1973

On non-normality in the Bayesian approach to the analysis of variance and regression theory.

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ON NON-NORMALITY IN THE BAYESIAN APPROACH TO THE ANALYSIS OF VARIANCE AND REGRESSION THEORY

BY

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A Thesis
Submitted to the Faculty of Graduate Studies through the
Department of Mathematics in Partial Fulfillment
of the Requirements for the Degree of
Doctor of Philosophy
at the University of Windsor

Windsor, Ontario

1973

ABSTRACT

In this paper, the posterior distributions of the variance components in the analysis of variance in the one-way random-effects model are developed. The distributions, first of the effects and then of the error, are assumed to be unknown but with the third and fourth moments known. The Edgeworth Series is then used to approximate these probability density functions. Approximate and asymptotic functions of the posterior distributions are also evolved in order to provide somewhat simplified probability distributions to work with.

The effects of varying the values of the third and fourth moments are studied through the aid of several computer-generated examples.

In addition, the posterior distributions of regression coefficients in a restricted case are calculated, again using the Edgeworth Series. Finally, the study of the impact of the third and fourth moments is carried out here in the same way as in the previous section.

ACKNOWLEDGEMENTS

The author is most sincerely grateful to Rev. D. T. Faught, C.S.B., for his advice and encouragement, and to Professor N. Shklov and Dr. S. K. Banerjee for their guidance in the preparation of this thesis.

I wish also to thank my wife Gita for her patience and good humour throughout this investigation.

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Chapter I

Introduction

We wish to consider two main topics. Firstly, we examine the analysis of variance in the one-way random-effects model, that is,

$$y_{ij} = \mu + a_i + e_{ij} \quad (i=1,2,\ldots k; \quad j=1,2,\ldots,n), \qquad (1.1)$$

where y_{ij} is the jth observation in the ith group, μ is a location parameter, a_i is the random-effect associated with the ith group and e_{ij} is the error in the (i, j)th observation. We will assume that the a_i are distributed independently of the e_{ij} and

$E(a_i) = 0,$

$E(e_{ij}) = 0,$

$\text{Variance } (a_i) = \sigma^2_a,$

$\text{Variance } (e_{ij}) = \sigma^2.$

We have, therefore, $E(y_{ij} - \mu)^2 = \sigma^2 + \sigma^2_a$.

The parameters σ^2 and σ_a^2 are called <u>variance-components</u> and the problem of estimating them has been attacked by many authors - see Bross (1950), Bulmer (1957), Bush and Anderson (1963), Crump (1946, 1951), Daniels (1939), Fisher (1935), Green (1954) and Healy (1963), etc. In most of these works the problem was analyzed from a sampling theory point of view. Two major difficulties arose and, in most of the above works, were left basically unresolved. One was the "negative estimated variance" problem. That is, using (1.1) and the assumption that the a_i and the e_{ij} are independent among themselves, the unbiased estimator of σ_a^2,

$$\hat{\sigma}_a^2 = \frac{\dfrac{S_2}{k-1} - \dfrac{S_1}{k(n-1)}}{n},$$

with $S_1 = \sum_{i=1}^{k} \sum_{j=1}^{n} (y_{ij} - \bar{y}_i)^2$,

$$S_2 = \sum_{i=1}^{k} n(\bar{y}_i - \bar{y})^2,$$

$$\bar{y}_i = \sum_{j=1}^{n} y_{ij}/n,$$

$$\bar{y} = \sum_{i=1}^{k} \sum_{j=1}^{n} y_{ij}/nk,$$

can clearly take on negative values. Attempts have been made to restrict the value of θ_a^2 to a positive range (see, for example, Herbach (1959) and Thompson (1962). In the work by Thompson the author uses a "restricted" maximum likelihood principle and the result is only slightly different from the traditional approach using the full maximum likelihood. However, this approach has the effect of destroying the unbiasedness property and very much complicating the distributions upon which one makes inferences.

The second difficulty using the traditional approach is the sensitivity to departures from underlying assumptions. Most writers presume that the error, e_{ij}, and the random-effects component, a_i, are normally distributed. However, Scheffé (1959) has shown that non-normality, particularly in the a_i and to a lesser degree in the e_{ij}, has a serious effect on the distributions of the criteria which one uses to make inferences about the parameters in the one-way model.

In an attempt to solve these problems, as well as others that occur, the Bayesian approach has been adopted by several authors in recent years. For example, Tiao and Tan (1965) and Hill (1965) developed the posterior distributions of the variance components under the assumptions of normality of errors and random-effects and a non-informative prior probability distribution. The advantage of such an approach is that it eliminates the negative estimated variance problem since the posterior probability of σ_a^2 takes on only positive values. The difficulty concerning departures from assumptions, however, is not solved by this method. In fact, it seems that sensitivity to non-normality may be increased rather than decreased.

The approaches used to analyze this second problem are many and varied. However, the underlying idea of most of these is the replacement of the normal distribution by a family of non-normal distributions or by an approximation of a distribution which is more general than the normal. In addition, most of the works place a heavy emphasis on the third and fourth moments as measures of non-normality. For example, E. S. Pearson (1928, 1929) has studied the effect of universal excess and skewness of a variable related to Student's t. R. C. Geary (1936) obtained an expression for the distribution of t in samples drawn from a slightly asymmetrical population. A. K. Gayen (1949, 1950a) used the Edgeworth Series to develop the distributions of t and the variance ratio in random samples of any size drawn from non-normal universes. The Edgeworth Series was also used by A. B. L. Srivastava (1958, 1960) to find the distribution of regression coefficients. An attempt to combine the Bayesian approach with a non-normal population was made by Box and Tiao (1964). In that work the authors used the following non-normal family of distributions to measure the effects of non-normality on the posterior distributions:

$$f(y;\theta,\ \sigma,\ \beta) = k \exp\left\{-\tfrac{1}{2}\left|\frac{y-\theta}{\sigma}\right|^{2/(1+\beta)}\right\},\ -\infty<y<\infty, \qquad (1.2)$$

where β is a "measure" of non-normality. This approach, however, has limitations. The most serious of these is that by using β as above, we take into consideration only non-zero fourth moments and assume a symmetric distribution. The result is that asymmetric (which are obviously non-normal) populations are not in the Box-Tiao family. In another paper (1962), the same authors state that they expect that kurtosis would have a much greater effect on the inference about variance components than would skewness. This expectation, then, leads to the use of (1.2). However, there is no evidence - either theoretical or practical - which suggests such a belief.

In this paper we, too, intend to analyze the effect of non-normality on the Bayesian method. In order to be more general, we will use the Edgeworth Series in place of the normal distribution. The objectives will be to develop the posterior distributions and, if possible, the approximations or asymptotic expansions (since the distributions will be quite complex and difficult to use). We would like to particularly study the effect of non-normal values for skewness and kurtosis. We are also curious about the posterior distributions in the special case when θ^2_a is negative.

The second main topic of concern to us is regression analysis. Even though we will spend a preponderance of our time on the analysis of variance, we nevertheless include this second subject because there is a strong relationship between the two and also because many of the formulae developed in the analysis of variance can be applied quite readily to regression theory. Our objectives will be basically the same as in the first topic.

Since it will play such an important role, a discussion of the Edgeworth Series seems appropriate at this time. H. Cramér (1928) has shown that this series provides an asymptotic expansion of the probability distribution in powers of $n^{-\frac{1}{2}}$, with a remainder term of the same order as the first term neglected. The terms of order $n^{-A/2}$ contain the moments $\mu_3, \mu_4 \ldots \mu_{A+2}$. In this paper we shall not go beyond the third and fourth moments. In order to simplify the notation, we introduce λ_1 and λ_2 where $\lambda_1 = \mu_3/\sigma^3$ and $\lambda_2 = \mu_4/\sigma^4 - 3$. Thus the Edgeworth Series we shall use is

$$f(x) = \phi(x) - (\lambda_1/3!)\,\phi^{(3)}(x) + (\lambda_2/4!)\,\phi^{(4)}(x) + (10\lambda_1^2/6!)\,\phi^{(6)}(x), \text{ where } \phi(x) = (1/\sqrt{2\pi})(1/\sigma)e^{-x^2/2\sigma^2}. \quad (1.3)$$

and $\phi^{(v)}$ is the vth derivative of $\phi(x)$.

A very elementary examination of the Edgeworth Series reveals that it is possible for the Series to take on negative values. In order to avoid that possibility, the values of λ_1 and λ_2 will be restricted to those which produce only positive Edgeworth Series. A computer program was composed (See Appendix III) to accomplish this, since analytical methods proved impossible (it would have involved solving a sixth degree polynomial). The results of that program are presented in the following table (λ_1 and λ_2 take on values in tenths of integers).

TABLE I

VALUES OF λ_1 AND λ_2 PRODUCING POSITIVE EDGEWORTH SERIES

Values of λ_2	Range of λ_1
0.0	-.1, +.1
0.1	-.2, +.2
0.2,0.3	-.3, +.3
0.4,0.6	-.4, +.4
0.7, 1.2	-.5, +.5
1.3, 3.5	-.6, +.6
3.6, 3.8	-.5, +.5
3.9	-.4, +.4
4.0	-.1, +.1

At first glance the above values for λ_1 and λ_2 seem quite restrictive. However, in a study (published by Pearson, 1931) of engineering data based on thousands of measurements at the Bell Telephone Laboratories, the estimated λ_1 varied from -0.7 to +0.9 and λ_2 from -0.4 to +1.8. Therefore in actual fact the values of λ_1 and λ_2 in Table I are not too restrictive.

Chapter II

Bayesian Methods in the Analysis of Variance: Non-Normal Effects

2.1 Joint Likelihood Function

In this part of our analysis we shall assume that the e_{ij} are normally distributed, with mean = 0 and variance = σ^2, and that the distribution of the a_i is unknown. It is further assumed that the mean of the a_i = 0, the variance = σ^2_a, $E(a_i^3)/\sigma^3_a = \lambda_1$ and $E(a_i^4)/\sigma^4_a - 3 = \lambda_2$. The first four terms of the Edgeworth series will be used to approximate the distribution of a_i. All higher moments will be assumed to be negligible and λ_1 and λ_2 taken as known constants. In the cases where λ_1 and λ_2 are unknown, it may be possible to estimate these parameters by using Fisher's k-statistics, as suggested by R. C. Geary (1947).

The distribution of a_i is approximated by

$$f(a_i)=(1/\sqrt{2\pi})(1/\sigma_a)\Big(1+\lambda_1/3!\{a_i^3/\sigma^3_a - 3a_i/\sigma_a\} + \lambda_2/4!\,\{a_i^4/\sigma^4_a - 6a_i^2/\sigma^2_a + 3\} + (10/6!)\lambda_1^2\, \{a_i^6/\sigma^6_a - 15a_i^4/\sigma^4_a + 45a_i^2/\sigma^2_a - 15\}\Big)\, \exp\{(-\tfrac{1}{2})(a_i^2/\sigma^2_a)\}. \qquad (2.1)$$

The joint likelihood function is then

$$L(\mu,\sigma^2,\sigma^2_a|y) = \int_{a_1}\int_{a_2}\cdots\int_{a_k} \prod_{i=1}^{k} f(y_{ij}|\mu,\sigma^2,\sigma^2_a,a_i)$$

$$f(a_i|\mu,\sigma^2,\sigma^2_a)\,da_1\,da_2\cdots da_k$$

$$= \int_{a_1}\int_{a_2}\cdots\int_{a_k} \{(1/\sigma)(1/\sqrt{2\pi})\}^{nk} \exp\{-\sum_{i=1}^{k}\sum_{j=1}^{n} e_{ij}^2/2\sigma^2\}$$

$$\{(1/\sigma_a)(1/\sqrt{2\pi})\}^{k} \exp\{-\sum_{i=1}^{k} (a_i^2/2\sigma^2_a)\} \cdot \prod_{i=1}^{k} H(a_i)$$

$$da_1\,da_2\cdots da_k, \qquad (2.2)$$

where $H(a_i) = \{ 1+\lambda_1/3!(a_i^3/\sigma^3_a - 3a_i/\sigma_a) + \lambda_2/4!$

$(a_i^4/\sigma^4_a - 6a_i^2/\sigma^2_a + 3) + 10\lambda_1^2/6!(a_i^6/\sigma^6_a - 15a_i^4/\sigma^4_a$

$+ 45a_i^2/\sigma^2_a - 15)\}$.

