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Discrete-time multiobjective filtering and control

Ali Tahmasebi pour
University of Windsor

Discrete-Time Multiobjective Filtering and Control

by

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Abstract

In this dissertation, discrete-time multiobjective filtering and control is studied to complete the theory on these subjects. A discrete-time filter is developed for systems subject to both white noise and bounded-power disturbance signals. Sufficient and necessary conditions for the robust optimal filter are presented and the resulting filter gain is characterized by a set of two coupled Riccati equations.

Furthermore, control design methods for discrete-time systems subject to both white noise and bounded-power disturbance signals are developed in the framework of two multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ designs. For these two methods, namely: '*Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control*', and '*$\mathcal{H}_\infty$ Gaussian Control*', after some standard assumptions on the system and defining performance indexes, sufficient and necessary conditions are obtained for existence of output-feedback controllers which are characterized by coupled Riccati equations. Numerical examples are presented to validate the designs. As an application, control of electric power-assisted steering system is considered and the multiobjective control designs are developed and compared with regular $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers.

To Mitra: The centre of my universe

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Chapter 1

Introduction

1.1 Multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ Control

In any engineering problem, the goal is often to attain some desired performance, defined by the problem statement. This performance can be in the form of a target behavior of the addressed system, maintaining a vital characteristic of the system such as stability, or the ability to perform effectively in the presence of unknown changes to the environment. However, as any experienced engineer knows, the price for achieving one type of performance is often sacrificing other aspects of system behavior.

One of the most logical measures to evaluate the quality of a system design and compare it with other possible solutions, is to asses if it can satisfy multiple objectives at the same time, hence the designation: '*multiobjective*'. Since it is almost impossible to have a single solution to an engineering challenge without any downsides, a good practitioner can instead attempt to accomplish as many objectives as possible with limited number of drawbacks. For some examples of multiobjective control designs see [16, 50, 47, 48, 20, 42, 45].

Multivariable control analysis and design tools have enjoyed a rapid progress during the past decades. Two of the major contributions to this field are the so-called Linear Quadratic Gaussian (LQG) or $\mathcal{H}_2$, and $\mathcal{H}_\infty$ control theories. These two fields of study, although related in nature, address two different concerns in the design process.

The primary differences between the $\mathcal{H}_2$ control design theory [31, 44, 18] and $\mathcal{H}_\infty$ control theory [18, 56, 22, 4, 60] are rooted in their treatments of exogenous disturbances. In LQG approach, for a linear plant given by its state-space description, it is assumed that the disturbance and measurement noise are Gaussian stochastic processes with known power spectral densities. The design specifications are then converted into a quadratic performance criterion consisting of state variables and control input signals. The goal of the designer is then to minimize this performance criterion by using a suitable state or measurement feedback controller and at the same time guaranteeing the closed-loop stability. However, in many practical problems, the covariance of the disturbance signal is not known and furthermore, the robustness is not guaranteed when dealing with model inaccuracy and changes in system parameters [21].

On the other hand, $\mathcal{H}_\infty$ theory is based on a deterministic disturbance model consisting of bounded-power signals, and it tries to minimize the worst-case disturbance attenuation. This method is applied successfully wherever a robust design is required. Nevertheless, the transient response of the system with $\mathcal{H}_\infty$ controller is not usually desirable and also it may be too conservative for the systems with well-known disturbance power spectral densities.

The question of designing a stabilizing, mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller that is able to address both types of disturbances and also produce a robust controller with a good transient response is then natural to consider, since it is obviously an example of a '*multiobjective design*' as presented before. It is therefore no surprise that this problem has attracted a great deal of attention from the researchers in the past decade. There has been a large number of works reported in the literature that address this question in continuous-time domain. Some examples are given here for various approaches to this problem (for more examples of other multiobjective control methods see [51]).

The Linear Matrix Inequalities (LMI) method is applied to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem in a wide variety of ways leading to a convex optimization. For some examples of this methodology see [29, 25, 46, 32]. The authors in [6] utilize a transfer function approach using Youla parameterization [55]. A unifying formulation and solution to the general LMI-based design, which also includes the multiobjective control is developed in [36].

Some of the methods mentioned above formulate the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem in the general form of minimizing an $\mathcal{H}_2$ performance criterion which is subject to a prespecified $\mathcal{H}_\infty$ constraint with the closed-loop system stable. For this problem, in [5, 28] an auxiliary problem is proposed with an upper bound on the $\mathcal{H}_2/\mathcal{H}_\infty$ performance index and is solved through three Riccati equations.

In [19, 61] a system with both white noise and bounded-power disturbance signal is considered and the problem involves minimizing a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ norm of the system.

The authors in [35] introduced a method based on the Nash game theory, where each of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ criteria are represented independently as the two pay-off functions in a two-player, nonzero sum game. The resulting Nash equilibrium consists of a controller, characterized by cross-coupled Riccati equations, which satisfies both LQG and $\mathcal{H}_\infty$ performance indexes. The main attraction of this approach is that it has a very clear $\mathcal{H}_2/\mathcal{H}_\infty$ interpretation and is solvable through some standard numerical algorithm. A state-feedback controller is solved in [35] and a more general output-feedback solution is given in [13]. More recently, the results in [35] have been generalized to the stochastic system with state-dependant noise [8], and state, input and disturbance-dependant noise [59].

1.2 Discrete-Time Multiobjective Filter

As a natural continuation to the methodology applied in this work, a discrete-time multiobjective filter is developed.

One of the most important problems in signals and systems analysis is the signal estimation for the dynamic systems [1, 41]. The optimal $\mathcal{H}_2$ filter (also known as Kalman filter) [2], which is based on the stochastic noise model with known power spectral densities is a popular signal estimator. However, this technique may be very sensitive to changes in system parameters or other disturbances with unknown spectral densities. For such cases, a better choice is to use an $\mathcal{H}_\infty$ filter, which is developed specifically to address model uncertainty [24, 39], and different techniques have been well developed and applied for different systems (see for example [27, 33, 54] and the references therein). Although $\mathcal{H}_\infty$ filter usually provides much better robustness than $\mathcal{H}_2$ filter, it may not be possible to use it for systems affected by stochastic noise. Clearly, a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter design scheme that can combine the strengths of these two estimation methods in a systematic way is highly desirable.

Several methods have been proposed to carry out the robust optimal filter design and a few examples are given here for different approaches to this problem. In [43] and [30], the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filters are obtained using convex programming characterization. For systems with norm-bounded parameter uncertainties, the problem is solved in [52] and [53] by using Riccati-like equations, where the transfer function from the noise inputs to error state outputs meets an $\mathcal{H}_\infty$-norm upper bound constraint. For discrete-time polytopic systems, [40] obtains the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filters by solving a set of linear matrix inequalities (LMIs), while [26] uses the parameter-dependent stability idea and finds a filter that depends on the parameters, which are assumed to reside in a polytope and be measurable online. A time domain game theoretic approach is proposed in [49] which improves the $\mathcal{H}_2$ performance of the central $\mathcal{H}_\infty$ filter while satisfying the required $\mathcal{H}_\infty$ performance.

In [14], utilizing the game approach, a new formulation called '$\mathcal{H}_\infty$ Gaussian filter' is proposed, and it is shown that the robust optimal filter can be obtained by solving a set of cross-coupled Riccati equations. The result is a Kalman-type filter for uncertain plants and is characterized by the choice of the disturbance attenuation level γ. One advantage of this approach is that optimal state estimation is achieved at the presence of the worst case model uncertainty. Therefore, it clearly reflects the trade-off between the inherently conflicting $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances.

Motivated by the approach in [14], in this dissertation, the Nash game methodology is adopted to derive a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter in discrete time. The design is based on a constrained optimization problem and is characterized by two cross-coupled Riccati equations. As it can be seen, obtaining the discrete-time counterpart of the continuous procedure is not so straightforward. An optimal filter gain is characterized by an equation consisting of the plant parameters and the solutions to the Riccati equations.

1.3 Discrete-Time Multiobjective Control

Most of the signals considered in control systems, such as tracking error or actuator output, are continuous in nature. Also, many performance specifications, such as bandwidth, rise time, etc, are formulated in continuous-time. However, because of the many benefits of the digital technology and the ever-decreasing cost, in many applications, controllers and sometimes sensors are realized using digital technology. Such a system, having both continuous and discrete signals is called a sampled data system.

A widely used approach to design a digital controller for the sampled data system is to first construct the controller in continuous time domain, where the performance specifications are most natural, and then discretize it in order to be implemented by a digital controller. It is expected, at least in theory, that the analog performance is recovered exactly as $T \to 0$, where T is the sampling period. Nevertheless, this method has several practical and theoretical problems. First, during the process of discretization, many desired characteristics of the continuous system, including some of the system norms, are not transferred to discrete time and even the stability is not guaranteed [23, 9]. Furthermore, smaller sampling period requires faster and more expensive hardware. Moreover, the sampling rate is usually limited or fixed by other implementation issues unrelated to the control scheme, which puts more restrictions on the original continuous-time design to be able to produce the desired performance without any distortions. Therefore, it is always advantageous to be able to design a controller directly in discrete-time domain. For these reasons and more, the field of discrete-time control design and analysis has become a significant and ever-growing part of the control systems theory.

Considering the vast amount of work that has been done on mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control in continuous time, only a few references seem to exist that deal with discrete-time domain. The work in [5] has been extended to discrete time in [28], where an LQG output feedback is designed, with a constraint on $\mathcal{H}_\infty$ disturbance attenuation. In [38], the discrete-time counterpart of [19] is carried out and also a special case solution for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Nash equilibrium is suggested. The authors of [59] extend their work to discrete-time systems with state and disturbance dependent noise in [58]. In [15] the $\mathcal{H}_2/\mathcal{H}_\infty$ control is considered for discrete-time, Markovian jump linear systems. Using the game theory approach, a state-feedback controller is derived in [12] which considers a system with a bounded power disturbance signal.

In this dissertation, the Nash game methodology is adopted to derive mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controllers in discrete time. We assume the observer-based structure for the controller and therefore, the measurement noise (characterized by the white noise signal) is considered along with a bounded-power disturbance that can be a representative of the system model uncertainty. As can be seen from this work, extending the continuous-time design procedure to discrete-time is not so straightforward, making the problem worthy to be considered on its own.

Two different frameworks are considered which are called "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control" and "$\mathcal{H}_\infty$ Gaussian Control". Output feedback controllers have been derived which are characterized by three cross-coupled Riccati equations. The plant is assumed to be as general as possible and only some standard assumptions are made on the system. Some numerical examples are included to demonstrate performance indexes and to illustrate the solvability of the proposed procedure.

1.4 Dissertation Overview

This dissertation is organized as follows: after the introduction and background information provided in this chapter, some preliminary definitions and results are collected in chapter 2; in chapter 3 a multiobjective filter problem is presented and solved; the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control is introduced in chapter 4; chapter 5 covers the $\mathcal{H}_\infty$ Gaussian control design method; the application of the proposed solutions is studied on an electric power-assisted steering system in chapter 6; chapter 7 summarizes the conclusions, final remarks and a few suggestions for the future research on this path.

Chapter 2

Preliminary Results

2.1 Signals and Systems

Consider a linear, time-invariant, discrete-time control system described by:

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = 0\,,$$
$$y(k) = Cx(k) + Du(k)\,,$$

where x is the system states vector, u is the control input, y is the output measurement, and the index k represents the value of a signal at the time instance kT, where T is the sampling period. From this point on, we will drop the time index k, and adopt the notation $\delta x := x(k+1)$. The system presented above can then be written as:

$$\begin{aligned} \delta x &= Ax + Bu\,, \quad x(0) = 0\,, \\ y &= Cx + Du\,, \end{aligned} \tag{2.1}$$

The following packed notation is used to define the system transfer function:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(zI - A)^{-1}B + D \tag{2.2}$$

2.1.1 Norms of signals and systems

Definition 1 (Bounded power signal) *Consider the given discrete-time real vector stochastic signal $u(k)$:*

$$u(k) = [u_1(k) \quad u_2(k) \quad \cdots \quad u_m(k)]^T \in \mathbb{R}^m \ \forall k \in \mathbb{Z},$$

where $u_i(k)$, $i = 1, \ldots, m$, are real discrete random processes. We define the mean and autocorrelation matrices, respectively, as:

$$E\{u\} := [E\{u_1(k)\} \quad E\{u_2(k)\} \quad \cdots \quad E\{u_m(k)\}]^T,$$

$$R_{uu}(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{u(k+n)u^T(k)\}.$$

The power spectral density of $u(k)$, is:

$$S_{uu}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} R_{uu}(k)e^{-j\omega k}.$$

A stationary stochastic vector signal is said to have bounded power if:

- *both R_{uu} and S_{uu} exist;*
- $lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} E\{\| u(k) \|_2^2\} < \infty$.

Definition 2 ($\mathcal{P}$-norm) *Let $\mathcal{P}$ be the set of all signals with bounded power, we define the seminorm:*

$$\| u \|_{\mathcal{P}}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\| u(k) \|^2\}. \tag{2.3}$$

Note that whenever the unsubscripted norm $\| \cdot \|$ is used, it refers to the standard euclidian norm on vectors.

Definition 3 (Mutually uncorrelated signals) *Two stochastic vector signals u_1 and u_2 are said to be mutually uncorrelated if:*

$$\big(E\{u_1\} = 0 \quad or \quad E\{u_2\} = 0\big) \quad and \quad E\{u_1(k_1)u_2^T(k_2)\} = 0, \quad \forall k_1, k_2 \in \mathbb{Z}.$$

Figure 2.1: A general system with input and output signals

Definition 4 ($\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of systems) *Consider a given discrete-time system $G(z)$ in Figure 2.1, with state space realization (A, B, C, D), and denote w and z as the input and output signals to the system, respectively. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of this system are defined as:*

$$\| G \|_2 = \left\{ \frac{1}{2\pi} \int_0^{2\pi} trace\big[G(e^{-j\theta})^* G(e^{-j\theta})\big] d\theta \right\}, \qquad \| G \|_\infty = \max_\theta \overline{\sigma}\big[G(e^{-j\theta})\big],$$

where $\overline{\sigma}$ is the maximum singular value of $G(z)$.

Note that if w is a bounded power signal, it can be shown that $\| G \|_\infty = \sup_w \frac{\|z\|_{\mathcal{P}}}{\|w\|_{\mathcal{P}}}$ (see [9, 60]). Moreover, if $w = w_0$ is a white noise signal, it can be shown that $\| G \|_2 = \| z \|_{\mathcal{P}}$. Therefore, it is easy to validate that:

$$\| G \|_\infty < \gamma \iff 0 < \gamma^2 \| w \|_{\mathcal{P}}^2 - \| z \|_{\mathcal{P}}^2, \quad \forall w \neq 0 .$$

2.1.2 Expected value lemmas

Lemma 2.1 *Consider a dynamic system described by:*

$$\delta x = Ax + B_0 w_0 + B_2 u, \quad x(0) = 0 \tag{2.4}$$

$$z = C_1 x + D_{12} u, \tag{2.5}$$

$$y = C_2 x + D_{20} w_0, \tag{2.6}$$

where w_0 is a white noise signal and the controller, u, is given by $K(z) = C_K(zI - A_K)^{-1} B_K$ with its associated state variable $\bar{x}$. Then we have:

$$E\{x(k) w_0^T(l)\} = \begin{cases} (e_{11} B_0 + e_{12} B_K D_{20}), & if\, k \geq l+1 \\ 0, & if\, k < l+1 \end{cases}$$

where

$$\hat{A} = \begin{bmatrix} A & B_2C_K \\ B_KC_2 & A_K \end{bmatrix}, \quad \hat{A}^{k-l-1} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}.$$

Proof: The closed-loop system consisting of the plant (2.4)-(2.6) and the controller $K(z) = C_K(zI - A_K)^{-1}B_K$ can be written as:

$$\delta\hat{x} = \hat{A}\hat{x} + \hat{B}_0 w_0\,,$$

where $\hat{x}$ is the state vector of the augmented system, $\hat{x} = [x^T \quad \bar{x}^T]^T$, and $\hat{B}_0 = [B_0^T \quad (B_K D_{20})^T]^T$.

The solution to this difference equation is:

$$\begin{aligned}\hat{x}(k) &= \hat{A}^k\hat{x}(0) + \sum_{j=0}^{k-1}\hat{A}^{k-j-1}\hat{B}_0 w_0(j) \\ &= \sum_{j=0}^{k-1}\hat{A}^{k-j-1}\hat{B}_0 w_0(j), \quad k = 1, 2, 3, \cdots\end{aligned}$$

and

$$\begin{aligned}E\{\hat{x}(k)w_0^T(l)\} &= E[w_0(l)x^T(k) \quad w_0(l)\bar{x}^T(k)]^T = E\Big\{\sum_{j=0}^{k-1}\hat{A}^{k-j-1}\hat{B}_0 w_0(j)w_0^T(l)\Big\} \\ &= \sum_{j=0}^{k-1}\hat{A}^{k-j-1}\hat{B}_0 E\left[w_0(j)w_0^T(l)\right] \\ &= \sum_{j=0}^{k-1}\hat{A}^{k-j-1}\hat{B}_0\delta(l-j) \\ &= \begin{cases}\hat{A}^{k-l-1}\hat{B}_0, & \text{if } k \geq l+1 \\ 0, & \text{if } k < l+1\end{cases}\end{aligned}$$

or

$$E\{x(k)w_0^T(l)\} = \begin{cases}(e_{11}B_0 + e_{12}B_K D_{20}), & \text{if } k \geq l+1 \\ 0, & \text{if } k < l+1\end{cases}$$

■

Lemma 2.2 *Consider a dynamic system described by:*

$$\delta x = Ax + B_0 w_0 + B_1 w + B_2 u, \quad x(0) = 0 \tag{2.7}$$

$$z = C_1 x + D_{12} u\,, \tag{2.8}$$

$$y = C_2 x + D_{20} w_0\,, \tag{2.9}$$

where w_0 is a white noise signal, w is a stochastic signal and w_0 and w are mutually uncorrelated. If the controller, u, is given by $K(z) = C_K(zI - A_K)^{-1}B_K$ with its associated state variable $\bar{x}$. Then we have:

$$E\{x(k)w_0^T(l)\} = \begin{cases} (e_{11}B_0 + e_{12}B_K D_{20}), & \text{if } k \geq l+1 \\ 0, & \text{if } k < l+1 \end{cases}$$

where

$$\hat{A} = \begin{bmatrix} A & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix}, \quad \hat{A}^{k-l-1} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}.$$

Proof: The closed-loop system is:

$$\delta \hat{x} = \hat{A}\hat{x} + \hat{B}_0 w_0 + \hat{B}_1 w\,,$$

where $\hat{x} = [x^T \quad \bar{x}^T]^T$, $\hat{B}_0 = [B_0^T \quad (B_K D_{20})^T]^T$, and $\hat{B}_1 = [B_1^T \quad 0]^T$.

Therefore,

$$\begin{aligned} \hat{x}(k) &= \hat{A}^k \hat{x}(0) + \sum_{j=0}^{k-1} \hat{A}^{k-j-1}[\hat{B}_0 w_0(j) + \hat{B}_1 w(j)] \\ &= \sum_{j=0}^{k-1} \hat{A}^{k-j-1}[\hat{B}_0 w_0(j) + \hat{B}_1 w(j)], \quad k = 1, 2, 3, \cdots \end{aligned}$$

and since w_0 and w are mutually uncorrelated, we can write

$$
\begin{aligned}
E\{\hat{x}(k)w_0^T(l)\} = E[w_0(l)x^T(k) \quad w_0(l)\bar{x}^T(k)]^T &= E\Big\{\sum_{j=0}^{k-1}\hat{A}^{k-j-1}[\hat{B}_0 w_0(j) + \hat{B}_1 w(j)]w_0^T(l)\Big\} \\
&= \sum_{j=0}^{k-1}\hat{A}^{k-j-1}\hat{B}_0 E\left[w_0(j)w_0^T(l)\right] + \sum_{j=0}^{k-1}\hat{A}^{k-j-1}\hat{B}_1 E\left[w(j)w_0^T(l)\right] \\
&= \sum_{j=0}^{k-1}\hat{A}^{k-j-1}\hat{B}_0\delta(l-j) \\
&= \begin{cases}\hat{A}^{k-l-1}\hat{B}_0, & \text{if } k \geq l+1 \\ 0, & \text{if } k < l+1\end{cases}
\end{aligned}
$$

or

$$
E\{x(k)w_0^T(l)\} = \begin{cases}(e_{11}B_0 + e_{12}B_K D_{20}), & \text{if } k \geq l+1 \\ 0, & \text{if } k < l+1\end{cases}
$$

■

2.2 Discrete-Time LQG Control

In this section the discrete-time $\mathcal{H}_2$ control problem is presented along with the solution to the state-feedback controller, which is used in the derivations of this work.

