34)$$

That is, there will always be a set of vectors  $(x^r, \dots, x^1)$  such that the above decomposition is possible<sup>27</sup>. We now have the tools to prove the following theorem.

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<sup>26</sup> Arms and Anderson 1986.

<sup>27</sup> Arms and Anderson 1986.

Theorem (3.2)<sup>28</sup>: Let  $g \in \text{EinCC}$  with Killing vector  $\gamma$  and let

$P = (g, h_1, \dots, h_r)$  be an  $r^{\text{th}}$  order solution of the field equations. Then the Taub numbers of order  $k \leq r + 1$  are gauge invariant.

Proof: Let  $F_\lambda$  represent the gauge transformation and let  $g(\lambda)$  be a curve of solutions such that,  $P = j^r(g(\lambda))$ . Let us write the gauge transformation as,

$$\Gamma = \Gamma_{x^r}^r \Gamma_{x^{r-1}}^{r-1} \dots \Gamma_{x^1}^1. \quad (3.35)$$

Let  $\Sigma$  and  $\Sigma'$  be two nonintersecting Cauchy surfaces. Define,

$$\tilde{x}^r = x^r \text{ in a neighborhood of } \Sigma, \quad (3.36)$$

$$\tilde{x}^r = 0 \text{ in a neighborhood of } \Sigma'. \quad (3.37)$$

Then,

$$\tilde{\Gamma} = \Gamma_{\tilde{x}^r}^r \Gamma_{\tilde{x}^{r-1}}^{r-1} \dots \Gamma_{\tilde{x}^1}^1. \quad (3.38)$$

and,

$$t_k(\tilde{\Gamma}(P)) = t_k(\Gamma(P)) \text{ on } \Sigma \quad (3.39)$$

$$= t_k(P) \text{ on } \Sigma' \quad (3.40)$$

$$\Rightarrow \tau_k(\Gamma(P), \Sigma) = \tau_k(\tilde{\Gamma}(P), \Sigma) \quad (3.41)$$

$$= \tau_k(\tilde{\Gamma}(P), \Sigma') \quad (3.42)$$

$$= \tau_k(P, \Sigma') \quad (3.43)$$

$$= \tau_k(P, \Sigma) \quad (3.44)$$

The proof would fail without  $g \in \text{EinCC}$  since the manipulations going from (3.41) to (3.42) and (3.43) to (3.44) would not be possible without compact Cauchy surfaces. Thus, Taub numbers of order  $n$  for  $g \in \text{EinCC}$  are invariant with

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<sup>28</sup> Arms and Anderson 1986.

respect to gauge transformations on  $h_n$  (if using (1.24)) and with respect to gauge transformations on  $(h_{n-1}, \dots, h_1)$  (if using (1.25)). At present there is no proof for theorem 3.2 for  $g \in Ein$  and having noncompact Cauchy surfaces.

For asymptotically flat spacetimes,  $g \in Ein.AF$ , we want the perturbation to preserve null infinity. We want the curve of solutions to stay in  $EinAf$ . This restriction is provided by the Geroch - Xanthopoulos (GX) theorem<sup>29</sup>.

Theorem (3.3): Let  $g \in Ein.AF$  and  $h$  be a solution of  $(D_g R_{ab}) \cdot (h) = 0$ . If  $\tilde{h} = \Omega^2 h$  satisfies the following conditions at null infinity,

- (1)  $\tilde{h}_{ab}|_{S^+} = 0,$
- (2)  $\left. \frac{\tilde{h}_{ab} \tilde{h}^a}{\Omega} \right|_{S^+} = 0,$
- (3)  $\left. \frac{\tilde{h}_{ab} \tilde{h}^a \tilde{h}^b}{\Omega^2} \right|_{S^+} = 0,$

then asymptotic simplicity is preserved. The perturbations studied in this document shall all obey the GX theorem.

Note that if we were generating a curve of solutions by inverting,

$$(D_g R_{ab}) \cdot \left( \begin{matrix} h \\ n \end{matrix} \right) = S_{ab} \quad (3.45)$$

for each coefficient  $h_n$  we would need only check that  $h_1$  obeyed the GX theorem.

This is because  $S$  is comprised of products of  $h_1, \dots, h_{n-1}$  and their derivatives, hence  $h_n$  would fall off faster than all preceding fields. Thus asymptotic simplicity would be preserved to all orders.

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<sup>29</sup> Geroch and Xanthopoulos 1978 and Glass 1993.

#### 4 Potentials and Interpretations

To develop a physical interpretation of the Taub numbers let us consider the construction of the Komar integral<sup>30</sup>. Consider an asymptotically flat spacetime,  $g \in \text{EinAF}$ , which possesses a Killing vector  $k$ . Then,

$$k^{(a;b)} = 0, \quad k^{a;b} = k^{(a;b)}. \quad (4.1)$$

In this case we can construct a covariant conservation law.

$$J^a_{;a} = k^{a;b}{}_{;a} = 0 \quad (4.2)$$

$$J^a = k^{a;b}{}_{;b} = -R^a_b k^b \quad (4.3)$$

Equation (4.3) is proportional to the zero-order Taub vectors (recall that for the zero order Taub vector we use the Ricci tensor instead of the Einstein tensor).

Thus, properly normalized, the zero order Taub numbers are the same as the Komar quantities. The normalization can be found as follows, let  $\mathcal{L}_0$  denote the Komar superpotential (density)

$$U_0^{ab} = 2\sqrt{-g} k^{b;a}. \quad (4.4)$$

Then the zero-order Taub vector (density) is,

$$t_0^a = -\frac{1}{2} U_0^{ab}{}_{;b}. \quad (4.5)$$

$$t_0^a{}_{;a} = 0. \quad (4.6)$$

Now we integrate this over a domain  $D$ , a 4-dimensional region bounded by two 3-surfaces  $\Sigma_1$  and  $\Sigma_2$  such that the two surfaces meet at the same slice of  $\mathfrak{S}^+$ . The first surface is a null surface in the vacuum region which becomes spacelike in the interior (source) region. The second surface lies to the future of the first. It is

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<sup>30</sup> Komar 1959.

spacelike in the source and vacuum regions and becomes null asymptotically. The second surface intersects  $\mathfrak{S}^+$  in the same cut as the first. This intersection forms a two sphere. Thus we have the following,

$$\int_D t_0^a{}_{;a} d^4x = 0, \quad (4.7)$$

$$\Rightarrow \int_{\partial D} t_0^a ds_a = \int_{\Sigma_1 - \Sigma_2} t_0^a ds_a = 0, \quad (4.8)$$

$$\Rightarrow \int_{\Sigma_1} t_0^a ds_a = -\frac{1}{2} \int_{\Sigma_1} U_0^{ab}{}_{;b} ds_a, \quad (4.9)$$

$$= -\frac{1}{4} \int_{\partial \Sigma_1} U_0^{ab} dS_{ab} = \text{Komar quantity} = \tau_0 \quad (4.10)$$

The zero-order Taub numbers (Komar quantities) for the Schwarzschild and Kerr solutions are well known<sup>31</sup> (see appendix 3 for these solutions).

$$\tau_0(\text{timelike Killing vector}) = 8\pi m, \text{ Schwarzschild} \quad (4.11)$$

$$\tau_0(\text{timelike Killing vector}) = 8\pi m, \text{ Kerr} \quad (4.12)$$

$$\tau_0(\text{rotational Killing vector}) = -16\pi ma, \text{ Kerr}. \quad (4.13)$$

The constant 'a' is the rotation parameter of the Kerr solution. Note that there is a factor of 2 difference between the normalization for the mass and angular momentum.

$$\frac{1}{8\pi_0} \tau_0(\text{timelike Killing vector}) = \text{Mass} \quad (4.14)$$

$$-\frac{1}{16\pi_0} \tau_0(\text{rotational Killing vector}) = \text{Angular Momentum} \quad (4.15)$$

A similar superpotential for the higher order Taub numbers can be found provided we use (1.24).

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<sup>31</sup> Wald 1984 and Glass 1993.

$$t_a = k^b (D_\sigma G_{ab}) \cdot \left( \frac{h}{n} \right) \quad (4.16)$$

$$= \frac{1}{2} k^b \left\{ h_n^m{}_{b;am} + h_n^m{}_{a;bm} - h_n^{b;a}{}_{m} - h_n{}_{;ab} \right\} - \frac{1}{2} k_a \left\{ h_n^{mb}{}_{;bm} - h_n{}_{;m} \right\}. \quad (4.17)$$

We need find an antisymmetric tensor such that,

$$U^{ab}{}_{;b} = t^a \Rightarrow U^{ab}{}_{;ba} = 0 \quad (4.18)$$

$$U^{ab} = U^{ab} \left( \frac{h}{n}, k \right). \quad (4.19)$$

From the functional form of (4.17) we may 'guess' at a superpotential without too much difficulty, or, we can be more systematic and write out all combinations of the various contractions of the Killing vector, the perturbation field and their derivatives. The search is thereby reduced to a problem in linear algebra. The possible terms are given below,

$$k^{[a} h^{b]}{}_{m;}, \quad (4.20)$$

$$k^m h_m^{[a;b]}, \quad (4.21)$$

$$k^{[a} h^{b]}, \quad (4.22)$$

$$k^{[a;b]} h, \quad (4.23)$$

$$k^{[a}{}_{;m} h^{b]m}, \quad (4.24)$$

$$k^{m; [a} h^{b]}{}_m. \quad (4.25)$$

After computing the divergence of each term and comparing with (4.17) the superpotential (Glass) is found to be<sup>32</sup>

$$U^{ab} = k^{[a} h^{b]}{}_{m;} - k^{[a} h^{b]} + \frac{h}{2} k^{a;b} + k^m h_m^{[a;b]} + k^{m; [a} h^{b]}{}_m. \quad (4.26)$$

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<sup>32</sup> Glass 1993. A similar superpotential given in Arms and Anderson 1986, which, however is incorrect.



We can now compute Taub numbers of all orders provided we know the perturbing field of the desired order.

Suppose we know the fields  $h_1, \dots, h_{n-1}$  and wish to compute  $\tau_n$ . Is it possible

to construct a potential for (1.25)? Let's consider the simplest case,  $\tau_2$  given  $h_1$ .

$$t_a = k^b S_{ab} - \frac{1}{2} k_a g^{lm} S_{lm} \quad (4.27)$$

$$= k^b (D_g^2 R_{ab}) \cdot \left( \begin{matrix} h \\ 1 \\ h \\ 1 \end{matrix} \right) - \frac{1}{2} k_a g^{lm} (D_g^2 R_{lm}) \cdot \left( \begin{matrix} h \\ 1 \\ h \\ 1 \end{matrix} \right) \quad (4.28)$$

In this case the superpotential must be quadratic in the field  $h_1$ . If we repeat the

process used to find the superpotential for (4.16) and write out all possible contractions of the Killing vector, quadratic fields and their derivatives (a somewhat longer calculation) we find that there is no simple potential of the form of (4.26). This is to be expected since it is quadratic in  $h_1$ , and  $h_2$  is in principle determined by a Green function.

To interpret the first order Taub numbers we shall consider the Schwarzschild solution as a perturbation of the Minkowski solution via the Xanthopoulos theorem<sup>33</sup>. We shall also consider the Kerr solution as a perturbation of Minkowski. These solutions are not related by the Xanthopoulos theorem since the Kerr solution has twist and Minkowski does not (in this case the perturbation term does not satisfy the linearized field equation). Solutions related by the Xanthopoulos theorem must have the same shear and twist<sup>34</sup>.

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<sup>33</sup> Xanthopoulos 1978.

<sup>34</sup> Xanthopoulos 1983.

Let us now derive the Xanthopoulos theorem. Let  $g \in LS$  so that we may freely construct a curve of solutions for any tangent which is a solution of the linearized field equations.

$$(D_g R_{ab}) \cdot \begin{pmatrix} h \\ l \end{pmatrix} = 0 \quad (4.29)$$

Let us consider only those solutions which we can write as a double null vector.

$$h_{ab} = l_a l_b \quad (4.30)$$

$$l^a l_a = 0 \quad (4.31)$$

Note: We are not claiming this is always possible. For purposes of this discussion we shall assume that for the point  $g \in LS$  chosen this is possible.

The following theorem will be useful.

Theorem (4.1)<sup>35</sup>: Let  $g \in Ein$  and let  $l$  be a null vector such that,  $h = ll$  satisfies the linearized field equations. Then,

$$(D_g^n R_{ab}) \cdot \begin{pmatrix} h \\ l \end{pmatrix}, \dots, \begin{pmatrix} h \\ l \end{pmatrix} = 0 \text{ for } n = 1, 2, \dots \quad (4.32)$$

Proof: We are given that this is true for  $n = 1$ . In fact this forces  $l$  to be geodesic with respect to  $g$ .<sup>36</sup> A straightforward calculation shows that this is also true for  $n = 2$  and 3. Consider the  $n^{\text{th}}$  term (see equation A1.36).

$$\begin{aligned} (D_g^n R_{ld}) \cdot \begin{pmatrix} h \\ l \end{pmatrix}, \dots, \begin{pmatrix} h \\ l \end{pmatrix} &= -n h_l^m (D_g^{n-1} R_{lmd}^b) \cdot \begin{pmatrix} h \\ l \end{pmatrix}, \dots, \begin{pmatrix} h \\ l \end{pmatrix} \\ &+ n (D_g^{n-1} \Gamma_{ld}^k) \cdot \begin{pmatrix} h \\ l \end{pmatrix}, \dots, \begin{pmatrix} h \\ l \end{pmatrix} \left\{ (D_g \Gamma_{km}^m) \cdot \begin{pmatrix} h \\ l \end{pmatrix} - h_{k:m}^m \right\} \\ &- n (D_g^{n-1} \Gamma_{lm}^k) \cdot \begin{pmatrix} h \\ l \end{pmatrix}, \dots, \begin{pmatrix} h \\ l \end{pmatrix} \left\{ (D_g \Gamma_{kd}^m) \cdot \begin{pmatrix} h \\ l \end{pmatrix} - h_{k:d}^m \right\} \end{aligned} \quad (4.33)$$

Let us apply the recursion relations from appendix 1.

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<sup>35</sup> Naber 1992.

<sup>36</sup> Xanthopoulos 1978.

$$\begin{aligned}
& (D_{\sigma}^n R_{ld}) \cdot \left( h_1^1, \dots, h_1^1 \right) = n(n-1) h_1^m h_1^b (D_{\sigma}^{n-2} R_{lmd}^c) \cdot \left( h_1^1, \dots, h_1^1 \right) \\
& + n(n-1)(n-2) h_1^k h_1^c h_1^b (D_{\sigma}^{n-3} \Gamma_{ld}^b) \cdot \left( h_1^1, \dots, h_1^1 \right) \left\{ (D_{\sigma} \Gamma_{km}^m) \cdot \left( h_1^1 \right) - h_1^{k:m} \right\} \\
& - n(n-1)(n-2) h_1^k h_1^c h_1^b (D_{\sigma}^{n-3} \Gamma_{lm}^b) \cdot \left( h_1^1, \dots, h_1^1 \right) \left\{ (D_{\sigma} \Gamma_{kd}^m) \cdot \left( h_1^1 \right) - h_1^{k:d} \right\} \\
& - n(n-1) h_1^m \left\{ (D_{\sigma}^{n-2} \Gamma_{ld}^k) \cdot \left( h_1^1, \dots, h_1^1 \right) \left\{ (D_{\sigma} \Gamma_{km}^b) \cdot \left( h_1^1 \right) - h_1^{k:m} \right\} \right. \\
& \quad \left. - (D_{\sigma}^{n-2} \Gamma_{lm}^k) \cdot \left( h_1^1, \dots, h_1^1 \right) \left\{ (D_{\sigma} \Gamma_{kd}^b) \cdot \left( h_1^1 \right) - h_1^{k:d} \right\} \right\} \quad (4.34)
\end{aligned}$$

We are given that  $h_1^1 = ll$  and that  $l$  is null, and thus, we have  $h_1^k h_1^c h_1^d = 0$ .