Simplifying, we have

$$L(\mu, \sigma^2, \sigma^2_a|y) = \{(1/\sigma)(1/\sqrt{2\pi})\}^{nk} \{(1/\sigma_a)(1/\sqrt{2\pi})\}^{k}$$

$$\exp\{-S_1/2\sigma^2 - S_2/2(\sigma^2+n\sigma^2_a) - nk(\mu-\bar{y})^2/2(\sigma^2+n\sigma^2_a)\}$$

$$\int_{a_1} H(a_1) \exp(-(a_1-P_1)^2/2Q) \, da_1 \int_{a_2} H(a_2)$$

$$\exp(-(a_2-P_2)^2/2Q) \, da_2 \ldots \int_{a_k} H(a_k) \exp(-(a_k-P_k)^2/2Q) \, da_k, \quad (2.3)$$

where $P_i = n\sigma^2_a/\sigma^2+n\sigma^2_a \, (\bar{y}_i-\mu)$, $\quad i=1, 2, \ldots, k$,

and $Q = \sigma^2\sigma^2_a/(\sigma^2+n\sigma^2_a)$. Now,

$$\int_{a_i} H(a_i) \exp \, (-(a_i-P_i)^2/2Q \;\; da_i$$

$$= \sqrt{2\pi Q} \, \{b_0+b_1M_i+b_2M_i^2 + b_3M_i^3 + b_4M_i^4 + b_6M_i^6\} ,$$

where $b_0 = 1 + (3\lambda_2)(n^2\sigma^4_a/4!(\sigma^2+n\sigma^2_a)^2 - (150\lambda^2_1)$

$(n^3\sigma^6_a)/6!(\sigma^2+n\sigma^2_a)^3$,

$b_1 = (3\lambda_1)(n^2\sigma^2_a)/3!(\sigma^2+n\sigma^2_a)^2$,

$b_2 = -(6\lambda_2)(n^3\sigma^4_a)/4!(\sigma^2+n\sigma^2_a)^3 + (450\lambda_1^2)(n^4\sigma^6_a)/$

$6!(\sigma^2+n\sigma^2_a)^4$,

$b_3 = (-\lambda_1)(n^3\sigma^3_a)/3!(\sigma^2+n\sigma^2_a)^3$,

$b_4 = (\lambda_2)(n^4\sigma^4_a)/4!(\sigma^2+n\sigma^2_a)^4 - (150\lambda_1^2)(n^5\sigma^6_a)/$

$6!(\sigma^2+n\sigma^2_a)^5$,

$$b_6 = (10\lambda_1^2/6!)(n^6\sigma^6{}_a)/(\sigma^2+n\sigma^2{}_a)^6$$

and $M_i = -P_i$, $\forall\, i = 1, 2, \ldots, k$.

Therefore,

$$L(\mu, \sigma^2, \sigma^2{}_a \mid y) = \{(1/\sigma)(1/\sqrt{2\pi})\}^{nk} \{(1/\sigma_a)(1/\sqrt{2\pi})\}^k$$

$$\exp\{-S_1/2\sigma^2 - S_2/2(\sigma^2+n\sigma^2{}_a) - nk(\mu-\bar{y})^2/2(\sigma^2+n\sigma^2{}_a)\}$$

$$\{(2\pi)(\sigma^2\sigma^2{}_a)/(\sigma^2+n\sigma^2{}_a)\}^{k/2} \cdot E, \qquad (2.4)$$

where

$$E = \prod_{i=1}^{k} \int_{a_i} H(a_i) \exp\{-(a_i-P_i)^2/2Q\}\, da_i$$

$$= \prod_{i=1}^{k} (b_0+\gamma_i),$$

where $\gamma_i = \sum_{t=1}^{6} b_t M_i^t$ and where the b_t are defined above,

with $b_5 = 0$.

Therefore

$$E = b_0^k + b_0^{k-1} \sum_{i=1}^{k} \gamma_i + b_0^{k-2} \sum_{i_1<i_2} \gamma_{i_1}\gamma_{i_2} + \ldots$$

$$+ \sum_{i_1<i_2<\ldots<i_k} \gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_k}$$

$$= b_0^{k} + b_0^{k-1} \sum_{A_1=1}^{6} b_{A_1} \sum_{S_1=0}^{A_1} (-1)^{S_i} \mu^{A_1-S_1} \sum_{l_1=1}^{k} \binom{A_1}{S_1} \bar{y}_{l_1}^{S_i} + b_0^{k-2} \sum_{A_1=1}^{6} \sum_{A_2=1}^{6} b_{A_1} b_{A_2} \sum_{S_1=0}^{A_1} (-1)^{\sum_{i=1}^{2} S_i}$$

$$\mu^{\sum_{i=1}^{2} A_i - \sum_{i=1}^{2} S_i} \sum_{l_1<l_2}^{k} \left\{ \binom{A_1}{S_1} \bar{y}_{l_1}^{S_1} \cdot \binom{A_2}{S_2} \bar{y}_{l_2}^{S_2} \right\}$$

$$+ \ldots + b_0^{0} \sum_{A_1=1}^{6} \sum_{A_2=1}^{6} \ldots \sum_{A_k=1}^{6} b_{A_1} b_{A_2} \ldots b_{A_k} \sum_{S_1=0}^{A_1} \sum_{S_2=0}^{A_2} \ldots$$

$$\sum_{S_k=0}^{A_k} (-1)^{\sum_{i=1}^{k} S_i} \mu^{\sum_{i=1}^{k} A_i - \sum_{i=1}^{k} S_i} \sum_{l_1<l_2<\ldots<l_k}^{k} \left\{ \binom{A_1}{S_1} \bar{y}_{l_1}^{S_1} \right.$$

$$\left. \binom{A_2}{S_2} \bar{y}_{l_2}^{S_2} \ldots \binom{A_k}{S_k} \bar{y}_{l_k}^{S_k} \right\}.$$

Let $g_{A_1 A_2 \ldots A_c}(S) = \sum_{S_1=0}^{A_1} \sum_{S_2=0}^{A_2} \ldots \sum_{S_c=0}^{A_c} \sum_{l_1<l_2<\ldots<l_c}^{k}$

$$\left\{ \binom{A_1}{S_1} \bar{y}_{l_1}^{S_1} \cdot \binom{A_2}{S_2} \bar{y}_{l_2}^{S_2} \ldots \binom{A_c}{S_c} \bar{y}_{l_c}^{S_c} \right\},$$

where $\sum_{i=1}^{c} S_i = S$. Therefore,

$$E = b_0^k + b_0^{k-1}\{ b_1(\mu g_1(0) - g_1(1)) + b_2(\mu^2 g_2(0) - \mu g_2(1) + g_2(2)) + b_3(\mu^3 g_3(0) - \mu^2 g_3(1) + \mu g_3(2) - g_3(3)) + \dots + b_6(\mu^6 g_6(0) - \dots + g_6(6))\} + b_0^{k-2}\{b_1 b_1(\mu^2 g_{11}(0) - \mu g_{11}(1) + g_{11}(2)) + b_1 b_2(\mu^3 g_{12}(0) - \mu^2 g_{12}(1) + \mu g_{12}(2) - g_{13}(3)) + \dots + b_6 b_6(\mu^{12} g_{66}(0) - \dots + g_{66}(12))\}$$

$$+ \dots + b_0^0\{b_1 b_1 \dots b_1(\mu^k g_{11\dots 1}(0) - \dots (-1)^k g_{66\dots 6}(6k))\}.$$

Collecting coefficients of μ^r and simplifying, we finally get:

$$E = b_0^k + \sum_{r=0}^{6k} \mu^r \sum_{t=1}^{k} b_0^{k-t} \sum_{A_1=1}^{6} \sum_{A_2=1}^{6} \dots \sum_{A_t=1}^{6} b_{A_1} b_{A_2} \dots b_{A_t} (-1)^{\sum_{i=1}^{t} A_i - r} g_{A_1 A_2 \dots A_t}\left(\sum_{i=1}^{t} A_i - r\right),$$

where $g_{A_1 A_2 \dots A_t}(\sum_{i=1}^{t} A_i - r) = 0$ if $\sum_{i=1}^{6} A_i - r = 0$.

2.2 The Prior and Posterior Distributions of $(\mu, \sigma^2, \sigma^2_a)$

The selection of the prior probability distribution in a Bayesian model normally reflects the subjective point of view of the experimenter. However, in this study, to produce a more general solution, we shall assume that little is known about the parameters μ, σ^2 and σ^2_a. Following the lead of Jeffreys (1961) and Tiao and Tan (1965), we take the "non-informative" prior distribution to be

$$f\{\mu, \sigma^2, (\sigma^2+n\sigma^2_a)/n\} \propto 1/\sigma^2(\sigma^2+n\sigma^2_a).$$

Thus, the joint posterior distribution is

$$f(\mu,\sigma^2, \sigma^2_a|y) \propto (\sigma^2)^{-\{k/2(n-1)+1\}} (\sigma^2+n\sigma^2_a)^{-(k/2+1)} \quad E$$

$$\exp\{-S_1/2\sigma^2 - S_2/2(\sigma^2+n\sigma^2_a) - nk(\mu-\bar{y})^2/2(\sigma^2+n\sigma^2_a). \qquad (2.5)$$

2.3 The Joint Posterior Distribution of σ^2_a and σ^2

To get the joint posterior distribution of (σ^2_a, σ^2) we integrate out μ from (2.5).

$$f(\sigma^2_a, \sigma^2|y) \propto (\sigma^2)^{-\{k/2(n-1)+1\}} (\sigma^2 + n\sigma^2_a)^{-(k/2+1)}$$

$$\exp\left(-S_1/2\sigma^2 - S_2/2(\sigma^2+n\sigma^2_a)\right) \int_\mu \exp\left(-nk(\mu-\bar{y})^2/\right.$$

$$\left.2(\sigma^2+n\sigma^2_a)\right) \cdot E \, d\mu.$$

Since E is a polynomial in powers of μ, where the powers range from 0 to 6k, we need

$$\int_\mu \exp\{-nk(\mu-\bar{Y})^2/2(\sigma^2+n\sigma^2_a)\}\,\mu^r d\mu$$

$$= \sum_{p=0}^{[r/2]} H_{r,p}\,\bar{Y}^{r-2p}\,(\sigma^2+n\sigma^2_a/nk)^p \cdot \sqrt{(\sigma^2+n\sigma^2_a)\,2\pi/nk}\,,$$

where $H_{r,p}$ is the absolute value of the p^{th} coefficient of the r^{th} Hermite polynomial and $[r/2]$ is the greatest integer less than or equal to $r/2$. Therefore

$$f(\sigma^2_a, \sigma^2|y) = c\,(\sigma^2)^{-\{k/2(n-1)+1\}}(\sigma^2+n\sigma^2_a)^{-(k/2+1)}$$

$$\exp\{-S_1/2\sigma^2 - S_2/2(\sigma^2+n\sigma^2_a)\}\,\sqrt{(\sigma^2+n\sigma^2_a)/nk}\;\cdot\beta\,, \qquad (2.6)$$

where $\beta = b_0^{\,k} + \sum_{r=0}^{6k}\sum_{p=0}^{[r/2]} H_{r,p}\,\bar{Y}^{r-2p}\,(\sigma^2+n\sigma^2_a/nk)^p$

$$\sum_{t=1}^{k} b_0^{\,k-t} \sum_{A_1=1}^{6}\sum_{A_2=1}^{6}\cdots\sum_{A_t=1}^{6} b_{A_1} b_{A_2}\cdots b_{A_t}\,(-1)^{\sum_{i=1}^{t} A_i - r}$$

$$g_{A_1\ldots A_t}\Big(\sum_{i=1}^{t} A_i - r\Big)$$ and c is the normalizing constant.

We postpone the calculation of c until after the next section.

2.4 Posterior Distribution of σ^2_a/σ^2

From (2.6) we make the transformation

$$w = 1 + n\sigma^2_a/\sigma^2, \quad v = \sigma^2$$

and get the joint distribution

$$f(w,v|y) = c\, v^{-\{k(n-1)/2+1\}} (wv)^{-\frac{1}{2}(k+1)} v/n\, \beta_1$$

$$\exp\{-S_1/2v - S_2/2wv\},$$

$$\text{where } \beta_1 = c_0^{\,k} + \sum_{r=0}^{6k} \sum_{p=0}^{[r/2]} H_{r,p}\, \bar{Y}^{\,r-2p} (wv/nk)^p$$

$$\sum_{t=1}^{k} c_0^{\,k-t} \sum_{A_1=1}^{6} \sum_{A_2=1}^{6} \cdots \sum_{A_t=1}^{6} c_{A_1} c_{A_2} \cdots c_{A_t} (-1)^{\sum_{i=1}^{t} A_i - r}$$

$$g_{A_1 \ldots A_t}\left(\sum_{i=1}^{t} A_i - r\right)$$

$$\text{and } c_0 = 1 + 3\lambda_2/4!\{(w-1)/w\}^2 - 150\lambda_1^{\,2}/6!\{(w-1)/w\}^3,$$

$$c_1 = (3\lambda_1/3!)\, n^{\frac{1}{2}} \{(w-1)^{3/2}/(w^2)\}\, v^{-\frac{1}{2}},$$

$$c_2 = (-6\lambda_2/4!)\, n\{(w-1)^2/w^3\}\, v^{-1} + (450\lambda_1^{\,2}/6!)\, n\{(w-1)^3/w^4\} v^{-1},$$

$$c_3 = (-\lambda_1/3!)\, n^{3/2} \{(w-1)^{3/2}/w^3\}\; v^{-3/2},$$

$$c_4 = (\lambda_2/4!)\, n^2 \{(w-1)^2/w^4\}\, v^{-2} - (150\lambda_1^2/6!)\, n^2$$

$$\{(w-1)^3/w^5\} v^{-2},$$

$c_5 = 0$, and

$$c_6 = (10\lambda_1^2/6!)\; n^3 \{(w-1)^3/w^6\}\; v^{-3}.$$

Integrating out v, we have

$$f(w|y) = c/n\; (w)^{-\frac{1}{2}(k+1)} \int_0^\infty v^{-\frac{1}{2}(kn+1)} \exp\{-\tfrac{1}{2} v(S_1 + S_2/w)\}\; \beta_1\, dv.$$

We can now further simplify β_1 by letting $c_J = D_J v^{-J/2}$,

where $J = 0, 1, \ldots 6$ and the D_J are independent of v.