Consider the general configuration of a plant G, connected to the controller C, depicted in Figure 2.2. In this setup, the plant measurement output y is fed to the controller, which in turn provides the control signal u to be applied to the plant. The exogenous signal is represented by w and the performance signal z is used for design purpose and is the signal to be controlled.

The plant G can be described by the set of difference equation:

$$
\begin{aligned}
\delta x &= Ax + B_1 w + B_2 u \\
z &= C_1 x + D_{11} w + D_{12} u \\
y &= C_2 x + D_{21} w + D_{22} u
\end{aligned}
\tag{2.10}
$$

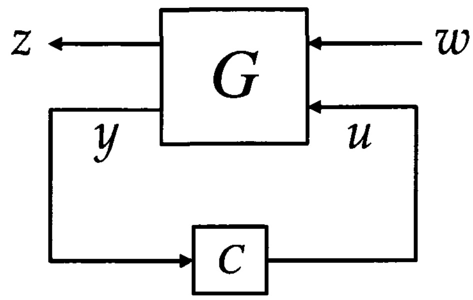

Figure 2.2: Plant and controller general configuration

It can be argued that (see [44]), one can set D_{11} and D_{22} to zero without any loss of generality. Therefore, the above equation can be rewritten as:

$$\begin{aligned} \delta x &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned} \tag{2.11}$$

The controller is then designed to have the general form:

$$\begin{aligned} \delta x_c &= A_c x_c + B_c y \\ u &= C_c x_c + D_c y \,, \end{aligned} \tag{2.12}$$

where x_c is the state vector of the controller.

The closed-loop system in Figure 2.2 consisting of G and C constructs a lower *linear fractional transformation* (LFT) and is represented by $\mathcal{F}_l(G, C)$. Furthermore, assuming T_{zw} to be the transfer function matrix from w to z, we have:

$$T_{zw} = \mathcal{F}_l(G, C) \,.$$

Definition 5 ($\mathcal{H}_2$ optimal control problem) *Consider a system G as given by (2.11). The $\mathcal{H}_2$ optimal control is defined as the problem of finding, if it exists, an admissible controller C, which minimizes the performance index $J = \| \mathcal{F}_l(G, C) \|_2^2$ over all the admissible controllers.*

The following theorem presents the solution to the state-feedback controller design problem:

Theorem 2.1 *Consider an $\mathcal{H}_2$ optimal control problem as defined by Definition 5 for a system G as in (2.11). Assume that the entire state is available for feedback, i.e. assume that $C_2 = I$ and $D_{21} = 0$. Let the class of controllers be taken as in (2.12) with $y = x$. Then, there exists a unique proper dynamic state feedback $\mathcal{H}_2$ optimal controller if and only if the following conditions hold:*

1. *(A, B_2) is stabilizable,*
2. *$M := D_{12}^T D_{12}$ is nonsingular,*
3. *the matrix $\begin{bmatrix} A - \lambda I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank, $\forall \lambda \in \partial\mathbb{D}$.*

Moreover, in this case, the unique $\mathcal{H}_2$ optimal controller for plant (2.11) is given by:

$$u = -(B_2 P B_2 + M)^{-1}(B_2^T P A + D_{12}^T C_1)\, x \,, \tag{2.13}$$

where P is the unique, positive semi-definite solution of:

$$P = A^T P A - (C_1^T D_{12} + A^T P B_2)(M + B_2^T P B_2)^{-1}(D_{12} C_1 + B_2^T P A) + C_1^T C_1 \,, \tag{2.14}$$

and $D_{12}^T D_{12} + B_2^T P B^2 > 0$.

Proof: A complete proof of this theorem can be found in many reference books (see for example [44]). ■

2.3 Discrete-Time $\mathcal{H}_\infty$ Control

Consider again the system in the general form of Figure 2.2. The plant G is described by (2.10). Let $\gamma > 0$ be a prescribed level of disturbance attenuation, the so-called suboptimal $\mathcal{H}_\infty$ control problem is defined as follows:

Definition 6 (Suboptimal $\mathcal{H}_\infty$ control problem) *Given a $\gamma > 0$, find an admissible controller, if it exists, such that $\| T_{zw} \|_\infty < \gamma$.*

The solution to the aforementioned problem is constructed on a fundamental concept known as the *bounded real lemma*, introduced in the next section.

2.3.1 Bounded Real Lemma

The discrete-time form of the bounded real lemma, as one of the most important building blocks of the $\mathcal{H}_\infty$ control theory as well as the work presented in this dissertation, is introduced in the following theorem.

Theorem 2.2 *Let $P(z)$ be a $p \times m$ real rational transfer function matrix of a proper linear discrete-time system with state-space realization (A, B, C, D), i.e.,*

$$P(z) = C(zI - A)^{-1}B + D\,.$$

The following statements are equivalent:

(a) A is a stable matrix and $\| P \|_\infty < \gamma$.

(b) There exists a stabilizing solution $P = P^T \geq 0$ to the Riccati equation:

$$P = A^T PA + \gamma^{-2}(A^T PB + C^T D)\left[I - \gamma^{-2}(D^T D + B^T PB)\right]^{-1}(B^T PA + D^T C) + C^T C\,, \tag{2.15}$$

such that $I - \gamma^{-2}(D^T D + B^T PB) > 0$.

Proof: See [17] for a comprehensive version of the proof of this lemma. ■

2.3.2 $\mathcal{H}_\infty$ State-Feedback Control

The direct design of a discrete-time $\mathcal{H}_\infty$ controller is much more complicated than its continuous-time counterpart. For this reason, in most of the literature, an easier way is suggested which is converting to a continuous-time problem via bilinear transformation. The reason that this works is that the bilinear transformation preserves $\mathcal{H}_\infty$ norms (where it does not preserves $\mathcal{H}_2$ norms, for instance).

However, the discrete-time problem has been addressed and solved directly by different approaches. The following lemma provides the solution to the state-feedback control design problem. This result is used in the derivations of this work.

Lemma 2.3 *For the dynamic system:*

$$\begin{aligned} \delta x &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \end{aligned}$$

where (A, B_2) *is stabilizable and* $D_{12}^T D_{12} > 0$*, an state-feedback controller* $u = Kx$ *that achieves the* $\mathcal{H}_\infty$ *performance, i.e:*

1. *the closed-loop matrix* $A_c = A + B_2 K$ *is stable,*
2. *the closed-loop transfer function matrix,* T_{zw}*, from* w *to* z *satisfies* $\| T_{zw} \|_\infty < \gamma$*,*

where $\gamma > 0$ *is a prescribed level of disturbance attenuation, can be written as:*

$$K = -S(B^T P_\infty B + \hat{R})^{-1}(B^T P_\infty A + D^T C_1)$$

where

$$S = [0 \quad I], \qquad B = [\gamma^{-1} B_1 \quad B_2],$$

$$D = [\gamma^{-1} D_{11} \quad D_{12}], \qquad \hat{R} = D^T D - [I \quad 0]^T [I \quad 0]$$

and P_∞ *is the positive semi-definite solution to the equation:*

$$P_\infty = A^T P_\infty A - (A^T P_\infty B + C_1^T D)(B^T P_\infty B + \hat{R})^{-1}(B^T P_\infty A + D^T C_1) + C_1^T C_1$$

Note that the above controller has the property:

$$I - \gamma^{-2} B_1^T P_\infty B_1 > 0 \tag{2.16}$$

Furthermore, the closed-loop matrix $A - B(B^T P_\infty B + \hat{R})^{-1}(B^T P_1 A + D^T C_1)$ *is stable.*

Proof: A proof of this lemma, derived from the discrete-time bounded real lemma can be found in [17]. ■

Lemma 2.4 *For the dynamic system:*

$$\begin{aligned} \delta x &= Ax + B_1 w + B_2 u, \quad x(0) = 0 \\ z &= C_1 x + D_{12} u \end{aligned} \tag{2.17}$$

Define the cost function:

$$J_1(u, w, w_0) = \gamma^2 \parallel w \parallel_{\mathcal{P}}^2 - \parallel z \parallel_{\mathcal{P}}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\gamma^2 \parallel w \parallel^2 - \parallel z \parallel^2\}$$

If the state-feedback controller is given by:

$$\tilde{u}_* = -S(B^T P_1 B + \hat{R})^{-1}(B^T P_1 A + D^T C_1)x = -K_1 x$$

where S,B,D and $\hat{R}$ *are defined similar to Lemma 2.3, with* $D_{11} = 0$, *and* P_1 *is the solution to:*

$$P_1 = A^T P_1 A - (A^T P_1 B + C_1^T D)(B^T P_1 B + R)^{-1}(B^T P_1 A + D^T C_1) + C_1^T C_1 \tag{2.18}$$

the worst-case signal $\tilde{w}_*$, *for which* $J_1(\tilde{u}_*, \tilde{w}_*, 0) \leq J_1(\tilde{u}_*, w, 0)$, *is:*

$$\tilde{w}_* = \gamma^{-2} B_1^T P_1 (I - \gamma^{-2} B_1 B_1^T P_1)^{-1} \tilde{A} x = K_2 x \tag{2.19}$$

where $\tilde{A} = A - B_2 K_1$.

Proof: After substituting the controller $\tilde{u}_* = -K_1 x$ into system (2.17), we have:

$$\begin{aligned} \delta x &= \tilde{A}x + B_1 w \\ z &= \tilde{C}_1 x \end{aligned} \tag{2.20}$$

where

$$\tilde{A} = A - B_2 K_1\,, \quad \tilde{C}_1 = C_1 - D_{12} K_1$$

To find the worst disturbance signal $\tilde{w}_*$, we apply the discrete-time bounded real lemma to system (2.20), for which we require $\| T_{zw} \|_\infty < \gamma$. Therefore, P_1, is the solution to:

$$\tilde{A}^T P_1 \tilde{A} - P_1 + \gamma^{-2} \tilde{A}^T P_1 B_1 (I - \gamma^{-2} B_1^T P_1 B_1)^{-1} B_1^T P_1 \tilde{A} + \tilde{C}_1^T \tilde{C}_1 = 0 \tag{2.21}$$

Now, completing the squares, using (2.21) and introducing the new variable $V = x^T P_1 x$, we have:

$$\begin{aligned} \delta V - V &= (\delta x)^T P_1 (\delta x) - x^T P_1 x \\ &= (\tilde{A}x + B_1 w)^T P_1 (\tilde{A}x + B_1 w) - x^T P_1 x \\ &= x^T (\tilde{A}^T P_1 \tilde{A} - P_1) x + 2w^T B_1^T P_1 \tilde{A} x + w^T B_1^T P_1 B_1 w \\ &= -\gamma^{-2} x^T \tilde{A}^T P_1 B_1 (I - \gamma^{-2} B_1^T P_1 B_1)^{-1} B_1^T P_1 \tilde{A} x - x^T \tilde{C}_1^T \tilde{C}_1 x + 2w^T B_1^T P_1 \tilde{A} x \\ &\quad + w^T B_1^T P_1 B_1 w \\ &= -\gamma^{-2} \| (I - \gamma^{-2} B_1^T P_1 B_1)^{-\frac{1}{2}} B_1^T P_1 \tilde{A} x \|^2 + 2w^T B_1^T P_1 \tilde{A} x + w^T B_1^T P_1 B_1 w \\ &\quad - \gamma^2 w^T w + \gamma^2 w^T w - x^T \tilde{C}_1^T \tilde{C}_1 x \end{aligned}$$

Noting that $w^T B_1^T P_1 B_1 w - \gamma^2 w^T w = -\gamma^2 w^T (I - \gamma^{-2} B_1^T P_1 B_1) w$, we proceed:

$$\begin{aligned} \delta V - V &= - \| \gamma^{-1} (I - \gamma^{-2} B_1^T P_1 B_1)^{-\frac{1}{2}} B_1^T P_1 \tilde{A} x \|^2 - \| \gamma (I - \gamma^{-2} B_1^T P_1 B_1)^{\frac{1}{2}} w \|^2 \\ &\quad + 2w^T B_1^T P_1 \tilde{A} x + \gamma^2 \| w \|^2 - \| z \|^2 \\ &= - \| \gamma (I - \gamma^{-2} B_1^T P_1 B_1)^{\frac{1}{2}} w - \gamma^{-1} (I - \gamma^{-2} B_1^T P_1 B_1)^{-\frac{1}{2}} B_1^T P_1 \tilde{A} x \|^2 \\ &\quad + \gamma^2 \| w \|^2 - \| z \|^2 \end{aligned}$$

Then the cost function will be:

$$\begin{aligned} J_1(\tilde{u}_*, w, 0) &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\gamma^2 \parallel w \parallel^2 - \parallel z \parallel^2\} \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\delta V - V + \parallel \gamma(I - \gamma^{-2} B_1^T P_1 B_1)^{\frac{1}{2}} w \\ &\qquad - \gamma^{-1}(I - \gamma^{-2} B_1^T P_1 B_1)^{-\frac{1}{2}} B_1^T P_1 \tilde{A} x \parallel^2\} \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\parallel \gamma(I - \gamma^{-2} B_1^T P_1 B_1)^{\frac{1}{2}} w \\ &\qquad - \gamma^{-1}(I - \gamma^{-2} B_1^T P_1 B_1)^{-\frac{1}{2}} B_1^T P_1 \tilde{A} x \parallel^2\} \end{aligned}$$

For $J_1(\tilde{u}_*, \tilde{w}_*, 0) \leq J_1(\tilde{u}_*, w, 0)$ to hold for all values of w, we have:

$$\tilde{w}_* = \gamma^{-2}(I - \gamma^{-2} B_1^T P_1 B_1)^{-1} B_1^T P_1 \tilde{A} x = K_2 x \tag{2.22}$$

or equivalently:

$$\tilde{w}_* = \gamma^{-2} B_1^T P_1 (I - \gamma^{-2} B_1 B_1^T P_1)^{-1} \tilde{A} x = K_2 x$$

■

2.4 Constrained Optimization

The constrained optimization problem presented in this section plays an important role in the main derivations of this work and is solved here in detail.

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times r}$ and $R = DD^T > 0$, define the index function:

$$J(L) = \text{trace}(QPQ^T) \tag{2.23}$$

where Q is any constant weighting matrix, $A + LC$ is Hurwitz, and $P = P^T \geq 0$ satisfies:

$$P = (A + LC)P(A + LC)^T + (B + LD)(B + LD)^T \quad (2.24)$$

The constrained optimization problem is stated as follows:

Problem: Find (L_*, P_*) where $A + L_*C$ is Hurwitz, such that $J(L)$ is minimized at L_*, i.e.

$$\min_L J(L) = \min_P \text{trace}(QPQ^T)$$

where (L, P) and (L_*, P_*) are all subject to constraint (2.24).

Theorem 2.3 *For the constrained optimization problem stated above, suppose (C, A) is detectable. If there is a solution P_* for*

$$P_* = AP_*A^T - (BD^T + AP_*C^T)(R + CP_*C^T)^{-1}(DB^T + CP_*A^T) + BB^T \quad (2.25)$$

where $R+CP_*C^T > 0$, i.e., $A-(AP_*C^T+BD^T)(R+CP_*C^T)^{-1}C$ is stable, then $J(L)$ achieves the minimum value at $L_* = -(AP_*C^T + BD^T)(R + CP_*C^T)^{-1}$.

Conversely, let (C, A) be detectable. If there are L_1 and $P_1 \geq 0$, where $A + L_1C$ is Hurwitz and P_1 solves

$$P_1 = (A + L_1C)P_1(A + L_1C)^T + (B + L_1D)(B + L_1D)^T$$

such that $J(L)$ is minimized at L_1, then there is a $P_* \geq 0$ solving

$$P_* = AP_*A^T - (BD^T + AP_*C^T)(R + CP_*C^T)^{-1}(DB^T + CP_*A^T) + BB^T$$

where $R + CP_*C^T > 0$.

Moreover, the optimal L_* can be found as $L_* = -(AP_*C^T+BD^T)(R+CP_*C^T)^{-1}$ if $A+L_*C$ is Hurwitz.

Proof: (*Sufficiency*) For any L for which $A+LC$ is Hurwitz, there is a $P \geq 0$ solving

$$P = (A+LC)P(A+LC)^T + (B+LD)(B+LD)^T$$

On the other hand, since P_* is a stabilizing solution, so $A + L_*C$ is Hurwitz, for $L_* = -(AP_*C^T + BD^T)(R + CP_*C^T)^{-1}$. Using (2.25) , (2.24) can be rewritten as

$$\begin{aligned} P = APA^T + APC^TL^T + LCPA^T + LCPC^TL^T + P_* - AP_*A^T - L(R + CP_*C^T)L_*^T \\ - LCP_*A^T + L_*(R + CP_*C^T)L_*^T - L_*(R + CP_*C^T)L^T - AP_*C^TL^T + LRL^T \end{aligned}$$

If we define $\Delta P = P - P_*$, then the above expression can be simplified into

$$\Delta P = (A+LC)\Delta P(A+LC)^T + (L-L_*)(R+CP_*C^T)(L-L_*)^T$$

From this Lyapunov equation, it is obvious that $\Delta P \geq 0$ and also $\Delta P = 0$ if and only if $L = L_*$. Hence

$$J(L) - J(L_*) = \text{trace}(Q\Delta PQ^T) \geq 0$$

or in other words, $J(L)$ achieves the minimum value at L_*, which concludes the proof for the sufficiency condition.

Before presenting the proof of the necessity condition, we need to set up some introductory definitions and results.

Definition 7 *Define the set* $S_L \subset \mathbb{R}^{n \times p}$ *as*

$$S_L = \{L : \; L \in \mathbb{R}^{n \times p}, \; A + LC \textit{ is Hurwitz}\}$$

and $S_P \subset \mathbb{R}^{n \times n}$ *as*

$$\begin{aligned} S_P = \{P : P \in \mathbb{R}^{n \times n}, \; P = P^T \textit{ and} \\ (A+LC)P(A+LC)^T + (B+LD)(B+LD)^T - P = 0, \textit{ for some } L \in S_L\} \end{aligned}$$

By inspecting these definitions, it can be concluded that $P \geq 0$ if $P \in S_P$.

Lemma 2.5 *For any $L \in S_L$, there is one and only one $P \in S_P$ solving*

$$(A+LC)P(A+LC)^T + (B+LD)(B+LD)^T = P$$

Proof: For any $L \in S_L$, there is a $P \geq 0$ solving

$$(A+LC)P(A+LC)^T + (B+LD)(B+LD)^T - P = 0$$

which leads to a $P \in S_P$. Now assume a $P_1 \in S_P$ also solves the above equation, or

$$(A+LC)P_1(A+LC)^T + (B+LD)(B+LD)^T - P_1 = 0$$

Now define $\Delta P = P - P_1$ and combine the above two equation to get

$$(A+LC)\Delta P(A+LC)^T = 0$$

This results in $\Delta P = 0$ or $P = P_1$.

