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The unitary group approach to nuclear magnetic resonance of higher spin nuclei.

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THE UNITARY GROUP APPROACH TO NUCLEAR MAGNETIC RESONANCE OF HIGHER SPIN NUCLEI

by

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ABSTRACT

The application of the Unitary Group Approach to atomic physics has been established in the literature. The extension of this approach to NMR is studied in detail. The spectra of A_4B_2 and A_3 systems are calculated, and new projection operator techniques for the treatment of mix-d configurations is presented.

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INTRODUCTION

The Unitary Group Approach is a relatively new technique in the theory of angular momentum. Although the mathematical foundation for it has been in existence for several years, its application to atomic, nuclear and condensed matter physics commenced recently.

In order to make the discussion of the approach to angular momentum pedagogically sound, we begin with a few comments about the role of angular momentum in physics.

Historically, one of the primary reasons for the study of angular momentum in classical physics is its conservation by a central force. When it was discovered that there is a relationship between conservation of angular momentum and rotational symmetry of the Lagrangian (or Hamiltonian), the foundation was laid for studying angular momentum in a more general context through the use of symmetry principles.

The interrelationship between symmetry and angular momentum prompted physicists, notably G. Racah and E.P. Wigner, to use group theory in the analysis of the quantum theory of angular momentum.

An exhaustive treatment of the Racah approach is not possible in this introduction, but a brief synopsis will be presented.

The Racah approach relies on irreducible tensors (and inner products of irreducible tensors) to analyze atomic structure. Specifically, multipole moment operators, spin-orbit operators, and various other operators of physical interest are expressed in terms of irreducible tensors to facilitate the analysis of complicated electronic configurations. The Wigner-Eckart theorem naturally proves to be useful in such analysis.

The state-labeling problem[1], addressed initially by Slater, necessitated the infusion of group theory into the Racah scheme. Slater used determinantal wavefunctions of single-particle states which indeed satisfied the Pauli Exclusion Principle, but led to hopelessly cumbersome calculations for complicated configurations.

Racah begins with a set of one body tensor operators from which several many body operators are defined through the use of the tensor inner product. Each such many body operator defines a quantum number and belongs to a representation of a chain of groups. For ℓ^N configurations the appropriate chain is[2]

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[1] The problem of labeling a complete basis for an N-Fermion quantum mechanical system within the constraints imposed by the Pauli Exclusion Principle. This term is also used (in a more general sense) for the problem of labeling and complete basis for an N-particle system.

[2] Deciding which chain is appropriate is not a trivial matter. As we shall see this problem does not exist in the Unitary Group Approach.

$$U(2\ell+1) \supset SO(2\ell+1) \supset \ldots \supset SO(3) \supset SO(2).$$

The classification of these many body operators is based on their transformation properties with respect to the basis of the irreducible representations of each group in the chain. These then provide labels for the irreducible representations. Such operators are called invariant operators. Each Racah chain terminates with $SO(3) \supset SO(2)$ since the invariant operator for SO(2) labels the total magnetic quantum number while the invariant operator for SO(3) labels the total orbital angular momentum. This provides identification of states in terms of conventional spectroscopic notation.

There are some difficulties with the Racah scheme. First and foremost is the fact that the labeling based on the Racah scheme is incomplete for $f^5 - f^9$ configurations [1]. Other difficulties include complicated expressions for operators and tedious calculations involving fractional parentage coefficients.

The Unitary Group Approach to atomic physics eliminates these problems by considering the representations of the chain of groups

$$U(2\ell+1) \supset U(2\ell) \ \ldots \supset U(2) \supset U(1)$$

for a pure ℓ^N configuration. Labeling based upon such a chain is unique and accounts for all states. More precise mathematical details will be forthcoming in the theoretical background section.

The purpose of this thesis is to apply the Unitary Group Approach to nuclear magnetic resonance. (As we shall see, this presents some crucial differences from its application to atomic physics.) We will evaluate the matrix elements of a (nuclear) spin-spin coupling Hamiltonian for an A_3 system and an A_4B_2 system.

CHAPTER I

REPRESENTATIONS OF THE UNITARY GROUP - THE GELFAND-ZETLIN REPRESENTATION AND THE WEYL REPRESENTATION.

The Unitary Group U(N) is defined as follows:

$$U(N) \equiv \{ A \in M_N(\mathbb{C}) \mid \langle Ax, Ay \rangle = \langle x, y \rangle \ \forall \ x, y \in \mathbb{C}^N \}.$$

U(N) is one of the classical Lie groups. It is a compact, connected Lie group of dimension N^2. The Lie algebra of U(N) is the set of all skew-Hermitian matrices. Explicitly,

$$L(U(N)) = \{ A \in M_N(\mathbb{C}) \mid A + \bar{A}^t = 0 \} .$$

The set $S = \{ A_{kk}, B_{\ell m}, C_{\ell m} \mid {}^{k = 1,2 - N}_{\ell < m = 1,2 - N} \}$

generates L(U(N)) as a sub-algebra of the universal enveloping algebra of L(U(N)). Here we use the definitions

$$A_{kk} = \begin{cases} i & \text{in the kk'th entry} \\ 0 & \text{otherwise} \end{cases}$$

$$B_{\ell m} = \begin{cases} 1 & \text{in the } \ell m\text{'th entry} \\ -1 & \text{in the } m\ell\text{'th entry} \\ 0 & \text{otherwise} \end{cases}$$

$$C_{\ell m} = \begin{cases} i & \text{in the } \ell m\text{'th entry} \\ i & \text{in the } m\ell\text{'th entry} \\ 0 & \text{otherwise} \end{cases}$$

The complexification of L(U(N)) yields an algebra generated by $\{E_{ij} \mid i,j = 1,2 \ldots N\}$ where

$$E_{ij} = \begin{cases} 1 & \text{in the } ij\text{'th entry} \\ 0 & \text{otherwise.} \end{cases}$$

This algebra is the Lie Algebra of GL(N,$\mathbb{C}$). A trivial computation shows that

(i) $[E_{ij}, E_{k\ell}] = \delta_{jk}E_{i\ell} - \delta_{i\ell}E_{kj}$; $1 \leq i,j,k,\ell \leq N$

(ii) $E_{ij}^{+} = E_{ji}$; $1 \leq i,j \leq N$.

Each unitary matrix can be written as a product of the following set of matrices (where $t \in \mathbb{R}$):

$$U_{kk}(t) = \text{EXP}\ (it\ E_{kk})\ ; \quad k = 1,2 - N.$$

$$U_{\ell m}(t) = \text{EXP}\ \left(\frac{it(E_{\ell m} + E_{m\ell})}{2}\right); \ell < m = 1,2-N.$$

$$U_{pr}(t) = \text{EXP}\ \left(\frac{t(E_{pr} - E_{rp})}{2}\right); \quad r < p = 1,2-N.$$

For this reason, the set $\{U_{kk}(t), U_{\ell m}(t), U_{pr}(t) \mid k = 1,2 - N;\ \ell<m = 1,2-N;\ r<p = 1,2-N\}$ is called a basic set of one parameter subgroups. [This result follows from the connectedness of U(N).] For further details on the basic properties of U(N) stated thus far, see [2,3,4].

In the analysis of (analytic finite-dimensional) representations of A Lie group G the following result is crucial: If Γ is a finite dimensional analytic representation of G, then

$$d\Gamma(x) \equiv \frac{d}{dt}(\Gamma(\mathrm{EXP}tx))|_{t=0} \; ; \quad t\varepsilon|R$$

(where x ε L(G)) is a representation of L(G). dΓ is called the differential representation of L(G). Henceforth, a representation of a Lie group is assumed to be finite dimensional and analytic unless otherwise stated. For connected Lie groups (such as U(N)) the converse of this result is also true. [The proof of this statement uses the fact that a connected Lie group is generated by a basic set of one parameter subgroups.] The preceding discussion establishes a one-one correspondence between representations of a connected Lie group and its Lie algebra.

The fact that GL(N,¢) is the regular complexification of U(N) establishes a one-one correspondence between representations of U(N) and representations of GL(N,¢). We summarize our results as follows:

Representations of U(N) ⟷ Representations of GL(N,¢)

↕ (left column) and ↕ (right column)

Differential Representations of L(U(N)) ⟷ Differential Representations L(GL(N,¢)).

The arrows indicate a one-one correspondence. Therefore, in order to study representations of U(N), it suffices to either [A] study representations of GL(N,¢) [Global Approach] or [B] study differential representations of L(GL(N,¢)) [Infinitesimal Approach].

Finally, a further restriction in our analysis is possible due to the principle of complete reducibility which states that the representations of a compact Lie group are equivalent to unitary representations and are completely reducible. We can thus confine our attention to irreducible representations of U(N).

We begin our discussion of representations of U(N) with the infinitesimal approach. For every differential representation of L(GL(N,¢)) we can define a linear functional on a maximal commutative sub-algebra [Cartan Sub-Algebra] called a weight. Each weight has an associated weight vector. If dΓ is an irreducible representation of L(GL(N,¢)) with representation space V, then there exists a set of weight vectors which form a basis for V. The highest weight vector is unique [up to a multiplicative constant] for irreducible representations. The highest weight vector determines irreducible representations uniquely [up to equivalence]. Finally, if dΓ is an irreducible representation of L(GL(N,¢)) and it induces a single valued representation of GL(N,¢), then the highest weight vector is characterized by the conditions

$$m_1 \geq m_2 \geq m_3 \geq \ldots \geq m_N: \quad m_j \in \mathbb{Z} \; j = 1,2,\ldots N.$$

Here $(m_1, m_2, \ldots, m_N)$ is the highest weight vector. For proofs of these statements, see reference 1.

Now suppose Γ is a (single-valued) irreducible representation of $GL(N,¢)$ with highest weight vector $(m_{1N}, m_{2N} \ldots, m_{NN})$. Then Γ restricted to $GL(N-1,¢)$ contains each irreducible representation of $GL(N-1,¢)$ exactly once. [Weyl's Branching Law.] Furthermore, if an irreducible representation of $GL(N-1,¢)$ has the highest weight vector $(m_{1N-1}, m_{2N-1}, \ldots, m_{N-1N-1})$ then the following conditions are satisfied:

$$m_{1N} \geq m_{1N-1} \geq m_{2N} \geq m_{2N-1} \geq \cdots \geq m_{N-1N-1} \geq m_{NN}$$

These conditions are called in betweenness conditions.

Now consider the following chain of subgroups

$$GL(N,¢) \supset GL(N-1,¢) \supset \ldots \supset GL(1)$$

where $GL(N-1,¢)$ is embedded in $GL(N,¢)$ in the following manner:

$$GL(N-1,\mathbb{C}) = \begin{bmatrix} GL(N-1,\mathbb{C}) & 0 \\ 0 & 1 \end{bmatrix} \leq GL(N,\mathbb{C})$$

Using Weyl's Branching Law, for an irreducible representation Γ of $GL(N,c)$, we have

$$\Gamma|_{GL(N-1,\mathbb{C})} = \bigoplus_{\mu_1 = 1}^{\alpha_1} \Gamma^{(\mu_1)}$$

where $\{\Gamma^{(\mu_1)} | 1 \leq \mu_1 \leq \alpha_1\}$ are irreducible representations of $Gl(N-1,\mathbb{C})$. We can iterate this procedure and obtain Γ^{μ_1} in terms of irreducible representations of $GL(N-2,\mathbb{C})$. It can be shown that [5,6] such a restriction yields a decomposition of a cyclic module R_α which is imbedded in the right regular representation of $GL(N,c)$. Explicitly,

$$R_\alpha = \bigoplus_{\mu_{1}=1}^{\alpha_1} \bigoplus_{\mu_{2}=1}^{\alpha_1} \cdots \bigoplus_{\mu_{k}=1}^{\alpha_k} R_\alpha^{(\mu_1,\mu_1,..\mu_k)}$$

Choosing a non-zero vector from each $R_\alpha^{(\mu_1,\mu_2,...\mu_k)}$ we obtain a basis for R_α. Each basis vector is uniquely specified by the following tableau:

$$(m) = \begin{bmatrix} m_{1N} & m_{2N} & \cdots\cdots\cdots & m_{NN} \\ & m_{1N-1} & m_{2N-1} \cdots & m_{N-1N-1} \\ & & & \\ & & m_{11} & \end{bmatrix}$$

By the in betweeness conditions we have,

$$m_{kj} \geq m_{kj-1} \geq m_{k+1j}; \quad j = 2,3,\text{--}N \quad k = 1,2,\text{--}j$$

The tableau (m) is called a Gelfand tableau.
The associated basis is called the Gelfand-Zetlin basis, and the associated representation of GL(N,¢) is called the Gelfand-Zetlin representation [5,6].

With each Gelfand tableau we can associate a set of operators $\{E_{ij}|i,j = 1,2\ldots N\}$ which we interpret as a differential represention of L(GL(N,¢)). These are partitioned into three sets:

(i) $\{ E_{ij}|i<j = 1,2\ldots N\}$ [raising operators]
(ii) $\{ E_{ij}|j<i = 1,2\ldots N\}$ [lowering operators]
(iii) $\{ E_{ii}|i = 1,2 \ldots N\}$ [diagonal operators].

The motivation for the partitioning and terminology arises from the definition of weights. The raising (lowering) operators raise (lower) the weight of |(m)>. By definition, the basis vector |(m)> is an eigenvector of the diagonal operators and the associated eigenvalue [the weight] is given by the following expression:

$$E_{\ell\ell}|(m)\rangle = \left(\sum_{k=1}^{\ell} m_{k\ell} - \sum_{k=1}^{\ell-1} m_{k\ell-1}\right)|(m)\rangle \equiv \omega_\ell|(m)\rangle.$$

The weight vector associated with (m) is $(\omega_1, \omega_2, \ldots, \omega_N)$. The action of E_{i+1i} on the basis is given by the following expression

$$E_{i+1i}|(m)\rangle = \sum_{k=1}^{i} \left[\frac{\prod_{q=1}^{i+1}(\ell_{qi+1}-\ell_{ki}+1)\prod_{q=1}^{i-1}(\ell_{qi-1}-\ell_{ki})}{\prod_{\substack{q=1\\q\neq k}}^{i}(\ell_{qi}-\ell_{ki})\prod_{q=1}^{i}(\ell_{qi}-\ell_{ki}+1)} \right]^{1/2} |(m)-\varepsilon_k(i)\rangle$$

where $\ell_{jp} = m_{jp}-j$ and $\varepsilon_k(i)$ is a unit vector of the i^{th} row with 1 in the k^{th} entry. The matrix elements for arbitrary E_{ij} can be derived from this expression [6,7].

An alternate description of the Gelfand-Zetlin basis is given by Biedenharn [7]. Assuming the existence of a differential representation of $L(GL(N,¢)$ on a finite dimensional inner product space V and assuming $\{E_{ij}|i<j = 1,2\ldots N\}$ is irreducible on V, we can construct the following set of basic invariants for $\{E_{ij}\ i,j = 1,2\ldots N\}$:

$$E_k^{(N)} = \sum_{i_1,i_2,\ldots,i_k=1}^{N} E_{i_1i_2}\, E_{i_2i_3}\, E_{i_3i_4} \ldots E_{i_ki_1}$$

$$k = 1,2,\ldots N$$

Since the set of generators are assumed to be irreducible and $\{E_1^{(N)}, E_2^{(N)}, \ldots, E_N^{(N)}\}$ commute with the generators, there exists a simultaneous eigenvector of $\{E_1^{(N)}, E_2^{(N)}, \ldots, E_N^{(N)}\}$

In the same manner, a basic set of invariants $\{E_1^{(k)}, E_2^{(k)}, \ldots E_N^{(k)}\}$ can be constructed for U(k) [k = 1,2,...,N-1]. The $\frac{N(N+1)}{2}$ Hermitian operators so defined mutually commute. The Gelfand basis can then be interpreted as the simultaneous eigenvectors of these operators. The action of the linear operators $\{E_{ij} | i,j = 1,2\ldots N\}$ take a slightly different [but equivalent] form:

$$E_{i+1\,i}|(m)\rangle = \sum_{k=1}^{i} \left[\frac{(-)\prod_{q=1}^{i+1}(P_{qi+1}-P_{ki}) \prod_{q=1}^{i-1}(P_{qi-1}-P_{ki}+1)}{\prod_{\substack{q=1\\q\neq k}}^{i}(P_{qi}-P_{ki}) \prod_{\substack{q=1\\q\neq k}}^{i}(P_{qi}-P_{ki}+1)} \right]^{1/2} |(m)-\epsilon_k(i)\rangle$$

where $P_{rs} = m_{rs} + s-r$.

The dimension of the vector space associated with the Gelfand tableau (M) is given by Weyl's dimension formula [7]

$$\dim[V(m)] = \prod_{i<j=1}^{N} \frac{(P_{iN} - P_{jN})}{(N-1)!(N-2)!\ldots 2!}$$

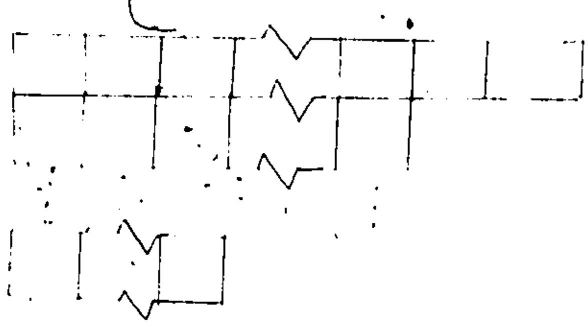
where the P_{rs} are as before. For the matrix elements of E_{ij}

for arbitrary i and j see [7].

For physical applications, it is more convenient to deal with Weyl tableaux than Gelfand tableaux. In order to describe Weyl tableaux, we will digress briefly and discuss Young tableaux.

The irreducible representations of S(k) are conveniently described by Young graphs. These irreducible representations are connected with those of GL(N,¢) and the other classical Lie groups. This provides the connection between Weyl and Gelfand tableaux.

We can associate with each class of S(k) a partition of k into positive integers. For a finite group [such as S(k)] the number of irreducible representations is equal to the number of classes. Thus all partitions of k into positive integers with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$ and $\sum_{\ell=1}^{r} \lambda_\ell = k$ generate an irreducible representation of S(k). Given a partition of S(k) the following graph is called a Young graph:



λ_1 boxes in first row.

λ_2 boxes in second row.

λ_r boxes in r'th row.

This graph is denoted by $[\lambda_1, \lambda_2, \ldots, \lambda_r]$ or simply $[\lambda]$

A Young tableaux of $S(k)$ is a Young graph of $S(k)$ filled with the integers $1,2,\ldots,k$ each occurring only once.

Permutations within a row (column) of a Young tableaux are called row (column) permutations. Let $R[\lambda]$ denote the set of all row permutations and let $C[\lambda]$ denote the set of all column permutations. The operator

$$P \equiv \sum_{\rho \varepsilon R[\lambda]} \rho$$

is called the row operator of the tableau while

$$Q \equiv \sum_{\gamma \varepsilon C[\lambda]} \delta_\gamma \gamma$$

is called the column operator of the tableau. Here $\delta_\gamma = 1$ for even permutations, and $\delta_\gamma = -1$ for odd permutations. The Young operator of the tableau is defined by

$$Y = PQ$$

The definition $Y' = QP$ also appears in the literature. Y' is the adjoint representation of Y and both definitions lead to the same results concerning irreducible representations, dimensionality and graphical rules.

Each Young operator [to within a multiplicative constant] is a primitive idempotent in the group algebra of S(k) and therefore generates an irreducible representation. Different Young tableaux belonging to the same graph generate equivalent irreducible representations, while different Young graphs generate inequivalent irreducible representations. The graphs generate a complete set of inequivalent irreducible representations.

A standard [or lexical] Young tableau is one in which the numbers increase from let to right in a row and top or bottom in a column. The number of standard Young tableaux for a given graph [λ] is the dimension of the associated irreducible representation $\Gamma[\lambda]$. [The dimension can also be determined from Robinson's formula which is particularly useful for higher order symmetric groups.]

Given a representation of GL(N,¢) on a vector space V, we can induce a tensor product representation on $V^{\otimes k}$ of GL(N,¢). This tensor product representation is, in general, reducible. The permutational symmetries of tensor indices induce a decomposition of a representation into irreducible representations of S(k). Simultaneously, the algebra of bisymmetric transformations induces a representation which is decomposable into irreducible representations of GL(N,¢). This naturally allows for the definition of a representation of GL(N,¢) $\otimes$ S(k) on $V^{\otimes k}$.

The basis states of an irreducible representation labeling. both these decompositions are contained in Weyl tableaux. Weyl tableaux are Young tableaux in which repetitions are allowed in a row but not in a column. Lexical Weyl tableaux are those in which numbers do not decrease in a row and strictly increase in a column. The Weyl tableaux thus carry information on both permutational symmetry and unitary symmetry while Young tableaux carry only permutational symmetry. To emphasize this fundamental distinction [which is often ignored in the literature] we will use the notation $[\eta]$ for a Weyl graph [which is the same as a Young graph] and $|[\eta]>$ for a Weyl basis state [which is distinct from a Young basis state]. The dimension of the irreducible representation of GL(N,¢) generated by a Weyl graph is the number of Lexical Weyl tableaux associated with that graph. The dimension can also be calculated from the formula

$$\dim \Gamma^{[\eta]} = \prod_{\substack{i,j=1 \\ i\neq j}}^{N} \frac{(N+j-i)}{H_{ij}}$$

where H_{ij} is the hook length of the ij'th box, i is the row index and j is the column index.

The weight of a Weyl tableau $[\eta]$ is $(\omega_1,\omega_2,\ldots,\omega_N)$ where ω_i is the number of times the label i appears in

[n]. There is a unique one-one correspondence between Gelfand tableaux and Weyl tableaux. Let (m) be a Gelfand tableau. Then the corresponding Weyl tableau has the graph $\{m_{1N}, m_{2N}, \ldots, m_{NN}\}$ and has the following entries:

Row k: m_{kk} k's, $(m_{kk+1} - m_{kk})$k+1's, ..., $(m_{kN} - m_{kN-1})$N's;

$$k = 1,2,3,\ldots, N.$$

It is important to note that the Gelfand and Weyl bases are not the same in general. [For the case of U(2), however, they are the same and all results which apply to Gelfand bases apply to Weyl bases. This fact is useful in atomic physics.] There are, however, two important conclusions which can be inferred. First, the unique one-to-one correspondence establishes the fact that the Weyl and Gelfand bases generate vector spaces of the same dimension. Second, since the highest weights of Gelfand basis vectors and Weyl basis vectors coincide we can conclude that they generate equivalent irreducible representations. In terms of the vector space decomposition of $V^{\otimes k}$ this means that the Weyl and Gelfand basis constructed from a given tableau will generate the same invariant subspace of $V^{\otimes k}$, so that we can characterize the properties of such invariant subspaces using the Weyl basis. The Weyl basis offers the advantage of physical insight, and the disadvantage of not being orthogonal [except for the simplest cases of U(1) and U(2)]. Weyl basis vectors are linear combinations of single particle states (with appropriate permutational symmetry) which carry the state labels.

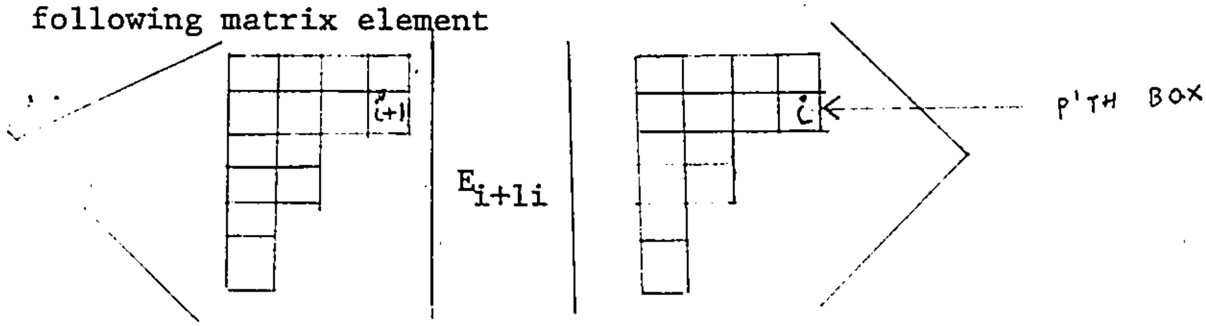
The specific form of a Weyl basis state depends upon the projection operators associated with a graph. For example, for the completely symmetric partition {k} of S(k),

$$| \; i_1 \;\; i_2 \;\; \ldots \;\; i_k> = \frac{1}{k!} \sum_P P|i_1;\; i_2;\; i_3; \ldots\ldots i_k>$$

where $|i_1;\; i_2; \ldots ; i_k>$ specifies the single-particle states of all k-particles. For the completely antisymmetric partition we obtain the Slater determinantal state. All other partitions have basis functions of mixed symmetry [2,8].

Evaluation of matrix elements of $\{E_{ij} | i,j = 1,2 \ldots N\}$ necessitates working with an orthonormal basis. This means that we either orthonormalize the Weyl basis or work with the Gelfand basis directly. The former approach yields a basis which is unitarily equivalent to the Gelfand basis so that the matrix elements or $\{E_{ij} | i,j = 1,2 \ldots N\}$ are preserved. If we use the latter approach, it is convenient to label the Gelfand basis using Weyl tableaux [such a labeling scheme is justified due to the unique one to one correspondence between Weyl and Gelfand tableaux]. Note that these difficulties do not occur in atomic physics since the Gelfand and Weyl basis are identical for U(2).

The advantage of labeling the Gelfand basis using Weyl tableaux is that the matrix elements of the operators E_{i+1i} are given by a simple algorithm called the Harter Jawbone Formula [9,10]. This algorithm is equivalent to the algebraic formulae we have stated before. Consider the following matrix element



It is assumed that all other labels except for the P'th box are the same in both tableaux. The Harter Jawbone formula consists of the following four steps:

STEP 1: Draw all arrows from the last box labeled i+1 in each row to the left of the P'th box. To account for all contributions, add a string of boxes of length N [for Weyl tableaux of U(N)] as the first column of the tableau. The product of all such arrow lengths yields the resultant factor of each step. If no arrow can be drawn, the factor is unity.

STEP 2: Draw all arrows from the left of the first box labeled i+1 in each row to the left of the P'th box.

STEP 3: Draw all arrows from the last box called i in each row to the P'th box itself.

STEP 4: Draw all arrows from the left of the first box labeled i in each row to the P'th box.

The matrix elements is then $\sqrt{\frac{F_1F_4}{F_2F_3}}$ where F_i denotes the result of step i for i = 1,2,3,4. The evaluation of the matrix elements of the operators E_{i+2i} is performed using the commutation relations

$$E_{i+2i} = [E_{i+2i+1}, E_{i+1i}]$$

Iteration clearly yields matrix elements of E_{i+qi} where $q > 2$. [These operators are called multistep operators.] Products of one body operators such as $E_{ij}E_{k\ell}$ are called two body operators and are evaluated using the following expression:

$$E_{ij}E_{k\ell} = \sum_{|T>} E_{ij}|T><T|E_{k\ell}.$$

Here the summation extends over all lexical tableaux. The physical interpretation of these operators and the physical applications of Weyl tableaux will be discussed in the next chapter.

CHAPTER II

APPLICATIONS OF THE UNITARY GROUP APPROACH TO NUCLEAR MAGNETIC RESONANCE SPECTRA

The application of the Unitary Group Approach to nuclear magnetic resonance (NMR) spectra was first discussed by Kent and Schlesinger [11]. We will follow the method in this reference, and analyze the spectra of an A_3 system with $\sigma_A = 1$ and an A_4B_2 system with $\sigma_A = 2$ and $\sigma_B = 1$. We will compare our results with calculations based upon the method of Siddall and Flurry [12, 13, 14].

Siddall and Flurry analyze NMR spectra for various nuclear spin configurations with spin greater than 1/2. They analyze A_3, A_2B, ABC, A_2X systems with individual nuclear spin 1, and A_2B_2 systems with arbitrary individual nuclear spin. They calculate matrix elements of a (nuclear) spin-spin coupling Hamiltonian using various bases, and discuss the advantages and disadvantages of these bases from a computational viewpoint. For further details, see the reference cited above.

The notation $A_{nA}B_{nB}$ appears frequently in the literature of NMR and deserves an explanation. Any set of nuclei for which the total spin squared (I^2) is a constant of the motion is said to be magnetically equivalent. If a system can be separated into two separate sets of magnetically equivalent nuclei with n_A in one set and n_B in the other, then the notation $A_{nA}B_{nB}$ is used to denote the entire system.

In the Unitary Group Approach, given a pure σ^N configuration we begin by relabeling the states in the following manner:

$$|\sigma m_\omega\rangle \rightarrow |\sigma\ \sigma+1 - m_\omega\rangle_{(\omega)};\quad \ell \leq m_\omega \leq -\ell$$
$$\omega = 1,2,\ldots, N.$$

Here σ is the individual nuclear spin, m_ω is the (spin) magnetic quantum number of the ω'th particle, and the subscript (ω) appears on the right hand side to distinguish among particles. Thus we have,

$$H_\omega(\sigma) \equiv \text{Span}\{|\ell m_\omega\rangle \mid \sigma \leq m_\omega \leq -\sigma\}$$

$$= \text{Span}\{|\ell i\rangle_{(\omega)} \mid 1 \leq i \leq 2\sigma+1\}$$

Here $H_\omega(\sigma)$ is the state space of the ω'th particle. This method of labeling proves to be a convenient choice.

Now we define one particle operators $e_{ij}^{(\omega)} : H_\omega(\sigma) \rightarrow H_\omega(\sigma)$ by

$$e_{ij}^{(\omega')}|\ell r\rangle_{(\omega)} = \delta_{\omega'\omega}\delta_{jr}|\ell i\rangle_{(\omega)};\quad 1 \leq i,i,r \leq 2\sigma+1$$
$$1 \leq \omega,\ \omega' \leq N$$

The physical interpretation of $e_{ij}^{(\omega)}$ is that it creates the state $|\ell i\rangle_{(\omega)}$ from the state $|\ell j\rangle_{(\omega)}$ when acting to the right. Observe that

$$[e_{ij}^{(\omega)}, e_{kp}^{(\omega)}] = \delta_{jk}e_{ip}^{(\omega)} - \delta_{ip}e_{kj}^{(\omega)}; \quad 1 \le i,j,k,p \le 2\sigma+1.$$

We conclude that $\{e_{ij}^{(\omega)} \ 1 \le i,j \le 2\sigma+1\}$ forms a representation of $L(U(2\sigma+1))$ on $H_\omega(\sigma)$. The state space of the overall σ^N system is $\bigotimes_{\omega=1}^{N} H_\omega(\sigma)$, and we can define many-particle operators $E_{ij}: \bigotimes_{\omega=1}^{N} H_\omega(\sigma) \to \bigotimes_{\omega=1}^{N} H_\omega(\sigma)$ as follows

$$E_{ij} = \sum_{\omega=1}^{N} e_{ij}^{(\omega)} \quad ; \quad 1 \le i,j \le 2\sigma+1 \cdot$$

We will defer the physical interpretation of E_{ij} for now. Observe that

$$[E_{ij}, E_{kp}] = \delta_{jk}E_{ip} - \delta_{ip}E_{kj} \quad ; \quad 1 \le i,j,k,p \le 2\sigma+1$$

Thus $\{E_{ij} \mid 1 \le i,j \le 2\sigma+1\}$ forms a representation of $L(U(2\sigma+1))$ on $\bigotimes_{\omega=1}^{N} H_\omega(\sigma)$. By restricting the range of indices i,j,k,p, we can also conclude that $\{E_{ij} \mid 1 \le i,j \le S\}$ forms a representation of $L(U(S))$ on $\bigotimes_{\omega=1}^{N} H_\omega(\sigma)$ for S = 1,2...,2σ. This means that we can construct a Gelfand basis for invariant subspaces of $\bigotimes_{\omega=1}^{N} H_\omega(\sigma)$ which we will label by lexical Weyl tableaux. Our labeling of states is based upon the decomposition

$$U(2\sigma+1) \supset U(2\sigma) \supset \ldots \supset U(1),$$

The top row of the Gelfand tableaux we use to label states has certain physical restrictions on it. Observe that

$$E_{ii} \bigotimes_{\omega=1}^{N} |\sigma r_\omega\rangle_{(\omega)} = (\sum_{j=1}^{N} \delta_i r_j) \bigotimes_{\omega=1}^{N} |\sigma r_\omega\rangle_{(\omega)}; \; 1 \le j \le 2\sigma+1.$$

Thus the weights corresponding to our tensor product states are given by $\omega = (\omega_1, \omega_2, \ldots, \omega_{2\sigma+1})$ where

$$\omega_i = \sum_{j=1}^{N} \delta_{ir_j}.$$

Actually, to be more precise we should consider E_{ii} acting on a linear combination of single particle states which are appropriately symmetrized, but the end result is the same. We now define $(\omega_1, \omega_2, \ldots \omega_{2\sigma+1}) \equiv (m_{1\;2\sigma+1}, m_{2\;2\sigma+1}, \ldots, m_{2\sigma+1\;2\sigma+1})$. This restricts our considerations to physically meaningful irreducible representations. Observe that $\sum_{k=1}^{2\sigma+1} m_{k2\sigma+1} = N$. Using the one to one correspondence between Gelfand and Weyl tableaux, this means that the only Weyl tableaux we consider are those with N boxes on them. A given Weyl tableau specifies the total spin magnetic quantum number of the associated state but not the total nuclear spin. The total magnetic quantum number M is obtained by simply adding the individual magnetic quantum numbers which appear in the tableau. For the systems we shall consider this is the only restriction possible. [In atomic physics, a great simplification is possible at this stage since no weight component could be greater than 2. This is due to the Pauli exclusion principle. Furthermore, the number of unpaired boxes in Weyl tableaux immediately yields the total (electronic) spin. Also, the spin basis states arise from the adjoint representation of the orbital basis states in atomic physics. None of these simplifications are possible for the systems we are considering.]

Let us clarify the difference between using the tensor product basis directly, and using the Gelfand basis. Let Γ be a representation of $U(2\sigma+1)$ on $X_{\omega=1}^{N} H_\omega(\sigma)$. Then

$$\Gamma \cong \oplus_{\mu=1}^{\alpha} n_\mu \, \Gamma^{(\mu)}$$

where $\{\Gamma^{(\mu)} | 1 \leq \mu \leq \alpha\}$ is a set of irreducible representations of $U(2\sigma+1)$. This induces a decomposition

$$X_{\omega=1}^{N} H_\omega(\sigma) \cong \sum_{k=1}^{n_\mu} + {}_{\mu=1}^{\alpha} V_k^{(\mu)}$$

where $V_k^{(\mu)}$ are invariant subspaces [with respect to $\Gamma^{(\mu)}$, and certain operations from S(N) which are dictated by the Weyl graph we associate with $V_k^{(\mu)}$]. When we use the Gelfand basis we are constructing a basis for each $V_k^{(\mu)}$. The tensor product basis, on the other hand, is a basis directly for $\bigotimes_{\omega=1}^{N} H_\omega(\sigma)$. These considerations also give us a clear physical interpretation for the operators $\{E_{ij} | i,j = 1,2,\ldots 2\sigma+1\}$. $\{V_k^{(\mu)} | 1 \leq \mu < \alpha,\ 1 \leq k \leq n_\mu\}$ are invariant under $\{E_{ij} |\ i,j = 1,2\ldots 2\sigma+1\}$. E_{ij} acting on a basis vector of $V_k^{(\mu)}$ changes the state label from i to j, possibly in steps of one, possibly for more than one particle, but does not alter the permutational symmetry of the particles.

Now let us define a nuclear spin unit irreducible tensor $i_q^k(\omega)$ of rank 2σ, which has the following representation in the basis $\{|\sigma m_\omega>|\ -\sigma \leq m_\omega \leq \sigma\}$:

$$i_q^k(\omega) = \sum_{m_\omega, m'_\omega = -\sigma}^{\sigma} |\sigma m_\omega><\sigma m_\omega| i_q^k(\omega) |\sigma m'_\omega><\sigma m'_\omega|$$

$$= \sum_{m_\omega, m'_\omega = -\sigma}^{\sigma} (-)^{\sigma - m_\omega} \begin{pmatrix} \sigma & k & \sigma \\ -m_\omega & q & m'_\omega \end{pmatrix} e_{ij}^{(\omega)}$$

where we have used the standard decomposition of the identity, the Wigner-Eckart theorem, and the definition of a unit tensor. We have also used the fact that $e_{ij}^{(\omega)} = |\sigma m_\omega\rangle\langle\sigma m_{\omega'}|$ is a realization of the operators we have defined before on $H_\omega(\sigma)$.

Now we define the many-particle nuclear spin tensor operator $I_q^k : \bigotimes_{\omega=1}^{N} H_\omega(\sigma) \to \bigotimes_{\omega=1}^{N} H_\omega(\sigma)$ as follows:

$$I_q^k = \sum_{\omega=1}^{N} i_q^k(\omega)$$

$$= \sum_{m,m'=-\sigma}^{\sigma} \begin{pmatrix} \sigma & k & \sigma \\ -m & q & m' \end{pmatrix} \sum_{\omega=1}^{N} e_{ij}^{(\omega)}$$

$$= \sum_{m,m'=-\sigma}^{\sigma} \begin{pmatrix} \sigma & k & \sigma \\ -m & q & m' \end{pmatrix} E_{ij}$$

where $j = \sigma + 1 - m'$, and $i = \sigma + 1 - m$.

We define the NMR (nuclear) spin-spin coupling operator by

$$H_{NMR} = - \sum_{k=1}^{2\sigma} T^k \; I^k \cdot I^k$$

where the T^k are tensor operator coupling constants which give the rank-dependent strength of the coupling between nuclear spins, and $I^k \cdot I^k$ is the tensor inner product. Note that

$$I^k \cdot I^k \equiv \sum_{q=-k}^{k} (I_q^k)^+ \, I_q^k$$

$$= \sum_{q=-k}^{k} (-)^q \, I_{-q}^k \, I_q^k \ [\text{since } I_q^{k+} = (-)^q \, I_{-q}^k]$$

$$= I_o^k \, I_o^k + \sum_{q=1}^{k} (-)^q \, [I_{-q}^k \, I_q^k + I_q^k \, I_{-q}^k].$$

Now observe that

$$I_o^k = \sum_{m,m'=-\sigma}^{\sigma} (-)^{\sigma-m'} \begin{pmatrix} \sigma & k & \sigma \\ m & o & m' \end{pmatrix} E_{ij}$$

$$= \sum_{m=-\sigma}^{\sigma} (-)^{\sigma-m} \begin{pmatrix} \sigma & k & \sigma \\ -m & o & m \end{pmatrix} E_{ii} \ .$$

Furthermore, we have

$$I_q^k = \sum_{m,m'=-\sigma}^{\sigma} (-)^{\sigma-m'} \begin{pmatrix} \sigma & k & \sigma \\ -m & q & m' \end{pmatrix} E_{ij}$$

$$= \sum_{m'=-\sigma}^{\sigma} (-)^{-m'} \begin{pmatrix} \sigma & k & \sigma \\ -(m'+q) & q & m' \end{pmatrix} E_{j-qj}$$

$$I_{-q}^k = \sum_{m'=-\sigma}^{\sigma} (-)^{\sigma-m'} \begin{pmatrix} \sigma & k & \sigma \\ -(m'-q) & -q & m' \end{pmatrix} E_{j+qj} \ .$$

We thus obtain the equation

$$H_{NMR} = -\sum_{k=1}^{2\sigma} T^k \{\sum_{m,m'=-\sigma}^{\sigma} (-)^{2\sigma-m-m'} \begin{pmatrix} \sigma & k & \sigma \\ -m & o & m \end{pmatrix} \begin{pmatrix} \sigma & k & \sigma \\ -m' & o & m' \end{pmatrix} E_{ii}E_{jj}$$

$$+ (\sum_{q=1}^{k} \sum_{m,m'=-\sigma}^{\sigma} (-)^q (-)^{2\sigma-m-m'} \begin{pmatrix} \sigma & k & \sigma \\ -(m'-q) & -q & m' \end{pmatrix} \begin{pmatrix} \sigma & k & \sigma \\ -(m+q) & q & m \end{pmatrix}$$

$$[E_{p+qp} E_{j-qj} + E_{j-qj} E_{p+qp}])\}$$

We can now evaluate matrix elements of the Hamiltonian using Harter's Jawbone formula. There are two methods of dealing with mixed configurations in the Unitary Group Approach. Suppose we are given a configuration $\sigma_A^{N_A} \sigma_B^{N_B}$. As for the pure configuration we can introduce the labeling

$$|\sigma_A m_\omega > \rightarrow |\sigma_A \sigma_A + 1 - m_\omega >_{(\omega)}; \quad -\sigma_A \leq m_\omega \leq \sigma_A$$

$$\omega = 1, 2, \ldots N_A$$

$$|\sigma_B m_\omega > \rightarrow |\sigma_B \sigma_B + 1 - m_\omega >_{(\omega)}; \quad -\sigma_B \leq m_\omega \leq \sigma_B$$

$$\omega = 1, 2, \ldots N_B.$$

The basis we construct is then a tensor product of the Gelfand basis for system A and system B. The Hamiltonian is defined as $H_{NMR} = H_{NMR}(A) + H_{NMR}(B)$ and matrix elements can again be calculated using Harter's Jawbone formula.

The alternative approach is to use a different scheme of labeling and follow the same procedure as in pure configurations. We use the labeling

$$|\sigma_A m_\omega> \quad \omega \rightarrow |\sigma_A \sigma_A + 1 - m_\omega>_{(\omega)} \quad -\sigma_A \leq m_\omega \leq \sigma_A$$

$$\omega = 1, 2, \ldots\ldots N_1 .$$

$$|\sigma_B m_\omega>_{(\omega)} \rightarrow |\sigma_B \sigma_B + 1 - m_\omega + 2\sigma_A + 1> \quad -\sigma_2 \leq m_\omega \leq \sigma_B$$

$$\omega = 1, 2, \ldots\ldots N_2 .$$

The calculations now proceed in the same manner as for pure configurations. We note that in this approach we need to define $[(2\sigma_A+1)(2\sigma_B+1)]^2$ operators whereas in the first approach we define only $(2\sigma_A+1)^2 + (2\sigma_B+1)^2$ operators. The physical meaning of the "extra" operators in this approach is not clear. Another problem with this approach is that it is not feasible to make a correlation between tableaux basis states and conventionally labeled states [this problem does not exist in atomic physics mixed configuration treatments].

The identification between tableaux basis states and conventionally labeled $|I_A M_A>$ states is based upon lowering and raising operator techniques [10,15]. First we observe that [for pure configurations],

$$I_1^1 = \sum_{m=-\sigma}^{\sigma} (-)^{\sigma-m} \begin{pmatrix} \sigma & 1 & \sigma \\ -(m+1) & 1 & m \end{pmatrix} E_{j-1j}$$

$$= \sum_{m=-\sigma}^{\sigma} (-)^{\sigma-m} (-)^{\sigma-m} \left[\frac{(\sigma-m) + (\sigma+m+1)}{\sigma(2\sigma+1)(2\sigma+2)}\right]^{1/2} E_{j-1}$$

$$= \sum_{m=-\sigma}^{\sigma} \frac{1}{\sqrt{2}} \left[\frac{(\sigma-m)(\sigma+m+1)}{\sigma(2\sigma+1)(\sigma+1)}\right]^{1/2} E_{j-1j}$$

Similarly, we obtain

$$I_{-1}^1 = -\sum_{m=-\sigma}^{\sigma} \frac{1}{\sqrt{2}} \left[\frac{(\sigma+m)(\sigma-m+1)}{\sigma(2\sigma+1)(\sigma+1)}\right]^{1/2} E_{j+1j} .$$

Therefore, we have

$$I_- = -I_{-1}^1 [\sigma(2\sigma+1)(\sigma+1)]^{1/2}$$

$$I_+ = I_1^1 [\sigma(2\sigma+1)(\sigma+1)]^{1/2}$$

where I_- and I_+ are the usual raising and lowering operators. Since the total magnetic quantum number is specified in tableaux states, and the highest M value can immediately be correlated with the total maximum spin I_{max}, we can generate all states $|I_{max} M\rangle$ with $-M \leq I_{max} \leq M$. The rest of the states are determined from the fact that the $Q_M - Q_{M+1}$ [where Q_M is the number of tableaux at the level M] yields the number of linearly independent vectors of I^2. Any possible remaining states at a given M are generated by either a projection operator technique or a raising operator technique [10,15]. In the former, we begin by calculating

$$Q_M = \sum_{I=M+1}^{I_{max}} |IM\rangle \langle IM| .$$

If there is no new state at the level M then $Q_M = 1$. If there is a new state at the level M, then any column of $P_M = 1-Q_M$ is an eigenvector of $|I=M,M\rangle$. The procedure is iterated until $Q_M = 1$.

The raising operator technique is similar in nature, but offers the advantage of determining the states at each M level directly. The reader is referred to [15] for further details.