Therefore

$$\beta_1 = D_0^k + \sum_{r=0}^{6k} \sum_{p=0}^{[r/2]} \sum_{t=1}^{k} \sum_{A_1=1}^{6} \sum_{A_2=1}^{6} \cdots \sum_{A_t=1}^{6} H_{r,p}\, \bar{y}^{r-2p}$$

$$(1/nk)^p D_0^{k-t} D_{A_1} D_{A_2} \cdots D_{A_t} (-1)^{\sum_{i=1}^{t} A_i - r}$$

$$g_{A_1 A_2 \ldots A_t} \left(\sum_{i=1}^{t} A_i - r\right) w^p v^{p - \sum_{i=1}^{t} A_i/2}.$$

Now $v^{-\frac{1}{2}(kn+1)} \exp\{-1/2v(S_1 + S_2/w)\}\ \beta_1$

is a polynomial in powers of v. The powers are in the form $-\{\frac{1}{2}(kn+1) - p + \sum_{i=1}^{t} A_i/2\}$. We know that the quantity in the brackets is positive, since

$$\sum_{i=1}^{t} A_i \geq r \quad \left(\text{otherwise } g_{A_1 \cdots A_t}\left(\sum_{i=1}^{t} A_i - r\right) = 0\right)$$

and $r \geq 2p$. Therefore $\sum_{i=1}^{t} A_i/2 \geq p$

and, of course, $\frac{1}{2}(kn+1)$ is positive. Therefore, when we integrate, we shall have terms in the following form:

$\int_0^\infty v^{-A} e^{-G/2v}\, dv$, where A is positive. The above is in the form of an inverted gamma integral and it is equal to $\Gamma(A-1)(2/G)^{A-1}$.

Finally we have,

$$f(w|y) = cn^{-1} w^{-\frac{1}{2}(k+1)} \qquad D_0^k\ \Gamma\{\tfrac{1}{2}(kn+1) - 1\}$$

$$\{S_1/2(1+\phi/w)\}^{1-\frac{1}{2}(kn+1)} + \sum_{r=0}^{6k} \sum_{p=0}^{[r/2]} \sum_{t=1}^{k} \sum_{A_1=1}^{6} \sum_{A_2=1}^{6} \cdots \sum_{A_t=1}^{6}$$

$$H_{r,p}\,\bar{Y}^{r-2p}\,(1/nk)^{p}\,D_0^{k-t}\,D_{A_1}D_{A_2}\cdots D_{A_t}\,(-1)^{\sum_{i=1}^{t}A_i-r}$$

$$g_{A_1A_2\ldots A_t}\Big(\sum_{i=1}^{t}A_i-r\Big)\,w^{p}\,\Gamma\{\tfrac{1}{2}(kn+1)-p+\sum_{i=1}^{t}A_i/2-1\}$$

$$\{S_1/2\,(1+\phi/w)\}^{-\frac{1}{2}(kn+1)+p-\sum_{i=1}^{t}A_i/2+1}, \quad 1 < \text{[illegible]}, \qquad (2.7)$$

where $\phi = S_2/S_1$.

2.5 Calculation of the Normalizing Constant

By straightforward integration we find:

$$c = \frac{n\,S_1^{\frac{1}{2}k(n-1)}\,S_2^{\frac{1}{2}(k-1)}\,2^{-\frac{1}{2}(kn-1)}}{\Gamma\{\frac{1}{2}(k-1)\}\Gamma\{\frac{1}{2}k(n-1)\}H_\phi\{\frac{1}{2}(k-1),\ \frac{1}{2}k(n-1)\}}\,D_0^{k}$$

$$+\sum_{r=0}^{6k}\sum_{p=0}^{[r/2]}\sum_{t=1}^{k}\sum_{A_1=1}^{6}\sum_{A_2=1}^{6}\cdots\sum_{A_t=1}^{6} H_{r,p}\,\bar{y}^{r-2p}\,(1/nk)^{p}$$

$$D_0^{k-t}\,D_{A_1}D_{A_2}\cdots D_{A_t}\,(-1)^{\sum_{i=1}^{t}A_i-r}\,g_{A_1}g_{A_2}\cdots_{A_t}\Big(\sum_{i=1}^{t}A_i-r\Big)$$

$$\frac{n\,S_1^{k/2(n-1)+\Sigma A_i/2}\,S_2^{\frac{1}{2}(k-1)+p}\,2^{-\frac{1}{2}(kn-1)-p-\sum_{i=1}^{t}A_i/2}}{\Gamma\{\frac{1}{2}(k-1)+p\}\,\Gamma\{k/2(n-1)+\sum_{i=1}^{t}A_i/2\}\,H_\phi\{\frac{1}{2}(k-1)+p\ ,k/2(n-1)+\sum_{i=1}^{t}A_i/2\}},$$

where $H_\phi(m_1 m_2)$ is the incomplete beta integral

$$H_\phi(m_1 m_2) = 1/B(m_1 m_2) \int_0^{\phi/1+\phi} x^{m_1-1} (1-x)^{m_2-1} dx.$$

If we let $\lambda_1 = \lambda_2 = 0$, the value of c reduces to the normalizing constant developed under the normal assumptions. However, in numerous examples generated, it turned out that the normalizing constant above differed by an extremely small amount from the Tiao-Tan constant. Therefore, from this point on, when developing the approximate or asymptotic formulae for the posterior distributions we shall use the latter constant. However, when working out posterior probabilities, we shall use the true constant.

2.6 Comparison with Normal Results

Notice that the individual terms in (2.7) are in the form of a truncated F distribution. It follows that the probability that the variance ratio $n\sigma^2_a/\sigma^2$ is greater than some constant R is

$$P(n\sigma^2_a/\sigma^2 > R) = P(w > 1 + R)$$

$$= \int_{1+R}^{\infty} f(w|y)$$

$$= \int_{1+R}^{\infty} \sum_i c_i \, f(F_i) \, d\, F_i,$$

where the c_i are constants and $f(F_i)$ are F-distribution functions. If we assume in (2.7) that $\lambda_1 = \lambda_2 = 0$, we get the same distribution as the one developed under normal assumptions. Under normal assumptions the posterior distribution (see Tiao and Tan, 1965) of the variance ratio is

$$f(w|y) = \text{Const.}\ \phi^{-\frac{1}{2}k(n-1)} \cdot w^{\frac{1}{2}k(n-1)-1}\ (1+w/\phi)^{-\frac{1}{2}(kn-1)} .$$

Therefore (2.7) is in the form of a simple truncated F-distribution plus correction terms containing λ_1 and λ_2.

In order to analyze the effects of non-normality several examples were generated (see Appendix I for methodology) and graphed. In general the graphs had the following shape.

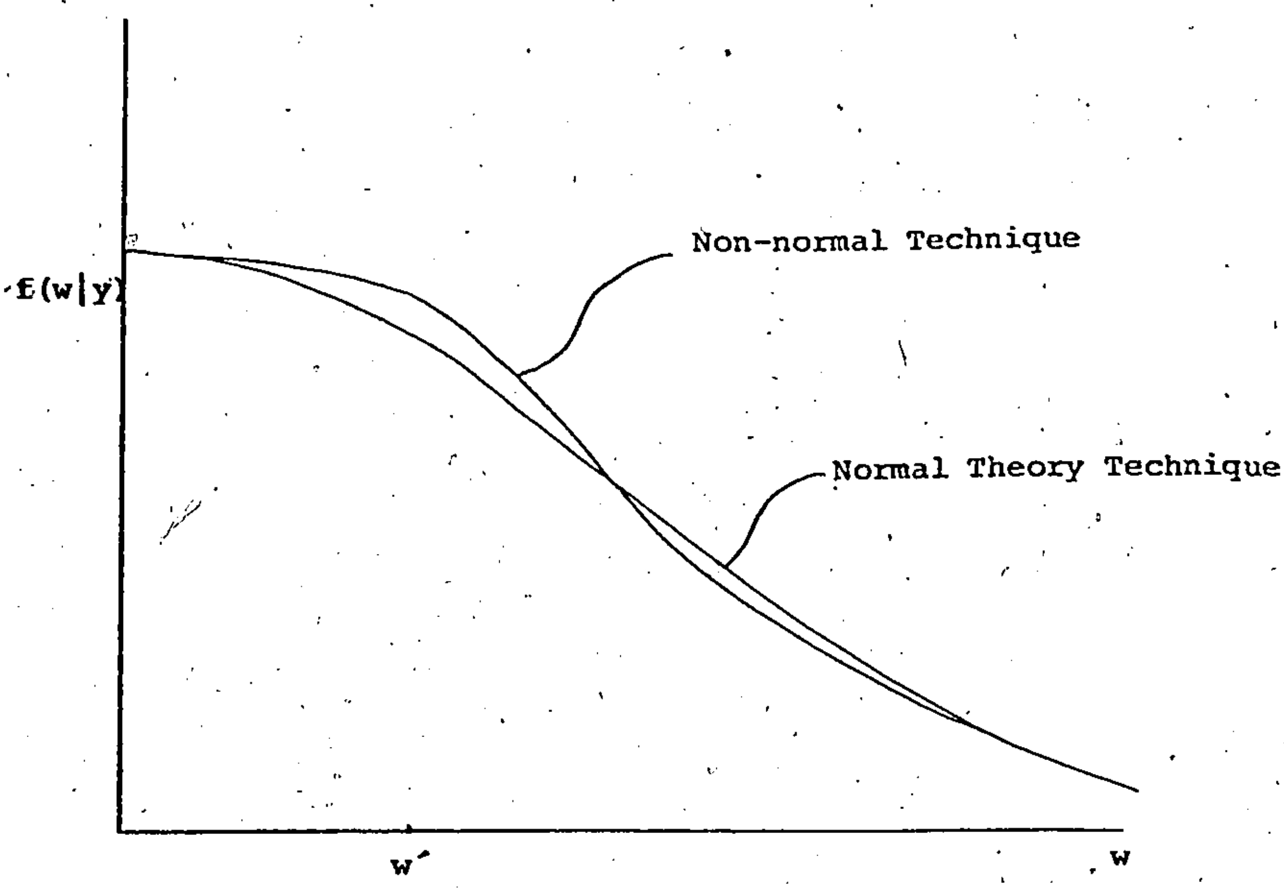

Fig. 1: Posterior Probability of Variance Ratio

w' represents the true value of $1+n\sigma^2_a/\sigma^2$. It would seem then that more area is clustered around the true value in the non-normal formula than in the normal formula when the actual distribution is non-normal. This was the case in almost all the examples generated. The exceptions most likely arose when the values of λ_1 and λ_2 used did not coincide with the values of the population from which the observations were drawn. In these examples the use of $\lambda_1 = 0$ and $\lambda_2 = 0$ provided a better distribution.

The behavior of the posterior distribution when w takes on extreme values provides another interesting aspect. Upon examination it is seen that $f(w|y)$ is in the form const. F_1 + Σ const. F_2, where,

$$F_1 = w^{-\frac{1}{2}(k+1)} (1+\phi/w)^{1-\frac{1}{2}(kn+1)},$$

and $F_2 = (w-1)^A \cdot w^B (1+\phi/w)^{-c}$, with A, B and c being positive constants. It should be noted that F_1 is the normal-theory formula while Σ const. F_2 is the summation of the correction terms. Now, when $w \to 1$, $F_1 \to 1$ $F_2 \to 0$; when $w \to \infty$, $F_1 \to 0$ $F_2 \to 0$. The effect, of course, is that the correction terms disappear in the tail regions of the distributions. Hence both tail-area probabilities were very similar for both techniques. The results obtained in the experiments generated concurred with this finding. It is evident that the extent of this similarity would depend on the parameters involved - particularly the value of ϕ.

In light of the above results it would be illuminating to examine the effect of various values of ϕ on $f(w|y)$, especially when w takes on intermediate values (and where one finds the greatest difference between the two techniques). When ϕ takes on large or intermediate values (and therefore S_1 is relatively small) F_2 does not disappear and, hence, the correction terms take on great importance. When ϕ is small the results are somewhat different. Firstly, a small ϕ ($\phi < k-1/k(n-1)$)

results in $\hat{\sigma}^2_a$ taking on negative values. In addition, S_1 is relatively large. The effect, then, is that $1+\phi/w$ is approximately equal to one (assuming an intermediate value for w) and $S_1/2$ is large. But $S_1/2$ is raised to a negative power and therefore approaches zero as $\phi \to 0$. Thus, when ϕ is small, F_2 tends to be small and (2.7) can then be approximated by the normal-theory result.

2.7 An Illustrative Example

We have shown in the previous section that the tail area probabilities of the posterior distribution of the variance ratio can be approximated quite closely by the normal-theory technique. However, in many cases, the experimenter may be more interested in the shape of the distribution in the central region. The shape of the distribution is illustrated by the following example.

A problem was generated with the following information :

$n=10$, $k=2$, $\bar{y}_1 = 1.2625$, $\bar{y}_2 = 2.878$,

$\bar{y} = 2.0702$, $S_1 = 14.6878$, $S_2 = 13.0492$, $\phi = .8884$,

$\lambda_1 = -.5$, $\lambda_2 = 2.5$, $\sigma^2 = 1$, $\sigma^2_a = 1$, The normally distributed e_{ij} were generated in the usual way. The a_i were generated by the following procedure:

A computer program produced the cumulative probability distribution (using the Trapezoid rule) for the appropriate values of λ_1 and λ_2. Then a three-digit random number table was used in the formula

$$\int_{-\infty}^{a} f(a_i) \doteq \frac{\text{random number}}{1000}$$ and a was calculated.

(For example, if $\lambda_1 = \lambda_2 = 0$ and random number = 500, then a = 0.) Then, using the computer program in Appendix I, the posterior probability density was calculated for both techniques. The following table summarizes the results.

TABLE II

COMPARISON OF NORMAL-THEORY VARIANCE RATIO POSTERIOR PROBABILITIES VS. NON-NORMAL APPROACH

	Normal	Non-normal
$w_{.05}$	1.253	1.254
$w_{.10}$	1.599	1.608
$w_{.25}$	3.001	3.193
$w_{.50}$	7.250	7.467
$w_{.75}$	20.112	20.330
$w_{.90}$	46.989	47.051
$w_{.95}$	71.651	71.639

Notice that the tail area probabilities were almost identical for both methods. The largest difference existed in the range $5 < w < 40$. The true value of w was equal to 11. The mean, the median and the mode were all approximately equal to 11 for both the normal and non-normal techniques. However, the non-normal approach produced a posterior distribution which was sharper and narrower. When using the Bayesian approach, a sharper, narrower distribution implies a more certain conclusion - a result which was entirely expected since more information was being utilized.