Since  $l$  is geodesic<sup>37</sup> (i.e. the spin coefficient  $\kappa$  is zero) the terms multiplying  $h_1^m$  are proportional to  $l_m$  or  $l^b$ . Therefore after contraction with  $h_1^m$  the remaining terms vanish. Thus the theorem is proved.

Let us now consider the set of field equations for the coefficients  $h_1, h_2, \dots$

given by equations (1.13) - (1.16). Let us assume we have been given a  $g \in LS$  and an  $h_1$  such that  $h_1^1 = ll$  for some vector  $l$  which is null on  $g$ . Consider equation (1.4).

$$(D_{\sigma}^2 R_{ab}) \cdot (ll, ll) + (D_{\sigma} R_{ab}) \cdot \left( h_2^1 \right) = 0 \quad (4.35)$$

The first term is zero by the previous theorem. This leaves,

$$(D_{\sigma} R_{ab}) \cdot \left( h_2^1 \right) = 0. \quad (4.36)$$

Since this is a linear, homogeneous second order hyperbolic equation we are perfectly free to chose  $h_2^1 = 0$  as the solution. Having done this let us move onto the next field equation and repeat the previous steps. Once again we are left with,

$$(D_{\sigma} R_{ab}) \cdot \left( h_3^1 \right) = 0, \quad (4.37)$$

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<sup>37</sup> Xanthopoulos 1978.

and we are free to set  $h_3$  to zero as well. We may repeat this procedure ad infini-

tum and set all  $h_n = 0$  for  $n > 1$ . This results in the Xanthopoulos theorem.

**Theorem (4.2)<sup>38</sup>:** Let  $(M, g_{ab})$  be an exact vacuum solution of Einstein's equations and let  $l_a$  be a null vector field such that  $h_{ab} = l_a l_b$  satisfies the linearized field equations. Then  $g_{ab} + l_a l_b$  is an exact vacuum solution.

A similar theorem can be produced for the electrovac, scalar field and perfect fluid spacetimes<sup>39</sup>.

As mentioned before the Schwarzschild and Minkowski solutions are related by the Xanthopoulos theorem. This can be shown by two separate methods: either by solving the linearized field equations and choosing the 'double null' solution which when added to the Minkowski solution can be transformed into a familiar form of the Schwarzschild solution; or we may follow Eddington<sup>40</sup> and apply a coordinate transformation to the time coordinate and split the solution into two pieces. This is carried out below. Consider the Schwarzschild solution in the usual polar coordinates.

$$ds^2 = \left( \frac{r-2m}{r} \right) dt^2 - \left( \frac{r}{r-2m} \right) dr^2 - r^2 (d\theta^2 + \sin^2(\theta) d\phi^2) \quad (4.38)$$

We then transform the time coordinate as follows,

$$t \rightarrow t - 2m \ln \left( \frac{r-2m}{2m} \right) \quad (4.39)$$

$$\Rightarrow ds^2 = dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2(\theta) d\phi^2) - \frac{2m}{r} (dt + dr)^2, \quad (4.40)$$

Thus, the Schwarzschild metric can be written as,

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**38 Xanthopoulos 1978.**

**39 Mastronikola and Xanthopoulos 1989, Mastronikola 1987 and Xanthopoulos 1986.**

**40 Eddington 1924 or chapter 7 of Adler, Bazin and Schiffer 1975.**

$$g_{\underline{s}} = \eta - 2m \frac{l l}{\underline{s}}, \quad (4.41)$$

$$l_{\underline{s}^a} = \frac{1}{\sqrt{r}}(1, 1, 0, 0). \quad (4.42)$$

The vector  $l_{\underline{s}}$  is null with respect to  $g_{\underline{s}}$  and  $\eta$ ; as well, a straight forward calculation shows that  $l_{\underline{s}} l_{\underline{s}}$  satisfies the linearized field equations at Minkowski. Minkowski and Schwarzschild are indeed Xanthopoulos-related.

The Minkowski metric represents flat spacetime and therefore has no energy or angular momentum. Hence we expect that all 'mass' information is carried in the perturbation  $l_{\underline{s}} l_{\underline{s}}$ . Let us consider the first order Taub number for this perturbation of the Minkowski solution. Let us transform to a null polar coordinate system and thereby use the tetrad and spin coefficients which are given in appendix 3;  $u = t - r$ , for coordinates  $(u, r, \theta, \phi)$ .

$$\Rightarrow l_{\underline{s}^a} = \frac{1}{\sqrt{r}}(1, 2, 0, 0) = \frac{2}{\sqrt{r}}n_a \quad (4.43)$$

$$\Rightarrow g_{\underline{s}ab} = \eta_{ab} - \frac{8m}{r}n_a n_b \quad (4.44)$$

The timelike Killing vector is given by,

$$k^a = n^a + \frac{1}{2}l^a = \partial_u. \quad (4.45)$$

There are three terms in the superpotential which are not manifestly zero.

$$U^{ab} = k^{[a} h_m^{b]} :^m + k^m h_m^{[a:b]} + k^{m:[a} h_m^{b]} \quad (4.46)$$

$$k^a h_m^{b :m} = \frac{m}{r^2}(n^b l^a - n^a l^b) \quad (4.47)$$

$$(k^m h_m^{[a :b]}) = \frac{m}{r^2}(n^b l^a - n^a l^b) \quad (4.48)$$

$$k^{m:[a} h_m^{b]} = 0 \quad (4.49)$$

$$\Rightarrow \tau_1 = \int \sqrt{-g} l^{ab} dS_{ab} \quad (4.50)$$

$$= \int r^2 U^{ab} l_{(a} n_{b)} \sin(\theta) d\theta d\phi \quad (4.51)$$

$$= 8\pi m \quad (4.52)$$

The Kerr solution can also be put in the form of the Minkowski metric plus a double null vector (in the literature this is called a G.K.S. metric, or transformation)<sup>41</sup>. See appendix 3 for details.

$$g_{ab} = \eta_{ab} - 2m l_a l_b \quad (4.53)$$

$$l_a = Pu l_a + Pr n_a + P m_a + \bar{P} \bar{m}_a \quad (4.54)$$

$$l_a l^a = 0 \Rightarrow PuPr = P\bar{P} \quad (4.55)$$

The functions Pu, Pr and P are given in appendix 3. The first order Taub numbers for the Kerr solution written as a perturbation of Minkowski will still have meaning since the perturbations for the first-order Taub number need only be symmetric to satisfy the covariant conservation equation.  $\frac{l l}{K K}$  is the sum of an infinite set of perturbations which obey the sequential set of field equations. We shall return to this later when we attempt to fix an understanding of the higher order Taub numbers. Essentially what we will show is that 'α' is a good curve parameter<sup>42</sup>. For this perturbation and the timelike Killing vector the superpotential reduces to two terms,

$$U^{ab} = k^{[a} h^{b]} :^m + (k^m h_m^{[a} :^b]). \quad (4.56)$$

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**41** Chapter 7 of Adler, Bazin and Schiffer 1975 and references therein.

**42** By a good curve parameter we mean that the Taylor expansion of a solution with respect to a parameter of the solution, say a or m, satisfies the sequential set of field equations (1.21). For example m is a good curve parameter for Schwarzschild but not for Kerr.

Keeping only terms which are proportional to  $l^{[a} n^{b]}$  we have<sup>43</sup>,

$$(k^m h_m^{[a} ]^{b]}) = -2m l^{[a} n^{b]} D \left[ \left( Pu + \frac{1}{2} Pr \right)^2 \right], \quad (4.57)$$

$$k^{[a} h^{b] m} = 2m l^{[a} n^{b]} \left[ D \left( Pu - \frac{1}{2} Pr \right)^2 + \frac{2}{r} \left( \left( Pu - \frac{1}{2} Pr \right)^2 \right) \right], \quad (4.58)$$

$$\tau = -m \int r^2 \left[ -2D(PuPr) + \frac{2}{r} \left( Pu^2 - 2PuPr + \frac{1}{4} Pr^2 \right) \right] \sin(\theta) d\theta d\phi, \quad (4.59)$$

$$\tau = 8\pi m. \quad (4.60)$$

This is the same result that was obtained for Schwarzschild viewed as a perturbation of Minkowski. The rotational Killing vector relevant to the Kerr solution,  $\partial_\phi$ , is given by

$$k^a = \frac{r \sin(\theta)}{\sqrt{2}i} (m^a - \bar{m}^a). \quad (4.61)$$

Once again there are only two terms in the superpotential that are not zero.

$$U^{ab} = (k^m h_m^{[a} ]^{b]}) + 2k^{m:[a} h^{b]}_m, \quad (4.62)$$

$$k^{m:[a} h^{b]}_m = -\frac{2m \sin(\theta)}{\sqrt{2}i} (\bar{P} - P) \left( Pu + \frac{1}{2} Pr \right) l^{[a} n^{b]}, \quad (4.63)$$

$$(k^m h_m^{[a} ]^{b]}) = -\frac{2m \sin(\theta)}{\sqrt{2}i} D \left[ r(P - \bar{P}) \left( Pu + \frac{1}{2} Pr \right) \right] l^{[a} n^{b]}. \quad (4.64)$$

Integrating this result over the two-sphere cut of  $\Sigma^+$  yields,

$$\tau = -8\pi m a. \quad (4.65)$$

As a further example we will write the Kerr solution as a perturbation of Schwarzschild. Once again the perturbation is not a solution of the linearized equa-

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<sup>43</sup> In the following 'D' is the NP intrinsic derivative.

tions but is the sum of an infinite set of perturbations which satisfy the full set of perturbation equations. Recall the previous decompositions of Schwarzschild and Kerr:

$$\underline{g}_S = \eta - 2m \begin{matrix} l & l \\ S & S \end{matrix}, \quad (4.66)$$

$$\underline{g}_K = \eta - 2m \begin{matrix} l & l \\ K & K \end{matrix}. \quad (4.67)$$

Now add and subtract the Schwarzschild perturbation to  $\underline{g}_K$ :

$$\underline{g}_K = \eta - 2m \begin{matrix} l & l \\ S & S \end{matrix} - 2m \left( \begin{matrix} l & l \\ K & K \end{matrix} - \begin{matrix} l & l \\ S & S \end{matrix} \right), \quad (4.68)$$

$$\underline{g}_K = \underline{g}_S - 2m \tilde{h}, \quad (4.69)$$

$$\text{where } \tilde{h} = \begin{matrix} l & l \\ K & K \end{matrix} - \begin{matrix} l & l \\ S & S \end{matrix}.$$

We now view  $\tilde{h}$  as an object on the Schwarzschild geometry. As such it does not obey the linearized field equation and is not trace free (it can not be written as a double null vector). The Schwarzschild metric and a tetrad are given in appendix 3.

Given the tetrad, the timelike and rotational Killing vectors are

$$k_t^a = \frac{1}{2} \left( 1 - \frac{2m}{r} \right) l^a + n^a, \quad (4.70)$$

$$k_\phi^a = \frac{r \sin(\theta)}{\sqrt{2}l} (m^a - \bar{m}^a). \quad (4.71)$$

Notice that when the mass goes to zero the timelike Killing vector reduces to the timelike Killing vector for Minkowski.

For this case the first-order Taub numbers are,

$$\tau_1(\text{timelike}) = 0, \quad (4.72)$$

$$\tau_1(\text{rotational}) = -8\pi m a. \quad (4.73)$$



If we use the normalization from the first two examples, the above Taub numbers are what we expect. The perturbation,  $\bar{h}$ , induces angular momentum ( $-ma$ ) but no mass. All mass information is contained in the background geometry.

The normalization for the first-order Taub numbers is  $1/8\pi$  for both the timelike and rotational Killing vectors. Whereas for the zero-order Taub numbers there is a factor of 2 between the normalization for the timelike and rotational Komar quantities. This may be explained as follows: the Komar superpotential generates only the Ricci tensor,

$$U_0^{ab}{}_{;b} = k^b R^a{}_b. \quad (4.74)$$

The 2-surface integral samples just the Ricci tensor not the full Einstein tensor. However angular momentum and energy are most properly defined using the energy momentum tensor, i.e. the Einstein tensor. The Glass superpotential generates the functional derivative of the Einstein tensor not just the Ricci tensor.

$$U^{ab}{}_{;b} = k^b (D_g G^a{}_b) \cdot (h) \quad (4.75)$$

Thus, it is expected that if a superpotential could be found that would generate the Einstein tensor, then, the factor of 2 between the angular momentum and the energy would vanish.

$$\tilde{U}^{ab}{}_{;b} = k^b G^a{}_b \quad (4.76)$$

$$\tilde{\tau}_0 = 8\pi \bar{m} \quad (4.77)$$

$$\tilde{\tau}_0 = -8\pi \bar{m} \bar{a} \quad (4.78)$$

## 5 Further Applications

We will now apply the method of Taub numbers to the perturbation results of Regge and Wheeler (1957), Vishveswara (1970) and Zerilli (1970). Regge and Wheeler<sup>44</sup> were interested in the stability of the Schwarzschild solution under metrical perturbations. Their results were inconclusive due to coordinate difficulties. The stability question was finally answered by Vishveshwara<sup>45</sup> who had the Kruskal coordinates<sup>46</sup> in which to work. Essentially what they wanted to know was, if the metric is perturbed by a solution of the linearized field equations, would the perturbed metric oscillate about the Schwarzschild solution or would the perturbations grow exponentially in time?

For the vacuum Schwarzschild geometry, solutions of the linearized field equations can be divided into two classes: axial and radial (in the literature these are alternatively called electric and magnetic or odd and even parity perturbations<sup>47</sup>). The under letters A and R shall differentiate between the axial and radial perturbations. The most general form of the perturbations can be written, in coordinates  $(t, r, \theta, \phi)$ , (after a gauge transformation  $h \rightarrow h + \mathcal{L}_x g$ ) as<sup>48</sup>

$$h_{Ab} = - \left\{ h_0(r) 2\delta_{(a}^0 \delta_{b)}^3 + h_1(r) 2\delta_{(a}^1 \delta_{b)}^3 \right\} e^{-ikt} \sin(\theta) \frac{\partial}{\partial \theta} P_l(\cos(\theta)), \quad (5.1)$$

$$h_{Rab} = \left\{ H_0 \left( \frac{2m}{r} - 1 \right) \delta_a^0 \delta_b^0 + H_2 \frac{r}{2m-r} \delta_a^1 \delta_b^1 - H_1 2\delta_{(a}^0 \delta_{b)}^1 \right. \\ \left. - r^2 K \delta_a^2 \delta_b^2 - r^2 K \sin^2(\theta) \delta_a^3 \delta_b^3 \right\} e^{-ikt} P_l(\cos(\theta)). \quad (5.2)$$

In terms of the tetrad we are using for the Schwarzschild metric the perturbation takes the following form,

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<sup>44</sup> Regge and Wheeler 1957.