For mixed configurations, the identification of states with conventionally labeled $|IM\rangle$ states depends upon which labeling scheme is used. If the A and B systems are treated separately $[|\sigma_A m\rangle \rightarrow |\sigma_A\sigma_A + 1 - m_\omega\rangle_{(\omega)}$, $|\sigma_B m_\omega\rangle \rightarrow |\sigma_B\sigma_B + 1 - m_\omega\rangle_{(\omega)}]$, then we can identify states within the A and B systems by the lowering or raising operator techniques. These are then coupled together using standard vector coupling techniques to yield the overall states. Explicitly,

$$|\{\eta_A\}\ I_A,\ \{\eta_B\}\ I_B,\ IM\rangle = \sum_{\substack{m_A, m_B \\ m_A+m_B = M}} \langle I_A I_B M_A M_B | IM\rangle\ |\{\eta_A\} I_A M_A\rangle \otimes |\{\eta_B\} I_B M_B\rangle$$

We have proposed a procedure for correlating tensor product states directly with SO(3) states [17]. This method will be examined in depth in conjunction with our treatment of the A_4B_2-system.

CHAPTER III

ANALYSIS OF AN A_3 SPIN-1 SYSTEM AND AN A_4B_2 SYSTEM WITH $\sigma_A = 2$ AND $\sigma_B = 1$

(i) A_3 SYSTEM

We begin with a system of three magnetically equivalent nuclei. The chain we use for labeling is U(3)$\supset$U(2)$\supset$U(1). The Gelfand basis states [labeled by Weyl tableaux] are listed in Table I, including correlation with SO(3) terms.

The operators in the Hamiltonian for a spin 1 system are as follows

$$I^1_o = \sum^1_{m=-1}(-)^{1-m}\begin{pmatrix}1 & 1 & 1\\ -m & o & m\end{pmatrix}E_{ii}$$

$$= -\frac{1}{\sqrt{6}}E_{33} + \frac{1}{\sqrt{6}}E_{11}$$

$$I^1_1 = \sum^1_{m=-1}(-)^{1-m}\begin{pmatrix}1 & 1 & 1\\ -(m+1) & 1 & m\end{pmatrix}E_{j-1j}$$

$$= \frac{1}{\sqrt{6}}(E_{23} + E_{12})$$

$$I^1_{-1} = -\frac{1}{\sqrt{6}}(E_{32} + E_{21}).$$

Thus we obtain, upon collecting terms and using the commutation relations $[E_{ij}, E_{k\ell}] = \delta_{jk}E_{i\ell} - \delta_{i\ell}E_{kj}$

$$I^1 \cdot I^1 \equiv (I^1_o)^2 - [I^1_1 I^1_{-1} + I^1_{-1} I^1_1]$$

$$= \frac{1}{6} [(E_{11} - E_{33})^2 + (E_{11} - E_{33})$$

$$+ 2(E_{32}E_{23} + E_{21}E_{12} + E_{12}E_{32} + E_{23}E_{21})].$$

Similarly, we have

$$I^2_o = \sum^1_{m=-1} (-)^{1-m} \begin{pmatrix} 1 & 2 & 1 \\ -m & o & m \end{pmatrix} E_{ii}$$

$$= \frac{1}{\sqrt{30}} E_{11} - \frac{2}{\sqrt{30}} E_{22} + \frac{1}{\sqrt{30}} E_{33}$$

$$I^2_1 = \sum^1_{m=-1} (-)^{1-m} \begin{pmatrix} 1 & 2 & 1 \\ -(m+1) & 1 & m \end{pmatrix} E_{j-1j}$$

$$= - \frac{1}{\sqrt{10}} E_{23} + \frac{1}{\sqrt{10}} E_{12}$$

$$I^2_{-1} = \sum^1_{m=-1} (-)^{1-m} \begin{pmatrix} 1 & 2 & 1 \\ -(m+1) & -1 & m \end{pmatrix} E_{j+1j}$$

$$= - \frac{1}{\sqrt{10}} E_{21} + \frac{1}{\sqrt{10}} E_{32}$$

$$I^2_2 = \sum^1_{m=-1} (-)^{1-m} \begin{pmatrix} 1 & 2 & 1 \\ -(m+2) & 2 & m \end{pmatrix} E_{j-2j}$$

$$= \frac{1}{\sqrt{5}} E_{13}$$

$$I^2_{-2} = \sum_{m=-1}^{1} (-)^{1-m} \begin{pmatrix} 1 & 2 & 1 \\ -(m-2) & -2 & m \end{pmatrix} E_{j+2j}$$

$$= \frac{1}{\sqrt{5}} E_{31}$$

Therefore, again simplifying, we obtain,

$$I^2 \cdot I^2 = \frac{1}{30} [(E_{11} - 2E_{22} + E_{33})^2]$$

$$+ \frac{1}{10} [E_{11} - E_{33} + 2(E_{23}E_{32} + E_{12}E_{21} - E_{32}E_{12} - E_{21}E_{23}) +$$

$$+ 2(E_{31}E_{13})].$$

The matrix elements of the one body operators E_{32}, E_{21} and E_{31} are listed in Table II. Note that the diagonal operators E_{ii} simply yield the number of i's in a tableau and do not alter the numbers in the tableau. Also the matrix elements of E_{23}, E_{12}, and E_{13} can be obtained from those of E_{32}, E_{21} and E_{31} by the use of the relation $E_{ij}{}^{+} = E_{ji}$. The matrix elements of the operators E_{32}, E_{21} and E_{31} are zero for the state $\left| \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \right\rangle$.

(ii) COMPARISON WITH RESULTS BASED UPON SIDDALL'S METHOD

In order to compare the results we have obtained with those based upon Siddall's method, we first note that there is a difference between the definition of $I^k \cdot I^k$ in the Unitary Group Approach and Siddall's approach. In the former we have,

$$I_q^k = \sum_{\omega=1}^{N} I_q^k(\omega) .$$

Consequently,

$$I^k \cdot I^k \equiv \sum_{q=-k}^{k} (-)^q \, I_q^k I_{-q}^k$$

$$= \sum_{\omega<\omega'=1}^{N} \sum_{q=-k}^{k} (-)^q \, I_q^k(\omega) \; I_{-q}^k(\omega') +$$

$$+ \sum_{\omega>\omega'=1}^{N} \sum_{q=-k}^{k} (-)^q \, I_q^k(\omega) \; I_{-q}^k(\omega') +$$

$$+ \sum_{\omega=1}^{N} \sum_{q=-k}^{k} (-)^q \, I_q^k(\omega) \; I_{-q}^k(\omega) .$$

On the other hand, in Siddall's approach,

$$[I^k \cdot I^k]_{Siddall} = \sum_{\omega<\omega'=1}^{N} \sum_{q=-k}^{k} (-)^q \, I_q^k(\omega) \; I_{-q}^k(\omega') .$$

We thus have the relation

$$I^k \cdot I^k = [I^k \cdot I^k]_{Siddall} + [I^k \cdot I^k]_{Siddall}^{+} + \sum_{\omega=1}^{N} \sum_{q=-k}^{k} (-)^q \, I_q^k(\omega) \; I_{-q}^k(\omega)$$

The last term represents the self-interaction of the species. Fortunately, it can be evaluated explicitly. Note that

$$I^k_q(\omega) = \sum^{\sigma}_{m_\omega m'_\omega=-\sigma} (-)^{\sigma-m'_\omega} \begin{pmatrix} \sigma & k & \sigma \\ -m_\omega & q & m'_\omega \end{pmatrix} e^{(\omega)}_{ij}$$

where $j = \sigma + 1 - m'_\omega$ and $i = \sigma + 1 - m_\omega$. Furthermore,

$$I^k_q(\omega)^+ = (-)^q I^k_{-q}(\omega) = \sum^{\sigma}_{m''_\omega, m'''_\omega=-\sigma} (-)^{\sigma-m'''_\omega} \begin{pmatrix} \sigma & k & \sigma \\ -m''_\omega & q & m'''_\omega \end{pmatrix} e^{(\omega)}_{sp}$$

where $p = \sigma+1-m''_\omega$, $s = \sigma+1-m'''_\omega$. Thus

$$I^k_q(\omega) I^{k^+}_q(\omega) \; |\sigma r\rangle = \sum^{\sigma}_{m''_\omega, m'''_\omega=-\sigma} (-)^{\sigma-m'''_\omega} \begin{pmatrix} \sigma & k & \sigma \\ -m''_\omega & q & m'''_\omega \end{pmatrix} \delta_{pr} I^k_q(\omega) |\sigma s\rangle$$

$$= \sum^{\sigma}_{m_\omega, m'_\omega, m''_\omega, m'''_\omega = -\sigma} (-)^{2\sigma-m'''_\omega-m'_\omega} \begin{pmatrix} \sigma & k & \sigma \\ -m''_\omega & q & m'''_\omega \end{pmatrix}$$

$$\begin{pmatrix} \sigma & k & \sigma \\ -m_\omega & q & m'_\omega \end{pmatrix} \delta_{pr} \delta_{sj} |\sigma i\rangle$$

where $1 \leq r \leq 2\sigma+1$. Therefore we have

$$\langle\sigma t| \; I^k_q(\omega) I^{k^+}_q(\omega) |\sigma r\rangle = \sum^{\sigma}_{m_\omega, m'_\omega, m''_\omega, m'''_\omega = -\sigma} \delta_{it} \delta_{pr} \delta_{sj} (-)^{2\sigma-m'''_\omega-m'_\omega}$$

$$\begin{pmatrix} \sigma & k & \sigma \\ -m''_\omega & q & m'''_\omega \end{pmatrix} \begin{pmatrix} \sigma & k & \sigma \\ -m_\omega & q & m'_\omega \end{pmatrix}$$

Note that $j = s \Rightarrow m'_\omega = m'''_\omega$. Thus

$$<\sigma t|I^k_q(\omega)I^{k^+}_q(\omega)|\sigma r>=\sum_{m_\omega,m'_\omega,m''_\omega=-\sigma}^{\sigma} \delta_{it}\delta_{pr}(-)^{2(\sigma-m'_\omega)}$$

$$\begin{pmatrix} \sigma & k & \sigma \\ -m''_\omega & q & m'_\omega \end{pmatrix} \begin{pmatrix} \sigma & k & \sigma \\ -m_\omega & q & m'_\omega \end{pmatrix}$$

$$= \sum_{q=-k}^{k} \sum_{m_\omega,m'_\omega,m''_\omega=-\sigma}^{\sigma} \delta_{it}\delta_{pr} \begin{pmatrix} \sigma & k & \sigma \\ -m''_\omega & q & m'_\omega \end{pmatrix} \begin{pmatrix} \sigma & k & \sigma \\ -m_\omega & q & m'_\omega \end{pmatrix}$$

$$= \sum_{m_\omega,m'_\omega,m''_\omega=-\sigma}^{\sigma} \delta_{it}\delta_{pr} \left[\sum_{q=-k}^{k} \sum_{m'_\omega=-\sigma}^{\sigma} \right.$$

$$\left. \begin{pmatrix} \sigma & k & \sigma \\ -m''_\omega & q & m'_\omega \end{pmatrix} \begin{pmatrix} \sigma & k & \sigma \\ -m_\omega & q & m'_\omega \end{pmatrix} \right] = \sum_{m''_\omega,m'_\omega=-\sigma}^{\sigma} \delta_{it}\delta_{pr} \frac{\delta_{m_\omega,m''_\omega}}{2\sigma+1}$$

[Using an orthogonality property of 3-j symbols.]

Finally, observe that $m_\omega = m''_\omega \Rightarrow i=p$. Hence our result is,

$$<\sigma t| \sum_{q=-k}^{k} I^k_q(\omega)\ I^{k^+}_q(\omega)|\sigma r> = \frac{\delta_{tr}}{2\sigma+1}.$$

Extending this result to the tensor product basis we obtain

$$\bigotimes_{\alpha=1}^{N} \langle \sigma i_\alpha | \sum_{\omega=1}^{N} \sum_{q=-k}^{k} I_q^k(\omega) I_q^{k^+}(\omega) | \bigotimes_{\alpha=1}^{N} | \sigma i'_\alpha \rangle = \frac{N}{2\sigma+1} \Pi_{\alpha=1}^{N} \delta_{i_\alpha, i'_\alpha}$$

This self-energy term has been derived by Harter and Patterson for atomic configurations [16]. The derivation we have presented appears to be simpler and is completely general. The relationship between $I^k \cdot I^k$ and $[I^k \cdot I^k]_{Siddall}$ now simplifies to

$$I^k \cdot I^k = [I^k \cdot I^k]_{Siddall} + [I^k \cdot I^k]^+_{Siddall} + \frac{N}{2\sigma+1} 1$$

where 1 is the identity operator.

In [11], we note that the A_2-spin 1 system matrix elements were calculated using Siddall's operator, while the A_2B_2 system matrix elements were not. To obtain matrix elements of $[I^k \cdot I^k]_{Siddall}$ we divide by 2 for off-diagonal elements and subtract $\frac{N}{2\sigma+1}$ and divide by 2 for diagonal elements. The matrix representation of $[I^1 \cdot I^1]_{Siddall}$ in the Gelfand basis for the A_3 spin-1 system and listed in Tables IIIA, IIIB, IVA and IVB.

Siddall [14] has classified his basis states according to permutational symmetry, total spin (I), and total magnetic quantum number (M). The transformation from $|[\lambda]IM\rangle$ to Gelfand basis states is performed through the lowering operator technique. We obtain the following results [note that $I_- = -\sqrt{6}\, I^1_{-1}$]:

$$|[3]\ 33\rangle = |111\rangle$$

$$|[3]\ 32\rangle = |112\rangle$$

$$|[3]\ 31\rangle = \frac{1}{\sqrt{5}}\,|113\rangle + \frac{2}{\sqrt{5}}\,|122\rangle$$

$$|[3]\ 30\rangle = \sqrt{\frac{3}{5}}\,|123\rangle + \sqrt{\frac{2}{5}}\,|222\rangle$$

$$|[3]\ 3\text{-}1\rangle = \frac{1}{\sqrt{5}}\,|133\rangle + \frac{2}{\sqrt{5}}\,|223\rangle$$

$$|[3]\ 3\text{-}2\rangle = |233\rangle$$

$$|[3]\ 3\text{-}3\rangle = |333\rangle$$

$$|[3]\ 11\rangle = \frac{2}{\sqrt{5}}\,|113\rangle - \frac{1}{\sqrt{5}}\,|122\rangle$$

$$|[3]\ 10\rangle = \sqrt{\frac{2}{5}}\,|123\rangle - \sqrt{\frac{3}{5}}\,|222\rangle$$

$$|[3]\ 1\text{-}1\rangle = \frac{2}{\sqrt{5}}\,|133\rangle - \frac{1}{\sqrt{5}}\,|223\rangle$$

$$|[21]\ 22\rangle = \left|\begin{smallmatrix}11\\2\end{smallmatrix}\right\rangle$$

$$|[21]\ 21\rangle = \frac{1}{\sqrt{5}}\,\left|\begin{smallmatrix}11\\3\end{smallmatrix}\right\rangle + \frac{1}{\sqrt{2}}\,\left|\begin{smallmatrix}12\\2\end{smallmatrix}\right\rangle$$

$$|[21]\ 20\rangle = \frac{1}{2}\sqrt{3}\,\left|\begin{smallmatrix}12\\3\end{smallmatrix}\right\rangle + \frac{1}{2}\,\left|\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle$$

$$|[21]\ 2\text{-}1\rangle = \frac{1}{\sqrt{2}}\,\left|\begin{smallmatrix}13\\3\end{smallmatrix}\right\rangle + \frac{1}{\sqrt{2}}\,\left|\begin{smallmatrix}22\\3\end{smallmatrix}\right\rangle$$

$$|[21]\ 2\,{-2}\rangle = \left|\begin{smallmatrix}2&3\\3\end{smallmatrix}\right\rangle$$

$$|[21]\ 11\rangle = \frac{1}{\sqrt{2}}\left|\begin{smallmatrix}1&1\\3\end{smallmatrix}\right\rangle - \frac{1}{\sqrt{2}}\left|\begin{smallmatrix}1&2\\2\end{smallmatrix}\right\rangle$$

$$|[21]\ 10\rangle = \frac{1}{2}\left|\begin{smallmatrix}1&2\\3\end{smallmatrix}\right\rangle - \frac{1}{2}\sqrt{3}\left|\begin{smallmatrix}1&3\\2\end{smallmatrix}\right\rangle$$

$$|[21]\ 1\,{-1}\rangle = -\frac{1}{\sqrt{2}}\left|\begin{smallmatrix}1&3\\3\end{smallmatrix}\right\rangle + \frac{1}{\sqrt{2}}\left|\begin{smallmatrix}2&2\\3\end{smallmatrix}\right\rangle$$

$$|[1^3]\ 00\rangle = \left|\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right\rangle\ .$$

These lead to the following matrix elements of $[I^1\cdot I^1]_{\text{Siddall}}$ and $[I^2\cdot I^2]_{\text{Siddall}}$ in the basis $|[\lambda]IM\rangle$:

$$\langle[3]\ 3M|\ [I^1\cdot I^1]_{\text{Siddall}}|[3]\ 3M\rangle = \frac{1}{2}\ ;\ -3 \le M \le 3$$

$$\langle[3]\ 1M|\ [I^1\cdot I^1]_{\text{Siddall}}|[13]\ 1M\rangle = -\frac{1}{3}\ ;\ -1 \le M \le 1$$

$$\langle[21]\ 2M|\ [I^1\cdot I^1]_{\text{Siddall}}|[21]\ 2M\rangle = 0;\quad -2 \le M \le 2$$

$$\langle[21]\ 1M|\ [I^1\cdot I^1]_{\text{Siddall}}|[21]\ 1M\rangle = -\frac{1}{3}\ ;\quad -1 \le M \le 1$$

$$\langle[1^3]\ 00|\ [I^1\cdot I^1]_{\text{Siddall}}|[1^3]\ 00\rangle = -\frac{1}{2}$$

$$\langle[3]\ 3M|\ [I^2\cdot I^2]_{\text{Siddall}}|[3]\ 3M\rangle = \frac{1}{10}\ ;\quad -3 \le M \le 3$$

$$\langle[3]\ 1M|\ [I^2\cdot I^2]_{\text{Siddall}}|[3]\ 1M\rangle = \frac{3}{5}\ ;\quad -1 \le M \le 1$$

$$<[21]\ 2M|\ [I^2\cdot I^2]_{Siddall}|[21]\ 2M> = -\frac{1}{5}\ ; \quad -2 \le M \le 2$$

$$<[21]\ 1M|\ [I^2\cdot I^2]_{Siddall}|[21]\ 1M> = 0\ ; \quad -1 \le M \le 1$$

$$<[1^3]\ 00|\ [I^2\cdot I^2]_{Siddall}|[1^3]\ 00> = -\frac{1}{2}$$

We can check these results using the basis $\{|\sigma m_1> \otimes |\sigma m_2> \otimes |\sigma m_3> \equiv |m_1 m_2 m_3> \ |-\sigma \le m_1, m_2, m_3 \le \sigma\}$. The basis is called the spin product basis in reference [14]. We will first calculate the Weyl basis states using standard techniques with Young operators [2]. For the partition {3}, we have

$$|i_1 i_2 i_3>_W = \frac{1}{6}\ [|m_1 m_2 m_3> + |m_3 m_1 m_2> + |m_2 m_1 m_3> +$$

$$+ |m_2 m_3 m_1> + |m_3 m_2 m_1> + |m_1 m_3 m_2>]$$

where $i_\alpha = \sigma+1-m_\alpha$ for $\alpha = 1,2,3$. Here we have introduced the subscript W in our basis states to indicate that these are Weyl basis states, not Gelfand basis states. We obtain, upon normalization,

$$|111\rangle_W = |111\rangle$$

$$|112\rangle_W = \frac{1}{\sqrt{5}}\,[|110\rangle + |011\rangle + |101\rangle]$$

$$|113\rangle_W = \frac{1}{\sqrt{3}}\,[|11\text{-}1\rangle + |1\text{-}11\rangle + |\text{-}111\rangle]$$

$$|122\rangle_W = \frac{1}{\sqrt{3}}\,[|100\rangle + |010\rangle + |001\rangle]$$

$$|123\rangle_W = \frac{1}{\sqrt{6}}\,[|10\text{-}1\rangle + |\text{-}110\rangle + |01\text{-}1\rangle + |0\text{-}11\rangle + |\text{-}101\rangle + |1\text{-}10\rangle]$$

$$|222\rangle_W = |000\rangle$$

$$|133\rangle_W = \frac{1}{\sqrt{3}}\,[|\text{-}1\text{-}11\rangle + |\text{-}11\text{-}1\rangle + |1\text{-}1\text{-}1\rangle]$$

$$|223\rangle_W = \frac{1}{\sqrt{3}}\,[|\text{-}100\rangle + |00\text{-}1\rangle + |0\text{-}10\rangle]$$

$$|233\rangle_W = \frac{1}{\sqrt{3}}\,[|\text{-}1\text{-}10\rangle + |0\text{-}1\text{-}1\rangle + |\text{-}10\text{-}1\rangle]$$

$$|333\rangle_W = |\text{-}1\text{-}1\text{-}1\rangle\ .$$

For the partition [21] we have two sets of basis functions corresponding to two different invariant spaces with respect to S(3) (but belonging to the same irreducible representation of U(2σ+1)). These are,

$$\left|\begin{matrix} i_1 & i_2 \\ i_3 & \end{matrix}\right\rangle^{(1)}_{W} = \frac{1}{3}\,[\,|m_1m_2m_3\rangle - |m_3m_2m_1\rangle + |m_2m_1m_3\rangle - |m_3m_1m_2\rangle]$$

$$\left|\begin{matrix} i_1 & i_2 \\ i_3 & \end{matrix}\right\rangle^{(2)}_{W} = \frac{1}{3}\,[\,|m_1m_2m_3\rangle - |m_2m_1m_3\rangle + |m_3m_2m_1\rangle - |m_2m_3m_1\rangle]$$

We choose the first set as our basis functions and obtain upon normalization

$$|{}^{11}_{2}\rangle_W = \frac{1}{\sqrt{2}}\,[\,|100\rangle - |011\rangle]$$

$$|{}^{11}_{3}\rangle_W = \frac{1}{\sqrt{2}}\,[\,|11\text{-}1\rangle - |\text{-}111\rangle]$$

$$|{}^{12}_{3}\rangle_W = \frac{1}{\sqrt{2}}\,[\,|100\rangle - |001\rangle]$$

$$|{}^{12}_{3}\rangle^{*}_W = \frac{1}{2}\,[\,|10\text{-}1\rangle - |\text{-}101\rangle + |01\text{-}1\rangle - |\text{-}110\rangle]$$

$$|{}^{13}_{2}\rangle^{*}_W = \frac{1}{2}\,[\,|1\text{-}10\rangle + |\text{-}110\rangle - |0\text{-}11\rangle - |01\text{-}1\rangle]$$

$$|{}^{13}_{3}\rangle_W = \frac{1}{\sqrt{2}}\,[\,|1\text{-}1\text{-}1\rangle - |\text{-}1\text{-}11\rangle]$$

$$|{}^{22}_{3}\rangle_W = \frac{1}{\sqrt{2}}\,[\,|00\text{-}1\rangle - |\text{-}100\rangle]$$

$$|{}^{23}_{3}\rangle_W = \frac{1}{\sqrt{2}}\,[\,|01\text{-}1\rangle - |\text{-}1\text{-}10\rangle]$$

Here we have placed an asterisk on $|{}^{12}_{3}\rangle$ and $|{}^{13}_{2}\rangle$ to indicate that they are not orthogonal. This incidentally illustrates our previous claim that the Weyl basis is not orthogonal in general. We set $|{}^{12}_{3}\rangle_W = |{}^{12}_{3}\rangle^*_W$ and orthogonalize $|{}^{13}_{2}\rangle^*$ using the Gram-Schmidt orthogonalization procedure. We obtain, upon normalization

$$|{}^{12}_{3}\rangle_W = |{}^{12}_{3}\rangle^*_W = \frac{1}{2}\,[\,|10\text{-}1\rangle - |\text{-}101\rangle + |01\text{-}1\rangle - |\text{-}110\rangle]$$

$$|{}^{13}_{2}\rangle_W = \frac{1}{\sqrt{6}}\,[\,|\text{-}110\rangle - |01\text{-}1\rangle + |1\text{-}10\rangle - |0\text{-}11\rangle + |10\text{-}1\rangle - |\text{-}101\rangle]$$

Finally, for the partition $[1^3]$ we obtain

$$\left|\begin{matrix}1\\2\\3\end{matrix}\right\rangle_W = \frac{1}{\sqrt{6}}\,[\,|10\text{-}1\rangle - |1\text{-}10\rangle - |01\text{-}1\rangle + |0\text{-}11\rangle - |\text{-}110\rangle - |\text{-}101\rangle]$$

This orthonormalized Weyl basis we have constructed need not be the same as the Gelfand basis; however, it is related to the Gelfand basis through a unitary transformation, which means that the matrix elements are preserved. Substituting into the transformation coefficients between $|[\lambda]IM\rangle$ and tableaux states yields the following results:

$$|[3]\ 33\rangle \doteq |111\rangle$$

$$|[3]\ 32\rangle \doteq \frac{1}{\sqrt{3}} [|110\rangle + |011\rangle + |101\rangle]$$

$$|[3]\ 31\rangle \doteq \frac{1}{\sqrt{15}} [|11\text{-}1\rangle + |1\text{-}11\rangle + |\text{-}111\rangle]$$

$$+ \frac{2}{\sqrt{15}} [|100\rangle + |010\rangle + |001\rangle]$$

$$|[3]\ 30\rangle \doteq \sqrt{\tfrac{2}{5}}\ |000\rangle + \frac{1}{\sqrt{10}} |10\text{-}1\rangle + \frac{1}{\sqrt{10}} |\text{-}100\rangle$$

$$+ \frac{1}{\sqrt{10}} |01\text{-}1\rangle + \frac{1}{\sqrt{10}} |0\text{-}11\rangle + \frac{1}{\sqrt{10}} |\text{-}101\rangle + \frac{1}{\sqrt{10}} |1\text{-}10\rangle$$

$$|[3]\ 3\text{-}1\rangle \doteq \frac{1}{\sqrt{15}} [|\text{-}1\text{-}11\rangle + |\text{-}11\text{-}1\rangle + |1\text{-}1\text{-}1\rangle + 2|\text{-}100\rangle +$$

$$+ 2|00\text{-}1\rangle + 2|0\text{-}10\rangle]$$

$$|[3]\ 3\text{-}2\rangle \doteq \frac{1}{\sqrt{3}} [|\text{-}1\text{-}10\rangle + |0\text{-}1\text{-}1\rangle + |\text{-}10\text{-}1\rangle]$$

$$|[3]\ 3\text{-}3\rangle \doteq |\text{-}1\text{-}1\text{-}1\rangle$$

$$|[3]\ 11\rangle \doteq \frac{2}{\sqrt{15}} |11\text{-}1\rangle + \frac{2}{\sqrt{15}} |1\text{-}11\rangle + \frac{2}{\sqrt{15}} |\text{-}111\rangle -$$

$$- \frac{1}{\sqrt{15}} [|100\rangle + |010\rangle - |001\rangle]$$

$$|[3]\ 10\rangle \doteq \frac{1}{\sqrt{15}} [|10\text{-}1\rangle + |\text{-}110\rangle + |01\text{-}1\rangle + |0\text{-}11\rangle +$$

$$+ |101\rangle + |1\text{-}10\rangle] - \sqrt{\tfrac{3}{5}}\, |000\rangle$$

$$|[3] 1\text{-}1\rangle \doteq \frac{2}{\sqrt{15}} [|\text{-}1\text{-}11\rangle + |\text{-}11\text{-}1\rangle + |1\text{-}1\text{-}1\rangle] -$$

$$- \frac{1}{\sqrt{15}} [|\text{-}100\rangle + |00\text{-}1\rangle + |0\text{-}10\rangle]$$

$$|[21]\ 22\rangle \doteq \frac{1}{\sqrt{2}} [|110\rangle - |011\rangle]$$

$$|[21]\ 21\rangle \doteq \frac{1}{2} [|11\text{-}1\rangle - |\text{-}111\rangle + |100\rangle - |001\rangle]$$

$$|[21]\ 20\rangle \doteq \frac{1}{\sqrt{3}} [|10\text{-}1\rangle - |\text{-}101\rangle] + \frac{1}{\sqrt{12}} [|01\text{-}1\rangle + |\text{-}110\rangle +$$

$$+ |1\text{-}10\rangle - |0\text{-}11\rangle]$$

$$|[21]\ 2\text{-}1\rangle \doteq \frac{1}{2} [|1\text{-}1\text{-}1\rangle - |\text{-}1\text{-}11\rangle + |00\text{-}1\rangle - |\text{-}100\rangle]$$

$$|[21]\ 11\rangle \doteq \frac{1}{2} [|11\text{-}1\rangle - |\text{-}111\rangle - |100\rangle + |001\rangle]$$

$$|[21]\ 1\text{-}1\rangle \doteq \frac{1}{2} [|00\text{-}1\rangle - |\text{-}100\rangle - |1\text{-}1\text{-}1\rangle + |\text{-}1\text{-}11\rangle]$$

$$|[1^3]\ 00\rangle \doteq \frac{1}{\sqrt{6}} [|10\text{-}1\rangle - |1\text{-}10\rangle - |01\text{-}1\rangle + |0\text{-}11\rangle +$$

$$+ |\text{-}110\rangle - |\text{-}101\rangle].$$

Here we have placed dots over the equality to signify that the basis vectors on the left and the right may differ by a unitary transformation.

Suppressing the labels σ_1, σ_2, σ_3, and introducing the notation $\delta(a,b,c,d,) \equiv \delta(\tilde{m}_\omega,a)\ \delta(m_\omega,b)\ \delta(\tilde{m}_{\omega'},c)\ \delta(m_{\omega'},d)$, the operators in [12, 13, 14], have the following matrix elements in the basis $\{|m_1m_2m_3\rangle \mid -\sigma \leq m_1m_2m_3 \leq \sigma\}$:

$$\langle \tilde{m}_1\tilde{m}_2\tilde{m}_3|[I^1\cdot I^1]_{\text{Siddall}}|m_1m_2m_3\rangle$$

$$= \sum_{\omega<\omega'=1}^{3} \frac{1}{6}[\delta(m_\omega,-1,m_{\omega'},1) - \delta(m_\omega,-1,m_{\omega'},1)$$

$$- \delta(m_\omega,1,m_{\omega'},-1) + \delta(m_\omega,1,m_{\omega'},1) + \delta(0,-1,-1,0)$$

$$+ \delta(0,-1,0,1) + \delta(1,0,0,1) + \delta(1,0,-1,0)$$

$$+ \delta(-1,0,0,-1) + \delta(-1,0,1,0) + \delta(0,1,0,-1) + \delta(0,1,1,0)]$$

$$\langle \tilde{m}_1\tilde{m}_2\tilde{m}_3|[I^2\cdot I^2]_{\text{Siddall}}|m_1m_2m_3\rangle$$

$$= \sum_{\omega<\omega'=1}^{3} \frac{1}{30}\ [\delta(m_\omega,1,m_{\omega'},1) - 2\delta(m_\omega,1,m_{\omega'},0)$$

$$+ \delta(m_\omega,1,m_{\omega'},-1) - 2\delta(m_\omega,0,m_{\omega'},1)$$

$$+ 4\delta(m_\omega,0,m_{\omega'},0) - 2\delta(m_\omega,0,m_{\omega'},-1) + \delta(m_\omega,-1,m_{\omega'},1)$$

$$- 2\delta(m_\omega,-1,m_{\omega'},0) + \delta(m_\omega,-1,m_{\omega'},-1)]$$

$$+ \frac{1}{10} [\delta(1,0,0,1) + \delta(0,-1,-1,0) - \delta(0,-1,0,1)$$

$$- \delta(1,0,-1,0) + \delta(-1,0,0,-1)$$

$$+ \delta(0,1,1,0) - \delta(-1,0,1,0) - \delta(0,1,-1,0)]$$

$$+ \frac{1}{5} [\delta(1,-1,-1,1) + \delta(-1,1,1,-1)] .$$

From these equations we obtain the following matrix elements of $[I^1 \cdot I^1]_{Siddall}$ and $[I^2 \cdot I^2]_{Siddall}$ in the basis $\{|m_1 m_2 m_3\rangle \mid -\sigma \le m_1, m_2, m_3 \le \sigma\}$:

Partition $[1^3]$

$$\langle [1^3]00|[I^1 \cdot I^1]_{Siddall}|[1^3]00\rangle = \frac{1}{6}\langle 10-1|[I^1 \cdot I^1]_{Siddall}|10-1\rangle +$$

$$+ \frac{1}{6} \langle 1-10|[I^1 \cdot I^1]_{Siddall}|1-10\rangle$$

$$+ \frac{1}{6}\langle 01-1|[I^1 \cdot I^1]_{Siddall}|01-1\rangle + \frac{1}{6}\langle 0-11|[I^1 \cdot I^1]_{Siddall}|0-11\rangle$$

$$+ \frac{1}{6} \langle -110|[I^1 \cdot I^1]_{Siddall}|-110\rangle + \frac{1}{6}\langle -101|[I^1 \cdot I^1]_{Siddall}|-101\rangle$$

$$-\frac{1}{3}\langle 10-1|[I^1\cdot I^1]_{Siddall}|{+1}-10\rangle - \frac{1}{3}\langle 10-1|[I^1\cdot I^1]_{Siddall}|01-1\rangle$$

$$+\frac{1}{3}\langle 10-1|[I^1\cdot I^1]_{Siddall}|-110\rangle - \frac{1}{3}\langle 10-1|[I^1\cdot I^1]_{Siddall}|-101\rangle$$

$$+\frac{1}{3}\langle 1-10|[I^1\cdot I^1]_{Siddall}|01-1\rangle - \frac{1}{3}\langle 1-10|[I^1\cdot I^1]_{Siddall}|0-11\rangle$$

$$-\frac{1}{3}\langle 1-10|[I^1\cdot I^1]_{Siddall}|-110\rangle + \frac{1}{3}\langle 1-10|[I^1\cdot I^1]_{Siddall}|-101\rangle$$

$$-\frac{1}{3}\langle 01-1|[I^1\cdot I^1]_{Siddall}|0-11\rangle - \frac{1}{3}\langle 01-1|[I^1\cdot I^1]_{Siddall}|-110\rangle$$

$$-\frac{1}{3}\langle -110|[I^1\cdot I^1]_{Siddall}|-101\rangle - \frac{1}{3}\langle 0-11|[I^1\cdot I^1]_{Siddall}|-101\rangle$$

$$= 6\left(\frac{1}{6}\right)\left(-\frac{1}{6}\right) - \frac{1}{18} - \frac{1}{18} - \frac{1}{18} - \frac{1}{18} - \frac{1}{18} - \frac{1}{18} = -\frac{1}{2}$$

$$\langle [1^3]00|[I^2\cdot I^2]_{Siddall}|[1^3]00\rangle = \frac{1}{6}\langle 10-1|[I^2\cdot I^2]_{Siddall}|10-1\rangle +$$

$$+\frac{1}{6}\langle 1-10|[I^2\cdot I^2]_{Siddall}|1-10\rangle + \frac{1}{6}\langle 01-1|[I^2\cdot I^2]_{Siddall}|01-1\rangle +$$

$$+\frac{1}{6}\langle 0-11|[I^2\cdot I^2]_{Siddall}|0-11\rangle + \frac{1}{6}\langle -110|[I^2\cdot I^2]_{Siddall}|-110\rangle +$$

$$+\frac{1}{6}\langle -101|[I^2\cdot I^2]_{Siddall}|-101\rangle - \frac{1}{3}\langle 10-1|[I^2\cdot I^2]_{Siddall}|1-10\rangle -$$

$$-\frac{1}{3}\langle 10-1|[I^2\cdot I^2]_{Siddall}|01-1\rangle + \frac{1}{3}\langle 10-1|[I^2\cdot I^2]_{Siddall}|-110\rangle -$$

$$- \frac{1}{3}\langle 10\text{-}1|[I^2\cdot I^2]_{Siddall}|\text{-}101\rangle + \frac{1}{3}\langle 1\text{-}10|[I^2\cdot I^2]_{Siddall}|01\text{-}1\rangle -$$

$$- \frac{1}{3}\langle 1\text{-}10|[I^2\cdot I^2]_{Siddall}|0\text{-}11\rangle - \frac{1}{3}\langle 1\text{-}10|[I^2\cdot I^2]_{Siddall}|\text{-}110\rangle +$$

$$+ \frac{1}{3}\langle 1\text{-}10|[I^2\cdot I^2]_{Siddall}|\text{-}101\rangle - \frac{1}{3}\langle 01\text{-}1|[I^2\cdot I^2]_{Siddall}|0\text{-}11\rangle -$$

$$- \frac{1}{3}\langle 01\text{-}1|[I^2\cdot I^2]_{Siddall}|\text{-}110\rangle + \frac{1}{3}\langle 01\text{-}1|[I^2\cdot I^2]_{Siddall}|\text{-}101\rangle +$$

$$+ \frac{1}{3}\langle 0\text{-}11|[I^2\cdot I^2]_{Siddall}|\text{-}110\rangle - \frac{1}{3}\langle 0\text{-}11|[I^2\cdot I^2]_{Siddall}|\text{-}101\rangle -$$

$$- \frac{1}{3}\langle \text{-}110|[I^2\cdot I^2]_{Siddall}|\text{-}101\rangle = -\frac{15}{30} = -\frac{1}{2}$$

Partition [3]

$$\langle [3]\ 33|[I^1\cdot I^1]_{Siddall}|[3]33\rangle = \langle 111|[I^1\cdot I^1]_{Siddall}|111\rangle =$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

$$\langle [3]\ 32|[I^1\cdot I^1]_{Siddall}|[3]\ 32\rangle = \frac{1}{3}\langle 110|[I^1\cdot I^1]_{Siddall}|110\rangle +$$

$$+ \frac{1}{3}\langle 011|[I^1\cdot I^1]_{Siddall}|011\rangle + \frac{1}{3}\langle 101|[I^1\cdot I^1]_{Siddall}|101\rangle +$$

$$+ \frac{2}{3}\langle 110|[I^1\cdot I^1]_{Siddall}|011\rangle + \frac{2}{3}\langle 110|[I^1\cdot I^1]_{Siddall}|101\rangle +$$

$$+ \frac{2}{3}\langle 011|[I^1\cdot I^1]_{Siddall}|101\rangle = 3\cdot\frac{1}{3}(\frac{1}{6}) + 3\cdot\frac{2}{3}(\frac{1}{6}) = \frac{1}{2}.$$

$$<[3]\ 31|[I^1\cdot I^1]_{Siddall}|[3]\ 31> = \frac{1}{15}<11\text{-}1|[I^1\cdot I^1]_{Siddall}|11\text{-}1> +$$

$$+ \frac{4}{15}<100|[I^1\cdot I^1]_{Siddall}|100> + \frac{1}{15}<1\text{-}11|[I^1\cdot I^1]_{Siddall}|1\text{-}11> +$$

$$+ \frac{4}{15}<010|[I^1\cdot I^1]_{Siddall}|010> + \frac{4}{15}<001|[I^1\cdot I^1]_{Siddall}|001> +$$

$$+ \frac{1}{15}<\text{-}111|[I^1\cdot I^1]_{Siddall}|\text{-}111> + \frac{4}{15}<11\text{-}1|[I^1\cdot I^1]_{Siddall}|100> +$$

$$+ \frac{2}{15}<11\text{-}1|[I^1\cdot I^1]_{Siddall}|1\text{-}11> + \frac{4}{15}<11\text{-}1|[I^1\cdot I^1]_{Siddall}|010> +$$

$$+ \frac{4}{15}<11\text{-}1|[I^1\cdot I^1]_{Siddall}|001> + \frac{2}{15}<11\text{-}1|[I^1\cdot I^1]_{Siddall}|\text{-}111> +$$

$$+ \frac{4}{15}<100|[I^1\cdot I^1]_{Siddall}|1\text{-}11> + \frac{8}{15}<100|[I^1\cdot I^1]_{Siddall}|010> +$$

$$+ \frac{8}{15}<100|[I^1\cdot I^1]_{Siddall}|001> + \frac{4}{15}<100|[I^1\cdot I^1]_{Siddall}|\text{-}111> +$$

$$+ \frac{4}{15}<1\text{-}11|[I^1\cdot I^1]_{Siddall}|010> + \frac{4}{15}<1\text{-}11|[I^1\cdot I^1]_{Siddall}|001> +$$

$$+ \frac{2}{15}<1\text{-}11|[I^1\cdot I^1]_{Siddall}|\text{-}111> + \frac{8}{15}<010|[I^1\cdot I^1]_{Siddall}|001> +$$

$$+ \frac{4}{15}<010|[I^1\cdot I^1]_{Siddall}|\text{-}111> + \frac{4}{15}<001|[I^1\cdot I^1]_{Siddall}|\text{-}111>$$

$$= \frac{1}{15}(-\frac{1}{6}) + \frac{4}{15}(0) + \frac{1}{15}(-\frac{1}{6}) + \frac{4}{15}(0) + \frac{1}{15}(-\frac{1}{6}) + \frac{4}{15}(\frac{1}{6}) + \frac{2}{15}(0) +$$

$$+ \frac{4}{15}(\frac{1}{6}) + \frac{4}{15}(0) + \frac{2}{15}(0) + \frac{4}{15}(\frac{1}{6}) + \frac{8}{15}(\frac{1}{6}) + \frac{8}{15}(\frac{1}{6}) + \frac{4}{15}(\frac{1}{6})$$

$$+ \frac{8}{15}(\frac{1}{6}) + \frac{4}{15}(\frac{1}{6}) + \frac{4}{15}(\frac{1}{6}) = \frac{1}{2}$$

$$\langle[3]\ 30|[I^1 \cdot I^1]_{Siddall}|[3]\ 30\rangle = \frac{1}{10}(-\frac{1}{6}) + \frac{1}{10}(-\frac{1}{6}) + \frac{1}{10}(-\frac{1}{6}) +$$

$$+ \frac{1}{10}(-\frac{1}{6}) + \frac{4}{10}(0) + \frac{1}{10}(-\frac{1}{6}) + \frac{1}{10}(-\frac{1}{6}) + \frac{2}{10}(\frac{1}{6}) + \frac{4}{10}(\frac{1}{6}) + \frac{4}{10}(\frac{1}{6})$$

$$+ \frac{2}{10}(\frac{1}{6}) + \frac{4}{10}(\frac{1}{6}) + \frac{2}{10}(\frac{1}{6}) + \frac{4}{10}(\frac{1}{6}) + \frac{4}{10}(\frac{1}{6}) + \frac{4}{10}(\frac{1}{6}) + \frac{2}{10}(\frac{1}{6}) +$$

$$+ \frac{2}{10}(\frac{1}{6}) = \frac{1}{2}.$$

For the operator $[I^2 \cdot I^2]_{Siddall}$, we find

$$\langle[3]\ 33|[I^2 \cdot I^2]_{Siddall}|[3]\ 33\rangle = \langle 111|[I^2 \cdot I^2]_{Siddall}|111\rangle =$$

$$\frac{1}{30} + \frac{1}{30} + \frac{1}{30} = \frac{1}{10} \quad .$$

$$\langle[3]\ 32|[I^2 \cdot I^2]_{Siddall}|[3]\ 32\rangle = \frac{1}{3}\langle 110|[I^2 \cdot I^2]_{Siddall}|110\rangle +$$

$$+ \frac{1}{3}\langle 011|[I^2 \cdot I^2]_{Siddall}|011\rangle + \frac{1}{3}\langle 101|[I^2 \cdot I^2]_{Siddall}|011\rangle +$$

$$+ \frac{2}{3}\langle 110|[I^2 \cdot I^2]_{Siddall}|101\rangle + \frac{2}{3}\langle 011|[I^2 \cdot I^2]_{Siddall}|101\rangle =$$

$$= \frac{1}{3}(-\frac{1}{10}) + \frac{1}{3}(-\frac{1}{10}) + \frac{1}{3}(-\frac{1}{10}) + \frac{2}{3}(\frac{1}{10}) + \frac{2}{3}(\frac{1}{10}) + \frac{2}{3}(\frac{1}{10}) = \frac{1}{10}.$$

$$\langle [3]\ 31|[I^2\cdot I^2]_{Siddall}|[3]\ 31\rangle = \frac{1}{15}\langle 11\text{-}1|[I^2\cdot I^2]_{Siddall}|11\text{-}1\rangle +$$

$$+ \frac{1}{15}\langle\text{-}111|[I^2\cdot I^2]_{Siddall}|\text{-}111\rangle + \frac{4}{15}\langle 11\text{-}1|[I^2\cdot I^2]_{Siddall}|100\rangle +$$

$$+ \frac{2}{15}\langle 11\text{-}1|[I^2\cdot I^2]_{Siddall}|1\text{-}11\rangle + \frac{4}{15}\langle 11\text{-}1|[I^2\cdot I^2]_{Siddall}|010\rangle +$$

$$+ \frac{2}{15}\langle 11\text{-}1|[I^2\cdot I^2]_{Siddall}|\text{-}111\rangle + \frac{4}{15}\langle 100|[I^2\cdot I^2]_{Siddall}|1\text{-}11\rangle +$$