2.8 Posterior Distribution of S_1/σ^2

From the joint posterior distribution of σ^2 and σ^2_a (2.6), we again make the transformation

$$\delta = S_2/\sigma^2 \quad \text{and} \quad \rho = 2n\sigma^2_a/S_2 .$$

The joint posterior distribution of δ and ρ is

$$f(\delta,\rho|\, y) = c\ (S_1/\delta)^{-\{k/2(n-1)+1\}} \{1/S_2(\phi\delta/1+\tfrac{1}{2}\phi\delta\rho)\}^{\frac{1}{2}(k+1)}$$

$$S_1S_2/2n\delta^2 \cdot \beta_2\ e^{-\delta/2}\ e^{-S_2/2\,.\{(1/S_2)(\phi\delta/1+\frac{1}{2}\phi\delta\rho)\}}, \tag{2.8}$$

$$\text{where} \quad \beta_2 = h_0^k + \sum_{r=0}^{6k}\sum_{p=0}^{[r/2]}\sum_{t=1}^{k}\sum_{A_1=1}^{6}\sum_{A_2=1}^{6}\cdots\sum_{A_t=1}^{6} H_{r,p}\,\bar{y}^{r-2p}$$

$$(1/nk)^p\{(1/S_2)(\phi\delta/1+\tfrac{1}{2}\phi\delta\rho)\}^{-p}\ h_0^{k-t}\ h_{A_1}h_{A_2}\cdots h_{A_t}$$

$$(-1)^{\sum_{i=1}^{t} A_i - r} \quad g_{A_1 A_2 \ldots A_t} \left(\sum_{i=1}^{t} A_i - r\right)$$

and $h_0 = 1 + (3\lambda_2/4!)(\rho S_2/2)^2\{(1/S_2)(\phi\delta)/1+\tfrac{1}{2}\phi\delta\rho\}^2$

$-150\lambda_1^{\ 2}/6!\ (\rho S_2/2)^3\ \{(1/S_2)(\phi\delta)/1+\tfrac{1}{2}\phi\delta\rho\}^3,$

$h_1 = (3\lambda_1/3!)n^{\frac{1}{2}}\ (\rho S_2\ /2)^2\ \{(1/S_2)(\phi\delta)/1+\tfrac{1}{2}\phi\delta\rho)\}^2,$

$h_2 = (-6\lambda_2/4!)\ n\ (\rho S_2\ /2)^2\ \{(1/S_2)(\phi\delta/1+\tfrac{1}{2}\phi\delta\rho)\}^3 + (450\lambda_1^{\ 2}\ /6!)$

$n_2(\rho S_2/2)^3\ \ \{(1/S_2)(\phi\delta/1+\tfrac{1}{2}\phi\delta\rho)\}^4,$

$h_3 = (-\lambda_1/3!)n^{3/2}\ \ (\rho S_2/2)^{3/2}\ \ 1/\ _2(\phi\delta/1+\tfrac{1}{2}\phi\delta\rho)^3,$

$h_4 = (\cdot\lambda_2/4!)n^2\ \ (\rho S_2/2)^2\ \ \{(1/S_2)(\phi\delta\ /\ 1+\tfrac{1}{2}\phi\delta\rho)\}^4$

$(-150\lambda_1^{\ 2}/6!)\ n^2(\rho S_2/2)^3\ \{(1/S_2)(\phi\delta/1+\tfrac{1}{2}\phi\delta\rho\ )\}^5,$

$h_5 = 0$, and

$h_6 = (10\lambda_1^{\ 2}/6!)\ n^3\ (\rho S_2/2)^3\ \{(1/S_2)(\phi\delta/1+\tfrac{1}{2}\phi\delta\rho)\ \}^6.$

To get the posterior distribution of S_1/σ^2 we integrate out ρ from (2.8) to arrive at

$$f(\delta|y) = \int_\rho f(\delta,\rho)d\rho\ .$$

To simplify, we make the following transformation in (2.8)

$x = (\frac{1}{2})\ (\phi\delta/1+\frac{1}{2}\phi\delta\rho)$. Therefore,

$$f(\delta|y) = \int_0^{\phi\delta/2} c\ (1/x^2)\ (S_1S_2/2n\delta^2)\ (S_1/\delta)^{-\{k/2(n-1)+1\}}$$

$$(2x/S_2)^{\frac{1}{2}(k+1)}\ \beta_2\ e^{-\delta/2}\ e^{-x}\ dx, \qquad (2.9)$$

where β_2 is defined above and

$$h_0 = 1 + 3\lambda_2/4!\ (1-2x/\phi\delta)^2 - 150\lambda_1{}^2/6!\ (1-2x/\phi\delta)^3,$$

$$h_1 = 3\lambda_1/3!\ (2n/S^2)^{\frac{1}{2}}\ (1-2x/\phi\delta)^{3/2}\ x^{\frac{1}{2}},$$

$$h_2 = (-6\lambda_2/4!)(2n/S_2)(1-2x/\phi\delta)^2\ x + 450\lambda_1{}^2/6!\ (2n/S_2)$$

$$(1-2x/\phi\delta)^3\ x,$$

$$h_3 = (-\lambda_1/3!)(2n/S_2)^{3/2}\ (1-2x/\phi\delta)^{3/2}\ x^{3/2},$$

$$h_4 = \lambda_2/4!\ (2n/S_2)^2(1-2x/\phi\delta)^2\ x^2 - 150\lambda_1{}^2/6!\ (2n/S_2)^2$$

$$(1-2x/\phi\delta)^3\ x^2,$$

$h_5 = 0$, and

$$h_6 = 10\lambda_1{}^2/6!\ (2n/S_2)^3\ (1-2x/\phi\delta)^3 x^3.$$

Therefore,

$$f(\delta|y) = c\,(S_1 S_2/2n\delta^2)(2/S_2)^{\frac{1}{2}(k+1)}(S_1)^{-\{k/2(n-1)+1\}}$$

$$\delta^{k/2(n-1)-1}\, e^{-\delta/2} \cdot \int_0^{\phi\delta/2} x^{\frac{1}{2}(k+1)-2}\, \beta_3\, e^{-x}\, dx, \quad (2.10)$$

where $\beta_3 = \sum_{d_1 d_2 d_3}^{k} k!/d_1!d_2!d_3!\,(h_{0,1})^{d_1}(h_{0,2})^{d_2}$

$$(1-2x/\phi\delta)^{2d_1+3d_2} + \sum_{r=0}^{6k}\sum_{p=0}^{[r/2]}\sum_{t=1}^{k}\sum_{A_1=1}^{6}\sum_{A_2=1}^{6}\cdots\sum_{A_t=1}^{6}$$

$$\sum_{L_1=1}^{2}\sum_{L_2=1}^{2}\cdots\sum_{L_t=1}^{2} H_{r,p}\,\bar{y}^{r-2p}(1/nk)^{p}(2x/S_2)^{-p}$$

$$\sum_{b_1 b_2 b_3}^{k-t}(k-t)!/b_1!b_2!b_3!\,(h_{0,1})^{b_1}(h_{0,2})^{b_2}$$

$$(1-2x/\phi\delta)^{2b_1+3b_2}\, h_{A_1L_1} h_{A_2L_2}\cdots h_{A_tL_t}\,(1-2x/\phi\delta)^{\sum_{i=1}^{6}\sum_{j=1}^{6} f(A_iL_j)}$$

$$x^{\sum_{i=1}^{t}A_i/2}\,(-1)^{\sum_{i=1}^{t}A_i-r}\, g_{A_1A_2\cdots A_t}\left(\sum_{i=1}^{t}A_i-r\right)$$

and $h_{0,1} = 3\lambda_2/4!$,

$h_{0,2} = -150\lambda_1^2/6!$,

$h_{1,1} = 3\lambda_1/3!\ (2n/S_2)^{\frac{1}{2}}$,

$h_{1,2} = 0$,

$h_{2,1} = (-6\lambda_2/4!)(2n/S_2)$,

$h_{2,2} = (450\lambda_1^2\ /6!)(2n/S_2)$,

$h_{3,1} = (-\lambda_1 3!)(2n/S_2)^{3/2}$,

$h_{3,2} = 0$,

$h_{4,1} = (\lambda_2/4!)(2n/S_2)^2$,

$h_{4,2} = (-150\lambda_1^2/6!)(2n/S_2)^2$,

$h_{5,1} = 0$,

$h_{5,2} = 0$,

$h_{6,1} = (10\lambda_1^2/6!)(2n/S_2)^2$,

$h_{6,2} = 0$ and

$f(0,1) = 2$,

$f(0,2) = 3$,

$f(1,1) = 3/2$,

$f(1,2) = 0$,

$f(2,1) = 2$,

$f(2,2) = 3$,

$f(3,1) = 3/2$,

$f(3,2) = 0$,

$f(4,1) = 2$,

$f(4,2) = 3$,

$f(5,1) = 1$,

$f(5,2) = 1$,

$f(6,1) = 3$,

$f(6,2) = 0$.

Now the integral part of $f(\delta|y)$ is of the form

$$\Sigma \int_0^{\phi\delta/2} \text{const.}\ (1-2x/\phi\delta)^A\ x^B\ e^{-x}\ dx.$$

Consider $2x/\phi\delta = 1/(1+\frac{1}{2}\phi\delta\rho)$. Since $\frac{1}{2}\phi\delta\rho = n\sigma^2/\sigma^2$, which ranges from 0 to ∞, $1/(1+\frac{1}{2}\phi\delta\rho)$ ranges from 0 to 1. Since $\frac{1}{2}\phi\delta\rho = n\sigma^2_a/\sigma^2$, which ranges from 0 to ∞ we have $1/(1+\frac{1}{2}\phi\delta\rho)$ ranging from 0 to 1.

Consider $(1-2x/\phi\delta)^A$. Since A may not necessarily be an integer, it is possible upon expansion to have an infinite series. However, since $2x/\phi\delta$ is a positive fraction between 0 and 1, we can approximate $(1-2x/\phi\delta)$ by the first $[A]$ terms of the binomial expansion (where $[A]$ is the greatest integer less than or equal to A). That is:

$$(1-2x/\phi\delta)^A = 1 - A\ (2x/\phi\delta) + (A(A-1)/2)(2x/\phi\delta)^2 + \ldots$$

$$+ (-1)^{[A]} \binom{[A]}{A} (2x/\phi\delta)^{[A]}.$$

Therefore the integral part of $f(\delta|y)$ is of the form:

$$\Sigma \int_0^{\phi\delta/2} \text{const.}\ x^c\ e^{-x}\ dx,$$

which is in the form of a sum of incomplete-gamma integrals.

Let $G_w(p) = 1/\Gamma(p) \int_0^w x^{p-1} e^{-x}\, dx.$

Thus, $f(\delta|y) = \sum_{d_1 d_2 d_3}^{k} k!/d_1!d_2!d_3! \sum_{i=0}^{[2d_1+3d_2]} h_{0,1}^{d_1} h_{0,2}^{d_2}$

$$(-\phi)^{-i} f_\delta\{k(n-1)-2i\}\ \Gamma(k-1/2 + i)/\Gamma(k-1)/2\ \ \Gamma\{k/2(n-1)-i\}/$$

$$\Gamma\{k/2(n-1)\}\ G_{\phi\delta/2}\{(k-1)/2 + i\}/\ H_\phi\{(k-1)/2,\ k/2(n-1)\}$$

$$\binom{2d_1 + 3d_2}{i} + \sum_{r=0}^{6k} \sum_{p=0}^{[r/2]} \sum_{t=1}^{k} \sum_{A_1=1} \cdots \sum_{A_t=1} \sum_{L_1=1} \cdots$$

$$\sum_{L_t=1}^{2} \sum_{b_1 b_2 b_3}^{k-t} \sum_{m=0}^{[2b_1+3b_2+\Sigma\Sigma f(A_i L)]} Hr,p\ \bar{Y}^{r-2p}$$

$$(S_2/2nk)^p\ h_{0,1}^{b_1}\ h_{0,2}^{b_2}\ (-1)^{\sum_{i=1}^{t} A_i - r}$$

$$g_{A_1 \ldots A_t}\ (\sum_{i-1}^{t} A_i - r)\, 2b_1 + 3b_2 + \Sigma\Sigma f(A,L)_m\ (-\phi)^{-m}$$

$$f_\delta\{k(n-1)-2m\}\ \Gamma\{(k-1)/2 + (\sum_{i=1}^{t} A_i/2 + m\}\ /\ \Gamma\{(k-1)/2\}$$

$$\Gamma\{k(n-1)/2-m\}/\Gamma\{k(n-1)/2\}\ G_{\phi\delta/2}\{(k-1)/2 + \sum_{i=1}^{t} A_i/2+m\}/$$

$$H_\phi\{(k-1)/2,\ k/2(n-1)\}, \tag{2.11}$$

where $f_\delta(\eta)$ is a chi-square distribution function with η degrees of freedom (we are using the Tiao-Tan constant).

