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ASYMPTOTIC FLATNESS AND PEELING

By

Mark G. Naber

A thesis submitted to the Fauculty of Graduate
Studies and Research through the Department of Physics
in partial fulfilment of the reqiurment for the Degree of
Master of Science at the University of Windsor.

Windsor, Ontario, Canada

1990

Canada

Mark C. Naber

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ABSTRACT ASYMPTOTIC FLATNESS AND PEELING

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The properties of Asymptotically flat space-times are considered with emphasis on Peeling. Penrose's method of conformal mapping is used to define Asymptotes, Asymptotic Simplicity and Asymptotically flat space-times. The boundary manifold is examined by the behavior of its geometrical properties and by group theoretic means. The behavior of massless fields on flat and curved space-times is considered with the final results being the Peeling theorems. The Peeling theorems are used to examine the asymptotic behavior of physical quantities associated with massless fields. Asymptotically flat space-times which do not peel are also briefly considered.

ACKNOWLEDGEMENT

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1 Introduction

In the attempt to understand nature it has been found expedient to postulate the existence of isolated systems. This usually simplifies the mathematics and allows an understanding of cause and effect as well as determining what physical features will characterize the system. In Newtonian gravitation an isolated system may be conceived by requiring that the source of the field (matter) vanish outside a compact set. From this system such concepts as mass, momentum, angular momentum and higher moments may readily be developed.

In General Relativity isolated systems are realized by studying spaces which are asymptotically flat. These are spaces which become, in some sense, Minkowskian at large distances from the source region of the gravitational field. Asymptotic flatness was initially studied by examining space-times in which the metric tensor approached the Minkowski metric as the distance from the source region became infinite. This method has coordinate difficulties since a coordinate transformation will change how the metric behaves in the limit. Similar problems are encountered by requiring the components of the Riemann tensor go to zero in the above limit. These difficulties were removed when Penrose introduced a conformal technique which adds a boundary to the space-time manifold and effectively makes infinity finite.

Asymptotically flat spaces have been studied in two different areas, spatial infinity and null infinity. Spatial infinity is achieved by 'traveling' to infinity along space-like directions. Spatial infinity is thus a time-like surface with a finite dimensional symmetry group. Null infinity is achieved by 'traveling' to infinity along a null direction. Null infinity is a null surface with an infinite dimensional symmetry group. One important difference between the two infinities is that the boundary manifold for spatial infinity has an invertable metric whereas the null case does not. This paper will primarily be concerned with the null case.

In the following section Penrose's definition of Asymptotes and asymptotic simplicity will be discussed and used to define asymptotically flat spaces. The Schwarzschild solution will also be presented as an example of an asymptotically flat space-time.

In section 3 the geometrical properties of the boundary manifold will be examined. This will involve the BMS group, the problem of defining a covariant derivative and the behavior of various quantities under a conformal transformation.

Section 4 will discuss the properties of massless fields on the space-time manifold. In particular the concepts of principle spinors (principle null directions) and the phenomena of Peeling will be examined. Examples from electromagnetism will be presented for conceptual clarification.

Section 5 will examine the physical implications of Peeling. This will make use of the Newman-Penrose formalism and present physical interpretations of the dyad components of the vacuum gravitational field. The Newman-Penrose constants will also be developed.

In section 6 asymptotically flat spaces which do not exhibit the Peeling property will be considered. These spaces will be found to posses many of the same properties as spaces which Peel, with the exception of the Newman-Penrose constants.

This thesis makes use of three different formalisms for discussing the topics presented. They are the Newman-Penrose formalism, spinor calculus and tensor calculus. The appendices are used to develop the Newman-Penrose formalism and spinor calculus to the level used in text. They are by no means meant to be complete but are given to fix notation and provide a convenient reference for the relations used in text. Hence the reader is encouraged to look over the appendices before beginning the main text.

2 Defining Flatness

There are three fundamental concepts which are instrumental for understanding asymptotically flat spaces. These are Asymptotes, Asymptotic Simplicity and the property of Peeling. Simplicity follows from the manifold's ability to be compactified (to have an asymptote), while Peeling is a property of the massless fields on the manifold. Peeling is analogous to the near field and far field approximations one encounters in the study of classical electrodynamics.

The concept of an asymptote arose out of the conformal approach due to Penrose (Penrose 1965). The conformal method is a geometrical approach to asymptotic flatness which is free of the 'choice of chart' difficulties one encounters when attempting to define flatness using the metric or Riemann tensor. The definition follows[1]:

Definition (2.1) Let $\{\tilde{M}, \tilde{g}_{ab}\}$ be a space-time. An asymptote of $\{\tilde{M}, \tilde{g}_{ab}\}$ is a manifold M with a boundary I, a smooth Lorentz metric g_{ab}, a smooth scalar function $\Omega > 0$ on $\{M, g_{ab}\}$, and a diffeomorphism $\Psi : \tilde{M} \to M - I$ such that

(1) On $\{M, g_{ab}\}$, $\qquad g_{ab} = \Omega^2 \tilde{g}_{ab}$

and at I

(2a) $\Omega = 0$

(2b) $\nabla_a \Omega \neq 0$

(2c) $g^{ab}(\nabla_a \Omega)(\nabla_b \Omega) = 0$

with ∇_a the gradient on $\{M, g_{ab}\}$.

Hence, to construct an asymptote one must scale down the metric (to bring infinity in) and imbed the space-time manifold in a new manifold with a boundary. This process

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[1] Page 8, Esposito and Witten 1977.

is called compactification. The new metric g_{ab} is called the unphysical metric in the literature. Notice that, by definition, the boundary is a null hypersurface (assuming $\{\tilde{M}, \tilde{g}_{ab}\}$ satisfies Einstein's equations, without cosmological constant, near $\mathcal{I}$).

The boundary can also be shown to be the disjoint union of two null surfaces, $\mathcal{I}^-$ and $\mathcal{I}^+$.[2] $\mathcal{I}^+$ represents future null infinity and $\mathcal{I}^-$ represents past null infinity. Each of these surfaces then has the topology of $S^2 \times \mathcal{R}^1$ (cylindrical). One should also note that the requirement of $\Omega \to 0$ at $\mathcal{I}$ forces g_{ab} to be non-invertable at $\mathcal{I}$ (to be proved explicitly later). Hence, there will not exist a unique universal derivative operator on $\mathcal{I}$ (Lie and Exterior derivatives are still well defined).

A useful property for an asymptote is that of regularity. An asymptote is said to be regular,[3] if for any $p \in \mathcal{I}$ and any non-zero null vector l^a at a point x of $\{\tilde{M}, \tilde{g}_{ab}\}$ such that the null geodesic in $\{M, g_{ab}\}$ generated by l^a does not meet p, then there exists neighborhoods L of p in $\{M, g_{ab}\}$ and Γ of l^a in the space of null vectors in $\{\tilde{M}, \tilde{g}_{ab}\}$ such that no null geodesic in $\{M, g_{ab}\}$ generated by an element of Γ enters L. This essentially says that null geodesics for a regular asymptote behave nicely, i.e. they do not interfere with each other.

The question of uniqueness can now be addressed. It has been found[4] that there are only two freedoms in which an asymptote may be non-unique, provided it is regular. These are equivalence and extension. Two asymptotes are equivalent[5] if one can be conformally mapped into the other. Let $\{\tilde{M}, \tilde{g}_{ab}\}$ be a space-time, $\{M, g_{ab}, \Omega\}$ an asymptote and ω a smooth positive scalar function on $\{M, g_{ab}\}$. Then $\{M, \omega^2 g_{ab}, \omega\Omega\}$ is also an asymptote of $\{\tilde{M}, \tilde{g}_{ab}\}$ (equivalence is also a relation in the algebraic sense).

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2 See page 201, Penrose 1965 for the proof.
3 Page 13-14, Esposito and Witten 1977.
4 Page 13, Esposito and Witten 1977.
5 Page 13, Esposito and Witten 1977.

The other freedom is that of extension. The term extension is used in the following sense: for any C (a closed subset of $\tilde{1}$) $\{ \tilde{M} - C, \tilde{g}_{ab}, \Omega \}$ (restricted to $\tilde{M} - C$) is also an asymptote[6] $\{ \tilde{M}, \tilde{g}_{ab}, \Omega \}$ is then an extension of $\{ \tilde{M} - C, \tilde{g}_{ab}, \Omega \}$. Thus, if an asymptote is regular it is unique up to equivalence and extension.

From the discussion above, the following theorem due to Geroch[7] may be obtained.

Theorem (2.1) Let $\{ \tilde{M}, \tilde{g}_{ab} \}$ be a space-time. Then there exists a regular asymptote

$\{ \tilde{M}, \tilde{g}_{ab}, \Omega \}$ unique up to equivalence. This asymptote is maximal,

any other regular asymptote of $\{ \tilde{M}, \tilde{g}_{ab} \}$ is equivalent to one of

which $\{ \tilde{M}, \tilde{g}_{ab}, \Omega \}$ is an extension.

The proof can be found in Esposito and Witten 1977.

Note that the asymptote guaranteed by theorem (2.1) may have an empty boundary. Hence, not every space-time will posses a 'physically' useful asymptote. At present there does not exist a means of determining if a space-time has an asymptote with a non-empty boundary.

The next step towards asymptotic flatness is that of asymptotic simplicity, and the somewhat less restrictive, weak simplicity.

Definition (2.2) A space-time is said to be asymptotically simple if it possesses a non-trivial asymptote. Weak simplicity[8] occurs if the manifold $\{ \tilde{M}, \tilde{g}_{ab} \}$ has a sub-manifold $\tilde{M}_0$ which is simple (corresponding to M_0) such that for some open subset K of M_0 including $\tilde{1}$, the region $M_0 \cap K$ is isometric with an open subset of $\{ \tilde{M}, \tilde{g}_{ab} \}$.

The difference between the two definitions can be attributed to the physical meaning of part (2c) of the definition of an asymptote. Recall (2c):

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6 Page 13, Esposito and Witten 1977.

7 Page 14, Esposito and Witten 1977.

8 Page 5, Newman and Todd 1980.

$$g^{ab}(\nabla_a \Omega)(\nabla_b \Omega) = 0 \quad \text{al} \quad | $$

$=>$ Every maximally extended null geodesic in $\{M, g_{ab}\}$ contains two end points on | (one on |⁻ and one on |⁻).

Strict adherence to this version of (2c) would disallow the existence of black holes and null orbits. Thereby greatly restricting the number of useful space-times within the class. Weak simplicity on the other hand is essentially the same as simplicity but with (2c) changed to read:

> (2d) Every null geodesic which does not fall inside an event horizon and which
>
> is not a null orbit has two end points on | .

Weak simplicity would then appear to be more applicable to physical space-times. Penrose has used this definition of weak simplicity to prove the following two theorems[9] (Theorem (2.3) will be discussed at length in a following chapter).

Theorem (2.2) If a manifold is weakly simple then the Weyl tensor, C^a_{bcd}, vanishes at

$$| \quad (\tilde{C}^a_{bcd} = C^a_{bcd} \ on \ \tilde{M}).$$

Theorem (2.3) If a manifold is weakly simple then all massless spin s >0 fields

exhibit the property of 'Peeling'.

Theorem (2.2) establishes that the manifold is indeed conformally flat at null infinity. Theorem (2.3) provides the rate at which the Weyl tensor goes to zero. Theorems (2.2) and (2.3) present the possibility of defining asymptotically flat space-times as follows[10]:

Definition (2.3) Let $\{\tilde{M}, \tilde{g}_{ab}\}$ be a space-time with smooth Lorentz metric $\tilde{g}_{ab}$.

Let $\{\tilde{M}, \tilde{g}_{ab}\}$ be (weakly) simple such that,

> (1) Einstein's equations without cosmological constant are valid.

---

9 Page 5, Newman and Todd 1980 (see also Penrose 1965 and Newman and Penrose 1968 for a further discussion).

10 Page 294, Ludwig 1976.

(2) The energy-momentum tensor does not approach a non-zero

multiple of the metric near I.

(3) The trace free part of the energy-momentum tensor remains

finite near I.

Then, $\{\tilde{M}, \tilde{g}_{ab}\}$ is asymptotically flat at null infinity.

Conditions (1) and (2) ensure that one is studying a manifold which obeys our present understanding of gravitation. Requiring the cosmological constant be zero causes I to be a null surface. Condition (3) together with the Weyl tensor being zero at I forces the energy-momentum to vanish at I.

This is not the only possible definition of asymptotic flatness, cf. Ashtekar and Hansen 1978 and Peresides 1979. The definition given is, however, one of the most commonly used. Despite the apparent completeness there still exist some ambiguities in the given definition. Specifically the word 'smooth'. In the literature there exist many examples of problems worked out with varying degrees of differentiability, C^∞ down to C^3 and C^4. Clearly C^∞ is too restrictive to be part of a general definition but, perhaps quite useful for some problems. At present there is no minimum order known. This problem will be discussed in greater detail later in the text. Another problem is that examples of asymptotically flat spaces which do not exhibit the peeling property for the Weyl tensor have been found. This topic will be discussed further in chapter 6.

An example of an asymptotically flat space-time is given by the Schwarzschild solution of the Einstein equation[11]. The line element is given by,

(2.1) $ds^2 = -(1 - (2m/r))dt^2 + (1 - 2m/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$

where   r = radial distance from source center (spherically symmetric),

t = time,

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11 Complete derivation of solution is given in Ray 1970, pages 206-212.

ϕ , θ = angular coordinates,

valid $\forall \ r > 2m$.

Following Geroch[12] let us make the following substitutions.

Let $u = t - r - 2m[\ \log(r - 2m)\ ]$,

$x = 1/r$.

The line element is now.

(2.2) $$ds^2 = x^{-2} \{ 2 du dx - x^2 (1 - 2mx) du^2 + d\theta^2 + \sin^2 \theta d\phi^2 \}$$

As a conformal factor choose $\Omega = x$. If the point $x = 0$ is included as part of the domain, the boundary $r = \infty$ is then part of the structure. This gives a manifold whose limit point $x = 0$ is mapped into the limit point $r = \infty$ of $\{ \tilde{M} , \tilde{g}_{ab} \}$, the original Scharzschild solution. The conditions for being an asymptote are quite clearly met, and since $\tilde{g}_{ab}$ is a static solution of Einstein's equation it may be concluded that $\{ \tilde{M} , \tilde{g}_{ab} \}$ is an asymptotically flat space-time, as expected. Note that if $m = 0$, we have an asymptote for Minkowski space-time.

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12 Page 9, Esposito and Witten 1977.

3 The Boundary manifold

This section is devoted to studying the geometrical properties of the boundary manifold and the behavior of physical fields at the boundary. The geometrical properties are unique because the boundary, when viewed as a manifold in it's own right, does not posses an invertible metric. The study of physical fields at the boundary is important because it will give an indication as to appropriateness of the given definition of asymptotic flatness and motivate a definition of total energy as well as angular momentum. (Please note that the study of physical fields will be initiated here and finished in the later chapters.)

A Conformal Properties

Before the boundary manifold is examined, a brief discussion of Einstein's equations and conformal transformations will be given (more can be found in Appendix III). Many of the following equations can be simplified if the unphysical Ricci tensor is replaced by[13]

$$(3.1) \qquad S_{ab} = R_{ab} - 1/6 R g_{ab}$$

with $R_{ab} = R^m{}_{amb}$. It is customary to write the physical version of S_{ab} as L_{ab}. The Riemann tensor now has the following decomposition:

$$(3.2) \qquad R_{abcd} = C_{abcd} + g_{a[c} S_{d]b} - g_{b[c} S_{d]a}.$$

Where the Riemann tensor is defined by $R_{abc}{}^d k_d = 2\nabla_{[a} \nabla_{b]} k_c$ and, C_{abcd} is the unphysical Weyl tensor.

With the given definitions of S_{ab} and L_{ab}, it is possible to construct an equation which relates some unphysical quantities to L_{ab} (found by examining the behavior of the Ricci tensor under a conformal transformation)[14].

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13 Same use of notation as Geroch in Esposito and Witten 1977.
14 Page 16, Esposito and Witten 1977.

$$(3.3) \qquad \Omega S_{ab} + 2\nabla_a(\nabla_b\Omega) - (\nabla^m\Omega)(\nabla_m\Omega)\Omega^{-1}g_{ab} = \Omega^{-1}L_{ab}.$$

As the vector $\nabla_a\Omega$ appears frequently, the following substitutions will be made:

$$(3.4) \qquad n_b = \nabla_b\Omega,$$

$$(3.5) \qquad f = (\nabla^m\Omega)(\nabla_m\Omega)\Omega^{-1}.$$

Thus, equation (3.3) becomes

$$(3.7) \qquad \Omega S_{ab} + 2\nabla_a n_b - f g_{ab} = \Omega^{-1}L_{ab}.$$

From this equation four other equations may be derived, by using the Bianchi identity, relating the unphysical metric g_{ab} and the conformal factor Ω to the curvature fields S_{ab} and C_{abcd}. The following derivation follows closely that of Geroch (Esposito and Witten 1977). First consider the curl of equation (3.7)

$$(3.8) \qquad \Omega\nabla_{[a}S_{b]c} + n_{[a}S_{b]c} + 2\nabla_{[a}\nabla_{b]}n_c - \nabla_{[a}fg_{b]c} = \nabla_{[a}\Omega^{-1}L_{b]c}.$$

Inserting the Riemann tensor where appropriate yields:

$$(3.9) \qquad \Omega\nabla_{[a}S_{b]c} + n_{[a}S_{b]c} + R_{abc}{}^d n_d - \nabla_{[a}fg_{b]c} = \nabla_{[a}\Omega^{-1}L_{b]c}.$$

Decomposing the Riemann tensor into it's Weyl and Ricci tensor components yields:

$$(3.10) \qquad \Omega\nabla_{[a}S_{b]c} + C_{abcd}n^d = \nabla_{[a}\Omega^{-1}L_{b]c} - \Omega^{-2}g_{c[a}L_{b]d}n^d.$$

If the Bianchi identity is now applied twice contracted, singly contracted and uncontracted the following three equations are obtained:

$$(3.11) \qquad \nabla^b S_{ab} - \nabla_a S^b{}_b = 0,$$

$$(3.12) \qquad \nabla^d C_{abcd} + \nabla_{[a}S_{b]c} = 0,$$

$$(3.13) \qquad \nabla_{[a}\Omega^{-1}C_{bc]}{}^{ed} = 2\Omega^{-2}\delta_{[a}{}^{[e}\nabla_{[b}\Omega^{-1}L_{c]}{}^{d]} - 2\Omega^{-4}\delta_{[a}{}^{[d}\delta_b{}^{e]}L_{c]m}n^m.$$

A fourth equation is obtained by contraction with n^b:

(3.14) $$S_{ab}n^b + 7_a f = l_{ab}n^b \Omega^{-4}.$$

B Boundary Manifold

With these preliminaries out of the way the boundary manifold I will now be examined explicitly. As seen in the previous section, the space-time manifold $\tilde{M}$ is diffeomorphic to the asymptote with the boundary removed (i.e. $M - I \approx \tilde{M}$). Also note that I may be regarded as a 3-manifold which is a null surface with respect to $\{M, g_{ab}, \Omega\}$. To study I by itself, a great deal of success has been made by mapping $\{M, g_{ab}, \Omega\}$ onto a new 3-manifold J which is diffeomorphic to I. The mapping is called a pullback, ζ^* with inverse ζ.[15]

$$\zeta^* : M \to J$$

$$\zeta : J \to M \text{ such that } Image(\zeta) = I$$

Definition (3.1) The pullback operator for the boundary manifold may be defined by the following four properties:

(1) $\forall \mu$ such that $\mu : M \to \mathcal{R}$: $\zeta^*(\mu) = \mu \circ \zeta$

(2) $\forall \mu$ such that $\mu : M \to \mathcal{R}$: $\zeta^*(7^a\mu) = 7^a\zeta^*(\mu)$.