<sup>45</sup> Vishveshwara 1970.

<sup>46</sup> Kruskal 1960.

<sup>47</sup> Zerilli 1970.

<sup>48</sup> Vishveshwara 1970.

$$h_{ab} = \left\{ \sqrt{2} \frac{n_{(a}[\bar{m}_b) - m_b]}{ir \sin(\theta)} \left( \frac{h_0 r}{2m-r} - h_1 \right) + \frac{l_{(a}[\bar{m}_b) - m_b]}{ir \sqrt{2} \sin(\theta)} \left( h_1 \left( \frac{r-2m}{r} \right) - h_0 \right) \right\} e^{-ikt} \sin(\theta) \frac{\partial}{\partial \theta} P_l(\cos(\theta)) \quad (5.3)$$

$$h_{ab} = \left\{ l_a l_b \left( \frac{2m-r}{2r} \right) [H_0/2 + H_2 - H_1] + n_a n_b \left( \frac{r}{2m-r} \right) [H_0 + H_2 + 2H_1] + l_{(a} n_{b)} [H_2 - H_0] - 2k m_{(a} \bar{m}_{b)} \right\} e^{-ikt} P_l(\cos(\theta)) \quad (5.4)$$

Let us first consider stationary axial perturbations for  $l = 1$ . In this case we will expect that the first order Taub numbers will be zero for the timelike Killing vector and non-zero for the rotational killing vector. The two unknown functions are

$$h_0 = \frac{c}{r}, \quad c = \text{constant}. \quad (5.5)$$

$$h_1 = 0 \quad \text{through a gauge transformation} \quad (5.6)$$

$$\Rightarrow h_{ab} = \frac{c \sin(\theta)}{ir} \left\{ n_{(a} [m_b) - \bar{m}_b] \left( \frac{\sqrt{2}}{r-2m} \right) + l_{(a} [m_b) - \bar{m}_b] \left( \frac{1}{\sqrt{2}r} \right) \right\} \quad (5.7)$$

$$h_a^a = 0 \quad (5.8)$$

Notice that the perturbation is trace free. This reduces the superpotential to three terms,

$$U^{ab} = k^{[a} h_m^{b]:m} + (k^m h_m^{[a]:b]) + 2k^{m:[a} h_m^{b]}. \quad (5.9)$$

For the timelike Killing vector none of the three pieces of the superpotential have components along  $l^{[a} n^{b]}$ , therefore

$$\tau_l(\text{timelike}) = 0 \quad (5.10)$$

as expected. For the rotational Killing vector we obtain,

$$\tau_l(\text{rotational}) = -2\pi c \quad (5.11)$$

Setting  $c = 4ma$  gives the Kerr value. One of Vishveswara's results was that there are no stationary axial perturbations with  $l$  values greater than 1. In fact the  $l = 1$  case is the only one possible since the  $l = 0$  solution is identically zero. Hence the above calculations represent the only stationary axial perturbation. This is quite interesting since, if we examine the Kerr solution viewed as a perturbation of Schwarzschild,

$$\tilde{h} = \begin{matrix} ll \\ KK \end{matrix} - \begin{matrix} ll \\ SS \end{matrix}, \quad (5.12)$$

we find that,

$$\tilde{h}(\alpha = 0) = 0, \quad (5.13)$$

$$\left. \frac{d\tilde{h}_{ab}}{d\alpha} \right|_{\alpha=0} = h_{ab}(l=1) + f_{(a;b)}, \quad (5.14)$$

where  $f_{(a;b)}$  is a gauge term.

$$f_a = ig(r)\sin(\theta)(m_a - \bar{m}_a) \quad (5.15)$$

$$g(r) = \frac{c}{2\sqrt{2}m} + \frac{cr}{4\sqrt{2}m^2} \ln\left(\frac{r-2m}{r}\right) \quad (5.16)$$

Thus, if we are constructing a curve of solutions beginning at Schwarzschild, such that the curve is a proper Taylor series (satisfies the sequential set of field equations), and we wish to go along the stationary axial direction, then the only direction in which we can go is towards the Kerr solution. This result might have been guessed since the Schwarzschild solution is the unique solution for non-rotating black holes<sup>49</sup> and the Kerr solution is the unique solution representing stationary axisymmetric black holes<sup>50</sup>.

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<sup>49</sup> Page 299, Chandrasekar 1983.

<sup>50</sup> Page 292, Chandrasekar 1983.

There are two stationary radial solutions,  $l = 0, 1$ . For the  $l = 0$  case we expect a mass contribution with no angular momentum. The  $l = 1$  solution has been shown to displace the center of attraction<sup>51</sup> and will not be treated by the Taub number method. For  $l = 0$  the solution is given by<sup>52</sup>

$$K = H_1 = 0, \quad (5.17)$$

$$H_0 = H_2 = \frac{c}{r-2m}, \quad (5.18)$$

$$\Rightarrow h_{ab} = l_a l_b \left( \frac{-3c}{4r} \right) + n_a n_b \left[ \frac{-2cr}{(r-2m)^2} \right], \quad (5.19)$$

$$h_{ab}^a = 0. \quad (5.20)$$

Once again the perturbation is trace free which simplifies the calculations. In this case we find that the rotational Taub number is indeed zero and the timelike Taub number is proportional to the constant  $c$ .

From these results it would appear that the rotation parameter 'a' is a good curve parameter. For Kerr viewed as a curve starting at Schwarzschild we have,

$$\frac{g}{k} = \frac{g}{s} + 4ma h_A(l=1) + gauge + \sum_{i=2}^{\infty} \frac{a^i}{i!} h_i \quad (5.21)$$

where the  $h_i$  are tabulated in appendix 3. The zero-order and first-order field equations are satisfied and, if the series is summed, the vacuum field equations are satisfied.

For Kerr viewed as a curve starting at Minkowski we have,

$$\frac{g}{k} = \eta - \frac{8m}{r} nn + 4ma h_A(l=1) + gauge + \sum_{i=2}^{\infty} \frac{a^i}{i!} h_i. \quad (5.22)$$

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<sup>51</sup> Vishveshwara 1970.

<sup>52</sup> Zerilli 1970.

The  $h$  are tabulated in appendix 3 and the tetrad used is the tetrad for Schwarzschild with the mass parameter set to zero. Here the zero, first and second order field equations are satisfied and, if the series is summed the vacuum field equations are satisfied. Thus it would appear that the rotation parameter is indeed a good curve parameter, and we would expect that the sequential set of field equations (1.21) are satisfied, for the two cases, by the two sets of fields as given in appendix 3.

## 6 Future Work

In this section some of the open questions raised in the preceding sections shall be discussed, as well as, some future applications of Taub numbers and the formalisms developed.

To begin let us return to the sequential set of field equations for curves of solution  $\hat{g}(\lambda) \in \text{EinAF}$ .

$$R_{ab}(g) = 0 \quad (6.1)$$

$$(D_g R_{ab}) \cdot \left( \frac{h}{n} \right) = S_{ab} \quad (6.2)$$

Inverting the field equations sequentially has been suggested, c.f. Lerner and Porter 1974. A closed form for the source term to all orders has not yet been presented (as is done here). Consider the functional form of  $D_g R_{ab}$ :

$$(D_g R_{ab}) \cdot (h) = \frac{1}{2} (h_{ab}{}^{;m} + h_{;ab} - h^m{}_{b;a} - h^m{}_{a;b}). \quad (6.3)$$

Consider a gauge transformation  $h \rightarrow h + \mathcal{L}_x g$  such that<sup>53</sup>

$$x_a{}^{;t} = h_a{}^{;t} - \frac{1}{2} h_{;a}, \quad (6.4)$$

$$x^t{}_{;t} = \frac{1}{2} h. \quad (6.5)$$

This transforms  $D_g R_{ab}$  into the following,

$$2(D_g R_{ab}) \cdot (h) = h_{ab;t}{}^t + 2R^t{}_{abm} h^m{}_{;t}. \quad (6.6)$$

This is a covariant wave equation, and for a flat space background we have

$$2(D_g R_{ab}) \cdot (h) = \left( \frac{\partial^2}{\partial t^2} - \nabla^2 \right) h_{ab}. \quad (6.7)$$

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<sup>53</sup> Chung 1973, Friedlander 1975 and Peterson and Ruffini 1977.

A Green function for flat space may readily be constructed<sup>54</sup>. Let

$$G_{abcd}(x, x') = G(x, x') \frac{1}{2} [\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}]. \quad (6.8)$$

Then,

$$h_{ab} = \int G_{abcd}(x, x') S_n^{cd}(x') d^4 x', \quad (6.9)$$

where,

$$\left( \nabla^2 - \frac{\partial^2}{\partial t^2} \right) G(x, x') = \delta^4(x - x'). \quad (6.10)$$

Let us rotate the time variable into the complex plane such that  $t \rightarrow i\tau$  then,

$$-\left( \nabla^2 + \frac{\partial^2}{\partial \tau^2} \right) G(x, x') = \delta^4(x - x'). \quad (6.11)$$

The solution for this operator, the 4 dimensional Laplacian, is well known.

$$G(x, x') = \lim_{\epsilon \rightarrow 0} \text{Im} \left[ \frac{1}{4\pi^2 [(\vec{r} - \vec{r}')^2 - (t - t')^2 - i\epsilon]} \right] \quad (6.12)$$

For a background geometry which is not flat Dewitt and Brehme (1960), Chung (1973) and Peterson and Ruffini (1977) have adopted Hadamard's method (1952) to generate a Green function for the covariant wave equation. This is more involved than the above derivation but for simple geometries it should not be too difficult.

The source term,  $S_n$ , will be somewhat tedious to compute for  $n > 2$ . Comput-

ing  $S_n$  only involves taking derivatives of the  $h_{n-1}, \dots, h_1$ , various contractions and summing the series. Hence it would seem that a program for a symbolic computer language<sup>55</sup> could be written which would evaluate  $S_n$  given  $h_{n-1}, \dots, h_1$ . Generating each new field would then involve 10 integrations. Depending upon the nature of

<sup>54</sup> Chung 1973, Friedlander 1975 and Peterson and Ruffini 1977.

<sup>55</sup> For example, Maple or Mathematica.



the preceding perturbations and the Green function, these integrations might also be handled by Maple or Mathematica. The fields  $h_1, \dots, h_n, \dots$  could then be analyzed using the Taub number method.

One comment: curves passing through (or perturbations of) Schwarzschild may be generated with the Green function for flat space, due to the Xanthopoulos theorem. Here one would choose the first field to be  $h_1 = \frac{8m}{r} n_a n_b$ . The theorem allows us to set all higher-order fields to zero; however we need not do this. We may, for example, choose to set  $h_2$  to be one of the time-dependent perturbations of Schwarzschild generated by Vishveswra or Zerrilli. We would simply let the mass parameter in the tetrad decomposition go to zero. We are then looking at a perturbation of Schwarzschild in terms of Minkowski geometry. The higher-order fields may then be computed with the flat space Green function. It is hoped that this method could be used to generate an exact radiative asymptotically flat vacuum solution. This would be an important result.

An interesting open question raised is that of a superpotential for the zero-order Taub number when the Einstein tensor is not replaced by the Ricci tensor. This question has a partial answer due to Penrose<sup>56</sup> 1986, Goldberg 1990 and is discussed by Glass 1993. A superpotential may be constructed using a Killing potential  $Q$ .

$$Q^{ab} = Q^{[a b]} \quad (6.13)$$

$$\frac{1}{3} Q^{ab}{}_{;b} = k^a = \text{Killing vector} \quad (6.14)$$

Consider the double dual of the Riemann tensor,

$${}^*R^{*mn}{}_{ab} = -R^{mn}{}_{ab} - 2\delta_{[a}^m R_{b]}^n - 2G_{[a}^m \delta_{b]}^n. \quad (6.15)$$

From the Bianchi identities we know,

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<sup>56</sup> Penrose and Rindler 1986.

$${}^*R^{mna}{}_{;b} = 0. \quad (6.16)$$

If we combine this with the Killing potential we obtain,

$$\frac{1}{3}\nabla_n({}^*R^{mn}{}_{ab}Q^{ab}) = 2G^m{}_ak^a. \quad (6.17)$$

This apparently solves the problem. Unfortunately it also introduces a new gauge, the gauge of the Killing potential. To Q we may add any antisymmetric 2-tensor which is divergence free.

$$Q^{ab} \rightarrow Q^{ab} + A^{[ab]} \quad (6.18)$$

$$\text{such that } A^{[a}{}_{;b]} = 0 \quad (6.19)$$

In particular, for vacuum spacetimes, we can add the divergence of a Killing vector.

$$k^{a;b} \quad (6.20)$$

$$k^{a;b}{}_{;b} = R^a{}_bk^b = 0 \quad (6.21)$$

It is doubtful that the Taub integral would remain invariant under such a transformation.

$$\int {}^*R^{mn}{}_{ab}Q^{ab}dS_{mn} \stackrel{?}{=} \int {}^*R^{mn}{}_{ab}[Q^{ab} + A^{ab}]dS_{mn} \quad (6.22)$$

To use this potential one may need to separate out that part of Q which is pure gauge and work with what is left. Clearly there need be further work in this direction.

Given that all vacuum spacetimes are locally linearization stable, another potentially useful application of Taub numbers may be constructed by redefining the region of integration such that the superpotential would throw the integration onto a sphere about the source region at a finite value of r. In this case  $\Sigma_1$  would remain the same (see chapter 4) but  $\Sigma_2$  would become null at a finite value of r and then intersect  $\Sigma_1$  in a two sphere about the source region:

$$\tau_1 = \int_{S_2(r)} U^{ab} dS_{ab}. \quad (6.22)$$

This type of construction should find use in astrophysical applications, where measurements are indeed at a finite value of the radial parameter (luminosity distance).

One topic not addressed here is that of gravitational radiation. Habisohn<sup>57</sup> has investigated the properties of

$$\tau_2 \left( \begin{matrix} h \\ 1 \end{matrix}, \begin{matrix} h \\ 1 \end{matrix} \right) = \int_{\sigma} (D_{\sigma}^2 G^{ab}) \cdot \left( \begin{matrix} h \\ 1 \end{matrix}, \begin{matrix} h \\ 1 \end{matrix} \right) k_a ds_b, \quad (6.23)$$

where  $\sigma$  is a spacelike hypersurface on the conformally rescaled manifold. It was found that this generated the second order contribution to the Bondi flux. Glass<sup>58</sup>, working in the physical spacetime, has used the time dependent perturbations of Vishveswara to confirm the Habisohn result. Both calculations are involved and somewhat messy. A further investigation computing  $h_2$  due to the initial  $h_1$  and then using the superpotential to compute  $\tau_2$  directly would be useful.