□

Consider the sequences $\{P_i\,,\ i = 1,2,3,\cdots\}$ in $\mathbb{R}^{n\times n}$ and $\{L_i\,,\ i = 2,3,\cdots\}$ in $\mathbb{R}^{n\times p}$, where $L_{i+1} = -(AP_iC^T + BD^T)(R + CP_iC^T)^{-1}$. The limits of these sequences are defined as follows:

Definition 8 *P_* and L_* are said to be the limits of $\{P_i\}$ and $\{L_i\}$, respectively, if for any $x \in \mathbb{R}^n$,*

$$x^T P_* x = \lim_{i\to\infty} x^T P_i x\,, \quad L_* = -(AP_*C^T + BD^T)(R + CP_*C^T)^{-1}$$

where $R + CP_*C^T > 0$.

If these limits exist, denote

$$P_* = \lim_{i\to\infty} P_i\,, \quad L_* = \lim_{i\to\infty} L_{i+1} = -\lim_{i\to\infty}(AP_iC^T + BD^T)(R + CP_iC^T)^{-1}$$

where $R + CP_iC^T > 0$ for $i = 1,2,\cdots$.

Proposition 1 *A sequence $\{P_i\}$ converges to some P_* if and only if the convergence is entry-wise, that is, if p^i_{kj} and p_{kj*} are entries to P_i and P_*, respectively, then*

$$p_{kj*} = \lim_{i\to\infty} p^i_{kj}\,, \quad k,j = 1,2,\cdots,n$$

Proof: First, if the convergence is entry-wise, or:

$$p_{kj*} = \lim_{i\to\infty} p^i_{kj}\,, \quad k,j = 1,2,\cdots,n$$

then for any $x \in \mathbb{R}^n$, we have

$$\lim_{i\to\infty} x^T P_i x = \lim_{i\to\infty} \sum_{k,j} p^i_{kj} x_k x_j = \sum_{k,j} \lim_{i\to\infty} p^i_{kj} x_k x_j = \sum_{k,j} p_{kj*} x_k x_j = x^T P_* x$$

Then, from the definition of P_*,

$$P_* = \lim_{i\to\infty} P_i$$

Conversely, if $\{P_i\}$ converges to a P_*, or

$$x^T P_* x = \lim_{i\to\infty} x^T P_i x\,, \quad \forall x \in \mathbb{R}^n$$

then

$$\sum_{j,q} p_{jq*} x_j x_q = \lim_{i\to\infty} \sum_{j,q} p^i_{jq} x_j x_q = \sum_{j,q} \lim_{i\to\infty} p^i_{jq} x_j x_q$$

By inspecting the above equation, bearing in mind that x is arbitrary, we can conclude:

$$P_{jq*} = \lim_{i\to\infty} P^i_{jq}\,, \quad j,q = 1,2,\cdots,n$$

In other words, the convergence of $\{P_i\}$ to P_* is entry-wise. □

The following procedures provides us with special sequences $\{P_i\}$ and $\{L_i\}$ that are particularly important to our proof:

Procedures:

1. Choose an L_1 from S_L,

2. Solve for P_i, $i = 1, 2, \cdots$:

$$(A + L_iC)P_i(A + L_iC)^T + (B + L_iD)(B + L_iD)^T - P_i = 0,$$

3. Set $L_{i+1} = -(AP_iC^T + BD^T)(R + CP_iC^T)^{-1}$, where $R + CP_iC^T > 0$ for $i = 1, 2, \cdots$.

Proposition 2 *The sequences $\{P_i\}$ and $\{L_i\}$, generated by Procedures 1-3, always have limits P_* and L_*, respectively.*

Proof: To prove that $\{P_i\}$ has a limit P_* for $i = 1, 2, \cdots$, consider:

$$P_i = (A + L_iC)P_i(A + L_iC)^T + (B + L_iD)(B + L_iD)^T$$

and

$$P_{i+1} = (A + L_{i+1}C)P_{i+1}(A + L_{i+1}C)^T + (B + L_{i+1}D)(B + L_{i+1}D)^T$$

where L_{i+1} is calculated by *Procedure 3.*

Define $\Delta P_i = P_{i+1} - P_i$ and $\Delta L_i = L_{i+1} - L_i$. Combining the above two equations, we get:

$$\Delta P = (A + L_{i+1}C)\Delta P(A + L_{i+1}C)^T - \Delta L(R + CP_iC^T)\Delta L^T$$

Since $R + CP_iC^T > 0$, the above equation gives that $\Delta P \leq 0$, which means that:

$$0 \leq \cdots \leq P_3 \leq P_2 \leq P_1 \ \Rightarrow\ 0 \leq \cdots \leq x^TP_{i+1}x \leq x^TP_ix \leq \cdots \leq x^TP_2x \leq x^TP_1x\,, \quad \forall x \in \mathbb{R}^n$$

Then, by definition, $\lim_{i\to\infty} x^TP_ix$ exists and

$$\lim_{i\to\infty} x^TP_ix = \lim_{i\to\infty}\sum_{k,j} p^i_{kj}x_kx_j = \sum_{k,j}\lim_{i\to\infty} p^i_{kj}x_kx_j = \sum_{k,j} p_{kj*}x_kx_j = x^TP_*x$$

where $p_{kj*} = \lim_{i\to\infty} p^i_{kj}$. This means that $\{P_i\}$ has a limit P_* and, obviously, so does $\{L_i\}$ with:

$$L_* = \lim_{i\to\infty} L_{i+1} = -\lim_{i\to\infty}(AP_iC^T+BD^T)(R+CP_iC^T)^{-1} = -(AP_*C^T+BD^T)(R+CP_*C^T)^{-1}$$

□

Lemma 2.6 *For the sequences $\{P_i\}$ and $\{L_i\}$, generated by Procedures 1-3, if P_* and L_* are the limits of these sequences, then P_* also solves*

$$P_* = (A+L_*C)P_*(A+L_*C)^T + (B+L_*D)(B+L_*D)^T \tag{2.26}$$

where $L_* = -(AP_*C^T + BD^T)(R+CP_*C^T)^{-1}$.

Furthermore, $A-(AP_*C^T+BD^T)(R+CP_*C^T)^{-1}C$ is stable and P_* also solves the Riccati equation:

$$P_* = AP_*A^T - (BD^T + AP_*C^T)(R+CP_*C^T)^{-1}(DB^T + CP_*A^T) + BB^T$$

where $R + CP_*C^T > 0$.

Proof: Let p^i_{kj} and p_{kj*} be entries of P_i and P_*, respectively. Also let l^i_{mq} and l_{mq*} be entries of L_i and L_*, respectively. By Proposition 1, we have the entry-wise convergence:

$$p_{kj*} = \lim_{i\to\infty} p^i_{kj}\,, \quad k,j = 1,2,\cdots,n$$

and accordingly, for any $m = 1,\cdots,n$ and $q = 1,\cdots,p$,

$$l_{mq*} = \lim_{i\to\infty} l^i_{mq}(p^i_{kj}\,,\ k,j=1,2,\cdots,n) = l^i_{mq}(\lim_{i\to\infty} p^i_{kj}\,,\ k,j=1,2,\cdots,n)$$

since l^i_{mq} is a continuous function of $p^i_{kj}\,,\ k,j = 1,2,\cdots,n$.

Now define

$$F(P_i,L_i) = (A+L_iC)P_i(A+L_iC)^T + (B+L_iD)(B+L_iD)^T - P_i$$

where, clearly $F(P_i,L_i) = 0,\ \forall i = 1,2,\cdots$. Let $f^i_{kj}\,, k,j = 1,2,\cdots,n$ be the entries of $F(P_i,L_i)$, which will be continuous functions of all p^i_{kj} and l^i_{mq}. Therefore

$$f_{kj*} = \lim_{i\to\infty} f_{kj}^i(p_{kj}^i, l_{mq}^i) = 0\,, \quad k, j = 1, 2, \cdots, n$$

Which means that $F(P_*, L_*) = 0$ or

$$P_* = (A + L_*C)P_*(A + L_*C)^T + (B + L_*D)(B + L_*D)^T$$

where $L_* = -(AP_*C^T + BD^T)(R + CP_*C^T)^{-1}$.

By the standard properties of Lyapunov equations, it is evident that $A + L_*C = A - (AP_*C^T + BD^T)(R + CP_*C^T)^{-1}$ is stable.

Now, substitute L_* in (2.26), to get, after some manipulation:

$$\begin{aligned} P_* &= AP_*A^T + AP_*C^TL_*^T + L_*CP_*A^T + L_*(CP_*C^T + R)L_*^T + BB^T + BD^TL_*^T + L_*DB^T \\ &= AP_*A^T + L_*(CP_*A^T + DB^T) + BB^T \end{aligned}$$

or

$$P_* = AP_*A^T - (BD^T + AP_*C^T)(R + CP_*C^T)^{-1}(DB^T + CP_*A^T) + BB^T$$

□

At this point, we are ready to complete the proof of the Theorem 2.3.

Proof of Theorem 2.3, cont'd: (Necessity) If there are $L_1 \in S_L$ and $P_1 \in S_P$ such that

$$P_1 = (A + L_1C)P_1(A + L_1C)^T + (B + L_1D)(B + L_1D)^T$$

and $J(L)$ achieves the minimum value at L_1, take L_1 as the initial value and generate the sequences $\{P_i\}$ and $\{L_i\}$ using the *Procedures 1-3*, then by Proposition 2, the following claims can be made:

1. $0 \le \cdots \le P_{i+1} \le P_i \le \cdots \le P_2 \le P_1$,

2. $\{P_i\}$ and $\{L_i\}$ have limit points P_* and L_*, respectively, where $P_* \le P_1$.

Furthermore, by Lemma 2.6, P_* and $L_* = -(AP_*C^T + BD^T)(R + CP_*C^T)^{-1}$ solve

$$P_* = (A + L_*C)P_*(A + L_*C)^T + (B + L_*D)(B + L_*D)^T$$

If $A + L_*C = A - (AP_*C^T + BD^T)(R + CP_*C^T)^{-1}$ is stable, then

$$J(L_*) = \text{trace}(QP_*Q^T) \leq \text{trace}(QP_1Q^T) = J(L_1)$$

On the other hand, it was first assumed that

$$J(L_1) \leq J(L_*)$$

which leads to $J(L_*) = J(L_1)$ or, in other words, $J(L)$ achieves the minimum value at $L_* = -(AP_*C^T + BD^T)(R + CP_*C^T)^{-1}$.

■

Chapter 3

Discrete-Time Multiobjective Filter

This chapter provides the problem formulation and solution to a discrete-time multiobjective filter design. An illustrative example is also included to show the solvability and performance of the proposed filter.

3.1 Problem Formulation

Consider the filter design problem in Figure 3.1. For the plant G, described by:

$$\begin{aligned} \delta x &= Ax + B_0 w_0 + B_1 w\,, \quad x(0) = 0\,, \\ z_0 &= C_0 x\,, \\ z &= C_1 x\,, \\ y &= C_2 x + D_{20} w_0\,, \end{aligned} \tag{3.1}$$

where w is a bounded power signal and w_0 is a white noise signal. The following standard assumptions are made:

(A1) (C_2, A) is detectable ;

(A2) $R_0 := D_{20}D_{20}^T > 0$;

(A3) $\begin{bmatrix} A - \lambda I & B_0 \\ C_2 & D_{20} \end{bmatrix}$ has full row rank, $\forall \lambda \in \partial\mathbb{D}$,

where $\partial\mathbb{D} := \{z : |z| = 1\}$ describes the points on the unit circle in the complex plane.

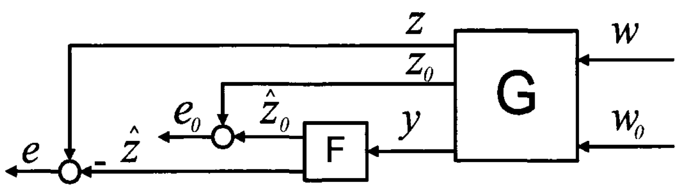

Figure 3.1: Multiobjective filter structure

The goal is to find a filter in the form:

$$F: \quad y \rightarrow \begin{bmatrix} \hat{z} \\ \hat{z}_0 \end{bmatrix}, \tag{3.2}$$

where $\hat{z}$ and $\hat{z}_0$ are estimates of z and z_0, respectively.

The filter is to be designed as:

$$\begin{aligned} \delta\hat{x} &= A\hat{x} + L(C_2\hat{x} - y), \\ \hat{z}_0 &= C_0\hat{x}, \\ \hat{z} &= C_1\hat{x}, \end{aligned} \tag{3.3}$$

where L is the filter gain to be calculated. Define the following variables:

$$e := z - \hat{z}, \quad e_0 := z_0 - \hat{z}_0, \quad e_x := x - \hat{x}, \tag{3.4}$$

and the cost functions as:

$$J_1(F, w, w_0) = \gamma^2 \parallel w \parallel_{\mathcal{P}}^2 - \parallel e \parallel_{\mathcal{P}}^2, \tag{3.5}$$

$$J_2(F, w, w_0) = \parallel e_0 \parallel_{\mathcal{P}}^2 . \tag{3.6}$$

The discrete-time multi-objective filter design problem is stated as follows:

Find an admissible filter F_* in the form (3.3) and a worst disturbance signal w_* such that they achieves:

$$
\begin{aligned}
J_1(F_*, w_*, w_0) &\leq J_1(F_*, w, w_0)\,,\\
J_2(F_*, w_*, w_0) &\leq J_2(F, w_*, w_0)\,.
\end{aligned}
$$

Combining the equations of the plant and the filter and implementing $e_x := x - \hat{x}$, derive:

$$
\begin{aligned}
\delta e_x &= (A + LC_2)e_x + (B_0 + LD_{20})w_0 + B_1 w\,,\\
e_0 &= C_0 e_x\,,\\
e &= C_1 e_x\,.
\end{aligned}
\tag{3.7}
$$

3.2 Discrete-Time Multiobjective Filter Design

.

The discrete-time multiobjective filter design problem is presented in the following theorem.

Theorem 3.1 *Let the plant G be described by the equation set (3.1), where w and w_0 are assumed to be uncorrelated, and the cost functions J_1 and J_2 are defined as (3.5) and (3.6), respectively. If there are stabilizing solutions $P_1 \geq 0$ and $P_2 \geq 0$ to:*

$$
P_1 = \bar{A}^T P_1 \bar{A} + \gamma^{-2}\bar{A}^T P_1 B_1 (I - \gamma^{-2} B_1^T P_1 B_1)^{-1} B_1^T P_1 \bar{A} + C_1^T C_1, \tag{3.8}
$$

$$
P_2 = A_F P_2 A_F^T - (B_0 D_{20}^T + A_F P_2 C_2^T)(R_0 + C_2 P_2 C_2^T)^{-1}(D_{20} B_0^T + C_2 P_2 A_F^T) + B_0 B_0^T\,, \tag{3.9}
$$

where $(I - \gamma^{-2} B_1^T P_1 B_1) > 0$, $R_0 + C_2 P_2 C_2^T > 0$ and:

$$
\bar{A} = A + L_* C_2\,, \qquad \Delta_1 = I - \gamma^{-2} B_1 B_1^T P_1\,,
$$

$$
A_F = (I + \gamma^{-2} B_1 B_1^T P_1 \Delta_1^{-1})A + \gamma^{-2} B_1 B_1^T P_1 \Delta_1^{-1} L_* C_2\,.
$$

Then by choosing L_* that satisfies:

$$L_* = -(A_F P_2 C_2^T + B_0 D_{20}^T)(R_0 + C_2 P_2 C_2^T)^{-1}, \tag{3.10}$$

the filter F_*:

$$\delta \hat{x} = (A + L_* C_2)\hat{x} - L_* y,$$
$$\hat{z}_0 = C_0 \hat{x},$$
$$\hat{z} = C_1 \hat{x},$$

and the worst disturbance signal:

$$w_* = \gamma^{-2} B_1^T P_1 \Delta_1^{-1} \bar{A} e_x,$$

achieve:

$$J_1(F_*, w_*, w_0) \le J_1(F_*, w, w_0),$$
$$J_2(F_*, w_*, w_0) \le J_2(F, w_*, w_0).$$

Conversely, if there exists a filter F_*, with a worst disturbance signal w'_*, such that for the system without white noise, we have:

$$0 < J_1(F_*, w'_*, 0) \le J_1(F_*, w, 0), \qquad \forall w \ne w'_*,$$

and a worst disturbance signal w_* at the presence of white noise, such that:

$$J_1(F_*, w_*, w_0) \le J_1(F_*, w, w_0),$$
$$J_2(F_*, w_*, w_0) \le J_2(F, w_*, w_0),$$

then, there exist stabilizing solutions $P_1 \ge 0$ and $P_2 \ge 0$ to:

$$P_1 = \bar{A}^T P_1 \bar{A} + \gamma^{-2} \bar{A}^T P_1 B_1 (I - \gamma^{-2} B_1^T P_1 B_1)^{-1} B_1^T P_1 \bar{A} + C_1^T C_1,$$

$$P_2 = A_F P_2 A_F^T - (B_0 D_{20}^T + A_F P_2 C_2^T)(R_0 + C_2 P_2 C_2^T)^{-1}(D_{20} B_0^T + C_2 P_2 A_F^T) + B_0 B_0^T,$$

where $(I - \gamma^{-2} B_1^T P_1 B_1) > 0$ and $R_0 + C_2 P_2 C_2^T > 0$.

Moreover, the optimal value of the filter gain L_* satisfies:

$$L_* = -(A_F P_2 C_2^T + B_0 D_{20}^T)(R_0 + C_2 P_2 C_2^T)^{-1} .$$

Proof: ***(Sufficiency)*** If we choose the filter gain L_*, that satisfies:

$$L_* = -(A_F P_2 C_2^T + B_0 D_{20}^T)(R_0 + C_2 P_2 C_2^T)^{-1} ,$$

completing the squares, using (3.8), we have:

$$\begin{aligned} J_1(F_*, w, w_0) =& \gamma^2 \parallel w \parallel_{\mathcal{P}}^2 - \parallel e \parallel_{\mathcal{P}}^2 \\ =& \gamma^2 \parallel w - w_* \parallel_{\mathcal{P}}^2 - \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \text{trace}\big((B_0 + L_* D_{20})^T P_1 A E\{x w_0^T\}\big) \\ =& \gamma^2 \parallel w - w_* \parallel_{\mathcal{P}}^2 , \end{aligned}$$

where

$$w_* = \gamma^{-2} B_1^T P_1 \Delta_1^{-1} \bar{A} e_x ,$$

which is bounded since $A + L_* C_2 + B_1 w_*$ is stable.

Next it is shown that J_1 achieves the minimum value at the given L_*. Let L_1 be any filter gain such that both $A + L_1 C$ and $A + L_1 C_2 + B_1 w_*$ are stable. Substituting the above w_* in the plant-filter equations (3.7), we get:

$$\begin{aligned} \delta e_x =& \big(A + L_1 C_2 + \gamma^{-2} B_1 B_1^T P_1 \Delta_1^{-1} (A + L_1 C_2)\big) e_x + (B_0 + L_1 D_{20}) w_0 \\ =& A_L e_x + B_L w_0 , \\ e_0 =& C_0 e_x . \end{aligned}$$

The first difference equation above can be solved as:

$$e_{x(k)} = \sum_{j=0}^{k-1} A_L^{k-j-1} B_L w_0(j) ,$$

and

$$
\begin{aligned}
J_2(F, w_*, w_0) &= \| e_0 \|_{\mathcal{P}}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{e_x^T C_0^T C_0 e_x\} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\left\{ \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} w_0(i)^T B_L^T (A_L^T)^{k-i-1} C_0^T C_0 A_L^{k-j-1} B_L w_0(j) \right\} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \text{trace}[C_0 A_L^{k-i-1} B_L \delta(i-j) B_L^T (A_L^T)^{k-j-1} C_0^T] \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{i=0}^{k-1} \text{trace}[C_0 A_L^{k-i-1} B_L B_L^T (A_L^T)^{k-i-1} C_0^T] \\
&= \text{trace}(C_0 Y C_0^T)
\end{aligned}
$$

where

$$
Y = \sum_{i=0}^{\infty} A_L^i B_L B_L^T (A_L^T)^i ,
$$

which is the solution of the Lyapunov equation $A_L Y A_L^T - Y + B_L B_L^T = 0$. Then, by Theorem 2.3, the solution to this constrained optimization problem, L_* is to satisfy:

$$
L_* = -(A_F P_2 C_2^T + B_0 D_{20}^T)(R_0 + C_2 P_2 C_2^T)^{-1} ,
$$

where P_2 is the solution to (3.9).