The expression of the distribution of δ in (2.11) is not very useful as it now stands. We can apply two different methods of approximation. For both we need to develop the expression for the rth moment and for the second we need the moment generating function. Consider $f_\delta(\eta) \,.\, G_{\phi\delta/2}(\gamma)$ where $f_\delta(\eta)$ and $G_{\phi\delta/2}(\gamma)$ are defined above.

$$\int_0^\infty \delta^r f_\delta(\eta) \,.\, G_{\phi\delta/2}(\gamma)\, d\delta$$

can be shown by straightforward integration to equal

$$2^r \{\Gamma(\eta/2 + r)/\Gamma(\eta/2)\} \; H_\phi(\gamma, \eta/2 + r),$$

and $\int_0^\infty e^{\delta t} f_\delta(\eta) \; G_{\phi\delta/2}(\gamma)$

$$= H_{\phi/1-2t}(\gamma, \eta/2) \;.\; (1-2t)^{-\eta/2}, \qquad |t| < \tfrac{1}{2}.$$

Since (2.11) can be expressed as $\Sigma c_\delta f_\delta(\eta) \,.\, G_{\phi\delta/2}(\gamma)$, we have,

$$E(\delta^R) = \sum_{d_1 d_2 d_3}^{k} k!/d_1!d_2!.d_3! \sum_{i=0}^{[2d_1+3d_2]} h_{0,1}^{d_1} \; h_{0,2}^{d_2}$$

$$(-\phi)^{-i} \binom{2d_1+3d_2}{i} \; 2^R \; \Gamma\{k(n-1)/2 - i + R\}/\;\Gamma\{k(n-1)/2 - i\}$$

$$\Gamma\{(k-1)/2 + i\}/\Gamma\{k-1\}/2 \qquad \Gamma\{k/2(n-1)-i\}/\Gamma\{k(n-1)/2\}$$

$$H_{\phi}\{(k-1)/2 + i,\ k(n-1)/2 - i + R\}/\ H_{\phi}\{(k-1)/2,\ k(n-1)/2\}$$

$$+ \sum_{r=0}^{6k} \sum_{p=0}^{[r/2]} \sum_{t=1}^{k} \sum_{A_1=1}^{6} \cdots \sum_{A_t=1}^{6} \sum_{L_1=1}^{2} \cdots \sum_{L_t=1}^{2} \sum_{b_1 b_2 b_3}^{k-t}$$

$$\sum_{m=0}^{[2b_1+3b_2+\Sigma\Sigma f(A,L)]} H_{r,p}\ \bar{Y}^{r-2p}\ (S_2/2nk)^{p}\ h_{0,1}^{b_1}$$

$$h_{0,2}^{b_2}\ (-1)^{\sum_{i=1}^{t} A_i - r}\ g_{A_1 \ldots A_t}\left(\sum_{i=1}^{t} A_i - r\right)$$

$$\binom{2b_1+3b_2+\Sigma\Sigma f(A,L)}{m} (-\phi)^{-m}\ \Gamma\{(k-1)/2 + \sum_{i=1}^{t} A_i/2 + m\}/$$

$$\Gamma\{k-1\}/2 \qquad \Gamma\{k(n-1)/2 - m\}/\Gamma\{k(n-1)/2\} \qquad 2^{R}$$

$$\Gamma\{k(n-1)/2 - m + R\}/\Gamma\{k(n-1)/2 - m\}.\ \ H_{\phi}\{(k-1)/2 +$$

$$\sum_{i=1}^{t} A_i/2 + m,\ k(n-1)/2 - m + R\}/\ H_{\phi}\{(k-1)/2,\ k/2(n-1)\}, \qquad (2.12)$$

and the moment generating function

$$M_{\delta}(t) = \sum_{d_1 d_2 d_3}^{k} k!/d_1!d_2!d_3! \sum_{i=0}^{[2d+3d]} h_{0,1}^{d_1}\ h_{0,2}^{d_2}$$

$$(-\phi)^{-i} \binom{2d_1+3d_2}{i} \Gamma\{(k-1)/2 + i\}/\Gamma(k-1)/2$$

$$\Gamma\{k(n-1)/2 - i\}/\Gamma\{k(n-1)/2\} \; H_{\phi/1-2t}\{(k-1)/2 + i,\ k(n-1)/2 - i\}$$

$$/H_{\phi}\{(k-1)/2,\ k(n-1)/2\} \quad (1-2t)^{-\{k(n-1)/2 - i\}}$$

$$+ \sum_{r=0}^{k} \sum_{p=0}^{[r/2]} \sum_{t=1}^{k} \sum_{A_1=1}^{6} \cdots \sum_{A_t=1}^{6} \sum_{L_1=1}^{2} \cdots \sum_{L_t=1}^{2} \sum_{b_1 b_2 b_3}^{k-t}$$

$$\sum_{m=0}^{[2b_1+3b_2+\Sigma\Sigma f(A,L)]} Hr,p\ \bar{Y}^{r-2p} (S_2/2nk)^{p}\ h_{0,1}^{b_1}\ h_{0,2}^{b_2}$$

$$(-1)^{\sum_{i=1}^{t} A_i - r}\ g_{A_1 \ldots A_t}\left(\sum_{i=1}^{t} A_i - r\right) \binom{2b_1+3b_2+\Sigma\Sigma f(A,L)}{m}$$

$$(-\phi)^{-m}\ \Gamma\{(k-1)/2 + \sum_{i=1}^{t} A_i/2 + m\}/\ \Gamma(k-1)/2$$

$$\Gamma\{k(n-1)/2 - m\}/\ \Gamma\{k(n-1)/2 \ .\ H_{\phi/1-2t}\{(k-1)/2+\sum_{i=1}^{t} A_i/2 + m,$$

$$k(n-1)/2 - m\}/\ H_{\phi}\{(k-1)/2,\ k(n-1)/2\}$$

$$(1-2t)^{-\{k(n-1)/2 - m\}}, \qquad |t| < \tfrac{1}{2}. \qquad (2.13)$$

To obtain the approximate distribution of δ, we can firstly apply a very simple technique. If we can calculate the first r moments from (2.12) and then plug them into an Edgeworth series, we can be fairly confident of having a function reasonably close to the true distribution. The second approach would be somewhat more complex. If we analyze the terms of (2.11) we see that each is approximately a chi-square variable function. What we shall now do is examine the moment generating function to see how close that assumption is. The study of $M_\delta(t)$ takes three forms: 1. when $\phi \to \infty$ 2. when $\phi \to 0$ and 3. intermediate values of ϕ.

When ϕ is very large, it is clear that both $\phi/1-2t+\phi$ and $\phi/1+\phi$ are close to 1. Therefore, both $H_{\phi/1-2t}\{A,B\}$ and $H_\phi\{C,D\}$ are close to 1. Also the presence of the terms $(-\phi)^{-i}$ and $(-\phi)^{-m}$ reduce $M_\delta(t)$ to the terms when i and m = 0.

Thus in this case

$M_\delta(t) = \Sigma_l\, c_l\, (1-2t)^{-k(n-1)/2}$, where the c_l are independend of ϕ, t and δ. Therefore, the terms of the distribution of δ tend to a chi-square function with k(n-1) degrees of freedom.

Now, when ϕ tends to zero, we can show by applying L'hôpital's rule that

$$H_{\phi/1-2t}\{(k-1)/2 + i,\ k(n-1)/2 - i\}/H_{\phi}\{(k-1)/2,\ k(n-1)/2\}$$

$$= \phi^{i}\ (1-2t)^{-kn/2 + 3/2} \qquad \text{and}$$

$$H_{\phi/1-2t}\{(k-1)/2 + \Sigma A_i/2 + m,\ k(n-1)/2 - m\}/H_{\phi}\{(k-1)/2,\ k(n-1)/2\}$$

$$= \phi^{\Sigma A_i/2 + m}\ (1-2t)^{-kn/2 - \Sigma A_i/2 + 3/2}.$$

Again the presence of $(-\phi)^{-i}$ and $(-\phi)^{-m}$ affect the function. We can now reduce $M_\delta(t)$ to the terms where $\Sigma A_i = o$. Therefore

$$M_\delta(t) = \Sigma_l\ c_i\ (1-2t)^{-kn/2 + 3/2},$$

where again the c_l are independent of ϕ, t and δ. Therefore, the terms of the distribution of δ again tends to the chi-square function now with kn-3 degrees of freedom. That is, the same as the first case, but with k-3 additional degrees of freedom. This result seems to indicate that when S_1 (the within group sum of square) is much larger than S_2 (the between group sum of squares), we should again base our decision on S_1 alone, but with degrees of freedom increased to kn-3.

Both of the above results are interesting for another reason. In the first case, when ϕ is large which implies S_2 much greater than S_1, the unbiased estimator of σ^2_a will be based mostly on S_2 and therefore will be rather large compared to the unbiased estimator of σ^2. The implication of the above results is that the corrective terms will be negligible and thus, this method reverts back to the normal-theory assumptions developed by Tiao and Tan. This assertion is born out by examining the moment generating function. In this case

$$M_\delta(t) = \sum_l c_l (1-2t)^{-k(n-1)/2}$$

$$= (1-2t)^{-k(n-1)/2} \sum_l c_l .$$

It is clear that $\sum_l c_l = 1$ since if we integrate (2.11) over δ we must have unity. And $M_\delta(t) = (1-2t)^{-k(n-1)/2}$ is the same result obtained by Tiao and Tan.

In the second case when ϕ is small; that is $S_1 >> S_2$, $\hat{\sigma}^2_a$ will be negative. And the moment-generating function is

$M_\delta(t) = (1-2t)^{-kn-3/2} \sum_l c_l$, where again $\Sigma c_l = 1$. This result implies that when $\hat{\sigma}^2_a$ is negative, it suffices to use the distribution of δ developed under the normal assumptions.

Continuing with our discussion of the effect of different values of ϕ on $M_\delta(t)$, we note that even for intermediate ϕ, we have an approximate chi-square function. This can be seen by noticing that when $k(n-1)/2 - i$ and $k(n-1)/2 - m$ are large, then

$$H_{\phi/1-2t}\{(k-1)/2 + i,\ k(n-1)/2 - i\} / H_\phi\{(k-1)/2,\ k(n-1)/2\}$$

and $$H_{\phi/1-2t}\{(k-1)/2 + \Sigma A_i/2 + m,\ k(n-1)/2 - m\} /$$

$$H_\phi\{(k-1)/2,\ k(n-1)/2\},$$

for t in some interval $(-\Delta,\Delta)$, are very close to unity. (Since i and m can be as high as 3k, n would have to be quite large). Therefore, again we have the terms of $f(\delta|y)$ tending toward chi-square functions.

From the above we can suggest that the individual terms of $f(\delta|y)$ might best be approximated by a scaled chi-square function $af(\chi^2_b)$ where a is a constant and b are the degrees of freedom. That is,

$$f(\delta|y) = \Sigma_l\, c_l\, a_l\, f(\chi^2_{b_l}).$$

We can solve for a_l and b_l by equating the first two moments of the terms of $f(\delta|y)$ with those of $\Sigma_l\, c_l\, a_l\, f(\chi^2_{b_l})$, as follows:

$$f(\delta|y) = \sum_{l_1} \sum_{i} c_{l} a_{i} f(\chi^2_{b_i}) + \sum_{l_2} \sum_{m,\Sigma A_i} c_{l_2} a_{m,\Sigma A_i} f(\chi^2_{b_{m,\Sigma A_i}}).$$

Therefore,

$$a_i = \{\tfrac{1}{2}k(n-1) - i + 1\} \; H_\phi\{(k-1)/2 + i,\; k/2(n-1) - 1 + 2\} / H_\phi\{(k-1)/2 + 1,\; k/2(n-1) - i + 1\},$$

$$b_i = k(n-1)-2i/2 \quad H_\phi\{(k-1)/2 + 1,\; k/2(n-1) - i + 1\} / H_\phi\{(k-1)/2,\; k(n-1)/2\},$$

$$a_{m,\Sigma A_i} = \{\tfrac{1}{2}k(n-1) - m + 1\} \; H_\phi\{(k-1)/2+\Sigma A_i/2+m,\; k/2(n-1)-m+2\} / H_\phi\{(k-1)/2+\Sigma A_i/2+m,\; k/2(n-1)-m+1\}$$

$$- (\tfrac{1}{2}k(n-1)-m) \quad H_\phi\{(k-1)/2+\Sigma A_i/2+m,\; k/2(n-1)-m+1\} / H_\phi\{(k-1)/2,\; k(n-1)/2 \quad \text{and}$$

$$b_{m,\Sigma A_i} = k(n-1)-2m/a \quad H_\phi\{(k-1)/2 + \Sigma A_i/2 + m,\; k/2(n-1)-m+1\} / H_\phi\{(k-1)/2,\; k/2(n-1)\}.$$