It should be noted that the higher order Taub numbers need not represent only radiation. For finite values of  $r$  the Kerr solution, viewed as a perturbation of Minkowski or Schwarzschild, has rotational Taub numbers of all orders. Each higher order goes like  $(a/r)^n$ . Thus, when the integral is pushed out to  $\Sigma^*$  we are left with - ma.

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<sup>57</sup> Habisohn 1986.

<sup>58</sup> Glass 1993.

## 7 Conclusion

In this work we have examined the construction of the Taub numbers from the point of view of curves of solution to the Einstein equations. This was a fruitful approach in that a formulation for Taub numbers of all orders was given, as well as, making explicit a solution generation technique and derivation of the Xanthopoulos theorem.

In developing the realm of applicability of the Taub number method a discussion of linearization stability was given. Taub numbers with zero value on spacetimes with compact Cauchy surfaces without boundary are used to exclude spurious solutions of the linearized field equations which are not tangent to the solution manifold. Asymptotically simple spacetimes are linearization stable, wherein Taub numbers are free to be non-zero and take on physical meaning.

The gauge invariance of Taub numbers was also presented. Gauge invariance was demonstrated for all cases with the exception of gauge transformations on the set of fields  $\{h_1, \dots, h_{n-1}\}$  for spacetimes with noncompact Cauchy surfaces. This remains an open question in the field.

From the results presented it would appear that given a solution  $\hat{g} \in EinAF$  we can determine the mass and angular momentum by splitting the metric into two pieces,

$$\hat{g} = g + h \quad (7.1)$$

where  $g$  is a background geometry which contains the appropriate Killing vectors and  $h$  is a symmetric 2-tensor which represents the sum of the set of fields  $\{h_1, \dots, h_n, \dots\}$  which satisfy the sequential set of field equations (1.21). Then,

$$Mass(\hat{g}) = \frac{1}{8\pi} \left[ \tau_0(g) + \tau_1(h) \right] (timelike), \quad (7.2)$$

$$Angular \ Momentum(\hat{g}) = \frac{-1}{8\pi} \left[ \frac{1}{2} \tau_0(g) + \tau_1(h) \right] (rotational). \quad (7.3)$$

This construction offers an explanation of the factor of 2 anomaly between the mass and angular momentum Komar quantities in that the Komar superpotential is incomplete. The Komar superpotential generates only the Ricci tensor, not the Einstein tensor (the energy momentum tensor)<sup>59</sup>. Once the gauge difficulties associated with the Penrose potential are understood the zero-order Taub numbers need not be computed through the Komar construction. In this case the formula for mass and angular momentum will be more symmetric.

$$Mass(\hat{g}) = \frac{1}{8\pi} \left[ \tau_0(g) + \tau_1(h) \right] (timelike), \quad (7.4)$$

$$Angular \ Momentum(\hat{g}) = -\frac{1}{8\pi} \left[ \tau_0(g) + \tau_1(h) \right] (rotational). \quad (7.5)$$

The author feels that this splitting of the spacetime metric into a background geometry plus an additional set of fields does not violate the spirit of general relativity, but rather, enhances it. From the Kerr examples we see that the manner of the splitting is not important, only that the background geometry contain the appropriate Killing vectors and that the sum of the perturbation and the background metric yield the metric one is studying. To put the statement in more mathematical terms; provided the curve of solution ends at the solution under investigation and begins at a solution where mass and angular momentum have their associated symmetries (Noether) the mass and angular momentum of the spacetime in question may be computed by the Taub number method.

There have also been a number of interesting questions raised in this work for future study. It would appear that much more remains to be understood about Taub numbers.

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<sup>59</sup> See Glass and Naber 1993 for a more detailed discussion.

## 8 Appendix 1 Notation and Conventions

This appendix fixes notation, conventions and provides a convenient list of formulas used in the text. The sign conventions are those of Chandrasekhar 1983.

Connection:

$$\Gamma_{bc}^a = \frac{1}{2} g^{al} \left\{ \frac{\partial g_{lb}}{\partial x^c} + \frac{\partial g_{lc}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^l} \right\}, \quad (A1.1)$$

$$\Gamma_{bc}^a = \frac{1}{2} g^{al} \{ g_{lb,c} + g_{lc,b} - g_{bc,l} \}. \quad (A1.2)$$

Covariant derivative:

$$\nabla_a A^b = A^b{}_{;a} = A^b{}_{,a} + A^l \Gamma_{la}^b, \quad (A1.3)$$

$$\nabla_a A_b = A_{b;a} - A_l \Gamma_{ba}^l, \quad (A1.4)$$

$$A^{ab}{}_{;cd} = A^{ab}{}_{,cd} + A^{mb} \Gamma_{md}^a + A^{am} \Gamma_{md}^b - A^{ab}{}_{,m} \Gamma_{cd}^m. \quad (A1.5)$$

Riemann Tensor:

$$R^a{}_{bcd} A^b = A^a{}_{;dc} - A^a{}_{;cd}, \quad (A1.6)$$

$$R^a{}_{bcd} A_a = A_{b;cd} - A_{b;dc}, \quad (A1.7)$$

$$R^a{}_{mdc} A^{mb} + R^b{}_{mdc} A^{am} = A^{ab}{}_{;cd} - A^{ab}{}_{;dc}, \quad (A1.8)$$

$$R^m{}_{bcd} A_{am} + R^m{}_{acd} A_{mb} = A_{ab;cd} - A_{ab;dc}, \quad (A1.9)$$

$$R^a{}_{bcd} = \Gamma_{bd,c}^a - \Gamma_{bc,d}^a + \Gamma_{lc}^a \Gamma_{bd}^l - \Gamma_{ld}^a \Gamma_{bc}^l. \quad (A1.10)$$

Ricci Tensor:

$$R_{ab} = R^l{}_{alb}, \quad (A1.11)$$

$$R_{ab} = \Gamma_{ab,l}^l - \Gamma_{al,b}^l + \Gamma_{ml}^l \Gamma_{ab}^m - \Gamma_{ma}^l \Gamma_{bl}^m. \quad (A1.12)$$

In the following  $D_g$  shall denote the functional derivative of an object with respect to the metric and  $D_f$  the functional derivative with respect to the connection.  $h_0$  and  $f_0$  are frequently used to denote the background metric and its inverse.

There is a difference in notation between this work and Glass 1993. Glass uses an under number to denote the order of functional derivative and an over dot to denote the order in a Taylor expansion.

$$R_{\dot{1}ab}(h) = (D_{\dot{g}} R_{ab}) \cdot (h) \quad (A1.13)$$

$$G_{\dot{2}ab}(h, h) = (D_{\dot{g}}^2 G_{ab}) \cdot (h, h) \quad (A1.14)$$

$$\hat{g}_{ab} = g_{ab} + \lambda \dot{g}_{ab} + \frac{\lambda^2}{2} \ddot{g}_{ab} + \dots \quad (A1.15)$$

Consider a curve of solutions in the form of a power series.

$$\hat{g}_{ab}(\lambda) = g_{ab} + \lambda h_{\dot{1}ab} + \frac{\lambda^2}{2} h_{\dot{2}ab} + \frac{\lambda^3}{3!} h_{\dot{3}ab} \dots \quad (A1.16)$$

The inverse curve of solutions is denoted by,

$$\hat{g}^{ab} = g^{ab} + \lambda f_{\dot{1}}^{ab} + \frac{\lambda^2}{2} f_{\dot{2}}^{ab} + \frac{\lambda^3}{3!} f_{\dot{3}}^{ab} + \dots \quad (A1.17)$$

Where the inverse metric coefficients are given by,

$$f_{\dot{1}}^{ab} = -h_{\dot{1}}^{ab}, \quad (A1.18)$$

$$f_{\dot{2}}^{ab} = -h_{\dot{2}}^{ab} + 2h_{\dot{1}i}^a h_{\dot{1}}^{ib}, \quad (A1.19)$$

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$$f_{\dot{n}}^{ab} = -\sum_{i=1}^n \frac{n!}{i!(n-i)!} h_{\dot{i}}^a f_{\dot{n-i}}^{ib}. \quad (A1.20)$$

Lerner and Porter have shown that the Riemann tensor and its contractions are analytic in a neighborhood of Minkowski<sup>60</sup> (it is expected that the proof may be generalized to an open sub set of  $E_{in}AF$ ). Hence we can expand these operators in a Taylor series as well.

$$\hat{\Gamma}_{ab}^c = \Gamma_{ab}^c + \lambda \Gamma_1^c{}_{ab} + \frac{\lambda^2}{2} \Gamma_2^c{}_{ab} + \dots \quad (A1.21)$$

$$\hat{R}_{bcd}^a = R_{bcd}^a + \lambda R_1^a{}_{bcd} + \frac{\lambda^2}{2} R_2^a{}_{bcd} + \dots \quad (A1.22)$$

$$\hat{R}_{bd} = R_{bd} + \lambda R_1{}_{bd} + \frac{\lambda^2}{2} R_2{}_{bd} + \dots \quad (A1.23)$$

Given the above notation the following relations can be obtained.

$$\Gamma_{ab}^c = \Gamma_{ab}^l \sum_{i=0}^n \frac{n!}{i!(n-i)!} f^{cm} h_{lm} + \sum_{i=1}^n f_d^c (D_g \Gamma_{ab}^d) \cdot \binom{h}{i} \frac{n!}{i!(n-i)!} \quad (A1.24)$$

$$\Gamma_{ab}^c = (D_g \Gamma_{ab}^c) \cdot \binom{h}{n} - \sum_{i=1}^{n-1} \frac{n!}{i!(n-i)!} h_i^c \Gamma_{n-i}^d{}_{ab} \quad (A1.25)$$

$$R_{bcd}^a = (D_\Gamma R_{bcd}^a) \cdot \binom{\Gamma}{n} + 2 \sum_{i=1}^{n-1} \frac{n!}{i!(n-i)!} \Gamma_{n-i}^a{}_{i[c} \Gamma_{d]b}^i \quad (A1.26)$$

$$R_{bd} = (D_\Gamma R_{bd}) \cdot \binom{\Gamma}{n} + 2 \sum_{i=1}^{n-1} \frac{n!}{i!(n-i)!} \Gamma_{n-i}^a{}_{i[a} \Gamma_{d]b}^i \quad (A1.27)$$

$$R_{bcd}^a = (D_g R_{bcd}^a) \cdot \binom{h}{n} + 2 \sum_{i=1}^{n-1} \frac{n!}{i!(n-i)!} \left\{ \Gamma_{n-i}^a{}_{i[c} \Gamma_{d]b}^i - \left( h_i^a \Gamma_{n-i}^i{}_{b(d):c} \right) \right\} \quad (A1.28)$$

$$R_{bd} = (D_g R_{bd}) \cdot \binom{h}{n} + 2 \sum_{i=1}^{n-1} \frac{n!}{i!(n-i)!} \left\{ \Gamma_{n-i}^a{}_{i[a} \Gamma_{d]b}^i - \left( h_i^a \Gamma_{n-i}^i{}_{b(d):a} \right) \right\} \quad (A1.29)$$

$$(D_g R_{bcd}^a) \cdot (h) = \frac{1}{2} (h_{b:dc}^a + h_{d:bc}^a - h_{bd}{}^a{}_c - h_{c:bd}^a) \quad (A1.30)$$

$$(D_g R_{bd}) \cdot (h) = \frac{1}{2} (h_{b:da}^a + h_{d:ba}^a - h_{bd}{}^a{}_a - h_{a:bd}^a) \quad (A1.31)$$

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<sup>60</sup> Lerner and Porter 1974.



$$(D_{\Gamma} R_{bcd}^a) \cdot (\Gamma) = 2\Gamma_{b[d:c]}^a \quad (A1.32)$$

$$(D_{\Gamma} R_{bd}) \cdot (\Gamma) = 2\Gamma_{b[d:a]}^a \quad (A1.33)$$

$$(D_{\sigma} \Gamma_{ab}^c) \cdot (h) = \frac{1}{2}(h_{a:b}^c + h_{b:a}^c - h_{ab}^{\quad ;c}) \quad (A1.34)$$

Thus for a curve of vacuum solutions the connection and metric coefficients can be obtained by solving the following,

$$(D_{\Gamma} R_{bd}) \cdot \left(\Gamma_n\right) = 2 \sum_{i=1}^{n-1} \frac{n!}{i!(n-i)!} \Gamma_{n-i}^a \Gamma_{[d]_i^l]_a]b}^l \quad (A1.35)$$

$$(D_{\sigma} R_{bd}) \cdot \left(h_n\right) = 2 \sum_{i=1}^{n-1} \frac{n!}{i!(n-i)!} \left\{ \left( h_{i \quad n-i}^a \Gamma_{b[d]}^l \right)_{;a] + \Gamma_{n-i}^a \Gamma_{[d]_i^l]_a]b}^l \right\} \quad (A1.36)$$

The following recursion relations can also be obtained.

$$(D_{\sigma}^n \Gamma_{ab}^c) \cdot (h, \dots, h) = -nh_i^c (D_{\sigma}^{n-1} \Gamma_{ab}^l) \cdot (h, \dots, h) \quad n \geq 2 \quad (A1.37)$$

$$\begin{aligned} (D_{\sigma}^n R_{imd}^a) \cdot (h, \dots, h) &= -nh_b^a (D_{\sigma}^{n-1} R_{imd}^b) \cdot (h, \dots, h) \quad n \geq 2 \\ &+ n(D_{\sigma}^{n-1} \Gamma_{id}^k) \cdot (h, \dots, h) \{ (D_{\sigma} \Gamma_{km}^a) \cdot (h) - h_{k:m}^a \} \\ &- n(D_{\sigma}^{n-1} \Gamma_{im}^k) \cdot (h, \dots, h) \{ (D_{\sigma} \Gamma_{kd}^a) \cdot (h) - h_{k:d}^a \}, \quad (A1.38) \end{aligned}$$

$$\begin{aligned} (D_{\sigma}^n R_{id}) \cdot (h, \dots, h) &= -nh_b^a (D_{\sigma}^{n-1} R_{ia d}^b) \cdot (h, \dots, h) \quad n \geq 2 \\ &+ n(D_{\sigma}^{n-1} \Gamma_{id}^k) \cdot (h, \dots, h) \{ (D_{\sigma} \Gamma_{ka}^a) \cdot (h) - h_{k:a}^a \} \\ &- n(D_{\sigma}^{n-1} \Gamma_{ia}^k) \cdot (h, \dots, h) \{ (D_{\sigma} \Gamma_{kd}^a) \cdot (h) - h_{k:d}^a \}. \quad (A1.39) \end{aligned}$$

## 9 Appendix 2 Newman-Penrose Formalism

The Newman-Penrose (NP) formalism is employed throughout this work for computations. This appendix is intended to provide a convenient reference of formulae and conventions. For a thorough discussion of its origins and use see Newman and Penrose 1962, Pirani 1965 and Chandrasekhar 1983. This appendix follows Chandrasekhar 1983.