$$+ \frac{8}{15}\langle 100|[I^2\cdot I^2]_{Siddall}|010\rangle + \frac{8}{15}\langle 100|[I^2\cdot I^2]_{Siddall}|001\rangle +$$

$$+ \frac{4}{15}\langle 1\text{-}11|[I^2\cdot I^2]_{Siddall}|001\rangle + \frac{2}{15}\langle 1\text{-}11|[I^2\cdot I^2]_{Siddall}|\text{-}111\rangle +$$

$$+ \frac{8}{15}\langle 010|[I^2\cdot I^2]_{Siddall}|001\rangle + \frac{4}{15}\langle 010|[I^2\cdot I^2]_{Siddall}|\text{-}111\rangle +$$

$$+ \frac{2}{15}\langle 001|[I^2\cdot I^2]_{Siddall}|\text{-}111\rangle = \frac{1}{10}$$

$$\langle [3]\ 30|[I^2\cdot I^2]_{Siddall}|[3]\ 30\rangle = \frac{1}{10}(-\frac{1}{10}) + \frac{1}{10}(-\frac{1}{10}) + \frac{1}{10}(-\frac{1}{10})$$

$$+ \frac{4}{10}(\frac{12}{30}) + \frac{1}{10}(-\frac{1}{10}) + \frac{1}{10}(-\frac{1}{10}) + \frac{1}{10}(-\frac{1}{10}) + \frac{2}{10}(\frac{1}{10}) + \frac{4}{10}(-\frac{1}{10})$$

$$+ \frac{2}{10}(\frac{1}{10}) + \frac{2}{10}(\frac{1}{5}) + \frac{4}{10}(-\frac{1}{10}) + \frac{2}{10}(\frac{1}{10}) + \frac{2}{10}(\frac{1}{5}) + \frac{4}{10}(-\frac{1}{10})$$

$$+ \frac{2}{10}(\frac{1}{5}) + \frac{2}{10}(\frac{1}{10}) + \frac{4}{10}(-\frac{1}{10} + \frac{4}{10}(-\frac{1}{10}) + \frac{4}{10}(-\frac{1}{10}) + \frac{2}{10}(\frac{1}{10})$$

$$+ \frac{2}{10}(\frac{1}{10}) = \frac{1}{10}$$

$$<[3]\ 11|[I^1\cdot I^1]_{Siddall}|[3]\ 11> = \frac{4}{15}<11-1|[I^1\cdot I^1]_{Siddall}|11-1> +$$

$$+ \frac{4}{15}<1-11|[I^1\cdot I^1]_{Siddall}|1-11> + \frac{4}{15}<-111|[I^1\cdot I^1]_{Siddall}|-111> +$$

$$+ \frac{1}{15}<-100|[I^1\cdot I^1]_{Siddall}|100> + \frac{1}{15}<010|[I^1\cdot I^1]_{Siddall}|010> +$$

$$+ \frac{1}{15}<001|[I^1\cdot I^1]_{Siddall}|001> + \frac{8}{15}<11-1|[I^1\cdot I^1]_{Siddall}|1-11> +$$

$$+ \frac{8}{15}<11-1|[I^1\cdot I^1]_{Siddall}|-111> - \frac{4}{15}<11-1|[I^1\cdot I^1]_{Siddall}|100> +$$

$$- \frac{4}{15}<11-1|[I^1\cdot I^1]_{Siddall}|010> - \frac{4}{15}<11-1|[I^1\cdot I^1]_{Siddall}|001> +$$

$$+ \frac{8}{15}<1-11|[I^1\cdot I^1]_{Siddall}|-111> - \frac{4}{15}<1-11|[I^1\cdot I^1]_{Siddall}|100> -$$

$$- \frac{4}{15}<1-11|[I^1\cdot I^1]_{Siddall}|010> - \frac{4}{15}<1-11|[I^1\cdot I^1]_{Siddall}|001> -$$

$$- \frac{4}{15}<-111|[I^1\cdot I^1]_{Siddall}|100> - \frac{4}{15}<-111|[I^1\cdot I^1]_{Siddall}|010> -$$

$$- \frac{4}{15}<-111|[I^1\cdot I^1]_{Siddall}|001> + \frac{2}{15}<100|[I^1\cdot I^1]_{Siddall}|010> +$$

$$+ \frac{2}{15}<100|[I^1\cdot I^1]_{Siddall}|001> + \frac{2}{15}<010|[I^1\cdot I^1]_{Siddall}|001> =$$

$$= \frac{4}{15}(-\frac{1}{6}) + \frac{4}{15}(-\frac{1}{6}) + \frac{4}{15}(-\frac{1}{6}) + \frac{1}{15}(0) + \frac{1}{15}(0) + \frac{1}{15}(0) + \frac{8}{15}(0) +$$

$$+ \frac{8}{15}(0) - \frac{4}{15}(\frac{1}{6}) - \frac{4}{15}(\frac{1}{6}) - \frac{4}{15}(0) + \frac{8}{15}(0) - \frac{4}{16}(\frac{1}{6}) - \frac{4}{15}(0) -$$

$$- \frac{4}{15}(\frac{1}{6}) - \frac{4}{15}(0) - \frac{4}{15}(\frac{1}{6}) - \frac{4}{15}(\frac{1}{6}) + \frac{2}{15}(\frac{1}{6}) + \frac{2}{15}(\frac{1}{6}) + \frac{2}{15}(\frac{1}{6}) =$$

$$= - \frac{1}{3}$$

$$<[3]\ 10|[I^1 \cdot I^1]_{Siddall}|[3]\ 10> = \frac{1}{15}\ [<10\text{-}1|[I^1 \cdot I^1]_{Siddall}|10\text{-}1> +$$

$$+ <\text{-}110|[I^1 \cdot I^1]_{Siddall}|\text{-}110> + <01\text{-}1|[I^1 \cdot I^1]_{Siddall}|01\text{-}1> +$$

$$+ <0\text{-}11|[I^1 \cdot I^1]_{Siddall}|0\text{-}11> + <\text{-}101|[I^1 \cdot I^1]_{Siddall}|\text{-}101> +$$

$$+ <1\text{-}10|[I^1 \cdot I^1]_{Siddall}|1\text{-}10>] + \frac{3}{5}<000|[I^1 \cdot I^1]_{Siddall}|000> +$$

$$+ \frac{2}{15}[<10\text{-}1|[I^1 \cdot I^1]_{Siddall}|\text{-}110> + <10\text{-}1|[I^1 \cdot I^1]_{Siddall}|01\text{-}1> +$$

$$+ <10\text{-}1|[I^1 \cdot I^1]_{Siddall}|0\text{-}11> + <10\text{-}1|[I^1 \cdot I^1]_{Siddall}|\text{-}101> +$$

$$+ <10\text{-}1|[I^1 \cdot I^1]_{Siddall}|1\text{-}10>]$$

$$- \frac{2}{5}<10\text{-}1|[I^1 \cdot I^1]_{Siddall}|000> + \frac{2}{15}\ [<\text{-}110|[I^1 \cdot I^1]_{Siddall}|01\text{-}1> +$$

$$+ <\text{-}110|[I^1 \cdot I^1]_{Siddall}|0\text{-}11> + <\text{-}110|[I^1 \cdot I^1]_{Siddall}|\text{-}101> +$$

$$+ \langle -110|[I^1 \cdot I^1]_{Siddall}|1-10\rangle] - \frac{2}{5} \langle -110|[I^1 \cdot I^1]_{Siddall}|000\rangle +$$

$$+ \frac{2}{15} [\langle 01-1|[I^1 \cdot I^1]_{Siddall}|0-11\rangle + \langle 01-1|[I^1 \cdot I^1]_{Siddall}|-101\rangle +$$

$$+ \langle 01-1|[I^1 \cdot I^1]_{Siddall}|1-10\rangle] - \frac{2}{5} \langle 01-1|[I^1 \cdot I^1]_{Siddall}|000\rangle +$$

$$+ \frac{2}{15} [\langle 0-11|[I^1 \cdot I^1]_{Siddall}|-101\rangle + \langle 0-11|[I^1 \cdot I^1]_{Siddall}|-101\rangle +$$

$$+ \langle 01-1|[I^1 \cdot I^1]_{Siddall}|1-10\rangle] - \frac{2}{5} \langle 01-1|[I^1 \cdot I^1]_{Siddall}|000\rangle +$$

$$+ \frac{2}{15} [\langle 0-11|[I^1 \cdot I^1]_{Siddall}|-101\rangle + \langle 0-11|[I^1 \cdot I^1]_{Siddall}|1-10\rangle +$$

$$+ \langle -101|[I^1 \cdot I^1]_{Siddall}|1-10\rangle] - \frac{2}{5} \langle 0-11|[I^1 \cdot I^1]_{Siddall}|000\rangle -$$

$$- \frac{2}{5} \langle -101|[I^1 \cdot I^1]_{Siddall}|000\rangle - \frac{2}{5} \langle 1-10|[I^1 \cdot I^1]_{Siddall}|000\rangle =$$

$$= \frac{6}{15}(-\frac{1}{6}) + \frac{2}{15}(\frac{1}{6} + \frac{1}{6}) - \frac{2}{5}(\frac{1}{6}) + \frac{2}{15}(\frac{1}{6}) - \frac{2}{5}(\frac{1}{6}) - \frac{2}{5}(\frac{1}{6}) +$$

$$+ \frac{2}{15}(\frac{1}{6} + \frac{1}{6}) - \frac{2}{5}(\frac{1}{6}) - \frac{2}{5}(\frac{1}{6}) - \frac{2}{5}(\frac{1}{6}) = -\frac{1}{3}$$

$$\langle [3]\ 11|[I^2 \cdot I^2]_{Siddall}|[3]\ 11\rangle = \frac{4}{15} \langle 1-11|[I^2 \cdot I^2]_{Siddall}|11-1\rangle +$$

$$+ \frac{4}{15} \langle 1-11|[I^2 \cdot I^2]_{Siddall}|1-11\rangle + \frac{4}{15} \langle -111|[I^2 \cdot I^2]_{Siddall}|-111\rangle +$$

$$+\ \frac{1}{15}\langle 100|[I^2\cdot I^2]_{Siddall}|100\rangle + \frac{1}{15}\langle 010|[I^2\cdot I^2]_{Siddall}|010\rangle +$$

$$+\ \frac{1}{15}\langle 001|[I^2\cdot I^2]_{Siddall}|001\rangle + \frac{8}{15}\langle 11\text{-}1|[I^2\cdot I^2]_{Siddall}|1\text{-}11\rangle +$$

$$+\ \frac{8}{15}\langle 11\text{-}1|[I^2\cdot I^2]_{Siddall}|\text{-}111\rangle - \frac{4}{15}\langle 11\text{-}1|[I^2\cdot I^2]_{Siddall}|100\rangle -$$

$$-\ \frac{4}{15}\langle 11\text{-}1|[I^2\cdot I^2]_{Siddall}|010\rangle + \frac{4}{15}\langle 11\text{-}1|[I^2\cdot I^2]_{Siddall}|001\rangle +$$

$$+\ \frac{8}{15}\langle 1\text{-}11|[I^2\cdot I^2]_{Siddall}|111\rangle - \frac{4}{15}\langle 1\text{-}11|[I^2\cdot I^2]_{Siddall}|100\rangle -$$

$$-\ \frac{4}{15}\langle 1\text{-}11|[I^2\cdot I^2]_{Siddall}|010\rangle - \frac{4}{15}\langle 1\text{-}11|[I^2\cdot I^2]_{Siddall}|001\rangle -$$

$$-\ \frac{4}{15}\langle \text{-}111|[I^2\cdot I^2]_{Siddall}|100\rangle - \frac{4}{15}\langle \text{-}111|[I^2\cdot I^2]_{Siddall}|010\rangle -$$

$$-\ \frac{4}{15}\langle \text{-}111|[I^2\cdot I^2]_{Siddall}|001\rangle + \frac{2}{15}\langle 100|[I^2\cdot I^2]_{Siddall}|010\rangle +$$

$$+\ \frac{2}{15}\langle 100|[I^2\cdot I^2]_{Siddall}|001\rangle + \frac{2}{15}\langle 010|[I^2\cdot I^2]_{Siddall}|001\rangle =$$

$$= \frac{4}{15}\left(\frac{1}{10}\right) + \frac{4}{150} + \frac{4}{150} + \frac{8}{75} + \frac{8}{75} + \frac{4}{150} + \frac{4}{150} + \frac{8}{75} + \frac{4}{150} +$$

$$+\ \frac{4}{150} + \frac{4}{150} + \frac{4}{150} + \frac{2}{150} + \frac{2}{150} + \frac{2}{150} = \frac{3}{5}$$

$$<[3]<10|[I^2\cdot I^2]_{Siddall}|[3]\,10> = \frac{1}{15}[<101|[I^2\cdot I^2]_{Siddall}|101> +$$

$$+ <110|[I^2\cdot I^2]_{Siddall}|-110> + <01-1|[I^2\cdot I^2]_{Siddall}|01-1> +$$

$$+ <0-11|[I^2\cdot I^2]_{Siddall}|011> + <-101|[I^2\cdot I^2]_{Siddall}|-101> +$$

$$+ <1-10|[I^2\cdot I^2]_{Siddall}|1-10>] + \frac{3}{5}<000|[I^2\cdot I^2]_{Siddall}|000> +$$

$$+ \frac{2}{15}\,[<10-1|[I^2\cdot I^2]_{Siddall}|-110> + <10-1|[I^2\cdot I^2]_{Siddall}|01-1> +$$

$$+ <10-1|[I^2\cdot I^2]_{Siddall}|0-11> + <10-1|[I^2\cdot I^2]_{Siddall}|-101> +$$

$$+ <10-1|[I^2\cdot I^2]_{Siddall}|1-10>] - \frac{2}{5}<10-1|[I^2\cdot I^2]_{Siddall}|000> +$$

$$+ \frac{2}{15}\,[<-110|[I^2\cdot I^2]_{Siddall}|01-1> + <-110|[I^2\cdot I^2]_{Siddall}|0-11> +$$

$$+ <-110|[I^2\cdot I^2]_{Siddall}|-101> + <-110|[I^2\cdot I^2]_{Siddall}|1-10>]$$

$$- \frac{2}{5}<-110|[I^2\cdot I^2]_{Siddall}|000> + \frac{2}{15}[<01-1|[I^2\cdot I^2]_{Siddall}|0-11> +$$

$$+ <011|[I^2\cdot I^2]_{Siddall}|-101> + <01-1|[I^2\cdot I^2]_{Siddall}|1-10>\,] -$$

$$- \frac{2}{5}<01-1|[I^2\cdot I^2]_{Siddall}|000> + \frac{2}{15}\,[<0-11|[I^2\cdot I^2]_{Siddall}|-101> +$$

$$+\langle 0\text{-}11|[I^2\cdot I^2]_{Siddall}|1\text{-}10\rangle + \langle -101|[I^2\cdot I^2]_{Siddall}|1\text{-}10\rangle] -$$

$$-\frac{2}{15}[\langle 0\text{-}11|[I^2\cdot I^2]_{Siddall}|000\rangle - \frac{2}{5}\langle -101|[I^2\cdot I^2]_{Siddall}|000\rangle -$$

$$-\frac{2}{5}\langle 1\text{-}10|[I^2\cdot I^2]_{Siddall}|000\rangle = \frac{1}{15}[-\frac{1}{10} + -\frac{1}{10} + -\frac{1}{10} + -\frac{1}{10} +$$

$$+ -\frac{1}{10}] + \frac{3}{5}(\frac{12}{30}) + \frac{2}{15}[\frac{1}{10} + \frac{1}{5} + \frac{1}{10}] + \frac{2}{5}(\frac{1}{10}) + \frac{2}{15}[\frac{1}{10} + \frac{1}{10} + \frac{1}{5}] +$$

$$+ \frac{2}{5}(\frac{1}{10}) + \frac{2}{15}(\frac{1}{15}) + \frac{2}{5}(\frac{1}{10}) + \frac{2}{15}[\frac{1}{10} + \frac{1}{10}] + \frac{2}{5}(\frac{1}{10}) + \frac{2}{5}(\frac{1}{10}) = \frac{3}{5}.$$

Similarly, for the partition [21] we find,

$$\langle [21]\langle 22|[I^1\cdot I^1]_{Siddall}|[21]22\rangle = \frac{1}{2}\langle 110|[I^1\cdot I^1]_{Siddall}|110\rangle +$$

$$+ \frac{1}{2}\langle 011|[I^1\cdot I^1]_{Siddall}|011\rangle - \langle 110|[I^1\cdot I^1]_{Siddall}|011\rangle =$$

$$= \frac{1}{2}(\frac{1}{6}) + \frac{1}{2}(\frac{1}{6}) - \frac{1}{6} = 0$$

$$\langle [21]\ 21|[I^1\cdot I^1]_{Siddall}|[21]21\rangle = \frac{1}{4}\langle 11\text{-}1|[I^1\cdot I^1]_{Siddall}|11\text{-}1\rangle +$$

$$+ \frac{1}{4}\langle -111|[I^1\cdot I^1]_{Siddall}|-111\rangle + \frac{1}{4}\langle 100|[I^1\cdot I^1]_{Siddall}|100\rangle +$$

$$+ \frac{1}{4}\langle 100|[I^1\cdot I^1]_{Siddall}|001\rangle + \frac{1}{2}\langle 11\text{-}1|[I^1\cdot I^1]_{Siddall}|100\rangle -$$

$$- \frac{1}{2}\langle 11\text{-}1|[I^1\cdot I^1]_{\text{Siddall}}|\text{-}111\rangle - \frac{1}{2}\langle 11\text{-}1|[I^1\cdot I^1]_{\text{Siddall}}|001\rangle -$$

$$- \frac{1}{2}\langle \text{-}111|[I^1\cdot I^1]_{\text{Siddall}}|100\rangle + \frac{1}{2}\langle 11\text{-}1|[I^1\cdot I^1]_{\text{Siddall}}|001\rangle -$$

$$- \frac{1}{2}\langle 100|[I^1\cdot I^1]_{\text{Siddall}}|001\rangle = \frac{1}{4}(-\frac{1}{6}) + \frac{1}{4}(-\frac{1}{6}) + \frac{1}{2}(\frac{1}{6}) - \frac{1}{2}(0) -$$

$$- \frac{1}{2}(0) - \frac{1}{2} + \frac{1}{2}(\frac{1}{6}) - \frac{1}{2}(\frac{1}{6}) = 0$$

$$\langle [21]\,20|[I^1\cdot I^1]_{\text{Siddall}}|[21]20\rangle = \frac{1}{3}\langle 10\text{-}1|[I^1\cdot I^1]_{\text{Siddall}}|10\text{-}1\rangle +$$

$$+ \frac{1}{3}\langle \text{-}101|[I^1\cdot I^1]_{\text{Siddall}}|\text{-}101\rangle + \frac{1}{12}\langle 01\text{-}1|[I^1\cdot I^1]_{\text{Siddall}}|01\text{-}1\rangle +$$

$$+ \frac{1}{12}\langle \text{-}110|[I^1\cdot I^1]_{\text{Siddall}}|\text{-}110\rangle + \frac{1}{12}\langle 1\text{-}10|[I^1\cdot I^1]_{\text{Siddall}}|1\text{-}10\rangle +$$

$$+ \frac{1}{12}\langle 0\text{-}11|[I^1\cdot I^1]_{\text{Siddall}}|0\text{-}11\rangle - \frac{2}{3}\langle 10\text{-}1|[I^1\cdot I^1]_{\text{Siddall}}|\text{-}101\rangle +$$

$$+ \frac{1}{3}\langle 10\text{-}1|[I^1\cdot I^1]_{\text{Siddall}}|1\text{-}10\rangle - \frac{1}{3}\langle 10\text{-}1|[I^1\cdot I^1]_{\text{Siddall}}|0\text{-}11\rangle -$$

$$- \frac{1}{3}\langle \text{-}101|[I^1\cdot I^1]_{\text{Siddall}}|01\text{-}1\rangle + \frac{1}{3}\langle \text{-}101|[I^1\cdot I^1]_{\text{Siddall}}|\text{-}110\rangle -$$

$$- \frac{1}{3}\langle 101|[I^1\cdot I^1]_{\text{Siddall}}|1\text{-}10\rangle + \frac{1}{3}\langle \text{-}101|[I^1\cdot I^1]_{\text{Siddall}}|0\text{-}11\rangle -$$

$$- \frac{1}{6}\langle 01\text{-}1|[I^1\cdot I^1]_{\text{Siddall}}|\text{-}110\rangle + \frac{1}{6}\langle 01\text{-}1|[I^1\cdot I^1]_{\text{Siddall}}|1\text{-}10\rangle -$$

$$- \frac{1}{6}\langle 01\text{-}1|[I^1\cdot I^1]_{Siddall}|0\text{-}11\rangle - \frac{1}{6}\langle -110|[I^1\cdot I^1]_{Siddall}|1\text{-}10\rangle +$$

$$+ \frac{1}{6}\langle -110|[I^1\cdot I^1]_{Siddall}|0\text{-}11\rangle - \frac{1}{6}\langle 1\text{-}10|[I^1\cdot I^1]_{Siddall}|0\text{-}11\rangle =$$

$$= 2\ \frac{1}{3}(-\frac{1}{6}) + 4\ \frac{1}{12}(-\frac{1}{6}) + \frac{1}{3}(\frac{1}{6}) + \frac{1}{3}(\frac{1}{6}) + \frac{1}{3}(\frac{1}{6}) + \frac{1}{3}(\frac{1}{6}) - \frac{1}{6}(\frac{1}{6}) -$$

$$- \frac{1}{6}(\frac{1}{6}) = 0$$

$$\langle [21]22|[I^2\cdot I^2]_{Siddall}|[21]22\rangle = \frac{1}{2}\langle 110|[I^2\cdot I^2]_{Siddall}|110\rangle +$$

$$+ \frac{1}{2}\langle 011|[I^2\cdot I^2]_{Siddall}|011\rangle - \langle 110|[I^2\cdot I^2]_{Siddall}|011\rangle =$$

$$= \frac{1}{2}(-\frac{1}{10}) + \frac{1}{2}(-\frac{1}{10}) - \frac{1}{10} = -\frac{1}{5}$$

$$\langle [21]21|[I^2\cdot I^2]_{Siddall}|[21]21\rangle = \frac{1}{4}\langle 11\text{-}1|[I^2\cdot I^2]_{Siddall}|11\text{-}1\rangle +$$

$$+ \frac{1}{4}\langle -111|[I^2\cdot I^2]_{Siddall}|-111\rangle + \frac{1}{4}\langle 100|[I^2\cdot I^2]_{Siddall}|100\rangle +$$

$$+ \frac{1}{4}\langle 001|[I^2\cdot I^2]_{Siddall}|001\rangle + \frac{1}{2}\langle 11\text{-}1|[I^2\cdot I^2]_{Siddall}|100\rangle -$$

$$- \frac{1}{2}\langle 11\text{-}1|[I^2\cdot I^2]_{Siddall}|-111\rangle - \frac{1}{2}\langle 11\text{-}1|[I^2\cdot I^2]_{Siddall}|001\rangle -$$

$$- \frac{1}{2}\langle -111|[I^2\cdot I^2]_{Siddall}|100\rangle + \frac{1}{2}\langle -111|[I^2\cdot I^2]_{Siddall}|001\rangle -$$

$$-\frac{1}{2}\langle 100|[I^2\cdot I^2]_{Siddall}|001\rangle = \frac{1}{4}(\frac{1}{10}) + \frac{1}{4}(\frac{1}{10}) + \frac{1}{2}(-\frac{1}{10}) -$$

$$-\frac{1}{2}(\frac{1}{5}) - \frac{1}{2}(0) - \frac{1}{2}(0) + \frac{1}{2}(-\frac{1}{10}) - \frac{1}{2}(\frac{1}{10}) = -\frac{1}{5}$$

$$\langle [21]20|[I^2\cdot I^2]_{Siddall}|[21]20\rangle = 2\,\frac{1}{3}(-\frac{1}{10}) + 4\,\frac{1}{12}(-\frac{1}{10}) -$$

$$-\frac{2}{3}(\frac{1}{5}) + \frac{1}{3}(\frac{1}{10}) + \frac{1}{3}(\frac{1}{10}) + \frac{1}{3}(\frac{1}{10}) + \frac{1}{3}(\frac{1}{10}) + \frac{1}{3}(\frac{1}{10}) -$$

$$-\frac{1}{6}(\frac{1}{10}) - \frac{1}{6}(\frac{1}{5}) - \frac{1}{6}(\frac{1}{5}) - \frac{1}{6}(\frac{1}{10}) = -\frac{1}{5}$$

$$\langle [21]11|[I^1\cdot I^1]_{Siddall}|[21]11\rangle = \frac{1}{4}[\langle 11\text{-}1|[I^1\cdot I^1]_{Siddall}|11\text{-}1\rangle +$$

$$+ \langle \text{-}111|[I^1\cdot I^1]_{Siddall}|\text{-}111\rangle + \langle 100|[I^1\cdot I^1]_{Siddall}|100\rangle +$$

$$+ \langle 001|[I^1\cdot I^1]_{Siddall}|001\rangle] - \frac{1}{2}\langle 11\text{-}1|[I^1\cdot I^1]_{Siddall}|\text{-}111\rangle -$$

$$-\frac{1}{2}\langle 11\text{-}1|[I^1\cdot I^1]_{Siddall}|100\rangle + \frac{1}{2}\langle 11\text{-}1|[I^1\cdot I^1]_{Siddall}|001\rangle +$$

$$+\frac{1}{2}\langle \text{-}111|[I^1\cdot I^1]_{Siddall}|100\rangle - \frac{1}{2}\langle \text{-}111|[I^1\cdot I^1]_{Siddall}|001\rangle =$$

$$= \frac{1}{4}[-\frac{1}{6} + -\frac{1}{6}] - \frac{1}{2}(\frac{1}{6}) - \frac{1}{2}(\frac{1}{6}) - \frac{1}{2}(\frac{1}{6}) = -\frac{1}{3}$$

$$\langle [21]10|[I^1\cdot I^1]_{Siddall}|[21]10\rangle = \frac{1}{4}[\langle 01\text{-}1|[I^1\cdot I^1]_{Siddall}|01\text{-}1\rangle +$$

$$+ \langle -110|[I^1 \cdot I^1]_{Siddall}|-110\rangle + \langle 1-10|[I^1 \cdot I^1]_{Siddall}|1-10\rangle +$$

$$+ \langle 0-11|[I^1 \cdot I^1]_{Siddall}|0-11\rangle] - \frac{1}{2} \langle 01-1|[I^1 \cdot I^1]_{Siddall}|-110\rangle +$$

$$+ \frac{1}{2} \langle 01-1|[I^1 \cdot I^1]_{Siddall}|1-10\rangle - \frac{1}{2} \langle 01-1|[I^1 \cdot I^1]_{Siddall}|0-11\rangle -$$

$$- \frac{1}{2} \langle -110|[I^1 \cdot I^1]_{Siddall}|1-10\rangle + \frac{1}{2} \langle -110|[I^1 \cdot I^1]_{Siddall}|0-11\rangle -$$

$$- \frac{1}{2} \langle 1-10|[I^1 \cdot I^1]_{Siddall}|0-11\rangle = \frac{1}{4} [-\frac{1}{6} + -\frac{1}{6} + -\frac{1}{6} + -\frac{1}{6}] -$$

$$- \frac{1}{2}(\frac{1}{6}) = -\frac{1}{3}$$

$$\langle [21]11|[I^2 \cdot I^2]_{Siddall}|[21]11\rangle = \frac{1}{4}[\langle 11-1|[I^2 \cdot I^2]_{Siddall}|11-1\rangle +$$

$$+ \langle -111|[I^2 \cdot I^2]_{Siddall}|-111\rangle + \langle 100|[I^2 \cdot I^2]_{Siddall}|100\rangle +$$

$$+ \langle 001|[I^2 \cdot I^2]_{Siddall}|001\rangle] - \frac{1}{2} \langle 11-1|[I^2 \cdot I^2]_{Siddall}|-111\rangle -$$

$$- \frac{1}{2} \langle 11-1|[I^2 \cdot I^2]_{Siddall}|100\rangle + \frac{1}{2} \langle 11-1|[I^2 \cdot I^2]_{Siddall}|001\rangle +$$

$$+ \frac{1}{2} \langle -111|[I^2 \cdot I^2]_{Siddall}|100\rangle - \frac{1}{2} \langle -111|[I^2 \cdot I^2]_{Siddall}|001\rangle -$$

$$- \frac{1}{2} \langle 100|[I^2 \cdot I^2]_{Siddall}|001\rangle = \frac{1}{4}[\frac{1}{10} + \frac{1}{10}] - \frac{1}{2}(\frac{1}{5}) + \frac{1}{2}(\frac{1}{10}) = 0$$

$$
\begin{aligned}
&<[21]10|[I^2\cdot I^2]_{Siddall}|[21]10> = \tfrac{1}{4}[<01\text{-}1|[I^2\cdot I^2]_{Siddall}|01\text{-}1> + \\
&+ <\text{-}110|[I^2\cdot I^2]_{Siddall}|\text{-}110> + <1\text{-}10|[I^2\cdot I^2]_{Siddall}|1\text{-}10> + \\
&+ <0\text{-}11|[I^2\cdot I^2]_{Siddall}|0\text{-}11>] - \tfrac{1}{2} <01\text{-}1|[I^2\cdot I^2]_{Siddall}|\text{-}110> + \\
&+ \tfrac{1}{2} <01\text{-}1|[I^2\cdot I^2]_{Siddall}|1\text{-}10> - \tfrac{1}{2} <01\text{-}1|[I^2\cdot I^2]_{Siddall}|0\text{-}11> - \\
&- \tfrac{1}{2} <\text{-}110|[I^2\cdot I^2]_{Siddall}|1\text{-}10> + \tfrac{1}{2} <\text{-}110|[I^2\cdot I^2]_{Siddall}|0\text{-}11> - \\
&- \tfrac{1}{2} <1\text{-}10|[I^2\cdot I^2]_{Siddall}|0\text{-}11> = \tfrac{1}{4}[\tfrac{1}{10} + \tfrac{1}{10} + \tfrac{1}{10} + \tfrac{1}{10}] + \\
&+ \tfrac{1}{2}(\tfrac{1}{10}) - \tfrac{1}{2}(\tfrac{1}{5}) - \tfrac{1}{2}(\tfrac{1}{5}) + \tfrac{1}{2}(\tfrac{1}{10}) = 0
\end{aligned}
$$

These results are in complete agreement with our previous results. Note that we have omitted the calculations for negative M values since they are the same calculations as for the positive M values.

(iii) A_4B_2 SYSTEM $\sigma_A = 2$, $\sigma_B = 1$

The analysis of an A_4B_2 system proceeds in a slightly different manner from the treatment of the A_3 system. First, we list all the states of the A_4 system in Table VI and those of the B_2 system in Table VII. We have omitted states with negative m values in Table VI due to the symmetry of the spectrum about m = 0. The reason for including states of negative m values in Table VII will become apparent later.

The matrix elements of all one body operators for the A_4 system are compiled in Table VIII. Although we will only use a portion of this table, it is useful for any desired calculations of the A_4 system. The matrix elements of the operators $I^1 \cdot I^1$, $I^2 \cdot I^2$, $I^3 \cdot I^3$ and $I^4 \cdot I^4$ appear in Table IX (for the partition $[2^2]$. Due to the relative complexity of $I^2 \cdot I^2$, $I^3 \cdot I^3$, $I^4 \cdot I^4$, we have calculated only matrix elements for m = 3,4,5,6 for these operators.

To obtain matrix elements of the Hamiltonian of the overall A_4B_2 system, one simply uses the operator $H_{NMR} = H_{NMR}(A) + H_{NMR}(B)$ on tensor product states of the A_4 and B_2 systems. Note that this Hamiltonian does not account for interaction between A and B systems; nevertheless, it is useful as a starting point for perturbation theory.

The correlation between tableaux states and $|I_A M_A\rangle$

states can be performed using the lowering and projection operator techniques. For the partition $[2^2]$ we have

$$|[2^2]66\rangle = |{}^{11}_{22}\rangle$$

$$|[2^2]65\rangle = |{}^{11}_{23}\rangle$$

$$|[2^2]64\rangle = \frac{1}{\sqrt{11}}\,[\sqrt{3}\,|{}^{11}_{24}\rangle + \sqrt{6}\,|{}^{11}_{33}\rangle + \sqrt{2}\,|{}^{12}_{23}\rangle]$$

$$|[2^2](1);44\rangle = \frac{1}{\sqrt{11}}\,[\sqrt{8}\,|{}^{11}_{24}\rangle - \frac{3}{2}|{}^{11}_{33}\rangle - \frac{\sqrt{3}}{2}\,|{}^{12}_{33}\rangle]$$

$$|[2^2](2);44\rangle = \frac{1}{2}\,|{}^{11}_{33}\rangle - \frac{\sqrt{3}}{2}\,|{}^{12}_{23}\rangle$$

$$|[2^2];63\rangle = \sqrt{\frac{2}{55}}\,|{}^{11}_{25}\rangle + 3\sqrt{\frac{3}{55}}\,|{}^{11}_{34}\rangle + 2\sqrt{\frac{2}{55}}\,|{}^{12}_{24}\rangle + 3\sqrt{\frac{2}{55}}\,|{}^{12}_{33}\rangle$$

$$|[2^2](1);43\rangle = \frac{2}{\sqrt{11}}|{}^{11}_{25}\rangle + \frac{1}{4}\sqrt{\frac{6}{11}}\,|{}^{11}_{34}\rangle + \frac{5}{4}\,\frac{1}{\sqrt{11}}\,|{}^{12}_{24}\rangle - \frac{9}{4}\,\frac{1}{\sqrt{11}}\,|{}^{12}_{33}\rangle$$

$$|[2^2](2);43\rangle = \frac{1}{2}\sqrt{\frac{3}{2}}\,|{}^{11}_{34}\rangle - \frac{1}{4}\,|{}^{12}_{33}\rangle - \frac{3}{4}\,|{}^{12}_{24}\rangle$$

$$|[2^2](1);33\rangle = \frac{1}{\sqrt{20}}\,[-2\sqrt{3}\,|{}^{11}_{25}\rangle + \sqrt{2}\,|{}^{11}_{34}\rangle + \sqrt{3}\,|{}^{12}_{24}\rangle - \sqrt{3}\,|{}^{12}_{33}\rangle\,]$$

$$|[2^2];62\rangle = \frac{4}{\sqrt{165}}\,|{}^{11}_{35}\rangle + \sqrt{\frac{2}{55}}\,|{}^{12}_{25}\rangle + \frac{3}{\sqrt{55}}\,|{}^{11}_{44}\rangle + \sqrt{\frac{98}{165}}\,|{}^{12}_{34}\rangle +$$

$$+ \frac{2}{\sqrt{55}} \left|{22 \atop 33}\right> + \sqrt{\frac{2}{55}} \left|{13 \atop 24}\right>$$

$$|[2^2](1);42\rangle = \frac{5}{2}\sqrt{\frac{3}{77}} \left|{11 \atop 35}\right> + \frac{13}{4}\sqrt{\frac{2}{77}} \left|{12 \atop 25}\right> + \frac{3}{2\sqrt{77}} \left|{11 \atop 44}\right>$$

$$- \frac{9}{8}\sqrt{\frac{6}{77}} \left|{12 \atop 34}\right> + \frac{15}{4\sqrt{154}} \left|{13 \atop 24}\right> - \frac{18}{4\sqrt{77}} \left|{22 \atop 33}\right>$$

$$|[2^2](2);42\rangle = \frac{1}{2}\sqrt{\frac{3}{7}} \left|{11 \atop 35}\right> + \frac{3}{2\sqrt{7}} \left|{11 \atop 44}\right> - \frac{1}{4}\sqrt{\frac{3}{14}} \left|{12 \atop 34}\right>$$

$$- \frac{1}{2\sqrt{7}} \left|{22 \atop 33}\right> - \frac{3}{4}\sqrt{\frac{2}{7}} \left|{12 \atop 25}\right> - \frac{9}{4\sqrt{114}} \left|{13 \atop 24}\right>$$

$$|[2^2](1);32\rangle = \frac{-2}{\sqrt{15}} \left|{11 \atop 35}\right> - \sqrt{\frac{1}{10}} \left|{12 \atop 25}\right> + \frac{1}{\sqrt{5}} \left|{11 \atop 44}\right> + \frac{1}{2\sqrt{30}} \left|{12 \atop 34}\right>$$

$$+ \frac{3}{\sqrt{40}} \left|{13 \atop 24}\right> - \frac{1}{\sqrt{5}} \left|{22 \atop 33}\right>$$

$$|[2^2](1);22\rangle = \frac{1}{\sqrt{21}} \left[\sqrt{6} \left|{11 \atop 35}\right> - 3 \left|{12 \atop 25}\right> - \sqrt{2} \left|{11 \atop 44}\right> + 2 \left|{13 \atop 24}\right> \right]$$

$$|[2^2](2);22\rangle = \frac{1}{\sqrt{84}} \left[4\left|{11 \atop 44}\right> - 2\sqrt{6} \left|{12 \atop 34}\right> + 6 \left|{22 \atop 33}\right> + 2\sqrt{2} \left|{13 \atop 24}\right> \right]$$

$$|[2^2]61\rangle = \sqrt{\frac{25}{132}} \left|{12 \atop 35}\right> + \frac{1}{\sqrt{11}} \left|{11 \atop 45}\right> + \frac{1}{\sqrt{44}} \left|{13 \atop 25}\right> + \sqrt{\frac{20}{25}} \left|{12 \atop 44}\right>$$

$$+ \frac{1}{3}\sqrt{\frac{24}{11}} \left|{22 \atop 34}\right> + \frac{1}{\sqrt{11}} \left|{13 \atop 34}\right>$$

$$|[2^2](1);41\rangle = 2\sqrt{\frac{3}{77}} \left|{12 \atop 35}\right> + \frac{7}{2\sqrt{77}} \left|{11 \atop 45}\right> + \frac{18}{4\sqrt{77}} \left|{13 \atop 25}\right> - \frac{5}{4\sqrt{77}} \left|{12 \atop 44}\right>$$

$$-\frac{9}{4}\sqrt{\frac{6}{77}}\,|{}^{22}_{34}\rangle + \frac{6}{8\sqrt{77}}\,|{}^{13}_{34}\rangle$$

$$|[2^2](2)41\rangle = \frac{3}{2}\sqrt{\frac{1}{7}}\,|{}^{11}_{45}\rangle + \frac{3}{4\sqrt{7}}\,|{}^{12}_{44}\rangle - \frac{1}{2}\sqrt{\frac{3}{14}}\,|{}^{22}_{34}\rangle$$

$$- \frac{6}{4\sqrt{7}}\,|{}^{13}_{25}\rangle - \frac{10}{4\sqrt{28}}\,|{}^{13}_{34}\rangle$$

$$|[2^2]31\rangle = -\frac{1}{\sqrt{3}}\,|{}^{12}_{35}\rangle + \frac{1}{2}\,|{}^{12}_{44}\rangle - \frac{1}{\sqrt{6}}\,|{}^{22}_{34}\rangle + \frac{1}{2}\,|{}^{13}_{34}\rangle$$

$$|[2^2](1)21\rangle = \frac{1}{2\sqrt{7}}\,|{}^{12}_{35}\rangle + \sqrt{\frac{1}{21}}\,|{}^{11}_{45}\rangle - \frac{5}{2\sqrt{21}}\,|{}^{13}_{25}\rangle$$

$$- \frac{2}{\sqrt{21}}\,|{}^{12}_{44}\rangle + \sqrt{\frac{3}{7}}\,|{}^{13}_{34}\rangle$$

$$|[2^2](2)21\rangle = -\frac{12}{\sqrt{42}}\,|{}^{12}_{44}\rangle + \sqrt{\frac{8}{21}}\,|{}^{11}_{45}\rangle + \frac{1}{\sqrt{7}}\,|{}^{22}_{34}\rangle$$

$$- \sqrt{\frac{2}{7}}\,|{}^{12}_{35}\rangle + \sqrt{\frac{2}{21}}\,|{}^{13}_{25}\rangle$$

$$|[2^2]60\rangle = \sqrt{\frac{9}{77}}\,|{}^{22}_{35}\rangle + \frac{12}{\sqrt{1848}}\,|{}^{13}_{35}\rangle + \frac{17}{2\sqrt{231}}\,|{}^{12}_{45}\rangle + \frac{2}{\sqrt{231}}\,|{}^{11}_{55}\rangle$$

$$+ \frac{1}{\sqrt{308}}\,|{}^{14}_{25}\rangle + \frac{8}{\sqrt{231}}\,|{}^{22}_{44}\rangle + \frac{3}{\sqrt{77}}\,|{}^{13}_{44}\rangle + 3\sqrt{\frac{2}{231}}\,|{}^{23}_{34}\rangle$$

$$|[2^2](1)40\rangle = -\frac{1}{2}\sqrt{\frac{3}{770}}\,|{}^{22}_{35}\rangle + \frac{21}{\sqrt{1540}}\,|{}^{13}_{35}\rangle + \frac{42}{4\sqrt{770}}\,|{}^{12}_{45}\rangle$$

$$+ \frac{7}{\sqrt{770}}\,|{}^{11}_{55}\rangle + \frac{18}{4}\sqrt{\frac{3}{770}}\,|{}^{14}_{25}\rangle - \frac{64}{4\sqrt{770}}\,|{}^{22}_{44}\rangle - 6\sqrt{\frac{2}{770}}\,|{}^{23}_{34}\rangle$$

$$- \frac{1}{2}\sqrt{\frac{3}{770}}\left|{}^{13}_{44}\right>$$

$$|[2^2](2\ )40> = \frac{9}{2\sqrt{70}}\left|{}^{12}_{45}\right> + \frac{3}{\sqrt{70}}\left|{}^{11}_{55}\right> - \frac{1}{2}\sqrt{\frac{3}{70}}\left|{}^{13}_{44}\right>$$

$$- \frac{4}{\sqrt{140}}\left|{}^{23}_{34}\right> - \frac{1}{2}\sqrt{\frac{3}{70}}\left|{}^{22}_{35}\right> - \frac{7}{\sqrt{140}}\left|{}^{13}_{35}\right> - \frac{3}{2}\sqrt{\frac{3}{70}}\left|{}^{14}_{25}\right>$$

$$|[2^2]30> = - \frac{1}{\sqrt{2}}\left|{}^{22}_{35}\right> + \frac{1}{\sqrt{2}}\left|{}^{13}_{44}\right>$$

$$|\ [2^2](1)20> = \frac{1}{\sqrt{21}}\left|{}^{22}_{35}\right> - \frac{1}{2\sqrt{63}}\left|{}^{12}_{45}\right> + \frac{2}{\sqrt{63}}\left|{}^{11}_{55}\right>$$

$$- \frac{5}{2\sqrt{21}}\left|{}^{14}_{25}\right> + \frac{1}{\sqrt{21}}\left|{}^{13}_{44}\right> - \frac{4}{\sqrt{63}}\left|{}^{22}_{44}\right> + \sqrt{\frac{6}{21}}\left|{}^{23}_{34}\right>$$

$$|[2^2](2)20> = \frac{2}{\sqrt{126}}\left|{}^{22}_{44}\right> - \frac{2}{\sqrt{42}}\left|{}^{13}_{44}\right> - \frac{2}{\sqrt{126}}\left|{}^{12}_{45}\right> + 2\sqrt{\frac{8}{63}}\left|{}^{11}_{55}\right>$$

$$+ \frac{1}{\sqrt{7}}\left|{}^{23}_{34}\right> - \sqrt{\frac{2}{21}}\left|{}^{22}_{35}\right> + \sqrt{\frac{2}{21}}\left|{}^{14}_{25}\right>.$$

$$|[2^2](1)00> = \sqrt{\frac{8}{35}}\left|{}^{22}_{35}\right> - \sqrt{\frac{6}{70}}\left|{}^{13}_{35}\right> - \sqrt{\frac{5}{42}}\left|{}^{12}_{45}\right>$$

$$+ \sqrt{\frac{5}{42}}\left|{}^{11}_{55}\right> + \sqrt{\frac{3}{70}}\left|{}^{14}_{25}\right> - \frac{1}{\sqrt{210}}\left|{}^{22}_{44}\right> + \sqrt{\frac{8}{35}}\left|{}^{13}_{44}\right> - \sqrt{\frac{6}{70}}\left|{}^{23}_{34}\right>$$

$$|[2^2](2)00> = - \frac{1}{\sqrt{5}}\left|{}^{13}_{35}\right> + \frac{1}{\sqrt{10}}\left|{}^{12}_{45}\right> - \frac{1}{\sqrt{10}}\left|{}^{11}_{55}\right>$$

$$+ \sqrt{\frac{3}{10}}\left|{}^{14}_{25}\right> - \frac{1}{\sqrt{10}}\left|{}^{22}_{44}\right> + \sqrt{\frac{1}{5}}\left|{}^{13}_{44}\right>$$