(3) ζ^* commutes with addition and outer product but not with contraction.

(4) Uniqueness, defined and discussed momentarily.

The following theorem is an immediate consequence of the given definition.

Theorem (3.1) Let $\alpha_{a..c}$ be a tensor on $\{M, g_{ab}, \Omega\}$. If $\alpha_{a..c}$ can be written as a sum of outer products of vectors such that each term contains a vector which is proportional to n_a then, $\zeta^*(\alpha_{a..c}) = 0$.

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15 Pullback operators are discussed in more generality in Bott and Tu 1982 and Choquet-Bruhat et.al. 1977.

The proof follows from $\zeta^*(n_a) = \zeta^*(\nabla_a \Omega) = \nabla_a \zeta^*(\Omega) = \nabla_a 0 = 0$ and commutivity with the outer product.

For covariant fields the action of ζ^* specifies a unique operation since any covariant field may be written as a sum of outer products of scalars and gradients of scalars. For contravariant or mixed tensor fields only a restricted uniqueness can be guaranteed (due to the non-invertible metric on $\mathcal{J}$).

To discuss uniqueness we need construct a set which algebraically could be considered as an ideal of the domain of ζ^*. Let[16] C denote the collection of all smooth tensor fields on $\{M, g_{ab}, \Omega\}$, say $\alpha^{a \cdots c}{}_{b \ldots d}$, such that if

$$\zeta^*(\nu_{a \ldots c}) = 0,$$

then $\qquad\qquad\qquad\qquad\qquad \zeta^*(\alpha^{a \cdots c}{}_{b \ldots d} \nu_{a \ldots c}) = 0.$

Geometrically, a vector $\alpha^c \in C$ if and only if $\alpha^c \in T_p \mathcal{J} \quad \forall \quad p \in \mathcal{J}$. A tensor of rank (1,1) $\alpha^a{}_b \in C$ if and only if $\alpha^a{}_b \mu^b \in T_p \mathcal{J} \quad \forall \quad \mu^a \in T_p \mathcal{J} \quad \forall \quad p \in \mathcal{J}$.

Similarly the same type of statement can be generated for tensors of higher rank. Due to the algebraic similarities between ζ^* and C, it is found that C is closed under addition, outer product and gradient (hence also closed under lie and exterior derivatives). Note also that C is not closed under contraction.

Now that C is fully defined, a uniqueness statement may be given for ζ^*. Let $\alpha^{a \cdots c}{}_{b \ldots d} \in C$ and let $\mu_{a \ldots c}$ be any element of the set of smooth tensor fields on $\{M, g_{ab}, \Omega\}$. If we require ζ^* obey the following relation:

(3.15) $\qquad\qquad \zeta^*(\alpha^{a \cdots c}{}_{b \ldots d} \mu_{a \ldots c}) = \zeta^*(\alpha^{a \cdots c}{}_{b \ldots d}) \zeta^*(\mu_{a \ldots c}).$

Then $\zeta^*(\alpha^{a \cdots c}{}_{b \ldots d})$ is unique.

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16   Page 20, Esposito and Witten 1977.

Uniqueness for the pullback is somewhat restrictive, however this is to be expected when projecting a 4-manifold onto a 3-manifold.

To understand the geometry of $\mathcal{J}$ a natural place to start is with its metric. Denote the image of the metric on $\{M, g_{ab}, \Omega\}$ under the pullback as $\underline{g}_{ab} = \zeta^*(g_{ab})$ and let $\underline{n}^a = \zeta^*(n^a)$ (recall $\zeta^*(n_a) = 0$). Both g_{ab} and n^a are in C. The non-invertability of the metric on $\mathcal{J}$ will cause an ambiguity in defining a universal derivative operator and in the act of raising indices. The proof for non-invertability is given below.

Theorem (3.2) $\underline{g}_{ab}$ is not invertible.

Proof: Since $g_{ab}, n^b \in C$ the following relation must be true:

$$\zeta^*(g_{ab} n^b) = \zeta^*(g_{ab}) \zeta^*(n^b)$$

$$\zeta^*(g_{ab} n^b) = \zeta^*(n_a) = 0$$

$$\therefore \ \zeta^*(g_{ab}) \zeta^*(n^b) = 0$$

$$\therefore \ \underline{g}_{ab} \underline{n}^b = 0 .$$

However $\underline{n}^b$ is nowhere zero on $\mathcal{J}$. Therefore $\underline{g}_{ab}$ has a non-empty null space and is therefore not invertible.

Clearly, lowering indices is well defined as $\underline{g}_{ab}$ is unique. When an index is raised the result will be unique only up to addition of a symmetric tensor whose decomposition has at least one vector in the null space of $\underline{g}_{ab}$. The pseudo-inverse may be defined by[17]

(3.16) $$\underline{g}_{am} \underline{g}^{mn} \underline{g}_{nb} = \underline{g}_{ab} .$$

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17 Pages 138 and 145, Strang 1980.

The arbitrary tensor introduced upon raising an index is of the form

(3.17) $$v^{(a} \underline{n}^{b)},$$

where v^a is arbitrary and $\underline{n}^a$ is given above as part of the null space of $\underline{g}_{ab}$:

(3.18) $$\underline{g}^{ab} A_b{}^c = A^{ac} + v^{(a} \underline{n}^{c)}.$$

Let us now examine the problem of defining a covariant derivative operator on $\mathcal{J}$. Consider the following equation which defines a covariant derivative operator D_a, on a manifold with invertible metric, in terms of the exterior and Lie derivatives (both well defined on $\mathcal{J}$)[18].

(3.19) $$D_a \alpha_b = D_{[a} \alpha_{b]} + 1/2 \mathcal{L}_a \underline{g}_{ab}$$

For D_a to be defined unambiguously it must be required that α_b be orthogonal to $\underline{n}^a$. Equation (3.19) may be expanded for the case of covariant tensors of arbitrary rank, provided orthogonality is maintained. Equation (3.19) also satisfies the Leibnitz rule and is additive on α_b.

Let α_a be orthogonal to $\underline{n}^a$ and let it also be Lie propagated along $\underline{n}^a$ (these are required to ensure existence of $D_a D_b \alpha_c$). The given definition may then be verified against the Riemann tensor for $\mathcal{J}$. Consider

(3.20) $$D_{[a} D_{b]} \alpha_c = 1/2 R_{abc}{}^d \alpha_d$$

Equation (3.20) is additive in α_c and commutative under scalar multiplication. $R_{abc}{}^d$ is determined up to a multiple of $\underline{n}^a$, hence R_{abcd} is unique on $\mathcal{J}$. R_{abcd} can be shown to have the same symmetries as the usual Riemann tensor. This may be done

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18 Page 23, Esposito and Witten 1977.

by examining $D_{[a}D_{b]}\underline{g}_{cd} = 0$ and $D_{[a}D_{b}\alpha_{c]} = 0$. Since $R_{abcd}\underline{n}^d = 0$ the Ricci tensor, $R_{ab} = \underline{g}^{mn}R_{ambn}$, and the curvature scalar, $R = \underline{g}^{mn}R_{mn}$, are also well defined and have their usual properties.

The Bianchi identities are also obtainable. These are found by examining $D_{[a}D_bD_{c]}\alpha_d$ for α_d orthogonal to $\underline{n}^d$. The Bianchi identities may also be found by using spinor calculus and looking at the projection of the spin 2 massless field equation on $\mathcal{J}$ (much faster than the above tensor method). Thus the covariant derivative operator on $\mathcal{J}$ is satisfactorily defined albeit somewhat restrictive.

To discuss the geometrical aspects of $\mathcal{J}$ further it is convenient to define a tensor which is gauge invariant (with respect to conformal transformations) and which contains in some sense, the geometrical information of the boundary manifold. Such a tensor is given by[19]

(3.21)
$$\Gamma^{ab}{}_{cd} = \underline{n}^a\underline{n}^b\underline{g}_{cd}.$$

$\Gamma^{ab}{}_{cd}$ is symmetric in both pairs of indices. Note also that $\Gamma^{[a|b}{}_{cd}\Gamma^{|c|d]}{}_{gh} = 0$; this ensures that $\Gamma^{ab}{}_{cd}$ is decomposable for some $\underline{n}^a$ and $\underline{g}_{ab}$. Since $\underline{n}^a$ is in the null space of $\underline{g}_{ab}$, we also have $\Gamma^{am}{}_{cm} = 0$. If $w_c v^{[a}\Gamma^{b]c}{}_{de} \neq 0$, then $w_a w_b v^c v^d \Gamma^{ab}{}_{cd} > 0$. This reflects the fact that $\underline{g}_{ab}$ has a positive signature. If $v^{[a}\Gamma^{b]c}{}_{de} = 0$ and $\mathcal{L}_v\Gamma^{ab}{}_{cd} = \lambda\Gamma^{ab}{}_{cd}$ for some λ it can be shown that $\underline{n}^a$ is a conformal killing field with respect to $\underline{g}_{ab}$.

From the above properties one may conclude that at least locally, every asymptotic geometry possesses a unique (up to conformal transformation) decomposition for $\Gamma^{ab}{}_{cd}$.

C Asymptotic Symmetries

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19 Page 22, Esposito and Witten 1977.

$\Gamma^{ab}{}_{cd}$ is also useful for looking at the global properties of $\mathcal{I}$. For example, asymptotic symmetries. To examine this aspect a base space must first be defined.

Definition (3.2) Let $\{\mathcal{I}, \Gamma^{ab}{}_{cd}\}$ be an asymptotic geometry having no almost closed curves[20] of $\underline{n}^a$. Let B represent the set of all maximally extended integral curves of $\underline{n}^a$. Let π map $\mathcal{I} \to B$ such that each point of $\mathcal{I}$ corresponds to the integral curve on which it lies. Let U be open in $\mathcal{I}$ and consider a chart such that no $\underline{n}^a$ integral curve passes through U more than once (this is always possible since we may choose U as small as we wish and there are no almost closed curves). Let us also choose the chart such that two of the coordinate functions are constant in U along the integral curves of $\underline{n}^a$. Using π the chart may then be projected into B to produce a chart for B based on $\pi[U]$. Since there are no almost closed curves we may cover B by choosing a sufficient number of open sets U to cover $\mathcal{I}$. If B is then Hausdorff, it is called the base space of the asymptotic geometry[21].

To study the symmetries of asymptotically flat spaces we shall begin by looking at the Lie algebra produced by the set of asymptotic symmetries. Following this the group structure will be examined directly from the point of view of conformal transformations.

Firstly, symmetries and infinitesimal symmetries need be defined with respect to the asymptotic geometry.

Definition (3.3) Let $\{\mathcal{I}, \Gamma^{ab}{}_{cd}, B\}$ be an asymptotic geometry with base B. Then a symmetry is an automorphism on $\mathcal{I}$ such that $\Gamma^{ab}{}_{cd}$ is mapped onto itself.

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20 An almost closed curve is a maximally extended integral curve γ of $\underline{n}^a$ such that, for some point p of γ, γ reenters every sufficiently small neighborhood of p.

21 Page 27, Esposito and Witten 1977.

Definition (3.4) An infinitesimal symmetry on $\{ \mathcal{J} , \Gamma^{ab}_{cd} , \mathcal{B} \}$ is a vector ξ^a such

that $\mathcal{L}_\xi \Gamma^{ab}_{cd} = 0$ (i.e. a vector which Lie propagates Γ^{ab}_{cd}).

A special subset of the infinitesimal symmetries are the infinitesimal super-

translations, which is an infinitesimal symmetry ξ^a that is proportional to $\underline{n}^a$. This

will be seen to form an important subalgebra of $\mathcal{L}_{is}$ (to be defined shortly).

In terms of the decomposition for Γ^{ab}_{cd} the infinitesimal symmetries may be

defined as follows:

$$\mathcal{L}_\xi \Gamma^{ab}_{cd} = \underline{n}^a \underline{n}^b \mathcal{L}_\xi \underline{g}_{cd} + 2\underline{g}_{cd} \mathcal{L}_\xi \underline{n}^a \underline{n}^b.$$

Requiring that the right hand side be zero yields:

(3.22) $$\mathcal{L}_\xi \underline{g}_{ab} = 2k\underline{g}_{ab} ,$$

(3.23) $$\mathcal{L}_\xi \underline{n}^a = -k\underline{n}^a ,$$

for some k on $\mathcal{J}$.

The set of symmetries forms a group and the set of infinitesimal symmetries form

it's Lie algebra. As one might anticipate, these will be similar to the Poincare group

and the set of Killing fields for Minkowski space. In fact an infinitesimal symmetry on

the physical space-time produces an infinitesimal symmetry on $\mathcal{J}$[22].

Theorem (3.3) Let $\{ \tilde{M} , \tilde{g}_{ab} \}$ be a space-time with asymptote $\{ M , g_{ab} , \Omega \}$ (with

$\{ \mathcal{J} , \Gamma^{ab}_{cd} \}$) and Killing field $\tilde{\eta}^a$. Then $\exists ! \ \eta^a \in C$ which is a smooth

extension of $\tilde{\eta}^a$ on $\{ M , g_{ab} , \Omega \}$. $\xi^*(\eta^a)$ is also an infinitesimal

symmetry.

Proof: On $\{ \tilde{M} , \tilde{g}_{ab} \}$; $\quad \mathcal{L}_{\tilde{\eta}} g_{ab} = \mathcal{L}_{\tilde{\eta}} (\Omega^2 \tilde{g}_{ab})$

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22 Page 30, Esposito and Witten 1977.

$$= 2\Omega \tilde{g}_{ab} \mathcal{L}_{\bar{\eta}}(\Omega)$$

$$= 2\Omega^{-1} g_{ab} \mathcal{L}_{\bar{\eta}}(\Omega).$$

Therefore $\bar{\eta}^a$ is a conformal Killing field in $\{\tilde{M}, \tilde{g}_{ab}\}$ with respect to g_{ab}, and hence has a smooth extension to η^a in $\{M, g_{ab}, \Omega\}$.

Let $\alpha = \Omega^{-1} \mathcal{L}_\eta(\Omega)$ which is smooth in $\{M, g_{ab}, \Omega\}$

$$\Rightarrow \quad \eta^a \nabla_a \Omega = \Omega \alpha \to 0 \quad \text{on} \quad \mathcal{J}.$$

We also have
$$\mathcal{L}_\eta(n^a) = \mathcal{L}_\eta(g^{ab} \nabla_b \Omega)$$

$$= -\alpha n^a + \Omega \nabla^a \alpha$$

$$\Rightarrow \quad \mathcal{L}_\eta(n^a n^b g_{cd}) = 0 \quad \text{on} \quad \mathcal{J}$$

$$\Rightarrow \quad \zeta^*(\mathcal{L}_\eta(n^a n^b g_{cd})) = 0$$

$$\mathcal{L}_{\zeta^*(\eta)}(\zeta^* n^a n^b g_{cd}) = 0$$

$$\mathcal{L}_{(\zeta^*(\eta))}(\Gamma^{ab}{}_{cd}) = 0.$$

Therefore $\zeta^*(\eta)$ is an infinitesimal symmetry.

As mentioned earlier, the set of infinitesimal symmetries form a Lie algebra with its binary operations being addition and the Lie product for vector fields. This Lie algebra shall be denoted by $\mathcal{L}_{is}$. Denote the subalgebra of the super translations by S_{st}. From the definitions of the infinitesimal symmetries and super translations the following theorem may be constructed.

Theorem (3.4) S_{st} is an ideal of $\mathcal{L}_{is}$.

Proof: What need be shown is that if $\xi^a = \alpha \underline{n}^a \in S_{st}$ and $\eta^a \in \mathcal{L}_{is}$. Then $\mathcal{L}_\xi \eta^a \in S_{st}$

or $\mathcal{L}_\eta \xi^a \in S_{st}$ (since the Lie product is antisymmetric).