2.9 Posterior Distribution of $2n\sigma^2_a/S_2$

If in the joint posterior distribution of ρ and δ (2.8) we make the transformation $T = \delta/2$, then we have

$$f(\rho) = 1/\Gamma\{(k-1)/2\}\Gamma\{k/2(n-1)\}\, H_{\phi}\{(k-1)/2,\ k/2(n-1)\}$$

$$\int_0^{\infty} T^{k/2(n-1)-1}\, e^{-T}\, \beta_4 ,$$

where $\beta_4 =$

$$\sum_{d_1 d_2 d_3}^{k} k!/d_1!d_2!d_3!\ h_{0,1}^{d_1}\, h_{0,2}^{d_2}\, \rho^{2d_1+3d_2}$$

$$h_{(k+1)/2+2d_1+3d_2}(\rho, T)$$

$$+ \sum_{r=0}^{6k} \sum_{p=0}^{[r/2]} \sum_{t=1}^{k} \sum_{A_1=1}^{6} \cdots \sum_{A_t=1}^{6} \sum_{L_1=1}^{2} \cdots \sum_{L_t=1}^{2} \sum_{b_1 b_2 b_3}^{k-t} Hr,p\, Y^{r-2p}$$

$$(S_2/2nk)^p\, h_{0,1}^{b_1}\, h_{0,2}^{b_2}\, h_{A_1,L_1} \cdots h_{A_t,L_t}\, (-1)^{\sum_{i=1}^{t} A_i - r}$$

$$g_{A_1 \ldots A_t}\left(\sum_{i=1}^{t} A_i - r\right) \rho^{-\sum_{i=1}^{t} A_i/2 + 2b_1 + 3b_2 + \Sigma\Sigma f(A,L)}$$

$$h_{(k+1)/2-p+2b_1+3b_2+\Sigma\Sigma f(A,L)}(\rho, T)\, dT , \qquad (2.14)$$

where

$$h_{0,1} = 3\lambda_2/4!,$$

$$h_{0,2} = -150\lambda_1^2/6!,$$

$$h_{1,1} = (3\lambda_1/3!)(2n/S_2)^{\frac{1}{2}},$$

$$h_{1,2} = 0,$$

$$h_{2,1} = (-6\lambda_2/4!)(2n/S_2),$$

$$h_{2,2} = 450\lambda_1^2/6!(2n/S_2),$$

$$h_{3,1} = (-\lambda_1/3!)(2n/S_2)^{3/2},$$

$$h_{3,2} = 0,$$

$$h_{4,1} = \lambda_2/4!(2n/S_2)^2,$$

$$h_{4,2} = -150\lambda_1^2/6!(2n/S_2)^2,$$

$$h_{5,1} = 0,$$

$$h_{5,2} = 0,$$

$$h_{6,1} = 10\lambda_1^2/6!\ (2n/S_2)^3,$$

$h_{6,2} = 0$,

$f(0,1) = 2$,

$f(0,2) = 3$,

$f(1,1) = 2$,

$f(1,2) = 0$,

$f(2,1) = 3$,

$f(2,2) = 4$,

$f(3,1) = 3$,

$f(3,2) = 0$,

$f(4,1) = 4$,

$f(4,2) = 5$,

$f(5,1) = 0$,

$f(5,2) = 0$,

$$f(6,1) = 6,$$

$$f(6,2) = 0$$

and

$$h_A(\rho\ T) = \{(\phi T)^{-1} + \rho\}^{-\frac{1}{2}(A)} \exp\{-1/\{(\phi T)^{-1} + \rho\}.$$

The posterior distribution in (2.14) is defined over the range 0 to ∞ and hence there is no negative estimated variance problem. However our problem is making any inference from the rather complicated function. We shall now simplify it.

The moments of ρ can be expressed as follows:

$$E(\rho^R) = 1/[H_\phi\{(k-1)/2,\ k/2(n-1)\}\Gamma(k-1)/2\ \ \Gamma\{k/2(n-1)\}]$$

$$\sum_{d_1 d_2 d_3}^{k} k!/d_1!d_2!d_3!\ \ h_{0,1}^{d_1}\ h_{0,2}^{d_2}$$

$$\sum_{l=0}^{R+2d_1+3d_2} \binom{R+2d_1+3d_2}{l} (-\phi)^{-l}$$

$$H_\phi\{((k-1)/2)-R+1,\ k/2(n-1)-1\}\Gamma\{(k-1)/2-R+1\}\Gamma\{k/2(n-1)-1\}$$

$$+\sum_{r=0}^{6k}\sum_{p=0}^{[r/2]}\sum_{t=1}^{k}\sum_{A_1=1}^{6}\cdots\sum_{A_t=1}^{6}\sum_{L_1=1}^{2}\cdots\sum_{L_t=1}^{2}\sum_{b_1 b_2 b_3}^{k-t} Hr,p\,\overline{y}^{r-2p}$$

$$(S_2/2nk)^p \; h_{0,1}^{b_1} \; h_{0,2}^{b_2} \; h_{A_1,L_1} \cdots h_{A_t,L_t} \; (-1)^{\sum_{i=1}^{t} A_i - r}$$

$$g_{A_1 \ldots A_t} \left(\sum_{i=1}^{t} A_i - r\right)$$

$$\sum_{l=0}^{R-\Sigma A_i/2+2b_1+3b_2+\Sigma\Sigma f(A,L)} (-\phi)^{-1} \left[R - \sum_{i=1}^{t} A_i/2 + 2b_1+3b_2+\Sigma\Sigma f(A,L)\right]_l$$

$$H_\phi \{ (k-1)/2 - P + \sum_{i=1}^{t} A_i/2 - R + 1, \; k/2(n-1) - 1\}$$

$$\Gamma\{(k-1/2 - P + \sum_{i=1}^{t} A_i/2 - R + 1\} \; \Gamma\{k/2(n-1) - 1\} . \qquad (2.15)$$

We can apply this formula by plugging the moments into an Edgeworth series. However, (2.15) is restricted to the case where $(k-1)/2>R$. So if we want the first four moments, then $k \geq 10$, obviously putting the usefulness of this method in doubt.

Another technique would be to use a similar approach to the one developed under the normal assumption and arrive at an asymptotic expansion. The expression

$\int_0^\infty T^{k/2(n-1)} e^{-T} h_A(\rho, T)$ suggests that when $k(n-1)/2$

is large, we can write

$$\int_0^\infty T^{k/2(n-1)} e^{-T} h_A(\rho,T) = h_A(\rho, z)\,\Gamma(z+1),$$

where $z = k/2(n-1) - 1$ is the value of T which maximizes the factor $T^{k/2(n-1)-1} \cdot e^{-T}$. Following the method discussed in Jeffreys and Swirlee (1956), we can use this quality as follows.

For fixed ρ, the function $h_A(\rho,T)$ is analytic in $0 < T < \infty$. Using Taylor's theorem, we can expand $h_A(\rho,T)$ around z.

$$\int_0^\infty h_A(\rho,T)\, T^{k/2(n-1)} e^{-T}\, dT$$

$$= \int_0^\infty \sum_{r=0}^\infty 1/r!\; h_A^{(r)}(\rho,z)\,(T-z)^r \cdot T^z e^{-T}\, dT$$

$$= \sum_{r=0}^\infty 1/r!\; h_A^{(r)}(\rho,z) \int_0^\infty (T-z)^r\, T^z\, e^{-T}\, dT,$$

where $h_A^{(0)}(\rho,z) = \exp\{-(\lambda^{-1} + \rho)^{-1}\}\,\lambda^{-1} + \rho)^{-A}$,

$$h_A^{(1)}(\rho,z) = -z^{-1}\, h_A^{(0)}(\rho,T)\; R_{1,A}(\lambda),$$

$$h_A^{(2)}(\rho,z) = +z^{-2}\, h_A^{(0)}(\rho,T)\,\{R_{2,A}(\lambda) + 2R_{1,A}(\lambda)\},$$

$$h_A^{(3)}(\rho,z) = -z^{-3}\, h_A^{(0)}(\rho,T)\,\{R_{3,A}(\lambda) + 6R_{2,A}(\lambda) + 6R_{1,A}(\lambda)\},$$

$$h_A^{(4)}(\rho,z) = +z^{-4} H_A^{(0)}(\rho,T) \{R_{4,A}(\lambda)+12R_{3,A}(\lambda)+36R_{2,A}(\lambda) + 24R_{1,A}(\lambda)\},$$

where $\lambda = \phi z$ and

$$R_{1,A}(\lambda) = (1+\lambda\rho)^{-1}\{((\lambda^{-1}+\rho)^{-1})-A\},$$

$$R_{2,A}(\lambda) = (1+\lambda\rho)^{-2}\{(\lambda^{-1}+\rho)^{-2}-2\{A+1\}(\lambda^{-1}+\rho)^{-1}+A(A+1)\},$$

$$R_{3,A}(\lambda) = (1+\lambda\rho)^{-3}\{(\lambda^{-1}+\rho)^{-3}-3\{A+2\}(\lambda^{-1}+\rho)^{-2}+3(A+1)(A+2)(\lambda^{-1}+\rho)^{-1}-A(A+1)(A+2)\},$$

$$R_{4,A}(\lambda) = (1+\lambda\rho)^{-4}\{(\lambda^{-1}+\rho)^{-4}-4\{A+3\}(\lambda^{-1}+\rho)^{-3}+6\{(A+2)(A+3)\}(\lambda^{-1}+\rho)^{-2}-4\{(A+1)(A+2)(A+3)\}(\lambda^{-1}+\rho)^{-1}+(A)(A+1)(A+2)(A+3)\}.$$

For fixed $\lambda=\phi z$, $h_A^{(r)}(\rho,z)$ is of order z^{-r}.

From the relationship between the gamma distribution and the normal distribution, we know that

$\int_0^\infty (x-z)^r x^z e^{-x} dx$ is a polynomial in z of degree

$\lceil \frac{1}{2}(r-1) \rceil'$, where $\lceil A \rceil'$ is the smallest non-negative integer greater than or equal to A. Therefore

$$\sum_{r=0}^{\infty} (1/r!)\, h_A^{(r)}(\rho,z) \int_0^\infty (T-2)^r T^z e^{-T} dt$$

$$= h_A^{(0)}(\rho,z) z! + h_A^{(1)}(\rho,z)\{(z+1)! - zz!\} + \tfrac{1}{2}h_A^{(2)}\{(z+2)!$$

$$-2z(z+1)! + z^2 z!\} + h_A^{(3)}/6\,\{(z+3)! - 3z(z+2)! + 3z^2(z+1)!$$

$$+z^3(z!)\} + h_A^{(4)}/24\,\{(z+4)! - 4z(z+3)! + 6z^2(z+2)!$$

$$-4z^3(z+1)! + z^4 z!\}$$

$$+ \ldots \qquad (2.16)$$

Substituting in the values for $h_A^{(1)}(\rho,z)$, $h_A^{(2)}(\rho,z)$, $h_A^{(3)}(\rho,z)$ and $h_A^{(4)}(\rho,z)$ and simplifying, we have (2.16) equal to:

$$z!\{ h_A^{(0)}(\rho,z) - z^{-1} h_A^{(0)}(\rho,z) R_{1,A}(\lambda) + z^{-1} h_A^{(0)}(\rho,z)/2$$

$$(R_{2,A}(\lambda) + 2R_{1,A}(\lambda)) + z^{-2} h_A^{(0)}(\rho,z)(R_{2,A}(\lambda) + 2R_{1,A}(\lambda))$$

$$-z^{-2}5/6\ h^{0}_{A}(\rho,z)(R_{3,A}(\lambda) + 6R_{2,A}(\lambda) + 6R_{1,A}(\lambda))$$

$$+z^{-2}\ 3/24\ h^{(0)}_{A}(\rho,z)(R_{4,A}(\lambda) + 12R_{3,A}(\lambda) + 36R_{2,A}(\lambda)+24R_{1,A}(\lambda)$$

$$+ 0\ (z^{-3})\ \}$$

$$= z!\ \{h^{(0)}_{A}(\rho,z)\ \{1+1/z\ \ R_{2,A}(\lambda)/2\ + 1/z^{2}\ (R_{4,A}(\lambda)/8$$

$$+2R_{3,A}(\lambda)/3\ + R_{2,A}(\lambda)/2\ .$$

Also $H_{\phi}\ \{(k-1)/2,\ k/2(n-1)\ \}$

$$=1/\beta\{(k-1)/2,\ z+1\}\ \int_{0}^{\phi/1+\phi} x^{\frac{1}{2}(k-1)-1}\ (1-x)^{\frac{1}{2}k(n-1)-1}\ dx\ .$$

If we apply the transformation $Y = x/1-x$, we obtain

$$H_{\phi}\{(k-1)/2,\ k/2(n-1)\} = 1/\beta\{(k-1)/2,z+1\}\ \int_{0}^{\infty} Y^{(k-1)/2\ -1}$$

$$(1+Y)^{-k/2\ -\ \frac{1}{2}}\ (1+Y)^{-z}\ dY\ .$$

Applying the transformation $T/z = Y$, we get

$$H_{\phi}\{(k-1)/2,\ k/2(n-1)\} = 1/\ \beta\{(k-1)/2,\ z+1\}\ \int_{0}^{\phi z} (T/z)^{k/2\ -\ 3/2}$$

$$(1+T/z)^{-k/2\ -\ \frac{1}{2}}\ (1+T/z)^{-z}\ dT/z$$

$$= z^{-k/2 + \frac{1}{2}} / \beta\{(k-1)/2,\ z+1\} \int_0^{\lambda} T^{(k-3)/2} (1 + T/z)^{-(k+1)/2}$$

$$-z \exp \{ \log (1+T/z) \} dT/z$$

$$= z^{-k/2 + \frac{1}{2}} / \beta\{(k-1)/2,\ z+1\} \int_0^{\lambda} T^{(k-3)/2} (1+T/2)^{-\{(k+1)/2\}}$$

$$-z \exp \{ \log (1+T/z) \} dT$$

$$= z^{-k/2 + \frac{1}{2}} / \beta\{(k-1)/2,\ z+1\} \int_0^{\lambda} T^{k-3/2} e^{-T} \{(1+T/z)^{-(k+1)/2}$$

$$\exp \left(T^2/2z - T^3/3z^2 + T^4/4z^3 \ldots \right) \} \, d\,T.$$

If we apply Stirling's formula to $\beta\{(k-1)/2,\ z+1\}$ and expand $\left(T^2/2z - T^3/3z^2 + T^4 / 4z^3 \ldots\right)$ in powers of z^{-1}, we obtain for fixed λ,

$$H_{\phi} \{ (k-1)/2,\ z+1\} = G_{\lambda} \left[(k-1)/2 \quad + 1/z \quad A_1(\lambda) + 1/z^2 \right.$$

$$\left. A_2(\lambda) \quad + O(z^{-3})\right],$$

where $A_1(\lambda) = g_{\lambda}\left[(k-1)/2 \quad \frac{1}{2}\{\frac{1}{2}(k+1)\lambda - \lambda^2\}\right]$,

$$A_2(\lambda) = g_{\lambda}\left[(k-1)/2 \quad 1/24 \ \{1/8\lambda(3k^3-5k^2-11k-3) - \tfrac{1}{4}\lambda^2(9k^2+8k-9)\right.$$

$$\left. + \tfrac{1}{2}\lambda^3(9k+13) - 3\lambda^4 \}\right],$$

and $g_{\theta}(k-1)/2 = G'(k-1)/2$.