For the partition $[21^2]$, we find

$$|[21^2]55\rangle = \left|\begin{matrix}1\,1\\2\\3\end{matrix}\right\rangle$$

$$|[21^2]54\rangle = \sqrt{\tfrac{2}{5}}\left|\begin{matrix}1\,2\\2\\3\end{matrix}\right\rangle + \sqrt{\tfrac{3}{5}}\left|\begin{matrix}1\,1\\2\\4\end{matrix}\right\rangle$$

$$|[21^2]44\rangle = \sqrt{\tfrac{3}{5}}\left|\begin{matrix}1\,2\\2\\3\end{matrix}\right\rangle - \sqrt{\tfrac{2}{5}}\left|\begin{matrix}1\,1\\2\\4\end{matrix}\right\rangle$$

$$|[21^2]53\rangle = 2\sqrt{\tfrac{2}{15}}\left|\begin{matrix}1\,2\\2\\4\end{matrix}\right\rangle + \sqrt{\tfrac{2}{15}}\left|\begin{matrix}1\,3\\2\\3\end{matrix}\right\rangle + \frac{1}{\sqrt{5}}\left|\begin{matrix}1\,1\\3\\4\end{matrix}\right\rangle + \sqrt{\tfrac{2}{15}}\left|\begin{matrix}1\,1\\2\\5\end{matrix}\right\rangle$$

$$|[21^2]43\rangle = \sqrt{\tfrac{1}{20}}\left|\begin{matrix}1\,2\\2\\4\end{matrix}\right\rangle + \frac{3}{\sqrt{20}}\left|\begin{matrix}1\,3\\2\\3\end{matrix}\right\rangle - \sqrt{\tfrac{3}{10}}\left|\begin{matrix}1\,1\\3\\4\end{matrix}\right\rangle - \sqrt{\tfrac{1}{5}}\left|\begin{matrix}1\,1\\2\\5\end{matrix}\right\rangle$$

$$|[21^2](1)33\rangle = \sqrt{\tfrac{5}{12}}\left|\begin{matrix}1\,2\\2\\4\end{matrix}\right\rangle - \sqrt{\tfrac{5}{12}}\left|\begin{matrix}1\,3\\2\\3\end{matrix}\right\rangle - \frac{1}{\sqrt{10}}\left|\begin{matrix}1\,1\\3\\4\end{matrix}\right\rangle - \frac{1}{\sqrt{15}}\left|\begin{matrix}1\,1\\2\\5\end{matrix}\right\rangle$$

$$|[21^2](2\ )33\rangle = \sqrt{\tfrac{2}{5}}\left|\begin{matrix}1\,1\\3\\4\end{matrix}\right\rangle - \sqrt{\tfrac{3}{5}}\left|\begin{matrix}1\,1\\2\\5\end{matrix}\right\rangle$$

$$|[21^2]52\rangle = \frac{1}{\sqrt{5}}\left|\begin{matrix}1\,2\\2\\5\end{matrix}\right\rangle + \sqrt{\tfrac{16}{45}}\left|\begin{matrix}1\,3\\2\\4\end{matrix}\right\rangle + \sqrt{\tfrac{4}{15}}\left|\begin{matrix}1\,2\\3\\4\end{matrix}\right\rangle$$

$$+ \sqrt{\tfrac{2}{45}}\left|\begin{matrix}1\,4\\2\\3\end{matrix}\right\rangle + \sqrt{\tfrac{2}{15}}\left|\begin{matrix}1\,1\\3\\5\end{matrix}\right\rangle$$

$$|[21^2]42\rangle = -\frac{1}{\sqrt{70}}\left|\begin{matrix}1\,2\\2\\5\end{matrix}\right\rangle + \frac{9}{\sqrt{280}}\left|\begin{matrix}1\,3\\2\\4\end{matrix}\right\rangle - 3\sqrt{\tfrac{3}{280}}\left|\begin{matrix}1\,2\\3\\4\end{matrix}\right\rangle$$

$$+ \frac{6}{\sqrt{140}}\left|\begin{matrix}1\,4\\2\\3\end{matrix}\right\rangle - 2\sqrt{\tfrac{3}{35}}\left|\begin{matrix}1\,1\\3\\5\end{matrix}\right\rangle$$

$$|[21^2](1)32\rangle = \frac{1}{\sqrt{10}}\left|\begin{matrix}12\\2\\5\end{matrix}\right\rangle + \frac{1}{2}\sqrt{\frac{5}{18}}\left|\begin{matrix}13\\2\\4\end{matrix}\right\rangle + \frac{1}{\sqrt{120}}\left|\begin{matrix}12\\3\\4\end{matrix}\right\rangle$$

$$- \sqrt{\frac{5}{9}}\left|\begin{matrix}14\\2\\3\end{matrix}\right\rangle - \frac{2}{\sqrt{15}}\left|\begin{matrix}11\\3\\5\end{matrix}\right\rangle$$

$$|[21^2](2)32\rangle = -\frac{1}{\sqrt{15}}\left|\begin{matrix}11\\3\\5\end{matrix}\right\rangle + 2\sqrt{\frac{2}{15}}\left|\begin{matrix}12\\3\\4\end{matrix}\right\rangle - \sqrt{\frac{6}{15}}\left|\begin{matrix}12\\2\\5\end{matrix}\right\rangle$$

$$|[21^2](1)22\rangle = \frac{1}{\sqrt{14}}\left[2\left|\begin{matrix}12\\2\\5\end{matrix}\right\rangle - 2\left|\begin{matrix}13\\2\\4\end{matrix}\right\rangle + \frac{2}{\sqrt{3}}\left|\begin{matrix}12\\3\\4\end{matrix}\right\rangle\right.$$

$$\left. + \sqrt{2}\left|\begin{matrix}14\\2\\3\end{matrix}\right\rangle - \frac{2\sqrt{6}}{3}\left|\begin{matrix}11\\3\\5\end{matrix}\right\rangle\right]$$

$$|[21^2]51\rangle = \frac{17}{\sqrt{1260}}\left|\begin{matrix}13\\2\\5\end{matrix}\right\rangle + \frac{11}{\sqrt{420}}\left|\begin{matrix}12\\3\\5\end{matrix}\right\rangle + 3\sqrt{\frac{4}{315}}\left|\begin{matrix}14\\2\\4\end{matrix}\right\rangle + \frac{9}{\sqrt{315}}\left|\begin{matrix}13\\3\\4\end{matrix}\right\rangle$$

$$+ \frac{4}{\sqrt{210}}\left|\begin{matrix}22\\3\\4\end{matrix}\right\rangle + \frac{2}{\sqrt{630}}\left|\begin{matrix}15\\2\\3\end{matrix}\right\rangle + \sqrt{\frac{6}{210}}\left|\begin{matrix}11\\4\\5\end{matrix}\right\rangle$$

$$|[21^2]41\rangle = \frac{2}{\sqrt{140}}\left|\begin{matrix}13\\2\\5\end{matrix}\right\rangle - 4\sqrt{\frac{3}{140}}\left|\begin{matrix}12\\3\\5\end{matrix}\right\rangle + \frac{7}{\sqrt{140}}\left|\begin{matrix}14\\2\\4\end{matrix}\right\rangle + \frac{6}{\sqrt{560}}\left|\begin{matrix}13\\3\\4\end{matrix}\right\rangle$$

$$- 2\sqrt{\frac{3}{280}}\left|\begin{matrix}22\\3\\4\end{matrix}\right\rangle + \frac{2}{\sqrt{70}}\left|\begin{matrix}15\\2\\3\end{matrix}\right\rangle - \frac{2}{\sqrt{35}}\left|\begin{matrix}11\\4\\5\end{matrix}\right\rangle$$

$$|[21^2](1)31\rangle = \frac{7}{15}\left|\begin{matrix}13\\2\\5\end{matrix}\right\rangle - \frac{2}{\sqrt{75}}\left|\begin{matrix}12\\3\\5\end{matrix}\right\rangle - \frac{1}{2}\left|\begin{matrix}14\\2\\4\end{matrix}\right\rangle + \frac{3}{10}\left|\begin{matrix}13\\3\\4\end{matrix}\right\rangle$$

$$+ \frac{1}{\sqrt{150}}\left|\begin{matrix}22\\3\\4\end{matrix}\right\rangle - \frac{2}{9}\left|\begin{matrix}15\\2\\3\end{matrix}\right\rangle - \frac{2}{5}\left|\begin{matrix}11\\4\\5\end{matrix}\right\rangle$$

$$|[21^2](1)21\rangle = \frac{1}{\sqrt{14}}\left|\begin{matrix}13\\2\\5\end{matrix}\right\rangle + \frac{1}{\sqrt{42}}\left|\begin{matrix}12\\3\\5\end{matrix}\right\rangle - \frac{2}{\sqrt{14}}\left|\begin{matrix}13\\3\\4\end{matrix}\right\rangle + \frac{2}{\sqrt{21}}\left|\begin{matrix}22\\3\\4\end{matrix}\right\rangle$$

$$+\frac{1}{\sqrt{7}}\left|\begin{matrix}15\\2\\3\end{matrix}\right\rangle-\frac{2}{\sqrt{14}}\left|\begin{matrix}11\\4\\5\end{matrix}\right\rangle$$

$$|[21^2](1)11\rangle=\sqrt{\frac{13}{140}}\left|\begin{matrix}13\\2\\5\end{matrix}\right\rangle-\sqrt{\frac{39}{140}}\left|\begin{matrix}12\\3\\5\end{matrix}\right\rangle-\frac{2}{\sqrt{455}}\left|\begin{matrix}14\\2\\4\end{matrix}\right\rangle-\frac{6}{\sqrt{1820}}\left|\begin{matrix}13\\3\\4\end{matrix}\right\rangle$$

$$+8\sqrt{\frac{6}{1820}}\left|\begin{matrix}22\\3\\4\end{matrix}\right\rangle+\frac{4}{\sqrt{910}}\left|\begin{matrix}15\\2\\3\end{matrix}\right\rangle+\sqrt{\frac{26}{70}}\left|\begin{matrix}11\\4\\5\end{matrix}\right\rangle$$

$$|[21^2](2)11\rangle=\frac{1}{\sqrt{350}}\Big[-\left|\begin{matrix}13\\2\\5\end{matrix}\right\rangle+\sqrt{3}\left|\begin{matrix}12\\3\\5\end{matrix}\right\rangle+10\left|\begin{matrix}14\\2\\4\end{matrix}\right\rangle-6\left|\begin{matrix}13\\3\\4\end{matrix}\right\rangle$$

$$+\sqrt{6}\left|\begin{matrix}22\\3\\4\end{matrix}\right\rangle-10\sqrt{2}\left|\begin{matrix}15\\2\\3\end{matrix}\right\rangle-2\cdot\left|\begin{matrix}11\\4\\5\end{matrix}\right\rangle\Big]$$

$$|[21^2]50\rangle=\frac{25}{\sqrt{6300}}\left|\begin{matrix}14\\2\\5\end{matrix}\right\rangle+\sqrt{\frac{120}{315}}\left|\begin{matrix}13\\3\\5\end{matrix}\right\rangle+\frac{1}{\sqrt{7}}\left|\begin{matrix}22\\3\\5\end{matrix}\right\rangle+\sqrt{\frac{15}{140}}\left|\begin{matrix}12\\4\\5\end{matrix}\right\rangle$$

$$+\sqrt{\frac{10}{315}}\left|\begin{matrix}15\\2\\4\end{matrix}\right\rangle+\frac{1}{\sqrt{7}}\left|\begin{matrix}14\\3\\4\end{matrix}\right\rangle+\sqrt{\frac{2}{21}}\left|\begin{matrix}23\\5\\4\end{matrix}\right\rangle$$

$$|[21^2]40\rangle=\frac{1}{3}\sqrt{\frac{6}{7}}\left|\begin{matrix}14\\2\\5\end{matrix}\right\rangle-\sqrt{\frac{2}{7}}\left|\begin{matrix}12\\4\\5\end{matrix}\right\rangle-\sqrt{\frac{3}{14}}\left|\begin{matrix}22\\3\\5\end{matrix}\right\rangle$$

$$+\frac{1}{3}\sqrt{\frac{12}{7}}\left|\begin{matrix}15\\2\\4\end{matrix}\right\rangle-\sqrt{\frac{3}{14}}\left|\begin{matrix}14\\3\\4\end{matrix}\right\rangle$$

$$|[21^2](1)30\rangle=\frac{4}{4\sqrt{3}}\left|\begin{matrix}13\\3\\5\end{matrix}\right\rangle+\frac{2}{15\sqrt{2}}\left|\begin{matrix}14\\2\\5\end{matrix}\right\rangle-\frac{\sqrt{6}}{5}\left|\begin{matrix}12\\4\\5\end{matrix}\right\rangle-\frac{1}{5\sqrt{2}}\left|\begin{matrix}22\\3\\5\end{matrix}\right\rangle$$

$$-\frac{2}{3}\left|\begin{matrix}15\\2\\4\end{matrix}\right\rangle-\frac{1}{5\sqrt{2}}\left|\begin{matrix}14\\3\\4\end{matrix}\right\rangle+\frac{2}{5\sqrt{3}}\left|\begin{matrix}22\\3\\4\end{matrix}\right\rangle$$

$$|[21^2](2)30\rangle = -\frac{\sqrt{3}}{5}\left|\begin{smallmatrix}13\\3\\5\end{smallmatrix}\right\rangle - \frac{3}{5\sqrt{2}}\left|\begin{smallmatrix}14\\2\\5\end{smallmatrix}\right\rangle - \frac{1}{5}\sqrt{\frac{3}{2}}\left|\begin{smallmatrix}12\\4\\5\end{smallmatrix}\right\rangle + \frac{\sqrt{2}}{5}\left|\begin{smallmatrix}22\\3\\5\end{smallmatrix}\right\rangle$$

$$+ \frac{6}{5\sqrt{3}}\left|\begin{smallmatrix}23\\3\\4\end{smallmatrix}\right\rangle + \frac{2}{5\sqrt{2}}\left|\begin{smallmatrix}14\\3\\4\end{smallmatrix}\right\rangle$$

$$|[21^2](1)10\rangle = -6\sqrt{\frac{2}{1820}}\left|\begin{smallmatrix}13\\3\\5\end{smallmatrix}\right\rangle + \frac{31}{3}\sqrt{\frac{3}{1820}}\left|\begin{smallmatrix}14\\2\\5\end{smallmatrix}\right\rangle$$

$$+ \sqrt{\frac{13}{140}}\left|\begin{smallmatrix}12\\4\\5\end{smallmatrix}\right\rangle - 10\sqrt{\frac{3}{1820}}\left|\begin{smallmatrix}22\\3\\5\end{smallmatrix}\right\rangle + \frac{4}{3}\sqrt{\frac{6}{1280}}\left|\begin{smallmatrix}15\\2\\4\end{smallmatrix}\right\rangle$$

$$- 10\sqrt{\frac{3}{1820}}\left|\begin{smallmatrix}14\\3\\4\end{smallmatrix}\right\rangle + 18\sqrt{\frac{2}{180}}\left|\begin{smallmatrix}23\\3\\4\end{smallmatrix}\right\rangle$$

$$|[21^2](2)10\rangle = -6\sqrt{\frac{2}{350}}\left|\begin{smallmatrix}13\\3\\5\end{smallmatrix}\right\rangle + \frac{17\sqrt{3}}{3\sqrt{350}}\left|\begin{smallmatrix}14\\2\\5\end{smallmatrix}\right\rangle - \frac{1}{\sqrt{350}}\left|\begin{smallmatrix}12\\4\\5\end{smallmatrix}\right\rangle$$

$$+ 4\sqrt{\frac{3}{350}}\left|\begin{smallmatrix}22\\3\\5\end{smallmatrix}\right\rangle - \frac{10}{3}\sqrt{\frac{6}{350}}\left|\begin{smallmatrix}15\\2\\4\end{smallmatrix}\right\rangle + 4\sqrt{\frac{3}{350}}\left|\begin{smallmatrix}14\\3\\4\end{smallmatrix}\right\rangle$$

$$- 3\sqrt{\frac{2}{350}}\left|\begin{smallmatrix}23\\3\\4\end{smallmatrix}\right\rangle$$

$$|[21^2](2)31\rangle = -\frac{1}{5}\left|\begin{smallmatrix}11\\4\\5\end{smallmatrix}\right\rangle - \frac{1}{\sqrt{75}}\left|\begin{smallmatrix}12\\3\\5\end{smallmatrix}\right\rangle + \frac{2}{5}\left|\begin{smallmatrix}13\\3\\4\end{smallmatrix}\right\rangle \; 4\sqrt{\frac{2}{75}}\left|\begin{smallmatrix}22\\3\\4\end{smallmatrix}\right\rangle - \frac{3}{5}\left|\begin{smallmatrix}13\\2\\5\end{smallmatrix}\right\rangle$$

The B_2 spin 1 system has been analyzed previously [11]. The terms which occur for a B_2 spin 1 system are as follows.

$$|[2]22\rangle = |66\rangle$$

$$|[2]21\rangle = |67\rangle$$

$$|[2]20\rangle = \frac{1}{\sqrt{3}}\,|68\rangle + \sqrt{\frac{2}{3}}\,|77\rangle$$

$$|[2]2{-1}\rangle = |78\rangle$$

$$|[2]2{-2}\rangle = |88\rangle$$

$$|[2]00\rangle = \sqrt{\frac{2}{3}}\,|68\rangle - \frac{1}{\sqrt{3}}\,|77\rangle$$

$$|[1^2]11\rangle = \left|\begin{smallmatrix}6\\7\end{smallmatrix}\right\rangle$$

$$|[1^2]10\rangle = \left|\begin{smallmatrix}6\\8\end{smallmatrix}\right\rangle$$

$$|[1^2]1{-1}\rangle = \left|\begin{smallmatrix}7\\8\end{smallmatrix}\right\rangle$$

Here we have used 6, 7, 8 for $m_B = 1, 0, -1$ respectively, in order to avoid confusion with the labeling of the A_4 system.

To obtain the states $|[n_A]I_A, [n_B]I_B, IM\rangle$ of the A_4B_2 system, we can vector couple the A_4 SO(3) states to the B_2 SO(3) states. For example, for the states $|[21^2]5, [2]2, 7M\rangle$ we obtain

$|[21^2]5,\ [2]2,77\rangle = |[21^2]55\rangle \times |[2]22\rangle$

$|[21^2]5,\ [2]2,76\rangle = \sqrt{\frac{5}{7}}\ |[21^2]54\rangle \times |[2]22\rangle$

$\quad + \sqrt{\frac{2}{7}}\ |[21^2]55\rangle \times |[2]21\rangle$

$|[21^2]5,\ [2]2,75\rangle = \sqrt{\frac{6}{91}}\ |[21^2]55\rangle \times |[2]20\rangle + \sqrt{\frac{40}{91}}\ |[21^2]54\rangle \times$

$\quad |[2]21\rangle + \sqrt{\frac{45}{91}}\ |[21^2]53\rangle \times |[2]22\rangle$

$|[21^2]5,\ [2]2,74\rangle = \frac{1}{\sqrt{91}}\ |[21^2]55\rangle \times |[2]2{-1}\rangle$

$\quad + \sqrt{\frac{15}{91}}|[21^2]54\rangle \times |[2]20\rangle + \sqrt{\frac{45}{91}}|[21^2]53\rangle \times |[2]21\rangle$

$\quad + \sqrt{\frac{30}{91}}\ [21^2]52\rangle \times |[2]22\rangle$

$|[21^2]5,[2]2,73\rangle = \frac{1}{\sqrt{1001}}\ |[21^2]55\rangle \times |[2]2{-2}\rangle$

$\quad + \sqrt{\frac{40}{1001}}\ |[21^2]54\rangle \times |[2]2{-1}\rangle$

$\quad + \sqrt{\frac{270}{1001}}\ |[21^2]53\rangle \times |[2]20\rangle$

$\quad + \sqrt{\frac{480}{1001}}\ |[21^2]52\rangle \times |[2]21\rangle$

$$+ \sqrt{\frac{210}{1001}}\,|[21^2]51\rangle \times |[2]22\rangle$$

$$|[21^2]5,[2]2,72\rangle = \sqrt{\frac{5}{1001}}\,|[21^2]54\rangle \times |[2]2-2\rangle$$

$$+ \sqrt{\frac{90}{1001}}\,|[21^2]53\rangle \times |[2]2-1\rangle + \sqrt{\frac{360}{1001}}\,|[21^2]52\rangle \times |[2]20\rangle$$

$$+ \sqrt{\frac{420}{1001}}\,|[21^2]51\rangle \times |[2]21\rangle + \sqrt{\frac{126}{1001}}\,|[21^2]\,50\rangle \times |[2]2-2\rangle$$

$$|[21^2]5,[2]2,71\rangle = \sqrt{\frac{15}{1001}}\,|[21^2]53\rangle \times |[2]2-2\rangle$$

$$+ \sqrt{\frac{160}{1001}}\,|[21^2]52\rangle \times |[2]-1\rangle + \sqrt{\frac{420}{1001}}\,|[21^2]51\rangle \times |[2]20\rangle$$

$$+ \sqrt{\frac{336}{1001}}\,|[21^2]50\rangle \times |[2]21\rangle + \sqrt{\frac{70}{1001}}\,|[21^2]5-1\rangle \times |[2]22\rangle$$

$$|[21^2]5,[2]2,70\rangle = \sqrt{\frac{35}{1001}}\,|[21^2]52\rangle \times |[2]2-2\rangle$$

$$+ \sqrt{\frac{245}{1001}}\,|[21^2]51\rangle \times |[2]2-1\rangle + \sqrt{\frac{441}{1001}}\,|[21^2]50\rangle \times |[2]20\rangle$$

$$+ \sqrt{\frac{245}{1001}}\,|[21^2]5-1\rangle \times |[2]21\rangle + \sqrt{\frac{35}{1001}}\,|[21^2]5-2\rangle \times |[2]22\rangle$$

Although there is no difficulty with the vector coupling approach for systems with three or four species, it is difficult to extend to more than four species.

An alternative is to list directly the tensor product states of the entire A_4B_2 system. Lowering of states in this method is achieved by operating with $I- = I-(A) + I-(B)$. This method, however, appears not to be satisfactory since the projection operator Q_M fails to resolve states that are properly labeled by the total spin I. The underlying cause of this is that a given state $|[\eta_A]> \otimes |[\eta_B]>$ is a basis vector of the tensor product representation $\Gamma^{[\eta_A]} \otimes \Gamma^{[\eta_B]}$ and such a representation is, in general, reducible. The basis vector we obtained by applying the Q_M projection operator will in general be a linear combination of properly labeled states.

This problem can be circumvented by the application of a spin eigenfunction projection operator proposed originally by Lowdin [18,19] to the tensor product states. This operator has the following form

$$P_I = \Pi_{k \neq I}(I^2 - k(k+1)).$$

This operator annihilates the contribution from all eigenfunctions except the desired I state. The extra quantum label we will demand to specify uniquely the tensor product states in terms of conventionally labeled states will be the intermediate value of the spins. The general spin projection operator will thus be expressed as

$$P_{I_1 \cdots I_T I_{12} \cdots I_{1 \cdots T}} = P_{I_{1 \cdots T}} \cdot P_{I_{1 \cdots T-1}} \cdots P_{I_{12}} P_{I_T} P_{I_{T-1}} \cdots P_{I_1} .$$

Here $I_{1 \cdots T}$ denotes the intermediate valued spin obtained when coupling $I_1, I_2, \ldots, I_T$. This operator ensures that the only surviving contribution from a given tensor product state has the desired intermediate and total spin value.

The use of this operator has incidental computational advantages [17]. The resulting states need to be normalized, as in the technique using the Q_M projection operator. It is possible that the application of P to a tensor product state results in a zero projection. This simply means that the desired state does not contribute to the chosen state. In terms of representations, this means that the decomposition of the tensor product representation does not contain the irreducible representation we seek. For systems with two species, this operator takes a simple form since there is no need to specify intermediate spin values. In order to illustrate this technique we first list the tensor product states belonging to [4] $\otimes$ [2] for the A_4B_2 system in Table X. In this table we have suppressed the tensor product symbol $\otimes$ to avoid cumbersome notation.

At the highest level M = 7, we have only one state, and only one possible combination of I_A and I_B

$$|5277\rangle = \left|\begin{smallmatrix}11\\2\\3\end{smallmatrix}\right\rangle |66\rangle \quad .$$

Here we suppressed partition labels, and have denoted states by $|I_A I_B I_M\rangle$. Note also that we have suppressed the tensor product symbol, and will henceforth do so for all states. Lowering the state $|5277\rangle$ will yield a state $|5276\rangle$ at the M = 6 level. We also have two new I = 6 states at this level. The possible I_A, I_B values for I = 6 can easily be determined from vector coupling considerations. These are $I_A = 5$, $I_B = 2$, $I_A = 5$, $I_B = 1$, and $I_A = 4$, $I_B = 2$. Since there are only two new I = 6 states, only two of these three combinations will contribute. Choosing $I_A = 5$, $I_B = 2$, and $I_A = 4$, $I_B = 2$, the relevant projection operators are:

$$P_{I_A = 5} = (I_A^2 - 20)$$

$$P_{I_A = 4} = (I_A^2 - 30)$$

$$P_{I_B = 2} = 1$$

$$P_{I = 6} = (I^2 - 56) .$$

The overall projection operators for the states $|5266\rangle$ and $|4266\rangle$ are then, respectively,

$$P_{526} = (I^2 - 56)(I_A^2 - 20)$$

$$P_{426} = (I^2 - 56)(I_A^2 - 30).$$

The operators I^2 and I_A^2 are calculated using the relations $I_A^2 = I_{A+}I_{A-} + I_{A-}I_{A+} + I_A$ and $I^2 = I^2_A + I_B^2 + 2I_AI_B$. This gives rise to the following expressions:

$$I_A^2\left|\begin{matrix}12\\2\\3\end{matrix}\right\rangle|66\rangle = 24\left|\begin{matrix}12\\2\\3\end{matrix}\right\rangle|66\rangle + 2\sqrt{6}\left|\begin{matrix}11\\2\\4\end{matrix}\right\rangle|66\rangle$$

$$2I_AI_B\left|\begin{matrix}12\\2\\3\end{matrix}\right\rangle|66\rangle = 16\left|\begin{matrix}12\\2\\3\end{matrix}\right\rangle|66\rangle - 4\left|\begin{matrix}11\\2\\3\end{matrix}\right\rangle 66\rangle$$

$$I_A^2\left|\begin{matrix}11\\2\\4\end{matrix}\right\rangle|66\rangle = 26\left|\begin{matrix}11\\2\\4\end{matrix}\right\rangle|66\rangle + 2\sqrt{6}\left|\begin{matrix}12\\2\\3\end{matrix}\right\rangle|66\rangle$$

$$2I_AI_B\left|\begin{matrix}11\\2\\4\end{matrix}\right\rangle|66\rangle = 16\left|\begin{matrix}11\\2\\4\end{matrix}\right\rangle|66\rangle - 2\sqrt{6}\left|\begin{matrix}11\\2\\3\end{matrix}\right\rangle|67\rangle$$

This leads to the following state, after normalization

$$|4266\rangle = \frac{1}{\sqrt{15}}\left[3\left|\begin{matrix}12\\2\\3\end{matrix}\right\rangle|66\rangle - \sqrt{6}\left|\begin{matrix}11\\2\\4\end{matrix}\right\rangle|66\rangle\right]$$

This result can be checked by computing $\langle 4266|5276\rangle$. We have previously obtained $|5276\rangle$ through vector coupling. Observe that

$$|5276\rangle = \sqrt{\frac{2}{7}}\left|\begin{matrix}12\\2\\3\end{matrix}\right\rangle|66\rangle + \sqrt{\frac{3}{7}}\left|\begin{matrix}11\\2\\4\end{matrix}\right\rangle|66\rangle + \sqrt{\frac{2}{7}}\left|\begin{matrix}11\\2\\3\end{matrix}\right\rangle|67\rangle$$

Therefore, we have

$$\langle 4266|5276\rangle = \frac{1}{\sqrt{15}\sqrt{7}}\left[3\sqrt{2} - \sqrt{18}\right] = 0.$$

Similarly, we obtain

$$|5266\rangle = \sqrt{\frac{6}{43}}\left|\begin{matrix}11\\2\\4\end{matrix}\right\rangle|66\rangle - \sqrt{\frac{25}{43}}\left|\begin{matrix}11\\2\\3\end{matrix}\right\rangle|67\rangle + \sqrt{\frac{4}{43}}\left|\begin{matrix}12\\2\\3\end{matrix}\right\rangle|66\rangle$$

The power of this technique is even more evident when configurations with more than two species are considered [17].

CONCLUSIONS

We have demonstrated in this thesis that the Unitary Group Approach is quite useful even when non-fermionic systems are considered. Many of the features which render the Unitary Group Approach to atomic physics less complex are lacking in the consideration of non-fermionic systems. Despite this, the unique labeling of states and the ease with which generalization from coupling within a system can be extended to coupling between systems stress the power of the Unitary Group Approach.

Another significant point in favour of the Unitary Group Approach is the consistent manner in which the infinitesimal operators are defined. This is a reflection of the fact that we employ a natural embedding of groups. In comparsion, Racah labeling is based upon decompositions of groups into some esoteric subgroups, and has to be altered for different angular momenta states. The Racah scheme also forces one to use branching rules, which are completely unnecessary in the Unitary Group Approach.

In our treatment of A_3 and A_4B_2 systems we have clarified the structure of the basis being used to label states. The projection operator technique we have introduced in the treatment of mixed configurations is perfectly suitable for computer implementation.

Although we have calculated only the matrix elements of a spin-spin coupling Hamiltonian, any physical operator which has two body interactions in it [such as the quadrupole moment operator] can be treated in the same manner.

The algorithm for multistep operators [Appendix 2] should be of interest, particularly since it contains Harter's Jawbone formula within it, and it does not refer to intermediate tableaux. The closed form expression we have derived for two body operators [Appendix 3] is also worth pursuing in greater detail for it appears that simplifications are possible for fermionic systems. An interesting point to note in conjunction with this discussion is that the matrix elements of the operators E_{i+1i} have a much less complicated form when expressed in an orthogonal basis which is related to the Gelfand basis [5]. If the appropriate normalization constants could be determined in a fairly simple manner, then this would mean that steps could be eliminated from Harter's Jawbone formula.

There is still further work required for mixed configurations. It would be desirable to use irreducible representations of $U(2\sigma_A + 1 + 2\sigma_B + 1)$ for an $A_{n_A}B_{n_B}$ configuration. If states could be classified according to such a scheme, the corresponding Hamiltonian would include interaction between species.

APPENDIX 1

NOTATION

$U(N)$	-	Unitary group
$L(U(N))$	-	Lie algebra of $U(N)$
$GL(N,\mathbb{C})$	-	General linear group over $\mathbb{C}$.
$L(GL(N,\mathbb{C}))$	-	Lie algebra of $GL(N,\mathbb{C})$.
$d\Gamma$	-	Differential representation of a Lie algebra associated with a Lie group.
$\Gamma\vert_{G_o}$	-	If G_o is a subgroup of G and Γ is a representation of G, the $\Gamma\vert_{G_o}$ denotes the representation Γ restricted to the subgroup G_o.
(m)	-	Gelfand tableau.
$\vert(m)\rangle$	-	Basis vector associated with (m).
V(m)	-	Vector space associated with the Gelfand tableau (m).
S(k)	-	Symmetric group.
$[\lambda]$	-	Young graph.
$\Gamma[\lambda]$	-	Irreducible representation of S(k) associated with $[\lambda]$.
$\otimes$	-	Tensor product.
$[\eta]$	-	Weyl graph.
$\vert[\eta]\rangle$	-	Basic vector associated with $[\eta]$.
$H_\omega(\sigma)$	-	State space of a single particle with spin σ.
$\otimes_{\omega=1}^{N} H_\omega(\sigma)$	-	State space of N identical particle each with spin σ.

$\Gamma^{[n]}$ - Irreducible representation of U(N) associated with the Weyl graph [n]. Each Weyl graph also has an irreducible representation of S(k) associated with it.

$i_q^k(\omega), I_q^k(\omega)$ - Single-particle nuclear spin tensor of rank 2σ.

I_q^k - Many-particle nuclear spins tensor of rank 2σ.

E_{ij} - Generators of GL(N,₵) or a faithful representation of generators of GL(N,₵).

SO(N) - Rotation group in N-dimension.

All tensors in this thesis are spherical tensors. When defining tensors such as $i_q^k(\omega)$ we have written them as linear maps from one Hilbert space to another [for example, $i_q^k(\omega):H_\omega(\sigma) \to H_\omega(\sigma)$], in the interest of brevity. This is not precisely true since tensor operators are defined on the space of all linear operators on the Hilbert space.

We used the Condon-Shortley phase convention in this thesis. All formulae used to evaluate 3-j symbols can be found in Condon and Odobasi.

APPENDIX 2

ALGORITHMS FOR MULTISTEP OPERATORS

We have the following equation for multistep operators E_{ij} (i<j) [7]:

$$E_{ij}|(m)> = \sum_{p=i}^{j-1} \sum_{k_p=1}^{P} \left[\frac{(-) \prod_{s=1}^{j} (P_{sj} - P_{k_{j-1}j-1} - 1)}{\prod_{\substack{s=1 \\ s \neq k_{j-1}}}^{j-1} (P_{sj-1} - P_{k_{j-1}j-1} - 1)} \right]^{1/2}$$

$$\times \prod_{q=i+1}^{j-1} \{S(k_{q-1} - k_q) \left[\prod_{\substack{s=1 \\ s \neq k_{q-1}}}^{q-1} \frac{(P_{sq-1} - P_{k_q q})}{(P_{sq-1} - P_{k_{q-1}q-1} - 1)} \right.$$

$$\left. \prod_{\substack{s=1 \\ s \neq k_q}}^{q} \frac{(P_{sq} - P_{k_{q-1}q-1} - 1)}{(P_{sq} - P_{k_q q})} \right]^{1/2} \}$$

$$\times \left[\frac{\prod_{s=1}^{i-1} (P_{si-1} - P_{k_i i})}{\prod_{\substack{s=1 \\ s \neq k_q}}^{i} (P_{si} - P_{k_i - i})} \right]^{1/2} |(m) + \varepsilon_{k_p}(P)>$$

Here $S(\alpha-\beta)$ is 1 for $\alpha \geq \beta$, -1 for $\alpha < \beta$, and $P_{st} = m_{st} + t - s$. This immediately gives [since $E_{ij} = E_{ji}^{+}$], for i<j,

$$E_{ji}|(m)\rangle = \sum_{P=1}^{j-1} \sum_{k_P=1}^{P} \left[\frac{(-)\prod_{s=1}^{j} (P_{sj} - P_{k_{j-1}j-1})}{\prod_{\substack{s=1 \\ s\neq k_{j-1}}}^{j-1} (P_{sj-1} - P_{k_{j-1}j-1})} \right]^{1/2}$$

$$\times \prod_{q=i+1}^{j-1} \{ S(k_{q-1} - k_q) \left[\prod_{\substack{s=1 \\ s\neq k_{q-1}}}^{q-1} \frac{(P_{sq-1} - P_{k_q q} +1)}{(P_{sq-1} - P_{k_{q-1}q-1})} \right.$$

$$\left. \prod_{\substack{s=1 \\ s\neq k_q}}^{q} \frac{(P_{sq} - P_{k_{q-1}q-1})}{(P_{sq} - P_{k_{q-1}q-1})} \right]^{1/2} \}$$

$$\times \left[\frac{\prod_{s=1}^{i-1} (P_{si-1} - P_{k_i i} + 1)}{\prod_{\substack{s=1 \\ s\neq k_i}}^{i} (P_{si} - P_{k_i i} + 1)} \right]^{1/2} |(m) - \varepsilon_{k_P}(P)\rangle$$

In particular, for $j = i+2$,

$$E_{i+2\ i}|(m)\rangle = \sum_{P=i}^{i+1} \sum_{k_P=1}^{P} \left[\frac{(-)\prod_{s=1}^{i+2} (P_{si+2} - P_{k_{i+1}i+1})}{\prod_{\substack{s=1 \\ s\neq k_{i+1}}}^{i+1} (P_{si+1} - P_{k_{i+1}i+1})} \right]^{1/2}$$

$$\times \left[\frac{\prod_{s=1}^{i-1} (P_{si-1} - P_{k_i i} + 1)}{\prod_{\substack{s=1 \\ s \neq k_i}}^{i} (P_{si} - P_{k_i i} + 1)} \right]^{1/2}$$

$$\times \; \{S(k_{i+1} - k_i) \left[\prod_{\substack{s=1 \\ s \neq k_i}}^{i} \frac{(P_{si} - P_{k_{i+1} i+1} + 1)}{(P_{si} - P_{k_i i})} \right.$$

$$\left. \prod_{\substack{s=1 \\ s \neq k_{i+1}}}^{i+1} \frac{(P_{si+1} - P_{k_i i})}{(P_{si+1} - P_{k_{i+1} i+1} + 1)} \right]^{1/2} \} \, |(m) - \varepsilon_{k_p}(P) .$$

This yields the following algorithm for matrix elements of E_{i+2i}:

(i) Draw all arrows from the left of the last box labeled i+1 in row k_{i+1} to the last box labeled i+2 in row 1 to row i+2. If there are no i+1's in row k_{i+1} and the highest label which occurs in row k_{i+1} is p where $p \in \{1, 2, \ldots, i\}$, then draw all arrows from the left of the last box labeled P in row k_{i+1} to the last box labeled i+2 in row 1 to row 1+2. If there are no i+2's in a given row and q is the maximum label which appears in this row, then draw all arrows to the last box labeled q in this row. These "if" statements apply to all subsequent steps with appropriate modifications.

(ii) Draw all arrows from the last box labeled $i+1$ in row k_{i+1} to the last box labeled $i+1$ in row 1 to row $i+1$. [Exclude row k_{i+1}.]

(iii) Draw all arrows from the last box labeled i in row k_i to the last box labeled $i-1$ in row 1 to row $i-1$.

(iv) Draw all arrows from the left of the last box labeled i in row k_i to the last box labeled i in frow 1 to row i. [Exclude row k_i.]

(v) Draw all arrows from the last box labeled $i+1$ in the k_{i+1}'th row to the last box labeled i in row 1 to row i. [Exclude row k_i.]

(vi) Draw all arrows from the last box labeled i in the k_i'th row to the last box labeled i in row 1 to row i. [Exclude row k_i.]

(vii) Draw all arrows from the left of the last box labeled i in row k_i to the last box labeled $i+1$ in row 1 to row $i+1$. [Exclude row k_{i+1}.]

(viii) Draw all arrows from the left of the last box labeled $i+1$ in row k_{i+1} to the last box labeled $i+1$ in row 1 to row $i+1$. [Exclude row k_{i+1}.]

The numerical factors from each step are counted in the same manner as in Harter's algorithm. This algorithm requires only 8 steps whereas iterating one-step operators requires 16 steps. As in Harter's algorithm, it is necessary to append a box of length $2\sigma+1$ to the graph. The interesting feature of this algorithm is that there is no need to know the intermediate tableaux [as in one-step iteration techniques]. This algorithm can easily be extended to E_{i+qi} where $q>2$. The allowed values of k_α are $1,2,\ldots\alpha$. The sign of the matrix element is negative, if row k_i is above row k_{i+1}, and positive otherwise. Due to the similarity of this algorithm to Harter's, it should be programmable [Harter's algorithm has been programmed in high-level languages].

If we denote by F_i the result of step (i), the the matrix element is $\sqrt{\dfrac{F_1F_3F_5F_7}{F_2F_4F_6F_8}}$. We will illustrate this algorithm with a few examples.

[1] <113|E_{31}|111> $k_1 = k_2 = 1$ $i = 1$

1 1 1 1 2 3	(i) 4·3	1 1 1 1 2 3	(iv) 1
1 1 1 1 2 3	(ii) 4	1 1 1 1 2 3	(v) 1
1 1 1 1 2 3	(iii) 1	1 1 1 1 2 3	(vi) 1

1 1 1 1 2 3	(vii) 3	1 1 1 1 2 3	(viii) 3

Matrix element is $\sqrt{3}$.

[2] $\langle 123|E_{31}|112\rangle$ $\quad k_1 = k_2 = 1 \quad i = 1$

(i) 4·3 (v) 1

(ii) 4 (vi) 1

(iii) 1 (vii) 2

(iv) 1 (viii) 3

Matrix element is $\sqrt{2}$.

[3] $\langle {}^{13}_{2}|E_{31}|{}^{11}_{2}\rangle$ $\quad k_1 = 1 \quad k_2 = 1$

(i) 3 (v) 1

(ii) 2 (vi) 1

(iii) 1 (vii) 1

(iv) 1 (viii) 1

Matrix element is $\sqrt{\frac{3}{2}}$.

[4] $\langle {}^{12}_{3}|E_{31}|{}^{11}_{2}\rangle$ $\quad k_1 = 1 \quad k_2 = 2$

(i) 3 (v) 1

(ii) 2 (vi) 1

(iii) 1 (vii) 3

Matrix element is $-\frac{1}{\sqrt{2}}$.

The matrix elements agree with those calculated on the basis of the commutation relations. This algorithm is valid for U(2) by virtue of the fact that the Gelfand and Weyl bases are the same for U(2). This should be of interest in atomic physics, particularly since it is possible to put a restriction on k_α values for two-rowed tableaux.

APPENDIX 3

CLOSED FORM EXPRESSIONS FOR TWO BODY OPERATORS

An outstanding problem in the Unitary Group Approach is the lack of closed form expressions for two body operators. For electronic systems, Drake and Schlesinger [21] derived such expressions based on a vector coupling approach. We can establish closed form expressions for two body operators in general by simply operating with the multistep operators in sequence. Let us introduce the notation

$$\alpha_{r\omega k_{\omega-1}} \equiv P_{r\omega} - P_{k_\omega} - P_{k_{\omega-1}\omega-1} - 1$$

$$\beta_{r\omega k_{\omega-1}} \equiv P_{r\omega-1} - P_{k_{\omega-1}\omega-1} - 1$$

$$\gamma_{r\omega k_\omega} \equiv P_{r\omega-1} - P_{k_\omega \omega}$$

$$\delta_{r\omega k_\omega} \equiv P_{r\omega} - P_{k_\omega \omega}$$

We first consider the matrix elements of $E_{t\ell}E_{ij}$ where $t<i<j<\ell$. We have

$$E_{t\ell}E_{ij}|(m)> = E_{t\ell}\sum_{p=i}^{j-1}\sum_{k_p=1}^{p}\left[(-)\frac{\prod_{s=1}^{j}\alpha_{sjk_{j-1}}}{\prod_{\substack{s=1\\ s\neq k_{j-1}}}^{j-1}\beta_{sjk_{j-1}}}\right]^{1/2}$$

$$\times\prod_{q=i+1}^{j-1}\left[S(k_{q-1}-k_q)\left[\prod_{\substack{s=1\\ s\neq k_{q-1}}}^{q-1}\frac{\alpha_{sqk_q}}{\beta_{sqk_{q-1}}}\prod_{\substack{s=1\\ s\neq k_q}}^{q}\frac{\alpha_{sqk_{q-1}}}{\delta_{sqk_q}}\right]^{1/2}\right]$$

$$\times\left[\prod_{\substack{s=1\\ s\neq k_i}}^{i-1}\frac{\gamma_{sik_i}}{\delta_{sik_i}}\right]^{1/2}|(m)+\varepsilon_{k_p}(p)>$$

$$=\sum_{p'=t}^{\ell-1}\sum_{k_{p'}=1}^{p'}\sum_{p=i}^{j-1}\sum_{k_p=1}^{p}\left[(-)\frac{\prod_{s=1}^{j}\alpha_{sjk_{j-1}}}{\prod_{\substack{s=1\\ s\neq k_{j-1}}}^{j-1}\beta_{sjk_{j-1}}}\right]^{1/2}$$

$$\times\prod_{q=i+1}^{j-1}\left[S(k_{q-1}-k_q)\left[\prod_{\substack{s=1\\ s\neq k_{q-1}}}^{q-1}\frac{\gamma_{sqk_q}}{\beta_{sqk_{q-1}}}\prod_{\substack{s=1\\ s\neq k_q}}^{q}\frac{\alpha_{sqk_{q-1}}}{\delta_{sqk_q}}\right]^{1/2}\right]$$

$$\times\left[\prod_{\substack{s=1\\ s\neq k_i}}^{i-1}\frac{\gamma_{sik_i}}{\delta_{sik_i}}\right]^{1/2}\left[(-)\frac{\prod_{s'=1}^{\ell}\alpha_{s'\ell k_{\ell-1}}}{\prod_{\substack{s'=1\\ s\neq k_{\ell-1}}}^{j-1}\beta_{s'\ell k_{\ell-1}}}\right]^{1/2}$$

$$\times \prod_{q'=t+1}^{\ell-1} \left[S(k_{q'-1}-k_{q'}) \left[\prod_{\substack{s'=1 \\ s'\neq k_{q'-1}}}^{q-1} \frac{\gamma_{s'qk_{q'}}}{\beta_{s'q'k_{q'-1}}} \prod_{\substack{s=1 \\ s'\neq k_{q'}}}^{q'} \frac{\alpha_{s'q'k_{q'-1}}}{\delta_{s'q'k_{q-1}}} \right]^{1/2} \right]$$

$$\times \left[\prod_{\substack{s'=1 \\ s\neq k_t}}^{t-1} \frac{\gamma_{s'tk_t}}{\delta_{s'tk_t}} \right]^{1/2} \quad |(m) + \varepsilon_{k_p}(p) + \varepsilon_{k_{p'}}(p') > .$$

If $s' \varepsilon \{1,2,\ldots\ldots,i-1\}$ or $\{i,j+1, j+2,\ldots\ldots, t-1\}$, the terms $\alpha_{s'\ell k_{\ell-1}}$, $\beta_{s'\ell k_{\ell-1}}$, and the other factors still refer to entries of the original tableau (m) since the operator E_{ij} does not affect these rows. Although this expression may seem complicated, there are certain simplifying features for electronic systems. The factors $\varepsilon_{k_i}(i)$, $\varepsilon_{k+1}(i+1),\ldots\ldots$, $\varepsilon_{k_{\ell-1}}(\ell-1)$ are all constrained to be less than or equal to 2. For electronic systems this may eventually lead to a simpler formula. Further work is in progress in this direction.