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$$\mathcal{L}_{\xi}\eta^{a} = \mathcal{L}_{\alpha \underline{n}}\eta^{a} = \alpha \mathcal{L}_{\underline{n}}\eta^{a}$$

$$= -\alpha \mathcal{L}_{\eta}\underline{n}^{a}$$

but by theorem (3.2) $\mathcal{L}_{\eta}\underline{n}^{a} \sim \underline{n}^{a}$

$$\Rightarrow \quad -\alpha \mathcal{L}_{\eta}\underline{n}^{a} \in S_{st}$$

$$\Rightarrow \mathcal{L}_{\xi}\eta^{a} \in S_{st}$$

$$\therefore \quad S_{st} \text{ is an ideal of } \mathcal{L}_{is}.$$

The subalgebra also has another interesting property in that S_{st} may be represented by the base space, B [23]. Consider $\xi^{a} \in S_{st}$. Then

$$\mathcal{L}_{\xi}\underline{g}_{ab} = 0$$

$$\mathcal{L}_{\xi}\underline{g}_{ab} = \alpha \mathcal{L}_{\underline{n}}\underline{g}_{ab} + 2\underline{n}^{m}\underline{g}_{m(a}D_{b)}\alpha$$

$$\Rightarrow \quad \mathcal{L}_{\xi}\underline{n}^{a} = -(\underline{n}^{m}D_{m}\alpha)\underline{n}^{a}$$

but $\mathcal{L}_{\xi}\underline{n}^{a}$ must be zero since $\xi \in S_{st}$. Therefore α is constant along the $\underline{n}$ integral

curves. Therefore $\exists \beta \in B$ such that $\alpha = \pi^{*}(\beta)$. Hence for any $\xi^{a} = \alpha \underline{n}^{a} \in S_{st}$, $\exists \beta \in B$

such that $\alpha = \pi^{*}(\beta)$.

To examine what else is in $\mathcal{L}_{is}$ besides S_{st} it is useful to look at the quotient

algebra $\mathcal{L}_{is}/S_{st}$. Members of the quotient algebra may be described by the following

two equations [24]:

(3.24) $$D_{(a}\xi_{b)} = k\underline{g}_{ab},$$

(3.25) $$\mathcal{L}_{\underline{n}}\xi_{a} = 0,$$

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23 Page 31, Esposito and Witten 1977.
24 Proof (sketch) may be found on page 32 of Esposito and Witten 1977.

with $\xi_a = g_{ab}\xi^b$ and $\xi^b c \mathcal{L}_{is}$.

From equation (3.25) and the fact that ξ_a is orthogonal to $\underline{n}^a$, there must exist an element μ_a in the base space B such that for each ξ_a in the quotient algebra we may write $\xi_a = \pi^{\cdot}(\mu_a)$ for some μ_a in B. Equation (3.24) may be shown[25] to be a pullback of the conformal killing field equation on the base. Thus the quotient algebra is isomorphic to B. This is very similar to the algebraic structure of Minkowski space-time. In this case $\mathcal{L}_{is}$ would correspond to the Poincare algebra, S_{si} to the infinitesimal symmetries on Minkowski space-time and the quotient group to the Lorentz algebra.

The asymptotic symmetry group (the BMS group) will now be examined. This group was first developed by Bondi et. al. (1962) and then generalized by Sachs (1962). Let u, r, θ and ϕ denote the null polar coordinate system.

Definition (3.5) A BMS transformation[26] is defined by

(3.26) $u^{\cdot} = K(\theta, \phi)\lfloor u - \alpha(\theta, \phi)\rfloor$

(3.27) $\phi^{\cdot} = \phi^{\cdot}(\theta, \phi)$

(3.28) $\theta^{\cdot} = \theta^{\cdot}(\theta, \phi)$

Thus the angular coordinates are conformally transformed and null hypersurfaces are mapped into null hypersurfaces. Unlike SL(2,C) (symmetry group for Minkowski space-time) the BMS group[27] has an infinite number of parameters and is not locally compact.[28]

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25 Page 33, Esposito and Witten 1977.

26 Page 863, Newman and Penrose 1966.

27 Proof that the BMS transformations form a group is given in Sachs 1962, please note that the BMS group is referred to as the GBM group in this article.

28 Page 292, Carmeli 1977.

The subgroup given by $\theta' = \theta$ and $\phi' = \phi$ is called the supertranslation subgroup (usually denoted by N in the literature). The study of this is facilitated by expanding $\alpha(\theta, \phi)$ into the spherical harmonics.

(3.29)
$$\alpha(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi)$$

(3.30)
$$a_{lm} \in \mathfrak{R} \quad \text{and} \quad a_{l,-m} = (-1)^m a_{lm}$$

Another important subgroup is obtained by the case of $a_{lm} = 0 \quad \forall \, l \geq 2$. This four parameter subgroup is called the translation subgroup.

(3.31)
$$\alpha = \epsilon_0 + \epsilon_1 \sin(\theta)\cos(\phi) + \epsilon_2 \sin(\theta)\sin(\phi) + \epsilon_3 \cos(\theta)$$

The following two theorems concern the above subgroups:

Theorem (3.5) The supertranslations form an Abelian normal subgroup (N) of the BMS group. The factor group is isomorphic to the orthochronous homogeneous Lorentz group.

Proof (sketch, Sachs 1962): The supertranslations leave the angles θ, ϕ invariant. Using this one can show that the supertranslations are normal in the BMS group. The quotient group would consist of only the conformal transformations which, for the 2 sphere, are indeed isomorphic the orthochronous homogeneous Lorentz group. Commutivity follows from the definition of the BMS transformations.

Theorem (3.6) If N' is a four dimensional normal subgroup of the BMS group then N' in contained in the supertranslation group N.

Proof (Sachs 1962): Consider the image N'/N of N'under the homomorphism BMS -> BMS/N. Since N' is normal in BMS, N'/N is normal in BMS/N. Hence N'/N is normal in the orthochronous homogeneous Lorentz group. However the only normal

subgroups of this group are itself and the identity. Since the orthochronous homogeneous Lorentz group has six parameters we must have N'/N = identity, to avoid a contradiction. Therefore N' is contained in N.

In later chapters weaker forms of the definition of asymptotic flatness will be examined with one of the criteria being that the symmetry group be the BMS group.

4 The Peeling Theorems

To properly discuss the Peeling theorems, a discussion of massless fields of integer and half integer spin must first be given. Spinor calculus will be used extensively in this chapter. The reader is referred to Appendix I for a brief introduction and summary of useful formulae as well as references for further reading.

Massless vacuum fields of spin $s > 0$ may be represented classically by a symmetric spinor with 2s indices which satisfies the following equation[29]:

$$(4.1) \qquad \nabla^{AM} \phi_{AB...K} = 0$$

where $\phi_{AB...K}$ has 2s indices.

For example, the case of $s = 1$ generates a representation of the electromagnetic field (source free):

$$(4.2) \qquad F_{\mu\nu} = \sigma_\mu{}^{AP} \sigma_\nu{}^{BQ} \{ \phi_{AB} \epsilon_{PQ} + \epsilon_{AB} \bar{\phi}_{PQ} \}.$$

With $F_{\mu\nu}$ the electromagnetic field tensor in vacuum, ϕ_{AB} satisfies:

$$(4.3) \qquad \nabla^{AM} \phi_{AB} = 0.$$

Similarly, for the vacuum gravitational field (if and only if the cosmological constant vanishes), the Weyl tensor describes the spin 2 massless field

$$(4.4) \qquad C_{\mu\nu\rho\tau} = \sigma_\mu{}^{AP} \sigma_\nu{}^{BQ} \sigma_\rho{}^{CR} \sigma_\tau{}^{DS} \{ \phi_{ABCD} \epsilon_{PQ} \epsilon_{RS} + \epsilon_{AB} \epsilon_{CD} \bar{\phi}_{PQRS} \},$$

with Bianchi $\nabla^{AM} \phi_{ABCD} = 0.$

Consider equation (4.1) and apply the operator ∇_{YM}. Then

$$\nabla_{YM} \nabla^{AM} \phi_{AB...K} = 0$$

$$1/2 b_Y^A \Box \phi_{AB...K} = 0$$

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[29] Page 162, Penrose 1965.

(4.5) $$\Box \phi_{AB..K} = 0 .$$

Thus $\phi_{AB..K}$ satisfies the wave equation as required for a massless field. Using the spinor representation it is not difficult to prove the following two theorems concerning solutions of equation (4.1):

Theorem (4.1) A solution of equation (4.1) of spin s is given by

(4.6) $$\phi_{AB..K} = \mu^M \nabla_{AM} \psi_{B..K}$$

where, (1) $\phi_{AB..K}$ has 2s indices.

(2) μ^M is a constant spinor not equal to zero.

(3) $\psi_{B..K}$ is a solution of the massless spin s - 1/2 field

equation.

The proof[30] may be constructed by choosing a coordinate system (Minkowskian) with constant σ_μ^{AB} and examining the components of equation (4.1). This yields a vanishing curl in both pairs of coordinates which can be reduced to one of the components of the spin s - 1/2 field equation. It remains to demonstrate symmetry and that the two curl equations may be satisfied without effecting one another.

The next theorem is proven by induction on theorem (4.1).

Theorem (4.2) A solution of equation (4.1) may be given in terms of a 'Hertz type'[31] complex potential χ:

(4.7) $$\phi_{AB..K} = \mu^M \nu^N \cdots \omega^W \nabla_{AM} \nabla_{BN} \cdots \nabla_{KW} \chi$$

where $\mu^M \nu^N \cdots \omega^W$ are constant spinors and χ satisfies

$$\Box \chi = 0.$$

30 See pages 165-166, Penrose 1965.
31 Page 165, Penrose 1965.

Clearly an inductive proof would not be difficult based upon theorem (4.1). As an example (of theorem (4.2)), consider the s = 1 coulomb field (monopole).

Let $\chi = -\alpha u \zeta^{-1} r^{-1}$, with $\zeta = e^{i\phi} \cos(\theta/2)$ (θ, ϕ are the usual angular coordinates)

$$u = t - r \text{ retarded time.}$$

$$R = r(1 + \zeta\bar{\zeta}) = \text{radial distance}$$

Then $\Box\chi = 0$ and[32] $\chi = O^{*}(r^{-1})$. From this we obtain, $\psi_1 = -(1/2)\alpha/(R\zeta)\xi_1$

and $\phi_{AB} = (\alpha/R^2)\xi_{(A}\lambda_{B)}$ where

$$\xi_A = \text{outward radial null direction},$$

$$\lambda_B = \text{opposite inward null direction}.$$

The field ϕ_{AB} is static, spherically symmetric (as required for a coulomb field)

and has the proper radial dependance. The real and imaginary components of α

determine the electric and magnetic components (respectively) of the field. The dipole

field is similarly obtained. In this case the field is represented by:

(4.8)
$$\phi_{AB} = \alpha\mu^M\nu^N\nabla_{AM}\nabla_{BN}(1/R).$$

If $\nu^M\mu_M = 0$ then there is an equal mixture of magnetic and electric dipoles with

perpendicular axes. If $\nu^M\mu_M = 1$ then the coefficient α determines how the field is

divided between electric and magnetic components (real for electric and imaginary for

magnetic).

The next topic on the path to the Peeling theorems is that of principle spinors

(principle null directions in tensor language).

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[32] Note: $f = O^{*}(r^{-k}) \Rightarrow$ Asymptotically smooth of order r^{-k};
$\partial/\partial r(f) = O^{*}(r^{-k-1})$, $\partial/\partial\zeta(f)$, $\partial/\partial t(f) = O(r^{-k})$ same notation as in Penrose 1962.

Definition (4.1) A principle spinor is a 1-spinor which is proportional to one of the components of the canonical decomposition of $\phi_{AB..K}$ where $\phi_{AB..K}$ represents a solution of equation (4.1) for a spin s massless field.

Explicitly, ξ^A in a principle spinor of $\phi_{AB..K}$ if and only if,

$$(4.9) \qquad \phi_{AB..K}\xi^A\xi^B \cdot \cdot \xi^K = 0 \, .$$

The corresponding principle null direction is given by:

$$(4.10) \qquad \xi_\mu = \sigma_\mu^{\ AB}\xi_A\xi_B \, .$$

Proof of existence is given by choosing $\xi^A = (1, \xi)$ and then considering the resulting polynomial in ξ (complex) and appealing to the fundamental theorem of algebra[33]. A more involved discussion including the Petrov classification may be found in Pirani 1965 and Penrose and Rindler 1984.

In general a necessary and sufficient condition[34] for at least j principle spinors to coincide with ξ^A is:

$$(4.11) \qquad \phi_{AB..FG..K}\xi^G \cdot \cdot \xi^K = 0$$

where ξ appears 2s - j + 1 times (the proof is similar to the previous theorem).

Intuitively one would expect that very near the source region of a massless field none of the principle spinors would coincide. Think of the example from electrodynamics: in the static and induction zones all terms of an expansion (in powers of 1/r) are important for computation. In the radiation zone however only the leading term,

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33 Page 224, Jacobson 1985.
34 Page 164, Penrose 1965.

$1/r$, is important[35]. The Peeling theorems describe the rate at which the principle spinors become coincident with respect to the radial distant from the source region. The Peeling theorem for flat space may be stated as follows[36].

Theorem (4.3) Let $\phi_{AB..K}$ be a symmetric spinor with 2s indices satisfying the spin s massless field equation. Then,

(4.12)
$$\phi_{AB..FG..K}\xi^G \cdot \cdot \xi^K = O(r^{-k-1})$$

along all lines of constant retarded time and angular coordinates, and where there are k ξ's on the left hand side of equation (4.12)

when $\xi = O^\cdot(r^{-1})$.

Proof: (Penrose 1965) Choose a coordinate system such that the origin is within the source region (assumed to be compact). Let u, r and ξ be the coordinates as defined earlier. Choose a particular line, $u = u_0$ and $\zeta = \zeta_0$ (fixed). Then in a neighborhood of the line let

(4.13)
$$\phi_{AB..K} = \mu^M \nabla_{AM} \psi_{B..K}.$$

As the inductive hypothesis assume

(4.14)
$$\psi_{B..GH..K}\xi^H \cdot \cdot \xi^K = O(r^{-k})$$

along the given line. The hypothesis is true for k = 1 as given in the statement of the theorem. Apply $\xi^A \mu^M \nabla_{AM}$ to (4.14) then:

$$\xi^A \mu^M \nabla_{AM} \psi_{B..GH..K}\xi^H \cdot \cdot \xi^K = O(r^{-k-1}),$$

(4.15)
$$\Rightarrow \xi^A \phi_{AB..GH..K}\xi^H \cdot \cdot \xi^K = O(r^{-k-1}).$$

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35 Pages 392-393, Jackson 1975.
36 Page 172, Penrose 1965.

Hence the relation is consistent for the inductive proof. The key restraint in the above theorem is the requirement of $\xi = O^*(r^{-1})$. This condition may be summarized physically as requiring that the incoming waves fall off reasonably fast. The previous examples for electromagnetism fit into theorem (4.3) (see Penrose 1962 for a discussion of the Schwarzschild solution and also a discussion of a solution of the Weyl neutrino equation). Further examples and discussion will be given later in the text.

To discuss the Peeling theorem for curved space-time some generalizations of the property of peeling must be made. In particular, the generalizations should be conformally invariant as the ultimate goal is to treat asymptotically flat spaces.

To begin with, let $\{\tilde{M}, \tilde{g}_{ab}\}$ be an asymptotically simple space-time with $\{M, g_{ab}\}$ as the compactified manifold. Let $\tilde{g}$ be a null geodesic in $\{\tilde{M}, \tilde{g}_{ab}\}$, hence g is a null geodesic in $\{M, g_{ab}\}$. Let ξ^A be a tangent spinor which is parallel propagated along $\tilde{g}$. Hence we may write

(4.16)
$$\xi^B \xi^C \nabla_{BC} \xi_A = 0$$

or equivalently

(4.17)
$$\xi^B \xi^C \nabla_{BC} \xi_A = 0.$$

Using the conformal invariance of null geodesics it can be shown that the conformal weight of ξ is -1/2 (i.e. $\tilde{\xi}_A = \Omega^{1/2} \xi_A$). An auxiliary spinor $\tilde{\eta}^A$ will also be needed such that $\xi^A \eta_A = 1$ and which is parallel propagated along g:

(4.18)
$$\xi^B \xi^C \nabla_{BC} \eta_A = 0.$$

Requiring conformal invariance of the parallel propagation equation yields

(4.19)
$$\tilde{\eta}_A = \Omega^{-1/2} \eta_A + b \Omega^{1/2} \xi_A,$$

where[37] $b = \int \Omega^{-2} \eta^B \xi^C \nabla_{BC} \Omega dr$.

For curved space-times the property of peeling may now be defined.

Definition (4.2) A solution of the massless spin s field equation $\tilde{\phi}$ is said to peel if

$$(4.20) \qquad \tilde{\phi}_{A..FG..K} \bar{\eta}^A .. \bar{\eta}^F \xi^G .. \xi^K = O(r^{-k-1})$$

along a null geodesic g where ξ appears k times and $\bar{\eta}$ appears 2s-k times. $\bar{\eta}_A$ and ξ_A are parallel propagated along g with $\xi^A \bar{\eta}_A = 1$.

The general Peeling theorem may now be stated and proved.

Theorem (4.4) Let $\{\tilde{M}, \tilde{g}_{ab}\}$ be an asymptotically simple manifold. Let $\tilde{\phi}_{A..K}$

satisfy the spin s massless field equation such that it is asymptotically

regular[38]. Then $\tilde{\phi}_{AB..K}$ satisfies equation (4.20).

Proof: (Penrose 1965) Let g meet I at a point G. Since the manifold is simple we have:

$$d/dr \, \Omega = \xi^A \xi^B \nabla_{AB} \Omega \neq 0 \text{ at } G^{[39]}.$$

$\Rightarrow \Omega / (r - r_0)$ exists at G and is not zero ($r = r_0$ at G).

$\Rightarrow \Omega / r$ exists at G and is not zero.

Let η^A be a null spinor in the plane spanned by ξ^A and the normal to I at G

(different from ξ^A). Hence, we may write $\nabla_{AB} \Omega$ in terms of $\xi_A \xi_B$ and $\eta_A \eta_B$ at G.