Therefore

$$f(\rho|y) = 1/\{\Gamma((k-1)/2)\Gamma(z+1)\ G_{\lambda}(k-1)/2+1/zA_1(\lambda)+1/z^2A_2(\lambda)\}$$

$$\sum_{d_1 d_2 d_3}^{k} k!/d_1!d_2!d_3!\ h_{0,1}^{d_1}\ h_{0,2}^{d_2}\ \rho^{2d_1+3d_2}\ \exp\{-(\lambda^{-1}+\rho)^{-1}\}$$

$$\cdot\ (\lambda^{-1}+\rho)^{-\{(k+1)/2\ +\ 2d_1+3d_2\}}\left[z!\ \ 1+1/2z\ R_{2,(k+1)/2+2d_1+3d_2}(\lambda)\right]$$

$$+\ 1/z^2\ \ R_{4,(k+1)/2\ +\ 2d_1+3d_2}(\lambda)/8\ +\ 2/3\ \ R_{3,(k+1)/2\ +\ 2d_1+3d_2}(\lambda)$$

$$+\ \tfrac{1}{3}R_{2,(k+1)/2\ +\ 2d_1+\ 3d_2}(\lambda)$$

$$+\sum_{r=0}^{6k}\ \sum_{p=0}^{[r/2]}\ \sum_{t=1}^{k}\ \sum_{A_1=1}^{6}\cdots\sum_{A_t=1}^{6}\ \sum_{L_1=1}^{2}\cdots\sum_{L_t=1}^{2}\ \sum_{b_1 b_2 b_3}^{k-t}\ Hr,p\ \bar{Y}^{r-2p}$$

$$(S_2/2nk)^p\ \ h_{0,1}^{b_1}\ h_{0,2}^{b_2}\ h_{A_1L_1}\cdots h_{A_tL_t}(-1)^{\sum_{i=1}^{t}A_i-r}\ \ g_{A_1\ldots A_t}(\textstyle\sum_{i=1}^{t}A_i-r)$$

$$\rho^{-\sum_{i=1}^{t}A_i/2\ +\ 2b_1+3b_2+\Sigma\Sigma f(A,L)}\ \exp\{\ -(\lambda^{-1}+\rho)^{-1}\}$$

$$\cdot\ (\lambda^{-1}+\rho)^{-\{(k+1)/2\ -\rho+2b_1+3b_2+\Sigma\Sigma f(A,L)\}}$$

$$z\ !\ \Big\{\ \{1+1/2z\ R_{2,(k+1)/2\ -\rho+\ 2b_1+3b_2+\Sigma\Sigma f(A,L)}(\lambda)$$

$$+\ 1/z^2\ \{\ 1/8\ R_{4,\ (k+1)/2\ -\ p\ +2b_1+3b_2+\Sigma\Sigma f(A,L)}(\lambda)$$

$$+\ 2/3\ R_{3,\ (k+1)/2\ -P\ +2b_1+3b_2+\Sigma\Sigma f(A,L)}(\lambda)$$

$$+\ \tfrac{1}{2}\ R_{2,\ (k+1)/2\ -P\ +2b_1+3b_2+\Sigma\Sigma f(A,L)}(\lambda)\ \Big\}\ . \qquad (2.17)$$

Now $G_\lambda((k-1)/2+1/z\ .\ A_1(\lambda)\ +\ 1/z^2\ .\ A_2(\lambda))$

$$=\big(z^2\ G_\lambda\ (k-1)/2\ \ +\ 2A_1(\lambda)\ +\ A_2(\lambda)\big)/\ z^2\ .$$

Therefore

$$1/\ G_\lambda(k-1)/2\ +\ 1/z\ A_1(\lambda)\ +\ 1/z^2\ A_2(\lambda)$$

$$=\ z^2\ \{\ 1/\big(z^2 G_\lambda(k-1)/2\ \big)\ -\ A_1\ (\lambda)/\big(\ z^3G_\lambda^{\ 2}(k-1)/2\ \big)$$

$$-A_2(\lambda)\ \big(z^4G_\lambda^{\ 2}(k-1)/2\big)\ \ +\ A_1^{\ 2}/\big(z^4G_\lambda^{\ 3}(k-1)/2\big)\ +\ 0(z^{-5}).$$

Substituting the above equation into $f(\rho|y)$ and truncating terms of order z^{-3} we obtain:

$$f(\rho|y) = (1/\Gamma(k-1)/2 \sum_{d_1 d_2 d_3}^{k} k/d_1!d_2!d_3! \; h_{0,1}^{d_1} h_{0,2}^{d_2}$$

$$\rho^{2d_1+3d_2} \exp\{-(\lambda^{-1}+\rho)^{-1}\} (\lambda^{-1}+\rho)^{-(k+1)/2+2d_1+3d_2}$$

$$\{1/G_\lambda((k-1)/2) + 1/z\,\{-A_1(\lambda)/G_\lambda^2((k-1)/2) +$$

$$R_{2,\,(k+1)/2+2d_1+3d_2}(\lambda)/2G_\lambda(k-1)/2\}$$

$$+ 1/z^2\, R_{4,(k+1)/2+2d_1+3d_2}(\lambda) / 8+2/3R_{3,(k+1)/2d_1+3d_2+2}(\lambda)$$

$$+\tfrac{1}{2}\, R_{2,\,(k+1)/2+2d_1+3d_2}(\lambda) - \{A_2(\lambda)/G_\lambda^2(k-1)/2\} +$$

$$\{A_1^2(\lambda)/G_\lambda^3(k-1)/2 - R_{2,\,(k+1)/2+2d_1+3d_2}(\lambda)$$

$$\{A_1(\lambda)/2G_\lambda^2(k-1)/2\} + 1/\Gamma(k-1)/2 \sum_{r=0}^{6k} \sum_{p=0}^{r/2}$$

$$\sum_{t=1}^{k} \sum_{A_1=1}^{6} \sum_{A_2=1}^{6} \cdots \sum_{A_k=1}^{6} \sum_{L_1=1}^{2} \sum_{L_2=1}^{2} \cdots \sum_{L_k=1}^{2}$$

$$\sum_{b_1 b_2 b_3}^{k-t} H_{r,p}\, \overline{y}^{\,r-2p} (S_2/2nk)^p\, h_{0,1}^{b_1} h_{0,2}^{b_2}\, h_{A_1L_1} h_{A_2L_2} \cdots$$

$$h_{A_tL_t} (-1)^{\sum_{i=1}^{t} A_i - r}\; g_{A_1 A_2 \ldots A_t} (\textstyle\sum_{i=1}^{t} A_i - r)$$

$$\rho^{-\Sigma A_i/2 + 2b_1 + 3b_2 + \Sigma\Sigma f(A,L)} \exp\{-(\lambda^{-1} + \rho)^{-1}\}$$

$$(\lambda^{-1} + \rho)^{-\{(k+1)/2 - P + 2b_1 + 3b_2 + \Sigma\Sigma f(A,L)\}}$$

$$\{1/G_\lambda(k-1)/2 + 1/z\{ -A_1(\lambda)/G_\lambda{}^2(k-1)/2 +$$

$$R_{2,\ (k+1)/2 - P + 2b_1 + 3b_2 + \Sigma\Sigma f(A,L)} / 2G_\lambda(k-1)/2 \quad (\lambda)$$

$$+ 1/z^2 \{ 1/8\, R_{4,(k+1)/2 - P + 2b_1 + 3b_2 + \Sigma\Sigma f(A,L)}(\lambda)$$

$$+ 2/3\, R_{3,\ (k+1)/2 - P + 2b_1 + 3b_2 + \Sigma\Sigma f(A,L)}(\lambda)$$

$$+ \tfrac{1}{2}\, R_{2,\ (k+1)/2 - P + 2b_1 + 3b_2 + \Sigma\Sigma f(A,L)}(\lambda)$$

$$-A_2(\lambda) / G_\lambda{}^2(k-1)/2 + A_1{}^2(\lambda)/G_\lambda{}^3(k-1)/2$$

$$-R_{2,\ (k+1)/2\ -\ P\ +\ 2b_1+3b_2+\Sigma\Sigma f(A,L)}^{(\lambda)}$$

$$A_1(\lambda)/2G_\lambda^{2}(k-1)/2\} . \qquad (2.18)$$

As in the previous posterior distributions, $f(\rho|y)$ can be approximated by the normal theory distribution when θ^2_a is negative. This is seen by considering $h_A(\rho,T)$ when $S_1 >> S_2$.

When $S_1 >> S_2$ ϕ is small and,

$\{(\phi T)^{-1}+ \rho\}$ is large and hence, $\{(\phi T)^{-1}+ \rho\}^{-A/2} \to 0$ as A grows large.

By examining (2.14),we see that the correction terms are small and can usually be ignored. This result holds true for the asymptotic expansion of $f(\rho|y)$.

2.10 Summary

The posterior distributions developed in this chapter have a number of things in common.

1. All three posterior distributions are in the form $F + \Sigma_i c_i F_i$, where F is the function developed in the normal theory counterpart and F_i are functions of the same kind. The summation and the c_i (constant terms) include the measures of non-normality λ_1 and λ_2.

2. The distributions react in basically the same way when θ_a^2 is negative - they all tend to the normal-theory formulae.

3. The distributions are extremely complex and the computations require a computer. Thus, in cases where the underlying distributions are only slightly non-normal, it is doubtful whether our approach is worth the increased work and additional cost of a computer.

Because of the complexity of the distributions, it was impossible to examine the effects of λ_1 and λ_2 analytically. However, a number of examples with different values of λ_1 and λ_2 were generated and compared with the normal-theory results. It turned out that all the examples showed differences. However, the largest of these occurred when λ_1 was not equal to zero. This suggests, of course, that λ_1 plays a greater rôle than λ_2 in determining the posterior probabilities. The result appears at first to contradict Box and Tiao (1964) who stated that kurtosis is more important than skewness. However, since their distribution did not take into consideration the possibility of asymmetric distributions, it is difficult to truly compare results. In a way our outcomes coincide rather than contradict Box and Tiao.

Chapter III

Bayesian Methods in the Analysis of Variance - Non-Normal Errors

3.1 Joint Likelihood Function of $(\mu, \sigma^2_a, \sigma^2)$

In this chapter we will have the same objectives in mind as we did in the last chapter, that is, to develop and analyze the posterior distributions of σ^2_a/σ^2, σ^2 and σ^2_a. However, this time, we will assume that the effects, a_i, are normally distributed with mean and variance 0 and σ^2_a, respectively and that the distribution of the errors, e_{ij}, will be approximated by an Edgeworth series. Similar assumptions about this series will be made, that is, mean and variance are equal to 0 and σ^2 and $E(e^3_{ij})/\sigma^3 = \gamma_1$ and $E(e^4_{ij})/\sigma^4 - 3 = \gamma_2$, where γ_1 and γ_2 are known constants. Therefore the likelihood function is

$$L(\mu,\sigma^2_a,\sigma^2,\mu|y) = \int_{a_1} \int_{a_2} \dots \int_{a_k} f(y_{ij}|\mu,\sigma^2,\sigma^2_a,a_i)$$

$$f(a_i|\mu,\sigma^2_a,\sigma^2) \; da_1 \, da_2 \dots da_k .$$

Let $H(e_{ij}) = \{ 1+\gamma_1/3!\ (e_{ij}^3/\sigma^3 - 3e_{ij}/\sigma) + \gamma_2/4!$

$(e_{ij}^4/\sigma^4 - 6e_{ij}^2/\sigma^2 + 3) + 10\gamma_1^2/6!$

$(e_{ij}^6/\sigma^6 - 15e_{ij}^4/\sigma^4 + 45e_{ij}^2/\sigma^2 - 15)\}$.

Then $L(\mu,\sigma^2_a,\sigma^2|y) \propto (\sigma^2+n\sigma^2_a)^{-k/2} (\sigma^2)^{-k/2(n-1)}$

$$e^{-S_1/2\sigma^2 - S_2/2(\sigma^2+n\sigma^2_a) - nk(\mu-y)^2/2(\sigma^2+n\sigma^2_a)} \prod_{i=1}^{k} E_i, \quad (3.1)$$

where $E_i = \int_{a_i} \exp\{-(a_i-(y_i-\mu)^2 n\sigma^2_a/\sigma^2+n\sigma^2_a)/(2\sigma^2\sigma^2_a)/$

$(\sigma^2+n\sigma^2_a) \prod_{j=1}^{n} H(y_{ij}-\mu-a_i)\, da_i$.

Now we can write

$$E_i = \int_{a_i} \exp\{-(a_i-(y_i-\mu)^2 n\sigma^2_a/\sigma^2+n\sigma^2_a)/2\sigma^2\sigma^2_a/\sigma^2+n\sigma^2_a\}$$

$$\prod_{j=1}^{n} \{t_0 + \sum_{m=1}^{6} t_m f_j^m\},$$

where $f_j = \mu + a_i - y_{ij}$,

and $t_0 = 1 + \gamma_2/8 - 15\gamma_1^2/72$,

$t_1 = (1/\sigma)(\gamma_1/2)$,

$t_2 = 1/\sigma^2\{-\gamma_2/4 + 45\gamma_1{}^2/72\}$,

$t_3 = -1/\sigma^3(\gamma_1/6)$,

$t_4 = 1/\sigma^4\{\gamma_2/24 - 15\gamma_1{}^2/72\}$,

$t_5 = 0$, and

$t_6 = 1/\sigma^6(\gamma_1{}^2/72)$.