To obtain matrix elements from this expression, we need only account for the various combinations of intermediate tableaux which yield the same final tableaux. This can be done easily for specific examples, but we are presently striving to do a general treatment. The matrix elements should ultimately be of the form $\sqrt{\frac{a}{b}}$ $a,b\varepsilon|R$ [22].

TABLE I

Tableau States for $\sigma = 1$

Including Correlation with SO(3)

$[\lambda]$	M	Tableau States	ΔQ_M	SO(3) Correlation
[3]	3	$\|111\rangle$	1	F
	2	$\|112\rangle$	0	F
	1	$\|113\rangle$, $\|122\rangle$	1	P,F
	0	$\|123\rangle$, $\|222\rangle$	0	P,F
	-1	$\|133\rangle$, $\|223\rangle$	1	P,F
	-2	$\|233\rangle$	0	F
	-3	$\|333\rangle$	1	F
[21]	2	$\left\|\begin{matrix}11\\2\end{matrix}\right\rangle$	1	D
	1	$\left\|\begin{matrix}11\\3\end{matrix}\right\rangle, \left\|\begin{matrix}12\\2\end{matrix}\right\rangle$	1	D,P
	0	$\left\|\begin{matrix}12\\3\end{matrix}\right\rangle, \left\|\begin{matrix}13\\2\end{matrix}\right\rangle$	0	D,P
	-1	$\left\|\begin{matrix}13\\3\end{matrix}\right\rangle, \left\|\begin{matrix}22\\3\end{matrix}\right\rangle$	1	D,P
	-2	$\left\|\begin{matrix}23\\3\end{matrix}\right\rangle$	1	D,P
$[1^3]$	0	$\left\|\begin{matrix}1\\2\\3\end{matrix}\right\rangle$	1	S

TABLE II

One Body Operator Matrix Elements for E_{ij} for $i = j + q$

$q = 1$	$q = 1$	2
$\langle 112 \| E_{21} \| 111 \rangle = \sqrt{3}$	$\langle {}^{11}_{3} \| E_{32} \| {}^{11}_{2} \rangle = 1$	$\langle 113 \| E_{31} \| 111 \rangle = \sqrt{3}$
$\langle 122 \| E_{21} \| 112 \rangle = 2$	$\langle {}^{12}_{2} \| E_{21} \| {}^{11}_{2} \rangle = 1$	$\langle 123 \| E_{31} \| 112 \rangle = \sqrt{2}$
$\langle 113 \| E_{32} \| 112 \rangle = 1$	$\langle {}^{12}_{3} \| E_{32} \| {}^{12}_{2} \rangle = 1/\sqrt{2}$	$\langle 133 \| E_{31} \| 11 \rangle = 2$
$\langle 123 \| E_{21} \| 113 \rangle = \sqrt{2}$	$\langle {}^{13}_{2} \| E_{32} \| {}^{12}_{2} \rangle = \sqrt{3}/2$	$\langle 233 \| E_{31} \| 122 \rangle = 1$
$\langle 222 \| E_{21} \| 122 \rangle = \sqrt{3}$	$\langle {}^{12}_{3} \| E_{21} \| {}^{11}_{3} \rangle = \sqrt{2}$	$\langle 233 \| E_{31} \| 123 \rangle = \sqrt{2}$
		$\langle 333 \| E_{31} \| 133 \rangle \quad \sqrt{3}$
$\langle 123 \| E_{32} \| 122 \rangle = 2$	$\langle {}^{22}_{3} \| E_{21} \| {}^{12}_{3} \rangle = \sqrt{2}$	$\langle {}^{12}_{3} \| E_{31} \| {}^{11}_{2} \rangle = -1/\sqrt{2}$
$\langle 133 \| E_{32} \| 123 \rangle = 2$	$\langle {}^{13}_{3} \| E_{32} \| {}^{12}_{3} \rangle = 1/\sqrt{2}$	$\langle {}^{13}_{2} \| E_{31} \| {}^{11}_{2} \rangle = \sqrt{3}/2$
$\langle 223 \| E_{21} \| 123 \rangle = 2$	$\langle {}^{13}_{3} \| E_{32} \| {}^{13}_{2} \rangle = \sqrt{3}/2$	$\langle {}^{13}_{3} \| E_{31} \| {}^{11}_{3} \rangle = 1$
$\langle 223 \| E_{32} \| 222 \rangle = 3$	$\langle {}^{23}_{3} \| E_{21} \| {}^{13}_{3} \rangle = 1$	$\langle {}^{22}_{3} \| E_{31} \| {}^{12}_{2} \rangle = -1$
$\langle 233 \| E_{21} \| 133 \rangle = 1$	$\langle {}^{23}_{3} \| E_{32} \| {}^{22}_{3} \rangle = 1$	$\langle {}^{23}_{3} \| E_{31} \| {}^{12}_{3} \rangle = 1/\sqrt{2}$
$\langle 233 \| E_{32} \| 223 \rangle = 2$	$\langle \begin{smallmatrix}1\\2\\3\end{smallmatrix} \| E_{ij} \| \begin{smallmatrix}1\\2\\3\end{smallmatrix} \rangle = 0$	$\langle {}^{23}_{3} \| E_{31} \| {}^{13}_{2} \rangle = -\sqrt{3}/2$
$\langle 333 \| E_{32} \| 233 \rangle = 3$		

TABLE IIIA

Matrix Representation of $[I^1 \cdot I^1]_{Siddall}$ operator

for $[\eta] = [3]$ in the Gelfand basis

	|111>	|112>	|113>	|122>	|123>	|222>	|133>	|223>	|233>	|333>
<111|	1/2	.	.	.	.	.	.	.	.	.
<112|		1/2	.	.	.	.	.	.	.	.
<113|			-1/6	1/3	.	.	.	.	.	.
<122|				1/3	.	.	.	.	.	.
<123|					1/6	$1/\sqrt{6}$	.	.	.	.
<222|						.	.	.	.	.
<133|							-1/6	1/3	.	.
<223|								1/3	.	.
<233|									1/2	.
<333|										1/2

In Tables IIIA to IVB, the dots represent zeroes, and the matrix is symmetric.

TABLE IIIB

Matrix Representation of $[I^1 \cdot I^1]_{Siddall}$ operator

for $[\eta] = [21]$ in the Gelfand basis

	$\vert{}^{1\,1}_{2}\rangle$	$\vert{}^{1\,1}_{3}\rangle$	$\vert{}^{1\,2}_{2}\rangle$	$\vert{}^{1\,2}_{3}\rangle$	$\vert{}^{1\,3}_{2}\rangle$	$\vert{}^{1\,3}_{3}\rangle$	$\vert{}^{2\,2}_{3}\rangle$	$\vert{}^{2\,3}_{3}\rangle$
$\langle{}^{1\,1}_{2}\vert$								
$\langle{}^{1\,1}_{3}\vert$		-1/6	1/6					
$\langle{}^{1\,2}_{2}\vert$			-1/6					
$\langle{}^{1\,2}_{3}\vert$				-1/12	$1/4\sqrt{3}$			
$\langle{}^{1\,3}_{2}\vert$					-1/4			
$\langle{}^{1\,3}_{3}\vert$						11/6	1/6	
$\langle{}^{2\,2}_{3}\vert$							-1/6	
$\langle{}^{2\,3}_{3}\vert$								0

TABLE IVA

Matrix Representation of $[I^2 \cdot I^2]_{Siddall}$ operator

for [n] = [3] in the Gelfand basis

	|111>	|112>	|113>	|122>	|123>	|222>	|133>	|223>	|233>	|333>
<111|	1/10	.	.	.	.	.	.	.	.	.
<112|		1/10	.	.	.	.	.	.	.	.
<113|			1/2	-1/5	.	.	.	.	.	.
<122|				1/5	.	.	.	.	.	.
<123|					3/10	$-1/10\sqrt{6}$	.	.	.	.
<222|						2/5	.	.	.	.
<133|							1/2	-1/5	.	.
<223|								1/5	.	.
<233|									1/10	.
<333|										1/10

TABLE IVB

Matrix Representation of $[I^2 \cdot I^2]_{Siddall}$ operator

for $[n] = [21]$ in the Gelfand basis

	$\vert{}^{1\ 1}_{2}\rangle$	$\vert{}^{1\ 1}_{3}\rangle$	$\vert{}^{1\ 2}_{2}\rangle$	$\vert{}^{1\ 2}_{3}\rangle$	$\vert{}^{1\ 3}_{2}\rangle$	$\vert{}^{1\ 3}_{3}\rangle$	$\vert{}^{2\ 2}_{3}\rangle$	$\vert{}^{2\ 3}_{3}\rangle$
$\langle{}^{1\ 1}_{2}\vert$	-1/5	·	·	·	·	·	·	·
$\langle{}^{1\ 1}_{3}\vert$		-1/10	-1/10	·	·	·	·	·
$\langle{}^{1\ 2}_{2}\vert$			-1/10	·	·	·	·	·
$\langle{}^{1\ 2}_{3}\vert$				-3/20	$-1/20\sqrt{3}$	·	·	·
$\langle{}^{1\ 3}_{2}\vert$					-1/20	·	·	·
$\langle{}^{1\ 3}_{2}\vert$						-1/10	-1/10	·
$\langle{}^{2\ 2}_{3}\vert$							-1/10	·
$\langle{}^{2\ 3}_{3}\vert$								0

TABLE V

Basis States of A_4 Systems (σ_A=2)

Including Correlation with SO(3).

$[\eta_A]$	Basis States		M	Q_M	SO(3) Correlation
[4]	$\vert 1111\rangle$		8	1	L
	$\vert 1112\rangle$		7	1	L
	$\vert 1122\rangle$	$\vert 1113\rangle$	6	2	L,I
	$\vert 1222\rangle$	$\vert 1123\rangle$	5	3	L,I,H
	$\vert 1114\rangle$				
	$\vert 2222\rangle$	$\vert 1133\rangle$	4	5	L,I,H
	$\vert 1223\rangle$	$\vert 1124\rangle$			2G,2G
	$\vert 1115\rangle$				
	$\vert 2223\rangle$	$\vert 1233\rangle$	3	5	L,I,H
	$\vert 1224\rangle$	$\vert 1134\rangle$			2G
	$\vert 1125\rangle$				
	$\vert 2233\rangle$	$\vert 1333\rangle$	2	7	L,I,H
	$\vert 2224\rangle$	$\vert 1234\rangle$			2G,2D
	$\vert 1144\rangle$	$\vert 1225\rangle$			
	$\vert 1135\rangle$				
	$\vert 2333\rangle$	$\vert 2234\rangle$	1	7	L,I,H
	$\vert 1334\rangle$	$\vert 1244\rangle$			2G,2D
	$\vert 2225\rangle$	$\vert 1235\rangle$			
	$\vert 1145\rangle$				

$[n_A]$	Basis States		M	Q_M	SO(3) Correlation
	$\|3333\rangle$	$\|2334\rangle$	0	8	L,I,H
	$\|2244\rangle$	$\|1344\rangle$			2G,2D,S
	$\|2235\rangle$	$\|1335\rangle$			
	$\|1245\rangle$	$\|1155\rangle$			
[31]	$\|{}^{111}_{2}\rangle$		7	1	K
	$\|{}^{112}_{2}\rangle$	$\|{}^{111}_{3}\rangle$	6	2	K,I
	$\|{}^{122}_{2}\rangle$	$\|{}^{112}_{3}\rangle$	5	4	K,I,2H
	$\|{}^{113}_{2}\rangle$	$\|{}^{111}_{4}\rangle$			
	$\|{}^{122}_{3}\rangle$	$\|{}^{113}_{3}\rangle$	4	6	K,I,2H,2G
	$\|{}^{123}_{2}\rangle$	$\|{}^{112}_{4}\rangle$			
	$\|{}^{114}_{2}\rangle$	$\|{}^{111}_{5}\rangle$			
	$\|{}^{222}_{3}\rangle$	$\|{}^{123}_{3}\rangle$	3	9	K,I,2H,2G,3F
	$\|{}^{133}_{2}\rangle$	$\|{}^{122}_{4}\rangle$			
	$\|{}^{113}_{4}\rangle$	$\|{}^{124}_{2}\rangle$			
	$\|{}^{114}_{3}\rangle$	$\|{}^{112}_{5}\rangle$			
	$\|{}^{115}_{2}\rangle$				

$[n_A]$	Basis States	M	Q_M	SO(3) Correlation
[31]	$|^{223}_{3}\rangle$ $|^{133}_{3}\rangle$ $|^{222}_{4}\rangle$ $|^{123}_{4}\rangle$ $|^{114}_{4}\rangle$ $|^{124}_{3}\rangle$ $|^{134}_{2}\rangle$ $|^{122}_{5}\rangle$ $|^{113}_{5}\rangle$ $|^{125}_{2}\rangle$ $|^{115}_{3}\rangle$	2	11	K,I,2H, 2G,3F,2D
[31]	$|^{233}_{3}\rangle$ $|^{223}_{4}\rangle$ $|^{133}_{4}\rangle$ $|^{124}_{4}\rangle$ $|^{125}_{3}\rangle$ $|^{224}_{3}\rangle$ $|^{144}_{2}\rangle$ $|^{134}_{3}\rangle$ $|^{222}_{5}\rangle$ $|^{123}_{5}\rangle$ $|^{114}_{5}\rangle$ $|^{135}_{2}\rangle$ $|^{115}_{4}\rangle$	1	13	K,I,2H,2G, 3F,2D,2P
	$|^{233}_{4}\rangle$ $|^{224}_{4}\rangle$ $|^{144}_{3}\rangle$ $|^{134}_{4}\rangle$ $|^{234}_{3}\rangle$ $|^{133}_{5}\rangle$ $|^{124}_{5}\rangle$ $|^{223}_{5}\rangle$ $|^{225}_{3}\rangle$ $|^{135}_{3}\rangle$ $|^{125}_{4}\rangle$ $|^{145}_{2}\rangle$ $|^{115}_{5}\rangle$	0	13	K,I,2H,2G, 3F,2D,2P

$[n_A]$	Basis States	M	Q_M	SO(3) Correlation
$[2^2]$	$\left\|\begin{smallmatrix}11\\22\end{smallmatrix}\right\rangle$	6	1	I
	$\left\|\begin{smallmatrix}11\\23\end{smallmatrix}\right\rangle$	5	1	I
	$\left\|\begin{smallmatrix}12\\23\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}11\\33\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}11\\24\end{smallmatrix}\right\rangle$	4	3	I,2G
	$\left\|\begin{smallmatrix}12\\33\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}11\\34\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}12\\24\end{smallmatrix}\right\rangle$	3	4	I,2G,F
	$\left\|\begin{smallmatrix}11\\25\end{smallmatrix}\right\rangle$			
	$\left\|\begin{smallmatrix}12\\34\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}13\\24\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}11\\44\end{smallmatrix}\right\rangle$	2	6	I,2G,F
	$\left\|\begin{smallmatrix}12\\25\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}11\\35\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}22\\33\end{smallmatrix}\right\rangle$			
	$\left\|\begin{smallmatrix}22\\34\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}13\\34\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}12\\35\end{smallmatrix}\right\rangle$	1	6	I,2G,F,2D
	$\left\|\begin{smallmatrix}13\\25\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}11\\45\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}12\\44\end{smallmatrix}\right\rangle$			
	$\left\|\begin{smallmatrix}23\\34\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}22\\44\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}13\\44\end{smallmatrix}\right\rangle$	0	8	I,2G,F,2D,2S
	$\left\|\begin{smallmatrix}22\\35\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}13\\35\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}12\\45\end{smallmatrix}\right\rangle$			
	$\left\|\begin{smallmatrix}14\\25\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}11\\55\end{smallmatrix}\right\rangle$			
$[21^2]$	$\left\|\begin{smallmatrix}11\\2\\3\end{smallmatrix}\right\rangle$	5	1	H
	$\left\|\begin{smallmatrix}12\\2\\3\end{smallmatrix}\right\rangle$ $\left\|\begin{smallmatrix}11\\2\\4\end{smallmatrix}\right\rangle$	4	2	H,G

$[n_A]$	Basis States	M	Q_M	SO(3) Correlation
$[21^2]$	$\left\lvert\begin{smallmatrix}13\\2\\3\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}12\\2\\4\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}11\\2\\5\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}11\\3\\4\end{smallmatrix}\right\rangle$	3	4	H,G.2F
	$\left\lvert\begin{smallmatrix}13\\2\\4\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}14\\2\\3\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}12\\2\\5\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}11\\3\\4\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}12\\3\\4\end{smallmatrix}\right\rangle$	2	5	H,G.2F,D
	$\left\lvert\begin{smallmatrix}13\\3\\4\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}14\\2\\4\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}12\\3\\5\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}13\\2\\5\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}11\\4\\5\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}15\\2\\3\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}22\\3\\4\end{smallmatrix}\right\rangle$	1	7	H,G.2F,D,2P
$[21^2]$	$\left\lvert\begin{smallmatrix}23\\3\\4\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}14\\3\\4\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}22\\3\\5\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}13\\3\\5\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}12\\4\\5\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}15\\2\\4\end{smallmatrix}\right\rangle$ $\left\lvert\begin{smallmatrix}14\\2\\5\end{smallmatrix}\right\rangle$	0	7	H,G,2F,D,2P
$[1^4]$	$\left\lvert\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix}\right\rangle$	2	1	D

$[\eta_A]$	Basis States	M	Q_M	SO(3) Correlation
$[1^4]$	$\left\vert \begin{smallmatrix}1\\2\\3\\5\end{smallmatrix} \right\rangle$	1	1	D
	$\left\vert \begin{smallmatrix}1\\2\\4\\5\end{smallmatrix} \right\rangle$	0	1	D

TABLE VI

Basis sets of B_2 System ($\sigma_B=1$)

Including Correlation with SO(3).

$[n_B]$	Basis States	M	Q_M	SO(3) Correlation
[2]	|66>	2	1	D
	|67>	1	1	D
	|68> |77>	0	1	D,S
	|78>	-1	1	D
	|88>	-2	1	D

TABLE VII

Matrix Elements of One Body Operators for A_4 System

$\langle 1112|E_{21}|1111\rangle = 2$

$\langle 1122|E_{21}|1112\rangle = \sqrt{6}$

$\langle 1222|E_{21}|1122\rangle = \sqrt{6}$

$\langle 2222|E_{21}|1111\rangle = 2$

$\langle 1123|E_{21}|1113\rangle = \sqrt{3}$

$\langle 1223|E_{21}|1123\rangle = 2$

$\langle 1233|E_{21}|1133\rangle = 2$

$\langle 2233|E_{21}|1233\rangle = 2$

$\langle 1124|E_{21}|1114\rangle = \sqrt{3}$

$\langle 1224|E_{21}|1124\rangle = 2$

$\langle 2223|E_{21}|1223\rangle = \sqrt{3}$

$\langle 2224|E_{21}|1224\rangle = \sqrt{3}$

$\langle 1234|E_{21}|1134\rangle = \sqrt{2}$

$\langle 2234|E_{21}|1234\rangle = \sqrt{2}$

$\langle 2334|E_{21}|1334\rangle = 1$

$\langle 1244|E_{21}|1144\rangle = \sqrt{2}$

$\langle 2333|E_{21}|1333\rangle = 1$

$\langle 2355|E_{21}|1344\rangle = 1$

$\langle 2455|E_{21}|1455\rangle = 1$

$\langle 2555|E_{21}|1555\rangle = 1$

$\langle 1113|E_{32}|1112\rangle = 1$

$\langle 1123|E_{32}|1122\rangle = \sqrt{2}$

$\langle 2244|E_{21}|1244\rangle = \sqrt{2}$

$\langle 2344|E_{21}|1344\rangle = 1$

$\langle 2444|E_{21}|1444\rangle = 1$

$\langle 1125|E_{21}|1115\rangle = \sqrt{3}$

$\langle 1225|E_{21}|1125\rangle = 2$

$\langle 2225|E_{21}|1225\rangle = \sqrt{3}$

$\langle 1234|E_{21}|1135\rangle = \sqrt{2}$

$\langle 2235|E_{21}|1235\rangle = \sqrt{2}$

$\langle 2335|E_{21}|1335\rangle = 1$

$\langle 1245|E_{21}|1145\rangle = \sqrt{2}$

$\langle 2245|E_{21}|1245\rangle = \sqrt{2}$

$\langle 2345|E_{21}|1345\rangle = 1$

$\langle 2445|E_{21}|1445\rangle = 1$

$\langle 1255|E_{21}|1155\rangle = \sqrt{2}$

$\langle 2255|E_{21}|1255\rangle = \sqrt{2}$

$\langle 1134|E_{32}|1124\rangle = 1$

$\langle 1344|E_{32}|1244\rangle = 1$

$\langle 2344|E_{32}|2244| = \sqrt{2}$

$\langle 3344|E_{32}|2344\rangle = \sqrt{2}$

$\langle 3444|E_{32}|2444\rangle = 1$

$\langle 1135|E_{32}|1125\rangle = 1$

$\langle 1235|E_{32}|1225\rangle = \sqrt{2}$

$\langle 1223|E_{32}|1222\rangle = \sqrt{3}$

$\langle 2223|E_{32}|2222\rangle = 2$

$\langle 1133|E_{32}|1123\rangle = \sqrt{2}$

$\langle 1233|E_{32}|1223\rangle = 2$

$\langle 2233|E_{32}|2223\rangle = \sqrt{6}$

$\langle 1333|E_{32}|1233\rangle = \sqrt{3}$

$\langle 2333|E_{32}|2233\rangle = \sqrt{6}$

$\langle 3333|E_{32}|2333\rangle = 2$

$\langle 1234|E_{32}|1224\rangle = \sqrt{2}$

$\langle 2234|E_{32}|2224\rangle = \sqrt{3}$

$\langle 1334|E_{32}|1234\rangle = \sqrt{2}$

$\langle 2334|E_{32}|2234\rangle = 2$

$\langle 3334|E_{32}|2334\rangle = \sqrt{3}$

$\langle 3555|E_{32}|2555\rangle = 1$

$\langle 1114|E_{43}|1113\rangle = 1$

$\langle 1124|E_{43}|1123\rangle = 1$

$\langle 1224|E_{43}|1223\rangle = 1$

$\langle 2224|E_{43}|2223\rangle = 1$

$\langle 1134|E_{43}|1133\rangle = \sqrt{2}$

$\langle 1234|E_{43}|1233\rangle = \sqrt{2}$

$\langle 2234|E_{43}|2233\rangle = \sqrt{2}$

$\langle 1334|E_{43}|1333\rangle = \sqrt{3}$

$\langle 2334|E_{43}|2333\rangle = \sqrt{3}$

$\langle 2235|E_{32}|2225\rangle = \sqrt{3}$

$\langle 1335|E_{32}|1235\rangle = \sqrt{2}$

$\langle 2335|E_{32}|2235\rangle = 2$

$\langle 3335|E_{32}|2335\rangle = \sqrt{3}$

$\langle 1345|E_{32}|1245\rangle = 1$

$\langle 2345|E_{32}|2245\rangle = \sqrt{2}$

$\langle 3345|E_{32}|2345\rangle = \sqrt{2}$

$\langle 1134|E_{32}|1124\rangle = 1$

$\langle 3445|E_{32}|2445\rangle = 1$

$\langle 1355|E_{32}|1255\rangle = 1$

$\langle 2355|E_{32}|2255\rangle = \sqrt{2}$

$\langle 3355|E_{32}|2355\rangle = \sqrt{2}$

$\langle 3455|E_{32}|2455\rangle = 1$

$\langle 3444|E_{43}|3344\rangle = \sqrt{6}$

$\langle 4444|E_{43}|3444\rangle = 2$

$\langle 1145|E_{43}|1135\rangle = 1$

$\langle 1245|E_{43}|1235\rangle = 1$

$\langle 2245|E_{43}|2235\rangle = 1$

$\langle 1345|E_{43}|1335\rangle = \sqrt{2}$

$\langle 2345|E_{43}|2335\rangle = \sqrt{2}$

$\langle 3345|E_{43}|3335\rangle = \sqrt{3}$

$\langle 1445|E_{43}|1345\rangle = \sqrt{2}$

$\langle 2445|E_{43}|2345\rangle = \sqrt{2}$

$\langle 3334|E_{43}|3333\rangle = 2$

$\langle 1144|E_{43}|1134\rangle = \sqrt{2}$

$\langle 1244|E_{43}|2234\rangle = \sqrt{2}$

$\langle 1344|E_{43}|1334\rangle = 2$

$\langle 2344|E_{43}|2334\rangle = 2$

$\langle 3344|E_{43}|3334\rangle = \sqrt{6}$

$\langle 1444|E_{43}|1344\rangle = \sqrt{3}$

$\langle 2444|E_{43}|2344\rangle = \sqrt{3}$

$\langle 1115|E_{54}|1114\rangle = 1$

$\langle 1125|E_{54}|1124\rangle = 1$

$\langle 2225|E_{54}|2224\rangle = 1$

$\langle 1225|E_{54}|1224\rangle = 1$

$\langle 1135|E_{54}|1134\rangle = 1$

$\langle 1235|E_{54}|1234\rangle = 1$

$\langle 2235|E_{54}|2234\rangle = 1$

$\langle 1335|E_{54}|1334\rangle = 1$

$\langle 2335|E_{54}|2334\rangle = 1$

$\langle 3335|E_{54}|3334\rangle = 1$

$\langle 1145|E_{54}|1144\rangle = \sqrt{2}$

$\langle 1245|E_{54}|1244\rangle = \sqrt{2}$

$\langle 2245|E_{54}|2244\rangle = \sqrt{2}$

$\langle 1345|E_{54}|1344\rangle = \sqrt{2}$

$\langle 2345|E_{54}|2344\rangle = \sqrt{2}$

$\langle 3345|E_{54}|3344\rangle = \sqrt{2}$

$\langle 2445|E_{43}|2345\rangle = \sqrt{2}$

$\langle 3445|E_{43}|3345\rangle = 2$

$\langle 4445|E_{43}|3445\rangle = \sqrt{3}$

$\langle 1455|E_{43}|1355\rangle = 1$

$\langle 2455|E_{43}|2355\rangle = 1$

$\langle 3455|E_{43}|3355\rangle = \sqrt{2}$

$\langle 4555|E_{43}|3555\rangle = 1$

$\langle 4455|E_{43}|3455\rangle = \sqrt{2}$

$\langle 3445|E_{54}|3444\rangle = \sqrt{3}$

$\langle 4445|E_{54}|4444\rangle = 2$

$\langle 1155|E_{54}|1145\rangle = \sqrt{2}$

$\langle 1255|E_{54}|1245\rangle = \sqrt{2}$

$\langle 2255|E_{54}|2245\rangle = \sqrt{2}$

$\langle 1355|E_{54}|1345\rangle = \sqrt{2}$

$\langle 2355|E_{54}|2345\rangle = \sqrt{2}$

$\langle 3355|E_{54}|3345\rangle = \sqrt{2}$

$\langle 1455|E_{54}|1445\rangle = 2$

$\langle 2455|E_{54}|2445\rangle = 2$

$\langle 3455|E_{54}|3445\rangle = 2$

$\langle 4455|E_{54}|4445\rangle = \sqrt{6}$

$\langle 1555|E_{54}|1455\rangle = \sqrt{3}$

$\langle 2555|E_{54}|2455\rangle = \sqrt{3}$

$\langle 3555|E_{54}|3455\rangle = \sqrt{3}$

$\langle 4555|E54|4455\rangle = \sqrt{6}$

$\langle 1445|E_{54}|1444\rangle = \sqrt{3}$

$\langle 2445|E_{54}|2444\rangle = \sqrt{3}$

$\langle 1113|E_{31}|1111\rangle = 2$

$\langle 1123|E_{31}|1112\rangle = \sqrt{3}$

$\langle 1223|E_{31}|1122\rangle = \sqrt{2}$

$\langle 2223|E_{31}|1222\rangle = 1$

$\langle 1133|E_{31}|1113\rangle = \sqrt{6}$

$\langle 1233|E_{31}|1123\rangle = 2$

$\langle 1134|E_{31}|1114\rangle = \sqrt{3}$

$\langle 1135|E_{31}|1115\rangle = \sqrt{3}$

$\langle 2333|E_{31}|1233\rangle = \sqrt{3}$

$\langle 2334|E_{31}|1234\rangle = \sqrt{2}$

$\langle 1344|E_{31}|1144\rangle = \sqrt{2}$

$\langle 2235|E_{31}|1225\rangle = 1$

$\langle 1335|E_{31}|1135\rangle = 2$

$\langle 3334|E_{31}|1334\rangle = \sqrt{3}$

$\langle 2344|E_{31}|1244\rangle = 1$

$\langle 2335|E_{31}|1235\rangle = \sqrt{2}$

$\langle 1345|E_{31}|1145\rangle = \sqrt{2}$

$\langle 1114|E_{41}|1111\rangle = 2$

$\langle 1124|E_{41}|1112\rangle = \sqrt{3}$

$\langle 1224|E_{41}|1122\rangle = \sqrt{2}$

$\langle 1134|E_{41}|1113\rangle = \sqrt{3}$

$\langle 2244|E_{41}|1224\rangle = \sqrt{2}$

$\langle 1134|E_{41}|1113\rangle = \sqrt{3}$

$\langle 5555|E_{54}|4555\rangle = 2$

$\langle 3344|E_{31}|1344\rangle = \sqrt{2}$

$\langle 3345|E_{31}|1345\rangle = \sqrt{2}$

$\langle 2355|E_{31}|1255\rangle = 1$

$\langle 3445|E_{31}|1445\rangle = 1$

$\langle 3355|E_{31}|1355\rangle = \sqrt{2}$

$\langle 3555|E_{31}|1555\rangle = 1$

$\langle 1333|E_{31}|1133\rangle = \sqrt{6}$

$\langle 2233|E_{31}|1223\rangle = \sqrt{2}$

$\langle 1234|E_{31}|1124\rangle = \sqrt{2}$

$\langle 1144|E_{31}|1114\rangle = \sqrt{6}$

$\langle 2234|E_{31}|1224\rangle = 1$

$\langle 1334|E_{31}|1134\rangle = 2$

$\langle 1235|E_{31}|1125\rangle = \sqrt{2}$

$\langle 3333|E_{31}|1333\rangle = 2$

$\langle 2345|E_{31}|1245\rangle = 1$

$\langle 2335|E_{31}|1335\rangle = \sqrt{3}$

$\langle 2345|E_{31}|1245\rangle = 1$

$\langle 1355|E_{31}|1155\rangle = \sqrt{2}$

$\langle 3444|E_{31}|1444\rangle = 1$

$\langle 3455|E_{31}|1455\rangle = 1$

$\langle 2344|E_{41}|1234\rangle = \sqrt{2}$

$\langle 2245|E_{41}|1444\rangle = \sqrt{6}$

$\langle 2245|E_{41}|1225\rangle = 1$

$\langle 1345|E_{41}|1135\rangle = \sqrt{2}$

$\langle 2224|E_{42}|1222\rangle = 1$

$\langle 1234|E_{41}|1123\rangle = \sqrt{2}$

$\langle 1144|E_{41}|1114\rangle = \sqrt{6}$

$\langle 1334|E_{41}|1133\rangle = \sqrt{2}$

$\langle 2234|E_{41}|1223\rangle = 1$

$\langle 1244|E_{41}|1124\rangle = 2$

$\langle 1145|E_{41}|1115\rangle = 3$

$\langle 2334|E_{41}|1233\rangle = 1$

$\langle 1344|E_{41}|1134\rangle = 2$

$\langle 1245|E_{41}|1125\rangle = \sqrt{2}$

$\langle 3334|E_{41}|1333\rangle = 1$

$\langle 4455|E_{41}|1455\rangle = \sqrt{2}$

$\langle 1115|E_{51}|1111\rangle = 2$

$\langle 1125|E_{51}|1112\rangle = \sqrt{3}$

$\langle 1225|E_{51}|1122\rangle = \sqrt{2}$

$\langle 1135|E_{51}|1113\rangle = \sqrt{3}$

$\langle 2225|E_{51}|1222\rangle = 1$

$\langle 1235|E_{51}|1123\rangle = \sqrt{2}$

$\langle 1145|E_{51}|1114\rangle = \sqrt{3}$

$\langle 1335|E_{51}|1133\rangle = \sqrt{2}$

$\langle 2235|E_{51}|1223\rangle = 1$

$\langle 1245|E_{51}|1124\rangle = \sqrt{2}$

$\langle 1155|E_{51}|1115\rangle = \sqrt{6}$

$\langle 2335|E_{51}|1233\rangle = 1$

$\langle 3344|E_{41}|2234\rangle = \sqrt{2}$

$\langle 2444|E_{41}|1244\rangle = \sqrt{3}$

$\langle 2345|E_{41}|1235\rangle = 1$

$\langle 1445|E_{41}|1145\rangle = 2$

$\langle 3444|E_{41}|1344\rangle = \sqrt{3}$

$\langle 3345|E_{41}|1335\rangle = 1$

$\langle 2445|E_{41}|1245\rangle = \sqrt{2}$

$\langle 1455|E_{41}|1155\rangle = \sqrt{2}$

$\langle 4444|E_{41}|1444\rangle = 2$

$\langle 3344|E_{41}|1334\rangle = \sqrt{2}$

$\langle 3445|E_{41}|1345\rangle = \sqrt{2}$

$\langle 2455|E_{41}|1255\rangle = 1$

$\langle 4445|E_{41}|1445\rangle = \sqrt{3}$

$\langle 3455|E_{41}|1355\rangle = 1$

$\langle 4555|E_{41}|1555\rangle = 1$

$\langle 1445|E_{51}|1144\rangle = \sqrt{2}$

$\langle 2255|E_{51}|1225\rangle = \sqrt{2}$

$\langle 1355|E_{51}|1135\rangle = 2$

$\langle 3345|E_{51}|1334\rangle = 1$

$\langle 2445|E_{51}|1244\rangle = 1$

$\langle 2355|E_{51}|1235\rangle = \sqrt{2}$

$\langle 1455|E_{51}|1145\rangle = 2$

$\langle 3445|E_{51}|1344\rangle = 1$

$\langle 3355|E_{51}|1335\rangle = \sqrt{2}$

$\langle 2245|E_{51}|1224\rangle = 1$

$\langle 1345|E_{51}|1134\rangle = \sqrt{2}$

$\langle 1255|E_{51}|1125\rangle = 2$

$\langle 3335|E_{51}|1333\rangle = 1$

$\langle 2345|E_{51}|1234\rangle = 1$

$\langle 4455|E_{51}|1445\rangle = \sqrt{2}$

$\langle 3555|E_{51}|1355\rangle = \sqrt{3}$

$\langle 4555|E_{51}|1455\rangle = \sqrt{3}$

$\langle 1114|E_{42}|1112\rangle = 1$

$\langle 1124|E_{42}|1122\rangle = \sqrt{2}$

$\langle 1224|E_{42}|1222\rangle = \sqrt{3}$

$\langle 1134|E_{42}|1123\rangle = 1$

$\langle 2224|E_{42}|2222\rangle = 2$

$\langle 1234|E_{42}|1223\rangle = \sqrt{2}$

$\langle 1144|E_{42}|1124\rangle = \sqrt{2}$

$\langle 2234|E_{42}|2223\rangle = \sqrt{3}$

$\langle 1334|E_{42}|1233\rangle = 1$

$\langle 1244|E_{42}|1224\rangle = \sqrt{6}$

$\langle 1145|E_{42}|1125\rangle = 1$

$\langle 2334|E_{42}|2233\rangle = \sqrt{2}$

$\langle 1145|E_{42}|1125\rangle = 1$

$\langle 4444|E_{42}|2444\rangle = 2$

$\langle 3445|E_{42}|2345\rangle = \sqrt{2}$

$\langle 2455|E_{42}|2255\rangle = \sqrt{2}$

$\langle 2455|E_{51}|1245\rangle = \sqrt{2}$

$\langle 1555|E_{51}|1155\rangle = \sqrt{6}$

$\langle 4445|E_{51}|1444\rangle = 1$

$\langle 3455|E_{51}|1345\rangle = \sqrt{2}$

$\langle 2555|E_{51}|1255\rangle = \sqrt{3}$

$\langle 2255|E_{51}|1444\rangle = \sqrt{3}$

$\langle 3455|E_{51}|1345\rangle = \sqrt{2}$

$\langle 2555|E_{51}|1255\rangle = \sqrt{3}$

$\langle 2334|E_{42}|2233\rangle = \sqrt{2}$

$\langle 2244|E_{42}|2224\rangle = \sqrt{6}$

$\langle 1344|E_{42}|1234\rangle = \sqrt{2}$

$\langle 1245|E_{42}|1225\rangle = \sqrt{2}$

$\langle 3334|E_{42}|2333\rangle = 1$

$\langle 2344|E_{42}|2234\rangle = 2$

$\langle 1444|E_{42}|1244\rangle = \sqrt{3}$

$\langle 2245|E_{42}|2225\rangle = \sqrt{3}$

$\langle 1345|E_{42}|1235\rangle = 1$

$\langle 3344|E_{42}|2334\rangle = \sqrt{2}$

$\langle 2444|E_{42}|2244\rangle = \sqrt{6}$

$\langle 2445|E_{42}|2245\rangle = 2$

$\langle 2345|E_{42}|2235\rangle = \sqrt{2}$

$\langle 1445|E_{42}|1245\rangle = \sqrt{2}$

$\langle 3444|E_{42}|2344\rangle = \sqrt{3}$

$\langle 2445|E_{42}|2245\rangle = 2$

$\langle 4445|E_{42}|2445\rangle = \sqrt{3}$

$\langle 3455|E_{42}|2355\rangle = 1$

$\langle 4455|E_{42}|2455\rangle = \sqrt{2}$

$\langle 4555|E_{42}|2555\rangle = 1$

$\langle 1115|E_{52}|1112\rangle = 1$

$\langle 1125|E_{52}|1122\rangle = \sqrt{2}$

$\langle 1225|E_{52}|1222\rangle = \sqrt{3}$

$\langle 1135|E_{52}|1123\rangle = 1$

$\langle 2225|E_{52}|2222\rangle = 2$

$\langle 1235|E_{52}|1223\rangle = \sqrt{2}$

$\langle 1145|E_{52}|1124\rangle = 1$

$\langle 2235|E_{52}|2223\rangle = \sqrt{3}$

$\langle 1335|E_{52}|1233\rangle = 1$

$\langle 1245|E_{52}|1224\rangle = \sqrt{2}$

$\langle 1155|E_{52}|1125\rangle = \sqrt{2}$

$\langle 4445|E_{52}|2444\rangle = 1$

$\langle 4455|E_{52}|2445\rangle = \sqrt{2}$

$\langle 3555|E_{52}|2355\rangle = \sqrt{3}$

$\langle 4555|E_{52}|2455\rangle = \sqrt{3}$

$\langle 5555|E_{52}|2555\rangle = 2$

$\langle 1115|E_{53}|1113\rangle = 1$

$\langle 1125|E_{53}|1123\rangle = 1$

$\langle 1135|E_{53}|1133\rangle = \sqrt{2}$

$\langle 1225|E_{53}|1223\rangle = 1$

$\langle 2345|E_{42}|2235\rangle = \sqrt{2}$

$\langle 1445|E_{42}|1245\rangle = \sqrt{3}$

$\langle 3444|E_{42}|2335\rangle = 1$

$\langle 1455|E_{42}|1255\rangle = 1$

$\langle 2335|E_{52}|2233\rangle = \sqrt{2}$

$\langle 2245|E_{52}|2224\rangle = \sqrt{3}$

$\langle 1345|E_{52}|1234\rangle = 1$

$\langle 1255|E_{52}|1225\rangle = 2$

$\langle 3335|E_{52}|2333\rangle = 1$

$\langle 2345|E_{52}|2234\rangle = \sqrt{2}$

$\langle 1445|E_{52}|1244\rangle = 1$

$\langle 2255|E_{52}|2225\rangle = \sqrt{6}$

$\langle 1355|E_{52}|1235\rangle = \sqrt{2}$

$\langle 3345|E_{52}|2334\rangle = 1$

$\langle 2445|E_{52}|2244\rangle = \sqrt{2}$

$\langle 2355|E_{52}|2235\rangle = 2$

$\langle 1455|E_{52}|1245\rangle = \sqrt{2}$

$\langle 3445|E_{52}|2344\rangle = 1$

$\langle 3355|E_{52}|2335\rangle = \sqrt{2}$

$\langle 2455|E_{52}|2245\rangle = 1$

$\langle 1555|E_{52}|1255\rangle = \sqrt{3}$

$\langle 1255|E_{53}|1235\rangle = \sqrt{2}$

$\langle 3335|E_{53}|3333\rangle = 2$

$\langle 2345|E_{53}|2334\rangle = \sqrt{2}$

$\langle 1145|E_{53}|1134\rangle = 1$

$\langle 2244|E_{53}|2233\rangle = 2$

$\langle 1335|E_{53}|1333\rangle = \sqrt{3}$

$\langle 1245|E_{53}|1234\rangle = 1$

$\langle 1155|E_{53}|1135\rangle = \sqrt{2}$

$\langle 2335|E_{53}|2333\rangle = \sqrt{3}$

$\langle 2245|E_{53}|2234\rangle = 1$

$\langle 1345|E_{53}|1334\rangle = \sqrt{2}$

$\langle 4555|E_{53}|3455\rangle = \sqrt{3}$

$\langle 5555|E_{53}|3555\rangle = 2$

$\langle 1445|E_{53}|1344\rangle = 1$

$\langle 2255|E_{53}|2235\rangle = \sqrt{2}$

$\langle 1355|E_{53}|1335\rangle = 2$

$\langle 3345|E_{53}|3334\rangle = \sqrt{3}$

$\langle 2445|E_{53}|2344\rangle = 1$

$\langle 2355|E_{53}|2335\rangle = 2$

$\langle 1455|E_{53}|1345\rangle = \sqrt{2}$

$\langle 3335|E_{53}|3355\rangle = \sqrt{6}$

$\langle 2455|E_{53}|2345\rangle = \sqrt{2}$

$\langle 3445|E_{53}|3344\rangle = \sqrt{2}$

$\langle 1555|E_{53}|1355\rangle = \sqrt{3}$

$\langle 4445|E_{53}|3444\rangle = 1$

$\langle 3455|E_{53}|3345\rangle = 2$

$\langle 3555|E_{53}|3355\rangle = \sqrt{6}$

$$\langle \begin{smallmatrix}12\\2\\3\end{smallmatrix} | E_{21} | \begin{smallmatrix}11\\2\\3\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}12\\2\\4\end{smallmatrix} | E_{21} | \begin{smallmatrix}11\\2\\4\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}23\\3\\4\end{smallmatrix} | E_{21} | \begin{smallmatrix}13\\3\\4\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}24\\3\\4\end{smallmatrix} | E_{21} | \begin{smallmatrix}14\\3\\4\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}12\\2\\5\end{smallmatrix} | E_{21} | \begin{smallmatrix}11\\2\\5\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}12\\3\\5\end{smallmatrix} | E_{21} | \begin{smallmatrix}11\\3\\5\end{smallmatrix} \rangle = \sqrt{2}$$

$$\langle \begin{smallmatrix}22\\3\\5\end{smallmatrix} | E_{21} | \begin{smallmatrix}12\\3\\5\end{smallmatrix} \rangle = \sqrt{2}$$

$$\langle \begin{smallmatrix}23\\3\\5\end{smallmatrix} | E_{21} | \begin{smallmatrix}13\\3\\5\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}12\\4\\5\end{smallmatrix} | E_{21} | \begin{smallmatrix}11\\4\\5\end{smallmatrix} \rangle = \sqrt{2}$$

$$\langle \begin{smallmatrix}22\\4\\5\end{smallmatrix} | E_{21} | \begin{smallmatrix}12\\4\\5\end{smallmatrix} \rangle = \sqrt{2}$$

$$\langle \begin{smallmatrix}23\\4\\5\end{smallmatrix} | E_{21} | \begin{smallmatrix}13\\4\\5\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}24\\4\\5\end{smallmatrix} | E_{21} | \begin{smallmatrix}14\\4\\5\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}13\\3\\5\end{smallmatrix} | E_{32} | \begin{smallmatrix}13\\2\\5\end{smallmatrix} \rangle = \sqrt{\tfrac{3}{2}}$$

$$\langle \begin{smallmatrix}22\\3\\4\end{smallmatrix} | E_{21} | \begin{smallmatrix}12\\3\\4\end{smallmatrix} \rangle = \sqrt{2}$$