(Necessity) First, for the system without white noise ($w_0 = 0$), suppose there exists a filter F_* and a signal w'_* such that they achieve:

$$
0 < J_1(F_*, w'_*, 0) \le J_1(F_*, w, 0) , \qquad \forall w \ne w'_* .
$$

In other words, for the linear operator $R_{e'w}$, defined as:

$$
\begin{aligned}
\delta e'_x &= \bar{A} e'_x + B_1 w , \\
e &= C_1 e'_x ,
\end{aligned}
$$

it holds that $\| R_{e'w} \|_\infty < \gamma$. Then, by the bounded real lemma [17], there exists a $P_1 \ge 0$, solving (3.8) and the worst disturbance signal is

$$w'_* = \gamma^{-2} B_1^T P_1 \Delta_1^{-1} \bar{A} e'_x \,.$$

Next, including the white noise signal into the system, it can be seen that:

$$\begin{aligned} J_1(F_*, w'_*, w_0) =& \gamma^2 \parallel w - w'_* \parallel_{\mathcal{P}}^2 - \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \text{trace}\big[(B_0 + L_* D_{20})^T P_1 A E\{x w_0^T\}\big] \\ =& \gamma^2 \parallel w - w'_* \parallel_{\mathcal{P}}^2 , \end{aligned}$$

which means that the worst disturbance signal at the presence of the white noise is

$$w_* = \gamma^{-2} B_1^T P_1 \Delta_1^{-1} \bar{A} e_x \,.$$

Now, substituting the w_* into the equation set (3.7), we get:

$$\begin{aligned} \delta e_x &= \big(\bar{A} + \gamma^{-2} B_1 B_1^T P_1 \Delta_1^{-1} \bar{A}\big) e_x + (B_0 + L_1 D_{20}) w_0 \\ &= A_L e_x + B_L w_0 \\ e_0 &= C_0 e_x \end{aligned}$$

Similar to the proof of sufficiency, we can write:

$$J_2(F, w_*, w_0) = \parallel e_0 \parallel_{\mathcal{P}}^2 = \text{trace}(C_0 Y C_0^T)$$

where

$$Y = \sum_{i=0}^{\infty} A_L^i B_L B_L^T (A_L^T)^i$$

and by Theorem 2.3, L_* is to satisfy:

$$L_* = -(A_F P_2 C_2^T + B_0 D_{20}^T)(R_0 + C_2 P_2 C_2^T)^{-1} \,,$$

and P_2 is the solution to (3.9).

■

3.3 Illustrative Example

Consider the following dynamic system:

$$\delta x = \begin{bmatrix} 1 & -0.1 \\ 0.12 & 0.95 \end{bmatrix} x + \begin{bmatrix} 0.05 & 0.1 \\ 0.1 & 0.01 \end{bmatrix} w_0 + \begin{bmatrix} -0.12 \\ 0.03 \end{bmatrix} w ,$$
$$z = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} x ,$$
$$y = \begin{bmatrix} 0.5 & -0.65 \end{bmatrix} x + \begin{bmatrix} 1.2 & 1.6 \end{bmatrix} w_0 .$$

The goal is to design a filter in the form:

$$\delta \hat{x} = A\hat{x} + L(C_2\hat{x} - y) ,$$
$$\hat{z} = C_1\hat{x} ,$$

which leads to the error performance dynamics for the closed-loop system as:

$$\delta e_x = (A + LC_2)e_x + (B_0 + LD_{20})w_0 + B_1 w ,$$
$$e = C_1 e_x .$$

First, considering only the $\mathcal{H}_\infty$ performance, assume the filter gain $L = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Fixing $\gamma = 1.5$, there exists a solution $P_1 \geq 0$ to:

$$P_1 = (A+LC_2)^T P_1 (A+LC_2) + \gamma^{-2}(A+LC_2)^T P_1 B_1 (I - \gamma^{-2} B_1^T P_1 B_1)^{-1} B_1^T P_1 (A+LC_2) + C_1^T C_1 .$$

This filter achieves $\| T_{ew} \| = 0.9663 \leq 1.5$, where T_{ew} represents the transfer function from w to e, and therefore satisfying the $\mathcal{H}_\infty$ requirement. In this case, the worst disturbance signal is characterized by $w_* = 0.444 B_1^T P_1 \Delta_1^{-1}(A + LC_2)e_x = K_w e_x$. However, when the noise signal w_0 is added, the optimal performance of the system in the worst case is then calculated by:

$$J_2 = \text{trace}(C_1 C_1^T P_2) = 35.463 ,$$

where P_2 is the solution to the discrete-time Lyapunov equation:

$$P_2 = (A + LC_2 + B_1K_w)P_2(A + LC_2 + B_1K_w)^T + (B_0 + LD_{20})(B_0 + LD_{20})^T .$$

It is clear that the performance of this filter in presence of noise is not desirable. On the other hand, a Kalman filter can be calculated that satisfies the $\mathcal{H}_2$ optimal performance requirement. This filter can be found by solving the Riccati equation:

$$P_2 = AP_2A^T - (B_0D_{20}^T + AP_2C_2^T)(R_0 + C_2P_2C_2^T)^{-1}(D_{20}B_0^T + C_2P_2A^T) + B_0B_0^T ,$$

leading to a filter gain:

$$\begin{aligned} L_* &= -(AP_2C_2^T + B_0D_{20}^T)(R_0 + C_2P_2C_2^T)^{-1} \\ &= \begin{bmatrix} -0.0622 \\ -0.0248 \end{bmatrix} . \end{aligned}$$

The optimal performance of the system with this filter then becomes:

$$J_2 = \text{trace}(C_1C_1^TP_2) = 0.0472 ,$$

which in fact is much lower than 35.463 obtained when only the $\mathcal{H}_\infty$ performance was considered.

Now, designing a multi-objective filter using Theorem 3.1 and fixing $\gamma = 2.5$, results to solutions to Riccati equations (3.8) and (3.9) as:

$$P_1 = \begin{bmatrix} 8.900 & 0.012 \\ 0.012 & 2.684 \end{bmatrix} > 0 , \quad P_2 = \begin{bmatrix} 0.066 & -0.004 \\ -0.004 & 0.086 \end{bmatrix} > 0 ,$$

and a filter gain:

$$L_* = \begin{bmatrix} -0.0647 \\ -0.0210 \end{bmatrix} , \tag{3.11}$$

that satisfies (3.10).

For the closed-loop system consisting this filter, the cost functions are:

$$J_1 = 1.8927 , \quad J_2 = 0.0788 .$$

Note that although the index J_2 is worse than the Kalman filter, but is still much improved compared to the system with only an $\mathcal{H}_\infty$ filter. On the other hand, as can be seen in Figure 3.2, the error performance of the closed-loop system is much better in the presence of the white noise signal w_0 when the multi-objective filter is used, compared to the filter that only satisfies the $\mathcal{H}_\infty$ performance.

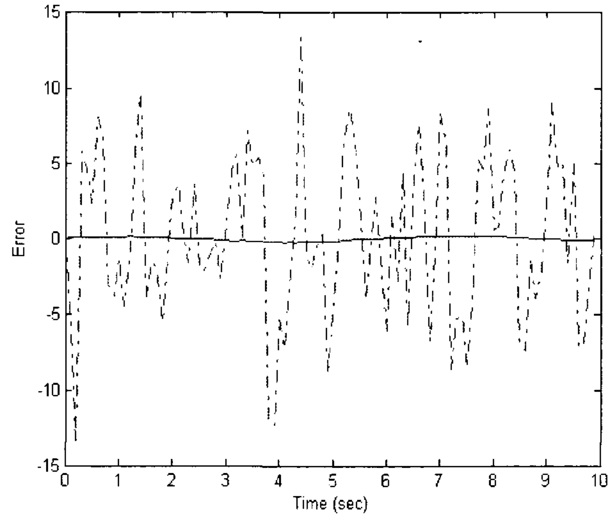

Figure 3.2: Error behavior at the presence of white noise signal w_0 for the closed -loop system with filter L_∞ (dashed) and with the multi-objective filter (solid).

Figure 3.3 shows the singular value diagram for the transfer function T_{ew} of the system with filter gain (3.11) which, as expected, meets the disturbance attenuation of $\gamma = 2.5$.

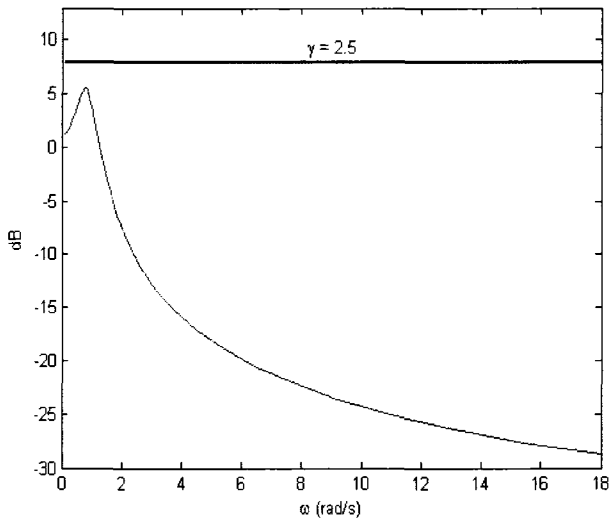

Figure 3.3: Singular value diagram for T_{ew} for the system with multi-objective filter.

Chapter 4

Discrete-Time Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control

4.1 Problem Formulation

Consider the discrete-time linear control system G in Figure 4.1 described by:

$$\begin{aligned} \delta x &= Ax + B_0 w_0 + B_1 w + B_2 u\,, \quad x(0) = 0 \\ z_0 &= C_0 x + D_{02} u\,, \\ z &= C_1 x + D_{12} u\,, \\ y &= C_2 x + D_{20} w_0\,, \end{aligned} \tag{4.1}$$

where w is a bounded power signal and w_0 is a white noise signal. The following standard assumptions are made:

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable;

(A2) $R_{02} := D_{02}^T D_{02} > 0$, $R_{12} := D_{12}^T D_{12} > 0$ and $R_{20} := D_{20} D_{20}^T > 0$;

(A3) $\begin{bmatrix} A - \lambda I & B_0 \\ C_2 & D_{20} \end{bmatrix}$ has full row rank, $\forall \lambda \in \partial\mathbb{D}$;

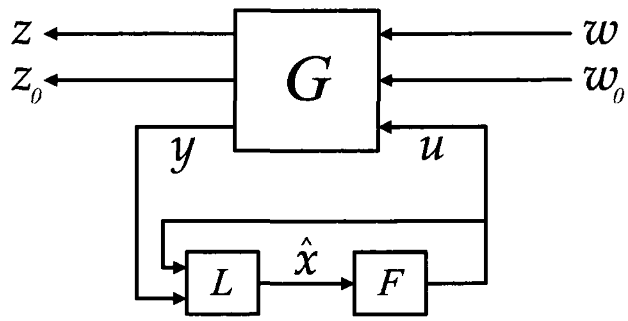

Figure 4.1: Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control setup

(A4) $\begin{bmatrix} A-\lambda I & B_2 \\ C_0 & D_{02} \end{bmatrix}$ has full column rank, $\forall \lambda \in \partial\mathbb{D}$;

(A5) $\begin{bmatrix} A-\lambda I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank, $\forall \lambda \in \partial\mathbb{D}$.

The cost functions are defined as:

$$J_1(u,w,w_0) = \gamma^2 \parallel w \parallel_{\mathcal{P}}^2 - \parallel z \parallel_{\mathcal{P}}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\gamma^2 \parallel w \parallel^2 - \parallel z \parallel^2\}, \tag{4.2}$$

$$J_2(u,w,w_0) = \parallel z_0 \parallel_{\mathcal{P}}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\parallel z_0 \parallel^2\}. \tag{4.3}$$

The discrete-time multiobjective control design problem is stated as follows:

Find an admissible output feedback control u_* and a worst disturbance signal w_* such that they achieve:

$$\begin{aligned} J_1(u_*,w_*,w_0) &\le J_1(u_*,w,w_0), \\ J_2(u_*,w_*,w_0) &\le J_2(u,w_*,w_0). \end{aligned}$$

Note that the control law u is supposed to have the observer-based form:

$$\begin{aligned} \delta\hat{x} &= \hat{A}\hat{x} + B_2 u - Ly, \quad \hat{x}(0) = 0, \\ u &= F\hat{x}. \end{aligned}$$

4.2 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control - State Feedback

First, we will present a state feedback problem for the system without white noise ($w_0 = 0$). The problem statement and the results are given in the following theorem.

Theorem 4.1 *Given the system equations in (4.1) and $w_0 = 0$, suppose there exist solutions $P_1 \geq 0$ and $P_2 \geq 0$ solving:*

$$P_1 = A_2^T P_1 A_2 + \gamma^{-2} A_2^T P_1 B_1 (I - \gamma^{-2} B_1^T P_1 B_1)^{-1} B_1^T P_1 A_2 + (C_0 + D_{02} F_2)^T (C_0 + D_{02} F_2)\,, \tag{4.4}$$

$$P_2 = A_1^T P_2 A_1 - (A_1^T P_2 B_2 + C_0^T D_{02})(R_{02} + B_2^T P_2 B_2)^{-1}(B_2^T P_2 A_1 + D_{02}^T C_0) + C_0^T C_0\,, \tag{4.5}$$

where $I - \gamma^{-2} B_1^T P_1 B_1 > 0$, $R_{02} + B_2^T P_2 B_2 > 0$ and

$$A_1 = A + B_1 F_1\,, \quad A_2 = A + B_2 F_2\,, \quad \Delta_1 = I - \gamma^{-2} B_1 B_1^T P_1\,,$$

$$F_1 = \gamma^{-2} B_1^T P_1 \Delta_1^{-1} A_2\,,$$

$$F_2 = -(R_{02} + B_2^T P_2 \Delta_1^{-1} B_2)^{-1}(B_2^T P_2 A + \gamma^{-2} B_2^T P_2 B_1 B_1^T P_1 \Delta_1^{-1} A + D_{02}^T C_0)\,,$$

then the strategies:

$$u_* = F_2 x\,,$$

$$w_* = F_1 x\,,$$

will result in:

1. *$A + B_1 F_1 + B_2 F_2$ is stable;*

2. *If (A_1, C_0) is detectable then*

   - *$\| T_{z_0 w} \|_\infty < \gamma$ when $u = u_*$;*

   - *$J_2(u_*, w_*, 0) \leq J_2(u, w_*, 0)$.*

Conversely, if (A, B_2) is stabilizable and the state feedback strategies $u_* = F_2 x$ and $w_* = F_1 x$ exist such that:

1. *A_2 is stable and (A_1, C_0) is detectable;*

2. $\| T_{z_0w} \|_\infty < \gamma$ *when* $u = u_*$;

3. $J_2(u_*, w_*, 0) \leq J_2(u, w_*, 0)$;

then equations (4.4) and (4.5) have solutions $P_1 \geq 0$ *and* $P_2 \geq 0$*, respectively.*

Proof: *(Sufficiency)* Since the Riccati equations (4.4) and (4.5) have solutions P_1 and P_2, then both A_2 and $A + B_1F_1 + B_2F_2$ are stable. Setting $u = u_* = F_2x$ gives:

$$\delta x = A_2x + B_1w\,,$$

$$z_0 = (C_0 + D_{02}F_2)x\,,$$

for which, (4.4) and the bounded real lemma result in $\| T_{z_0w} \|_\infty < \gamma$. After completing the squares using (4.4), it is found:

$$w_* = \gamma^{-2}(I - \gamma^{-2}B_1^TP_1B_1)^{-1}B_1^TP_1A_2x\,,$$

or equivalently

$$w_* = \gamma^{-2}B_1^TP_1\Delta_1^{-1}A_2x = F_1x\,.$$

Setting $w = w_* = F_1x$ gives:

$$\delta x = A_1x + B_2u\,,$$

$$z_0 = C_0x + D_{02}u\,.$$

Note that:

$$\min_u J_2(u, w_*, 0) = \min_u \| z_0 \|_{\mathcal{P}}^2 = \min_u \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} E\{\| z_0 \|^2\} = \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} E\{\min_u \| z_0 \|^2\}\,,$$

is a standard optimal control problem where, with P_2 solving (4.5), u_* is found as:

$$u_* = -(B_2^TP_2B_2 + R_{02})^{-1}(B_2^TP_2A_1 + D_{02}^TC_0)x = F_2x\,.$$

Further simplification of F_2, after substituting the expression for F_1, can be conducted as

$$\begin{aligned}F_2 = &-(B_2^TP_2B_2 + R_{02})^{-1}(B_2^TP_2A + \gamma^{-2}B_2^TP_2B_1B_1^TP_1\Delta_1^{-1}A \\ &\qquad + \gamma^{-2}B_2^TP_2B_1B_1^TP_1\Delta_1^{-1}B_2F_2 + D_{20}^TC_0) \\ = &-(B_2^TP_2B_2 + R_{02} + \gamma^{-2}B_2^TP_2B_1B_1^TP_1\Delta_1^{-1}B_2)^{-1}(B_2^TP_2A \\ &\qquad + \gamma^{-2}B_2^TP_2B_1B_1^TP_1\Delta_1^{-1}A + D_{20}^TC_0) \\ = &-(R_{02} + B_2^TP_2\Delta_1^{-1}B_2)^{-1}(B_2^TP_2A + \gamma^{-2}B_2^TP_2B_1B_1^TP_1\Delta_1^{-1}A + D_{20}^TC_0)\,,\end{aligned}$$

which will result in $J_2(u_*, w_*, 0) \leq J_2(u, w_*, 0)$ as required.

(Necessity) Implementing $u_* = F_2 x$ into the system equations gives:

$$\delta x = A_2 x + B_1 w\,,$$
$$z_0 = (C_0 + D_{02}F_2)x\,,$$

where, by the theorem assumption, A_2 is stable and $\| T_{z_0 w} \|_\infty < \gamma$. Then, by bounded real lemma, there exists a solution $P_1 \geq 0$ to (4.4) such that $A + B_2F_2 + B_1F_1$ is stable. The worst disturbance signal can similarly be found as:

$$w_* = \gamma^{-2} B_1^T P_1 \Delta_1^{-1} A_2 x = F_1 x\,.$$

Next, implement the above $w_* = F_1 x$ into the system equations to get:

$$\delta x = A_1 x + B_2 u\,,$$
$$z_0 = C_0 x + D_{02} u\,.$$

Then, for this system,

$$\min_u J_2(u, w_*, 0) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\min_u \| z_0 \|^2\}\,,$$

is a standard optimal control problem and u_* can be found as:

$$u_* = -(B_2^T P_2 B_2 + R_{02})^{-1}(B_2^T P_2 A_1 + D_{02}^T C_0)x = F_2 x\,,$$

which can be modified into:

$$F_2 = -(R_{02} + B_2^T P_2 \Delta_1^{-1} B_2)^{-1}(B_2^T P_2 A + \gamma^{-2} B_2^T P_2 B_1 B_1^T P_1 \Delta_1^{-1} A + D_{02}^T C_0)\,,$$

where P_2 solves (4.5).