The NP formalism is a tetrad formalism with the basis vectors chosen to be null, two of which are real and two of which are complex. The null tetrad shall be denoted as follows:

$$e_{(a)}^b = (l^b, n^b, m^b, \bar{m}^b). \quad (A2.1)$$

The tetrad vectors obey the following relationships.

$$l^a l_a = l^a m_a = l^a \bar{m}_a = 0 \quad (A2.2)$$

$$n^a n_a = n^a m_a = n^a \bar{m}_a = 0 \quad (A2.3)$$

$$m^a m_a = \bar{m}^a \bar{m}_a = 0 \quad (A2.4)$$

$$l^a n_a = -m^a \bar{m}_a = 1 \quad (A2.5)$$

The tensor index of the tetrad is raised and lowered with the metric tensor.

$$e_{(a)b} = g_{bc} e_{(a)}^c \quad (A2.6)$$

$$e_{(a)}^b = g^{bc} e_{(a)c} \quad (A2.7)$$

We may construct a constant symmetric matrix by taking the trace of the square of the tetrad.

$$e_{(a)}^c e_{(b)c} = \eta_{(a)(b)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (A2.8)$$

This matrix can be used to lower tetrad indices, while its inverse  $\eta^{(a)(b)}$  can be used to raise tetrad indices.

$$e_c^{(a)} = \eta^{(a)(b)} e_{(b)c} \quad (A2.9)$$

$$e_{(a)c} = \eta_{(a)(b)} e_c^{(b)} \quad (A2.10)$$

$$\eta^{(a)(c)} \eta_{(b)(c)} = \delta_{(b)}^{(a)} \quad (A2.11)$$

The metric tensor can be constructed from the tetrad by taking the trace of the square of the tetrad over the tetrad indices.

$$g_{ab} = e_{(c)a} e_b^{(c)} = l_a n_b + l_b n_a - m_a \bar{m}_b - \bar{m}_a m_b \quad (A2.12)$$

We can now write any tensor object in terms of its tetrad components and conversely.

$$A_{(a_1)(a_2)\dots(a_n)} = e_{(a_1)b_1} e_{(a_2)b_2} \dots e_{(a_n)b_n} A^{b_1 b_2 \dots b_n} \quad (A2.13)$$

$$A_{b_1 b_2 \dots b_n} = e_{(a_1)b_1} e_{(a_2)b_2} \dots e_{(a_n)b_n} A^{(a_1)(a_2)\dots(a_n)} \quad (A2.14)$$

The Ricci rotation coefficients can now be defined.

$$\Upsilon_{(a)(b)(c)} = e_{(a)}^k e_{(b)k;i} e_c^i \quad (A2.15)$$

$$\Upsilon_{(a)(b)(c)} = -\Upsilon_{(b)(a)(c)} \quad (A2.16)$$

The spin coefficients are then given by the following relationships<sup>61 62 63</sup>.

$$\kappa = \Upsilon_{311} = l_{i;j} m^i l^j = -m_{i;j} l^i l^j = 2l_{[a,b]} m^a l^b \quad (A2.17)$$

$$\begin{aligned} \epsilon &= \frac{1}{2}(\Upsilon_{211} + \Upsilon_{341}) = \frac{1}{2}(l_{i;j} n^i l^j + \bar{m}_{i;j} m^i l^j) \\ &= l_{[a,b]} n^a l^b + \frac{1}{2}(l_{[a,b]} m^a \bar{m}^b + \bar{m}_{[a,b]} m^a l^b + m_{[a,b]} l^a \bar{m}^b) \end{aligned} \quad (A2.18)$$

$$\begin{aligned} \pi &= \Upsilon_{241} = \bar{m}_{i;j} n^i l^j = -n_{i;j} \bar{m}^i l^j \\ &= l_{[a,b]} n^a \bar{m}^b + \bar{m}_{[a,b]} n^a l^b + n_{[a,b]} l^a \bar{m}^b \end{aligned} \quad (A2.19)$$

$$\sigma = \Upsilon_{313} = l_{i;j} m^i m^j = -m_{i;j} l^i m^j = 2m_{[a,b]} m^a l^b \quad (A2.20)$$

<sup>61</sup> W.J. Cocke 1989.

<sup>62</sup> Chandrasekhar in A. Schild lectures 1982.

<sup>63</sup> Page 42, Chandrasekar 1983.

$$\begin{aligned}\beta &= \frac{1}{2}(\gamma_{213} + \gamma_{343}) = \frac{1}{2}(l_{i,j}n^i m^j + \bar{m}_{i,j}m^i m^j) \\ &= m_{[a,b]}m^a \bar{m}^b + \frac{1}{2}(m_{[a,b]}n^a l^b + n_{[a,b]}m^a l^b + l_{[a,b]}n^a m^b) \quad (A2.21)\end{aligned}$$

$$\begin{aligned}\mu &= \gamma_{243} = \bar{m}_{i,j}n^i m^j = -n_{i,j}\bar{m}^i m^j \\ &= n_{[a,b]}m^a \bar{m}^b + m_{[a,b]}n^a \bar{m}^b + \bar{m}_{[a,b]}n^a \bar{m}^b \quad (A2.22)\end{aligned}$$

$$\begin{aligned}\rho &= \gamma_{314} = l_{i,j}m^i \bar{m}^j = -m_{i,j}l^i \bar{m}^j \\ &= \bar{m}_{[a,b]}m^a l^b + m_{[a,b]}\bar{m}^a l^b + l_{[a,b]}m^a \bar{m}^b \quad (A2.23)\end{aligned}$$

$$\begin{aligned}\alpha &= \frac{1}{2}(\gamma_{214} + \gamma_{344}) = \frac{1}{2}(l_{i,j}n^i \bar{m}^j + \bar{m}_{i,j}m^i \bar{m}^j) \\ &= \bar{m}_{[a,b]}m^a \bar{m}^b + \frac{1}{2}(\bar{m}_{[a,b]}n^a l^b + n_{[a,b]}\bar{m}^a l^b + l_{[a,b]}n^a \bar{m}^b) \quad (A2.24)\end{aligned}$$

$$\lambda = \gamma_{244} = \bar{m}_{i,j}n^i \bar{m}^j = -n_{i,j}\bar{m}^i \bar{m}^j = 2\bar{m}_{[a,b]}n^a \bar{m}^b \quad (A2.25)$$

$$\begin{aligned}\tau &= \gamma_{312} = l_{i,j}m^i n^j = -m_{i,j}l^i n^j \\ &= n_{[a,b]}m^a l^b + l_{[a,b]}m^a n^b + m_{[a,b]}n^a l^b \quad (A2.26)\end{aligned}$$

$$\begin{aligned}\gamma &= \frac{1}{2}(\gamma_{212} + \gamma_{342}) = \frac{1}{2}(l_{i,j}n^i n^j + \bar{m}_{i,j}m^i n^j) \\ &= n_{[a,b]}n^a l^b + \frac{1}{2}(n_{[a,b]}m^a \bar{m}^b + \bar{m}_{[a,b]}m^a n^b + m_{[a,b]}n^a \bar{m}^b) \quad (A2.27)\end{aligned}$$

$$\nu = \gamma_{242} = \bar{m}_{i,j}n^i n^j = -n_{i,j}\bar{m}^i n^j = 2n_{[a,b]}n^a \bar{m}^b \quad (A2.28)$$

Many of the above spin coefficients carry a specific geometrical significance. For example<sup>64</sup>: if  $l^a$  is geodesic, then  $\kappa = 0$ ; if  $l^a$  is expressible in terms of an affine parameter, then  $\epsilon + \bar{\epsilon} = 0$ ; if  $l^a$  is hypersurface orthogonal (proportional to a gradient), then  $\rho = \bar{\rho}$ ; if  $l^a$  is equal to a gradient, then  $\tau = \bar{\alpha} + \beta$ ; if  $n^a$  is parallel propagated along  $l^a$ , then  $\pi = 0$ ; if  $m^a$  and  $\bar{m}^a$  are parallel propagated along

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<sup>64</sup> Couch and Torrence 1968.

$l^a$ , then  $\epsilon - \bar{\epsilon} = 0$ ; if  $m^a$  and  $\bar{m}^a$  are surface forming, then  $\mu = \bar{\mu}$ ; if  $l^a$  is parallel propagated along  $n^a$ , then  $\tau = 0$ ; if  $n^a$  is geodesic, then  $\nu = 0$ ; if  $n^a$  is affine, then  $\gamma + \bar{\gamma} = 0$ ; if  $m^a$  and  $\bar{m}^a$  are parallel propagated along  $n^a$ , then  $\gamma - \bar{\gamma} = 0$ .

Using the tetrad, four intrinsic derivatives can be defined.

$$\begin{aligned} D &= l^a \nabla_a & \Delta &= n^a \nabla_a \\ \delta &= m^a \nabla_a & \bar{\delta} &= \bar{m}^a \nabla_a \end{aligned} \quad (A2.29)$$

The scalar components of the Weyl and Ricci tensors and Ricci scalar are given below.  $C_{abcd}$  is the Weyl tensor.

$$\Psi_0 = -C_{pqrs} l^p m^q l^r m^s \quad (A2.30)$$

$$\Psi_1 = -C_{pqrs} l^p n^q l^r m^s \quad (A2.31)$$

$$\Psi_2 = -C_{pqrs} l^p m^q \bar{m}^r n^s \quad (A2.32)$$

$$\Psi_3 = -C_{pqrs} l^p n^q \bar{m}^r n^s \quad (A2.33)$$

$$\Psi_4 = -C_{pqrs} l^p \bar{m}^q n^r \bar{m}^s \quad (A2.34)$$

$$\Phi_{00} = -\frac{1}{2} R_{pq} l^p l^q \quad (A2.35)$$

$$\Phi_{01} = -\frac{1}{2} R_{pq} l^p m^q \quad (A2.36)$$

$$\Phi_{11} = -\frac{1}{4} R_{pq} (l^p n^q + m^p \bar{m}^q) \quad (A2.37)$$

$$\Phi_{12} = -\frac{1}{2} R_{pq} n^p m^q \quad (A2.38)$$

$$\Phi_{22} = -\frac{1}{2} R_{pq} n^p n^q \quad (A2.39)$$

$$\Phi_{02} = -\frac{1}{2} R_{pq} m^p m^q \quad (A2.40)$$

$$\Lambda = \frac{1}{24} R \quad (A2.41)$$

Given these definitions we have the following sets of equations<sup>65</sup>.

Commutation relations:

$$\Delta D - D\Delta = (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\bar{\tau} + \pi)\delta - (\tau + \bar{\pi})\bar{\delta}, \quad (A2.42)$$

$$\delta D - D\delta = (\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta - \sigma\bar{\delta}, \quad (A2.43)$$

$$\delta\Delta - \Delta\delta = -\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + (\mu - \gamma + \bar{\gamma})\delta + \bar{\lambda}\bar{\delta}, \quad (A2.44)$$

$$\bar{\delta}\delta - \delta\bar{\delta} = (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta + (\alpha - \bar{\beta})\delta + (\beta - \bar{\alpha})\bar{\delta}. \quad (A2.45)$$

Ricci identities:

$$D\rho - \bar{\delta}\kappa = (\rho^2 + \sigma\bar{\sigma}) + \rho(\epsilon + \bar{\epsilon}) - \bar{\kappa}\tau - \kappa(3\alpha + \bar{\beta} - \pi) + \Phi_{00}, \quad (A2.46)$$

$$D\sigma - \delta\kappa = \sigma(\rho + \bar{\rho} + 3\epsilon - \bar{\epsilon}) - \kappa(\tau - \bar{\pi} + \bar{\alpha} + 3\beta) + \Psi_0, \quad (A2.47)$$

$$\begin{aligned} D\tau - \Delta\kappa &= \rho(\tau + \bar{\pi}) + \sigma(\bar{\tau} + \pi) \\ &+ \tau(\epsilon - \bar{\epsilon}) - \kappa(3\gamma + \bar{\gamma}) + \Psi_1 + \Phi_{01}, \end{aligned} \quad (A2.48)$$

$$D\alpha - \bar{\delta}\epsilon = \alpha(\rho + \bar{\epsilon} - 2\epsilon) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + \pi(\epsilon + \rho) + \Phi_{01}, \quad (A2.49)$$

$$D\beta - \delta\epsilon = \sigma(\alpha + \pi) + \beta(\bar{\rho} - \bar{\epsilon}) - \kappa(\mu + \gamma) - \epsilon(\bar{\alpha} - \bar{\pi}) + \Psi_1, \quad (A2.50)$$

$$\begin{aligned} D\gamma - \Delta\epsilon &= \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) - \gamma(\epsilon + \bar{\epsilon}) \\ &- \epsilon(\gamma + \bar{\gamma}) + \tau\pi - \nu\kappa + \Psi_2 + \Phi_{11} - \Lambda, \end{aligned} \quad (A2.51)$$

$$D\lambda - \bar{\delta}\pi = (\rho\lambda + \bar{\sigma}\mu) + \pi(\pi + \alpha - \beta) - \nu\bar{\kappa} - \lambda(3\epsilon - \bar{\epsilon}) + \Phi_{20}, \quad (A2.52)$$

$$D\mu - \delta\pi = (\bar{\rho}\mu + \sigma\lambda) + \pi(\bar{\pi} - \bar{\alpha} + \beta) - \mu(\epsilon + \bar{\epsilon}) - \nu\kappa + \Psi_2 + 2\Lambda, \quad (A2.53)$$

$$\begin{aligned} D\nu - \Delta\pi &= \mu(\pi + \bar{\tau}) + \lambda(\bar{\pi} + \tau) + \pi(\gamma - \bar{\gamma}) \\ &- \nu(3\epsilon + \bar{\epsilon}) + \Psi_3 + \Phi_{21}, \end{aligned} \quad (A2.54)$$

$$\Delta\lambda - \bar{\delta}\nu = -\lambda(\mu + \bar{\mu} + 3\gamma - \bar{\gamma}) + \nu(3\alpha + \bar{\beta} + \pi - \bar{\tau}) - \Psi_4, \quad (A2.55)$$

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}) - \Psi_1 + \Phi_{01}, \quad (A2.56)$$

$$\begin{aligned} \delta\alpha - \bar{\delta}\beta &= (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta \\ &+ \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2 + \Phi_{11} + \Lambda, \end{aligned} \quad (A2.57)$$

$$\delta\lambda - \bar{\delta}\mu = \nu(\rho - \bar{\rho}) + \pi(\mu - \bar{\mu}) + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) - \Psi_3 + \Phi_{21}, \quad (A2.58)$$

$$\delta\nu - \Delta\mu = (\mu^2 + \lambda\bar{\lambda}) + \mu(\gamma + \bar{\gamma}) - \bar{\nu}\pi + \nu(\tau - 3\beta - \bar{\alpha}) + \Phi_{22}, \quad (A2.59)$$

$$\delta\gamma - \Delta\beta = \gamma(\tau - \bar{\alpha} - \beta) + \mu\tau - \sigma\nu - \epsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda} + \Phi_{12}, \quad (A2.60)$$

$$\delta\tau - \Delta\sigma = (\mu\sigma + \bar{\lambda}\rho) + \tau(\tau + \beta - \bar{\alpha}) - \sigma(3\gamma - \bar{\gamma}) - \kappa\bar{\nu} + \Phi_{02}, \quad (A2.61)$$

$$\Delta\rho - \bar{\delta}\tau = -(\rho\bar{\mu} + \sigma\lambda) + \tau(\bar{\beta} - \alpha - \bar{\tau}) + \rho(\gamma + \bar{\gamma}) + \nu\kappa - \Psi_2 - 2\Lambda, \quad (A2.62)$$

$$\Delta\alpha - \bar{\delta}\gamma = \nu(\rho + \epsilon) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) - \Psi_3, \quad (A2.63)$$