$$\langle \begin{smallmatrix}24\\3\\5\end{smallmatrix} | E_{21} | \begin{smallmatrix}14\\3\\5\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}13\\2\\3\end{smallmatrix} | E_{32} | \begin{smallmatrix}12\\2\\3\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}11\\3\\4\end{smallmatrix} | E_{32} | \begin{smallmatrix}11\\2\\4\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}13\\2\\4\end{smallmatrix} | E_{32} | \begin{smallmatrix}12\\2\\4\end{smallmatrix} \rangle = \sqrt{\tfrac{3}{2}}$$

$$\langle \begin{smallmatrix}12\\3\\4\end{smallmatrix} | E_{32} | \begin{smallmatrix}12\\2\\4\end{smallmatrix} \rangle = \frac{1}{\sqrt{2}}$$

$$\langle \begin{smallmatrix}13\\3\\4\end{smallmatrix} | E_{32} | \begin{smallmatrix}13\\2\\4\end{smallmatrix} \rangle = \sqrt{\tfrac{3}{2}}$$

$$\langle \begin{smallmatrix}14\\3\\4\end{smallmatrix} | E_{32} | \begin{smallmatrix}14\\2\\4\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}11\\3\\5\end{smallmatrix} | E_{32} | \begin{smallmatrix}11\\2\\5\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}13\\2\\5\end{smallmatrix} | E_{32} | \begin{smallmatrix}12\\2\\5\end{smallmatrix} \rangle = \sqrt{\tfrac{3}{2}}$$

$$\langle \begin{smallmatrix}12\\3\\5\end{smallmatrix} | E_{32} | \begin{smallmatrix}12\\2\\5\end{smallmatrix} \rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix} 25 \\ 3 \\ 4 \end{smallmatrix} \right| E_{21} \left| \begin{smallmatrix} 15 \\ 3 \\ 4 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 25 \\ 3 \\ 5 \end{smallmatrix} \right| E_{21} \left| \begin{smallmatrix} 15 \\ 3 \\ 5 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 12 \\ 3 \\ 4 \end{smallmatrix} \right| E_{21} \left| \begin{smallmatrix} 11 \\ 3 \\ 4 \end{smallmatrix} \right\rangle = \sqrt{2}$$

$$\left\langle \begin{smallmatrix} 23 \\ 3 \\ 5 \end{smallmatrix} \right| E_{32} \left| \begin{smallmatrix} 22 \\ 3 \\ 5 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 15 \\ 3 \\ 4 \end{smallmatrix} \right| E_{32} \left| \begin{smallmatrix} 15 \\ 2 \\ 4 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 15 \\ 3 \\ 5 \end{smallmatrix} \right| E_{32} \left| \begin{smallmatrix} 15 \\ 2 \\ 5 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 35 \\ 4 \\ 5 \end{smallmatrix} \right| E_{32} \left| \begin{smallmatrix} 25 \\ 4 \\ 5 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 13 \\ 3 \\ 4 \end{smallmatrix} \right| E_{32} \left| \begin{smallmatrix} 12 \\ 3 \\ 4 \end{smallmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix} 14 \\ 3 \\ 5 \end{smallmatrix} \right| E_{32} \left| \begin{smallmatrix} 14 \\ 2 \\ 5 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 23 \\ 3 \\ 4 \end{smallmatrix} \right| E_{32} \left| \begin{smallmatrix} 22 \\ 3 \\ 4 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 11 \\ 2 \\ 4 \end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix} 11 \\ 2 \\ 3 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 12 \\ 2 \\ 4 \end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix} 12 \\ 2 \\ 3 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 13 \\ 3 \\ 5 \end{smallmatrix} \right| E_{32} \left| \begin{smallmatrix} 12 \\ 3 \\ 5 \end{smallmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix} 13 \\ 4 \\ 5 \end{smallmatrix} \right| E_{32} \left| \begin{smallmatrix} 12 \\ 4 \\ 5 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 23 \\ 4 \\ 5 \end{smallmatrix} \right| E_{32} \left| \begin{smallmatrix} 22 \\ 4 \\ 5 \end{smallmatrix} \right\rangle = \sqrt{2}$$

$$\left\langle \begin{smallmatrix} 33 \\ 4 \\ 5 \end{smallmatrix} \right| E_{32} \left| \begin{smallmatrix} 23 \\ 4 \\ 5 \end{smallmatrix} \right\rangle = \sqrt{2}$$

$$\left\langle \begin{smallmatrix} 34 \\ 4 \\ 5 \end{smallmatrix} \right| E_{32} \left| \begin{smallmatrix} 24 \\ 4 \\ 5 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 34 \\ 4 \\ 5 \end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix} 33 \\ 4 \\ 5 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 11 \\ 4 \\ 5 \end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix} 11 \\ 3 \\ 5 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 12 \\ 4 \\ 5 \end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix} 12 \\ 3 \\ 5 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 22 \\ 4 \\ 5 \end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix} 22 \\ 3 \\ 5 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 14 \\ 2 \\ 5 \end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix} 13 \\ 2 \\ 5 \end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix} 14 \\ 3 \\ 5 \end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix} 13 \\ 3 \\ 5 \end{smallmatrix} \right\rangle = \sqrt{\frac{3}{2}}$$

$$\left\langle \begin{smallmatrix} 13 \\ 4 \\ 5 \end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix} 13 \\ 3 \\ 5 \end{smallmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}1&4\\2\\3\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}1&3\\2\\3\end{smallmatrix} \right\rangle = \sqrt{\tfrac{4}{3}}$$

$$\left\langle \begin{smallmatrix}1&3\\2\\4\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}1&3\\2\\3\end{smallmatrix} \right\rangle = \sqrt{\tfrac{2}{3}}$$

$$\left\langle \begin{smallmatrix}1&4\\2\\4\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}1&3\\2\\4\end{smallmatrix} \right\rangle = \sqrt{\tfrac{2}{3}}$$

$$\left\langle \begin{smallmatrix}1&4\\3\\4\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}1&3\\3\\4\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}2&4\\3\\4\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}2&3\\3\\4\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1&4\\2\\4\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}1&4\\2\\3\end{smallmatrix} \right\rangle = \sqrt{\tfrac{4}{3}}$$

$$\left\langle \begin{smallmatrix}1&1\\2\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}1&1\\2\\5\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1&2\\2\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}1&2\\2\\4\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1&3\\2\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}1&3\\2\\4\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1&3\\3\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}1&3\\3\\4\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1&5\\2\\3\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}2&3\\2\\3\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1&5\\2\\4\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}1&4\\2\\4\end{smallmatrix} \right\rangle = \sqrt{\tfrac{4}{3}}$$

$$\left\langle \begin{smallmatrix}1&4\\2\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}1&4\\2\\4\end{smallmatrix} \right\rangle = \sqrt{\tfrac{2}{3}}$$

$$\left\langle \begin{smallmatrix}2&4\\3\\5\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}2&3\\3\\5\end{smallmatrix} \right\rangle = \sqrt{\tfrac{3}{2}}$$

$$\left\langle \begin{smallmatrix}2&3\\4\\5\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}2&3\\3\\5\end{smallmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}1&4\\4\\5\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}1&3\\4\\5\end{smallmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}2&4\\4\\5\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}2&3\\4\\5\end{smallmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}1&5\\2\\4\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}1&5\\2\\3\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1&5\\4\\5\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}1&5\\3\\5\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}2&5\\4\\5\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}2&5\\3\\5\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1&4\\4\\5\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}1&4\\3\\5\end{smallmatrix} \right\rangle = \sqrt{\tfrac{3}{2}}$$

$$\left\langle \begin{smallmatrix}1&4\\4\\5\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}1&4\\3\\5\end{smallmatrix} \right\rangle = \sqrt{\tfrac{3}{2}}$$

$$\left\langle \begin{smallmatrix}2&4\\4\\5\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}2&4\\3\\5\end{smallmatrix} \right\rangle = \sqrt{\tfrac{3}{2}}$$

$$\left\langle \begin{smallmatrix}1&5\\2\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}1&5\\2\\5\end{smallmatrix} \right\rangle = \sqrt{\tfrac{4}{3}}$$

$$\left\langle \begin{smallmatrix}1&5\\3\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}1&5\\3\\4\end{smallmatrix} \right\rangle = \sqrt{\tfrac{4}{3}}$$

$$\left\langle \begin{smallmatrix}2&5\\3\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}2&5\\3\\4\end{smallmatrix} \right\rangle = \sqrt{\tfrac{4}{3}}$$

$$\left\langle \begin{smallmatrix}15\\3\\4\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}14\\3\\4\end{smallmatrix} \right\rangle = \sqrt{\tfrac{4}{3}}$$

$$\left\langle \begin{smallmatrix}14\\3\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}14\\3\\4\end{smallmatrix} \right\rangle = \sqrt{\tfrac{2}{3}}$$

$$\left\langle \begin{smallmatrix}24\\3\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}24\\3\\4\end{smallmatrix} \right\rangle = \sqrt{\tfrac{2}{3}}$$

$$\left\langle \begin{smallmatrix}25\\3\\4\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}25\\3\\4\end{smallmatrix} \right\rangle = \sqrt{\tfrac{4}{3}}$$

$$\left\langle \begin{smallmatrix}15\\4\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}14\\4\\5\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}25\\4\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}24\\4\\5\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}35\\4\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}34\\4\\5\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}13\\2\\5\end{smallmatrix} \right| E_{31} \left| \begin{smallmatrix}11\\2\\5\end{smallmatrix} \right\rangle = \sqrt{\tfrac{3}{2}}$$

$$\left\langle \begin{smallmatrix}13\\2\\3\end{smallmatrix} \right| E_{31} \left| \begin{smallmatrix}11\\2\\3\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}12\\3\\4\end{smallmatrix} \right| E_{31} \left| \begin{smallmatrix}11\\2\\4\end{smallmatrix} \right\rangle = -\frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}13\\2\\4\end{smallmatrix} \right| E_{31} \left| \begin{smallmatrix}11\\2\\4\end{smallmatrix} \right\rangle = \sqrt{\tfrac{3}{2}}$$

$$\left\langle \begin{smallmatrix}22\\3\\4\end{smallmatrix} \right| E_{31} \left| \begin{smallmatrix}12\\2\\4\end{smallmatrix} \right\rangle = -1$$

$$\left\langle \begin{smallmatrix}11\\2\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}11\\3\\4\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}11\\3\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}11\\3\\4\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}12\\3\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}12\\3\\4\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}22\\3\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}22\\3\\4\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}15\\2\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}14\\2\\5\end{smallmatrix} \right\rangle = \sqrt{\tfrac{2}{3}}$$

$$\left\langle \begin{smallmatrix}15\\3\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}14\\3\\5\end{smallmatrix} \right\rangle = \sqrt{\tfrac{2}{3}}$$

$$\left\langle \begin{smallmatrix}25\\3\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}24\\3\\5\end{smallmatrix} \right\rangle = \sqrt{\tfrac{2}{3}}$$

$$\left\langle \begin{smallmatrix}24\\3\\4\end{smallmatrix} \right| E_{31} \left| \begin{smallmatrix}14\\2\\4\end{smallmatrix} \right\rangle = -1$$

$$\left\langle \begin{smallmatrix}24\\3\\5\end{smallmatrix} \right| E_{31} \left| \begin{smallmatrix}14\\2\\5\end{smallmatrix} \right\rangle = -1$$

$$\left\langle \begin{smallmatrix}33\\4\\5\end{smallmatrix} \right| E_{31} \left| \begin{smallmatrix}13\\4\\5\end{smallmatrix} \right\rangle = \sqrt{2}$$

$$\left\langle \begin{smallmatrix}25\\3\\5\end{smallmatrix} \right| E_{31} \left| \begin{smallmatrix}15\\2\\5\end{smallmatrix} \right\rangle = -1$$

$$\left\langle \begin{smallmatrix}34\\4\\5\end{smallmatrix} \right| E_{31} \left| \begin{smallmatrix}14\\4\\5\end{smallmatrix} \right\rangle = 1$$

$$\langle \begin{smallmatrix}12\\3\\5\end{smallmatrix} | E_{31} | \begin{smallmatrix}11\\2\\5\end{smallmatrix} \rangle = -\frac{1}{\sqrt{2}}$$

$$\langle \begin{smallmatrix}13\\3\\4\end{smallmatrix} | E_{31} | \begin{smallmatrix}11\\3\\4\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}23\\3\\4\end{smallmatrix} | E_{31} | \begin{smallmatrix}13\\2\\4\end{smallmatrix} \rangle = -\sqrt{\frac{3}{2}}$$

$$\langle \begin{smallmatrix}22\\3\\5\end{smallmatrix} | E_{31} | \begin{smallmatrix}12\\2\\5\end{smallmatrix} \rangle = -1$$

$$\langle \begin{smallmatrix}13\\3\\5\end{smallmatrix} | E_{31} | \begin{smallmatrix}11\\3\\5\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}23\\3\\4\end{smallmatrix} | E_{31} | \begin{smallmatrix}11\\3\\4\end{smallmatrix} \rangle = \frac{1}{\sqrt{2}}$$

$$\langle \begin{smallmatrix}24\\3\\4\end{smallmatrix} | E_{31} | \begin{smallmatrix}14\\2\\4\end{smallmatrix} \rangle = -1$$

$$\langle \begin{smallmatrix}23\\3\\5\end{smallmatrix} | E_{31} | \begin{smallmatrix}12\\3\\5\end{smallmatrix} \rangle = -\frac{1}{\sqrt{2}}$$

$$\langle \begin{smallmatrix}23\\3\\5\end{smallmatrix} | E_{31} | \begin{smallmatrix}13\\2\\5\end{smallmatrix} \rangle = -\sqrt{\frac{3}{2}}$$

$$\langle \begin{smallmatrix}13\\4\\5\end{smallmatrix} | E_{31} | \begin{smallmatrix}11\\4\\5\end{smallmatrix} \rangle = \sqrt{2}$$

$$\langle \begin{smallmatrix}23\\4\\5\end{smallmatrix} | E_{31} | \begin{smallmatrix}12\\4\\5\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}25\\3\\4\end{smallmatrix} | E_{31} | \begin{smallmatrix}15\\2\\4\end{smallmatrix} \rangle = -1$$

$$\langle \begin{smallmatrix}35\\4\\5\end{smallmatrix} | E_{32} | \begin{smallmatrix}15\\4\\5\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}35\\4\\5\end{smallmatrix} | E_{31} | \begin{smallmatrix}15\\4\\5\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}13\\2\\4\end{smallmatrix} | E_{41} | \begin{smallmatrix}11\\2\\3\end{smallmatrix} \rangle = -\frac{1}{\sqrt{6}}$$

$$\langle \begin{smallmatrix}12\\3\\4\end{smallmatrix} | E_{41} | \begin{smallmatrix}11\\2\\3\end{smallmatrix} \rangle = \frac{1}{\sqrt{2}}$$

$$\langle \begin{smallmatrix}22\\3\\4\end{smallmatrix} | E_{41} | \begin{smallmatrix}12\\2\\3\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}14\\2\\4\end{smallmatrix} | E_{41} | \begin{smallmatrix}11\\2\\4\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}23\\3\\4\end{smallmatrix} | E_{41} | \begin{smallmatrix}13\\2\\3\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}12\\4\\5\end{smallmatrix} | E_{41} | \begin{smallmatrix}11\\2\\5\end{smallmatrix} \rangle = -\frac{1}{\sqrt{2}}$$

$$\langle \begin{smallmatrix}14\\3\\4\end{smallmatrix} | E_{41} | \begin{smallmatrix}11\\3\\4\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}24\\3\\4\end{smallmatrix} | E_{41} | \begin{smallmatrix}13\\2\\4\end{smallmatrix} \rangle = \frac{1}{\sqrt{6}}$$

$$\langle \begin{smallmatrix}24\\3\\4\end{smallmatrix} | E_{41} | \begin{smallmatrix}14\\2\\3\end{smallmatrix} \rangle = \sqrt{\frac{4}{3}}$$

$$\langle \begin{smallmatrix}22\\4\\4\end{smallmatrix} | E_{41} | \begin{smallmatrix}12\\2\\5\end{smallmatrix} \rangle = -1$$

$$\left\langle \begin{smallmatrix}1&4\\2\\5\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1&1\\2\\5\end{smallmatrix} \right\rangle = \sqrt{\tfrac{3}{2}}$$

$$\left\langle \begin{smallmatrix}2&4\\3\\4\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1&2\\3\\4\end{smallmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}2&3\\4\\5\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1&2\\3\\5\end{smallmatrix} \right\rangle = -\frac{1}{2}$$

$$\left\langle \begin{smallmatrix}2&4\\3\\5\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1&2\\3\\5\end{smallmatrix} \right\rangle = \frac{\sqrt{3}}{2}$$

$$\left\langle \begin{smallmatrix}2&4\\3\\5\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1&3\\2\\5\end{smallmatrix} \right\rangle = -\frac{1}{2}$$

$$\left\langle \begin{smallmatrix}2&3\\4\\5\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1&3\\2\\5\end{smallmatrix} \right\rangle = -\frac{\sqrt{3}}{2}$$

$$\left\langle \begin{smallmatrix}1&4\\4\\5\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1&1\\4\\5\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}3&3\\4\\5\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1&3\\3\\5\end{smallmatrix} \right\rangle = -1$$

$$\left\langle \begin{smallmatrix}2&4\\4\\5\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1&2\\4\\5\end{smallmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}2&4\\4\\5\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1&4\\2\\5\end{smallmatrix} \right\rangle = -\sqrt{\tfrac{3}{2}}$$

$$\left\langle \begin{smallmatrix}3&4\\4\\5\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1&3\\4\\5\end{smallmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}2&5\\4\\5\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1&5\\2\\5\end{smallmatrix} \right\rangle = -1$$

$$\left\langle \begin{smallmatrix}1&3\\4\\5\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1&1\\3\\5\end{smallmatrix} \right\rangle = -\frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}1&4\\3\\5\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1&1\\3\\5\end{smallmatrix} \right\rangle = \sqrt{\tfrac{3}{2}}$$

$$\left\langle \begin{smallmatrix}1&2\\4\\5\end{smallmatrix} \right| E_{51} \left| \begin{smallmatrix}1&1\\2\\4\end{smallmatrix} \right\rangle = -\frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}2&2\\4\\5\end{smallmatrix} \right| E_{51} \left| \begin{smallmatrix}1&2\\2\\4\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1&5\\2\\5\end{smallmatrix} \right| E_{51} \left| \begin{smallmatrix}1&1\\2\\5\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1&5\\3\\4\end{smallmatrix} \right| E_{51} \left| \begin{smallmatrix}1&1\\3\\4\end{smallmatrix} \right\rangle = \sqrt{\tfrac{4}{3}}$$

$$\left\langle \begin{smallmatrix}1&3\\4\\5\end{smallmatrix} \right| E_{51} \left| \begin{smallmatrix}1&1\\3\\4\end{smallmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}1&4\\3\\5\end{smallmatrix} \right| E_{51} \left| \begin{smallmatrix}1&1\\3\\4\end{smallmatrix} \right\rangle = -\frac{1}{\sqrt{6}}$$

$$\left\langle \begin{smallmatrix}2&4\\3\\5\end{smallmatrix} \right| E_{51} \left| \begin{smallmatrix}1&1\\3\\4\end{smallmatrix} \right\rangle = -\frac{\sqrt{3}}{6}$$

$$\left\langle \begin{smallmatrix}2&5\\3\\4\end{smallmatrix} \right| E_{51} \left| \begin{smallmatrix}1&2\\3\\4\end{smallmatrix} \right\rangle = \sqrt{\tfrac{2}{3}}$$

$$\left\langle \begin{smallmatrix}2&3\\4\\5\end{smallmatrix} \right| E_{51} \left| \begin{smallmatrix}1&2\\3\\4\end{smallmatrix} \right\rangle = \frac{1}{2}$$

$$\left\langle \begin{smallmatrix}3&3\\4\\5\end{smallmatrix} \right| E_{51} \left| \begin{smallmatrix}1&3\\3\\4\end{smallmatrix} \right\rangle = 1$$

$$\langle \begin{smallmatrix}35\\4\\5\end{smallmatrix} | E_{41} | \begin{smallmatrix}15\\3\\5\end{smallmatrix} \rangle = -1$$

$$\langle \begin{smallmatrix}13\\2\\5\end{smallmatrix} | E_{51} | \begin{smallmatrix}11\\2\\3\end{smallmatrix} \rangle = -\frac{1}{\sqrt{6}}$$

$$\langle \begin{smallmatrix}22\\3\\5\end{smallmatrix} | E_{51} | \begin{smallmatrix}12\\2\\3\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}14\\2\\5\end{smallmatrix} | E_{51} | \begin{smallmatrix}11\\2\\4\end{smallmatrix} \rangle = -\frac{1}{\sqrt{6}}$$

$$\langle \begin{smallmatrix}15\\2\\4\end{smallmatrix} | E_{51} | \begin{smallmatrix}11\\2\\4\end{smallmatrix} \rangle = \sqrt{\frac{4}{3}}$$

$$\langle \begin{smallmatrix}35\\4\\5\end{smallmatrix} | E_{51} | \begin{smallmatrix}13\\4\\5\end{smallmatrix} \rangle = \frac{1}{\sqrt{2}}$$

$$\langle \begin{smallmatrix}35\\4\\4\end{smallmatrix} | E_{51} | \begin{smallmatrix}15\\3\\4\end{smallmatrix} \rangle = \sqrt{\frac{4}{3}}$$

$$\langle \begin{smallmatrix}35\\4\\5\end{smallmatrix} | E_{51} | \begin{smallmatrix}14\\3\\5\end{smallmatrix} \rangle = \sqrt{\frac{2}{3}}$$

$$\langle \begin{smallmatrix}11\\3\\4\end{smallmatrix} | E_{42} | \begin{smallmatrix}11\\2\\3\end{smallmatrix} \rangle = -1$$

$$\langle \begin{smallmatrix}14\\2\\3\end{smallmatrix} | E_{42} | \begin{smallmatrix}12\\2\\3\end{smallmatrix} \rangle = \sqrt{\frac{4}{3}}$$

$$\langle \begin{smallmatrix}12\\3\\4\end{smallmatrix} | E_{42} | \begin{smallmatrix}12\\2\\3\end{smallmatrix} \rangle = -\frac{1}{\sqrt{2}}$$

$$\langle \begin{smallmatrix}13\\2\\4\end{smallmatrix} | E_{42} | \begin{smallmatrix}12\\2\\3\end{smallmatrix} \rangle = -\frac{1}{\sqrt{6}}$$

$$\langle \begin{smallmatrix}24\\4\\5\end{smallmatrix} | E_{51} | \begin{smallmatrix}14\\2\\4\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}25\\3\\5\end{smallmatrix} | E_{51} | \begin{smallmatrix}12\\4\\5\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}25\\3\\5\end{smallmatrix} | E_{51} | \begin{smallmatrix}13\\2\\5\end{smallmatrix} \rangle = -\frac{1}{\sqrt{6}}$$

$$\langle \begin{smallmatrix}15\\4\\5\end{smallmatrix} | E_{51} | \begin{smallmatrix}11\\4\\5\end{smallmatrix} \rangle = 1$$

$$\langle \begin{smallmatrix}25\\4\\5\end{smallmatrix} | E_{51} | \begin{smallmatrix}12\\4\\5\end{smallmatrix} \rangle = \frac{1}{\sqrt{2}}$$

$$\langle \begin{smallmatrix}25\\4\\5\end{smallmatrix} | E_{51} | \begin{smallmatrix}15\\2\\5\end{smallmatrix} \rangle = \sqrt{\frac{4}{3}}$$

$$\langle \begin{smallmatrix}25\\4\\5\end{smallmatrix} | E_{51} | \begin{smallmatrix}14\\2\\5\end{smallmatrix} \rangle = -\frac{1}{\sqrt{6}}$$

$$\langle \begin{smallmatrix}14\\3\\5\end{smallmatrix} | E_{42} | \begin{smallmatrix}12\\3\\5\end{smallmatrix} \rangle = \frac{1}{2}\sqrt{3}$$

$$\langle \begin{smallmatrix}13\\4\\5\end{smallmatrix} | E_{42} | \begin{smallmatrix}13\\2\\5\end{smallmatrix} \rangle = -\frac{\sqrt{3}}{2}$$

$$\langle \begin{smallmatrix}14\\3\\5\end{smallmatrix} | E_{42} | \begin{smallmatrix}13\\2\\5\end{smallmatrix} \rangle = \frac{1}{2}$$

$$\langle \begin{smallmatrix}15\\3\\4\end{smallmatrix} | E_{42} | \begin{smallmatrix}15\\2\\3\end{smallmatrix} \rangle = -1$$

$$\langle \begin{smallmatrix}24\\3\\4\end{smallmatrix} | E_{42} | \begin{smallmatrix}22\\3\\4\end{smallmatrix} \rangle = 1$$

$$\left\langle \begin{matrix}13\\3\\4\end{matrix} \middle| E_{42} \middle| \begin{matrix}13\\2\\3\end{matrix} \right\rangle = -1$$

$$\left\langle \begin{matrix}14\\2\\4\end{matrix} \middle| E_{42} \middle| \begin{matrix}12\\2\\4\end{matrix} \right\rangle = 1$$

$$\left\langle \begin{matrix}11\\4\\5\end{matrix} \middle| E_{42} \middle| \begin{matrix}11\\2\\5\end{matrix} \right\rangle = 1$$

$$\left\langle \begin{matrix}14\\3\\4\end{matrix} \middle| E_{42} \middle| \begin{matrix}13\\2\\4\end{matrix} \right\rangle = \frac{1}{\sqrt{6}}$$

$$\left\langle \begin{matrix}14\\3\\4\end{matrix} \middle| E_{42} \middle| \begin{matrix}14\\2\\3\end{matrix} \right\rangle = -\sqrt{\frac{4}{3}}$$

$$\left\langle \begin{matrix}12\\4\\5\end{matrix} \middle| E_{42} \middle| \begin{matrix}12\\2\\5\end{matrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{matrix}14\\2\\5\end{matrix} \middle| E_{42} \middle| \begin{matrix}12\\2\\5\end{matrix} \right\rangle = \sqrt{\frac{3}{2}}$$

$$\left\langle \begin{matrix}14\\3\\4\end{matrix} \middle| E_{42} \middle| \begin{matrix}12\\3\\4\end{matrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{matrix}13\\4\\5\end{matrix} \middle| E_{42} \middle| \begin{matrix}12\\3\\5\end{matrix} \right\rangle = -\frac{1}{2}$$

$$\left\langle \begin{matrix}11\\3\\5\end{matrix} \middle| E_{52} \middle| \begin{matrix}11\\2\\3\end{matrix} \right\rangle = -1$$

$$\left\langle \begin{matrix}15\\2\\3\end{matrix} \middle| E_{52} \middle| \begin{matrix}12\\2\\3\end{matrix} \right\rangle = \sqrt{\frac{4}{3}}$$

$$\left\langle \begin{matrix}12\\3\\5\end{matrix} \middle| E_{52} \middle| \begin{matrix}12\\2\\3\end{matrix} \right\rangle = -\frac{1}{\sqrt{2}}$$

$$\left\langle \begin{matrix}13\\2\\5\end{matrix} \middle| E_{52} \middle| \begin{matrix}12\\2\\3\end{matrix} \right\rangle = -\frac{1}{\sqrt{6}}$$

$$\left\langle \begin{matrix}23\\4\\5\end{matrix} \middle| E_{42} \middle| \begin{matrix}22\\3\\5\end{matrix} \right\rangle = -\frac{1}{\sqrt{2}}$$

$$\left\langle \begin{matrix}24\\3\\5\end{matrix} \middle| E_{42} \middle| \begin{matrix}22\\3\\5\end{matrix} \right\rangle = \sqrt{\frac{3}{2}}$$

$$\left\langle \begin{matrix}14\\4\\5\end{matrix} \middle| E_{42} \middle| \begin{matrix}12\\4\\5\end{matrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{matrix}14\\4\\5\end{matrix} \middle| E_{42} \middle| \begin{matrix}14\\2\\5\end{matrix} \right\rangle = \sqrt{\frac{3}{2}}$$

$$\left\langle \begin{matrix}33\\4\\5\end{matrix} \middle| E_{42} \middle| \begin{matrix}23\\3\\5\end{matrix} \right\rangle = -1$$

$$\left\langle \begin{matrix}24\\4\\5\end{matrix} \middle| E_{42} \middle| \begin{matrix}22\\4\\5\end{matrix} \right\rangle = 1$$

$$\left\langle \begin{matrix}15\\4\\5\end{matrix} \middle| E_{42} \middle| \begin{matrix}15\\2\\5\end{matrix} \right\rangle = 1$$

$$\left\langle \begin{matrix}34\\4\\5\end{matrix} \middle| E_{42} \middle| \begin{matrix}23\\4\\5\end{matrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{matrix}34\\4\\5\end{matrix} \middle| E_{42} \middle| \begin{matrix}24\\3\\5\end{matrix} \right\rangle = -\sqrt{\frac{3}{2}}$$

$$\left\langle \begin{matrix}35\\4\\5\end{matrix} \middle| E_{42} \middle| \begin{matrix}25\\3\\5\end{matrix} \right\rangle = -1$$

$$\left\langle \begin{matrix}14\\3\\5\end{matrix} \middle| E_{52} \middle| \begin{matrix}12\\3\\4\end{matrix} \right\rangle = -\frac{\sqrt{3}}{6}$$

$$\left\langle \begin{matrix}15\\3\\4\end{matrix} \middle| E_{52} \middle| \begin{matrix}12\\3\\4\end{matrix} \right\rangle = \sqrt{\frac{2}{3}}$$

$$\left\langle \begin{matrix}14\\4\\5\end{matrix} \middle| E_{52} \middle| \begin{matrix}14\\2\\5\end{matrix} \right\rangle = -1$$

$$\left\langle \begin{smallmatrix}1&1\\4\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&1\\2\\4\end{smallmatrix} \right\rangle = -1$$

$$\left\langle \begin{smallmatrix}1&3\\3\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&3\\2\\3\end{smallmatrix} \right\rangle = -1$$

$$\left\langle \begin{smallmatrix}1&5\\2\\4\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&1\\2\\4\end{smallmatrix} \right\rangle = \sqrt{\tfrac{4}{3}}$$

$$\left\langle \begin{smallmatrix}1&4\\2\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&2\\2\\4\end{smallmatrix} \right\rangle = -\frac{1}{\sqrt{6}}$$

$$\left\langle \begin{smallmatrix}1&2\\4\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&2\\2\\4\end{smallmatrix} \right\rangle = -\frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}1&5\\3\\4\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&3\\2\\4\end{smallmatrix} \right\rangle = \frac{\sqrt{2}}{3}$$

$$\left\langle \begin{smallmatrix}1&4\\3\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&3\\2\\4\end{smallmatrix} \right\rangle = -\frac{1}{6}$$

$$\left\langle \begin{smallmatrix}1&3\\4\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&3\\2\\4\end{smallmatrix} \right\rangle = \frac{\sqrt{3}}{2}$$

$$\left\langle \begin{smallmatrix}1&4\\3\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&4\\2\\3\end{smallmatrix} \right\rangle = -\frac{2}{3}\sqrt{2}$$

$$\left\langle \begin{smallmatrix}1&5\\3\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&4\\2\\5\end{smallmatrix} \right\rangle = -\frac{1}{3}$$

$$\left\langle \begin{smallmatrix}1&5\\2\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&2\\2\\5\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1&3\\4\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&2\\3\\4\end{smallmatrix} \right\rangle = \frac{1}{2}$$

$$\left\langle \begin{smallmatrix}1&5\\3\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&2\\3\\5\end{smallmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}1&5\\3\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&3\\2\\5\end{smallmatrix} \right\rangle = \frac{1}{\sqrt{6}}$$

$$\left\langle \begin{smallmatrix}1&5\\3\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&5\\2\\3\end{smallmatrix} \right\rangle = -\sqrt{\tfrac{4}{3}}$$

$$\left\langle \begin{smallmatrix}2&3\\4\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}2&2\\3\\4\end{smallmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}2&4\\3\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}2&2\\3\\4\end{smallmatrix} \right\rangle = -\frac{1}{\sqrt{6}}$$

$$\left\langle \begin{smallmatrix}2&5\\3\\4\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}2&2\\3\\4\end{smallmatrix} \right\rangle = \sqrt{\tfrac{4}{3}}$$

$$\left\langle \begin{smallmatrix}3&3\\4\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}2&3\\3\\4\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}2&5\\3\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}2&2\\3\\5\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1&5\\4\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&2\\4\\5\end{smallmatrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{smallmatrix}1&5\\4\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&5\\2\\4\end{smallmatrix} \right\rangle = -\sqrt{\tfrac{4}{3}}$$

$$\left\langle \begin{smallmatrix}1&5\\4\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1&4\\2\\5\end{smallmatrix} \right\rangle = \frac{1}{\sqrt{6}}$$

$$\left\langle \begin{smallmatrix}3&4\\4\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}2&4\\3\\5\end{smallmatrix} \right\rangle = \frac{1}{\sqrt{3}}$$

$$\left\langle \begin{array}{ll}3&5\\4&\\5&\end{array} \right| E_{52} \left| \begin{array}{ll}2&3\\4&\\5&\end{array} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{array}{ll}3&5\\4&\\5&\end{array} \right| E_{52} \left| \begin{array}{ll}2&5\\3&\\4&\end{array} \right\rangle = \sqrt{\frac{4}{3}}$$

$$\left\langle \begin{array}{ll}3&5\\4&\\5&\end{array} \right| E_{52} \left| \begin{array}{ll}2&4\\3&\\5&\end{array} \right\rangle = -\frac{1}{\sqrt{6}}$$

$$\left\langle \begin{array}{ll}1&1\\2&\\5&\end{array} \right| E_{52} \left| \begin{array}{ll}1&1\\2&\\5&\end{array} \right\rangle = 1$$

$$\left\langle \begin{array}{ll}1&2\\2&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}1&2\\2&\\3&\end{array} \right\rangle = 1$$

$$\left\langle \begin{array}{ll}1&5\\2&\\3&\end{array} \right| E_{53} \left| \begin{array}{ll}1&3\\2&\\3&\end{array} \right\rangle = \sqrt{\frac{4}{3}}$$

$$\left\langle \begin{array}{ll}1&3\\2&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}1&3\\2&\\3&\end{array} \right\rangle = \sqrt{\frac{2}{3}}$$

$$\left\langle \begin{array}{ll}1&1\\4&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}1&1\\3&\\4&\end{array} \right\rangle = -1$$

$$\left\langle \begin{array}{ll}1&4\\2&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}1&3\\2&\\4&\end{array} \right\rangle = -\frac{1}{3}$$

$$\left\langle \begin{array}{ll}1&5\\2&\\4&\end{array} \right| E_{53} \left| \begin{array}{ll}1&3\\2&\\4&\end{array} \right\rangle = \frac{2\sqrt{2}}{3}$$

$$\left\langle \begin{array}{ll}1&4\\2&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}1&4\\2&\\3&\end{array} \right\rangle = \frac{2\sqrt{2}}{3}$$

$$\left\langle \begin{array}{ll}1&5\\2&\\4&\end{array} \right| E_{53} \left| \begin{array}{ll}1&4\\2&\\3&\end{array} \right\rangle = \frac{1}{3}$$

$$\left\langle \begin{array}{ll}2&5\\4&\\5&\end{array} \right| E_{52} \left| \begin{array}{ll}2&2\\4&\\5&\end{array} \right\rangle = 1$$

$$\left\langle \begin{array}{ll}1&5\\2&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}1&3\\2&\\5&\end{array} \right\rangle = \sqrt{\frac{2}{3}}$$

$$\left\langle \begin{array}{ll}1&5\\2&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}1&5\\2&\\3&\end{array} \right\rangle = \sqrt{\frac{4}{3}}$$

$$\left\langle \begin{array}{ll}2&2\\4&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}2&2\\3&\\4&\end{array} \right\rangle = -1$$

$$\left\langle \begin{array}{ll}2&3\\4&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}2&3\\3&\\4&\end{array} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{array}{ll}2&5\\3&\\4&\end{array} \right| E_{53} \left| \begin{array}{ll}2&3\\3&\\4&\end{array} \right\rangle = \sqrt{\frac{4}{3}}$$

$$\left\langle \begin{array}{ll}2&4\\3&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}2&3\\3&\\4&\end{array} \right\rangle = -\frac{1}{\sqrt{6}}$$

$$\left\langle \begin{array}{ll}1&4\\4&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}1&4\\3&\\4&\end{array} \right\rangle = -1$$

$$\left\langle \begin{array}{ll}1&5\\3&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}1&3\\3&\\5&\end{array} \right\rangle = 1$$

$$\left\langle \begin{array}{ll}2&4\\4&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}2&4\\3&\\4&\end{array} \right\rangle = -1$$

$$\left\langle \begin{array}{ll}2&5\\3&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}2&3\\3&\\5&\end{array} \right\rangle = 1$$

$$\left\langle \begin{array}{ll}1&5\\4&\\5&\end{array} \right| E_{53} \left| \begin{array}{ll}1&3\\4&\\5&\end{array} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{matrix}12\\4\\5\end{matrix} \middle| E_{53} \middle| \begin{matrix}11\\3\\4\end{matrix} \right\rangle = -1$$

$$\left\langle \begin{matrix}13\\4\\5\end{matrix} \middle| E_{53} \middle| \begin{matrix}13\\3\\4\end{matrix} \right\rangle = -\frac{1}{\sqrt{2}}$$

$$\left\langle \begin{matrix}14\\3\\5\end{matrix} \middle| E_{53} \middle| \begin{matrix}13\\3\\4\end{matrix} \right\rangle = -\frac{1}{\sqrt{6}}$$

$$\left\langle \begin{matrix}15\\3\\4\end{matrix} \middle| E_{53} \middle| \begin{matrix}13\\3\\4\end{matrix} \right\rangle = \sqrt{\frac{4}{3}}$$

$$\left\langle \begin{matrix}15\\4\\5\end{matrix} \middle| E_{53} \middle| \begin{matrix}15\\3\\4\end{matrix} \right\rangle = -\sqrt{\frac{4}{3}}$$

$$\left\langle \begin{matrix}15\\4\\5\end{matrix} \middle| E_{53} \middle| \begin{matrix}14\\3\\5\end{matrix} \right\rangle = \frac{1}{\sqrt{6}}$$

$$\left\langle \begin{matrix}25\\4\\5\end{matrix} \middle| E_{53} \middle| \begin{matrix}23\\4\\5\end{matrix} \right\rangle = \frac{1}{\sqrt{2}}$$

$$\left\langle \begin{matrix}25\\4\\5\end{matrix} \middle| E_{53} \middle| \begin{matrix}25\\3\\4\end{matrix} \right\rangle = \sqrt{\frac{4}{3}}$$

$$\left\langle \begin{matrix}25\\4\\5\end{matrix} \middle| E_{53} \middle| \begin{matrix}24\\3\\5\end{matrix} \right\rangle = \frac{1}{\sqrt{6}}$$