∎

Corollary 1 *If we assume*

$$D_{02}^T [C_0 \quad D_{02}] = [0 \quad I]\,,$$

then the strategies in the above theorem will reduce to:

$$F_1 = \gamma^{-2} B_1^T P_1 (I - \gamma^{-2} B_1 B_1^T P_1 + B_2 B_2^T P_2)^{-1} A \,,$$
$$F_2 = -B_2^T P_2 (I - \gamma^{-2} B_1 B_1^T P_1 + B_2 B_2^T P_2)^{-1} A \,,$$

which are the equivalents to the results found in [12].

Proof: The proof is trivial, using the results already developed in Theorem 4.1. ∎

4.3 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control - Output Feedback

The output-feedback control is given in the following theorem for discrete-time multiobjective control.

Theorem 4.2 *For the system given by (4.1), where w and w_0 are assumed to be uncorrelated, optimal strategies u_* and w_* exist such that:*

$$J_1(u_*, w_*, w_0) \leq J_1(u_*, w, w_0) \,,$$
$$J_2(u_*, w_*, w_0) \leq J_2(u, w_*, w_0) \,,$$

if the cross-coupled Riccati equations:

$$P_1 = A_2^T P_1 A_2 + \gamma^{-2} A_2^T P_1 B_1 (I - \gamma^{-2} B_1^T P_1 B_1)^{-1} B_1^T P_1 A_2 + (C_0 + D_{02} F_2)^T (C_0 + D_{02} F_2) \,, \tag{4.6}$$

$$P_2 = A_1^T P_2 A_1 - (A_1^T P_2 B_2 + C_0^T D_{02})(R_{02} + B_2^T P_2 B_2)^{-1} (B_2^T P_2 A_1 + D_{02}^T C_0) + C_0^T C_0 \,, \tag{4.7}$$

$$P_3 = A_1 P_3 A_1^T - (A_1 P_3 C_2^T + B_0 D_{20}^T)(R_{20} + C_2 P_3 C_2^T)^{-1} (C_2 P_3 A_1^T + D_{20} B_0^T) + B_0 B_0^T \,, \tag{4.8}$$

have stabilizing solutions $P_1 \geq 0$, $P_2 \geq 0$ and $P_3 \geq 0$, where:

$$I - \gamma^{-2} B_1^T P_1 B_1 > 0 \,, \quad R_{02} + B_2^T P_2 B_2 > 0 \,, \quad R_{20} + C_2 P_3 C_2^T > 0 \,,$$

$$A_1 = A + B_1 F_1 \,, \quad A_2 = A + B_2 F_2 \,, \quad F_1 = \gamma^{-2} B_1^T P_1 \Delta_1^{-1} (A + B_2 F_*) \,,$$

$$F_2 = -(R_{02} + B_2^T P_2 \Delta_1^{-1} B_2)^{-1} (B_2^T P_2 A + \gamma^{-2} B_2^T P_2 B_1 B_1^T P_1 \Delta_1^{-1} A + D_{02}^T C_0) \,.$$

If these solutions exist, we have $w_* = F_1 x$ and u_* is in observer-based form:

$$\delta\hat{x} = (A + B_1F_1 + B_2F_2)\hat{x} + L_*(C_2\hat{x} - y)\,,$$
$$u_* = F_2\hat{x}\,,$$

and

$$L_* = -(A_1P_3C_2^T + B_0D_{20}^T)(C_2P_3C_2^T + R_{20})^{-1}\,.$$

Conversely, if the state-feedback control problem is solvable, i.e., $P_1 \geq 0$ and $P_2 \geq 0$ exist and solve (4.6) and (4.7) and there exist an optimal controller u_* in the form:

$$\delta\hat{x} = (A + B_1F_1 + B_2F_2)\hat{x} + L_*(C_2\hat{x} - y)\,,$$
$$u_* = F_2\hat{x}\,,$$

where $F_2 = -(R_{02} + B_2^TP_2\Delta_1^{-1}B_2)^{-1}(B_2^TP_2A + \gamma^{-2}B_2^TP_2B_1B_1^TP_1\Delta_1^{-1}A + D_{02}^TC_0)$ and a w_* that achieve:

$$\begin{aligned} J_1(u_*, w_*, w_0) &\leq J_1(u_*, w, w_0)\,, \\ J_2(u_*, w_*, w_0) &\leq J_2(u, w_*, w_0)\,, \end{aligned}$$

then, a $P_3 \geq 0$ exists that solves (4.8).

Moreover, $L_* = -(A_1P_3C_2^T + B_0D_{20}^T)(C_2P_3C_2^T + R_{20})^{-1}$, if $A + B_1F_1 + L_*C_2$ is stable.

Proof: (Sufficiency) Suppose $P_1 \geq 0$, $P_2 \geq 0$ and $P_3 \geq 0$ exist and solve (4.6), (4.7) and (4.8), respectively. If u is any stabilizing control law, we can get:

$$\begin{aligned} &\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} E\{\gamma^2 \parallel w \parallel^2 - \parallel z \parallel^2\} \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} E\{\gamma^2 \parallel w \parallel^2 - x^TC_1^TC_1x - 2x^TC_1^TD_{12}u - u^TR_{12}u\}\,. \end{aligned}$$

Using the first Riccati equation, derive:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}E\Big\{\gamma^2 \parallel w \parallel^2 - \parallel z \parallel^2\Big\}$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}E\Big\{\gamma^2 \parallel w - w_* \parallel^2 - 2x^T(B_2^TP_1B_2 + C_1D_{12})(u-\tilde{u}_*) - u^TR_{12}u + \tilde{u}_*^TR_{12}\tilde{u}_*\Big\}$$

$$+ \operatorname{trace}\big(A^TP_1B_0E\{x^Tw_0\}\big) + \operatorname{trace}\big(B_0^TP_1AE\{xw_0^T\}\big)$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}E\Big\{\gamma^2 \parallel w - w_* \parallel^2 - 2x^T(B_2^TP_1B_2 + C_1D_{12})(u-\tilde{u}_*) - u^TR_{12}u + \tilde{u}_*^TR_{12}\tilde{u}_*\Big\},$$

where $w_* = F_1x$ and $\tilde{u}_* = F_2x$ (the optimal strategies for the state-feedback case). Note that Lemma 2.1 is used to draw this conclusion. From this expression for J_1, it is clear that for any u, including $u = u_* = F_2\hat{x}$, we have:

$$J_1(u_*, w_*, w_0) \leq J_1(u_*, w, w_0).$$

Next we minimize J_2. Substitute w_* into the system equation to get:

$$\delta x = A_1x + B_0w_0 + B_2u_*,$$

$$z_0 = C_0x + D_{02}u_*,$$

$$y = C_2x + D_{20}w_0.$$

For this system,

$$\min_u J_2(u, w_*, w_0) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}E\{\min_u \parallel z_0 \parallel^2\},$$

is a standard LQG control problem and the controller can be found in the form:

$$\delta\hat{x} = A_1\hat{x} + B_2u_* + L_*(C_2\hat{x} - y),$$

$$u_* = F_2\hat{x},$$

where:

$$F_2 = -(R_{02} + B_2^TP_2\Delta_1^{-1}B_2)^{-1}(B_2^TP_2A + \gamma^{-2}B_2^TP_2B_1B_1^TP_1\Delta_1^{-1}A + D_{02}^TC_0),$$

$$L_* = -(A_1P_3C_2^T + B_0D_{20}^T)(C_2P_3C_2^T + R_{20})^{-1},$$

achieve:

$$J_2(u_*, w_*, w_0) \leq J_2(u, w_*, w_0).$$

(Necessity) Suppose that the state-feedback control problem is solvable, in other words, there exist stabilizing solutions to (4.6) and (4.7) and assume the controller to be in the form:

$$\delta\hat{x} = (A + B_1F_1 + B_2F_2 + L_*C_2)\hat{x} - L_*y\,,$$
$$u_* = F_2\hat{x}\,,$$

where $F_1 = \gamma^{-2}B_1^TP_1\Delta_1^{-1}A_2$ and

$$F_2 = -(R_{02} + B_2^TP_2\Delta_1^{-1}B_2)^{-1}(B_2^TP_2A + \gamma^{-2}B_2^TP_2B_1B_1^TP_1\Delta_1^{-1}A + D_{02}^TC_0)\,.$$

Let u_* and w_* achieve:

$$J_1(u_*, w_*, w_0) \leq J_1(u_*, w, w_0)\,,$$
$$J_2(u_*, w_*, w_0) \leq J_2(u, w_*, w_0)\,.$$

From the proof of the sufficiency, $w_* = F_1x$. Implementing u_* and w_* into the system equations gives:

$$\delta x = A_1x + B_0w_0 + B_2u_*\,,$$
$$z_0 = C_0x + D_{02}u_*\,,$$
$$y = C_2x + D_{20}w_0\,,$$

where we can construct:

$$J_2(u_*, w_*, w_0) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}E\{\| z_0 \|^2\} = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}E\{x^TC_0^TC_0x + 2x^TC_0^TD_{02}u_* + u_*^TR_{02}u_*\}\,.$$

$$J_2(u_*, w_*, w_0) = \lim_{N\to\infty}\frac{1}{N}\sum E\{(u_* - \tilde{u}_*)^TR_{02}(u_* - \tilde{u}_*)\} + \text{trace}\big(A^TP_2B_0E\{x^Tw_0\}\big)$$

$$J_2(u_*, w_*, w_0) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}E\{(u_* - \tilde{u}_*)^TR_{02}(u_* - \tilde{u}_*)\} + \text{trace}\big(A^TP_2B_0E\{x^Tw_0\}\big)$$
$$+\,\text{trace}\big(B_0^TP_2AE\{xw_0^T\}\big)$$
$$= \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}E\{(u_* - \tilde{u}_*)^TR_{02}(u_* - \tilde{u}_*)\}\,,$$

where $\tilde{u}_* = F_*x$. Note that Lemma 2.1 is used to derive this conclusion. Define $e_x = x - \hat{x}$, then:

$$J_2(u_*, w_*, w_0) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}E\{e_x^TF_*^TR_{02}F_*e_x\}\,,$$

where:

$$\delta e_x = \delta x - \delta\hat{x} = (A_1 + L_* C_2)e_x + (B_0 + L_* D_{20})w_0 = A_{L_*} e_x + B_{L_*} w_0 \,,$$

which can be solved as:

$$e_x(k) = \sum_{j=0}^{k-1} A_{L*}^{k-j-1} B_{L*} w_{0(j)} \,.$$

Therefore:

$$\begin{aligned}
J_2(u_*, w_*, w_0) &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{e_x^T F_*^T R_{02} F_* e_x\} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\Big\{ \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} w_0(i)^T B_{L*}^T (A_{L*}^T)^{k-i-1} F_*^T D_{02}^T D_{02} F_* A_{L*}^{k-j-1} B_{L*} w_0(j) \Big\} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \text{trace}[D_{02} F_* A_{L*}^{k-i-1} B_{L*} \delta(i-j) B_{L*}^T (A_{L*}^T)^{k-j-1} F_*^T D_{02}^T] \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{i=0}^{k-1} \text{trace}[D_{02} F_* A_{L*}^{k-i-1} B_{L*} B_{L*}^T (A_{L*}^T)^{k-j-1} F_*^T D_{02}^T] \\
&= \text{trace}[D_{02} F_* Y F_*^T D_{02}^T] \,,
\end{aligned}$$

where:

$$Y = \sum_{i=0}^{\infty} A_{L*}^i B_{L*} B_{L*}^T (A_{L*}^T)^i \,,$$

which is the solution of the Lyapunov equation $A_{L*} Y A_{L*}^T - Y + B_{L*} B_{L*}^T = 0$. Then, by Theorem 2.3, the solution to this constrained optimization problem, L_* is:

$$L_* = -(A_1 P_3 C_2^T + B_0 D_{20}^T)(C_2 P_3 C_2^T + R_{20})^{-1} \,,$$

and P_3 is the solution to (4.8).

This concludes the proof. ∎

4.4 Numerical Example

In this section we present an example which is solved by the algorithms developed in this chapter. This example was studied for $\mathcal{H}_\infty$ control design in [17], and is a second-order, discrete time system given by:

$$\begin{aligned}
\delta x &= \begin{bmatrix} 0 & 2 \\ 4 & 0.2 \end{bmatrix} x + \begin{bmatrix} 0.5 & 1 \\ 1 & 0 \end{bmatrix} w_0 + \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix} w + \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} u \,, \\
z_0 &= \begin{bmatrix} 0.5 & 0.6 \end{bmatrix} x + 0.8u \,, \\
z &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \,, \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1.2 & 1.6 \end{bmatrix} w_0 \,.
\end{aligned}$$

For the above system, assume $\gamma = 2.5$. A standard $\mathcal{H}_\infty$ control design will result to cost functions:

$$J_1 = 2.4637 \,, \qquad J_2 = 3468.2 \,.$$

On the other hand, an LQG controller can easily be calculated and the cost functions of the system will become:

$$J_1 = 93.1317 \,, \qquad J_2 = 3.714 \,.$$

Now, we apply the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ output-feedback controller proposed in Theorem 4.2. Solving the Riccati equations (4.6)-(4.8), with $\gamma = 2.5$, we have:

$$P_1 = \begin{bmatrix} -667.07 & 258.67 \\ 258.67 & -121.44 \end{bmatrix} , P_2 = \begin{bmatrix} 0 & -6.353 \\ -6.353 & -2.6419 \end{bmatrix} , P_3 = \begin{bmatrix} 32.699 & -36.03 \\ -36.03 & 290.81 \end{bmatrix} .$$

The resulting gains for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control strategy are then calculated as:

$$\begin{aligned}
w_* &= F_1 x = \begin{bmatrix} 4.5616 & -3.3577 \end{bmatrix} x \,, \\
u_* &= F_* \hat{x} = \begin{bmatrix} -0.625 & -0.75 \end{bmatrix} \hat{x} \,,
\end{aligned}$$

and the optimal state estimator gain is:

$$L_* = \begin{bmatrix} -1.7768 \\ -5.6086 \end{bmatrix} .$$

The cost function as defined in (5.2) and (5.3) will then be:

$$J_1 = 53.586\,, \qquad J_2 = 23.705\,.$$

Note that although J_1 has increased from the system with only an $\mathcal{H}_\infty$ control, it has improved considerably from the case where only an LQG controller is applied. The same observation can be made for the J_2 performance index of the system with a multiobjective control compared to the cases where only one of the methods is applied.

Chapter 5

Discrete-Time $\mathcal{H}_\infty$ Gaussian Control

5.1 Problem Formulation

Consider a linear system in Figure 5.1 given by:

$$\begin{aligned} \delta x &= Ax + B_0 w_0 + B_1 w + B_2 u\,, \quad x(0) = 0 \\ z &= C_1 x + D_{12} u\,, \\ y &= C_2 x + D_{20} w_0\,, \end{aligned} \tag{5.1}$$

where w is a bounded power signal and w_0 is a white noise signal. The following assumptions are made for this system:

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable;

(A2) $R_0 := D_{20} D_{20}^T > 0$ and $R_1 := D_{12}^T D_{12} > 0$;

(A3) $\begin{bmatrix} A - \lambda I & B_0 \\ C_2 & D_{20} \end{bmatrix}$ has full row rank, $\forall \lambda \in \partial \mathbb{D}$;

(A4) $\begin{bmatrix} A - \lambda I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank, $\forall \lambda \in \partial \mathbb{D}$.

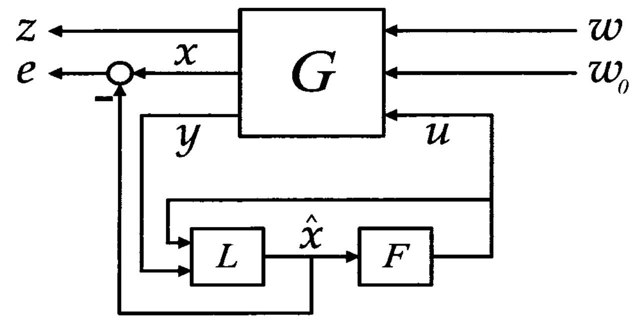

Figure 5.1: $\mathcal{H}_\infty$ Gaussian control setup

The controller is to have the observer-based form:

$$\begin{aligned} \delta\hat{x} &= \hat{A}\hat{x} + B_2 u - Ly\,, \quad \hat{x}(0) = 0, \\ u &= F\hat{x}. \end{aligned}$$

Let $e = x - \hat{x}$, define the cost functions as:

$$J_1(u, w, w_0) = \gamma^2 \parallel w \parallel_{\mathcal{P}}^2 - \parallel z \parallel_{\mathcal{P}}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\gamma^2 \parallel w \parallel^2 - \parallel z \parallel^2\}\,, \tag{5.2}$$

$$J_2(u, w, w_0) = \parallel e \parallel_{\mathcal{P}}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\parallel e \parallel^2\}\,. \tag{5.3}$$

The discrete-time $\mathcal{H}_\infty$ Gaussian design problem is stated as follows:

Find an admissible observer-based control law u_* and a worst disturbance signal w_*, such that they achieve:

$$\begin{aligned} J_1(u_*, w_*, w_0) &\leq J_1(u_*, w, w_0)\,, \\ J_2(u_*, w_*, w_0) &\leq J_2(u, w_*, w_0)\,. \end{aligned}$$

5.2 $\mathcal{H}_\infty$ Gaussian Control Design

The following theorem presents the results for the general form of output feedback, discrete-time $\mathcal{H}_\infty$ Gaussian control.

Theorem 5.1 *Let the dynamical system be described by equation set (5.1), where w and w_0 are assumed to be uncorrelated. If there are stabilizing solutions $P_1 \geq 0$, $P_2 \geq 0$ and $P_3 \geq 0$ solving the Riccati equations:*

$$P_1 = A^T P_1 A - (A^T P_1 B + C_1^T D)(B^T P_1 B + R)^{-1}(B^T P_1 A + D^T C_1) + C_1^T C_1 , \quad (5.4)$$

$$P_2 = \bar{A}^T P_2 \bar{A} + \gamma^{-2} \bar{A}^T P_2 B_1 (I - \gamma^{-2} B_1^T P_2 B_1)^{-1} B_1^T P_2 \bar{A} + K_1^T R_1 K_1 , \quad (5.5)$$

$$P_3 = A_M P_3 A_M^T - (B_0 D_{20}^T + A_M P_3 C_2^T)(R_0 + C_2 P_3 C_2^T)^{-1}(D_{20} B_0^T + C_2 P_3 A_M^T) + B_0 B_0^T , \quad (5.6)$$

where $I - \gamma^{-2} B_1^T P_1 B_1 > 0$, $I - \gamma^{-2} B_1^T P_2 B_1 > 0$, $R_0 + C_2 P_3 C_2^T > 0$, and:

$$B = [\gamma^{-1} B_1 \quad B_2], \qquad D = [0 \quad D_{12}], \qquad R = D^T D - [I \quad 0]^T [I \quad 0] , \quad S = [0 \quad I] ,$$

$$\Delta_1 = I - \gamma^{-2} B_1 B_1^T P_1 , \quad \Delta_2 = I - \gamma^{-2} B_1 B_1^T P_2 , \quad \bar{A} = A + B_1 K_2 + L_* C_2 ,$$

$$A_M = (I + \gamma^{-2} B_1 B_1^T P_2 \Delta_2^{-1}) A + B_1 K_2 + \gamma^{-2} B_1 B_1^T P_2 \Delta_2^{-1} L_* C_2 ,$$

$$K_1 = -S(B^T P_1 B + R)^{-1}(B^T P_1 A + D^T C_1) ,$$

$$K_2 = \gamma^{-2} B_1^T P_1 \Delta_1^{-1} (A + B_2 K_1) ,$$

$$K_3 = \gamma^{-2} B_1^T P_2 \Delta_2^{-1} \bar{A} ,$$

i.e., if $A - B(B^T P_1 B + R)^{-1}(B^T P_1 A + D^T C_1)$ and $\bar{A}$ are both stable, then there exists an optimal controller u_* and a worst disturbance signal w_*, such that:

$$J_1(u_*, w_*, w_0) \leq J_1(u_*, w, w_0) , \quad J_2(u_*, w_*, w_0) \leq J_2(u, w_*, w_0).$$

If the solutions exist, then

$$w_* = \gamma^{-2} B_1^T [P_1 \Delta_1^{-1} (A + B_2 K_1) x + P_2 \Delta_2^{-1} \bar{A} e] = K_2 x + K_3 e ,$$

and an optimal controller is given by:

$$\delta \hat{x} = [A + \gamma^{-2} B_1 B_1^T P_1 \Delta_1^{-1} (A + B_2 K_1) + B_2 F_* + L_* C_2] \hat{x} - L_* y ,$$

$$u_* = F_* \hat{x} , \quad \hat{x}(0) = 0,$$

where $F_* = K_1$ and L_* satisfies:

$$L_* = -(A_M P_3 C_2^T + B_0 D_{20}^T)(R_0 + C_2 P_3 C_2^T)^{-1} . \quad (5.7)$$

Conversely, let P_1 be a stabilizing solution to (5.4) and suppose there exists a w'_* and a controller u_* (hence an L_*):

$$\delta\hat{x} = [A + \gamma^{-2}B_1B_1^TP_1\Delta_1^{-1}(A + B_2K_1) + B_2F_* + L_*C_2]\hat{x} - L_*y\,,$$
$$u_* = F_*\hat{x}\,, \quad F_* = -S(B^TP_1B + R)^{-1}(B^TP_1A + D^TC_1)\,,$$

achieving $0 < J_1(u_*, w'_*, 0) \leq J_1(u_*, w, 0)$. Then there exists a w_* achieving $J_1(u_*, w_*, w_0) \leq J_1(u_*, w, w_0)$. If furthermore, this w_* also achieves $J_2(u_*, w_*, w_0) \leq J_2(u, w_*, w_0)$, then there exist $P_2 \geq 0$ and $P_3 \geq 0$ solving (5.5) and (5.6), respectively.