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37 Note: r is an affine parameter defined along g such that $\xi^B \xi^C \nabla_{BC} r = 1$ and $r = \int \Omega^{-2} dr$.

38 $\tilde{\phi}$ is asymptotically regular if ϕ exists throughout $\{M, g_{ab}\}$ and is continuous at I.

39 Due to $\nabla_{AB} \Omega \neq 0$ on I or along g.

$$\therefore \quad \eta^A \xi^B \nabla_{AB} \Omega = 0 \text{ at } G^{40}$$

$$\therefore \quad \eta^A \xi^B \nabla_{AB} \Omega = O(r - r_0)^{41}$$

$$\Rightarrow \quad b = O(\ln \Omega)$$

$$\therefore \quad \tilde{\eta}_A = \Omega^{-1/2} \eta_A + \xi_A O(\Omega^{1/2} \ln \Omega)$$

$$(4.21) \quad \therefore \quad \tilde{\phi}_{A..FG..K} \tilde{\eta}^A .. \tilde{\eta}^F \xi^G .. \xi^K = \Omega^{k-1} \phi_{A. FG..K} \eta^A .. \eta^F \xi^G .. \xi^K + O(\Omega^{k-2} \ln \Omega) .$$

$\phi_{A..K}$ is bounded near G by asymptotic regularity. And, $\Omega^{k-1} r^{-1-k}$ exists at G and is

not zero.

$$\therefore \quad R.H.S. \ of \ (4.21) \ is \ O(r^{-k-1}),$$

$$\therefore \quad \tilde{\phi}_{A..FG..K} \tilde{\eta}^A .. \tilde{\eta}^F \xi^G .. \xi^K = O(r^{-k-1}) .$$

For the particular cases of electromagnetism and gravity there exists an alternate proof which does not require asymptotic regularity, only that the field equations hold in a neighborhood about I and that the conformal factor be at least C^6 on $\{M, g_{ab}, \Omega\}$. This case is examined in detail in section 14 of Penrose 1965.

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40 Since $\xi^A \xi_A = 0$, and $\eta^A \eta_A = 0$.

41 Ω is at least C^2 at G.

5 The Physics of Peeling

The physical implications of Peeling are most readily discussed by examining the dyad components of the field spinors. These components will each have different radial dependence and yield information about the sources and radiative properties of the field. In general for a massless field of spin s there will be 2s + 1 dyad components (complex).

$$(5.1) \qquad \Phi_n = \xi^A_{i_{2s}} \xi^B_{i_{2s-1}} \cdots \xi^K_{i_0} \phi_{AB\ldots K}$$

Where $\xi^A_{i_j}$ is a dyad for the spinor basis such that $\xi^A_0 = \xi^A$, $\xi^A_1 = \eta^A$ and $\xi^A \eta_A = 1$.

With $i_0 \ldots i_n = 1$ and $i_{n-1} \ldots i_{2s} = 0$ $\forall$ $n+1 \le 2s$ such that $n \in \{0, \ldots, 2s\}$. In the cases of gravitation and electromagnetism Ψ_n and ϕ_n are used instead of Φ_n to denote the dyad components.

Applying the Peeling theorem to these components yields the following relation.

$$(5.2) \qquad \Phi_n = O^*(r^{n-2s-1}), \quad n = 0, \ldots, 2s$$

For gravity the components peel as:

$$(5.3) \qquad \Psi_0 = O^*(r^{-5})$$

$$(5.4) \qquad \Psi_1 = O^*(r^{-4})$$

$$(5.5) \qquad \Psi_2 = O^*(r^{-3})$$

$$(5.6) \qquad \Psi_3 = O^*(r^{-2})$$

$$(5.7) \qquad \Psi_4 = O^*(r^{-1}).$$

For electromagnetism the components fall off as:

$$(5.8) \qquad \phi_0 = O^*(r^{-3})$$

$$(5.9) \qquad \phi_1 = O^*(r^{-2})$$

(5.10) $$\phi_2 = O^*(r^{-1}).$$

To prepare for the gravitational analysis let us examine the Maxwell field, whose properties are well understood. Consider the null polar coordinate system for flat space

(5.11) $$ds^2 = du^2 + 2du\,dr - r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

$$\text{with} \quad u = t - r.$$

In this case the spin coefficients (see Appendix II) reduce to

(5.12) $$\rho = -1/r \qquad \alpha = -(1/2\sqrt{2}r)\cot(\theta)$$

$$\beta = (1/2\sqrt{2}r)\cot(\theta) \qquad \mu = -1/2r$$

$$\pi = \kappa = \epsilon = \lambda = \gamma = \nu = \tau = \sigma = 0.$$

The source-free Maxwell equations become

(5.13) $$(\partial/\partial r + 2/r)\phi_1 = (\sqrt{2}r)^{-1}\bar{\eth}\phi_0$$

(5.14) $$(\partial/\partial r + r^{-1})\phi_2 = (\sqrt{2}r)^{-1}\bar{\eth}\phi_1$$

(5.15) $$(\partial/\partial u - 1/2\,\partial/\partial r - (2r)^{-1})\phi_0 = (\sqrt{2}r)^{-1}\eth\phi_1$$

(5.16) $$(\partial/\partial u - 1/2\,\partial/\partial r - r^{-1})\phi_1 = (\sqrt{2}r)^{-1}\eth\phi_2.$$

A solution[42] may be obtained as follows. Let ϕ_0 be given as a function of r, θ and ϕ on a null surface $u = u_0$. Integrating equations (5.13) and (5.14) produces

(5.17) $$\phi_0 = \phi_0(r, \theta, \phi)$$

(5.18) $$\phi_1 = r^{-2}\phi_1^0(\theta, \phi) + r^{-2}\int (r/\sqrt{2})\bar{\eth}\phi_0\,dr$$

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42 Page 904, Janis and Newman 1965, slightly different notation.

(5.19) $\qquad \phi_2 = r^{-1}\phi_2^0(0,\phi) + r^{-1}\int \bar{\delta}(\phi_1/\sqrt{2})dr$

ϕ_1^0 and ϕ_2^0 are functions of integration.

The u dependence of ϕ_0 and ϕ_1^0 may be determined by substituting equations (5.17), (5.18) and (5.19) into equations (5.15) and (5.16). ϕ_2^0 is usually called a news function[43] due to the fact that it is the radiative initial data and determines the u dependence of ϕ_0 and ϕ_1^0. Three solutions are given below in the null polar coordinate system[44] as examples.

Monopole field:

$$\phi_0 = \phi_2 = 0$$

$$\phi_1 = a_0 r^{-2} \qquad a_0 = \text{constant}$$

Dipole field:

$$\phi_0 = a_1 \sin(\theta) r^{-3}$$

$$\phi_1 = -\sqrt{2}\dot{a}_1 \cos(\theta) r^{-2} - \sqrt{2} a_1 \cos(\theta) r^{-3}$$

$$\phi_2 = -\ddot{a}_1 \sin(\theta) r^{-1} - \dot{a}_1 \sin(\theta) r^{-2} - a_1 \sin(\theta) 2^{-1} r^{-3}$$

$$a_1 = a_1(u) \in C^2 \quad \text{with} \quad u = t - r.$$

Quadrupole field:

$$\phi_0 = \dot{a}_2 \sin(\theta)\cos(\theta) 2^{-1} r^{-3} + a_2 \sin(\theta)\cos(\theta) r^{-4}$$

$$\phi_1 = -\ddot{a}_2(3\cos^2(\theta)-1)(6\sqrt{2})^{-1} r^{-2} - \dot{a}_2(3\cos^2(\theta)-1)(2\sqrt{2})^{-1} r^{-3} - a_2(3\cos^2(\theta)-1)(2\sqrt{2})^{-1} r^{-4}$$

$$\phi_2 = -\ddot{a}_2 \sin(\theta)\cos(\theta) 6^{-1} r^{-1} - \dot{a}_2 \sin(\theta)\cos(\theta) 2^{-1} r^{-2}$$

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43 Page 905, Janis and Newman 1965.
44 Page 906, Janis and Newman 1965.

$$-(3/4)\dot{a}_2\sin(0)\cos(0)r^{-3}-(1/2)a_2\sin(0)\cos(0)r^{-4}$$

$$a_2 = a_2(u)\epsilon C^3 \quad \text{with} \quad u = t - r$$

In the above examples the real part of a_i ($i = 0, 1, 2$) is proportional to the electric charge, dipole or quadrupole moment for $i = 0, 1$ or 2 respectively. The imaginary part is proportional to the magnetic charge, dipole or quadrupole moment for $i = 0, 1$ or 2 respectively. Notice as well that the fields do indeed obey the Peeling theorem.

For the case of a general Maxwell field the expression for total charge may be given by the following. Let S denote a sphere of constant radius R on a null cone of constant retarded time u. Then

(5.20) $$electric \quad charge \quad \sim \quad \lim_{R \to \infty} \int_S Re(\phi_1)ds,$$

(5.21) $$magnetic \quad charge \quad \sim \quad \lim_{R \to \infty} \int_S Im(\phi_1)ds.$$

The proportionality may be made exact once units are chosen. Similarly, the radiative power of the field is given by[45]

(5.22) $$rate \quad of \quad energy \quad emmission \quad \sim \quad \lim_{R \to \infty} \int_S \phi_2\bar{\phi}_2 ds.$$

To examine the gravitational case we shall use a null foliation for curved spacetime. Following Janis and Newman (1965) a null tetrad is defined as follows: consider a null hypersurface u = constant. Then the first null vector may be taken as

(5.23) $$l_\mu = u_{,\mu} = \delta_\mu^0 \quad (u = x^0)$$

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45 Page 103, Norman 1964.

Choose the other three null vectors to obey the orthogonality conditions as presented in Appendix II. Letting U and X^i be arbitrary real functions and, ω and ξ^i be arbitrary complex functions (i = 2,3). The null tetrad may then be written as follows:

$$(5.24) \qquad\qquad l^\mu = \delta_0^\mu \, ,$$

$$(5.25) \qquad\qquad n^\mu = \delta_0^\mu + U \delta_1^\mu + X^i \delta_i^\mu \, ,$$

$$(5.26) \qquad\qquad m^\mu = \omega \delta_1^\mu + \xi^i \delta_i^\mu \, ,$$

$$(5.27) \qquad\qquad \overline{m}^\mu = \overline{\omega} \delta_1^\mu + \overline{\xi}^i \delta_i^\mu \, .$$

For this system the intrinsic derivatives take the form:

$$(5.28, .29) \qquad D = \partial/\partial r, \qquad \delta = \omega \partial/\partial r + \xi^i \partial/\partial x^i \, ,$$

$$(5.30) \qquad \Delta = U \partial/\partial r + \partial/\partial u + X^i \partial/\partial x^i .$$

The spin coefficients are also simplified:

$$(5.31) \qquad\qquad \pi = \epsilon = \kappa = 0 \, ,$$

$$(5.32) \qquad\qquad \rho = \overline{\rho} \, , \qquad \tau = \overline{\alpha} + \beta \, .$$

The field equations may then be written as follows[46]:

$$(5.33, .34) \qquad D\xi^i = \rho \xi^i + \sigma \overline{\xi}^i \qquad DX^i = \tau \overline{\xi}^i + \overline{\tau}\xi^i$$

$$(5.35, .36) \qquad D\rho = \rho^2 + \sigma\overline{\sigma} \qquad D\sigma = 2\rho\sigma + \Psi_0$$

$$(5.37, .38) \qquad D\alpha = \alpha\rho + \beta\overline{\sigma} \qquad D\lambda = \lambda\rho + \mu\overline{\sigma}$$

$$(5.39, .40) \qquad D\omega = \rho\omega + \sigma\overline{\omega} - \tau \qquad DU = \tau\overline{\omega} + \overline{\tau}\omega - (\gamma + \overline{\gamma})$$

$$(5.41, .42) \qquad D\beta = \beta\rho + \alpha\sigma + \Psi_1 \qquad D\gamma = \tau\alpha + \overline{\tau}\beta + \Psi_2$$

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46 Pages 893 and 894, Newman and Unti 1962.

$$(5.43, .44) \qquad D\mu = \mu\rho + \lambda\sigma + \Psi_2 \qquad D\nu = \tau\lambda + \bar{\tau}\mu + \Psi_3$$

$$(5.45) \qquad D\Psi_1 - \bar{\delta}\Psi_0 = 4\rho\Psi_1 - 4\alpha\Psi_0$$

$$(5.46) \qquad D\Psi_2 - \bar{\delta}\Psi_1 = 3\rho\Psi_2 - 2\alpha\Psi_1 - \lambda\Psi_0$$

$$(5.47) \qquad D\Psi_3 - \bar{\delta}\Psi_2 = 2\rho\Psi_3 - 2\lambda\Psi_1$$

$$(5.48) \qquad D\Psi_4 - \bar{\delta}\Psi_3 = \rho\Psi_4 + 2\alpha\Psi_3 - 3\lambda\Psi_2$$

$$(5.49) \qquad \Delta\Psi_0 - \delta\Psi_1 = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2$$

$$(5.50) \qquad \Delta\Psi_1 - \delta\Psi_2 = \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3$$

$$(5.51) \qquad \Delta\Psi_2 - \delta\Psi_3 = 2\nu\Psi_1 - 3\mu\Psi_2 - 2\bar{\alpha}\Psi_3 + \sigma\Psi_4$$

$$(5.52) \qquad \Delta\Psi_3 - \delta\Psi_4 = 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 - (\tau - 4\beta)\Psi_4$$

$$(5.53) \qquad \delta X^i - \Delta\xi^i = (\mu + \bar{\gamma} - \gamma)\xi^i + \bar{\lambda}\bar{\xi}^i$$

$$(5.54) \qquad \delta\bar{\xi}^i - \bar{\delta}\xi^i = (\bar{\beta} - \alpha)\xi^i + (\bar{\alpha} - \beta)\bar{\xi}^i$$

$$(5.55) \qquad \delta\bar{\omega} - \bar{\delta}\omega = (\bar{\beta} - \alpha)\omega + (\bar{\alpha} - \beta)\bar{\omega} + \mu - \bar{\mu}$$

$$(5.56) \qquad \delta U - \Delta\omega = (\mu + \bar{\gamma} - \gamma)\omega + \bar{\lambda}\bar{\omega} - \bar{\nu}$$

$$(5.57) \qquad \Delta\lambda - \bar{\delta}\nu = 2\alpha\nu + (\bar{\gamma} - 3\gamma - \mu - \bar{\mu})\lambda - \Psi_4$$

$$(5.58) \qquad \delta\rho - \bar{\delta}\sigma = \tau\rho + (\bar{\beta} - 3\alpha)\sigma - \Psi_1$$

$$(5.59) \qquad \delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma - 2\alpha\beta + \alpha\bar{\alpha} + \beta\bar{\beta} - \Psi_2$$

$$(5.60) \qquad \delta\lambda - \bar{\delta}\mu = \bar{\tau}\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3$$

$$(5.61) \qquad \delta\nu - \Delta\mu = \gamma\mu - 2\nu\beta + \bar{\gamma}\mu + \mu^2 + \lambda\bar{\lambda}$$

$$(5.62) \qquad \delta\gamma - \Delta\beta = \tau\mu - \sigma\nu + (\mu - \gamma + \bar{\gamma})\beta + \bar{\lambda}\alpha$$

$$(5.63) \qquad \delta\tau - \Delta\sigma = 2\tau\beta + (\bar{\gamma} + \mu - 3\gamma)\sigma + \bar{\lambda}\rho$$

$$(5.64) \qquad \Delta\rho - \bar{\delta}\tau = (\gamma + \bar{\gamma} - \bar{\mu})\rho - 2\alpha\tau - \lambda\sigma - \Psi_2$$

$$(5.65) \qquad \Delta\alpha - \bar{\delta}\gamma = \rho\nu - \tau\lambda - \lambda\beta + (\bar{\gamma} - \gamma - \bar{\mu})\alpha - \Psi_3$$

Assuming that Peeling holds each component may be expanded as follows[47]:

$$(5.66) \qquad \Psi_n = \Psi_n^0 r^{-5-n} + O(r^{-6-n}) \qquad n \in \{0, 1, 2, 3, 4\}.$$

To obtain a unique solution there must be given 5 pieces of initial data [48].

(1) Ψ_0 given on a surface u = constant.

(2) $\sigma^0 = \lim_{r \to \infty} r^2 \sigma$ on a time-like world tube at spatial infinity.

(3) $\Psi_1^0 = \lim_{r \to \infty} r^4 \Psi_1$ on the intersection of the u = constant surface and the

time-like world tube for part (2).

(4) $Re(\Psi_2^0) = \lim_{r \to \infty} r^3 Re(\Psi_2)$ on the same surface as for part (3).

(5) The purely spatial components of the metric on the same surface as for part (3).

Consider the initial data given by part (2). For gravity the news function is σ^0. As in the electromagnetic case the news will determine the radiative aspects of the field.

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47 For linearized gravity we may write, $\Psi_n = \sum_{i=0}^{\infty} \Psi_n^i r^{-n-5-i}$ with the multipole moments being given by further expanding the Ψ_0^i into the spherical harmonics.