**Eliminant relations:**

$$\begin{aligned} D(\rho - \bar{\rho}) + \delta\bar{\kappa} - \bar{\delta}\kappa &= (\rho - \bar{\rho})(\rho + \bar{\rho} + \epsilon + \bar{\epsilon}) \\ &+ \kappa(\bar{\tau} + \pi - 3\alpha - \bar{\beta}) - \bar{\kappa}(\tau + \bar{\pi} - 3\bar{\alpha}), \end{aligned} \quad (A2.64)$$

$$\begin{aligned} D(\mu - \bar{\mu}) + \delta(\alpha + \bar{\beta} - \pi) - \bar{\delta}(\bar{\alpha} + \beta - \bar{\pi}) \\ &= (\gamma + \bar{\gamma})(\rho - \bar{\rho}) + \alpha(\bar{\pi} - 2\beta) - \bar{\alpha}(\pi - 2\bar{\beta}) \\ &+ \bar{\kappa}\bar{\nu} - \kappa\nu + \beta\pi - \bar{\beta}\bar{\pi} + (\rho + \bar{\rho})(\mu - \bar{\mu}), \end{aligned} \quad (A2.65)$$

$$\begin{aligned} D(\mu - \bar{\mu} - \gamma + \bar{\gamma}) + \Delta(\epsilon - \bar{\epsilon}) - \delta\pi + \bar{\delta}\bar{\pi} &= (\epsilon + \bar{\epsilon})(\bar{\mu} - \mu) \\ \bar{\tau}(\bar{\alpha} + \bar{\pi} - \beta) - \tau(\alpha + \pi - \bar{\beta}) + \lambda\sigma - \bar{\lambda}\bar{\sigma} + \bar{\rho}\mu - \rho\bar{\mu} + 2(\epsilon\gamma - \bar{\epsilon}\bar{\gamma}), \end{aligned} \quad (A2.66)$$

$$\begin{aligned} \Delta(\bar{\mu} - \mu) + \delta\nu - \bar{\delta}\bar{\nu} &= (\mu - \bar{\mu})(\mu + \bar{\mu} + \gamma + \bar{\gamma}) + \nu(\tau - 3\beta - \bar{\alpha} + \bar{\pi}) \\ &- \bar{\nu}(\bar{\tau} + \pi - 3\bar{\beta} - \alpha), \end{aligned} \quad (A2.67)$$

$$\begin{aligned} D(\tau - \bar{\alpha} - \beta) - \Delta\kappa + \delta(\epsilon + \bar{\epsilon}) &= \rho(\bar{\tau} + \pi) + \bar{\kappa}\bar{\lambda} + \sigma(\bar{\tau} - \alpha - \bar{\beta}) \\ \epsilon(\tau - \bar{\pi}) - \bar{\rho}(\beta + \bar{\alpha} + \bar{\pi}) + \bar{\epsilon}(2\bar{\alpha} + 2\beta - \tau - \bar{\pi}) + \kappa(\mu - 2\gamma), \end{aligned} \quad (A2.68)$$

$$\begin{aligned} \delta(\rho - \epsilon + \bar{\epsilon}) - \bar{\delta}\sigma + D(\beta - \bar{\alpha}) &= \rho(\bar{\alpha} + \beta + \tau) - \bar{\rho}(\tau - \beta + \bar{\alpha} + \bar{\pi}) \\ &+ (\bar{\epsilon} - \epsilon)(2\bar{\alpha} - \bar{\pi}) + \sigma(\pi - 2\alpha) + \kappa(\bar{\gamma} - \gamma - \bar{\mu}) + \bar{\kappa}\bar{\lambda}, \end{aligned} \quad (A2.69)$$

$$D\lambda + \Delta\bar{\sigma} - \bar{\delta}(\bar{\tau} + \pi) = \bar{\sigma}(3\bar{\gamma} - \gamma + \mu - \bar{\mu}) + (\pi + \bar{\tau})(\pi - \bar{\tau} + \alpha) \\ + \lambda(\rho - \bar{\rho} - 3\epsilon + \bar{\epsilon}) - \beta\pi - \bar{\tau}\bar{\beta}, \quad (A2.70)$$

$$D\nu + \Delta(\alpha + \bar{\beta} - \pi) - \bar{\delta}(\gamma + \bar{\gamma}) = \nu(\rho - 2\epsilon) + \lambda(\bar{\pi} - \bar{\alpha} - \beta) + \mu(\pi + \bar{\tau}) \\ - \bar{\mu}(\alpha + \bar{\beta} + \bar{\tau}) + \gamma(\pi - \bar{\tau}) + \bar{\gamma}(2\alpha + 2\bar{\beta} - \pi - \bar{\tau}) + \bar{\sigma}\bar{\nu}, \quad (A2.71)$$

$$\Delta(\bar{\beta} - \alpha) + \delta\lambda + \bar{\delta}(\gamma - \bar{\gamma} - \mu) = \nu(\bar{\epsilon} - \epsilon - \bar{\rho}) + \lambda(\tau - 2\beta) + \alpha(\mu + \bar{\mu}) \\ - \bar{\mu}(\pi + \bar{\tau} + \bar{\beta}) + \mu(\pi + \bar{\beta}) + (\gamma - \bar{\gamma})(\bar{\tau} - 2\bar{\beta}) + \bar{\sigma}\bar{\nu}, \quad (A2.72)$$

$$D\mu + \Delta\rho - \delta\pi - \bar{\delta}\tau = \bar{\rho}\mu - \rho\bar{\mu} + \pi(\bar{\pi} - \bar{\alpha} + \beta) + \tau(\bar{\beta} - \alpha - \bar{\tau}) \\ \rho(\gamma + \bar{\gamma}) - \mu(\epsilon - \bar{\epsilon}). \quad (A2.73)$$

**Bianchi identities:**

$$-\bar{\delta}\Psi_0 + D\Psi_1 + (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2 - D\Phi_{01} + \delta\Phi_{00} \\ + 2(\epsilon + \bar{\rho})\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02} + (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} = 0, \quad (A2.74)$$

$$\bar{\delta}\Psi_1 - D\Psi_2 - \lambda\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3 + \bar{\delta}\Phi_{01} - \Delta\Phi_{00} - 2(\alpha + \bar{\tau})\Phi_{01} \\ + 2\rho\Phi_{11} + \bar{\sigma}\Phi_{02} - (\bar{\mu} - 2\gamma - 2\bar{\gamma})\Phi_{00} - 2\tau\Phi_{10} - 2D\Lambda = 0, \quad (A2.75)$$

$$-\bar{\delta}\Psi_2 + D\Psi_3 + 2\lambda\Psi_1 - 3\pi\Psi_2 + 2(\epsilon - \rho)\Psi_3 + \kappa\Psi_4 - D\Phi_{21} + \delta\Phi_{20} + 2(\bar{\rho} - \epsilon)\Phi_{21} \\ - 2\mu\Phi_{10} + 2\pi\Phi_{11} - \bar{\kappa}\Phi_{22} - (2\bar{\alpha} - 2\beta - \bar{\pi})\Phi_{20} - 2\bar{\delta}\Lambda = 0, \quad (A2.76)$$

$$\bar{\delta}\Psi_3 - D\Psi_4 - 3\lambda\Psi_2 + 2(2\pi + \alpha)\Psi_3 - (4\epsilon - \rho)\Psi_4 - \Delta\Phi_{20} + \bar{\delta}\Phi_{21} + 2(\alpha - \bar{\tau})\Phi_{21} \\ + 2\nu\Phi_{10} + \bar{\tau}\Phi_{22} - 2\lambda\Phi_{11} - (\bar{\mu} + 2\gamma - 2\bar{\gamma})\Phi_{02} = 0, \quad (A2.77)$$

$$-\Delta\Psi_0 + \delta\Psi_1 + (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 - D\Phi_{02} + \delta\Phi_{01} + 2(\bar{\pi} - \beta)\Phi_{01} \\ - 2\kappa\Phi_{12} - \bar{\lambda}\Phi_{00} + 2\sigma\Phi_{11} + (\bar{\rho} + 2\epsilon - 2\bar{\epsilon})\Phi_{02} = 0, \quad (A2.78)$$

$$-\Delta\Psi_1 + \delta\Psi_2 + \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 + \Delta\Phi_{01} - \bar{\delta}\Phi_{02} + 2(\bar{\mu} - \gamma)\Phi_{01} \\ - 2\rho\Phi_{12} - \bar{\nu}\Phi_{00} + 2\tau\Phi_{11} + (\bar{\tau} - 2\beta + 2\alpha)\Phi_{02} + 2\delta\Lambda = 0, \quad (A2.79)$$

$$-\Delta\Psi_2 + \delta\Psi_3 + 2\nu\Psi_1 - 3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 + \sigma\Psi_4 - D\Phi_{22} + \delta\Phi_{21} + 2(\bar{\pi} + \beta)\Phi_{21} \\ - 2\mu\Phi_{11} - \bar{\lambda}\Phi_{20} + 2\pi\Phi_{12} + (\bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22} - 2\Delta\Lambda = 0, \quad (A2.80)$$



$$\begin{aligned}
& -\Delta\Psi_3 + \delta\Psi_4 + 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 - (\tau - 4\beta)\Psi_4 + \Delta\Phi_{21} - \bar{\delta}\Phi_{22} \\
& + 2(\bar{\mu} + \gamma)\Phi_{21} - 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} + 2\lambda\Phi_{12} + (\bar{\tau} - 2\alpha - 2\bar{\beta})\Phi_{22} = 0. \quad (A2.81)
\end{aligned}$$

Contracted Bianchi identities:

$$\begin{aligned}
& \bar{\delta}\Phi_{01} + \delta\Phi_{10} - D(\Phi_{11} + 3\Lambda) - \Delta\Phi_{00} = \bar{\kappa}\Phi_{12} + \kappa\Phi_{21} \\
& + (2\alpha + 2\bar{\tau} - \pi)\Phi_{01} + (2\bar{\alpha} + 2\tau - \bar{\pi})\Phi_{10} - 2(\rho + \bar{\rho})\Phi_{11} \\
& - \bar{\sigma}\Phi_{02} - \sigma\Phi_{20} + \{\mu + \bar{\mu} - 2(\gamma + \bar{\gamma})\}\Phi_{00}, \quad (A2.82)
\end{aligned}$$

$$\begin{aligned}
& \bar{\delta}\Phi_{12} + \delta\Phi_{21} - \Delta(\Phi_{11} + 3\Lambda) - D\Phi_{22} = -\nu\Phi_{01} - \bar{\nu}\Phi_{10} \\
& + (\bar{\tau} - 2\bar{\beta} - 2\pi)\Phi_{12}(\tau - 2\beta - 2\bar{\pi})\Phi_{21} + 2(\mu + \bar{\mu})\Phi_{11} \\
& - (\rho + \bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22} + \lambda\Phi_{02} + \bar{\lambda}\Phi_{20}, \quad (A2.83)
\end{aligned}$$

$$\begin{aligned}
& \delta(\Phi_{11} - 3\lambda) - D\Phi_{12} - \Delta\Phi_{01} + \bar{\delta}\Phi_{02} = \kappa\Phi_{22} - \bar{\nu}\Phi_{00} \\
& + (\bar{\tau} - \pi + 2\alpha - 2\bar{\beta})\Phi_{02} - \sigma\Phi_{21} + \bar{\lambda}\Phi_{10} + 2(\tau - \bar{\pi})\Phi_{11} \\
& - (2\rho + \bar{\rho} - 2\bar{\epsilon})\Phi_{12} + (2\bar{\mu} + \mu - 2\gamma)\Phi_{01}. \quad (A2.84)
\end{aligned}$$

The derivatives of the tetrad legs are given below.

$$\begin{aligned}
l_{a;b} = & (\gamma + \bar{\gamma})l_a l_b - (\alpha + \bar{\beta})l_a m_b - (\bar{\alpha} + \beta)l_a \bar{m}_b - \tau m_a l_b - \bar{\tau} \bar{m}_a l_b + \sigma m_a m_b + \bar{\sigma} \bar{m}_a \bar{m}_b \\
& \rho \bar{m}_a m_b + \bar{\rho} m_a \bar{m}_b - \kappa \bar{m}_a n_b - \bar{\kappa} m_a n_b + (\epsilon + \bar{\epsilon})l_a n_b \quad (A2.85)
\end{aligned}$$

$$\begin{aligned}
n_{a;b} = & -(\epsilon + \bar{\epsilon})n_a n_b - (\gamma + \bar{\gamma})n_a l_b + (\bar{\alpha} + \beta)n_a \bar{m}_b \\
& + (\alpha + \bar{\beta})n_a m_b + \pi m_a n_b + \bar{\pi} \bar{m}_a n_b + \nu m_a l_b + \bar{\nu} \bar{m}_a l_b \\
& - \mu m_a \bar{m}_b - \bar{\mu} \bar{m}_a m_b - \lambda m_a m_b - \bar{\lambda} \bar{m}_a \bar{m}_b \quad (A2.86)
\end{aligned}$$

$$\begin{aligned}
m_{a;b} = & \bar{\pi}l_a n_b + \bar{\nu}l_a l_b - \bar{\lambda}l_a \bar{m}_b - \bar{\mu}l_a m_b - \kappa n_a n_b \\
& - \tau n_a l_b + \sigma n_a \bar{m}_b + \rho n_a m_b + (\epsilon - \bar{\epsilon})m_a n_b \\
& + (\gamma - \bar{\gamma})m_a l_b + (\bar{\alpha} - \beta)m_a \bar{m}_b - (\alpha - \bar{\beta})m_a m_b \quad (A2.87)
\end{aligned}$$

$$l_{;a}^a = -\rho - \bar{\rho} + \epsilon + \bar{\epsilon} \quad (A2.88)$$

$$n_{;a}^a = -\gamma - \bar{\gamma} + \mu + \bar{\mu} \quad (A2.89)$$

$$m_{;\alpha}^{\alpha} = \bar{\pi} - \tau - \bar{\alpha} + \beta \quad (12.90)$$

$$Dl_{\alpha} = -\kappa \bar{m}_{\alpha} - \bar{\kappa} m_{\alpha} + (\epsilon + \bar{\epsilon}) l_{\alpha} \quad (12.91)$$

$$Dn_{\alpha} = -(\epsilon + \bar{\epsilon}) n_{\alpha} + \pi m_{\alpha} + \bar{\pi} \bar{m}_{\alpha} \quad (12.92)$$

$$Dm_{\alpha} = \bar{\pi} l_{\alpha} - \kappa n_{\alpha} + (\epsilon - \bar{\epsilon}) m_{\alpha} \quad (12.93)$$

$$\Delta l_{\alpha} = (\gamma + \bar{\gamma}) l_{\alpha} - \bar{\tau} m_{\alpha} - \tau \bar{m}_{\alpha} \quad (12.94)$$

$$\Delta n_{\alpha} = -(\gamma + \bar{\gamma}) n_{\alpha} + \nu m_{\alpha} + \bar{\nu} \bar{m}_{\alpha} \quad (12.95)$$

$$\Delta m_{\alpha} = \bar{\nu} l_{\alpha} - \tau n_{\alpha} + (\gamma - \bar{\gamma}) m_{\alpha} \quad (12.96)$$

$$\delta l_{\alpha} = (\bar{\alpha} + \beta) l_{\alpha} - \sigma \bar{m}_{\alpha} - \bar{\rho} m_{\alpha} \quad (12.97)$$

$$\delta n_{\alpha} = -(\bar{\alpha} + \beta) n_{\alpha} + \mu m_{\alpha} + \bar{\lambda} \bar{m}_{\alpha} \quad (12.98)$$

$$\delta m_{\alpha} = \bar{\lambda} l_{\alpha} - \sigma n_{\alpha} - (\bar{\alpha} - \beta) m_{\alpha} \quad (12.99)$$

$$\bar{\delta} m_{\alpha} = \bar{\mu} l_{\alpha} - \rho n_{\alpha} + (\alpha - \bar{\beta}) m_{\alpha} \quad (12.100)$$

The last three sets of equations make the earlier geometrical interpretations of the spin coefficients manifest.