Moreover, if $A + B_1K_2$ is stable, then L_* satisfies:

$$L_* = -(A_MP_3C_2^T + B_0D_{20}^T)(R_0 + C_2P_3C_2^T)^{-1}.$$

Proof: (Sufficiency) Suppose that there are $P_1 \geq 0$, $P_2 \geq 0$ and $P_3 \geq 0$ solving (5.4), (5.5) and (5.6), respectively. Let u be any control law. Define $r := w - K_2x$ and $v := D_{12}(u + K_1x)$. Then the system equations can be written as:

$$\delta x = (A + B_1K_2)x + B_0w_0 + B_1r + B_2u\,,$$
$$v = D_{12}(u + K_1x)\,,$$
$$y = C_2x + D_{20}w_0\,,$$

and the performance index $J_1(u, w, w_0)$ becomes:

$$J_1(u,w,w_0) = \gamma^2 \parallel w \parallel_{\mathcal{P}}^2 - \parallel z \parallel_{\mathcal{P}}^2 = \gamma^2 \parallel r \parallel_{\mathcal{P}}^2 - \parallel v \parallel_{\mathcal{P}}^2 - \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\left[\text{trace}(B_0^TP_1A\,E\{xw_0^T\})\right]$$
$$= \gamma^2 \parallel r \parallel_{\mathcal{P}}^2 - \parallel v \parallel_{\mathcal{P}}^2\,.$$

Note that the first Riccati equation and Lemma 2.2 are used to derive this equation.

If we use L_* and build the state estimator as:

$$\delta\hat{x} = (A + B_1K_2)\hat{x} + B_2u + L_*(C_2\hat{x} - y)\,, \quad \hat{x}(0) = 0$$

a logical choice for the controller is $u_* = -S(B^TP_1B + R)^{-1}(B^TP_1A + D^TC_1)\hat{x}$, since the state information is not available.

Take $e = x - \hat{x}$ and modify the system equations into:

$$\delta e = \bar{A}e + (B_0 + L_* D_{20})w_0 + B_1 r\,,$$
$$v = D_{12}K_1 e\,.$$

By using Lemma 2.2 and the second Riccati equation, the index $J_1(u, w, w_0)$ becomes:

$$\begin{aligned} J_1(u_*, w, w_0) =& \gamma^2 \parallel r - \gamma^{-2}B_1^T P_2 \Delta_2^{-1}\bar{A}e \parallel_{\mathcal{P}}^2 \\ & - \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1}\Big[\text{trace}(B_0^T P_1 A\, E\{xw_0^T\}) + \text{trace}\big((B_0 + L_* D_{20})^T P_2 A\, E\{xw_0^T\}\big)\Big] \\ =& \gamma^2 \parallel r - \gamma^{-2}B_1^T P_2 \Delta_2^{-1}\bar{A}e \parallel_{\mathcal{P}}^2\,. \end{aligned}$$

And it follows that:

$$r_* = \gamma^{-2}B_1^T P_2 \Delta_2^{-1}\bar{A}e\,,$$

or

$$w_* = r_* + K_2 x = \gamma^{-2}B_1^T[P_1\Delta_1^{-1}(A + B_2K_1)x + P_2\Delta_2^{-1}\bar{A}e]\,.$$

Then we have $J_1(u_*, w_*, w_0) \leq J_1(u_*, w, w_0)$.

Next, consider the index $J_2(u, w, w_0)$. Let L_1 be any estimator gain such that $A + B_1K_2 + L_1C_2$ is stable. Substitute $w_*' = K_2x + P_2\Delta_2^{-1}(A + B_1K_2 + L_1C_2)e$ into the system to get:

$$\delta e = A_L e + B_L w_0\,,$$

where:

$$A_L = A + L_1C_2 + B_1K_2 + \gamma^{-2}B_1B_1^T P_2\Delta_2^{-1}(A + L_1C_2)\,, \quad B_L = B_0 + L_1D_{20}\,.$$

The above difference equation, when solved, results to:

$$e = \sum_{j=0}^{k-1} A_L^{k-j-1}B_L w_0(j)\,,$$

and

$$
\begin{aligned}
J_2(u, w'_*, w_0) = \| e \|_{\mathcal{P}}^2 &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\| e \|^2\} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\Big\{ \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} w_0(i)^T B_L^T (A_L^T)^{k-i-1} A_L^{k-j-1} B_L w_0(j) \Big\} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \text{trace}[A_L^{k-i-1} B_L \delta(i-j) B_L^T (A_L^T)^{k-j-1}] \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{i=0}^{k-1} \text{trace}[A_L^{k-i-1} B_L B_L^T (A_L^T)^{k-i-1}] = \text{trace}(Y) \,,
\end{aligned}
$$

where:

$$
Y = \sum_{i=0}^{\infty} A_L^i B_L B_L^T (A_L^T)^i \,,
$$

which is the solution of the Lyapunov equation $A_L Y A_L^T - Y + B_L B_L^T = 0$. By Theorem 2.3 and using the third Riccati equation, it can be seen that $J_2(u, w'_*, w_0)$ achieves the minimum value at L_*, where $L_* = -(A_M P_3 C_2^T + B_0 D_{20}^T)(R_0 + C_2 P_3 C_2^T)^{-1}$. Therefore, u_* is the desired optimal control.

Proof: (Necessity) Let a $P_1 \geq 0$ solve (5.4) and suppose there exists a controller u_*:

$$
\begin{aligned}
&\delta\hat{x} = [A + \gamma^{-2} B_1 B_1^T P_1 \Delta_1^{-1}(A + B_2 K_1) + B_2 F_* + L_* C_2]\hat{x} - L_* y \,, \\
&u_* = F_* \hat{x} \,, \quad F_* = -S(B^T P_1 B + R)^{-1}(B^T P_1 A + D^T C_1) \,,
\end{aligned}
$$

and a w'_* achieving $0 < J_1(u_*, w'_*, 0) \leq J_1(u_*, w, 0)$. In other words, the system without white noise:

$$
\begin{aligned}
\delta x &= Ax + B_1 w + B_2 u \,, \quad x(0) = 0 \\
z &= C_1 x + D_{12} u \,, \\
y &= C_2 x \,,
\end{aligned}
$$

achieves the $\mathcal{H}_\infty$ performance. Define:

$$
e := x - \hat{x} \,, \quad r := w - K_2 x \,, \quad v_* := D_{12}(u_* + K_1 x) \,.
$$

The system can be converted into:

$$\delta e = \bar{A}e + B_1 r\,, \quad e(0) = 0$$
$$v_* = D_{12}K_1 e\,,$$

and the cost function J_1 will become:

$$J_1(u_*, w, 0) = \gamma^2 \parallel w - \tilde{w}_* \parallel_{\mathcal{P}}^2 - \parallel D_{12}(u_* - \tilde{u}_*) \parallel_{\mathcal{P}}^2 = \gamma^2 \parallel r \parallel_{\mathcal{P}}^2 - \parallel v_* \parallel_{\mathcal{P}}^2\,,$$

where $\tilde{w}_* = K_2 x$ and $\tilde{u}_* = K_1 x$. Therefore, by the bounded real lemma, there exists a P_2 solving (5.5) and accordingly, w'_* (or r'_*) is:

$$w'_* = r'_* + K_2 x = \gamma^{-2} B_1^T [P_1 \Delta_1^{-1}(A + B_2 K_1)x + P_2 \Delta_2^{-1} \bar{A} e]\,,$$
$$r'_* = \gamma^{-2} B_1^T P_2 \Delta_2^{-1} \bar{A} e\,.$$

Next, for the system with white noise:

$$\delta x = Ax + B_0 w_0 + B_1 w + B_2 u\,, \quad x(0) = 0$$
$$z = C_1 x + D_{12} u\,,$$
$$y = C_2 x + D_{20} w_0\,,$$

under the same changes of variables, we have:

$$\delta e = \bar{A}e + (B_0 + L_* D_{20}) w_0 + B_1 r\,, \quad e(0) = 0$$
$$v_* = D_{12} K_1 e\,,$$

and, similar to the proof for the sufficiency, the performance index $J_1(u_*, w, w_0)$ becomes:

$$J_1(u_*, w, w_0) = \gamma^2 \parallel w - \tilde{w}_* \parallel_{\mathcal{P}}^2 - \parallel D_{12}(u_* - \tilde{u}_*) \parallel_{\mathcal{P}}^2 - \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} [\text{trace}(B_0^T P_1 A\, E\{x w_0^T\})]$$
$$= \gamma^2 \parallel r - \gamma^{-2} B_1^T P_2 \Delta_2^{-1} \bar{A} e \parallel_{\mathcal{P}}^2$$
$$- \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \Big[\text{trace}(B_0^T P_1 A\, E\{x w_0^T\}) + \text{trace}\big((B_0 + L_* D_{20})^T P_2 A\, E\{x w_0^T\}\big)\Big]$$
$$= \gamma^2 \parallel r - \gamma^{-2} B_1^T P_2 \Delta_2^{-1} \bar{A} e \parallel_{\mathcal{P}}^2\,.$$

Therefore, if we choose:

$$w_* = r_* + K_2 x = \gamma^{-2} B_1^T [P_1 \Delta_1^{-1}(A + B_2 K_1)x + P_2 \Delta_2^{-1} \bar{A} e]\,,$$
$$r_* = \gamma^{-2} B_1^T P_2 \Delta_2^{-1} \bar{A} e\,,$$

then the condition $J_1(u_*, w_*, w_0) \leq J_1(u_*, w, w_0)$ is satisfied.

If this w_*, along with u_* achieves $J_2(u_*, w_*, w_0) \leq J_2(u, w_*, w_0)$, substitute w_* into the system equations to get:

$$\delta e = A_{L_*} e + B_{L_*} w_0 \,,$$

where:

$$A_{L_*} = A + L_* C_2 + B_1 K_2 + \gamma^{-2} B_1 B_1^T P_2 \Delta_2^{-1}(A + L_* C_2) \,, \quad B_{L_*} = B_0 + L_* D_{20} \,.$$

So

$$e = \sum_{j=0}^{k-1} A_{L_*}^{k-j-1} B_{L_*} w_0(j) \,,$$

and $J_2(u_*, w_*, w_0) = \text{trace}(Y)$ is the minimum value, where:

$$Y = \sum_{i=0}^{\infty} A_{L_*}^i B_{L_*} B_{L_*}^T (A_{L_*}^T)^i \geq 0 \,,$$

satisfies $A_{L_*} Y A_{L_*}^T - Y + B_{L_*} B_{L_*}^T = 0$. Hence, by Theorem 2.3, there is a $P_3 \geq 0$ solving (5.6) and L_* is to satisfy:

$$L_* = -(A_M P_3 C_2^T + B_0 D_{20}^T)(R_0 + C_2 P_3 C_2^T)^{-1}.$$

which concludes the proof.

■

5.2.1 Special Case

If the problem formulation is altered slightly, an explicit expression for the state estimator gain L_* can be found as stated in the following theorem.

Theorem 5.2 *For the linear, discrete-time system given by the equation set (5.1), where w and w_0 are assumed to be mutually uncorrelated, consider the performance index functions J_1 and J_2 defined by (5.2) and (5.3), respectively. If there are stabilizing solutions $P_1 \geq 0$, $P_2 \geq 0$ and $P_3 \geq 0$ solving the Riccati equations:*

$$P_1 = A^T P_1 A - (A^T P_1 B + C_1^T D)(B^T P_1 B + R)^{-1}(B^T P_1 A + D^T C_1) + C_1^T C_1 \,, \tag{5.8}$$

$$P_2 = \bar{A}^T(P_2 + \gamma^{-2}P_2B_1B_1^TP_2\Delta_2^{-1})\bar{A} + K_1^TR_1K_1\,, \tag{5.9}$$

$$P_3 = A_2P_3A_2^T - (A_2P_3C_2^T + B_0D_{20}^T)(R_0 + C_2P_3C_2^T)^{-1}(C_2P_3A_2^T + D_{20}B_0^T) + B_0B_0^T\,, \tag{5.10}$$

where $I - \gamma^{-2}B_1^TP_1B_1 > 0$, $I - \gamma^{-2}B_1^TP_2B_1 > 0$, $R_0 + C_2P_3C_2^T > 0$ *and* $A_2 = A + B_1K_2$. *i.e., if* $A - B(B^TP_1B + R)^{-1}(B^TP_1A + D^TC_1)$ *and* $A + B_1K_2 + L_*C_2$ *are both stable, then there exists an optimal controller* u_*, *a worst disturbance signal* w_* *and a disturbance signal* $\tilde{w}_*$, *such that*

$$J_1(u_*, w_*, w_0) \le J_1(u_*, w, w_0)\,,$$

$$J_2(u_*, \tilde{w}_*, w_0) \le J_2(u, \tilde{w}_*, w_0)\,.$$

If the solutions exist, then:

$$\tilde{w}_* = K_2x\,, \quad w_* = K_2x + K_3e\,,$$

where $K_3 = \gamma^{-2}B_1^TP_2\Delta_2^{-1}\bar{A}$ *and an optimal controller is given by:*

$$\delta\hat{x} = (\bar{A} + B_2F_*)\hat{x} - L_*y\,,$$

$$u_* = F_*\hat{x}\,, \quad \hat{x}(0) = 0,$$

where $F_* = K_1$ *and:*

$$L_* = -(A_2P_3C_2^T + B_0D_{20}^T)(R_0 + C_2P_3C_2^T)^{-1}.$$

Conversely, let P_1 *be a stabilizing solution to (5.8), and suppose there exists a* w_1 *and a controller* u_* *(hence an* L_**) in the form:*

$$\delta\hat{x} = (\bar{A} + B_2F_*)\hat{x} - L_*y\,,$$

$$u_* = F_*\hat{x}\,,$$

$$F_* = -S(B^TP_1B + R)^{-1}(B^TP_1A + D^TC_1)\,,$$

satisfying $0 < J_1(u_*, w_1, 0) \le J_1(u_*, w, 0)$. *Then there exists a* w_* *achieving* $J_1(u_*, w_*, w_0) \le J_1(u_*, w, w_0)$. *If, furthermore, there exists a* $\tilde{w}_*$ *that achieves* $J_2(u_*, \tilde{w}_*, w_0) \le J_2(u, \tilde{w}_*, w_0)$, *then there exist* $P_2 \ge 0$ *and* $P_3 \ge 0$ *solving (5.9) and (5.10), respectively.*

Moreover, if A_2 *is stable, then:*

$$L_* = -(A_2P_3C_2^T + B_0D_{20}^T)(R_0 + C_2P_3C_2^T)^{-1}.$$

Proof: The proof can be carried out easily similar to the proof of Theorem 5.1, keeping in mind that when studying index function J_2, the disturbance signal $\tilde{w}_* = K_2 x$ is to be considered.

■

5.3 Numerical Example

In this section, we develop a controller as presented by Theorem 5.1 for the same system studied in the example of the previous chapter, which is repeated here:

$$\begin{aligned}
\delta x &= \begin{bmatrix} 0 & 2 \\ 4 & 0.2 \end{bmatrix} x + \begin{bmatrix} 0.5 & 1 \\ 1 & 0 \end{bmatrix} w_0 + \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix} w + \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} u\,, \\
z_0 &= \begin{bmatrix} 0.5 & 0.6 \end{bmatrix} x + 0.8u\,, \\
z &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\,, \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1.2 & 1.6 \end{bmatrix} w_0\,.
\end{aligned}$$

The desired controller gain is derived by solving the Riccati equation (5.4), resulting to:

$$P_1 = \begin{bmatrix} 39.97 & -15.763 \\ -15.763 & 7.2577 \end{bmatrix},$$

$$u_* = F_* \hat{x} = \begin{bmatrix} 2.1238 & -2.7406 \end{bmatrix} \hat{x}\,,$$

and the worst disturbance signal can be characterized as:

$$w_* = K_2 x + K_3 e = \begin{bmatrix} -0.3258 & 0.2023 \end{bmatrix} x + \begin{bmatrix} 3.9178 & -4.7487 \end{bmatrix} e\,.$$

The Riccati equations (5.5) and (5.6) can then be solves as:

$$P_2 = \begin{bmatrix} -1959.1 & 581.25 \\ 581.25 & -253.74 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0.1896 & 0.5609 \\ 0.5609 & 4.6507 \end{bmatrix}.$$

A state estimator satisfying (5.7) is then calculated to be:

$$L_* = \begin{bmatrix} -0.5697 \\ -0.3603 \end{bmatrix}.$$

For the resulting closed-loop system, the cost functions as defined by (5.2) and (5.3) are computed as:

$$J_1 = 7.7597\,, \qquad J_2 = 6.5146\,.$$

It can be observed that the performance index functions show a trade-off for the system with the $\mathcal{H}_\infty$ Gaussian controller compared to cases where only $\mathcal{H}_\infty$ or $\mathcal{H}_2$ controller is applied.

Chapter 6

Advanced Control of Electric Power-Assisted Steering (EPS) System

In this chapter, the electric power assisted steering (EPS) system is considered as an application example for the control design schemes proposed in previous chapters.

6.1 Introduction

The essential function of an EPS system is to provide assisting steering torque to the driver using an electric motor which is electronically controlled. This method is more flexible than the conventional hydraulic power assisted steering (HPS) and offers numerous advantages. For instance, better fuel economy, reduced development time as well as system weight and volume, and a much improved functionality [7]. Therefore, it is not surprising that EPS is already starting to be used in high-volume, lead-vehicle applications.

The main components of an EPS are shown in the schematic diagram of Figure 6.1. They include a hand wheel (HW), an intermediate shaft (I-Shaft), an electric motor (actuator), a torque sensor, a reduction gear, rack/pinion structure, and an electronic control unit (ECU) where control and diagnostic algorithms are implemented in software.