$$\langle {12 \atop 23} | E_{21} | {11 \atop 23} \rangle = 1$$

$$\langle {12 \atop 33} | E_{21} | {11 \atop 33} \rangle = \sqrt{2}$$

$$\langle {22 \atop 33} | E_{21} | {12 \atop 33} \rangle = \sqrt{2}$$

$$\langle {12 \atop 24} | E_{21} | {11 \atop 24} \rangle = 1$$

$$\langle {12 \atop 34} | E_{21} | {11 \atop 34} \rangle = \sqrt{2}$$

$$\langle {22 \atop 34} | E_{21} | {12 \atop 34} \rangle = \sqrt{2}$$

$$\langle {23 \atop 34} | E_{21} | {13 \atop 34} \rangle = 1$$

$$\langle {12 \atop 44} | E_{21} | {11 \atop 44} \rangle = \sqrt{2}$$

$$\langle {22 \atop 44} | E_{21} | {12 \atop 44} \rangle = 1$$

$$\langle {12 \atop 25} | E_{21} | {11 \atop 25} \rangle = 1$$

$$\langle {12 \atop 35} | E_{21} | {11 \atop 35} \rangle = \sqrt{2}$$

$$\langle {22 \atop 35} | E_{21} | {12 \atop 35} \rangle = \sqrt{2}$$

$$\langle {23 \atop 35} | E_{21} | {13 \atop 35} \rangle = 1$$

$$\langle {12 \atop 45} | E_{21} | {11 \atop 45} \rangle = \sqrt{2}$$

$$\langle {23 \atop 45} | E_{21} | {13 \atop 45} \rangle = 1$$

$$\langle {24 \atop 45} | E_{21} | {14 \atop 45} \rangle = 1$$

$$\langle {22 \atop 45} | E_{21} | {12 \atop 45} \rangle = \sqrt{2}$$

$$\langle {24 \atop 35} | E_{21} | {14 \atop 35} \rangle = 1$$

$$\langle {12 \atop 55} | E_{21} | {11 \atop 55} \rangle = \sqrt{2}$$

$$\langle {22 \atop 55} | E_{21} | {12 \atop 55} \rangle = \sqrt{2}$$

$$\langle {23 \atop 55} | E_{21} | {13 \atop 55} \rangle = 1$$

$$\langle {24 \atop 55} | E_{21} | {14 \atop 55} \rangle = 1$$

$$\langle {11 \atop 23} | E_{32} | {11 \atop 22} \rangle = \sqrt{2}$$

$$\langle {11 \atop 33} | E_{32} | {11 \atop 23} \rangle = \sqrt{2}$$

$$\langle {12 \atop 33} | E_{32} | {12 \atop 23} \rangle = 1$$

$$\langle {13 \atop 24} | E_{32} | {12 \atop 24} \rangle = \sqrt{\tfrac{3}{2}}$$

$$\langle {12 \atop 34} | E_{32} | {12 \atop 24} \rangle = \frac{1}{\sqrt{2}}$$

$$\langle {11 \atop 34} | E_{32} | {11 \atop 24} \rangle = 1$$

$$\langle {13 \atop 34} | E_{32} | {12 \atop 34} \rangle = \frac{1}{\sqrt{2}}$$

$$\langle {23 \atop 34} | E_{32} | {22 \atop 34} \rangle = 1$$

$$\langle {13 \atop 34} | E_{32} | {13 \atop 24} \rangle = \sqrt{\tfrac{3}{2}}$$

$$\langle {13 \atop 44} | E_{32} | {12 \atop 44} \rangle = 1$$

$\langle {}^{13}_{25} | E_{32} | {}^{12}_{25} \rangle = \sqrt{\tfrac{3}{2}}$

$\langle {}^{12}_{35} | E_{32} | {}^{12}_{25} \rangle = \frac{1}{\sqrt{2}}$

$\langle {}^{13}_{35} | E_{32} | {}^{12}_{35} \rangle = \frac{1}{\sqrt{2}}$

$\langle {}^{23}_{35} | E_{32} | {}^{22}_{35} \rangle = 1$

$\langle {}^{13}_{35} | E_{32} | {}^{13}_{25} \rangle = \sqrt{\tfrac{3}{2}}$

$\langle {}^{13}_{45} | E_{32} | {}^{12}_{45} \rangle = 1$

$\langle {}^{23}_{45} | E_{32} | {}^{22}_{45} \rangle = \sqrt{2}$

$\langle {}^{34}_{45} | E_{32} | {}^{24}_{45} \rangle = 1$

$\langle {}^{14}_{35} | E_{32} | {}^{14}_{25} \rangle = 1$

$\langle {}^{13}_{55} | E_{32} | {}^{12}_{45} \rangle = \sqrt{2}$

$\langle {}^{23}_{55} | E_{32} | {}^{22}_{55} \rangle = \sqrt{2}$

$\langle {}^{33}_{55} | E_{32} | {}^{23}_{55} \rangle = \sqrt{2}$

$\langle {}^{34}_{55} | E_{43} | {}^{24}_{23} \rangle = 1$

$\langle {}^{11}_{24} | E_{43} | {}^{11}_{23} \rangle = 1$

$\langle {}^{12}_{24} | E_{43} | {}^{12}_{23} \rangle = 1$

$\langle {}^{14}_{45} | E_{43} | {}^{13}_{45} \rangle = \frac{1}{\sqrt{2}}$

$\langle {}^{33}_{44} | E_{32} | {}^{23}_{44} \rangle = \sqrt{2}$

$\langle {}^{11}_{35} | E_{32} | {}^{11}_{25} \rangle = 1$

$\langle {}^{23}_{44} | E_{32} | {}^{22}_{44} \rangle = \sqrt{2}$

$\langle {}^{11}_{34} | E_{43} | {}^{11}_{33} \rangle = \sqrt{2}$

$\langle {}^{12}_{34} | E_{43} | {}^{12}_{33} \rangle = \sqrt{2}$

$\langle {}^{22}_{34} | E_{43} | {}^{22}_{33} \rangle = \sqrt{2}$

$\langle {}^{11}_{44} | E_{43} | {}^{11}_{34} \rangle = \sqrt{2}$

$\langle {}^{12}_{44} | E_{43} | {}^{12}_{34} \rangle = \sqrt{2}$

$\langle {}^{22}_{44} | E_{43} | {}^{22}_{34} \rangle = \sqrt{2}$

$\langle {}^{13}_{44} | E_{43} | {}^{13}_{34} \rangle = 1$

$\langle {}^{23}_{44} | E_{43} | {}^{23}_{34} \rangle = 1$

$\langle {}^{11}_{45} | E_{43} | {}^{11}_{35} \rangle = 1$

$\langle {}^{12}_{45} | E_{43} | {}^{12}_{35} \rangle = 1$

$\langle {}^{22}_{45} | E_{43} | {}^{22}_{35} \rangle = 1$

$\langle {}^{14}_{25} | E_{43} | {}^{13}_{25} \rangle = 1$

$\langle {}^{14}_{35} | E_{43} | {}^{13}_{35} \rangle = \sqrt{\tfrac{3}{2}}$

$$\langle {}^{24}_{45}|E_{43}|{}^{23}_{45}\rangle = \frac{1}{\sqrt{2}}$$

$$\langle {}^{34}_{45}|E_{43}|{}^{33}_{45}\rangle = 1$$

$$\langle {}^{14}_{45}|E_{43}|{}^{14}_{35}\rangle = \sqrt{\tfrac{3}{2}}$$

$$\langle {}^{24}_{45}|E_{43}|{}^{24}_{35}\rangle = \sqrt{\tfrac{3}{2}}$$

$$\langle {}^{15}_{55}|E_{43}|{}^{13}_{55}\rangle = 1$$

$$\langle {}^{24}_{55}|E_{43}|{}^{23}_{55}\rangle = 1$$

$$\langle {}^{34}_{55}|E_{43}|{}^{33}_{55}\rangle = \sqrt{2}$$

$$\langle {}^{44}_{55}|E_{43}|{}^{34}_{55}\rangle = \sqrt{2}$$

$$\langle {}^{12}_{25}|E_{54}|{}^{12}_{24}\rangle = 1$$

$$\langle {}^{11}_{35}|E_{54}|{}^{11}_{34}\rangle = 1$$

$$\langle {}^{12}_{35}|E_{54}|{}^{12}_{34}\rangle = 1$$

$$\langle {}^{22}_{35}|E_{54}|{}^{22}_{34}\rangle = 1$$

$$\langle {}^{13}_{25}|E_{54}|{}^{13}_{24}\rangle = 1$$

$$\langle {}^{13}_{35}|E_{54}|{}^{13}_{34}\rangle = 1$$

$$\langle {}^{23}_{35}|E_{54}|{}^{23}_{34}\rangle = 1$$

$$\langle {}^{44}_{55}|E_{52}|{}^{24}_{45}\rangle = -\sqrt{2}$$

$$\langle {}^{13}_{45}|E_{43}|{}^{13}_{35}\rangle = \sqrt{\tfrac{1}{2}}$$

$$\langle {}^{24}_{35}|E_{43}|{}^{23}_{35}\rangle = \sqrt{\tfrac{3}{2}}$$

$$\langle {}^{23}_{45}|E_{43}|{}^{23}_{35}\rangle = \frac{1}{\sqrt{2}}$$

$$\langle {}^{11}_{45}|E_{54}|{}^{11}_{44}\rangle = \sqrt{2}$$

$$\langle {}^{12}_{45}|E_{54}|{}^{12}_{44}\rangle = \sqrt{2}$$

$$\langle {}^{22}_{45}|E_{54}|{}^{22}_{44}\rangle = \sqrt{2}$$

$$\langle {}^{13}_{45}|E_{54}|{}^{13}_{44}\rangle = \sqrt{2}$$

$$\langle {}^{23}_{45}|E_{54}|{}^{23}_{44}\rangle = \sqrt{2}$$

$$\langle {}^{33}_{45}|E_{54}|{}^{33}_{44}\rangle = \sqrt{2}$$

$$\langle {}^{11}_{55}|E_{54}|{}^{11}_{45}\rangle = \sqrt{2}$$

$$\langle {}^{12}_{55}|E_{54}|{}^{12}_{45}\rangle = \sqrt{2}$$

$$\langle {}^{13}_{55}|E_{54}|{}^{13}_{45}\rangle = \sqrt{2}$$

$$\langle {}^{23}_{55}|E_{54}|{}^{23}_{45}\rangle = \sqrt{2}$$

$$\langle {}^{33}_{55}|E_{54}|{}^{33}_{45}\rangle = \sqrt{2}$$

$$\langle {}^{14}_{55}|E_{54}|{}^{14}_{45}\rangle = 1$$

$$\langle {}^{24}_{55}|E_{54}|{}^{24}_{45}\rangle = 1$$

$\langle {11 \atop 25} | E_{53} | {11 \atop 23} \rangle = 1$

$\langle {12 \atop 25} | E_{53} | {12 \atop 23} \rangle = 1$

$\langle {11 \atop 35} | E_{53} | {11 \atop 33} \rangle = \sqrt{2}$

$\langle {12 \atop 35} | E_{53} | {12 \atop 33} \rangle = \sqrt{2}$

$\langle {11 \atop 45} | E_{53} | {11 \atop 34} \rangle = 1$

$\langle {12 \atop 45} | E_{53} | {12 \atop 34} \rangle = 1$

$\langle {14 \atop 25} | E_{53} | {13 \atop 24} \rangle = -1$

$\langle {11 \atop 55} | E_{53} | {11 \atop 35} \rangle = \sqrt{2}$

$\langle {22 \atop 35} | E_{53} | {22 \atop 33} \rangle = \sqrt{2}$

$\langle {22 \atop 45} | E_{53} | {22 \atop 34} \rangle = 1$

$\langle {13 \atop 45} | E_{53} | {13 \atop 34} \rangle = \frac{1}{\sqrt{2}}$

$\langle {14 \atop 35} | E_{53} | {13 \atop 34} \rangle = -\sqrt{\frac{3}{2}}$

$\langle {12 \atop 55} | E_{53} | {12 \atop 35} \rangle = \sqrt{2}$

$\langle {23 \atop 45} | E_{53} | {23 \atop 34} \rangle = -\frac{1}{\sqrt{2}}$

$\langle {14 \atop 45} | E_{53} | {13 \atop 44} \rangle = -1$

$\langle {12 \atop 23} | E_{31} | {11 \atop 22} \rangle = -\sqrt{2}$

$\langle {34 \atop 55} | E_{54} | {34 \atop 45} \rangle = 1$

$\langle {22 \atop 55} | E_{54} | {22 \atop 45} \rangle = \sqrt{2}$

$\langle {22 \atop 55} | E_{53} | {22 \atop 35} \rangle = \sqrt{2}$

$\langle {13 \atop 55} | E_{53} | {13 \atop 35} \rangle = 1$

$\langle {24 \atop 45} | E_{53} | {23 \atop 44} \rangle = -1$

$\langle {23 \atop 55} | E_{53} | {23 \atop 35} \rangle = 1$

$\langle {14 \atop 55} | E_{53} | {13 \atop 45} \rangle = -\frac{1}{\sqrt{2}}$

$\langle {14 \atop 55} | E_{53} | {14 \atop 35} \rangle = \sqrt{\frac{3}{2}}$

$\langle {24 \atop 55} | E_{53} | {23 \atop 45} \rangle = -\frac{1}{\sqrt{2}}$

$\langle {24 \atop 55} | E_{53} | {24 \atop 35} \rangle = \sqrt{\frac{3}{2}}$

$\langle {34 \atop 45} | E_{53} | {33 \atop 44} \rangle = -\sqrt{2}$

$\langle {34 \atop 55} | E_{53} | {33 \atop 45} \rangle = -1$

$\langle {44 \atop 55} | E_{53} | {34 \atop 45} \rangle = -\sqrt{2}$

$\langle {23 \atop 44} | E_{31} | {12 \atop 44} \rangle = 1$

$\langle {33 \atop 44} | E_{31} | {13 \atop 44} \rangle = \sqrt{2}$

$\langle {23 \atop 45} | E_{31} | {12 \atop 45} \rangle = 1$

$$\langle {12 \atop 33} | E_{31} | {11 \atop 22} \rangle = -1$$

$$\langle {22 \atop 33} | E_{31} | {12 \atop 23} \rangle = -\sqrt{2}$$

$$\langle {12 \atop 34} | E_{31} | {11 \atop 24} \rangle = -\frac{1}{\sqrt{2}}$$

$$\langle {13 \atop 24} | E_{31} | {11 \atop 24} \rangle = \sqrt{\tfrac{3}{2}}$$

$$\langle {13 \atop 34} | E_{31} | {11 \atop 34} \rangle = 1$$

$$\langle {22 \atop 34} | E_{31} | {12 \atop 25} \rangle = \sqrt{\tfrac{3}{2}}$$

$$\langle {12 \atop 35} | E_{31} | {11 \atop 25} \rangle = -\frac{1}{\sqrt{2}}$$

$$\langle {23 \atop 34} | E_{31} | {12 \atop 24} \rangle = \frac{1}{\sqrt{2}}$$

$$\langle {23 \atop 34} | E_{31} | {13 \atop 24} \rangle = -\sqrt{\tfrac{3}{2}}$$

$$\langle {23 \atop 34} | E_{31} | {12 \atop 34} \rangle = \frac{1}{\sqrt{2}}$$

$$\langle {13 \atop 44} | E_{31} | {11 \atop 44} \rangle = \sqrt{2}$$

$$\langle {22 \atop 35} | E_{31} | {11 \atop 25} \rangle = -1$$

$$\langle {13 \atop 35} | E_{31} | {11 \atop 35} \rangle = 1$$

$$\langle {23 \atop 35} | E_{31} | {12 \atop 35} \rangle = \frac{1}{\sqrt{2}}$$

$$\langle {23 \atop 35} | E_{31} | {13 \atop 25} \rangle = -\sqrt{\tfrac{3}{2}}$$

$$\langle {13 \atop 45} | E_{31} | {11 \atop 45} \rangle = \sqrt{2}$$

$$\langle {24 \atop 35} | E_{31} | {14 \atop 25} \rangle = -1$$

$$\langle {13 \atop 55} | E_{31} | {11 \atop 55} \rangle = \sqrt{2}$$

$$\langle {33 \atop 45} | E_{31} | {13 \atop 45} \rangle = 2$$

$$\langle {23 \atop 55} | E_{31} | {12 \atop 55} \rangle = 1$$

$$\langle {34 \atop 45} | E_{31} | {14 \atop 45} \rangle = 1$$

$$\langle {33 \atop 55} | E_{31} | {13 \atop 55} \rangle = \sqrt{2}$$

$$\langle {34 \atop 55} | E_{31} | {14 \atop 55} \rangle = 1$$

$$\langle {12 \atop 24} | E_{41} | {11 \atop 22} \rangle = -\sqrt{2}$$

$$\langle {13 \atop 24} | E_{41} | {11 \atop 23} \rangle = -\sqrt{\tfrac{3}{2}}$$

$$\langle {12 \atop 34} | E_{41} | {11 \atop 23} \rangle = \frac{1}{\sqrt{2}}$$

$$\langle {22 \atop 34} | E_{41} | {12 \atop 23} \rangle = -1$$

$$\langle {13 \atop 34} | E_{41} | {11 \atop 33} \rangle = -\sqrt{2}$$

$$\langle {12 \atop 44} | E_{41} | {11 \atop 24} \rangle = -1$$

$$\langle {23 \atop 34} | E_{41} | {12 \atop 33} \rangle = -1$$

$$\langle {13 \atop 44} | E_{41} | {11 \atop 34} \rangle = -1$$

$$\langle {24 \atop 55} | E_{41} | {12 \atop 55} \rangle = 1$$

$\langle {14 \atop 25} | E_{41} | {11 \atop 25} \rangle = \sqrt{\frac{3}{2}}$

$\langle {22 \atop 44} | E_{41} | {11 \atop 24} \rangle = -\sqrt{2}$

$\langle {12 \atop 45} | E_{41} | {11 \atop 25} \rangle = -\frac{1}{\sqrt{2}}$

$\langle {23 \atop 44} | E_{41} | {12 \atop 34} \rangle = -\sqrt{\frac{3}{2}}$

$\langle {22 \atop 45} | E_{41} | {12 \atop 25} \rangle = -1$

$\langle {13 \atop 45} | E_{41} | {11 \atop 35} \rangle = -\frac{1}{\sqrt{2}}$

$\langle {14 \atop 35} | E_{41} | {11 \atop 35} \rangle = \sqrt{\frac{3}{2}}$

$\langle {33 \atop 44} | E_{41} | {13 \atop 34} \rangle = -\sqrt{2}$

$\langle {23 \atop 45} | E_{41} | {12 \atop 35} \rangle = -\frac{1}{2}$

$\langle {24 \atop 35} | E_{41} | {12 \atop 25} \rangle = \frac{1}{2}\sqrt{3}$

$\langle {24 \atop 35} | E_{41} | {13 \atop 25} \rangle = -\frac{1}{2}$

$\langle {23 \atop 45} | E_{41} | {23 \atop 25} \rangle = -\frac{1}{2}\sqrt{3}$

$\langle {14 \atop 45} | E_{41} | {11 \atop 45} \rangle = 1$

$\langle {33 \atop 45} | E_{41} | {13 \atop 35} \rangle = -1$

$\langle {24 \atop 45} | E_{41} | {12 \atop 45} \rangle = \frac{1}{\sqrt{2}}$

$\langle {24 \atop 45} | E_{41} | {14 \atop 25} \rangle = -\sqrt{\frac{3}{2}}$

$\langle {14 \atop 55} | E_{41} | {11 \atop 55} \rangle = \sqrt{2}$

$\langle {34 \atop 45} | E_{41} | {13 \atop 45} \rangle = \frac{1}{\sqrt{2}}$

$\langle {34 \atop 45} | E_{41} | {13 \atop 45} \rangle = \frac{1}{\sqrt{2}}$

$\langle {34 \atop 55} | E_{41} | {13 \atop 55} \rangle = 1$

$\langle {44 \atop 55} | E_{41} | {14 \atop 55} \rangle = \sqrt{2}$

$\langle {12 \atop 25} | E_{51} | {11 \atop 22} \rangle = -\sqrt{2}$

$\langle {12 \atop 34} | E_{51} | {11 \atop 23} \rangle = \frac{1}{\sqrt{2}}$

$\langle {13 \atop 25} | E_{51} | {11 \atop 23} \rangle = -\sqrt{\frac{3}{2}}$

$\langle {22 \atop 35} | E_{51} | {12 \atop 23} \rangle = -1$

$\langle {13 \atop 35} | E_{51} | {11 \atop 33} \rangle = -\sqrt{2}$

$\langle {12 \atop 45} | E_{51} | {11 \atop 24} \rangle = -\frac{1}{\sqrt{2}}$

$\langle {23 \atop 35} | E_{51} | {11 \atop 33} \rangle = -1$

$\langle {13 \atop 45} | E_{51} | {11 \atop 34} \rangle = -\frac{1}{\sqrt{2}}$

$\langle {14 \atop 35} | E_{51} | {11 \atop 34} \rangle = -\sqrt{\frac{3}{2}}$

$\langle {22 \atop 45} | E_{51} | {12 \atop 24} \rangle = -1$

$\langle {12 \atop 55} | E_{51} | {11 \atop 25} \rangle = -1$

$\langle {24 \atop 35} | E_{51} | {12 \atop 34} \rangle = -\frac{\sqrt{3}}{2}$

$\langle {23 \atop 45} | E_{51} | {12 \atop 34} \rangle = -\frac{1}{2}$

$$\langle {}^{14}_{45}|E_{51}|{}^{11}_{44}\rangle = -\sqrt{2}$$

$$\langle {}^{22}_{44}|E_{51}|{}^{12}_{25}\rangle = -\sqrt{2}$$

$$\langle {}^{13}_{55}|E_{51}|{}^{11}_{35}\rangle = \sqrt{3}$$

$$\langle {}^{33}_{45}|E_{51}|{}^{13}_{34}\rangle = -1$$

$$\langle {}^{23}_{55}|E_{51}|{}^{12}_{35}\rangle = -\frac{1}{\sqrt{2}}$$

$$\langle {}^{23}_{55}|E_{51}|{}^{13}_{25}\rangle = -\sqrt{\frac{3}{2}}$$

$$\langle {}^{14}_{55}|E_{51}|{}^{11}_{45}\rangle = -1$$

$$\langle {}^{24}_{45}|E_{51}|{}^{12}_{44}\rangle = -1$$

$$\langle {}^{34}_{45}|E_{51}|{}^{13}_{44}\rangle = -1$$

$$\langle {}^{33}_{55}|E_{51}|{}^{13}_{35}\rangle = -\sqrt{2}$$

$$\langle {}^{24}_{55}|E_{51}|{}^{12}_{55}\rangle = -\frac{1}{\sqrt{2}}$$

$$\langle {}^{44}_{55}|E_{51}|{}^{14}_{55}\rangle = -\sqrt{2}$$

$$\langle {}^{11}_{24}|E_{42}|{}^{11}_{22}\rangle = \sqrt{2}$$

$$\langle {}^{11}_{34}|E_{42}|{}^{11}_{23}\rangle = 1$$

$$\langle {}^{12}_{34}|E_{42}|{}^{12}_{23}\rangle = \frac{1}{\sqrt{2}}$$

$$\langle {}^{11}_{44}|E_{42}|{}^{11}_{24}\rangle = \sqrt{2}$$

$$\langle {}^{24}_{35}|E_{51}|{}^{13}_{24}\rangle = \frac{1}{2}$$

$$\langle {}^{23}_{45}|E_{51}|{}^{13}_{24}\rangle = -\sqrt{\frac{3}{2}}$$

$$\langle {}^{14}_{55}|E_{42}|{}^{12}_{55}\rangle = 1$$

$$\langle {}^{12}_{45}|E_{42}|{}^{12}_{25}\rangle = \frac{1}{\sqrt{2}}$$

$$\langle {}^{11}_{45}|E_{42}|{}^{11}_{25}\rangle = 1$$

$$\langle {}^{13}_{34}|E_{42}|{}^{12}_{33}\rangle = -1$$

$$\langle {}^{13}_{44}|E_{42}|{}^{12}_{34}\rangle = -\frac{1}{\sqrt{2}}$$

$$\langle {}^{13}_{44}|E_{42}|{}^{13}_{24}\rangle = \sqrt{\frac{3}{2}}$$

$$\langle {}^{14}_{24}|E_{42}|{}^{12}_{25}\rangle = \sqrt{\frac{3}{2}}$$

$$\langle {}^{23}_{34}|E_{42}|{}^{22}_{33}\rangle = -\sqrt{2}$$

$$\langle {}^{23}_{44}|E_{42}|{}^{22}_{34}\rangle = -1$$

$$\langle {}^{14}_{35}|E_{42}|{}^{12}_{35}\rangle = \frac{1}{2}\sqrt{3}$$

$$\langle {}^{13}_{45}|E_{42}|{}^{12}_{35}\rangle = -\frac{1}{2}$$

$$\langle {}^{14}_{35}|E_{42}|{}^{13}_{25}\rangle = \frac{1}{2}$$

$$\langle {}^{13}_{45}|E_{42}|{}^{13}_{25}\rangle = \frac{1}{2}\sqrt{3}$$

$$\langle {}^{33}_{44}|E_{42}|{}^{23}_{34}\rangle = -\sqrt{2}$$

$$\langle {14 \atop 55} | E_{42} | {13 \atop 55} \rangle = 1$$

$$\langle {34 \atop 45} | E_{42} | {23 \atop 45} \rangle = \frac{1}{\sqrt{2}}$$

$$\langle {34 \atop 45} | E_{42} | {24 \atop 35} \rangle = -\sqrt{\frac{3}{2}}$$

$$\langle {24 \atop 55} | E_{42} | {22 \atop 55} \rangle = \sqrt{2}$$

$$\langle {34 \atop 55} | E_{42} | {23 \atop 55} \rangle = 1$$

$$\langle {44 \atop 55} | E_{42} | {24 \atop 55} \rangle = \sqrt{2}$$

$$\langle {11 \atop 25} | E_{52} | {11 \atop 22} \rangle = \sqrt{2}$$

$$\langle {11 \atop 35} | E_{52} | {11 \atop 23} \rangle = 1$$

$$\langle {12 \atop 35} | E_{52} | {12 \atop 23} \rangle = \frac{1}{\sqrt{2}}$$

$$\langle {11 \atop 45} | E_{52} | {11 \atop 24} \rangle = 1$$

$$\langle {13 \atop 35} | E_{52} | {12 \atop 33} \rangle = -1$$

$$\langle {12 \atop 45} | E_{52} | {12 \atop 24} \rangle = \frac{1}{\sqrt{2}}$$

$$\langle {14 \atop 25} | E_{52} | {12 \atop 24} \rangle = -\sqrt{\frac{3}{2}}$$

$$\langle {11 \atop 55} | E_{52} | {11 \atop 25} \rangle = \sqrt{2}$$

$$\langle {13 \atop 45} | E_{52} | {12 \atop 34} \rangle = -\frac{1}{2}$$

$$\langle {13 \atop 45} | E_{52} | {13 \atop 24} \rangle = \frac{\sqrt{3}}{2}$$

$$\langle {12 \atop 44} | E_{42} | {12 \atop 24} \rangle = 1$$

$$\langle {23 \atop 45} | E_{42} | {22 \atop 35} \rangle = -\frac{1}{\sqrt{2}}$$

$$\langle {24 \atop 35} | E_{42} | {22 \atop 35} \rangle = \sqrt{\frac{3}{2}}$$

$$\langle {14 \atop 45} | E_{42} | {12 \atop 45} \rangle = \sqrt{\frac{3}{2}}$$

$$\langle {33 \atop 45} | E_{42} | {23 \atop 35} \rangle = -1$$

$$\langle {24 \atop 45} | E_{42} | {22 \atop 45} \rangle = 1$$

$$\langle {12 \atop 55} | E_{52} | {12 \atop 25} \rangle = 1$$

$$\langle {23 \atop 35} | E_{52} | {22 \atop 33} \rangle = -\sqrt{2}$$

$$\langle {23 \atop 45} | E_{52} | {22 \atop 34} \rangle = -\frac{1}{\sqrt{2}}$$

$$\langle {24 \atop 35} | E_{52} | {22 \atop 34} \rangle = -\sqrt{\frac{3}{2}}$$

$$\langle {13 \atop 55} | E_{52} | {12 \atop 35} \rangle = -\frac{1}{\sqrt{2}}$$

$$\langle {13 \atop 55} | E_{52} | {13 \atop 25} \rangle = \sqrt{\frac{3}{2}}$$

$$\langle {14 \atop 45} | E_{52} | {11 \atop 45} \rangle = -1$$

$$\langle {33 \atop 45} | E_{52} | {23 \atop 34} \rangle = 1$$

$$\langle {24 \atop 45} | E_{52} | {22 \atop 44} \rangle = -\sqrt{2}$$

$$\langle {23 \atop 55} | E_{52} | {22 \atop 35} \rangle = -1$$

$$\langle {}^{14}_{35}|E_{51}|{}^{13}_{24}\rangle = -\frac{1}{2}$$

$$\langle {}^{14}_{55}|E_{52}|{}^{12}_{45}\rangle = -\frac{1}{\sqrt{2}}$$

$$\langle {}^{14}_{55}|E_{52}|{}^{14}_{25}\rangle = \sqrt{\frac{3}{2}}$$

$$\langle {}^{34}_{45}|E_{52}|{}^{11}_{55}\rangle = -1$$

$$\langle {}^{33}_{55}|E_{52}|{}^{23}_{55}\rangle = -\sqrt{2}$$

$$\langle {}^{24}_{55}|E_{52}|{}^{22}_{45}\rangle = -1$$

$$\langle {}^{34}_{55}|E_{52}|{}^{23}_{45}\rangle = -\frac{1}{\sqrt{2}}$$

$$\langle {}^{34}_{55}|E_{52}|{}^{24}_{35}\rangle = -\sqrt{\frac{3}{2}}$$

$$\langle {}^{112}_{2} | E_{21} | {}^{111}_{2} \rangle = \sqrt{2}$$

$$\langle {}^{122}_{2} | E_{21} | {}^{112}_{2} \rangle = \sqrt{2}$$

$$\langle {}^{112}_{3} | E_{21} | {}^{111}_{3} \rangle = \sqrt{3}$$

$$\langle {}^{122}_{3} | E_{21} | {}^{112}_{3} \rangle = 2$$

$$\langle {}^{222}_{3} | E_{21} | {}^{122}_{3} \rangle = \sqrt{3}$$

$$\langle {}^{123}_{2} | E_{21} | {}^{113}_{2} \rangle = 1$$

$$\langle {}^{123}_{3} | E_{21} | {}^{113}_{3} \rangle = \sqrt{2}$$

$$\langle {}^{223}_{3} | E_{21} | {}^{123}_{3} \rangle = \sqrt{2}$$

$$\langle {}^{233}_{3} | E_{21} | {}^{133}_{3} \rangle = 1$$

$$\langle {}^{112}_{4} | E_{21} | {}^{111}_{4} \rangle = \sqrt{3}$$

$$\langle {}^{122}_{4} | E_{21} | {}^{112}_{4} \rangle = 2$$

$$\langle {}^{222}_{4} | E_{21} | {}^{122}_{4} \rangle = \sqrt{3}$$

$$\langle {}^{123}_{4} | E_{21} | {}^{113}_{4} \rangle = \sqrt{2}$$

$$\langle {}^{223}_{4} | E_{21} | {}^{123}_{4} \rangle = \sqrt{2}$$

$$\langle {}^{233}_{4} | E_{21} | {}^{133}_{4} \rangle = \sqrt{2}$$

$$\langle {}^{224}_{4} | E_{21} | {}^{124}_{4} \rangle = \sqrt{2}$$

$$\langle {}^{234}_{4} | E_{21} | {}^{134}_{4} \rangle = 1$$

$$\langle {}^{124}_{2} | E_{21} | {}^{114}_{2} \rangle = 1$$

$$\langle {}^{124}_{3} | E_{21} | {}^{114}_{3} \rangle = \sqrt{2}$$

$$\langle {}^{224}_{3} | E_{21} | {}^{124}_{3} \rangle = \sqrt{2}$$

$$\langle {}^{234}_{3} | E_{21} | {}^{134}_{3} \rangle = 1$$

$$\langle {}^{244}_{3} | E_{21} | {}^{144}_{3} \rangle = 1$$

$$\langle {}^{244}_{4} | E_{21} | {}^{144}_{4} \rangle = 1$$

$$\langle {}^{112}_{5} | E_{21} | {}^{111}_{5} \rangle = \sqrt{3}$$

$$\langle {}^{122}_{5} | E_{21} | {}^{112}_{5} \rangle = 2$$

$$\langle {}^{222}_{5} | E_{21} | {}^{122}_{5} \rangle = \sqrt{3}$$

$$\langle {}^{123}_{5} | E_{21} | {}^{113}_{5} \rangle = \sqrt{2}$$

$$\langle {}^{223}_{5} | E_{21} | {}^{123}_{5} \rangle = \sqrt{2}$$

$$\langle {}^{233}_{5} | E_{21} | {}^{133}_{5} \rangle = 1$$

$$\langle {}^{124}_{5} | E_{21} | {}^{114}_{5} \rangle = \sqrt{2}$$

$$\langle {}^{224}_{5} | E_{21} | {}^{124}_{5} \rangle = \sqrt{2}$$

$$\langle {}^{244}_{5} | E_{21} | {}^{144}_{5} \rangle = 1$$

$\langle {}^{125}_{2} | E_{21} | {}^{115}_{2} \rangle = 1$

$\langle {}^{125}_{3} | E_{21} | {}^{115}_{3} \rangle = \sqrt{2}$

$\langle {}^{225}_{3} | E_{21} | {}^{125}_{3} \rangle = \sqrt{2}$

$\langle {}^{235}_{3} | E_{21} | {}^{135}_{3} \rangle = 1$

$\langle {}^{125}_{4} | E_{21} | {}^{115}_{4} \rangle = \sqrt{2}$

$\langle {}^{225}_{4} | E_{21} | {}^{125}_{4} \rangle = \sqrt{2}$

$\langle {}^{235}_{4} | E_{21} | {}^{135}_{4} \rangle = 1$

$\langle {}^{245}_{3} | E_{21} | {}^{145}_{3} \rangle = 1$

$\langle {}^{245}_{4} | E_{21} | {}^{145}_{4} \rangle = 1$

$\langle {}^{125}_{5} | E_{21} | {}^{115}_{5} \rangle = \sqrt{2}$

$\langle {}^{225}_{5} | E_{21} | {}^{125}_{5} \rangle = \sqrt{2}$

$\langle {}^{235}_{5} | E_{21} | {}^{135}_{5} \rangle = 1$

$\langle {}^{245}_{5} | E_{21} | {}^{145}_{5} \rangle = 1$

$\langle {}^{255}_{3} | E_{21} | {}^{155}_{3} \rangle = 1$

$\langle {}^{255}_{4} | E_{21} | {}^{155}_{4} \rangle = 1$

$\langle {}^{255}_{5} | E_{21} | {}^{155}_{5} \rangle = 1$

$\langle {}^{234}_{5} | E_{21} | {}^{134}_{5} \rangle = 1$

$\langle {}^{111}_{3} | E_{32} | {}^{111}_{2} \rangle = 1$

$\langle {}^{113}_{2} | E_{32} | {}^{112}_{2} \rangle = \sqrt{\tfrac{4}{3}}$

$\langle {}^{112}_{3} | E_{32} | {}^{112}_{2} \rangle = \sqrt{\tfrac{2}{3}}$

$\langle {}^{123}_{2} | E_{32} | {}^{122}_{2} \rangle = \sqrt{\tfrac{8}{3}}$

$\langle {}^{122}_{3} | E_{32} | {}^{122}_{2} \rangle = \sqrt{\tfrac{1}{3}}$

$\langle {}^{113}_{3} | E_{32} | {}^{112}_{3} \rangle = \sqrt{\tfrac{2}{3}}$

$\langle {}^{123}_{3} | E_{32} | {}^{122}_{3} \rangle = \sqrt{\tfrac{4}{3}}$

$\langle {}^{223}_{3} | E_{32} | {}^{222}_{3} \rangle = \sqrt{2}$

$\langle {}^{113}_{3} | E_{32} | {}^{113}_{2} \rangle = \sqrt{\tfrac{4}{3}}$

$\langle {}^{133}_{2} | E_{32} | {}^{123}_{2} \rangle = \sqrt{3}$

$\langle {}^{123}_{3} | E_{32} | {}^{123}_{2} \rangle = \sqrt{\tfrac{2}{3}}$

$\langle {}^{133}_{3} | E_{32} | {}^{123}_{3} \rangle = 1$

$\langle {}^{233}_{3} | E_{32} | {}^{223}_{3} \rangle = \sqrt{2}$

$\langle {}^{133}_{3} | E_{32} | {}^{133}_{2} \rangle = \sqrt{2}$

$\langle {}^{344}_{4} | E_{32} | {}^{244}_{4} \rangle = 1$

$\langle {}^{113}_{5} | E_{32} | {}^{112}_{5} \rangle = 1$

$\langle {}^{123}_{5} | E_{32} | {}^{122}_{5} \rangle = 1$

$\langle {}^{113}_{4} | E_{32} | {}^{112}_{4} \rangle = 1$

$\langle {}^{123}_{4} | E_{32} | {}^{122}_{4} \rangle = \sqrt{2}$

$\langle {}^{223}_{4} | E_{32} | {}^{222}_{4} \rangle = \sqrt{3}$

$\langle {}^{133}_{4} | E_{32} | {}^{123}_{4} \rangle = \sqrt{2}$

$\langle {}^{233}_{4} | E_{32} | {}^{223}_{4} \rangle = 2$

$\langle {}^{333}_{4} | E_{32} | {}^{233}_{4} \rangle = \sqrt{3}$

$\langle {}^{134}_{4} | E_{32} | {}^{124}_{4} \rangle = 1$

$\langle {}^{234}_{4} | E_{32} | {}^{224}_{4} \rangle = \sqrt{2}$

$\langle {}^{334}_{4} | E_{32} | {}^{234}_{4} \rangle = \sqrt{2}$

$\langle {}^{114}_{3} | E_{32} | {}^{114}_{2} \rangle = 1$

$\langle {}^{134}_{2} | E_{32} | {}^{124}_{2} \rangle = \sqrt{\frac{3}{2}}$

$\langle {}^{124}_{3} | E_{32} | {}^{124}_{2} \rangle = \frac{1}{\sqrt{2}}$

$\langle {}^{134}_{3} | E_{32} | {}^{124}_{3} \rangle = \frac{1}{\sqrt{2}}$

$\langle {}^{234}_{3} | E_{32} | {}^{224}_{3} \rangle = 1$

$\langle {}^{134}_{3} | E_{32} | {}^{134}_{2} \rangle = \sqrt{\frac{3}{2}}$

$\langle {}^{144}_{3} | E_{32} | {}^{144}_{2} \rangle = 1$

$\langle {}^{223}_{5} | E_{32} | {}^{123}_{5} \rangle = \sqrt{3}$

$\langle {}^{133}_{5} | E_{32} | {}^{123}_{5} \rangle = \sqrt{2}$

$\langle {}^{233}_{5} | E_{32} | {}^{223}_{5} \rangle = 2$

$\langle {}^{333}_{5} | E_{32} | {}^{233}_{5} \rangle = 3$

$\langle {}^{134}_{5} | E_{32} | {}^{124}_{5} \rangle = 1$

$\langle {}^{234}_{4} | E_{32} | {}^{224}_{5} \rangle = \sqrt{2}$

$\langle {}^{344}_{5} | E_{32} | {}^{244}_{5} \rangle = 1$

$\langle {}^{115}_{3} | E_{32} | {}^{115}_{2} \rangle = 1$

$\langle {}^{135}_{2} | E_{32} | {}^{125}_{2} \rangle = \sqrt{\frac{3}{2}}$

$\langle {}^{125}_{3} | E_{32} | {}^{125}_{2} \rangle = \frac{1}{\sqrt{2}}$

$\langle {}^{135}_{5} | E_{32} | {}^{125}_{3} \rangle = \frac{1}{\sqrt{2}}$

$\langle {}^{235}_{3} | E_{32} | {}^{225}_{3} \rangle = 1$

$\langle {}^{135}_{3} | E_{32} | {}^{135}_{2} \rangle = \sqrt{\frac{3}{2}}$

$\langle {}^{344}_{5} | E_{43} | {}^{334}_{5} \rangle = 2$

$\langle {}^{111}_{4} | E_{43} | {}^{111}_{3} \rangle = 1$

$\langle {}^{112}_{4} | E_{43} | {}^{112}_{3} \rangle = 1$

$\langle {}^{135}_{4} | E_{32} | {}^{125}_{4} \rangle = 1$

$\langle {}^{235}_{4} | E_{32} | {}^{225}_{4} \rangle = \sqrt{2}$

$\langle {}^{335}_{4} | E_{32} | {}^{235}_{4} \rangle = \sqrt{2}$

$\langle {}^{145}_{3} | E_{32} | {}^{145}_{2} \rangle = 1$

$\langle {}^{345}_{4} | E_{32} | {}^{245}_{4} \rangle = 1$

$\langle {}^{135}_{5} | E_{32} | {}^{125}_{5} \rangle = 1$

$\langle {}^{235}_{5} | E_{32} | {}^{225}_{5} \rangle = \sqrt{2}$

$\langle {}^{335}_{5} | E_{32} | {}^{235}_{5} \rangle = \sqrt{2}$

$\langle {}^{345}_{5} | E_{32} | {}^{245}_{5} \rangle = 1$

$\langle {}^{155}_{3} | E_{32} | {}^{155}_{2} \rangle = 1$

$\langle {}^{355}_{4} | E_{32} | {}^{255}_{4} \rangle = 1$

$\langle {}^{355}_{4} | E_{32} | {}^{255}_{5} \rangle = 1$

$\langle {}^{334}_{5} | E_{32} | {}^{234}_{5} \rangle = \sqrt{2}$

$\langle {}^{144}_{5} | E_{43} | {}^{134}_{5} \rangle = \sqrt{2}$

$\langle {}^{244}_{5} | E_{43} | {}^{234}_{5} \rangle = \sqrt{2}$

$\langle {}^{222}_{4} | E_{43} | {}^{222}_{3} \rangle = 1$

$\langle {}^{114}_{2} | E_{43} | {}^{113}_{2} \rangle = 1$

$\langle {}^{124}_{2} | E_{43} | {}^{123}_{2} \rangle = 1$

$\langle {}^{113}_{4} | E_{43} | {}^{113}_{3} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{114}_{3} | E_{43} | {}^{113}_{3} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{123}_{4} | E_{43} | {}^{123}_{3} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{124}_{3} | E_{43} | {}^{123}_{3} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{224}_{3} | E_{43} | {}^{223}_{3} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{223}_{4} | E_{43} | {}^{223}_{3} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{134}_{2} | E_{43} | {}^{133}_{2} \rangle = \sqrt{2}$

$\langle {}^{134}_{3} | E_{43} | {}^{133}_{3} \rangle = \sqrt{\frac{8}{3}}$

$\langle {}^{133}_{4} | E_{43} | {}^{133}_{3} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{234}_{4} | E_{43} | {}^{234}_{3} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{144}_{4} | E_{43} | {}^{144}_{3} \rangle = \sqrt{2}$

$\langle {}^{244}_{4} | E_{43} | {}^{244}_{3} \rangle = \sqrt{2}$

$\langle {}^{234}_{3} | E_{43} | {}^{233}_{3} \rangle = \sqrt{\frac{8}{3}}$

$\langle {}^{233}_{4} | E_{43} | {}^{233}_{3} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{114}_{4} | E_{43} | {}^{113}_{4} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{124}_{4} | E_{43} | {}^{123}_{4} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{234}_{4} | E_{43} | {}^{235}_{4} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{334}_{4} | E_{43} | {}^{333}_{4} \rangle = \sqrt{2}$

$\langle {}^{144}_{4} | E_{43} | {}^{134}_{4} \rangle = 1$

$\langle {}^{234}_{4} | E_{43} | {}^{234}_{4} \rangle = 1$

$\langle {}^{344}_{4} | E_{43} | {}^{334}_{4} \rangle = \sqrt{2}$

$\langle {}^{114}_{4} | E_{43} | {}^{114}_{3} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{124}_{4} | E_{43} | {}^{124}_{3} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{224}_{4} | E_{43} | {}^{224}_{3} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{144}_{2} | E_{43} | {}^{134}_{2} \rangle = \sqrt{2}$

$\langle {}^{144}_{3} | E_{43} | {}^{134}_{3} \rangle = \sqrt{3}$

$\langle {}^{134}_{4} | E_{43} | {}^{134}_{3} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{244}_{3} | E_{43} | {}^{234}_{3} \rangle = \sqrt{3}$

$\langle {}^{114}_{5} | E_{43} | {}^{113}_{5} \rangle = 1$

$\langle {}^{124}_{4} | E_{43} | {}^{123}_{5} \rangle = 1$

$\langle {}^{124}_{4} | E_{43} | {}^{223}_{5} \rangle = 1$

$\langle {}^{134}_{4} | E_{43} | {}^{133}_{5} \rangle = \sqrt{2}$

$\langle {}^{234}_{4} | E_{43} | {}^{233}_{5} \rangle = \sqrt{2}$

$\langle {}^{334}_{5} | E_{43} | {}^{333}_{5} \rangle = \sqrt{3}$

$\langle {}^{444}_{5} | E_{43} | {}^{344}_{5} \rangle = \sqrt{3}$

$\langle {}^{115}_{4} | E_{43} | {}^{115}_{3} \rangle = \sqrt{1}$

$\langle {}^{125}_{4} | E_{43} | {}^{125}_{3} \rangle = 1$

$\langle {}^{225}_{4} | E_{43} | {}^{225}_{3} \rangle = 1$

$\langle {}^{145}_{2} | E_{43} | {}^{135}_{2} \rangle = 1$

$\langle {}^{145}_{3} | E_{43} | {}^{135}_{3} \rangle = \sqrt{\frac{3}{2}}$

$\langle {}^{135}_{4} | E_{43} | {}^{135}_{3} \rangle = \frac{1}{\sqrt{2}}$

$\langle {}^{235}_{4} | E_{43} | {}^{235}_{3} \rangle = \frac{1}{\sqrt{2}}$

$\langle {}^{112}_{5} | E_{54} | {}^{112}_{4} \rangle = 1$

$\langle {}^{122}_{5} | E_{54} | {}^{122}_{4} \rangle = 1$

$\langle {}^{245}_{3} | E_{43} | {}^{234}_{3} \rangle = \sqrt{\tfrac{3}{2}}$

$\langle {}^{145}_{4} | E_{43} | {}^{135}_{4} \rangle = \frac{1}{\sqrt{2}}$

$\langle {}^{245}_{4} | E_{43} | {}^{235}_{4} \rangle = \frac{1}{\sqrt{2}}$

$\langle {}^{345}_{4} | E_{43} | {}^{335}_{4} \rangle = 1$

$\langle {}^{145}_{4} | E_{43} | {}^{145}_{3} \rangle = \sqrt{\tfrac{3}{2}}$

$\langle {}^{245}_{4} | E_{43} | {}^{245}_{3} \rangle = \sqrt{\tfrac{3}{2}}$

$\langle {}^{145}_{4} | E_{43} | {}^{135}_{5} \rangle = 1$

$\langle {}^{245}_{5} | E_{43} | {}^{235}_{5} \rangle = 1$

$\langle {}^{345}_{5} | E_{43} | {}^{335}_{5} \rangle = \sqrt{2}$

$\langle {}^{445}_{5} | E_{43} | {}^{345}_{5} \rangle = \sqrt{2}$

$\langle {}^{155}_{4} | E_{43} | {}^{155}_{3} \rangle = 1$

$\langle {}^{255}_{4} | E_{43} | {}^{255}_{3} \rangle = 1$

$\langle {}^{455}_{5} | E_{43} | {}^{355}_{5} \rangle = 1$

$\langle {}^{234}_{5} | E_{54} | {}^{234}_{5} \rangle = \sqrt{\tfrac{2}{3}}$

$\langle {}^{335}_{5} | E_{54} | {}^{334}_{5} \rangle = \sqrt{\tfrac{2}{3}}$

$\langle {}^{111}_{5} | E_{54} | {}^{111}_{4} \rangle = 1$

$\langle {}^{222}_{5} | E_{54} | {}^{222}_{4} \rangle = 1$

$\langle {}^{113}_{5} | E_{54} | {}^{113}_{4} \rangle = 1$

$\langle {}^{123}_{5} | E_{54} | {}^{123}_{4} \rangle = 1$

$\langle {}^{223}_{5} | E_{54} | {}^{223}_{4} \rangle = 1$

$\langle {}^{133}_{5} | E_{54} | {}^{133}_{4} \rangle = 1$

$\langle {}^{233}_{5} | E_{54} | {}^{233}_{4} \rangle = 1$

$\langle {}^{333}_{5} | E_{54} | {}^{333}_{4} \rangle = 1$

$\langle {}^{115}_{4} | E_{54} | {}^{114}_{4} \rangle = \sqrt{\tfrac{4}{3}}$

$\langle {}^{125}_{4} | E_{54} | {}^{124}_{4} \rangle = \sqrt{\tfrac{4}{3}}$

$\langle {}^{225}_{4} | E_{54} | {}^{224}_{4} \rangle = \sqrt{\tfrac{4}{3}}$

$\langle {}^{135}_{4} | E_{54} | {}^{134}_{4} \rangle = \sqrt{\tfrac{4}{3}}$

$\langle {}^{235}_{4} | E_{54} | {}^{234}_{4} \rangle = \sqrt{\tfrac{4}{3}}$

$\langle {}^{335}_{4} | E_{54} | {}^{334}_{4} \rangle = \sqrt{\tfrac{4}{3}}$

$\langle {}^{114}_{5} | E_{54} | {}^{114}_{4} \rangle = \sqrt{\tfrac{2}{3}}$

$\langle {}^{124}_{5} | E_{54} | {}^{124}_{4} \rangle = \sqrt{\tfrac{2}{3}}$

$\langle {}^{145}_{4} | E_{54} | {}^{144}_{4} \rangle = \sqrt{\tfrac{8}{3}}$

$\langle {}^{224}_{5} | E_{54} | {}^{224}_{4} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{134}_{5} | E_{54} | {}^{134}_{4} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{234}_{5} | E_{54} | {}^{234}_{4} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{334}_{5} | E_{54} | {}^{334}_{5} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{115}_{2} | E_{54} | {}^{114}_{2} \rangle = 1$

$\langle {}^{125}_{2} | E_{54} | {}^{124}_{2} \rangle = 1$

$\langle {}^{115}_{3} | E_{54} | {}^{114}_{3} \rangle = 1$

$\langle {}^{125}_{3} | E_{54} | {}^{124}_{3} \rangle = 1$

$\langle {}^{225}_{3} | E_{54} | {}^{224}_{3} \rangle = 1$

$\langle {}^{135}_{2} | E_{54} | {}^{134}_{2} \rangle = 1$

$\langle {}^{135}_{3} | E_{54} | {}^{134}_{3} \rangle = 1$

$\langle {}^{235}_{3} | E_{54} | {}^{234}_{3} \rangle = 1$

$\langle {}^{145}_{3} | E_{54} | {}^{144}_{2} \rangle = \sqrt{2}$

$\langle {}^{145}_{3} | E_{54} | {}^{144}_{3} \rangle = \sqrt{2}$

$\langle {}^{245}_{3} | E_{54} | {}^{244}_{3} \rangle = \sqrt{2}$

$\langle {}^{335}_{5} | E_{54} | {}^{335}_{4} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{144}_{5} | E_{54} | {}^{144}_{4} \rangle = \frac{1}{\sqrt{3}}$

$\langle {}^{244}_{5} | E_{54} | {}^{244}_{4} \rangle = -\frac{1}{\sqrt{3}}$

$\langle {}^{245}_{4} | E_{54} | {}^{244}_{4} \rangle = \sqrt{\frac{8}{3}}$

$\langle {}^{345}_{4} | E_{54} | {}^{344}_{4} \rangle = \sqrt{\frac{8}{3}}$

$\langle {}^{344}_{5} | E_{54} | {}^{344}_{4} \rangle = \frac{1}{\sqrt{3}}$

$\langle {}^{115}_{5} | E_{54} | {}^{114}_{5} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{125}_{5} | E_{54} | {}^{124}_{5} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{225}_{5} | E_{54} | {}^{224}_{5} \rangle = \sqrt{\frac{2}{3}}$

$\langle {}^{145}_{5} | E_{54} | {}^{144}_{5} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{245}_{5} | E_{54} | {}^{244}_{5} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{345}_{5} | E_{54} | {}^{344}_{5} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{445}_{5} | E_{54} | {}^{444}_{5} \rangle = \sqrt{2}$

$\langle {}^{115}_{5} | E_{54} | {}^{115}_{4} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{225}_{5} | E_{54} | {}^{225}_{4} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{135}_{5} | E_{54} | {}^{135}_{4} \rangle = \sqrt{\frac{4}{3}}$

$\langle {}^{235}_{5} | E_{54} | {}^{235}_{4} \rangle = \sqrt{\frac{4}{3}}$

$$\langle {}^{155}_{2} | E_{54} | {}^{145}_{2} \rangle = \sqrt{2}$$

$$\langle {}^{155}_{2} | E_{54} | {}^{145}_{2} \rangle = \sqrt{2}$$

$$\langle {}^{255}_{3} | E_{54} | {}^{245}_{3} \rangle = \sqrt{2}$$

$$\langle {}^{155}_{4} | E_{54} | {}^{145}_{4} \rangle = \sqrt{3}$$

$$\langle {}^{145}_{5} | E_{54} | {}^{145}_{4} \rangle = \sqrt{\tfrac{2}{3}}$$

$$\langle {}^{245}_{5} | E_{54} | {}^{245}_{4} \rangle = \sqrt{\tfrac{2}{3}}$$

$$\langle {}^{255}_{4} | E_{54} | {}^{245}_{4} \rangle = \sqrt{3}$$

$$\langle {}^{355}_{4} | E_{54} | {}^{345}_{4} \rangle = \sqrt{3}$$

$$\langle {}^{345}_{5} | E_{54} | {}^{345}_{4} \rangle = \sqrt{\tfrac{2}{3}}$$

$$\langle {}^{155}_{5} | E_{54} | {}^{145}_{5} \rangle = 1$$

$$\langle {}^{255}_{5} | E_{54} | {}^{245}_{5} \rangle = 1$$

$$\langle {}^{355}_{5} | E_{54} | {}^{345}_{5} \rangle = 1$$

$$\langle {}^{455}_{5} | E_{54} | {}^{445}_{5} \rangle = \sqrt{2}$$

$$\langle {}^{155}_{5} | E_{54} | {}^{155}_{4} \rangle = \sqrt{2}$$

$$\langle {}^{255}_{5} | E_{54} | {}^{255}_{4} \rangle = \sqrt{2}$$

$$\langle {}^{355}_{5} | E_{54} | {}^{355}_{4} \rangle = \sqrt{2}$$

$$\left\langle \begin{smallmatrix}1\\2\\3\\5\end{smallmatrix} \right| E_{54} \left| \begin{smallmatrix}1\\2\\3\\4\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1\\2\\4\\5\end{smallmatrix} \right| E_{43} \left| \begin{smallmatrix}1\\2\\3\\5\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1\\3\\4\\5\end{smallmatrix} \right| E_{32} \left| \begin{smallmatrix}1\\2\\4\\5\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}2\\3\\4\\5\end{smallmatrix} \right| E_{32} \left| \begin{smallmatrix}1\\3\\4\\5\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}2\\3\\4\\5\end{smallmatrix} \right| E_{31} \left| \begin{smallmatrix}1\\2\\4\\5\end{smallmatrix} \right\rangle = -1$$

$$\left\langle \begin{smallmatrix}2\\3\\4\\5\end{smallmatrix} \right| E_{41} \left| \begin{smallmatrix}1\\2\\3\\5\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}2\\3\\4\\5\end{smallmatrix} \right| E_{51} \left| \begin{smallmatrix}1\\2\\3\\4\end{smallmatrix} \right\rangle = -1$$

$$\left\langle \begin{smallmatrix}1\\3\\4\\5\end{smallmatrix} \right| E_{42} \left| \begin{smallmatrix}1\\2\\3\\5\end{smallmatrix} \right\rangle = -1$$

$$\left\langle \begin{smallmatrix}1\\3\\4\\5\end{smallmatrix} \right| E_{52} \left| \begin{smallmatrix}1\\2\\3\\4\end{smallmatrix} \right\rangle = 1$$

$$\left\langle \begin{smallmatrix}1\\2\\4\\5\end{smallmatrix} \right| E_{53} \left| \begin{smallmatrix}1\\2\\3\\4\end{smallmatrix} \right\rangle = -1$$

TABLE VIII

Matrix Elements of $I^k \cdot I^k$ for k = 1,2,3,4.