48 Page 909, Janis and Newman 1965.

The physical meaning of the dyad components for gravity have the following interpretations[49].

$$\Psi_4 \quad \text{and} \quad \Psi_3 \;\rightarrow\; \text{Gravitational radiation,}$$

$$\Psi_2 \;\rightarrow\; \text{Momentum and intermediate zone radiation,}$$

$$\Psi_1 \;\rightarrow\; \text{Angular momentum and near zone radiation,}$$

$$\Psi_0 \;\rightarrow\; \text{Quadrupole and higher moments.}$$

The derivatives of Ψ_0 produce the higher multipole moments of the system. The news function determines the radiation as follows: consider the first components of the expansions for $\Psi_{2,3,4}$. Then, the following relationships may be produced[50]:

$$(5.67) \qquad Im(\Psi_2^0) = Im(\eth^2 \bar{\sigma}^0) + Im(\bar{\sigma}^0 \dot{\sigma}^0),$$

$$(5.68) \qquad \Psi_3^0 = \eth \dot{\bar{\sigma}}^0,$$

$$(5.69) \qquad \Psi_4^0 = -\ddot{\bar{\sigma}}^0.$$

Correspondingly, the news appears in the mass-loss equation. The Bondi mass of the system is given by

$$(5.70) \qquad M = -(8\pi\sqrt{2})^{-1} \int_s (\Psi_2^0 + \sigma^0 \dot{\bar{\sigma}}^0) ds.$$

Taking the derivative with respect to u (the retarded time) and then using equations (5.67)-(5.69) yields the mass-loss equation.

$$(5.71) \qquad \dot{M} = -(8\pi\sqrt{2})^{-1} \int_s \dot{\sigma}^0 \dot{\bar{\sigma}}^0 ds$$

49 Pages 302 and 303, Esposito and Witten 1977.
50 Page 454, Bramson 1975.

Another interesting property of space-times which Peel is that of the Newman-Penrose (NP) constants. These constants were first reported by Newman and Penrose in 1965. They found that for asymptotically flat space-times there exist quantities, expressed as integrals on any null hypersurface at infinity, which are absolutely conserved. In flat space-times the number of constants is infinite. In curved space-times there are precisely 2s + 1 complex constants for a field of spin s. The existence of these constants leads to an interesting property for asymptotically flat space-times. A stationary field cannot become non-stationary and then stationary in a finite period of time unless the Newman-Penrose constants return to their original values[51].

To obtain the Newman-Penrose constants let us assume Peeling and that the Weyl components are reasonably smooth so that

$$(5.72) \qquad \Psi_0 = \Psi_0^0 r^{-5} + \Psi_0^1 r^{-6} + \Psi_0^2 r^{-7} + O(r^{-8}) \ ,$$

$$(5.73) \qquad \Psi_1 = \Psi_1^0 r^{-4} + O(r^{-5}) \ ,$$

$$(5.74) \qquad \Psi_2 = \Psi_2^0 r^{-3} + O(r^{-4}) \ ,$$

$$(5.75) \qquad \Psi_3 = \Psi_3^0 r^{-2} + O(r^{-3}) \ ,$$

$$(5.76) \qquad \Psi_4 = \Psi_4^0 r^{-1} + O(r^{-2}) \ ,$$

$$(5.77) \qquad \sigma = \sigma^0 r^{-2} + O(r^{-4}).$$

Applying the Bianchi identity yields the following[52]:

$$(5.78) \qquad \dot{\Psi}_3^0 = -\eth \Psi_4^0 \ ,$$

$$(5.79) \qquad \dot{\Psi}_2^0 = -\eth \Psi_3^0 + \sigma^0 \Psi_4^0 \ ,$$

$$(5.80) \qquad \dot{\Psi}_1^0 = -\eth \Psi_2^0 + 2\sigma^0 \Psi_3^0 \ ,$$

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51 Page 179, Newman and Penrose 1968.
52 Page 188, Newman and Penrose 1968.

(5.81) $\qquad$ $\Psi_0^0 = -\eth' \Psi_1^0 + 3\sigma^0 \Psi_2^0$.

Consider the following equation, where the integration is carried out over a null hypersurface.

(5.82) $\qquad$ $G_m = \int_S {}_2\overline{Y}_{2m} \Psi_0^1 ds$ $\qquad$ $m \in \{-2,-1,0,1,2\}$

The quantities G_m will be absolutely conserved if $\dot{G}_m = 0$.

$$\dot{G}_m = \int_S {}_2\overline{Y}_{2m} \dot{\Psi}_0^1 ds$$

$$\dot{\Psi}_0^1 = -\eth'(\eth' \Psi_0^0 + 4\sigma^0 \Psi_1^0)$$

$$\therefore \quad \dot{G}_m = -\int_S {}_2\overline{Y}_{2s} \eth'\{\eth' \Psi_0^0 + 4\sigma^0 \Psi_1^0\} ds$$

The quantity within the brackets in the integrand has spin weight 3, therefore from equation (A4.16) we have $\dot{G}_m = 0$. Hence the G_m are absolutely conserved.

The similar manipulation of equations (5.78), (5.79) and (5.81) does not provide any further conserved quantities. Hence it is believed that the G_m are the only conserved quantities for gravity[53]. Similar quantities may be derived for the neutrino (N_m) and electromagnetic (F_m) fields.

(5.83) $\qquad$ $N_m = \int_S {}_{1/2}\overline{Y}_{1/2m} v_0^1 ds$ $\qquad$ $m \in \{-1/2, 1/2\}$

with $\quad v_0 = v_0^0 r^{-2} + v_0^1 r^{-3} + O(r^{-4})$

v_0 = dyad component of the neutrino field.

(5.84) $\qquad$ $F_m = \int_S {}_1\overline{Y}_{1m} \phi_0^1 ds$ $\qquad$ $m \in \{-1,0,1\}$

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53 Page 189, Newman and Penrose 1968.

When all three fields are present the N_m and F_m are the same as given above. However, the gravitational constants receive contributions from the neutrino and electromagnetic fields. This amounts to extra terms in the integrand for G_m. The total number of conserved quantities is cumulative for each additional field. Several authors have attempted to shed light on the physical meaning of the constants with limited success, cf. Carmeli 1969, Robinson 1969, Exton et. al. 1969 and, Glass and Goldberg 1970. The NP constants remain an interesting and outstanding problem in gravitational physics.

6 Asymptotic Flatness without Peeling

From the discussion of the previous chapter it is clear that Peeling of the gravitational field is a sufficient condition to ensure asymptotic flatness. Couch and Torrence (1972) have shown that an even weaker version of Peeling will still provide asymptotic flatness. They suggested the following modifications[54].

$$(6.1) \qquad \Psi_0 = O^*(r^{-2-\epsilon_0})$$

$$(6.2) \qquad \Psi_1 = O(r^{-2-\epsilon_1})$$

$$(6.3) \qquad \Psi_2 = O(r^{-2-\epsilon_2})$$

$$(6.4) \qquad \Psi_3 = O(r^{-2})$$

$$(6.5) \qquad \Psi_4 = O(r^{-1})$$

$$\text{with} \quad \epsilon_0 > 0 \quad \epsilon_0 > \epsilon_1 > \epsilon_2$$

$$\text{and} \quad \epsilon_1 \leq 2 \quad \epsilon_2 \leq 1 \, .$$

Thus Ψ_3 and Ψ_4 obey the Peeling theorem but Ψ_0, Ψ_1, and Ψ_2 are allowed to fall off much slower than Peeling requires. Physically the modifications allow for incoming radiation which dies off as $t \to \infty$. The solutions of Couch and Torrence are however not well defined. Indeed, only the radial dependance has been explicitly determined and the Winicour-Tamburino energy-momentum and angular momentum integrals[55] diverge[56]. The asymptotic symmetry group can still, however, be shown to be the BMS group[57].

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54 Page 69, Couch and Torrence 1972.

55 Complete discussion and derivation of the integrals is given in Tamburino and Winicour 1966 and Winicour 1968.

56 Page 79, Novak and Goldberg 1981.

57 Page 657, Novak and Goldberg 1982.

Novak and Goldberg (1981 and 1982) used a slightly stronger version of weakened Peeling in which they regarded the gravitational components to obey the following[58]:

$$(6.6) \qquad \Psi_0 = O(r^{-3-\epsilon_0})$$

$$(6.7) \qquad \Psi_1 = O(r^{-3-\epsilon_1})$$

$$\epsilon_1 < \epsilon_0 \quad \text{and} \quad \epsilon_1 \leq 1$$

$$\text{with} \quad \Psi_2, \Psi_3 \quad \text{and} \quad \Psi_4 \quad \text{Peeling.}$$

These solutions may be shown to be well defined and having finite expressions for the energy-momentum and angular momentum integrals (and are in fact the same as those for Peeling space-times) as well as having the BMS group as the asymptotic symmetry group (this group structure is to be expected since the Novak-Goldberg solutions are a subclass of the Couch-Torrence solutions). They may also be shown to be the weakest possible solution for finite outgoing radiation at future null infinity[59]. Given Ψ_0 the solutions may be written as follows[60]:

$$(6.8) \qquad \Psi_0 = O(r^{-3-\epsilon_0}),$$

$$(6.9) \qquad \Psi_1 = \eth A + \Psi_1^0/r^4 + O(r^{-4-\epsilon_0}),$$

$$(6.10) \qquad \Psi_2 = \Psi_2^0/r^3 + O(r^{-3-\epsilon_1}),$$

$$(6.11) \qquad \Psi_3 = \Psi_3^0/r^2 - \bar{\eth}\Psi_2^0/r^3 + O(r^{-3-\epsilon_1}),$$

$$(6.12) \qquad \Psi_4 = (\bar{\eth}^2\Psi_2^0 + \bar{\sigma}^0\eth\Psi_3^0 + \sigma^0\bar{\sigma}^0\Psi_4^0 + 3\lambda^0\Psi_2^0 + 2\bar{\omega}^0\Psi_3^0)/(2r^3),$$

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58 Page 81, Novak and Goldberg 1981.

59 Page 95, Novak and Goldberg 1981.

60 Page 81, Novak and Goldberg 1981.

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$$+ \Psi_4^0/r - \bar{\eth}\Psi_3^0/r^2 + O(r^{-3-\epsilon_1}).$$

Where the necessary spin coefficients and functions are defined by,

(6.13)
$$A = r^{-4}\int r^3 \Psi_0 dr = O(r^{-3-\epsilon_1})$$

(6.14)
$$B = -r^{-1}\int(\sigma - \sigma^0 r^{-1})dr = O(r^{-2-\epsilon_0})$$

(6.15)
$$C = r^{-1}\int r A dr = O(r^{-2-\epsilon_1})$$

(6.16)
$$D = -r^{-1}\int r C dr = O(r^{-1-\epsilon_1})$$

(6.17)
$$\mu = \mu^0/r + \sigma^0\lambda^0/r^2 - \Psi_2^0/r^2 + O(r^{-2-\epsilon_1})$$

(6.18)
$$\omega = \omega^0/r + \bar{\eth}D - (\sigma^0\bar{\omega}^0 + (1/2)\Psi_1^0)/r^2 + O(r^{-2-\epsilon_1})$$

(6.19)
$$\lambda = \lambda^0/r - \bar{\sigma}^0\mu^0/r^2 + O(r^{-2-\epsilon_0})$$

(6.20)
$$\sigma = \sigma^0/r^2 + O(r^{-2-\epsilon_0})$$

(6.21)
$$\rho = -r^{-1} - \bar{\sigma}^0\sigma^0 r^{-3} + O(r^{-3-\epsilon_0})$$

(6.22)
$$\alpha = \alpha^0 r^{-1} + \bar{\sigma}^0\bar{\alpha}^0 r^{-2} + \bar{B}\bar{\alpha}^0 + \bar{\sigma}^0\sigma^0\alpha^0 r^{-3} + O(r^{-3-\epsilon_1})$$

(6.23)
$$\beta = -\bar{\alpha}^0 r^{-1}\sigma^0\alpha^0 r^{-2} - B\alpha^0 - \sigma^0\bar{\sigma}^0\bar{\alpha}^0 r^{-3} + \bar{\eth}C - 1/2r^{-3}\Psi_1^0 + O(r^{-3-\epsilon_1})$$

(6.24)
$$\tau = \bar{\eth}C - - 1/2r^{-3}\Psi_1^0 + O(r^{-3-\epsilon_1})$$

(6.25)
$$\gamma = \gamma^0 - 1/2\Psi_2^1 r^{-2} + O(r^{-2-\epsilon_1})$$

(6.26)
$$\nu = \nu^0 - r^{-1}\Psi_3^0 + 1/2r^{-2}\bar{\eth}\Psi_2^0 + O(r^{-2-\epsilon_1})$$

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That the solution is well defined may be demonstrated by showing that the non-radial components (from the field equations) produce a consistent set of equations with proper time dependence. To obtain a solution the same initial data may be used as for peeling space-times (see previous chapter). It is possible to produce time evolution equations for the coefficients of the 'dyad' components similar to those given for peeling spaces.

(6.27)
$$\Psi_0 = O(r^{-4-\epsilon_1})$$

(6.28)
$$\Psi_1 = \Psi_1^0 r^{-1} + O(r^{-4-\epsilon_1})$$

(6.29)
$$\dot{\Psi}_1^0 = \eth\Psi_2^0 - 2\sigma^0\Psi_3^0$$

(6.30)
$$\dot{\Psi}_2^0 = \eth\Psi_3^0 - \sigma^0\Psi_4^0$$

(6.31)
$$\dot{\lambda}^0 = \Psi_4^0$$

(6.32)
$$Im(\dot{\Psi}_2^0) = -Im(\sigma^0\lambda^0) + Im(\eth\bar{\omega}^0)$$

(6.33)
$$\Psi_3^0 = -\eth\lambda^0 + \bar{\eth}\mu^0$$

These equations do not appear to produce any quantities similar to the NP constants[61]. Physically this does seem reasonable since the weakened Peeling may be associated with incoming radiation fields (which would imply the NP constants are continually changing), whereas the NP constants are derived under the assumption of no incoming radiation.

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61 There does not appear to be any such result in the literature, as well, the author has made several attempts to produce similar constants with no success.

7 Conclusion

The aim of this tthesis was to give a discussion of the properties of asymptotically flat space-times and the implications of peeling. We began with Penrose's definition of asymptotic flatness. This lead to an investigation of the properties of the boundary manifold I. It was found that this manifold has many interesting and 'non-standard' geometrical properties due to its non-invertible metric. From this structure the BMS group emerged. It is interesting to note that when the BMS group was first investigated there was hope for a connection with particle physics. There are two main reasons for this: (1) the group structure is metric independent hence there was hope of using it to create an S-matrix theory for gravitational waves[62]; (2) both the Poincare and the BMS group are the semidirect product of the proper orthochronus Lorentz group with an Abelian group.

Spinor representation of massless fields were also examined. It is striking that classically all massless fields (of integer and half integer spin s) may be represented by the same equation, namely:

$$\nabla^{AA'} \phi_{A..Q} = 0$$

with $\phi_{A..Q}$ having 2s indicies.

These fields were also shown to have very similar asymptotic behavior. Namely that they peel and have NP constants.

Asymptotic flatness without peeling was also briefly examined. In this section it was found that the BMS group was still the asymptotic symmetry group (for both cases presented) and that the Novak-Goldberg solution shared many of the same properties of peeling spaces with the differences attributed to the presence of incoming fields.

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62 Page 2863, Sachs 1962.

8 Appendix I   Summary of Spinor Calculus

This appendix provides a brief summary of spinor calculus relevant to the text. A more thorough treatment can be found in Pirani 1965, Penrose and Rindler 1984 and Benn and Tucker 1987.

Definition (A1.1) A spinor is an element of the vector space defined by the group of unimodular 2x2 matrices over C (hence summations are over 0,1 rather than 0,1,2,3). It should also be noted that as a group, spinors form a double valued representation of the proper homogeneous Lorentz group.

Spinors will be written using Greek letters with Latin indices (standard notation in the literature)

$$1\text{-spinor} \quad \xi^A, \eta^B,$$

$$2\text{-spinor} \quad \xi^{AB}, \eta^{BC}.$$

Complex conjugation will be denoted as

(A1.1)
$$\overline{\eta^A} = \bar{\eta}^A.$$

Spinor indices may be raised or lowered by use of the Levi-Civita symbol

(A1.2)
$$\epsilon^{AB}\xi_B = \xi^A$$

(A1.3)
$$[\epsilon^{AB}] = [\epsilon_{AB}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Since the metric is antisymmetric care should be taken with regard to the position of the indices which are being summed over (think of an arrow pointing down and to the right)

(A1.4)
$$\xi^B \eta_B = \epsilon^{BA}\xi_A \eta_B$$

$$= -\epsilon^{AB}\xi_A \eta_B$$

$$= -\xi_A \eta^A$$

$$= -\xi_B \eta^B.$$

Theorem (A1.1)[63] Let ξ_{AB} be any 2-spinor then

(A1.5)
$$\xi_{AB} = \xi_{(AB)} + 1/2\epsilon_{AB}\xi_C{}^C.$$

Proof: We begin with the identity, $\epsilon_{A[B}\epsilon_{CD]} = 0$

$$\epsilon_{AB}\epsilon_{CD} + \epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{BC} = 0$$

$$\epsilon_{AB}\epsilon^{CD} = \delta_A{}^C\delta_B{}^D - \delta_A{}^D\delta_B{}^C$$

$$\epsilon_{AB}\epsilon^{CD}\xi_{CD} = \delta_A{}^C\delta_B{}^D\xi_{CD} - \delta_A{}^D\delta_B{}^C\xi_{CD}$$

$$\epsilon_{AB}\xi_C{}^C = \xi_{AB} - \xi_{BA}$$

$$\therefore \xi_{[AB]} = 1/2\epsilon_{AB}\xi_C{}^C$$

$$\therefore \xi_{AB} = \xi_{(AB)} + 1/2\epsilon_{AB}\xi_C{}^C.$$

This is an important result used extensively in Spinor Calculus.

Theorem (A1.2) Let ξ^A and η^A be arbitrary 1-spinors. Then ξ^A is proportional

to η^A if and only if their inner product vanishes.

Proof: Define a 2-spinor $\beta^{AB} = \xi^A\eta^B$ and apply theorem (A1.1). The desired result

follows immediately.

Theorem (A1.3) Let ξ^A and η^A be 1-spinors such that $\xi_A\eta^A = 1$ then

(A1.6)
$$\epsilon_{AB} = \xi_A\eta_B - \eta_A\xi_B$$

and

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63 Page 310, Pirani 1965.