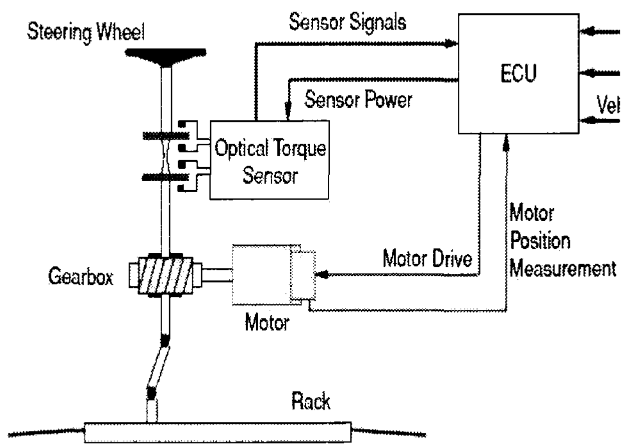

Figure 6.1: Schematic diagram of an electric power assisted system

The applied torque by driver is estimated by a torque sensor, which in turn communicates with the ECU. The control module is then drives the electric motor to provide the required assisting steering torque. This torque is then upgraded through the gearing mechanism and the tires are turned by the rack/pinion structure.

Different issues in EPS have been studied in the literature (see for example [3, 57, 34, 10] and references therein). Generally, there are two key requirements to be addressed in an EPS control system:

1. Sufficient torque has to be generated by the motor to perform the steering,

2. The driver's feeling during steering has to be smooth and comfortable.

6.2 Control-Oriented Dynamic Model for the EPS System

A generic, control-oriented dynamic model for the EPS system, as proposed in [11, 10], is given in Figure 6.2. In constructing this model, it is assumed that all mechanical connections are rigid, and also an armature-controlled DC motor [37] is used in the EPS system. The descriptions of the blocks are given in Table 6.1, where:

T_d: steering torque command from the driver;

T_r: road reaction torque on the rack and pinion;

θ_{rp}: rack/pinion angle (proportional to the tire angle);

θ_s: shaft angle;

J_{hw} , B_{hw}: inertia and damping constants of the hand wheel and I-shaft;

T_{ts}: torque sensor output with K_{ts} and B_{ts}torsion bar stiffness and damping constants;

θ_{ts}: torsion deformation of the torque sensor;

θ_m: motor shaft angle;

J_m , B_m: inertia and damping ratios of the motor;

L_a , R_a: inductance and resistance of the motor armature winding;

i_a: armature current;

K_e , K_t: counterbalance electromotive force and torque constants of the motor;

T_m: electromagnetic drive torque on the motor shaft;

J_{rp} , B_{rp}: inertia and damping constants of the rack and pinion structure.

It can be seen from the block diagram that this model involves a regulating operation as: $V = C_1(V_r - R_f i_a)$, and V_r is the reference signal in voltage generated by the torque sensor and through a mechanical/electric conversion ratio K.

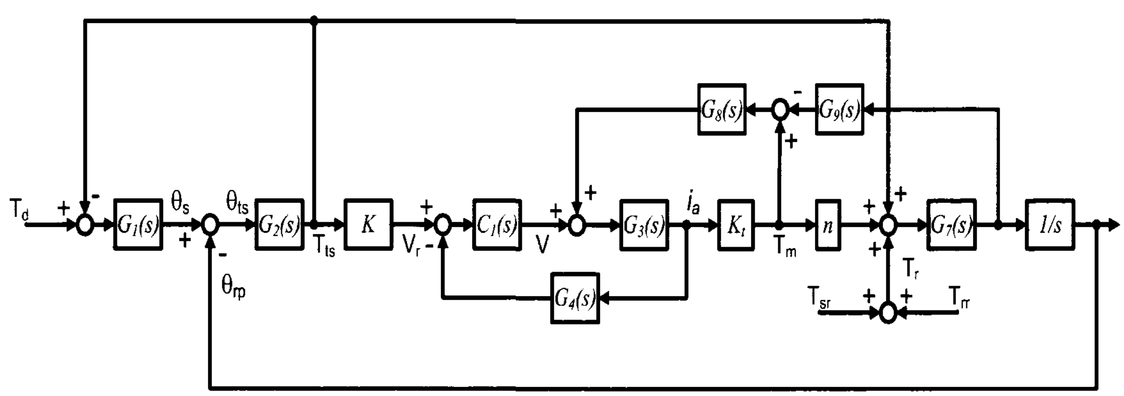

Figure 6.2: Block diagram of an EPS system

It is assumed that the road reaction torque on the rack and pinion structure consists of two parts as:

$$T_r = T_{sr} + T_{rr}$$

Block Description	Expression
Hand wheel and I-Shaft	$G_1 = \frac{1}{J_{hw}s^2+B_{hw}s}$
Torque sensor	$G_2 = B_{ts}s + K_{ts}$
Motor dynamics	$G_3 = \frac{J_m s+B_m}{(J_m s+B_m)(L_a s+R_a+R_f)+K_e K_t}$
Current feedback resistor	$G_4 = R_f$
Equivalent rack and pinion dynamics	$G_7 = \frac{1}{(J_{rp}+n^2 J_m)s+(B_{rp}+n^2 B_m)}$
Torque controller for the motor	C_1
Reduction gear	$n = \frac{\theta_m}{\theta_{rp}}$
Torque/voltage conversion	$G_8 = \frac{K_e}{J_m s+B_m}$
Speed/torque conversion	$G_9 = n(J_m s + B_m)$

Table 6.1: Blocks of the EPS system model

where T_{sr} and T_{rr} represent the tire reaction torque from a smooth and rough road surfaces respectively. It is also assumed that T_{sr} is in low frequency range while T_{rr} is in high frequency range [10].

6.3 Advanced Control Design

In this chapter, a model-based approach is employed and motivated by the works in [11, 10], a two-controller method is adopted. In this technique, the control action is considered in two parts: motion control and motor control. The function of the motor controller is to regulate the dynamics of the electric motor to produce the assisting torque with desired transients. The motion controller, on the other hand, provides an acceptable system performance and determines the driver's feeling during the steering. The motivation to apply an $\mathcal{H}_\infty$ or $\mathcal{H}_2$ controller as the motion controller is mainly the presence of the road disturbance, noisy measurements and the system model uncertainty.

The main purpose of the control design process is to first guarantee that most of torque required to turn the tires comes from the motor. From the EPS system model, it can be seen that the torque sensor output, which is in fact the difference between the driver input command and the rack/pinion position, acts as a reference value which determines the amount of torque that the motor has to provide. Following this view, the motor component loop can be seen as an input command follower and the motor controller C_1 can be designed to facilitate this following performance. On the other hand, since the road reaction torque depends on the road condition and it is very likely to contain undesired components, the torque sensor output alone cannot be a good reflection of the uncertainties in the system and furthermore, there might be noise in the measurements of the torque sensor mechanism itself. Therefore, the reference command sent to the motor needs to be regulated as well and the motor controller C_1 is not a good candidate to perform this task since it is inside the motor loop and cannot be too sensitive to the system uncertainties and disturbances.

To address these issues in the EPS system control problem, a new regulating control $C_2(s)$ is added to the system. Figure 6.3 shows this new controller which is to be designed as a discrete-time controller $C_2(z)$ via the algorithms developed in previous chapters and then included in the EPS system through a proper sampler and a zero-order holder (ZOH). To formulate this problem as a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design, as introduced in Chapter 4, the signals w, w_0, z and z_0 are chosen as shown in Figure 6.3. The weighting functions W and W_0 are chosen by trial and error to facilitate the performance requirements. W is a low pass Butterworth filter chosen to be sensitive to signals up to $100Hz$ frequency, which includes the major low frequency part of the road reaction torque, and W_0 is chosen to make the design problem regular:

$$W = \frac{35 \times 10^6}{s^3 + 200s^2 + 2 \times 10^4 s + 10^6}, \quad W_0 = 0.01 .$$

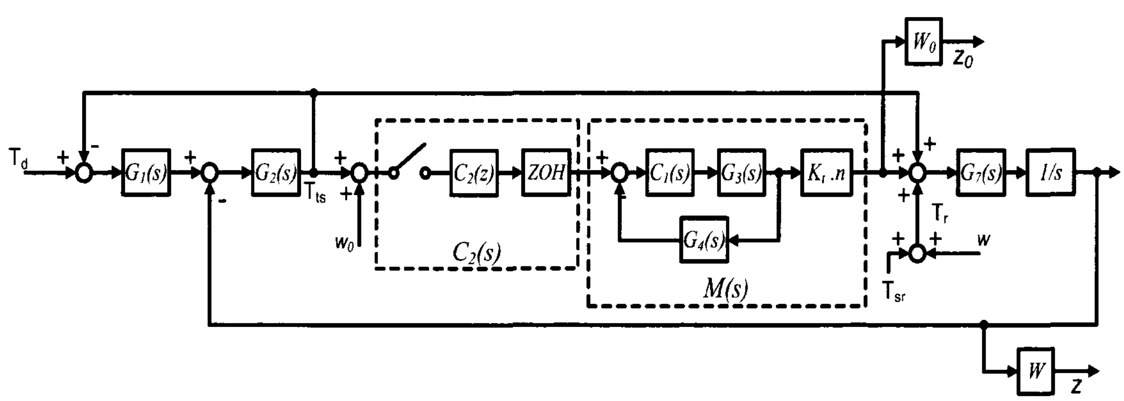

Figure 6.3: Block diagram of an EPS system with additional control C_2

In [10], full-order and reduced-order $\mathcal{H}_\infty$ and $\mathcal{H}_2$ controllers are developed and simulated with desirable performance results. However, only the disturbance signal w is considered in [10], representing the road reaction torque from a rough road. For this work, the measurement noise w_0 is also considered as a white noise signal added to the torque sensor output. The system is then converted to the standard setup shown in Figure 6.4, where $G(s) = C_G(sI - A_G)^{-1}B_G + D_G$ is found to be described by:

$$A_G = \begin{bmatrix} -200 & -20000 & -1e^6 & 0 & 0 & 0 & 0.6208 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.606 & -2424.2 & 80 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6208 \\ 0 & 0 & 0 & 0 & 2424.2 & -80 & -9.0017 \end{bmatrix},$$

$$B_G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 10000 & 1 \end{bmatrix}, \quad C_G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3.5e^7 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2424.2 \\ 0 & 0 & -80 \\ 0 & 0 & 0 \end{bmatrix}^T, \quad D_G = \begin{bmatrix} 0 & 0 & 0.01 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The block $M(s)$ in Figure 6.3 can be considered as a torque signal (generated by C_2) follower. Therefore, the whole controller would be found as $\bar{C}(s) = M(s)C_2$, with $M(s)$ being known after C_1 is designed. Clearly, C_2 can then be obtained by $\frac{\bar{C}}{M}$.

In order to implement the algorithms presented in previous chapters, this plant is converted to discrete time domain using the standard zero-order hold method, where $G(z) = C(sI-A)^{-1}B+D$ is obtained by:

$$A = e^{A_G T}, \quad B = \Big(\int_0^T e^{A_G \lambda} d\lambda\Big) B_G, \quad C = C_G, \quad D = D_G,$$

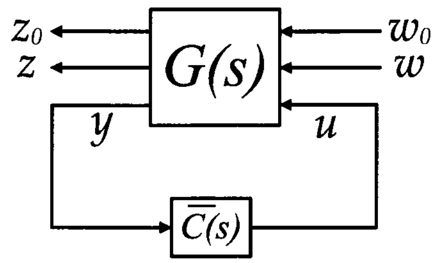

Figure 6.4: Standard setup for multiobjective control design

where T is the sampling period. Selecting $T = 0.1$, we obtain:

$$A = \begin{bmatrix} -5.34e^{-3} & -0.32 & 22.16 & -7.97e^{-4} & 0.06 & -1.96e^{-3} & 1.76e^{-5} \\ -2.22e^{-5} & -9.77e^{-3} & -0.76 & -2.62e^{-5} & -8.13e^{-4} & 2.68e^{-5} & -1.63e^{-6} \\ 7.6e^{-7} & 1.3e^{-4} & 5.43e^{-3} & 6.28e^{-7} & -2.58e^{-5} & 8.51e^{-7} & 2.92e^{-7} \\ 0 & 0 & 0 & 0.252 & 45.34 & -1.49 & 1.52e^{-4} \\ 0 & 0 & 0 & -1.7e^{2} & 0.24 & 0.025 & 0.0017 \\ 0 & 0 & 0 & 5.2e^{-2} & 3.6e^{-2} & 0.99 & 0.04 \\ 0 & 0 & 0 & 7.43e^{-3} & -45.4 & 1.49 & 0.416 \end{bmatrix},$$

$$B = \begin{bmatrix} 2.45e^{-5} & -0.016 & -1.6e^{-6} \\ -3.35e^{-7} & 2.92e^{-3} & 2.92e^{-7} \\ -1.06e^{-8} & 3.46e^{-4} & 3.46e^{-8} \\ 0.019 & 17.1 & 1.71e^{-3} \\ -3.13e^{-4} & 0.759 & 7.59e^{-5} \\ 1.49e^{-5} & 23.11 & 2.3e^{-3} \\ -0.019 & 647.3 & 0.0647 \end{bmatrix}.$$

At this point, for the EPS system represented by the block diagram of Figure 6.3, discrete-time controller $C_2(z) = C_c(zI - A_c)^{-1}B_c + D_c$ is designed.

First, a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller is calculated as developed in Chapter 4. Recall that the problem is formulated as:

$$\begin{aligned}\delta x &= Ax + B_0 w_0 + B_1 w + B_2 u\,, \quad x(0) = 0\\ z_0 &= C_0 x + D_{02} u\,,\\ z &= C_1 x + D_{12} u\,,\\ y &= C_2 x + D_{20} w_0\,,\end{aligned}$$

and the cost functions defined as:

$$J_1(u, w, w_0) = \gamma^2 \parallel w \parallel_{\mathcal{P}}^2 - \parallel z \parallel_{\mathcal{P}}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\gamma^2 \parallel w \parallel^2 - \parallel z \parallel^2\}\,,$$

$$J_2(u, w, w_0) = \parallel z_0 \parallel_{\mathcal{P}}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\parallel z_0 \parallel^2\}\,.$$

Assuming $\gamma = 3.5$ and $T = 0.1$, the resulted controller that can achieve:

$$\begin{aligned}J_1(u_*, w_*, w_0) &\leq J_1(u_*, w, w_0)\,,\\ J_2(u_*, w_*, w_0) &\leq J_2(u, w_*, w_0)\,,\end{aligned}$$

is given by:

$$A_{c1} = \begin{bmatrix} -0.153 & -32.22 & 7342 & 1.059e^{6} & 5.686e^{7} & 4.563e^{8} \\ -7.183e^{-4} & -0.5746 & -152.2 & -9391 & -3.103e^{5} & -2.258e^{6} \\ 3.525e^{-6} & 1.351e^{-3} & 0.01416 & -70.08 & -2673 & -2.015e^{4} \\ 3.156e^{-8} & 2.205e^{-5} & 6.623e^{-3} & 0.7494 & -9.931 & -75.51 \\ 1.183e^{-10} & 1.01e^{-7} & 4.182e^{-5} & 9.38e^{-3} & 0.9749 & -0.1913 \\ 2.999e^{-13} & 2.944e^{-10} & 1.511e^{-7} & 4.881e^{-5} & 9.951e^{-3} & 0.9996 \\ 5.857e^{-16} & 6.438e^{-13} & 3.922e^{-10} & 1.648e^{-7} & 4.992e^{-5} & 9.999e^{-3} \\ 9.382e^{-19} & 1.137e^{-15} & 8.005e^{-13} & 4.141e^{-10} & 1.666e^{-7} & 5e^{-5} \end{bmatrix},$$

$$A_{c2} = \begin{bmatrix} 2.704e^{8} & 8.743e^{6} \\ -1.328e^{6} & -4.291e^{4} \\ -1.188e^{4} & -384.1 \\ -44.56 & -1.44 \\ -0.1129 & -3.651e^{-3} \\ -2.205e^{-4} & -7.13e^{-6} \\ 1 & -1.142e^{-8} \\ 0.01 & 1 \end{bmatrix}, \qquad A_c = [A_{c1} \quad A_{c2}],$$

$$B_c = \begin{bmatrix} -7.183e^{-4} & 3.525e^{-6} & 3.156e^{-8} & 1.18e^{-10} & 3e^{-13} & 5.857e^{-16} & 9.38e^{-19} & 1.28e^{-21} \end{bmatrix}^T,$$

$$C_c = \begin{bmatrix} 334.6 & 7.804e^{4} & 6.85e^{6} & 1.222e^{8} & 1.762e^{9} & 1.801e^{10} & 6.821e^{10} & 2.286e^{9} \end{bmatrix},$$

$$D_c = 9.844e^{-4}.$$

Next, an $\mathcal{H}_\infty$ Gaussian controller is found by the method presented in Chapter 5. The problem is formulated as:

$$\begin{aligned} \delta x &= Ax + B_0 w_0 + B_1 w + B_2 u, \quad x(0) = 0 \\ z &= C_1 x + D_{12} u, \\ y &= C_2 x + D_{20} w_0, \end{aligned}$$

with cost functions:

$$J_1(u, w, w_0) = \gamma^2 \parallel w \parallel_{\mathcal{P}}^2 - \parallel z \parallel_{\mathcal{P}}^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\gamma^2 \parallel w \parallel^2 - \parallel z \parallel^2\},$$

$$J_2(u, w, w_0) = \parallel e \parallel_{\mathcal{P}}^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} E\{\parallel e \parallel^2\},$$

where $e = x - \hat{x}$.

The controller $C_2(z) = C_c(zI - A_c)^{-1}B_c + D_c$ that can achieve:

$$\begin{aligned} J_1(u_*, w_*, w_0) &\leq J_1(u_*, w, w_0)\,, \\ J_2(u_*, w_*, w_0) &\leq J_2(u, w_*, w_0)\,, \end{aligned}$$

can be characterized as:

$$A_{c1} = \begin{bmatrix} -0.0382 & -52.27 & -1561 & -2.75e^{4} & -2.22e^{5} & -6.63e^{5} \\ 1.45e^{-5} & -0.01617 & -48.51 & -1353 & -2.12e^{4} & -1.17e^{5} \\ 2.8e^{-6} & 4.277e^{-3} & 0.7101 & -8.379 & -135.3 & -754.8 \\ 1.812e^{-8} & 3.04e^{-5} & 8.984e^{-3} & 0.9702 & -0.4863 & -2.728 \\ 6.55e^{-11} & 1.18e^{-7} & 4.74e^{-5} & 9.924e^{-3} & 0.9988 & -6.984e^{-3} \\ 1.68e^{-13} & 3.21e^{-10} & 1.62e^{-7} & 4.985e^{-5} & 9.998e^{-3} & 1 \\ 3.352e^{-16} & 6.788e^{-13} & 4.08e^{-10} & 1.66e^{-7} & 5e^{-5} & 0.01 \\ 5.51e^{-19} & 1.175e^{-15} & 8.22e^{-13} & 4.16e^{-10} & 1.67e^{-7} & 5e^{-5} \end{bmatrix},$$

$$A_{c2} = \begin{bmatrix} -3.23e^{5} & -1.03e^{4} \\ -6.214e^{4} & -1996 \\ -402.6 & -12.93 \\ -1.456 & -4.677e^{-2} \\ -3.728e^{-3} & 1.198e^{-4} \\ -7.45e^{-6} & -2.39e^{-7} \\ 1 & -3.93e^{-10} \\ 0.01 & 1 \end{bmatrix}, \quad A_c = [A_{c1} \quad A_{c2}]\,,$$

$$B_c = \begin{bmatrix} 1.45e^{-5} & 2.8e^{-6} & 1.81e^{-8} & 6.56e^{-11} & 1.68e^{-13} & 3.35e^{-16} & 5.51e^{-19} & 7.71e^{-22} \end{bmatrix}^T,$$

$$C_c = \begin{bmatrix} 333.4 & 2.95e^{4} & 1.1e^{6} & 1.86e^{7} & 2.47e^{8} & 2.17e^{9} & 7.38e^{9} & 2.47e^{8} \end{bmatrix},$$

$$D_c = 9.841e^{-4}$$

6.4 Simulation Results

The controllers designed in the previous section is based on a simplified dynamic model of the EPS system. In order to validate the results, a high-fidelity simulation platform is developed using the model and $CarSim^{TM}$, a software package capable of simulating vehicle dynamics. The use of the CarSim serves two purpose: to generate the road reaction torque of a smooth road (T_{sr}); and to provide a benchmark rack/pinion angle for comparison with the controlled EPS system model. From this comparison, we can justify the desirable action of the power assisted steering system if it is able to deliver most of the steering torque to the motor, no matter what road condition is and also in the presence of the measurement noise. The simulation setup is illustrated in Figure 6.5.