Once the tetrad is chosen there still remains six degrees of freedom by which we may rotate the tetrad. These degrees of freedom are partitioned into three groups. Rotations about  $l$ , rotations about  $n$  and rotations which leave the directions of  $n$  and  $l$  fixed but rotate  $m$  and  $\bar{m}$ <sup>66 67</sup>.

Case I:

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<sup>66</sup> Janis and Newman 1965.

<sup>67</sup> Chandrasekhar 1983.

$$l \rightarrow l, \quad (A2.101)$$

$$n \rightarrow n + \bar{a}m + a\bar{m} + a\bar{a}l, \quad (A2.102)$$

$$m \rightarrow m + al, \quad (A2.103)$$

$$\Psi_0 \rightarrow \Psi_0, \quad (A2.104)$$

$$\Psi_1 \rightarrow \Psi_1 + \bar{a}\Psi_0, \quad (A2.105)$$

$$\Psi_2 \rightarrow \Psi_2 + 2\bar{a}\Psi_1 + \bar{a}^2\Psi_0, \quad (A2.106)$$

$$\Psi_3 \rightarrow \Psi_3 + 3\bar{a}\Psi_2 + 3\bar{a}^2\Psi_1 + \bar{a}^3\Psi_0, \quad (A2.107)$$

$$\Psi_4 \rightarrow \Psi_4 + 4\bar{a}\Psi_3 + 6\bar{a}^2\Psi_2 + 4\bar{a}^3\Psi_1 + \bar{a}^4\Psi_0, \quad (A2.108)$$

$$\kappa \rightarrow \kappa, \quad (A2.109)$$

$$\sigma \rightarrow \sigma + a\kappa, \quad (A2.110)$$

$$\rho \rightarrow \rho + \bar{a}\kappa, \quad (A2.111)$$

$$\epsilon \rightarrow \epsilon + \bar{a}\kappa, \quad (A2.112)$$

$$\tau \rightarrow \tau + a\rho + \bar{a}\sigma + a\bar{a}\kappa, \quad (A2.113)$$

$$\pi \rightarrow \pi + 2\bar{a}\epsilon + \bar{a}^2\kappa + D(\bar{a}), \quad (A2.114)$$

$$\alpha \rightarrow \alpha + \bar{a}(\rho + \epsilon) + \bar{a}^2\kappa, \quad (A2.115)$$

$$\beta \rightarrow \beta + a\epsilon + \bar{a}\sigma + a\bar{a}\kappa, \quad (A2.116)$$

$$\gamma \rightarrow \gamma + a\alpha + \bar{a}(\beta + \tau) + a\bar{a}(\rho + \epsilon) + \bar{a}^2\sigma + a\bar{a}^2\kappa, \quad (A2.117)$$

$$\lambda \rightarrow \lambda + \bar{a}(2\alpha + \pi) + \bar{a}^2(\rho + 2\epsilon) + \bar{a}^3\kappa + \bar{\delta}(\bar{a}) + \bar{a}D(\bar{a}), \quad (A2.118)$$

$$\mu \rightarrow \mu + a\pi + 2\bar{a}\beta + 2a\bar{a}\epsilon + \bar{a}^2\sigma + a\bar{a}^2\kappa + \delta(\bar{a}) + aD(\bar{a}), \quad (A2.119)$$

$$\begin{aligned} \nu \rightarrow \nu + a\lambda + \bar{a}(\mu + 2\gamma) + \bar{a}^2(\tau + 2\beta) + \bar{a}^3\sigma + a\bar{a}(\pi + 2\alpha) + a\bar{a}^2(\rho + 2\epsilon) \\ + a\bar{a}^3\kappa + (\Delta + \bar{a}\delta + a\bar{\delta} + a\bar{a}D)\bar{a}. \end{aligned} \quad (A2.120)$$

Case II:

$$l \rightarrow l + \bar{a}m + a\bar{m} + a\bar{a}n, \quad (A2.121)$$

$$n \rightarrow n, \quad (A2.122)$$

$$m \rightarrow m + an, \quad (A2.123)$$

$$\Psi_0 \rightarrow \Psi_0 + 4a\Psi_1 + 6a^2\Psi_2 + 4a^3\Psi_3 + a^4\Psi_4, \quad (A2.124)$$

$$\Psi_1 \rightarrow \Psi_1 + 3a\Psi_2 + 3a^2\Psi_3 + a^3\Psi_4, \quad (A2.125)$$

$$\Psi_2 \rightarrow \Psi_2 + 2a\Psi_3 + a^2\Psi_4, \quad (A2.126)$$

$$\Psi_3 \rightarrow \Psi_3 + a\Psi_4, \quad (A2.127)$$

$$\Psi_4 \rightarrow \Psi_4, \quad (A2.128)$$

$$\begin{aligned} \kappa \rightarrow & \kappa + \bar{a}\sigma + a(\rho + 2\sigma) + a^2(\pi + 2\alpha) + a^3\lambda + a\bar{a}(\tau + 2\beta) \\ & \bar{a}a^2(\mu + 2\gamma) + \bar{a}a^3\nu - (\Delta + \bar{a}\delta + a\bar{\delta} + a\bar{a}D)a, \end{aligned} \quad (A2.129)$$

$$\sigma \rightarrow \sigma + a(2\beta + \tau) + a^2(\mu + 2\gamma) + a^3\nu - \delta a - aDa, \quad (A2.130)$$

$$\rho \rightarrow \rho + \bar{a}\tau + 2a\alpha + a\bar{a}\gamma + a^2\lambda + \bar{a}a^2\nu - \bar{\delta}a - \bar{a}Da, \quad (A2.131)$$

$$\epsilon \rightarrow \epsilon + \bar{a}\beta + a(\alpha + \pi) + a\bar{a}(\mu + \gamma) + a^2\lambda + \bar{a}a^2\nu, \quad (A2.132)$$

$$\tau \rightarrow \tau + 2a\gamma + a^2\nu - Da, \quad (A2.133)$$

$$\pi \rightarrow \pi + \bar{a}\mu + a\lambda + a\bar{a}\nu, \quad (A2.134)$$

$$\alpha \rightarrow \alpha + \bar{a}\gamma + a\lambda + a\bar{a}\nu, \quad (A2.135)$$

$$\beta \rightarrow \beta + a(\mu\gamma) + a^2\nu, \quad (A2.136)$$

$$\gamma \rightarrow \gamma + a\nu, \quad (A2.137)$$

$$\lambda \rightarrow \lambda + \bar{a}\nu, \quad (A2.138)$$

$$\mu \rightarrow \mu + a\nu, \quad (A2.139)$$

$$v \rightarrow v, \quad (A2.140)$$

Case III:

$$l \rightarrow \frac{1}{A}l, \quad (A2.141)$$

$$n \rightarrow An, \quad (A2.142)$$

$$m \rightarrow e^{i\theta}m, \quad (A2.143)$$

$$\Psi_0 \rightarrow \frac{e^{2i\theta}}{A^2}\Psi_0, \quad (A2.144)$$

$$\Psi_1 \rightarrow \frac{e^{i\theta}}{A}\Psi_1, \quad (A2.145)$$

$$\Psi_2 \rightarrow \Psi_2, \quad (A2.146)$$

$$\Psi_3 \rightarrow Ae^{i\theta}\Psi_3, \quad (A2.147)$$

$$\Psi_4 \rightarrow A^2e^{-2i\theta}\Psi_4, \quad (A2.148)$$

$$\kappa \rightarrow \frac{e^{i\theta}}{A^2}\kappa, \quad (A2.149)$$

$$\sigma \rightarrow \frac{e^{2i\theta}}{A}\sigma, \quad (A2.150)$$

$$\rho \rightarrow \frac{1}{A}\rho, \quad (A2.151)$$

$$\tau \rightarrow e^{i\theta}\tau, \quad (A2.152)$$

$$\pi \rightarrow e^{-i\theta}\pi, \quad (A2.153)$$

$$\lambda \rightarrow Ae^{-2i\theta}\lambda, \quad (A2.154)$$

$$\mu \rightarrow A\mu, \quad (A2.155)$$

$$v \rightarrow A^2e^{-i\theta}, \quad (A2.156)$$

$$\gamma \rightarrow A\gamma - \frac{1}{2}\Delta A + \frac{1}{2}iA\Delta\theta, \quad (A2.157)$$

$$\epsilon \rightarrow .A^{-1}\epsilon - \frac{1}{2}.A^{-2}D.A + \frac{1}{2}i.A^{-1}D\theta, \quad (.12.158)$$

$$\alpha \rightarrow e^{-i\theta}\alpha + \frac{1}{2}ie^{-i\theta}\bar{\delta}\theta - \frac{1}{2}.A^{-1}e^{-i\theta}\bar{\delta}.A, \quad (.12.159)$$

$$\beta \rightarrow e^{i\theta}\beta + \frac{1}{2}ie^{i\theta}\delta\theta - \frac{1}{2}.A^{-1}e^{i\theta}\delta.A, \quad (.12.160)$$

## 10 Appendix 3 Solutions and Perturbations

For the calculations, a null tetrad is chosen for each geometry, such that  $m^a$  and  $\bar{m}^a$  are surface forming and the differential surface element for an  $S^2$  cut of  $\mathcal{S}^-$  is proportional to  $l^a n^b$ .

In Cartesian coordinates the Minkowski metric is

$\eta = \text{diag}(1, -1, -1, -1)$ . If we transform to a null polar coordinate system  $(u, r, \theta, \phi)$  we have the following,

$$u = t - r, \quad (A3.1)$$

$$r^2 = x^2 + y^2 + z^2, \quad (A3.2)$$

$$x = r \sin(\theta) \cos(\phi), \quad (A3.3)$$

$$y = r \sin(\theta) \sin(\phi), \quad (A3.4)$$

$$z = r \cos(\theta), \quad (A3.5)$$

$$ds^2 = du^2 + 2dudr - r^2(d\theta^2 + \sin^2(\theta)d\phi^2), \quad (A3.6)$$

$$\sqrt{-g} = r^2 \sin(\theta). \quad (A3.7)$$

A null tetrad is then given by,

$$l_a = \delta_a^0, \quad (A3.8)$$

$$n_a = \delta_a^1 + \frac{1}{2}\delta_a^2, \quad (A3.9)$$

$$m_a = -\frac{r}{2}(\delta_a^2 + i \sin(\theta)\delta_a^3), \quad (A3.10)$$

$$l^a = \delta_1^a, \quad (A3.11)$$

$$n^a = \delta_0^a - \frac{1}{2}\delta_1^a, \quad (A3.12)$$

$$m^a = \frac{1}{r\sqrt{2}}\left(\delta_2^a + \frac{i}{\sin(\theta)}\delta_3^a\right). \quad (A3.13)$$

In these coordinates,  $u = \text{constant}$  defines a null hypersurface and  $r$  is a radial parameter which is a luminosity distance. The non-zero spin coefficients are,

$$\mu = -\frac{1}{2r}, \quad \rho = -\frac{1}{r}, \quad \dots \quad (A3.14)$$

$$\alpha = -\frac{\cos(\theta)}{2\sqrt{2}r\sin(\theta)}, \quad \beta = \frac{\cos(\theta)}{2\sqrt{2}r\sin(\theta)}. \quad (A3.15)$$

$m$  and  $\bar{m}$  are surface forming. The timelike Killing vector is given by,

$$k_u^a = n^a + \frac{1}{2}l^a. \quad (A3.16)$$

The rotational Killing vector is given by,

$$k_\phi^a = \frac{r\sin(\theta)}{i\sqrt{2}}(m^a - \bar{m}^a). \quad (A3.17)$$

The tetrad derivatives are given below.

$$Dl_a = Dn_a = Dm_a = 0 \quad (A3.18)$$

$$\Delta l_a = \Delta n_a = \Delta m_a = 0 \quad (A3.19)$$

$$\delta l_a = \frac{1}{r}m_a \quad (A3.20)$$

$$\delta n_a = -\frac{1}{2r}m_a \quad (A3.21)$$

$$\delta m_a = \frac{\cot(\theta)}{\sqrt{2}r}m_a \quad (A3.22)$$

$$\delta \bar{m}_a = -\frac{1}{2r}l_a + \frac{1}{r}n_a - \frac{\cot(\theta)}{\sqrt{2}r}\bar{m}_a \quad (A3.23)$$

$$l_{a:b} = -\frac{1}{r}\bar{m}_a m_b - \frac{1}{r}m_a \bar{m}_b \quad (A3.24)$$

$$n_{a:b} = \frac{1}{2r}m_a \bar{m}_b + \frac{1}{2r}\bar{m}_a m_b \quad (A3.25)$$



$$m_{a;b} = \frac{1}{2r} l_a m_b - \frac{1}{r} n_a m_b - \frac{\cot(\theta)}{\sqrt{2r}} m_a \bar{m}_b + \frac{\cot(\theta)}{\sqrt{2r}} m_a m_b \quad (A3.26)$$

$$\therefore l^a_{;a} = \frac{2}{r} \quad (A3.27)$$

$$n^a_{;a} = -\frac{1}{r} \quad (A3.28)$$

$$m^a_{;a} = \frac{\cot(\theta)}{\sqrt{2r}} \quad (A3.29)$$

$$\delta^0_a = l_a \quad (A3.30)$$

$$\delta^1_a = n_a - \frac{1}{2} l_a \quad (A3.31)$$

$$\delta^2_a = -\frac{(m_a + \bar{m}_a)}{\sqrt{2r}} \quad (A3.32)$$

$$\delta^3_a = -\frac{(m_a - \bar{m}_a)}{r i \sqrt{2} \sin(\theta)} \quad (A3.33)$$

$$\delta^a_0 = n^a + \frac{1}{2} l^a \quad (A3.34)$$

$$\delta^a_1 = l^a \quad (A3.35)$$

$$\delta^a_2 = \frac{r}{\sqrt{2}} (m^a + \bar{m}^a) \quad (A3.36)$$

$$\delta^a_3 = \frac{r \sin(\theta)}{\sqrt{2} i} (m^a - \bar{m}^a) \quad (A3.37)$$

$$D = \partial_r \quad (A3.38)$$

$$\Delta = \partial_u - \frac{1}{2} \partial_r = \partial_u - \frac{1}{2} D \quad (A3.39)$$

$$\delta = \frac{1}{\sqrt{2} r} \left( \partial_\theta + \frac{i}{\sin(\theta)} \partial_\phi \right) \quad (A3.40)$$