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(A1.7) $$\delta^B{}_A = \xi_A \eta^C - \eta_A \xi^B.$$

Proof: Equation (A1.6) follows immediately from theorem (A1.1). From equation (A1.6) we have

$$C_{AC} = \xi_A \eta_C - \eta_A \xi_C$$

$$c^{BC} C_{AC} = c^{BC} \xi_A \eta_C - c^{BC} \eta_A \xi_C$$

$$\delta^B{}_A = \xi_A \eta^B - \eta_A \xi^B.$$

Theorem (A1.4)[64] Let $\xi_{AB..K}$ and η^A be arbitrary spinors. Then $\xi_{AB..K} \eta^K = 0$

if and only if $\xi_{AB..JK} = \lambda_{AB..J} \eta^K$ for some $\lambda_{AB..J}$.

Proof: $\Rightarrow$

Recall $\phi_{AB} - \phi_{BA} = \epsilon_{AB} \phi_C{}^C$, thus $\xi_{AB..J} \eta_L - \xi_{AB..L} \eta_J = 0$. Since this is true

$\forall J, K \in \{0, 1\}$ with $\eta_L \neq 0$ and $\xi_{AB..J} \neq 0$, it must therefore be concluded that ξ is

proportional to η in the last index. Hence we may decompose ξ into a product of

two spinors, the last of which would be η.

$\Leftarrow$

$$\xi_{AB..JK} = \lambda_{AB..J} \eta_K$$

$$\Rightarrow \xi_{AB..JK} \eta^K = \lambda_{AB..J} \eta_K \eta^K$$

However $\eta_K \eta^K = 0$ by theorem (A1.2). Therefore $\xi_{AB..JK} \eta^K = 0$ and the theorem is

proved.

The above theorem may now be used to prove a much more general result.

Theorem (A1.5)[65] Every spinor may be decomposed into a sum of products of

1-spinors.

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64 Page 311, Pirani 1965.
65 Page 311, Pirani 1965.

Proof: Without loss of generality we need only consider a spinor whose indices are all covariant. In this case there will only be one product of 1-spinors as the final result. Let us consider an arbitrary n-spinor $\phi_{AB..JK}$ which is not identically zero. Then a 1-spinor ξ_L may be found such that it is proportional to $\phi_{AB..JK}$ in the last index (a solution of $\phi_{AB..JK}\xi^K = 0$). If the previous theorem is now applied we can decompose ϕ such that,

$$\phi_{AB..JK} = \lambda_{AB..J}\xi_K \text{ for some } \lambda.$$

The above process is then repeated on the successive λ's until we arrive at a product of n 1-spinors (the process will terminate as n is finite). Thus $\phi_{AB..JK}$ has been completely decomposed into a product of 1-spinors.

The connection between spinors and tensors will now be discussed. The mapping between them is denoted by $\sigma_a{}^{BX}$. In flat space the following may be used[66]:

$$\sigma_1{}^{BX} = 1/\sqrt{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2{}^{BX} = 1/\sqrt{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma_3{}^{BX} = 1/\sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; \sigma_0{}^{BX} = 1/\sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The tensor indices maybe raised or lowered with the usual space-time metric and the spinor indices may be raised or lowered with the Levi-Civita symbol. The mapping is applied below

(A1.8) $$\xi^{AW'} = \sigma_a{}^{AW'}\xi^a,$$

(A1.9) $$\eta^{ABW'X} = \sigma_a{}^{AW'}\sigma_b{}^{BX}\eta^{ab}.$$

The inverse mapping is given by

(A1.10) $$\xi^a = \xi^{AX}\sigma^a{}_{AX}.$$

The metric tensor is easily seen to correspond with the Levi-Civita symbol

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66 Pages 305 and 309, Pirani 1965.

(A1.11)
$$g_{ab} <-> \epsilon_{AB}\epsilon_{W'X'}$$

(A1.12)
$$\delta^a{}_b <-> \delta^A{}_B\delta^{W'}{}_{X'}$$

Theorem (A1.6)[67] $\quad \xi^a \xi_a = 2 \det(\xi^{AX'})$.

Proof:
$$\xi^a \xi_a = \xi^a \xi^b g_{ab}$$

$$= \sigma^a{}_{BX'}\xi^{BX'}\sigma^b{}_{DW'}\xi^{DW'}g_{ab}$$

$$= \sigma_b{}^{CY'}\epsilon_{CB}\epsilon_{X'Y'}\xi^{BX'}\sigma^b{}_{DW'}\xi^{DW'}$$

$$= \delta^C{}_D\delta^{Y'}{}_{W'}\epsilon_{CB}\epsilon_{X'Y'}\xi^{BX'}\xi^{DW'}$$

$$= \epsilon_{DB}\epsilon_{X'W'}\xi^{BX'}\xi^{DW'}$$

(A1.13)
$$= 2\det(\xi^{BX'})$$

Thus if ξ^a is real and null the following simplification can be made[68]

(A1.14)
$$\xi^a <-> \pm \xi^A \bar{\xi}^{X'} .$$

This is due to the fact that $\det(\xi^{BX'}) = 0$ requires $\xi^{BX'}$ to be singular. This has an interesting geometrical interpretation which is discussed at some length in Pirani 1965.

Covariant differentiation may be defined for spinors as follows[69]:

(A1.15)
$$\nabla_a(\xi^{AX'\cdots} + \eta^{AX'\cdots}) = \nabla_a\xi^{AX'\cdots} + \nabla_a\eta^{AX'\cdots},$$

(A1.16)
$$\nabla_a(\xi^{AX'\cdots}\eta^{BY'\cdots}) = (\nabla_a\xi^{AX'\cdots})\eta^{BY'\cdots} + \xi^{AX'\cdots}(\nabla_a\eta^{BY'\cdots}),$$

(A1.17)
$$\nabla_a\phi = \partial_a\phi,$$

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67 Page 313, Pirani 1965.
68 Page 313, Pirani 1965.
69 Page 324, Pirani 1965.

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$$(A1.18) \qquad \overline{\nabla_a \xi^{AX}} = \nabla_a \overline{\xi^{AX}},$$

$$(A1.19) \qquad \nabla_a \epsilon_{AB} = \nabla_a \sigma_b{}^{BX} = 0,$$

$$(A1.20) \qquad \nabla_a \xi_A = \partial_a \xi_A - \Gamma_a{}^B{}_A \xi_B,$$

$$(A1.21) \qquad \Gamma_a{}^B{}_A = 1/2 \sigma_d{}^{BX}(\sigma^b{}_{AX}\Gamma^d{}_{ab} + \partial_a \sigma^d{}_{AX}) .$$

As an example electromagnetism will be examined in spinor form. The electromagnetic field tensor is defined by[70]

$$(A1.22) \qquad F_{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

which is antisymmetric. Transforming to a spinor yields

$$(A1.23) \qquad F_{ABWX} = \sigma^a{}_{AW}\sigma^b{}_{BX}F_{ab} .$$

If a symmetric 2-spinor is defined such that[71]

$$(A1.24) \qquad \phi_{AB} = 1/2 F_{BAW}{}^{W}$$

we may then write

$$(A1.25) \qquad F_{ABWX} = \epsilon_{AB}\overline{\phi}_{WX} + \phi_{AB}\epsilon_{WX}.$$

Note that there are 6 real components in the tensor and 3 complex components in the spinor. Thus a real bivector defines a symmetric 2-spinor and conversely.

A further simplification can be effected if we consider a linear combination of the electromagnetic field tensor and its dual. With the dual being defined as

$$(A1.26) \qquad F^*{}_{ab} = 1/2 \epsilon_{ad}{}^{bc}F_{bc}.$$

This tensor has the spinor correspondence

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70 Page 74, Misner, Thorne and Wheeler 1973.

71 Page 313 Pirani 1965.

(A1.27) $\quad F^{*}{}_{ab} <-> i(\epsilon_{AB}\bar{\phi}_{W'X'} - \phi_{AB}\epsilon_{W'X'}).$

Taking a linear combination between the field tensor and it's dual the following properties are obtained:

(A1.28) $\quad \mathcal{F}_{ab} = F_{ab} + iF^{*}{}_{ab} <-> \epsilon_{W'X'}\phi_{AB},$

(A1.29) $\quad \mathcal{F}^{*}{}_{ab} = -i\mathcal{F}_{ab}.$

Maxwell's equations then become[72]

(A1.30) $\quad \nabla^{D}{}_{W'}\phi_{AD} = J_{AW'},$

with $J_{AW'}$ being the spinor representation of the four current.

The following is a list of some useful spinor identities, most of which can be found in Pirani 1965 and Penrose and Rindler 1984.

(A1.31, .32) $\quad \xi_A = \xi^B \epsilon_{BA} \qquad \xi^B = \epsilon^{BA}\xi_A$

(A1.33, .34) $\quad \xi_A \eta^A = -\eta_A \xi^A \qquad \epsilon^{AC}\epsilon_{BC} = \delta^A{}_B$

(A1.35, .36) $\quad \epsilon_{A[B}\epsilon_{CD]} = 0 \qquad \epsilon_{AB}\epsilon^{CD} = \delta_A{}^C \delta^D{}_B - \delta_A{}^D \delta^C{}_B$

(A1.37, .38) $\quad \epsilon_{AB}\xi_C{}^C = \xi_{AB} - \xi_{BA} \qquad \xi_{AB} = \xi_{(AB)} - 1/2\epsilon_{AB}\xi_C{}^C$

(A1.39, .40) $\quad \sigma_a{}^{BX} = \overline{\sigma_a{}^{BX}} \qquad \sigma_{aBX}\sigma^a{}_{CY} = \delta^B{}_C \delta^X{}_Y$

(A1.41, .42) $\quad \xi^{BX} = \sigma_a{}^{BX}\xi^a \qquad \sigma_a{}^{BX}\sigma^a{}_{CY}\sigma_b{}^{CY} = \sigma_b{}^{BX}$

(A1.43, .44) $\quad \sigma^a{}_{CY}\sigma_b{}^{CY} = \delta^a{}_b \qquad g_{ab} <-> \epsilon_{AB}\epsilon_{W'X'}$

(A1.45, .46) $\quad \delta^a{}_b <-> \delta^A{}_B \delta^{W'}{}_{X'} \qquad \xi^a\xi_a = 2det(\xi^{BX})$

(A1.47, .48) $\quad \nabla_a \epsilon_{AB} = 0 \qquad \nabla_a \sigma_b{}^{BX} = 0$

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72 Page 329, Pirani 1965.

$$(A1.49) \qquad \nabla_a \xi_A = \partial_a \xi_A - \Gamma_a{}^B{}_A \xi_B$$

$$(A1.50) \qquad \Gamma_a{}^A{}_B = 1/2 \sigma_d{}^{AX}(\sigma^b{}_{BX}\Gamma^d{}_{ab} + \partial_a \sigma^d{}_{BX})$$

(A1.51) Riemann Spinor $= R_{AWBXCYDZ} = \Psi_{ABCD}\epsilon_{WX}\epsilon_{YZ} + \epsilon_{AB}\epsilon_{CD}\overline{\Psi}_{WXYZ}$

$$+ \epsilon_{AB}\epsilon_{YZ}\Phi_{CDWX} + \epsilon_{CD}\epsilon_{WX}\Phi_{ABYZ} + 2\Lambda(\epsilon_{AC}\epsilon_{BD}\epsilon_{WX}\epsilon_{YZ} + \epsilon_{AB}\epsilon_{CD}\epsilon_{WX}\epsilon_{YZ})$$

where the Weyl Spinor $= \Psi_{ABCD} = \Psi_{(ABCD)}$, curvature scalar $= 24\Lambda$ and the trace free

Ricci spinor $= \Phi_{ABYZ} = \Phi_{(AB)(YZ)} = \overline{\Phi_{ABYZ}}$.

$$(A1.52) \qquad (\nabla_{AW}\nabla_{BX} - \nabla_{BX}\nabla_{AW})X_{CDYZ} = R^{EQ}{}_{CYAWBX}X_{EQDZ} + R^{EQ}{}_{DZAWBX}X_{CYEQ}$$

$$(A1.53) \qquad \nabla_{AW}\nabla_{BX} - \nabla_{BX}\nabla_{AW} = \epsilon_{AB}\nabla_{H(W}\nabla^H{}_{X)} - \epsilon_{WX}\nabla_{(A}{}^P\nabla_{B)P}$$

$$(A1.54) \qquad \nabla_{(A}{}^P\nabla_{B)P}(\xi_C \xi_D) = -4\Lambda \xi_{(C}\epsilon_{D)(A}\xi_{B)} - 2\Psi_{ABE(C}\xi_{D)}\xi^E$$

$$(A1.55) \qquad \nabla_{(A}{}^P\nabla_{B)P}\xi_C = -\Psi_{ABCD}\xi^D + 2\Lambda \xi_{(A}\epsilon_{B)C}$$

$$(A1.56) \qquad \nabla_{H(W}\nabla^H{}_{X)}\xi_D = \Phi_{DEWX}\xi^E$$

$$(A1.57) \qquad \nabla_{(A}{}^P\nabla_{B)P}\xi^B = 3\Lambda \xi_A$$

$$(A1.58) \qquad \nabla_{(A}{}^P\nabla_{B|P|)}\xi_C = -\Psi_{ABCD}\xi^D$$

$$(A1.59) \qquad \nabla_{(A}{}^P\nabla_{B)P}\xi_C = -\Psi_{ABCD}\xi^D + 2\Lambda \xi_{(A}\epsilon_{B)C}$$

$$(A1.60) \qquad \nabla_{(A}{}^P\nabla_{B)P}(\xi_C \xi_D) = -2\Psi_{ABE(C}\xi_{D)}\xi^E - 4\Lambda \xi_{(C}\epsilon_{D)(A}\xi_{B)}$$

$$(A1.61) \qquad \nabla_{H(W}\nabla^H{}_{X)}\xi_D = \Phi_{DEWX}\xi^D$$

$$(A1.62) \qquad \nabla_{(A}{}^P\nabla_{B)P}\xi^B = 3\Lambda \xi_A$$

$$(A1.63) \qquad \nabla^D{}_Y\Psi_{ABCD} - \nabla_C{}^Z\Phi_{ABYZ} + 2\epsilon_{C(A}\nabla_{B)Y}\Lambda = 0$$

$$(A1.64) \qquad \nabla^D{}_Y\Psi_{ABCD} = \nabla_{(C}{}^Z\Phi_{AB)YZ}$$

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(A1.65)
$$\nabla^{BZ}\phi_{ABYZ} = -3\nabla_{AY}\Lambda$$

(A1.66)
$$\Box = \nabla_{FQ}\nabla^{FQ}$$

(A1.67)
$$\Box\Psi_{ABCD} = 6\Psi_{(AB}{}^{HK}\Psi_{CD)HK}$$

9 Appendix II   Null tetrad and Newman-Penrose Formalism

This appendix provides a summary of the use of a null tetrad, Newman-Penrose equations and spin coefficients. Let us begin by considering a null tetrad $(l_\mu, n_\mu, m_\mu, \overline{m}_\mu)$ such that

(A2.1) $\qquad\qquad l_\mu l^\mu = m_\mu m^\mu = \overline{m}_\mu m^\mu = n_\mu n^\mu = 0,$

(A2.2) $\qquad\qquad l_\mu n^\mu = -m_\mu \overline{m}^\mu = 1,$

(A2.3) $\qquad\qquad l_\mu m^\mu = l_\mu \overline{m}^\mu = n_\mu m^\mu = n_\mu \overline{m}^\mu = 0.$

The tetrad shall be denoted by

(A2.4) $\qquad\qquad T_{m\mu} = (l_\mu, n_\mu, m_\mu, \overline{m}_\mu).$

The tetrad indices are raised and lowered with n_{mn}

(A2.5) $\qquad\qquad [\eta]_{mn} = [\eta]^{mn} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$

There exists two types of null rotations which transform the tetrad components into linear combinations of themselves.

(1) Null rotations about l_μ [73]:

(A2.6) $\qquad\qquad l_\mu \rightarrow l_\mu,$

(A2.7) $\qquad\qquad m_\mu \rightarrow m_\mu + B\, l_\mu,$

(A2.8) $\qquad\qquad n_\mu \rightarrow n_\mu + \overline{B}\, m_\mu + B\, \overline{m}_\mu + B\overline{B}\, l_\mu,$

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[73] Page 7, Held 1980.

(A2.9)   with $B \epsilon C$.

(2) Rotate m_μ with l_μ and n_μ fixed[74]:

(A2.10) $$m_\mu -> e^{i\lambda} m_\mu \quad \text{with} \quad \lambda \epsilon \mathcal{R}.$$

Using the null tetrad the Einstein equations can be cast into a very compact and elegant form. If the null tetrad is viewed as a tetrad of 1-forms we may write:

(A2.11) $$dl = 1/2(u + \bar{u}) \wedge l + \bar{v} \wedge m + v \wedge \bar{m},$$

(A2.12) $$dn = -1/2(u + \bar{u}) \wedge n + u \wedge m + \bar{u} \wedge \bar{m},$$

(A2.13) $$dm = -\bar{u} \wedge l - v \wedge n + 1/2(u - \bar{u}) \wedge m,$$

(A2.14) $$dv - u \wedge v = C_2 n \wedge \bar{m} + C_1(l \wedge n + m \wedge \bar{m}) + C_0 l \wedge m - R/12 l \wedge m +$$

$$S_{ll}/2 n \wedge m + 1/2 S_{lm}(l \wedge n - m \wedge \bar{m}) + S_{mm} l \wedge \bar{m},$$

(A2.15) $$du - 2w \wedge v = -2C_1 n \wedge \bar{m} - C_0(l \wedge n + m \wedge \bar{m}) - 2C_{-1} l \wedge m -$$

$$R/12(l \wedge n + m \wedge \bar{m}) + S_{l\bar{m}}/2 n \wedge \bar{m} + S_{m\bar{m}}/2(l \wedge n - m \wedge \bar{m}) - S_{nm}/2 l \wedge \bar{m},$$

(A2.16) $$dw - w \wedge u = C_0 n \wedge \bar{m} + C_{-1}(l \wedge n + m \wedge \bar{m}) + C_{-2} l \wedge m -$$

$$R/12 n \wedge \bar{m} + S_{\bar{m}\bar{m}}/2 n \wedge m - S_{n\bar{m}}/2(l \wedge n - m \wedge \bar{m}) + S_{nn}/2 l \wedge \bar{m},$$

with C_i being the Weyl tensor components, S_{ij} the traceless Ricci tensor components, and R the curvature scalar. u, v, and w are the curvature 1-forms. Let us also define the following matrices with 1-form components[75]

(A2.17, .18) $$\eta = \begin{pmatrix} -l & m \\ \bar{m} & n \end{pmatrix} \qquad\qquad \gamma = \begin{pmatrix} u/2 & -v \\ w & -u/2 \end{pmatrix}.$$

The Einstein equations may then be written as:

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74 See 'spin' weight in appendix III on spin spherical harmonics.
75 Page 322-324, Chinea and Guerrero 1985 (slightly different notation).