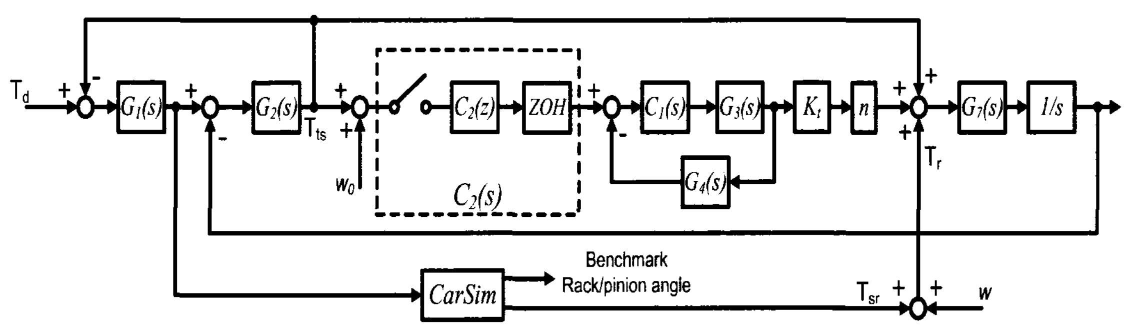

Figure 6.5: Simulation setup for the controlled EPS system

Models for the blocks of the EPS system are assumed as follows:

$$G_1(s) = \frac{1}{0.04s^2 + 0.03s}, \quad G_2(s) = 0.03s + 65, \quad G_7(s) = \frac{1}{3.18s + 28.5}, \quad n = 20,$$

$$G_3(s) = \frac{0.00395s + 0.035}{(0.00395s + 0.035)(0.0015s + 0.37) + 0.25}, \quad G_4(s) = \frac{1}{40},$$

In all of the simulations, it is assumed that the driver turns the steering wheel to the left with 1 Newton-meter torque for 5 seconds and then releases the wheel, as seen in Figure 6.6.

First, the motor controller C_1 is chosen to be, after trial and error, a PI controller $C_1(s) = K_P + \frac{K_I}{s}$ with $K_P = 20400, K_I = 700$. The tracking response of the closed-loop motor loop with this C_1 to a unit step input is shown in Figure 6.7 and Figure 6.8 illustrates the road reaction torque and the torque sensor output without applying any motion controller ($C_2 = I$). It can be seen that, although the motor control C_1 is capable of following the input command and also smoothing the

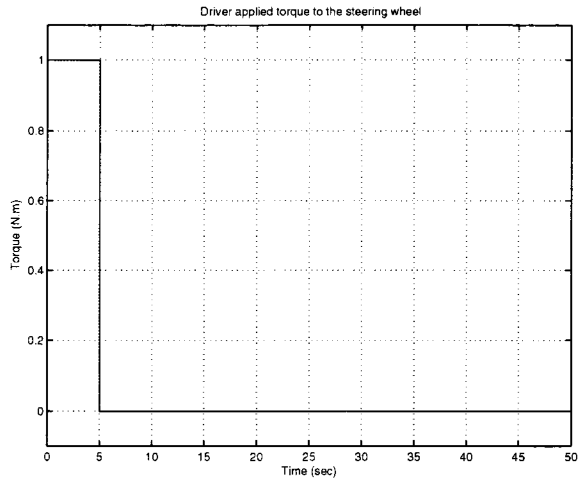

Figure 6.6: Driver's turning torque applied to the steering wheel

effect of the rough road, the transient response is slow and furthermore, there exists an overshoot in the steering torque which may be harmful to the mechanical parts.

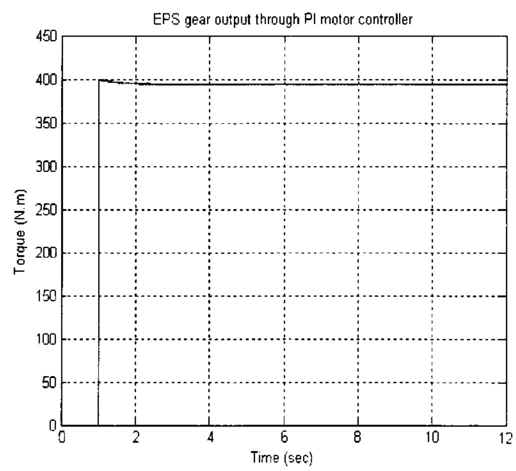

Figure 6.7: Motor response with PI controller

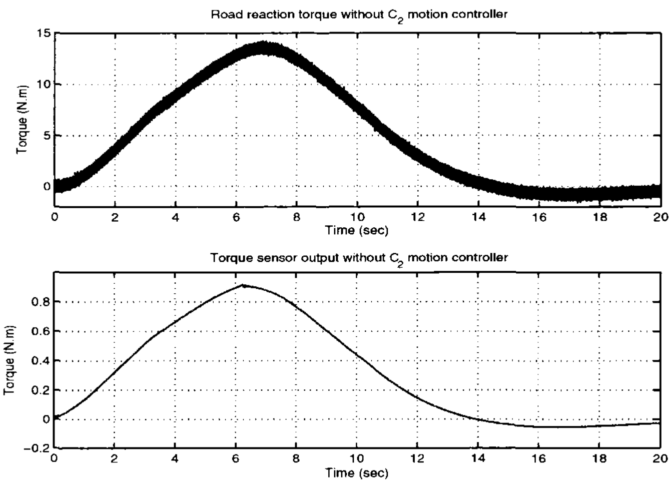

Figure 6.8: Road reaction torque and the torque sensor output without $C_2(z)$

Next we consider the system with only a disturbance signal from a rough road surface ($w_0 = 0$). For this case, as mentioned before, $\mathcal{H}_\infty$ and $\mathcal{H}_2$ controllers are developed in [10] and the simulation results of these controllers are given in Figures 6.9 and 6.10. It can be seen that, both of these schemes can suppress the effect of the rough road surface on how the driver feels. However, as can be expected, the transient response of the $\mathcal{H}_\infty$ controller takes longer, comparing with the $\mathcal{H}_2$ design.

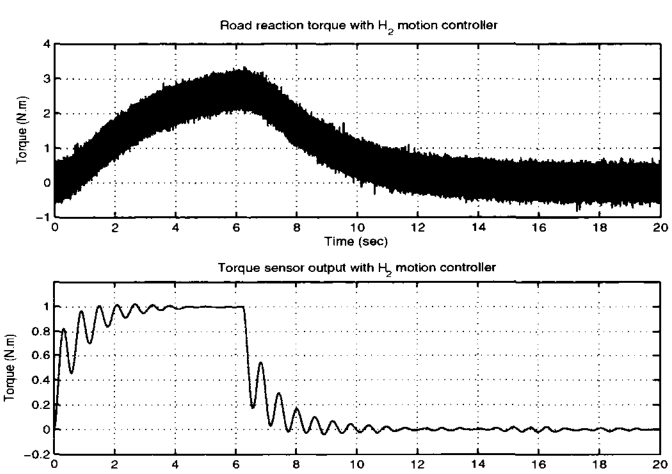

Figure 6.9: Road reaction torque and the torque sensor output with $w_0 = 0$ and $\mathcal{H}_2$ controller

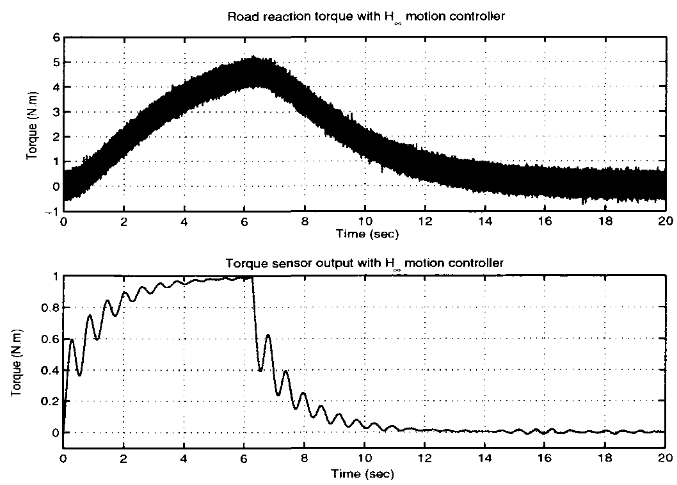

Figure 6.10: Road reaction torque and the torque sensor output with $w_0 = 0$ and $\mathcal{H}_\infty$ controller

If it is assumed that both disturbance signals w and w_0 are present, i.e. the rough road surface and the measurement noise are considered, as can be seen from Figures 6.11 and 6.12, neither an $\mathcal{H}_\infty$ nor an $\mathcal{H}_2$ controller can handle the added disturbance individually.

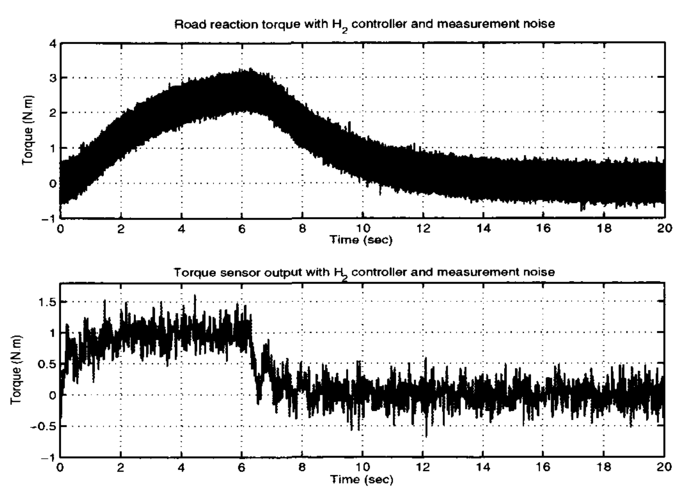

Figure 6.11: Road reaction torque and the torque sensor output with w_0 present and $\mathcal{H}_2$ controller

Now, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller derived in the previous section is applied for the motion control C_2. The simulation results for this system is shown in Figure 6.13. It can be seen that this controller is capable of overcoming both disturbances and provide a good output. To verify

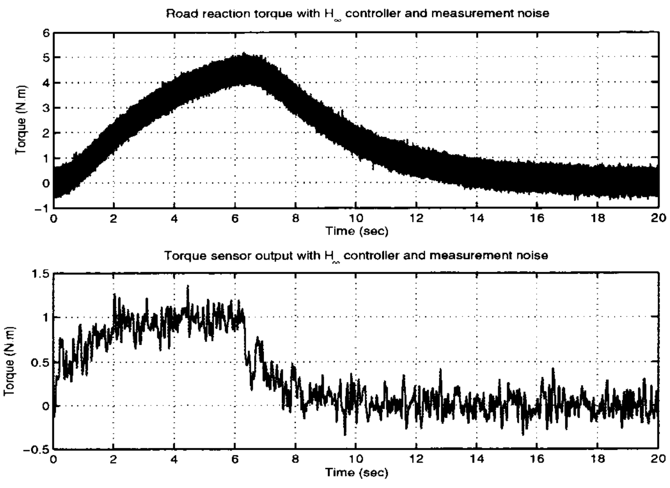

Figure 6.12: Road reaction torque and the torque sensor output with w_0 present and $\mathcal{H}_\infty$ controller

the performance further, the rack/pinion angle is compared with the benchmark angle generated by CarSim in Figure 6.14. It is shown that C_2 is now successful in providing a steering dynamics on a rough road and with a measurement noise close to that of a smooth road with perfect measurement.

Finally, if the $\mathcal{H}_\infty$ Gaussian controller calculated before is applied for C_2, a good system dynamics performance is achieved as well. Figure 6.15 shows the road reaction torque and the torque sensor output, and Figure 6.16 is a comparison of the rack/pinion angle with the CarSim benchmark with a system with this controller.

It can be concluded that although the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers are capable of providing a good performance when only a single disturbance from the rough road surface is considered, they are not able to handle an extra disturbance introduced to the system. This example illustrates the advantage of utilizing a multiobjective controller very clearly.

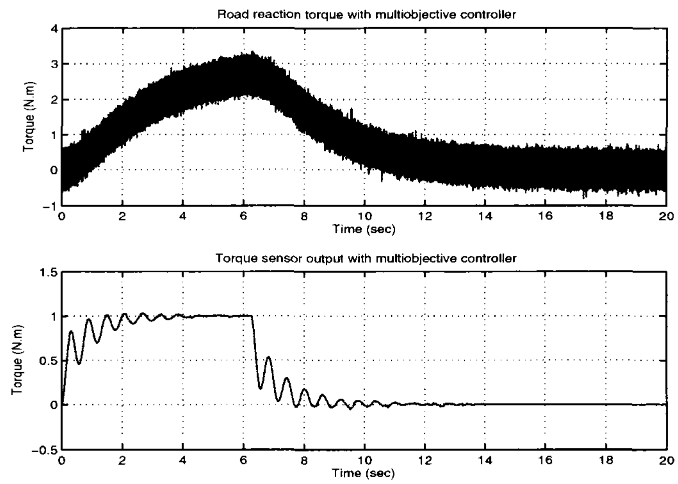

Figure 6.13: Road reaction torque and the torque sensor output with w_0 present and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller

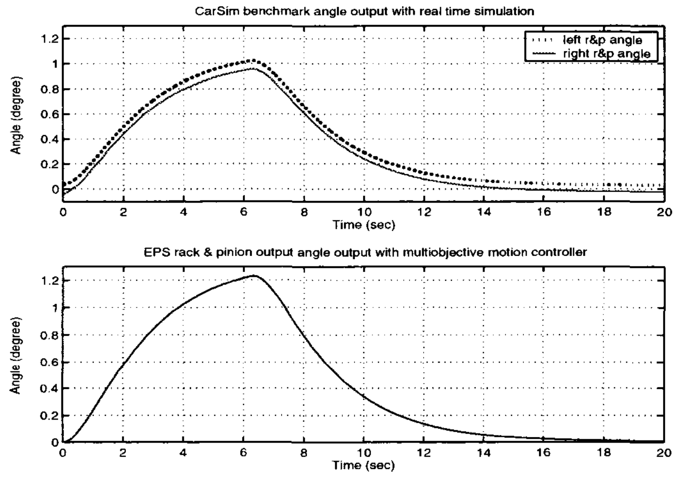

Figure 6.14: Benchmark comparison for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ controller

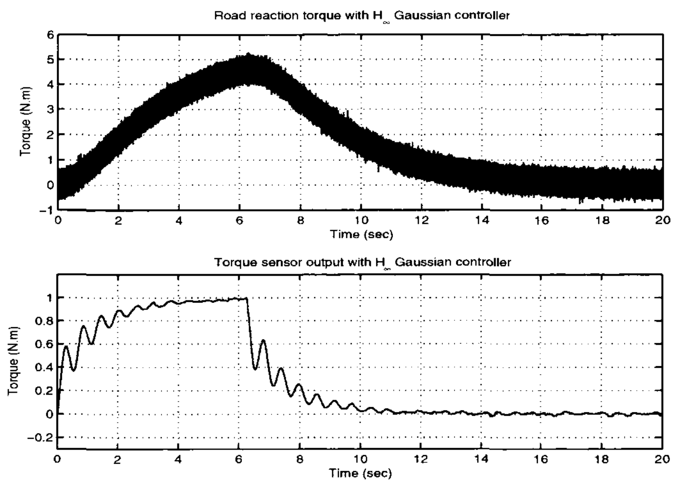

Figure 6.15: Road reaction torque and the torque sensor output with w_0 present and $\mathcal{H}_\infty$ Gaussian controller

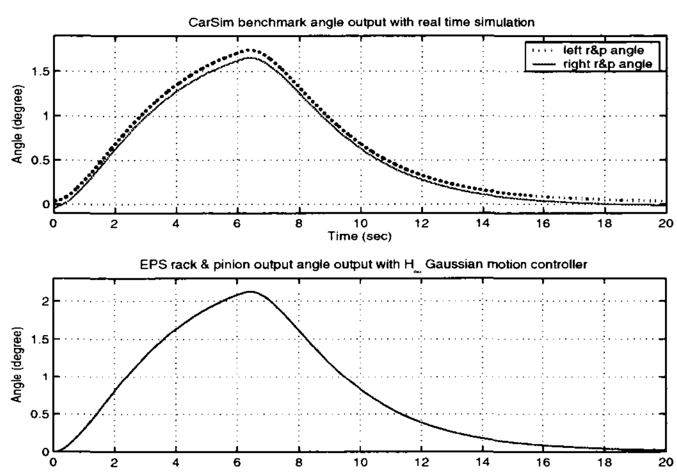

Figure 6.16: Benchmark comparison for $\mathcal{H}_\infty$ Gaussian controller

Chapter 7

Conclusions

Engineers are required to deal with problems of real world, whether they are overcoming a natural limitation of human beings and opening new horizons by way of creating new devices, or improving an already existing technical development in some way. Mathematics is one of the tools that is at the disposal of an engineer, which perhaps is more evident than any other field, in control engineering. Nevertheless, scientists today still have a long way to go to even get close to making an accurate impression of the real world by mathematical models. Then again, it is possible for engineers to develop models of the real world problems that are '*close enough*' for what is required of them to achieve. On this path, however, there is always room for improvement, and the closer a mathematical model is to real life, the more desirable results will be possible from an engineering mechanism. This is the main idea behind the problem defined in this work and the approach adopted in attempting to solve it.

In this dissertation, direct design methods for a multiobjective filter as well as two mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control strategies in discrete-time domain were developed to complete the theory on these problems. In all of these derivations, it was observed that carrying out the discrete-time design is not as routine as expected. A robust optimal signal estimator in discrete-time domain was introduced. This method provides a filter that can be capable of achieving robust performance against system model uncertainties, as well as optimal performance against white noise. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filter can be obtained by solving a set of coupled Riccati equations.

Furthermore, both necessary and sufficient conditions for existence of two output-feedback, mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control laws were given in the form of Riccati equations which are solvable through some standard algorithms. These methods provide solutions to achieve a trade-off between $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances and can address the plants affected by both white noise and bounded power disturbance signals, which in turn have the potential to be applied to more realistic engineering applications utilizing digital controllers. In addition to numerical examples provided for each design method, the electric power-assisted steering system was studied as an application for this multiobjective control scheme. These examples demonstrated system performances having the trade-off property as expected, and the ability to handle two different types of disturbances, namely a white gaussian noise and the bounded power disturbance.

7.1 Future Work

In the problem formulations considered in this dissertation, only some standard assumptions have been made on the system and the goal was to find the most general solutions possible. However, more considerations can now be added to the plant and the problem statement. For instance, time delay can be introduced to the system, which raises the question of what type of time delay the existing designs can handle and if they need to be modified depending on the nature of the delay. Also, new disturbances that can be dependent on the system states or output measurement can be considered, making the assumptions more realistic for dealing with some engineering problems.

Because of the discrete-time nature of the method adopted in this work, it can be a much better foundation for applying the idea of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control in Networked Control Systems (NCS), which is one of the fastest growing areas in Control Engineering and extensive time and interest is devoted to it by both industry and academic researchers.

Although the problem of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control has been addressed from many aspects in the literature, a comprehensive study seems to be needed to combine different solutions into a more unified approach. Also, the question of comparing these different solutions in terms of performance parameters such as transient response, stability margins, robustness and so on can be conducted to be used as a selection criteria when applying to a specific problem requirements.

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