For most of the calculations there is no time or '  $\phi$  ' dependence. In which case,

$$D = \partial_r, \quad (A3.41)$$

$$\Delta = -\frac{1}{2}D, \quad (A3.42)$$

$$\delta = \bar{\delta} = \frac{1}{\sqrt{2r}}\partial_\theta. \quad (A3.43)$$

The Schwarzschild metric in coordinates  $(t, r, \theta, \phi)$  is,

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2m}{r}} - r^2(d\theta^2 + \sin^2(\theta)d\phi^2), \quad (A3.44)$$

$$\sqrt{-g} = r^2 \sin(\theta). \quad (A3.45)$$

A null tetrad is given by,

$$l_a = \delta_a^0 - \frac{r}{r-2m}\delta_a^1, \quad (A3.46)$$

$$n_a = \frac{r-2m}{2r}\delta_a^0 + \frac{1}{2}\delta_a^1, \quad (A3.47)$$

$$m_a = -\frac{r}{\sqrt{2}}(\delta_a^2 + i\sin(\theta)\delta_a^3), \quad (A3.48)$$

$$l^a = \frac{r}{r-2m}\delta_0^a + \delta_1^a, \quad (A3.49)$$

$$n^a = \frac{1}{2}\delta_0^a - \frac{r-2m}{2r}\delta_1^a, \quad (A3.50)$$

$$m^a = \frac{1}{r\sqrt{2}}\left(\delta_2^a + \frac{i}{\sin(\theta)}\delta_3^a\right). \quad (A3.51)$$

$$D = \frac{r}{r-2m}\partial_t + \partial_r, \quad (A3.52)$$

$$\Delta = \frac{1}{2}\partial_t - \left(\frac{r-2m}{2r}\right)\partial_r, \quad (A3.53)$$

$$\delta = \frac{1}{r\sqrt{2}}\left(\partial_\theta + \frac{i}{\sin(\theta)}\partial_\phi\right) \quad (A3.54)$$

Note that an alternate tetrad for Minkowski spacetime can be obtained by letting the mass parameter go to zero.

The timelike and rotational Killing vectors are given by,

$$k^a = n^a + \frac{r-2m}{2r} l^a, \quad (A3.55)$$

$$k^a = \frac{r \sin(\theta)}{l\sqrt{2}} (m^a - \bar{m}^a). \quad (A3.56)$$

The non-zero spin coefficients are,

$$\rho = -\frac{1}{r}, \quad \mu = \frac{2m-r}{2r^2}, \quad (A3.57)$$

$$\alpha = -\frac{\cos(\theta)}{2\sqrt{2}r \sin(\theta)}, \quad \beta = \frac{\cos(\theta)}{2\sqrt{2}r \sin(\theta)}, \quad (A3.58)$$

$$\gamma = \frac{m}{2r^2}. \quad (A3.59)$$

The tetrad derivatives are given below.

$$l_{a;b} = \frac{m}{r^2} l_a l_b - \frac{1}{r} \bar{m}_a m_b - \frac{1}{r} m_a \bar{m}_b \quad (A3.60)$$

$$n_{a;b} = -\frac{m}{r^2} n_a l_b + \frac{r-2m}{2r^2} m_a \bar{m}_b + \frac{r-2m}{2r^2} \bar{m}_a m_b \quad (A3.61)$$

$$m_{a;b} = \frac{r-2m}{2r^2} l_a m_b - \frac{1}{r} n_a m_b - \frac{\cot(\theta)}{2r} (m_a \bar{m}_b - m_a m_b) \quad (A3.62)$$

$$l^a_{;a} = \frac{2}{r} \quad (A3.63)$$

$$n^a_{;a} = \frac{m}{r^2} - \frac{1}{r} \quad (A3.64)$$

$$m^a_{;a} = \frac{\cot(\theta)}{\sqrt{2}r} \quad (A3.65)$$

$$Dl_a = Dn_a = Dm_a = 0 \quad (A3.66)$$

$$\Delta m_a = 0 \quad (A3.67)$$

$$\Delta l_a = \frac{m}{r^2} l_a \quad (A3.68)$$

$$\Delta n_a = -\frac{m}{r^2} n_a \quad (A3.69)$$

$$\delta l_a = \frac{1}{r} m_a \quad (A3.70)$$

$$\delta n_a = \frac{2m-r}{2r^2} m_a \quad (A3.71)$$

$$\delta m_a = \frac{\cot(\theta)}{\sqrt{2}r} m_a \quad (A3.72)$$

$$\delta \bar{m}_a = \frac{2m-r}{2r^2} l_a + \frac{1}{r} n_a - \frac{\cot(\theta)}{\sqrt{2}r} \bar{m}_a \quad (A3.73)$$

$$\delta_a^0 = \frac{1}{2} l_a + \frac{r}{r-2m} n_a \quad (A3.74)$$

$$\delta_a^1 = n_a - \frac{r-2m}{2r} l_a \quad (A3.75)$$

$$\delta_a^2 = -\frac{(m_a + \bar{m}_a)}{\sqrt{2}r} \quad (A3.76)$$

$$\delta_a^3 = -\frac{(m_a - \bar{m}_a)}{ri\sqrt{2}\sin(\theta)} \quad (A3.77)$$

$$\delta_0^a = n^a + \frac{r-2m}{2r} l^a \quad (A3.78)$$

$$\delta_1^a = \frac{1}{2} l^a - \frac{r}{r-2m} n^a \quad (A3.79)$$

$$\delta_2^a = \frac{r}{\sqrt{2}} (m^a + \bar{m}^a) \quad (A3.80)$$

$$\delta_3^a = \frac{r\sin(\theta)}{\sqrt{2}i} (m^a - \bar{m}^a) \quad (A3.81)$$

For most calculations there is no time or '  $\phi$  ' dependence. In which case,

$$D = \partial_r \quad (A3.82)$$

$$\Delta = \frac{2m-r}{2r} D \quad (A3.83)$$

$$\delta = \bar{\delta} = \frac{1}{\sqrt{2r}} \partial_\theta \quad (A3.84)$$

The Kerr metric as a G.K.S. transformation is derived in chapter 7 of Adler, Bazin and Schiffer 1975. The tetrad used here is the tetrad for Minkowski.

$$g_{\kappa ab} = \eta_{ab} - 2m \frac{l_a l_b}{r} \quad (A3.85)$$

$$l_a = P u l_a + P r n_a + P m_a + \bar{P} \bar{m}_a \quad (A3.86)$$

$$P = P_\theta + iP_\phi \quad (A3.87)$$

$$P u = \frac{l_0}{2} \left( 1 - \frac{r \rho^2 + a^2 \cos^2(\theta)}{\rho (a^2 + \rho^2)} \right) \quad (A3.88)$$

$$P r = l_0 \left( 1 + \frac{r \rho^2 + a^2 \cos^2(\theta)}{\rho (a^2 + \rho^2)} \right) \quad (A3.89)$$

$$P_\theta = -\frac{l_0}{\sqrt{2}} \sin(\theta) \cos(\theta) \frac{r a}{a^2 + \rho^2} \quad (A3.90)$$

$$P_\phi = -\frac{l_0}{\sqrt{2}} \sin(\theta) \frac{r a}{a^2 + \rho^2} \quad (A3.91)$$

$$\rho^2 = \frac{r^2 - a^2}{2} + \sqrt{\frac{(r^2 - a^2)^2}{4} + a^2 r^2 \cos^2(\theta)} \quad (A3.92)$$

$$l_0^2 = \frac{\rho^3}{\rho^4 + a^2 r^2 \cos^2(\theta)} \quad (A3.93)$$

The Kerr metric viewed as a perturbation of Schwarzschild is given below.

$$ds_\kappa^2 = g_{ab} dx^a dx^b - 2m \left( \frac{l_a l_b}{r} - \frac{l_a l_b}{s} \right) dx^a dx^b \quad (A3.94)$$

Where  $g_s$  is the Schwarzschild metric as given previously. In terms of the Schwarz-

schild tetrad the perturbation terms are as follows.

$$l_k^a = K_l l_a + K_n n_a + K m_a + \bar{K} \bar{m}_a \quad (A3.95)$$

$$K_l = \frac{l_0}{2} \left( 1 - A_1 \frac{r-2m}{r} \right) \quad (A3.96)$$

$$K_n = l_0 \left( \frac{r}{r-2m} + A_1 \right) \quad (A3.97)$$

$$K = -\frac{l_0}{\sqrt{2}r} \left( A_2 - \frac{iA_3}{\sin(\theta)} \right) \quad (A3.98)$$

$$A_1 = \frac{r}{\rho} \left( \frac{\rho^2 + \alpha^2 \cos^2 \theta}{\alpha^2 + \rho^2} \right) + \frac{2m}{r-2m} \quad (A3.99)$$

$$A_2 = -\frac{r^2 \alpha^2}{\rho} \left( \frac{\sin \theta \cos \theta}{\alpha^2 + \rho^2} \right) \quad (A3.100)$$

$$A_3 = -\frac{r^2 \alpha \sin^2 \theta}{\alpha^2 + \rho^2} \quad (A3.101)$$

$$l_s^a l_s^b = -\frac{4r}{(r-2m)^2} n_a n_b \quad (A3.102)$$

The Taylor expansion in the rotation parameter  $\alpha$  is given below.

$$\tilde{h}_{ab} = l_k^a l_k^b - l_s^a l_s^b \quad (A3.103)$$

$$\tilde{h}_{ab} = \sum_{n=0}^{\infty} h_{ab}^{(n)} \frac{\alpha^n}{n!} \quad (A3.104)$$

$$h_{0ab} = 0 \quad (A3.10521)$$

$$h_{1ab} = 2n_{(a} [\bar{m}_{b)} - m_{b)}] \frac{\sqrt{2} i \sin(\theta)}{r(r-2m)} \quad (A3.106)$$

$$h_{2ab} = \frac{1}{r^3} \left\{ 4r \left( \frac{m \sin^2(\theta) - r \cos^2(\theta)}{(r-2m)^2} \right) n_a n_b \right. \\ \left. + \frac{\sin^2(\theta)}{2} (2l_{(a} n_{b)} + 2m_{(a} \bar{m}_{b)} + m_a m_b + \bar{m}_a \bar{m}_b) \right. \\ \left. + \frac{2\sqrt{2}r \sin(\theta) \cos(\theta)}{(r-2m)} n_{(a} [m_{b)} + \bar{m}_{b)}] \right\} \quad (A3.107)$$

$$h_{3ab} = \frac{i \sin(\theta)}{r^4} \{ \sin(\theta) \cos(\theta) (\bar{m}_a \bar{m}_b - m_a m_b) \\ + \frac{\sin^2(\theta)}{2\sqrt{2}} \left( \frac{r-2m}{r} \right) l_{(a} [\bar{m}_{b)} - m_{b)}] \\ + \frac{r(9 \cos^2(\theta) - 1) - 2m(\cos^2(\theta) + 1)}{2\sqrt{2}(r-2m)} n_{(a} [m_{b)} - \bar{m}_{b)}] \} \quad (A3.108)$$

etc...

It is also useful to expand the perturbation of Minkowski to get Kerr in a Taylor series. In this case, we use the Schwarzschild tetrad with the mass parameter set to zero. The perturbation is given below<sup>68</sup>.

$$l_{k_a} l_{k_b} dx^a dx^b = l_0^2 \left[ dt + \rho \frac{x dx + y dy}{a^2 + \rho^2} + a \frac{y dx - x dy}{a^2 + \rho^2} + \frac{z}{\rho} dz \right]^2 \quad (A3.109)$$

Transforming to polar coordinates we have,

$$l_0^2 \left[ dt + \frac{(\rho^2 + a^2 \cos^2 \theta) r dr}{\rho(a^2 + \rho^2)} - \frac{a^2 r^2 \sin \theta \cos \theta d\theta}{\rho(a^2 + \rho^2)} - \frac{a r^2 \sin^2 \theta d\phi}{a^2 + \rho^2} \right]^2 \quad (A3.110)$$

$$l_{k_a} = B_l l_a + B_n n_a + B m_a + \bar{B} \bar{m}_a \quad (A3.111)$$

$$B_l = \frac{l_0}{2} (1 - A_r) \quad (A3.112)$$

$$B_n = l_0 (1 + A_r) \quad (A3.113)$$

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<sup>68</sup> Page 253 Adler, Bazin and Schiffer 1975.

$$B = -\frac{l_0}{\sqrt{2}r} \left[ A_\theta - i \frac{A_\phi}{\sin\theta} \right] \quad (A3.114)$$

$$A_r = \frac{r(\rho^2 + a^2 \cos^2\theta)}{\rho \cdot \alpha^2 + \rho^2} \quad (A3.115)$$

$$A_\theta = -\frac{a^2 r^2 \sin(\theta) \cos(\theta)}{\rho(\alpha^2 + \rho^2)} \quad (A3.116)$$

$$A_\phi = -\frac{a r^2 \sin^2(\theta)}{\alpha^2 + \rho^2} \quad (A3.117)$$

$$\tilde{h}_{ab} = l_K^a l_K^b = \sum_{n=0}^{\infty} \frac{a^n}{n!} h_n^{ab} \quad (A3.118)$$

$$h_0 = \frac{4}{r} n_a n_b \quad (A3.119)$$

$$h_1 = \frac{2\sqrt{2}i \sin(\theta)}{r^2} n_{(a} [\bar{m}_b) - m_b] \quad (A3.120)$$

$$h_2 = \frac{1}{r^3} \{-4 \cos^2(\theta) n_a n_b$$

$$+ 2\sqrt{2} \sin(\theta) \cos(\theta) n_{(a} [m_b) + \bar{m}_b] \}$$

$$\frac{\sin^2(\theta)}{2} (2l_{(a} n_b) + 2m_{(a} \bar{m}_b) - m_a m_b - \bar{m}_a \bar{m}_b) \} \quad (A3.121)$$

$$h_3 = \frac{i \sin(\theta)}{r^4} \{ \sin(\theta) \cos(\theta) (\bar{m}_a \bar{m}_b - m_a m_b)$$

$$+ \frac{\sin^2(\theta)}{2\sqrt{2}} l_{(a} [\bar{m}_b) - m_b] \}$$

$$+ \frac{(9 \cos^2(\theta) - 1)}{\sqrt{2}} n_{(a} [m_b) - \bar{m}_b] \} \quad (A3.122)$$

etc...



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## **12 Vita Auctoris**

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