(A2.19) $d\eta = \gamma \wedge \eta - \eta \wedge \gamma^-$,

(A2.20) $d\gamma = \gamma \wedge \gamma + w \wedge \eta - X \wedge \eta + \eta \wedge Y$,

with

(A2.21, .22) $X = R/24 \begin{pmatrix} l+n & m \\ \overline{m} & n-l \end{pmatrix}$ $Y = \begin{pmatrix} a \cdot T & d \cdot T \\ b \cdot T & e \cdot T \end{pmatrix}$

(A2.23) $a = (-S_{n\overline{m}}, S_{nm}, -S_{nn}, -S_{m\overline{m}})$,

(A2.24) $b = (S_{m\overline{m}}, -S_{m\overline{m}}, S_{n\overline{m}}, -S_{lm})$,

(A2.25) $d = (-S_{m\overline{m}}, -S_{mm}, -S_{n\overline{m}}, -S_{lm})$,

(A2.26) $e = (-S_{l\overline{m}}, -S_{lm}, S_{m\overline{m}}, -S_{ll})$.

In vacuum the equations reduce to:

(A2.27) $d\eta = \gamma \wedge \eta - \eta \wedge \gamma^-$,

(A2.28) $d\gamma = \gamma \wedge \gamma + w \wedge \eta$.

In flat space the equations again reduce to form:

(A2.29) $d\eta = \gamma \wedge \eta - \eta \wedge \gamma^-$,

(A2.30) $d\gamma = \gamma \wedge \gamma$.

The exterior product sign may be slightly misleading, in the above three sets of equations it is meant to imply matrix multiplication such that the product between the matrix elements is the exterior product.

The set of equations for flat space have been solved with the solution being given by[76]:

(A2.31) $\phi\eta\phi^- = d\xi$

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76 Page 324, Chinea and Guerrero 1985.

(A2.32) $$d\phi = -\phi\, d\gamma,$$

where ϕ and ξ are matrices whose components are 0-forms, $\phi \in SL(2,C)$ and $\xi = \xi^*$.

The Ricci rotation coefficients (complex) may also be defined in terms of the null tetrad:

(A2.33) $$\gamma_m{}^{np} = T_{m\mu:\nu} T^{n\mu} T^{p\nu} .$$

To discuss the Newman-Penrose formalism[77] let us introduce a spinor basis $\{o^A, \iota^A\}$ such that

(A2.34) $$o^A \iota_A = -o_A \iota^A = 1 .$$

The spinor basis is related to the tetrad basis by the following:

(A2.35) $$l^\mu = \sigma^\mu{}_{AB} o^A o^B,$$

(A2.36) $$n^\mu = \sigma^\mu{}_{AB} \iota^A \iota^B,$$

(A2.37) $$m^\mu = \sigma^\mu{}_{AB} o^A \iota^B.$$

We may also define a dyad (similar to the tetrad notation presented earlier) for the spinor basis.

(A2.38) $$\xi_a{}^A \Rightarrow \xi_0{}^A = o^A \text{ and } \xi_1{}^A = \iota^A .$$

The dyad analog of the Ricci rotation coefficients is then given by

(A2.39) $$\Gamma_{abcd} = \xi_{aA:\mu} \xi_b{}^A \sigma^\mu{}_{cd} .$$

Γ_{abcd} is symmetric in the first two indices. The spin coefficients are defined as components of Γ_{abcd}:

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77 This presentation will follow closely that of Newman and Penrose 1962.

(A2.40)

$$\Gamma_{abcd} = \begin{array}{c|ccc} cd\backslash ab & 00 & 01 & 11 \\ \hline 0\dot{0} & \kappa & \epsilon & \pi \\ 1\dot{0} & \rho & \alpha & \lambda \\ 0\dot{1} & \sigma & \beta & \mu \\ 1\dot{1} & \tau & \gamma & \nu \end{array}$$

The spin coefficients may also be defined in terms of the Ricci rotation coefficients, and in the null tetrad notation:

(A2.41, .42) $\quad \kappa = \gamma_{131} = l_{\mu:\nu} m^{\mu} l^{\nu} . \qquad \pi = -\gamma_{241} = -n_{\mu:\nu}\overline{m}^{\mu} l^{\nu} .$

(A2.43) $\quad \epsilon = 1/2(\gamma_{121} - \gamma_{341}) = 1/2(l_{\mu:\nu} n^{\mu} l^{\nu} - m_{\mu:\nu}\overline{m}^{\mu} l^{\nu} ,$

(A2.44, .45) $\quad \rho = \gamma_{134} = l_{\mu:\nu} m^{\mu}\overline{m}^{\nu} . \qquad \lambda = -\gamma_{244} = -n_{\mu:\nu}\overline{m}^{\mu}\overline{m}^{\nu} .$

(A2.46) $\quad \alpha = 1/2(\gamma_{124} - \gamma_{344}) = 1/2\left(l_{\mu:\nu}\overline{m}^{\nu} n^{\mu} - m_{\mu:\nu}\overline{m}^{\mu}\overline{m}^{\nu}\right)$

(A2.47, .48) $\quad \sigma = \gamma_{133} = l_{\mu:\nu} m^{\mu} m^{\nu} . \qquad \mu = -\gamma_{243} = -n_{\mu:\nu}\overline{m}^{\mu} m^{\nu} .$

(A2.49) $\quad \beta = 1/2(\gamma_{123} - \gamma_{343}) = 1/2\left(l_{\mu:\nu} n^{\mu} m^{\nu} - m_{\mu:\nu}\overline{m}^{\mu} m^{\nu}\right)$

(A2.50, .51) $\quad \nu = -\gamma_{242} = -n_{\mu:\nu}\overline{m}^{\mu} n^{\nu} . \qquad \tau = \gamma_{132} = l_{\mu:\nu} m^{\mu} n^{\nu} .$

(A2.52) $\quad \gamma = 1/2(\gamma_{122} - \gamma_{342}) = 1/2\left(l_{\mu:\nu} n^{\mu} n^{\nu} - m_{\mu:\nu}\overline{m}^{\mu} n^{\nu}\right) .$

Many of the above coefficients carry a specific geometrical significance. A thorough discussion of this (and the application to ray optics) may be found in chapter 4 of Pirani 1965.

Four intrinsic derivatives may also be defined which are central to the Newman-Penrose formalism:

(A2.53, .54) $D = l^\mu \nabla_\mu$, $\delta = m^\mu \nabla_\mu$,

(A2.55, .56) $\Delta = n^\mu \nabla_\mu$, $\bar{\delta} = \bar{m}^\mu \nabla_\mu$.

In dyad notation they have the form:

(A2.57, .58) $D = \partial_{00}$, $\delta = \partial_{01}$,

(A2.59, .60) $\Delta = \partial_{11}$, $\bar{\delta} = \partial_{10}$.

Below is a list of identities involving the spin coefficients, intrinsic derivatives and the null tetrad. The Einstein equations are also given in spin coefficient form.

(A2.61) $l_\mu l^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = n_\mu n^\mu = 0$

(A2.62) $l_\mu m^\mu = l_\mu \bar{m}^\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0$

(A2.63) $T_{m\mu} = (l_\mu, n_\mu, m_\mu, \bar{m}_\mu)$

(A2.64) $\eta_{mn} = T_{m\mu} T_{n\nu} g^{\mu\nu}$

(A2.65)   Ricci Rotation coefficients $= \gamma_m{}^{np} = T_{m\mu;\nu} T^{n\mu} T^{p\nu}$

(A2.66) $\gamma^{mnp} = -\gamma^{nmp}$

(A2.67)   Reimann tensor $= R_{mnpq} = R_{\alpha\beta\gamma\delta} T^\alpha{}_m T^\beta{}_n T^\gamma{}_p T^\delta{}_q$

(A2.68) $R^{mnpq} = \gamma^{mnp;q} - \gamma^{mnq;p} + \gamma_l{}^{mq}\gamma^{lnp} - \gamma_l{}^{mp}\gamma^{lnq} + \gamma^{mnl}(\gamma_l{}^{pq} - \gamma_l{}^{qp})$

(A2.69) $\gamma^{mnp;q} = \gamma^{mnp}{}_{;\mu} T^{q\mu}$

(A2.70)   Ricci identity $\Rightarrow T_{m\mu;[\alpha\beta]} = 1/2 T_{m\nu} R^\nu{}_{\mu\alpha\beta}$

$(A2.71)$ Bianchi identity $\Rightarrow R_{mn[pq:r]} = \gamma_m{}^l{}_{[r}R_{pq]ln} - \gamma_n{}^l{}_{[r}R_{pq]lm}$

$$+ 2R_{mnl[p}\gamma_r{}^l{}_{q]}$$

$(A2.72)$ $l_{\mu;\nu} = (\gamma + \bar{\gamma})l_\mu l_\nu - (\alpha + \bar{\beta})l_\mu \bar{m}_\nu + \bar{\tau} m_\mu l_\nu - \tau \bar{m}_\mu l_\nu + \bar{\sigma} m_\mu m_\nu +$

$$\sigma \bar{m}_\mu \bar{m}_\nu + \rho \bar{m}_\mu m_\nu + \bar{\rho} m_\mu \bar{m}_\nu - \kappa \bar{m}_\mu n_\nu - \bar{\kappa} m_\mu n_\nu + (\epsilon + \bar{\epsilon})l_\mu n_\nu$$

$(A2.73)$ $n_{\mu;\nu} = -(\epsilon + \bar{\epsilon})n_\mu n_\nu - (\gamma + \bar{\gamma})n_\mu l_\nu + (\bar{\alpha} + \beta)n_\mu \bar{m}_\nu + (\alpha + \bar{\beta})n_\mu m_\nu +$

$$\pi m_\mu n_\nu + \bar{\pi}\bar{m}_\mu n_\nu + \nu m_\mu l_\nu + \bar{\nu}\bar{m}_\mu l_\nu - \mu m_\mu \bar{m}_\nu - \bar{\mu}\bar{m}_\mu m_\nu - \lambda m_\mu m_\nu - \bar{\lambda}\bar{m}_\mu \bar{m}_\nu$$

$(A2.74)$ $m_{\mu;\nu} = \bar{\pi} l_\mu n_\nu + \bar{\nu} l_\mu l_\nu - \bar{\lambda} l_\mu \bar{m}_\nu - \bar{\mu} l_\mu m_\nu - \kappa n_\mu n_\nu - \tau n_\mu l_\nu +$

$$\sigma n_\mu \bar{m}_\nu + \rho n_\mu m_\nu + (\epsilon - \bar{\epsilon})m_\mu n_\nu + (\gamma - \bar{\gamma})m_\mu l_\nu + (\bar{\alpha} - \beta)m_\mu \bar{m}_\nu - (\alpha - \bar{\beta})m_\mu m_\nu$$

$(A2.75)$ $$l^\alpha{}_{;\alpha} = -\rho - \bar{\rho} + (\epsilon + \bar{\epsilon})$$

$(A2.76)$ $$n^\alpha{}_{;\alpha} = -(\gamma + \bar{\gamma}) + \mu + \bar{\mu}$$

$(A2.77)$ $$m^\mu{}_{;\mu} = \bar{\pi} - \tau - (\bar{\alpha} - \beta)$$

Dyad components of the Weyl Spinor.

$(A2.78)$ $$\Psi_0 = \Psi_{0000} = \xi^A{}_0\xi^B{}_0\xi^C{}_0\xi^D{}_0\Psi_{ABCD} = -C_{\mu\nu\rho\sigma}l^\mu m^\nu l^\rho m^\sigma$$

$(A2.79)$ $$\Psi_1 = \Psi_{0001} = \xi^A{}_0\xi^B{}_0\xi^C{}_0\xi^D{}_1\Psi_{ABCD} = -C_{\mu\nu\rho\sigma}l^\mu n^\nu l^\rho m^\sigma$$

$(A2.80)$ $$\Psi_2 = \Psi_{0011} = \xi^A{}_0\xi^B{}_0\xi^C{}_1\xi^D{}_1\Psi_{ABCD} =$$

$$1/2 C_{\mu\nu\rho\sigma}(l^\mu n^\nu m^\rho \bar{m}^\sigma - l^\mu n^\nu l^\rho n^\sigma)$$

$(A2.81)$ $$\Psi_3 = \Psi_{0111} = \xi^A{}_0\xi^B{}_1\xi^C{}_1\xi^D{}_1\Psi_{ABCD} = C_{\mu\nu\rho\sigma}\bar{m}^\mu n^\nu l^\rho n^\sigma$$

$(A2.82)$ $$\Psi_4 = \Psi_{1111} = \xi^A{}_1\xi^B{}_1\xi^C{}_1\xi^D{}_1\Psi_{ABCD} = -C_{\mu\nu\rho\sigma}\bar{m}^\mu n^\nu \bar{m}^\rho n^\sigma$$

Dyad components of the traceless Ricci spinor.

(A2.83) $\quad \Phi_{00} = \Phi_{ABPQ} \xi^A{}_0 \xi^B{}_0 \xi^P{}_0 \xi^Q{}_0$

(A2.84) $\quad \Phi_{01} = \Phi_{ABPQ} \xi^A{}_0 \xi^B{}_0 \xi^P{}_0 \xi^Q{}_1$

(A2.85) $\quad \Phi_{02} = \Phi_{ABPQ} \xi^A{}_0 \xi^B{}_0 \xi^P{}_1 \xi^Q{}_1$

(A2.86) $\quad \Phi_{10} = \Phi_{ABPQ} \xi^A{}_0 \xi^B{}_0 \xi^P{}_1 \xi^Q{}_0$

(A2.87) $\quad \Phi_{11} = \Phi_{ABPQ} \xi^A{}_0 \xi^B{}_0 \xi^P{}_1 \xi^Q{}_1$

(A2.88) $\quad \Phi_{12} = \Phi_{ABPQ} \xi^A{}_0 \xi^B{}_0 \xi^P{}_1 \xi^Q{}_1$

(A2.89) $\quad \Phi_{20} = \Phi_{ABPQ} \xi^A{}_1 \xi^B{}_1 \xi^P{}_1 \xi^Q{}_0$

(A2.90) $\quad \Phi_{21} = \Phi_{ABPQ} \xi^A{}_1 \xi^B{}_1 \xi^P{}_1 \xi^Q{}_0$

(A2.91) $\quad \Phi_{22} = \Phi_{ABPQ} \xi^A{}_1 \xi^B{}_1 \xi^P{}_1 \xi^Q{}_1$

Dyad components of the Maxwell Spinor.

(A2.92) $\quad \phi_0 = F_{\mu\nu} l^\mu m^\nu = \phi_{AB} \xi^A \xi^B{}_0$

(A2.93) $\quad \phi_1 = 1/2 F_{\mu\nu} (l^\mu n^\nu + \overline{m}^\mu m^\nu) = \phi_{AB} \xi^A \xi^B{}_1$

(A2.94) $\quad \phi_2 = F_{\mu\nu} \overline{m}^\mu n^\nu = \phi_{AB} \xi^A{}_1 \xi^B{}_1$

Maxwell's equations.

(A2.95) $\quad \overline{\delta}\phi_0 - D\phi_1 = (2\alpha - \pi)\phi_0 - 2\rho\phi_1 + \kappa\phi_2 - 2\pi J_{00}$

(A2.96) $\quad \overline{\delta}\phi_1 - D\phi_2 = \lambda\phi_0 - 2\pi\phi_1 + (2\epsilon - \rho)\phi_2 - 2\pi J_{01}$

(A2.97) $\quad \Delta\phi_0 - \delta\phi_1 = (2\gamma - \mu)\phi_0 - 2\tau\phi_1 + \sigma\phi_2 - 2\pi J_{10}$

(A2.98) $\quad \Lambda\phi_1 - \delta\phi_2 = \nu\phi_0 - 2\mu\phi_1 + (2\beta - \tau)\phi_2 - 2\pi J_{11}$

Commutation relations where ϕ is any scalar function.