$\langle {11 \atop 22} \vert\, I^1 \cdot I^1 \,\vert {11 \atop 22} \rangle = \frac{7}{5}$	$\langle {12 \atop 33} \vert\, I^1 \cdot I^1 \,\vert {11 \atop 34} \rangle = \frac{2\sqrt{6}}{15}$
$\langle {11 \atop 23} \vert\, I^1 \cdot I^1 \,\vert {11 \atop 23} \rangle = \frac{7}{5}$	$\langle {12 \atop 24} \vert\, I^1 \cdot I^1 \,\vert {11 \atop 34} \rangle = \frac{\sqrt{6}}{15}$
$\langle {12 \atop 23} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 23} \rangle = \frac{4}{5}$	$\langle {11 \atop 25} \vert\, I^1 \cdot I^1 \,\vert {11 \atop 34} \rangle = \frac{\sqrt{6}}{15}$
$\langle {12 \atop 23} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 23} \rangle = \frac{4}{5}$	$\langle {11 \atop 25} \vert\, I^1 \cdot I^1 \,\vert {11 \atop 34} \rangle = \frac{\sqrt{6}}{15}$
$\langle {11 \atop 24} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 23} \rangle = \frac{\sqrt{6}}{15}$	$\langle {11 \atop 34} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 24} \rangle = \frac{\sqrt{6}}{15}$
$\langle {11 \atop 33} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 23} \rangle = \frac{2\sqrt{3}}{15}$	$\langle {12 \atop 33} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 24} \rangle = \frac{1}{5}$
$\langle {11 \atop 33} \vert\, I^1 \cdot I^1 \,\vert {11 \atop 33} \rangle = \frac{16}{15}$	$\langle {11 \atop 25} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 24} \rangle = \frac{2}{15}$
$\langle {12 \atop 23} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 33} \rangle = \frac{2\sqrt{3}}{15}$	$\langle {12 \atop 24} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 24} \rangle = \frac{11}{15}$
$\langle {11 \atop 24} \vert\, I^1 \cdot I^1 \,\vert {11 \atop 33} \rangle = \frac{\sqrt{2}}{5}$	$\langle {11 \atop 34} \vert\, I^1 \cdot I^1 \,\vert {11 \atop 25} \rangle = \frac{\sqrt{6}}{15}$
$\langle {11 \atop 33} \vert\, I^1 \cdot I^1 \,\vert {11 \atop 24} \rangle = \frac{\sqrt{2}}{5}$	$\langle {12 \atop 24} \vert\, I^1 \cdot I^1 \,\vert {11 \atop 25} \rangle = \frac{2}{15}$
$\langle {12 \atop 23} \vert\, I^1 \cdot I^1 \,\vert {11 \atop 24} \rangle = \frac{\sqrt{6}}{15}$	$\langle {11 \atop 25} \vert\, I^1 \cdot I^1 \,\vert {11 \atop 25} \rangle = \frac{3}{5}$
$\langle {11 \atop 24} \vert\, I^1 \cdot I^1 \,\vert {11 \atop 24} \rangle = \frac{13}{15}$	$\langle {11 \atop 44} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 34} \rangle = \frac{2\sqrt{6}}{15}$
$\langle {11 \atop 34} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 33} \rangle = \frac{2\sqrt{6}}{15}$	$\langle {12 \atop 25} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 34} \rangle = \frac{2\sqrt{3}}{30}$
$\langle {12 \atop 24} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 33} \rangle = \frac{1}{5}$	$\langle {22 \atop 33} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 34} \rangle = \frac{2\sqrt{6}}{15}$
$\langle {12 \atop 33} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 33} \rangle = \frac{13}{15}$	$\langle {11 \atop 35} \vert\, I^1 \cdot I^1 \,\vert {12 \atop 34} \rangle = \frac{2\sqrt{2}}{15}$

$\langle \substack{12\\34} | I^1 \cdot I^1 | \substack{12\\34} \rangle = \frac{28}{15}$

$\langle \substack{12\\25} | I^1 \cdot I^1 | \substack{13\\24} \rangle = \frac{1}{5}$

$\langle \substack{13\\24} | I^1 \cdot I^1 | \substack{13\\24} \rangle = \frac{1}{2}$

$\langle \substack{12\\34} | I^1 \cdot I^1 | \substack{11\\34} \rangle = \frac{2\sqrt{6}}{15}$

$\langle \substack{11\\35} | I^1 \cdot I^1 | \substack{11\\44} \rangle = \frac{2\sqrt{3}}{15}$

$\langle \substack{11\\44} | I^1 \cdot I^1 | \substack{11\\44} \rangle = \frac{3}{5}$

$\langle \substack{11\\35} | I^1 \cdot I^1 | \substack{12\\25} \rangle = \frac{\sqrt{6}}{15}$

$\langle \substack{13\\24} | I^1 \cdot I^1 | \substack{12\\25} \rangle = \frac{1}{5}$

$\langle \substack{12\\34} | I^1 \cdot I^1 | \substack{12\\25} \rangle = \frac{\sqrt{3}}{15}$

$\langle \substack{12\\25} | I^1 \cdot I^1 | \substack{12\\25} \rangle = \frac{7}{15}$

$\langle \substack{11\\44} | I^1 \cdot I^1 | \substack{11\\35} \rangle = \frac{2\sqrt{3}}{15}$

$\langle \substack{12\\25} | I^1 \cdot I^1 | \substack{11\\35} \rangle = \frac{\sqrt{6}}{15}$

$\langle \substack{12\\34} | I^1 \cdot I^1 | \substack{11\\35} \rangle = \frac{2\sqrt{2}}{15}$

$\langle \substack{11\\35} | I^1 \cdot I^1 | \substack{11\\35} \rangle = \frac{8}{15}$

$\langle \substack{12\\24} | I^1 \cdot I^1 | \substack{22\\33} \rangle = \frac{2\sqrt{6}}{15}$

$\langle \substack{22\\33} | I^1 \cdot I^1 | \substack{22\\33} \rangle = \frac{7}{15}$

$\langle \substack{12\\35} | I^1 \cdot I^1 | \substack{22\\34} \rangle = \frac{2\sqrt{2}}{15}$

$\langle \substack{12\\44} | I^1 \cdot I^1 | \substack{22\\34} \rangle = \frac{2\sqrt{6}}{15}$

$\langle \substack{13\\34} | I^1 \cdot I^1 | \substack{22\\34} \rangle = \frac{\sqrt{6}}{15}$

$\langle \substack{22\\34} | I^1 \cdot I^1 | \substack{22\\34} \rangle = \frac{11}{15}$

$\langle \substack{12\\44} | I^1 \cdot I^1 | \substack{13\\34} \rangle = \frac{1}{5}$

$\langle \substack{12\\35} | I^1 \cdot I^1 | \substack{13\\34} \rangle = \frac{\sqrt{3}}{15}$

$\langle \substack{12\\25} | I^1 \cdot I^1 | \substack{13\\34} \rangle = \frac{1}{5}$

$\langle \substack{22\\34} | I^1 \cdot I^1 | \substack{13\\34} \rangle = \frac{\sqrt{6}}{15}$

$\langle \substack{13\\34} | I^1 \cdot I^1 | \substack{13\\34} \rangle = \frac{7}{15}$

$\langle \substack{12\\44} | I^1 \cdot I^1 | \substack{12\\35} \rangle = \frac{2\sqrt{3}}{15}$

$\langle \substack{11\\45} | I^1 \cdot I^1 | \substack{12\\35} \rangle = \frac{\sqrt{3}}{15}$

$\langle \substack{12\\34} | I^1 \cdot I^1 | \substack{12\\35} \rangle = \frac{\sqrt{3}}{15}$

$\langle \substack{22\\34} | I^1 \cdot I^1 | \substack{12\\35} \rangle = \frac{2\sqrt{2}}{15}$

$\langle \substack{12\\35} | I^1 \cdot I^1 | \substack{12\\35} \rangle = \frac{17}{30}$

$\langle \substack{13\\34} | I^1 \cdot I^1 | \substack{13\\25} \rangle = \frac{1}{5}$

$\langle \substack{13\\25} | I^1 \cdot I^1 | \substack{13\\25} \rangle = \frac{1}{2}$

$\langle {12 \atop 34} | \; I^1 \cdot I^1 \; | {11 \atop 45} \rangle = \frac{2\sqrt{3}}{15}$

$\langle {12 \atop 44} | \; I^1 \cdot I^1 \; | {11 \atop 45} \rangle = \frac{4}{15}$

$\langle {11 \atop 45} | \; I^1 \cdot I^1 \; | {11 \atop 45} \rangle = \frac{8}{15}$

$\langle {12 \atop 35} | \; I^1 \cdot I^1 \; | {12 \atop 44} \rangle = \frac{2\sqrt{3}}{15}$

$\langle {22 \atop 34} | \; I^1 \cdot I^1 \; | {12 \atop 44} \rangle = \frac{2\sqrt{6}}{15}$

$\langle {11 \atop 45} | \; I^1 \cdot I^1 \; | {12 \atop 44} \rangle = \frac{4}{15}$

$\langle {13 \atop 34} | \; I^1 \cdot I^1 \; | {12 \atop 44} \rangle = \frac{1}{5}$

$\langle {12 \atop 44} | \; I^1 \cdot I^1 \; | {12 \atop 44} \rangle = \frac{11}{15}$

$\langle {13 \atop 44} | \; I^1 \cdot I^1 \; | {23 \atop 34} \rangle = \frac{\sqrt{6}}{15}$

$\langle {22 \atop 34} | \; I^1 \cdot I^1 \; | {23 \atop 34} \rangle = \frac{\sqrt{6}}{15}$

$\langle {22 \atop 44} | \; I^1 \cdot I^1 \; | {23 \atop 34} \rangle = \frac{\sqrt{2}}{5}$

$\langle {13 \atop 35} | \; I^1 \cdot I^1 \; | {23 \atop 34} \rangle = \frac{2}{15}$

$\langle {23 \atop 34} | \; I^1 \cdot I^1 \; | {23 \atop 34} \rangle = \frac{1}{3}$

$\langle {23 \atop 34} | \; I^1 \cdot I^1 \; | {22 \atop 44} \rangle = \frac{\sqrt{2}}{5}$

$\langle {22 \atop 35} | \; I^1 \cdot I^1 \; | {22 \atop 44} \rangle = \frac{2\sqrt{3}}{15}$

$\langle {13 \atop 44} | \; I^1 \cdot I^1 \; | {22 \atop 44} \rangle = \frac{2\sqrt{3}}{15}$

$\langle {12 \atop 45} | \; I^1 \cdot I^1 \; | {22 \atop 44} \rangle = \frac{4}{15}$

$\langle {22 \atop 44} | \; I^1 \cdot I^1 \; | {22 \atop 44} \rangle = \frac{17}{30}$

$\langle {22 \atop 44} | \; I^1 \cdot I^1 \; | {13 \atop 44} \rangle = \frac{2\sqrt{3}}{15}$

$\langle {12 \atop 45} | \; I^1 \cdot I^1 \; | {13 \atop 44} \rangle = \frac{2\sqrt{3}}{15}$

$\langle {23 \atop 34} | \; I^1 \cdot I^1 \; | {13 \atop 44} \rangle = \frac{\sqrt{6}}{15}$

$\langle {13 \atop 35} | \; I^1 \cdot I^1 \; | {13 \atop 44} \rangle = \frac{\sqrt{6}}{15}$

$\langle {13 \atop 44} | \; I^1 \cdot I^1 \; | {13 \atop 44} \rangle = \frac{6}{15}$

$\langle {12 \atop 45} | \; I^1 \cdot I^1 \; | {22 \atop 35} \rangle = \frac{2\sqrt{3}}{15}$

$\langle {22 \atop 44} | \; I^1 \cdot I^1 \; | {22 \atop 35} \rangle = \frac{2\sqrt{3}}{15}$

$\langle {23 \atop 34} | \; I^1 \cdot I^1 \; | {22 \atop 35} \rangle = \frac{\sqrt{6}}{15}$

$\langle {13 \atop 35} | \; I^1 \cdot I^1 \; | {22 \atop 35} \rangle = \frac{\sqrt{6}}{15}$

$\langle {22 \atop 35} | \; I^I \cdot I^1 \; | {22 \atop 35} \rangle = \frac{6}{15}$

$\langle {14 \atop 25} | \; I^1 \cdot I^1 \; | {13 \atop 35} \rangle = \frac{\sqrt{6}}{10}$

$\langle {12 \atop 45} | \; I^1 \cdot I^1 \; | {13 \atop 35} \rangle = \frac{\sqrt{2}}{10}$

$\langle {23 \atop 34} | \; I^1 \cdot I^1 \; | {13 \atop 35} \rangle = \frac{2}{15}$

$\langle {22 \atop 35} | \; I^1 \cdot I^1 \; | {13 \atop 35} \rangle = \frac{\sqrt{6}}{15}$

$\langle {}^{13}_{35}|\ I^1 \cdot I^1\ |{}^{13}_{35}\rangle = \frac{8}{15}$

$\langle {}^{22}_{35}|\ I^1 \cdot I^1\ |{}^{12}_{45}\rangle = \frac{2\sqrt{3}}{15}$

$\langle {}^{13}_{44}|\ I^1 \cdot I^1\ |{}^{12}_{45}\rangle = \frac{2\sqrt{3}}{15}$

$\langle {}^{13}_{35}|\ I^1 \cdot I^1\ |{}^{13}_{35}\rangle = \frac{\sqrt{2}}{20}$

$\langle {}^{11}_{55}|\ I^1 \cdot I^1\ |{}^{13}_{35}\rangle = \frac{4}{15}$

$\langle {}^{22}_{44}|\ I^1 \cdot I^1\ |{}^{13}_{35}\rangle = \frac{4}{15}$

$\langle {}^{12}_{45}|\ I^1 \cdot I^1\ |{}^{12}_{45}\rangle = \frac{19}{30}$

$\langle {}^{13}_{35}|\ I^1 \cdot I^1\ |{}^{14}_{25}\rangle = \frac{\sqrt{6}}{10}$

$\langle {}^{14}_{25}|\ I^1 \cdot I^1\ |{}^{14}_{25}\rangle = \frac{7}{30}$

$\langle {}^{12}_{45}|\ I^1 \cdot I^1\ |{}^{11}_{55}\rangle = \frac{4}{15}$

$\langle {}^{11}_{55}|\ I^1 \cdot I^1\ |{}^{11}_{55}\rangle = \frac{4}{15}$

$\langle {}^{11}_{22}|\ I^2 \cdot I^2\ |{}^{11}_{22}\rangle = \frac{9}{35}$

$\langle {}^{11}_{23}|\ I^2 \cdot I^2\ |{}^{11}_{23}\rangle = \frac{13}{35}$

$\langle {}^{12}_{24}|\ I^2 \cdot I^2\ |{}^{11}_{34}\rangle = \frac{\sqrt{6}}{35}$

$\langle {}^{11}_{34}|\ I^2 \cdot I^2\ |{}^{11}_{34}\rangle = \frac{19}{35}$

$\langle {}^{12}_{33}|\ I^2 \cdot I^2\ |{}^{12}_{24}\rangle = \frac{-1}{35}$

$\langle {}^{13}_{44}|\ I^1 \cdot I^1\ |{}^{13}_{35}\rangle = \frac{\sqrt{6}}{15}$

$\langle {}^{11}_{33}|\ I^2 \cdot I^2\ |{}^{12}_{23}\rangle = \frac{2\sqrt{3}}{35}$

$\langle {}^{11}_{24}|\ I^2 \cdot I^2\ |{}^{12}_{23}\rangle = \frac{-\sqrt{6}}{7}$

$\langle {}^{12}_{23}|\ I^2 \cdot I^2\ |{}^{12}_{23}\rangle = \frac{13}{35}$

$\langle {}^{12}_{23}|\ I^2 \cdot I^2\ |{}^{11}_{24}\rangle = \frac{-\sqrt{6}}{7}$

$\langle {}^{11}_{24}|\ I^2 \cdot I^2\ |{}^{11}_{24}\rangle = \frac{23}{35}$

$\langle {}^{11}_{33}|\ I^2 \cdot I^2\ |{}^{11}_{24}\rangle = \frac{-\sqrt{2}}{35}$

$\langle {}^{11}_{24}|\ I^2 \cdot I^2\ |{}^{11}_{33}\rangle = \frac{1\sqrt{2}}{35}$

$\langle {}^{11}_{33}|\ I^2 \cdot I^2\ |{}^{11}_{33}\rangle = \frac{12}{35}$

$\langle {}^{12}_{23}|\ I^2 \cdot I^2\ |{}^{11}_{33}\rangle = \frac{2\sqrt{3}}{35}$

$\langle {}^{12}_{33}|\ I^2 \cdot I^2\ |{}^{12}_{33}\rangle = \frac{26}{35}$

$\langle {}^{11}_{25}|\ I^2 \cdot I^2\ |{}^{12}_{33}\rangle = \frac{-4}{35}$

$\langle {}^{11}_{34}|\ I^2 \cdot I^2\ |{}^{12}_{33}\rangle = \frac{-4\sqrt{6}}{35}$

$\langle {}^{12}_{24}|\ I^2 \cdot I^2\ |{}^{12}_{33}\rangle = \frac{-1}{35}$

$\langle {}^{12}_{33}|\ I^2 \cdot I^2\ |{}^{11}_{34}\rangle = \frac{-4\sqrt{6}}{35}$

$\langle {}^{11}_{25}|\ I^2 \cdot I^2\ |{}^{11}_{34}\rangle = \frac{\sqrt{6}}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{34} | \, I^2 \cdot I^2 \, | \genfrac{}{}{0pt}{}{12}{24} \rangle = \frac{\sqrt{6}}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{25} | \, I^2 \cdot I^2 \, | \genfrac{}{}{0pt}{}{12}{24} \rangle = \frac{-6}{35}$

$\langle \genfrac{}{}{0pt}{}{12}{24} | \, I^2 \cdot I^2 \, | \genfrac{}{}{0pt}{}{12}{24} \rangle = \frac{13}{35}$

$\langle \genfrac{}{}{0pt}{}{12}{24} | \, I^2 \cdot I^2 \, | \genfrac{}{}{0pt}{}{11}{25} \rangle = \frac{-6}{35}$

$\langle \genfrac{}{}{0pt}{}{12}{33} | \, I^2 \cdot I^2 \, | \genfrac{}{}{0pt}{}{11}{25} \rangle = \frac{-4}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{34} | \, I^2 \cdot I^2 \, | \genfrac{}{}{0pt}{}{11}{25} \rangle = \frac{\sqrt{6}}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{25} | \, I^2 \cdot I^2 \, | \genfrac{}{}{0pt}{}{11}{25} \rangle = \frac{4}{5}$

$\langle \genfrac{}{}{0pt}{}{11}{22} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{22} \rangle = \frac{19}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{23} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{23} \rangle = \frac{19}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{33} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{12}{23} \rangle = \frac{-2\sqrt{3}}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{24} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{12}{23} \rangle = \frac{-\sqrt{6}}{35}$

$\langle \genfrac{}{}{0pt}{}{12}{23} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{12}{23} \rangle = \frac{41}{70}$

$\langle \genfrac{}{}{0pt}{}{11}{33} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{33} \rangle = \frac{9}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{25} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{25} \rangle = \frac{43}{70}$

$\langle \genfrac{}{}{0pt}{}{12}{24} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{25} \rangle = \frac{-1}{5}$

$\langle \genfrac{}{}{0pt}{}{11}{34} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{25} \rangle = \frac{-\sqrt{6}}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{24} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{33} \rangle = \frac{2\sqrt{2}}{35}$

$\langle \genfrac{}{}{0pt}{}{12}{23} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{33} \rangle = \frac{-2\sqrt{3}}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{24} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{24} \rangle = \frac{17}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{33} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{24} \rangle = \frac{2\sqrt{2}}{35}$

$\langle \genfrac{}{}{0pt}{}{12}{23} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{24} \rangle = \frac{-\sqrt{6}}{35}$

$\langle \genfrac{}{}{0pt}{}{12}{33} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{33} \rangle = \frac{22}{35}$

$\langle \genfrac{}{}{0pt}{}{12}{24} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{12}{33} \rangle = \frac{2}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{34} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{12}{33} \rangle = \frac{-2\sqrt{6}}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{25} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{33} \rangle = \frac{1}{7}$

$\langle \genfrac{}{}{0pt}{}{12}{33} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{34} \rangle = \frac{-2\sqrt{6}}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{25} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{34} \rangle = \frac{-\sqrt{6}}{35}$

$\langle \genfrac{}{}{0pt}{}{12}{24} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{34} \rangle = \frac{-\sqrt{6}}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{34} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{11}{34} \rangle = \frac{17}{35}$

$\langle \genfrac{}{}{0pt}{}{12}{33} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{12}{24} \rangle = \frac{2}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{34} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{12}{24} \rangle = \frac{-\sqrt{6}}{35}$

$\langle \genfrac{}{}{0pt}{}{11}{25} | \, I^3 \cdot I^3 \, | \genfrac{}{}{0pt}{}{12}{24} \rangle = \frac{-1}{5}$

$$\langle {11 \atop 22} | \; I^4 \cdot I^4 \; | {11 \atop 22} \rangle = \frac{27}{35}$$

$$\langle {11 \atop 23} | \; I^4 \cdot I^4 \; | {11 \atop 23} \rangle = \frac{27}{35}$$

$$\langle {12 \atop 23} | \; I^4 \cdot I^4 \; | {12 \atop 23} \rangle = \frac{19}{70}$$

$$\langle {11 \atop 24} | \; I^4 \cdot I^4 \; | {12 \atop 23} \rangle = \frac{2\sqrt{6}}{63}$$

$$\langle {11 \atop 33} | \; I^4 \cdot I^4 \; | {12 \atop 23} \rangle = \frac{2\sqrt{3}}{63}$$

$$\langle {11 \atop 33} | \; I^4 \cdot I^4 \; | {11 \atop 33} \rangle = \frac{71}{105}$$

$$\langle {12 \atop 23} | \; I^4 \cdot I^4 \; | {11 \atop 33} \rangle = \frac{-2\sqrt{3}}{63}$$

$$\langle {11 \atop 24} | \; I^4 \cdot I^4 \; | {11 \atop 33} \rangle = \frac{-2\sqrt{2}}{21}$$

$$\langle {11 \atop 33} | \; I^4 \cdot I^4 \; | {11 \atop 24} \rangle = \frac{-2\sqrt{2}}{21}$$

$$\langle {12 \atop 23} | \; I^4 \cdot I^4 \; | {11 \atop 24} \rangle = \frac{2\sqrt{6}}{63}$$

$$\langle {11 \atop 24} | \; I^4 \cdot I^4 \; | {11 \atop 24} \rangle = \frac{91}{105}$$

$$\langle {11 \atop 34} | \; I^4 \cdot I^4 \; | {11 \atop 25} \rangle = \frac{-\sqrt{6}}{63}$$

$$\langle {12 \atop 24} | \; I^4 \cdot I^4 \; | {11 \atop 25} \rangle = \frac{13}{63}$$

$$\langle {12 \atop 33} | \; I^4 \cdot I^4 \; | {11 \atop 25} \rangle = \frac{-1}{21}$$

$$\langle {11 \atop 25} | \; I^4 \cdot I^4 \; | {11 \atop 25} \rangle = \frac{127}{210}$$

$$\langle {12 \atop 24} | \; I^3 \cdot I^3 \; | {12 \atop 24} \rangle = \frac{24}{35}$$

$$\langle {11 \atop 34} | \; I^4 \cdot I^4 \; | {12 \atop 33} \rangle = \frac{2\sqrt{6}}{63}$$

$$\langle {12 \atop 24} | \; I^4 \cdot I^4 \; | {11 \atop 33} \rangle = \frac{-2}{21}$$

$$\langle {11 \atop 25} | \; I^4 \cdot I^4 \; | {12 \atop 33} \rangle = \frac{-1}{21}$$

$$\langle {11 \atop 33} | \; I^4 \cdot I^4 \; | {12 \atop 33} \rangle = \frac{208}{315}$$

$$\langle {11 \atop 33} | \; I^4 \cdot I^4 \; | {11 \atop 34} \rangle = \frac{2\sqrt{6}}{63}$$

$$\langle {11 \atop 25} | \; I^4 \cdot I^4 \; | {11 \atop 34} \rangle = \frac{-\sqrt{6}}{63}$$

$$\langle {12 \atop 24} | \; I^4 \cdot I^4 \; | {11 \atop 34} \rangle = \frac{-\sqrt{6}}{63}$$

$$\langle {11 \atop 34} | \; I^4 \cdot I^4 \; | {11 \atop 34} \rangle = \frac{71}{105}$$

$$\langle {12 \atop 33} | \; I^4 \cdot I^4 \; | {12 \atop 24} \rangle = \frac{-2}{21}$$

$$\langle {11 \atop 34} | \; I^4 \cdot I^4 \; | {12 \atop 24} \rangle = \frac{-\sqrt{6}}{63}$$

$$\langle {12 \atop 25} | \; I^4 \cdot I^4 \; | {12 \atop 24} \rangle = \frac{13}{63}$$

$$\langle {12 \atop 24} | \; I^4 \cdot I^4 \; | {12 \atop 24} \rangle = \frac{278}{315}$$

TABLE IX

Tensor Product States of A_4B_2 For M>, 0, $\sigma_A=2$, $\sigma_B=1$.

$[\eta_A] \otimes [\eta_B]$	Basis States	M	Q_M	SO(3) Correlation
$[21^2] \otimes [2]$	$\vert\begin{smallmatrix}11\\2\\3\end{smallmatrix}\rangle\vert 66\rangle$			
	$\vert\begin{smallmatrix}11\\2\\3\end{smallmatrix}\rangle\vert 67\rangle\vert\begin{smallmatrix}12\\2\\3\end{smallmatrix}\rangle\vert 66\rangle\vert\begin{smallmatrix}11\\2\\4\end{smallmatrix}\rangle\vert 66\rangle$	6	3	K,2I
	$\vert\begin{smallmatrix}11\\2\\3\end{smallmatrix}\rangle\vert 68\rangle\vert\begin{smallmatrix}12\\2\\3\end{smallmatrix}\rangle\vert 77\rangle\vert\begin{smallmatrix}11\\2\\3\end{smallmatrix}\rangle\vert 67\rangle$	5	8	K,2I,5H
	$\vert\begin{smallmatrix}11\\2\\4\end{smallmatrix}\rangle\vert 67\rangle\vert\begin{smallmatrix}13\\2\\3\end{smallmatrix}\rangle\vert 66\rangle\vert\begin{smallmatrix}12\\2\\4\end{smallmatrix}\rangle\vert 66\rangle$			
	$\vert\begin{smallmatrix}11\\2\\5\end{smallmatrix}\rangle\vert 66\rangle\vert\begin{smallmatrix}11\\3\\4\end{smallmatrix}\rangle\vert 66\rangle$			
	$\vert\begin{smallmatrix}12\\2\\3\end{smallmatrix}\rangle\vert 77\rangle\vert\begin{smallmatrix}12\\2\\3\end{smallmatrix}\rangle\vert 68\rangle\vert\begin{smallmatrix}11\\2\\4\end{smallmatrix}\rangle\vert 68\rangle$	4	13	K,2I,5H,5G
	$\vert\begin{smallmatrix}11\\2\\4\end{smallmatrix}\rangle\vert 77\rangle\vert\begin{smallmatrix}13\\2\\3\end{smallmatrix}\rangle\vert 67\rangle\vert\begin{smallmatrix}12\\2\\4\end{smallmatrix}\rangle\vert 67\rangle$			
	$\vert\begin{smallmatrix}13\\2\\4\end{smallmatrix}\rangle\vert 66\rangle\vert\begin{smallmatrix}11\\2\\5\end{smallmatrix}\rangle\vert 67\rangle\vert\begin{smallmatrix}11\\3\\4\end{smallmatrix}\rangle\vert 67\rangle$			
	$\vert\begin{smallmatrix}14\\2\\3\end{smallmatrix}\rangle\vert 66\rangle\vert\begin{smallmatrix}12\\2\\5\end{smallmatrix}\rangle\vert 66\rangle\vert\begin{smallmatrix}11\\3\\5\end{smallmatrix}\rangle\vert 66\rangle$			
	$\vert\begin{smallmatrix}12\\3\\4\end{smallmatrix}\rangle$			

$[n_A] \times [n_B]$	Basis States	M	Q_M	SO(3) Correlation
$[21^2] \times [2]$	$\left\|\begin{smallmatrix}11\\2\\3\end{smallmatrix}\right\rangle\|88\rangle\left\|\begin{smallmatrix}11\\2\\4\end{smallmatrix}\right\rangle\|78\rangle$	3		
	$\left\|\begin{smallmatrix}13\\2\\3\end{smallmatrix}\right\rangle\|68\rangle\left\|\begin{smallmatrix}13\\2\\4\end{smallmatrix}\right\rangle\|77\rangle\left\|\begin{smallmatrix}12\\2\\4\end{smallmatrix}\right\rangle\|68\rangle$			
	$\left\|\begin{smallmatrix}12\\2\\4\end{smallmatrix}\right\rangle\|77\rangle\left\|\begin{smallmatrix}13\\2\\4\end{smallmatrix}\right\rangle\|67\rangle\left\|\begin{smallmatrix}11\\2\\5\end{smallmatrix}\right\rangle\|68\rangle$			
	$\left\|\begin{smallmatrix}11\\2\\5\end{smallmatrix}\right\rangle\|77\rangle\left\|\begin{smallmatrix}11\\3\\5\end{smallmatrix}\right\rangle\|68\rangle\left\|\begin{smallmatrix}11\\3\\5\end{smallmatrix}\right\rangle\|77\rangle$			
	$\left\|\begin{smallmatrix}14\\2\\3\end{smallmatrix}\right\rangle\|67\rangle\left\|\begin{smallmatrix}12\\2\\5\end{smallmatrix}\right\rangle\|67\rangle\left\|\begin{smallmatrix}11\\3\\5\end{smallmatrix}\right\rangle\|67\rangle$			
	$\left\|\begin{smallmatrix}12\\3\\4\end{smallmatrix}\right\rangle\|67\rangle\left\|\begin{smallmatrix}13\\3\\4\end{smallmatrix}\right\rangle\|66\rangle\left\|\begin{smallmatrix}14\\2\\4\end{smallmatrix}\right\rangle\|66\rangle$			
	$\left\|\begin{smallmatrix}12\\3\\5\end{smallmatrix}\right\rangle\|66\rangle\left\|\begin{smallmatrix}15\\2\\3\end{smallmatrix}\right\rangle\|66\rangle$	3	23	K,2I,5H, 5G,10F
	$\left\|\begin{smallmatrix}13\\2\\5\end{smallmatrix}\right\rangle\|66\rangle\left\|\begin{smallmatrix}22\\3\\4\end{smallmatrix}\right\rangle\|66\rangle$			
	$\left\|\begin{smallmatrix}11\\4\\5\end{smallmatrix}\right\rangle\|66\rangle\left\|\begin{smallmatrix}12\\2\\3\end{smallmatrix}\right\rangle\|78\rangle$			
	$\left\|\begin{smallmatrix}12\\2\\3\end{smallmatrix}\right\rangle\|88\rangle\left\|\begin{smallmatrix}11\\2\\4\end{smallmatrix}\right\rangle\|88\rangle\left\|\begin{smallmatrix}13\\2\\3\end{smallmatrix}\right\rangle\|78\rangle$	2	29	K,2I,5H,5G, 10F,6D
	$\left\|\begin{smallmatrix}12\\2\\4\end{smallmatrix}\right\rangle\|78\rangle\left\|\begin{smallmatrix}13\\2\\4\end{smallmatrix}\right\rangle\|68\rangle\left\|\begin{smallmatrix}13\\2\\4\end{smallmatrix}\right\rangle\|77\rangle$			

$[n_A] \times [n_B]$	Basis States	M	Q_M	SO(3) Correlation
$[21^2] \times [2]$	|12/2/5 >|78>|13/3/4 >|78>|14/2/3 >|68>			
	|14/2/3 >|77>|12/2/5 >|68>|12/2/5 >|77>			
	|13/3/4 >|67>|14/2/4 >|67>|11/2/5 >|68>			
	|11/3/5 >|77>|12/3/4 >|68>|12/3/4 >|77>			
	|12/3/5 >|67>|13/2/5 >|67>|11/4 >|67>			
	|15/2/3 >|67>|23/3/4 >|66>|14/3/4 >|66>			
	|22/3/5 >|66>|13/3/5 >|66>|12/4/5 >|66>			
	|15/2/4 >|66>|14/2/5 >|66>			
	|13/2/3 >|88>|12/2/4 >|88>|13/2/4 >|78>	1		
	|11/2/5 >|88>|11/3/4 >|88>|14/2/3 >|78>			
	|12/2/5 >|78>|12/3/4 >|78>|11/3/5 >|78>			

$[n_A] \times [n_B]$	Basis States	M	Q_M	SO(3) Correlation
$[21^2] \times [2]$	$\left\|\begin{smallmatrix}14\\2\\4\end{smallmatrix}\right\rangle\|68\rangle\left\|\begin{smallmatrix}14\\2\\4\end{smallmatrix}\right\rangle\|77\rangle\left\|\begin{smallmatrix}12\\3\\5\end{smallmatrix}\right\rangle\|68\rangle$			
	$\left\|\begin{smallmatrix}12\\3\\5\end{smallmatrix}\right\rangle\|77\rangle\left\|\begin{smallmatrix}13\\2\\5\end{smallmatrix}\right\rangle\|68\rangle\left\|\begin{smallmatrix}13\\2\end{smallmatrix}\right\rangle\|77\rangle$			
	$\left\|\begin{smallmatrix}11\\4\\5\end{smallmatrix}\right\rangle\|68\rangle\left\|\begin{smallmatrix}11\\4\\5\end{smallmatrix}\right\rangle\|77\rangle\left\|\begin{smallmatrix}15\\2\\3\end{smallmatrix}\right\rangle\|68\rangle$	1	35	K,2I,5H,5G, 10F,6D,6P
	$\left\|\begin{smallmatrix}15\\2\\3\end{smallmatrix}\right\rangle\|77\rangle\left\|\begin{smallmatrix}22\\3\\4\end{smallmatrix}\right\rangle\|68\rangle\left\|\begin{smallmatrix}22\\3\\4\end{smallmatrix}\right\rangle\|77\rangle$			
	$\left\|\begin{smallmatrix}23\\3\\4\end{smallmatrix}\right\rangle\|67\rangle\left\|\begin{smallmatrix}14\\3\\4\end{smallmatrix}\right\rangle\|67\rangle\left\|\begin{smallmatrix}22\\3\\5\end{smallmatrix}\right\rangle\|67\rangle$			
	$\left\|\begin{smallmatrix}13\\3\\5\end{smallmatrix}\right\rangle\|67\rangle\left\|\begin{smallmatrix}12\\4\\5\end{smallmatrix}\right\rangle\|67\rangle\left\|\begin{smallmatrix}15\\2\\4\end{smallmatrix}\right\rangle\|67\rangle$			
	$\left\|\begin{smallmatrix}14\\2\\5\end{smallmatrix}\right\rangle\|67\rangle\left\|\begin{smallmatrix}24\\3\\4\end{smallmatrix}\right\rangle\|66\rangle\left\|\begin{smallmatrix}23\\3\\5\end{smallmatrix}\right\rangle\|66\rangle$			
	$\left\|\begin{smallmatrix}22\\4\\5\end{smallmatrix}\right\rangle\|66\rangle\left\|\begin{smallmatrix}13\\4\\5\end{smallmatrix}\right\rangle\|66\rangle\left\|\begin{smallmatrix}15\\3\\4\end{smallmatrix}\right\rangle\|66\rangle$			
	$\left\|\begin{smallmatrix}15\\2\\5\end{smallmatrix}\right\rangle\|66\rangle\left\|\begin{smallmatrix}14\\3\\5\end{smallmatrix}\right\rangle\|66\rangle$			
	$\left\|\begin{smallmatrix}12\\3\\5\end{smallmatrix}\right\rangle\|78\rangle\left\|\begin{smallmatrix}13\\2\\5\end{smallmatrix}\right\rangle\|78\rangle\left\|\begin{smallmatrix}22\\4\\5\end{smallmatrix}\right\rangle\|78\rangle$	0	35	K,2I,5H, 5G,10F,6D, 6P
	$\left\|\begin{smallmatrix}15\\2\\3\end{smallmatrix}\right\rangle\|78\rangle\left\|\begin{smallmatrix}22\\3\\4\end{smallmatrix}\right\rangle\|78\rangle\left\|\begin{smallmatrix}23\\3\\4\end{smallmatrix}\right\rangle\|68\rangle$			

$[\eta_A] \times [\eta_B]$	Basis States	M	Q_M	SO(3) Correlation
	$\left\|\begin{matrix}23\\3\\4\end{matrix}\right\rangle\|77\rangle \left\|\begin{matrix}14\\2\\3\end{matrix}\right\rangle\|88\rangle \left\|\begin{matrix}12\\2\\5\end{matrix}\right\rangle\|88\rangle$			
	$\left\|\begin{matrix}12\\3\\4\end{matrix}\right\rangle\|88\rangle \left\|\begin{matrix}13\\3\\4\end{matrix}\right\rangle\|78\rangle \left\|\begin{matrix}11\\3\\5\end{matrix}\right\rangle\|88\rangle$			
	$\left\|\begin{matrix}14\\2\\4\end{matrix}\right\rangle\|78\rangle \left\|\begin{matrix}13\\2\\4\end{matrix}\right\rangle\|88\rangle \left\|\begin{matrix}14\\3\\4\end{matrix}\right\rangle\|68\rangle$			
	$\left\|\begin{matrix}14\\3\\4\end{matrix}\right\rangle\|77\rangle \left\|\begin{matrix}13\\3\\5\end{matrix}\right\rangle\|77\rangle \left\|\begin{matrix}12\\4\\5\end{matrix}\right\rangle\|68\rangle$			
	$\left\|\begin{matrix}12\\4\\5\end{matrix}\right\rangle\|77\rangle \left\|\begin{matrix}15\\2\\4\end{matrix}\right\rangle\|68\rangle \left\|\begin{matrix}15\\2\\4\end{matrix}\right\rangle\|77\rangle$			
	$\left\|\begin{matrix}14\\2\\5\end{matrix}\right\rangle\|68\rangle \left\|\begin{matrix}14\\2\\5\end{matrix}\right\rangle\|77\rangle \left\|\begin{matrix}24\\3\\4\end{matrix}\right\rangle\|67\rangle$			
	$\left\|\begin{matrix}23\\3\\5\end{matrix}\right\rangle\|67\rangle \left\|\begin{matrix}22\\4\\7\end{matrix}\right\rangle\|67\rangle \left\|\begin{matrix}13\\4\\5\end{matrix}\right\rangle\|67\rangle$			
	$\left\|\begin{matrix}15\\3\\4\end{matrix}\right\rangle\|67\rangle \left\|\begin{matrix}15\\2\\5\end{matrix}\right\rangle\|67\rangle \left\|\begin{matrix}14\\3\\5\end{matrix}\right\rangle\|67\rangle$			
	$\left\|\begin{matrix}23\\4\\5\end{matrix}\right\rangle\|66\rangle \left\|\begin{matrix}14\\4\\5\end{matrix}\right\rangle\|66\rangle \left\|\begin{matrix}25\\3\\4\end{matrix}\right\rangle\|66\rangle$			
	$\left\|\begin{matrix}15\\3\\5\end{matrix}\right\rangle\|66\rangle \left\|\begin{matrix}24\\3\\5\end{matrix}\right\rangle\|66\rangle$			

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