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$$(A2.99) \qquad [\Delta, D]\phi = \{(\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\tau + \bar{\pi})\bar{\delta}\}\phi$$

$$(A2.100) \qquad [\delta, D]\phi = \{(\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta - \sigma\bar{\delta} - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta\}\phi$$

$$(A2.101) \qquad [\delta, \Delta]\phi = \{-\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + \bar{\lambda}\bar{\delta} + (\mu - \gamma - \bar{\gamma})\delta\}\phi$$

$$(A2.102) \qquad [\delta, \bar{\delta}]\phi = \{(\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta - (\bar{\alpha} - \beta)\bar{\delta} - (\bar{\beta} - \alpha)\delta\}\phi$$

Bianchi identities.

$$(A2.103) \qquad \bar{\delta}\Psi_0 - D\Psi_1 + D\Phi_{01} - \delta\Phi_{00} = (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2$$
$$+ (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} + 2(\epsilon + \bar{\rho})\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02}$$

$$(A2.104) \qquad \Delta\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2$$
$$- \bar{\lambda}\Phi_{00} + 2(\bar{\pi} - \beta)\Phi_{01} + 2\sigma\Phi_{11} + (2\epsilon - 2\bar{\epsilon} + \bar{\rho})\Phi_{02} - 2\kappa\Phi_{12}$$

$$(A2.105) \qquad 3(\bar{\delta}\Psi_1 - D\Psi_2) + 2(D\Phi_{11} - \delta\Phi_{10}) + \bar{\delta}\Phi_{01} - \Delta\Phi_{00} = 3\lambda\Psi_0 - 9\rho\Psi_2$$
$$+ 6(\alpha - \pi)\Psi_1 + 6\kappa\Psi_3 + (\bar{\mu} - 2\mu - 2\gamma - 2\bar{\gamma})\Phi_{00} + (2\alpha + 2\pi + 2\bar{\tau})\Phi_{01}$$
$$+ 2(\tau - 2\bar{\alpha} + \bar{\pi})\Phi_{10} + 2(2\bar{\rho} - \rho)\Phi_{11} + 2\sigma\Phi_{20} - \bar{\sigma}\Phi_{02} - 2\bar{\kappa}\Phi_{12} - 2\kappa\Phi_{21}$$

$$(A2.106) \qquad 3(\Delta\Psi_1 - \delta\Psi_2) + 2(D\Phi_{12} - \delta\Phi_{11}) + (\bar{\delta}\Phi_{02} - \Delta\Phi_{01}) = 3\nu\Psi_0 +$$
$$6(\gamma - \mu)\Psi_1 - 9\tau\Psi_2 + 6\sigma\Psi_3 - \bar{\nu}\Phi_{00} + 2(\bar{\mu} - \mu - \gamma)\Phi_{01} - 2\bar{\lambda}\Phi_{10} + 2(\tau + 2\bar{\pi})\Phi_{11} +$$
$$(2\alpha + 2\pi + \bar{\tau} - 2\bar{\beta})\Phi_{02} + (2\bar{\rho} - 2\rho - 4\epsilon)\Phi_{12} + 2\sigma\Phi_{21} - 2\kappa\Phi_{22}$$

$$(A2.107) \qquad 3(\bar{\delta}\Psi_2 - D\Psi_3) + D\Phi_{21} - \delta\Phi_{20} + 2(\bar{\delta}\Phi_{11} - \Delta\Phi_{10}) = 6\lambda\Psi_1 - 9\pi\Psi_2$$
$$+ 6(\epsilon - \rho)\Psi_3 + 3\kappa\Psi_4 - 2\nu\Phi_{00} + 2\lambda\Phi_{01} + 2(\bar{\mu} - \mu - 2\bar{\gamma})\Phi_{10} + (2\pi + 4\bar{\tau})\Phi_{11}$$
$$+ (2\beta + 2\tau + \bar{\pi} - 2\alpha)\Phi_{20} - 2\bar{\sigma}\Phi_{12} + 2(\bar{\rho} - \rho - \epsilon)\Phi_{21} - \bar{\kappa}\Phi_{22}$$

$$(A2.108) \qquad 3(\Delta\Psi_2 - \delta\Psi_3) + D\Phi_{22} - \delta\Phi_{21} + 2(\bar{\delta}\Phi_{12} - \Delta\Phi_{11}) =$$

$$6\nu\Psi_1 - 9\mu\Psi_2 + 6(\beta - \tau)\Psi_3 + 3\sigma\Psi_4 - 2\nu\Phi_{01} - 2\bar{\nu}\Phi_{10} + 2(2\bar{\mu} - \mu)\Phi_{11}$$

$$+ 2\lambda\Phi_{02} - \bar{\lambda}\Phi_{20} + 2(\pi + \bar{\tau} - 2\bar{\beta})\Phi_{12} + 2(\beta + \tau + \bar{\pi})\Phi_{21} + (\bar{\rho} - 2\epsilon - 2\bar{\epsilon} - 2\rho)\Phi_{22}$$

(A2.109)
$$\bar{\delta}\Psi_3 - D\Psi_4 + \bar{\delta}\Phi_{21} - \Delta\Phi_{20} = 3\lambda\Psi_2 - 2(\alpha + 2\pi)\Psi_3 + (4\epsilon - \rho)\Psi_4$$

$$- 2\nu\Phi_{10} + 2\lambda\Phi_{11} + (2\gamma - 2\bar{\gamma} + \bar{\mu})\Phi_{20} + 2(\bar{\tau} - \alpha)\Phi_{21} - \bar{\sigma}\Phi_{22}$$

(A2.110)
$$\Delta\Psi_3 - \delta\Psi_4 + \bar{\delta}\Phi_{22} - \Delta\Phi_{21} = 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (4\beta - \tau)\Psi_4$$

$$- 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} + 2\lambda\Phi_{12} + 2(\gamma + \bar{\mu})\Phi_{21} + (\bar{\tau} - 2\bar{\beta} - 2\alpha)\Phi_{22}$$

Contracted Bianchi identities.

(A2.111)
$$D\Phi_{11} - \delta\Phi_{10} - \bar{\delta}\Phi_{01} + \Delta\Phi_{00} + 3D\Lambda = (2\gamma - \mu + 2\bar{\gamma} - \bar{\mu})\Phi_{00} +$$

$$(\pi - 2\alpha - 2\bar{\tau})\Phi_{01} + (\bar{\pi} - 2\bar{\alpha} - 2\tau)\Phi_{10} + 2(\rho + \bar{\rho})\Phi_{11} + \bar{\sigma}\Phi_{02} + \sigma\Phi_{20} - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21}$$

(A2.112)
$$D\Phi_{12} - \delta\Phi_{11} - \bar{\delta}\Phi_{02} + \Delta\Phi_{01} + 3\delta\Lambda = (2\gamma - \mu - 2\bar{\mu})\Phi_{01} +$$

$$\bar{\nu}\Phi_{00} - \bar{\lambda}\Phi_{01} + 2(\bar{\pi} - \tau)\Phi_{11} + (\pi + 2\bar{\beta} - 2\alpha - \bar{\tau})\Phi_{02} + (2\rho + \bar{\rho} - 2\epsilon)\Phi_{12} + \sigma\Phi_{21} - \kappa\Phi_{22}$$

(A2.113)
$$D\Phi_{22} - \delta\Phi_{21} - \bar{\delta}\Phi_{12} + \Delta\Phi_{11} + 3\Delta\Lambda = \nu\Phi_{01} + \bar{\nu}\Phi_{10} - \lambda\Phi_{02} -$$

$$2(\mu + \bar{\mu})\Phi_{11} - \bar{\lambda}\Phi_{20} + (2\pi - \bar{\tau} + \beta 2\bar{\beta})\Phi_{12} + (2\beta - \tau + 2\bar{\pi})\Phi_{21} + (\rho + \bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22}$$

10 Appendix III   Conformal Transformations

This appendix will provide a brief summary of the Conformal behavior of various spinors and tensors relevant to General Relativity. Most of the results can be found in Esposito and Witten 1977 and Penrose 1964 and 1967.

Let $\{\tilde{M}, \tilde{g}_{ab}\}$ be a 4-manifold with metric. Let Ω be a smooth positive scalar function on $\{\tilde{M}, \tilde{g}_{ab}\}$. A Conformal Transformation of $\{\tilde{M}, \tilde{g}_{ab}\}$ is given by introducing a new metric, $g_{ab} = \Omega^2 \tilde{g}_{ab}$ on $\{\tilde{M}, \tilde{g}_{ab}\}$ and regarding the conformally transformed manifold as $\{M, g_{ab}\}$.

For each tensor field on $\{\tilde{M}, \tilde{g}_{ab}\}$ there exists a real number w called the dimension of the tensor field. For example let $\tilde{\alpha}^{a\cdots c}{}_{b\cdots d}$ be a tensor field on $\{\tilde{M}, \tilde{g}_{ab}\}$ with u contravariant indices and d covariant indices. Then under a conformal transformation the tensor field becomes:

$$(A3.1) \qquad \alpha^{a\cdots c}{}_{b\cdots d} = \Omega^{w-u-d}\tilde{\alpha}^{a\cdots c}{}_{b\cdots d}.$$

The indices of the transformed field are raised and lowered with the new metric g_{ab}. The dimension is unchanged under contraction and is additive under the exterior product.

To study the behavior of the derivative operator the following tensor is defined:

$$(A3.2) \qquad C^m{}_{ab} = -\Omega^{-1}(2\delta^m{}_{(a}\nabla_{b)}\Omega - g_{ab}g^{mn}\nabla_n\Omega).$$

Let $\alpha^{b\cdots c}{}_{d\cdots e}$ be any tensor field on $\{\tilde{M}, \tilde{g}_{ab}\}$. Then

$$(A3.3) \qquad \nabla_a\alpha^{b\cdots c}{}_{d\cdots e} = \tilde{\nabla}_a\alpha^{b\cdots c}{}_{d\cdots e} + C^b{}_{am}\alpha^{m\cdots c}{}_{d\cdots e} + \ldots + C^c{}_{am}\alpha^{b\cdots m}{}_{d\cdots e}$$

$$- C^m{}_{ad}\alpha^{b\cdots c}{}_{m\cdots e} - \ldots - C^m{}_{ae}\alpha^{b\cdots c}{}_{d\cdots m}.$$

The Riemann tensor may now rewritten as follows: let k_c be any field on $\{\tilde{M}, \tilde{g}_{ab}\}$ then

$$(A3.4) \qquad 1/2 \tilde{R}_{abc}{}^d k_d = \tilde{\nabla}_{[a} \tilde{\nabla}_{b]} k_c$$

$$(A3.5) \qquad = \tilde{\nabla}_{[a} (\nabla_{b]} + C^m{}_{b]} k_m)$$

$$(A3.6) \qquad = \nabla_{[a} (\nabla_{b]} k_c + C^m{}_{b]c} k_m) + C^n{}_{c[a}(\nabla_{b]} k_n + C^m{}_{b]n} k_m)$$

$$(A3.7) \qquad = 1/2 R_{abc}{}^d k_d + \nabla_{[a} C^m{}_{b]c} k_m + C^n{}_{c[a} C^m{}_{b]n} k_m$$

$$(A3.8) \qquad \therefore \quad \tilde{R}_{abc}{}^d = R_{abc}{}^d + 2\nabla_{[a} C^d{}_{b]c} + 2 C^n{}_{c[a} C^d{}_{b]n}$$

$$(A3.9) \qquad \Rightarrow \tilde{C}_{abc}{}^d = C_{abc}{}^d$$

$$(A3.10) \qquad \Rightarrow \tilde{R}_{ab} = R_{ab} + 2\Omega^{-1} \nabla_a \nabla_b \Omega + \Omega^{-1} g_{ab} \nabla^m \nabla_m \Omega$$

$$- 3\Omega^{-2} g_{ab}(\nabla^m \Omega)(\nabla_m \Omega)$$

$$(A3.11) \qquad \Rightarrow \tilde{R} = R\Omega^2 + 6\Omega \nabla^m \nabla_m \Omega - 12(\nabla^m \Omega)(\nabla_m \Omega).$$

The behavior of the spinor representations of massless fields of spin s is particularly simple. Consider a symmetric spinor with 2s indices which satisfies the spin s massless field equation.

$$(A3.12) \qquad \nabla^{AM} \tilde{\phi}_{AB..K} = 0$$

Recall that if there are 4 indices on $\tilde{\phi}_{AB..K}$ then the above equation may be interpreted as the Bianchi identity. Hence, this equation should be invariant under all conformal transformations (see Penrose 1967 for a proof constructed by examining the behavior of solution, written as a Kirkoff integral, under a conformal transformation). By requiring this we find that $\tilde{\phi}_{AB..K}$ must have a conformal density of -s-1

$$(A3.13) \qquad \tilde{\phi}_{AB..K} = \Omega^{s-1} \phi_{AB..K}.$$

Which yields the desired result $\nabla^{AM} \phi_{AB..K} = 0$.

11 Appendix IV Spin Spherical Harmonics

Before we define the spin spherical harmonics a definition of 'spin weight' and the action of the two operators $\eth$ and $\bar{\eth}$ (edth and edth-bar) must be given.

A quantity η has a spin weight s if it transforms under a rotation of angle ξ, by

(A4.1) $$\eta' = e^{is\xi}\eta.$$

If η has spin weight s ($sw(\eta) = s$) the operator $\eth$ may be defined as

(A4.2) $$\eth\eta = -(\sin\theta)^s(\partial/\partial\theta + i/\sin\theta\,\partial/\partial\phi)\{(\sin\theta)^{-s}\eta\}.$$

Note that the quantity $\eth\eta$ now transforms as

(A4.3) $$\eth\eta' = e^{i(s-1)\xi}\eth\eta,$$

$$\text{with}\quad sw(\eth\eta) = s + 1.$$

Similarly the operator $\bar{\eth}$ is defined as

(A4.4) $$\bar{\eth}\eta = -(\sin\theta)^{-s}(\partial/\partial\theta - i/\sin\theta\,\partial/\partial\phi)\{(\sin\theta)^s\eta\}.$$

$$\text{with}\quad sw(\bar{\eth}\eta) = s - 1.$$

We may now define the spin spherical harmonics as,

(A4.5) $$_sY_{lm} = \sqrt{(l-s)!/(l+s)!}\;\eth^sY_{lm}\qquad 0 \le s \le l$$

(A4.6) $$= (-1)^s\sqrt{(l+s)!/(l-s)!}\;\bar{\eth}^{-s}Y_{lm}\qquad -l \le s \le 0$$

$$s = 0 \;\Rightarrow\; _0Y_{lm} = Y_{lm} = \text{usual spherical harmonics.}$$

The harmonics are defined in what is called the standard gauge. A definition given in an arbitrary gauge can be found in Dray 1985. Below is a tabulation of some of the properties of the spin spherical harmonics which will be useful in the text (the proofs and/or derivations can be found in Newman and Penrose 1968, Dray 1985 and 1986, Moreshi 1986 and Ivancovich et. al. 1989). Note that all integrals are over the 2-sphere.

$$(A4.7) \qquad {}_sY_{lm} = \sqrt{(l-s)!/(l+s)!}\, \eth^s Y_{lm} \qquad 0 \le s \le l$$

$$(A4.8) \qquad {}_sY_{lm} = (-1)^s \sqrt{(l+s)!/(l-s)!}\, \bar{\eth}^s Y_{lm} \qquad -l \le s \le 0$$

$$(A4.9) \qquad {}_sY_{lm} = (-1)^{m-s}\, {}_{-s}\bar{Y}_{l,-m}$$

$$(A4.10) \qquad \eth\, {}_sY_{lm} = \sqrt{(l-s)(l+s+1)}\, {}_{s+1}Y_{lm}$$

$$(A4.11) \qquad \bar{\eth}\, {}_sY_{lm} = -\sqrt{(l-s)(l-s+1)}\, {}_{s-1}Y_{lm}$$

$$(A4.12) \qquad \bar{\eth}\eth\, {}_sY_{lm} = -(l-s)(l+s+1)\, {}_sY_{lm}$$

$$(A4.13, .14) \qquad \eth_l\, {}_lY_{lm} = 0 \qquad \bar{\eth}_{-l}\, {}_{-l}Y_{lm} = 0$$

$$(A4.15) \qquad \int {}_sY_{lm}\, \eth^{l-s-1}\xi\, d\omega = 0 \qquad \text{with} \quad sw(\xi) = -l-1.$$

$$(A4.16) \qquad \int {}_s\bar{Y}_{lm}\, \bar{\eth}^{l-s-1}\xi\, d\omega = 0 \qquad \text{with} \quad sw(\xi) = l+1.$$

$$(A4.17) \qquad \int A\eth B\, d\omega = -\int B\eth A\, d\omega \qquad \text{with} \quad sw(A) + sw(B) = -1$$

$$(A4.18) \qquad \int {}_s\bar{Y}_{lm}\, \bar{\eth}\eth\eta\, d\omega = (s-l)(l+s+1) \int {}_s\bar{Y}_{lm}\, \eta\, d\omega$$

$$(A4.19) \qquad \bar{\eth}\eth\eta - \eth\bar{\eth}\eta = 2s\eta$$

$$(A4.20) \qquad sw(\eth) = 1 = -sw(\bar{\eth})$$

If we let ζ and $\bar{\zeta}$ be the complex stereographic coordinates on the 2-sphere. Then the following identities may be used (Let η have a spin weight of s.):

$$(A4.21) \qquad \eth_\zeta\alpha = (1+\zeta\bar{\zeta})^{1-s}[(1+\zeta\bar{\zeta})^s\alpha]_{,\zeta}$$

$$(A4.22) \qquad \eth_\zeta\, {}_sY_{lm} = \sqrt{(l-s)(l-s+1)}\, {}_{s-1}Y_{lm}$$

$$(A4.23) \qquad {}_sY_{lm} = (-1)^{l-s}\sqrt{(l+m)!(l-m)!(2l+1)[4\pi(l-s)!(l+s)!]^{-1}}$$

$$\times \quad \sum_{\rho=0}^{l} (-1)^{\rho} \binom{l-s}{\rho} \binom{l+s}{\rho+s-m} \frac{\zeta^{\rho} \bar{\xi}^{\rho-s-m}}{(1+\zeta\bar{\xi})^{\rho}}$$

(A4.24)
$$\int {}_s Y_{lm} \overline{{}_s Y}_{l'm'} \cdot d\omega = \delta_{ll'} \cdot \delta_{mm'},$$

(A4.25)
$$\eth^2 \bar{\eth}^2 Y_{lm} = (l-1)(l+1)(l+2)(l/4) Y_{lm}.$$

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13 Vita Auctoris

Mark Naber born on September 2 1964 in Mineral Wells Texas. He graduated from Fremont Senior High School, Fremont Nebraska in 1982. He then obtained a D.E.C. in Pure and Applied Sciences in 1984 from Champlain Regional College in Lennoxville Quebec. He obtained a B.S.C. in Physics and Mathematics from McGill University, Montreal Quebec in 1989. Currently he is a candidate for the Masters degree in Physics at the University of Windsor, Windsor Ontario and hopes to graduate in 1990.