By a method similar to one used in the previous chapter, we have

$$\prod_{j=1}^{n}\{t_0 + \sum_{m=1}^{6} t_m f_j^m\} = t_0^n + t_0^{n-1}\sum_{A_1=1}^{6} t_{A_1}\sum_{S_1=0}^{A_1}$$

$$(-1)^{S_1} M_i^{A_1-S}\sum_{l_1=1}^{n}\binom{A_1}{S_1} Y_{i,l_1}^{S_1}$$

$$+ t_0^{n-2}\sum_{A_1=1}^{6}\sum_{A_2=1}^{6} t_{A_1} t_{A_2}\sum_{S_1=0}^{A}\sum_{S_2=0}^{A}(-1)^{\sum_{i=1}^{2} S_i} M_i^{\sum_{i=1}^{2} A_i - \sum_{i=1}^{2} S_i}$$

$$\sum_{l_1<l_2}^{n}\binom{A_1}{S_1}\binom{A_2}{S_2} y_{il_1}^{S_1}\, y_{il_2}^{S_2}$$

$$+ \ldots$$

$$+ t_0^0 \sum_{A_1=1}^{6} \sum_{A_2=1}^{6} \ldots \sum_{A_n=1}^{6} t_{A_1} t_{A_2} \ldots t_{A_n} \sum_{S_1=0}^{A_1} \sum_{S_2=0}^{A_2} \ldots \sum_{S_n=0}^{A_n}$$

$$(-1)^{\Sigma S_i} M_i^{\sum_{i=1}^{n} A_i - \sum_{i=1}^{n} S_i} \sum_{l_1 < l_2 < \ldots < l_n}^{n} \binom{A_1}{S_1} y_{il_1}^{S_1}$$

$$\binom{A_2}{S_2} y_{il_2}^{S_2} \ldots \binom{A_n}{S_n} y_{il_n}^{S_n}, \qquad (3.2)$$

where $M_i = \mu + a_i$.

Let $g_{iA_1A_2\ldots A_c}(S) = \sum_{S_1=0}^{A_1} \sum_{S_2=0}^{A_2} \ldots \sum_{S_c=0}^{A_c} \sum_{l_1<l_2<\ldots<l_c}^{n}$

$$\left\{ \binom{A_1}{S_1} y_{il_1}^{S_1} \binom{A_2}{S_2} y_{il_2}^{S_2} \ldots \binom{A_c}{S_c} y_{il_c}^{S_c} \right\},$$

where $S = \sum_{i=1}^{c} S_i$.

We also have

$$E_i = t_0^n + \sum_{r=0}^{6n} E(M_i)^r \sum_{c=1}^{n} t_0^{n-[illegible]} [illegible] \sum_{A_2=1}^{6} \ldots \sum_{A_c=1}^{6}$$

$$t_{A_1} t_{A_2} \ldots t_{A_c} (-1)^{\sum_{i=1}^{c} A_i - r} \quad g_{iA_1A_2\ldots A_c} \left(\sum_{i=1}^{c} A_i - r \right),$$

with $g_{iA_1A_2\ldots A_c}(\sum_{i=1}^{c} A_i - r) = 0$, if $\sum_{i=1}^{c} A_i - r < 0$,

and where $E(M_i^r) = \int_{a_i} M_i^r \exp \{(a_i-(y_i-\mu)^2 n\sigma^2{}_a/\sigma^2 + n\sigma^2{}_a)/$

$$(2\sigma^2\sigma^2{}_a/\sigma^2+n\sigma^2{}_a)\} \quad d_{a_i}$$

Let $Q_i = (y_i - \mu)^2 n\sigma^2{}_a / \sigma^2 + n\sigma^2{}_a$,

and $v = \sigma^2\sigma^2{}_a/\sigma^2+n\sigma^2{}_a$,

and consider $E(M_i^r)$ for $r = 1, 2, \ldots$

$$E(M_i) = \mu + E(a_i),$$

$$E(M_i^2) = \mu^2 + 2\mu E(a_i) + E(a_i^2),$$

$$\ldots,$$

where $E(a_i^q) - \int_{a_i} a_i^q \exp - \{ (a_i-Q_i)^2/2v \} da_i$.

Therefore $E(M_i) = \mu + Q_i$,

$$E(M_i^2) = \mu^2 + 2\mu Q_i^2 + v ,$$

...

Simplifying we have

$$E(M_i) = \mu + Q_i ,$$

$$E(M_i^2) = (\mu+Q_i)^2 + v ,$$

$$E(M_i^3) = (\mu+Q_i)^3 + 3v(\mu+Q_i) ,$$

...

In general we have

$$E(M_i^r) = H_{r,0}(\mu+Q_i)^r v^0 + H_{r,1}(\mu+Q_i)^{r-2} v^1$$

$$+ H_{r,2}(\mu+Q_i)^{r-4} v^2 + \ldots ,$$

where $H_{r,p}$ is defined in the previous chapter.

Therefore we have

$$E_i = t_0^n + \sum_{r=0}^{6n} \sum_{p=0}^{r/2} Hr,p\ (\mu+Q_i)^{r-2p}\ v^p \sum_{i=1}^{n} t_0^{n-c}$$

$$\sum_{A_1=1}^{6} \sum_{A_2=1}^{6} \cdots \sum_{A_c=1}^{6} t_{A_1} t_{A_2} \ldots t_{A_c} (-1)^{\sum_{i=1}^{6} A_i - r}$$

$$g_{iA_1A_2\ldots A_c} \left(\sum_{c=1}^{c} A_i - r\right) \tag{3.3}$$

$$= t_0^n + \sum_{r=0}^{6n} \sum_{p=0}^{r/2} Hr,p\ (\mu+Q_i)^{r-2p}\ B_{i,r,p},$$

where

$$B_{i,r,p,} = v^p \sum_{c=1}^{n} t_0^{n-c} \sum_{A_1=1}^{6} \sum_{A_2=1}^{6} \cdots \sum_{A_c=1}^{6} t_{A_1} t_{A_2} \ldots t_{A_c}$$

$$(-1)^{\sum_{i=1}^{c} A_i - r}\ g_{iA_1A_2\ldots A_c} \left(\sum_{i=1}^{c} A_i - r\right).$$

Collecting coefficients of $(\mu + Q_i)$ and again simplifying ,

$$E_i = \sum_{x=0}^{6n} c_{ix}\ (\mu+Q_i)^x ,$$

where

$$c_{i0} = t_0^n + H_{0,0} B_{i,0,0} + H_{2,1} B_{i,2,1} + H_{4,2} B_{i,4,2} + \dots$$
$$+ H_{6n,3n} B_{i,6n,3n} ,$$

$$c_{i_1} = H_{1,0} B_{i,1,0} + H_{3,1} B_{i,3,1} + \dots + H_{6n-1,3n-1} B_{i,6n-1,3n-1} ,$$

$$\vdots$$

$$c_{i_r} = H_{r,0} B_{i,r,0} + H_{r+2,1} B_{i,r+2,1} + \dots H_{r+6n-2[(r+1)/2],}$$
$$B_{i,r+6n-2[(r+1)/2],\ 3n-[(r+1)/2]} ,$$

$$\vdots$$

$$c_{i,6n} = H_{6n,3n} B_{i,6n,3n} . \qquad (3.4)$$

Now $\mu + Q_i = \mu + \{ (Y_i - \mu)^2 n\sigma^2{}_a/\sigma^2 + n\sigma^2{}_a\}$

$$= b_{2,i}\,\mu^2 + b_{1,i}\,\mu + b_{0,i} ,$$

where $b_{2,i} = n\sigma^2_a/(\sigma^2 + n\sigma^2_a)$,

$$b_{1,i} = (\sigma^2 + n\sigma_a^2 - 2Y_i n\sigma_a^2)/\sigma^2 + n\sigma_a^2 ,$$

$$b_{0,i} = Y_i^{\,2} n\sigma_a^2 / \sigma^2 + n\sigma_a^2 .$$

Therefore

$$(\mu + Q_i)^x = \sum_{e_1 e_2 e_3} b_{2,i}^{e_2} \, b_{1,i}^{e_1} \, b_{0,i}^{e_0} \, \mu^{2e_2+e_1}$$

$$x!/e_2!e_1!e_0!,$$

and $E_i = \sum_{Y=0}^{12n} D_{i_Y} \mu^Y ,$

where $D_{i_0} = c_{i_0} + c_{i_1} b_{0,i} + c_{i_2} b_{0,i}^{\;2} + \ldots c_{i,6n} b_{0,i}^{6n}$

$$D_{i_1} = c_{i_1} b_{1,i} + c_{1_2}(2b_{1,i} b_{0,i}) + c_{i_3}(3b_{0,i}^{\;2} b_{1,i}) + \ldots$$

In general

$$D_{i_Y} = \sum_{\eta = [(Y+1)/2]}^{6n} c_{i,\eta} \, \theta_{i,\eta,\,Y} ,$$

where $\theta_{i_Y P_i, z_i} = I(z_i) \sum_{e_1=0}^{2} \sum_{e_2=0}^{2} \ldots \sum_{e_{P_i}=0}^{2}$

$$b_{e_1,i}\, b_{e_2,i} \cdots b_{e_{p_i},i} \; ,$$

where $I(z_i) = 1$ if $e_1 + e_2 + \ldots + e_{p_i} = z_i$

$= 0$ otherwise .

Finally putting all the components together, we have

$$E_1 E_2 \ldots E_k = \sum_{z_1=0}^{12n} \sum_{z_2=0}^{12n} \cdots \sum_{z_k=0}^{12} D_{1,z_1} D_{2,z_2} \cdots D_{k,z_k}$$

$$D_{1,z_1} D_{2,z_2} \ldots D_{k,z_k} = \sum_{p_1=[(z_1+1)/2]}^{6n} \sum_{p_2=[(z_2+1)/2]}^{6n} \cdots \sum_{p_k=[z_k+1)/2]}^{6n}$$

$$c_{1,p_1} c_{2,p_2} \ldots c_{k,p_k} \quad \theta_{1,p_1,z_1} \theta_{1,p_1,z_1} \ldots \theta_{k,p_k,z_k}$$

$$\mu^{\sum_{i=1}^{k} z_i} \quad , \qquad (3.5)$$

and $$c_{i,p_i} = \sum_{w=0}^{3n-[(p_i+1)/2]} H_{p_i+2w,w} \; B_{i,p_i+2w,w} \; ,$$

and $$B_{i,p_i+2w,w} = \{(\sigma^2\sigma^2_a)/\sigma^2+n\sigma^2_a)\}^w \sum_{i=1}^{n} \sum_{A_1=1}^{6} \sum_{A_2=1}^{6}$$

$$\ldots \sum_{A_i=1}^{6} t_0^{n-c} \, t_{A_1} t_{A_2} \ldots t_{A_c} (-1)^{\sum_{i=1}^{c}(A_i) - P_i - 2w}$$

$$g_{i A_1 A_2 \ldots A_c} \left(\sum_{i=1}^{c}(A_i) - P_i - 2w \right).$$

Hence

$$L(\mu, \sigma^2_a, \sigma^2 | y) \propto (\sigma^2 + n\sigma^2_a)^{-k/2} (\sigma^2)^{-k/2(n-1)}$$

$$e^{-S_1/2\sigma^2 - S_2/2(\sigma^2+n\sigma^2_a) - nk(\mu-\overline{y})^2/2(\sigma^2+n\sigma^2_a)} \, E_1 E_2 \ldots E_k. \tag{3.6}$$

3.2 Posterior Distribution of (σ^2_a, σ^2)

Again we have the joint posterior distribution $f(\mu, \sigma^2_a, \sigma^2 | y) = L(\mu, \sigma^2_a, \sigma^2 | y) f'(\mu, \sigma^2_a, \sigma^2)$, where $f'(\mu, \sigma^2_a, \sigma^2)$ is the joint prior probability distribution. Therefore, the joint posterior distribution of $(\mu, \sigma^2_a, \sigma^2)$ is

$$f(\mu, \sigma^2_a, \sigma^2 | y) \propto (\sigma^2 + n\sigma^2_a)^{-k/2-1} (\sigma^2)^{-k/2(n-1)-1} \, E_1 E_2 \ldots E_k$$

$$e^{-S_1/2\sigma^2 - S_2/2(\sigma^2+n\sigma^2_a) - nk(\mu-\overline{y})^2/2(\sigma^2+n\sigma^2_a)}. \tag{3.7}$$

When we integrate out μ we arrive at the joint posterior distribution of σ^2_a and σ^2:

$$f(\sigma^2_a, \sigma^2 | y) = c(\sigma^2+n\sigma^2_a)^{-\frac{1}{2}(k+1)} (\sigma^2)^{-(k/2(n-1)+1)} \quad \xi$$

$$e^{-S_1/2\sigma^2 - S_2/2(\sigma^2+n\sigma^2_a)}, \qquad (3.8)$$

where

$$\xi = \sum_{z_1=0}^{12n} \sum_{z_2=0}^{12n} \cdots \sum_{z_k=0}^{12n} \sum_{p_1=[(z_1+1)/2]}^{6n} \sum_{p_2=[(z_2+1)/2]}^{6n} \cdots \sum_{p_k=[(z_k+1)/2]}^{6n}$$

$$\sum_{w_1=0}^{3n-[(p_1+1)/2]} \sum_{w_2=0}^{3n-[(p_2+1)/2]} \cdots \sum_{w_k=0}^{3n-[(p_k+1)/2]} \sum_{B=0}^{[\Sigma z_i/2]} H_{\Sigma z_i, B}$$

$$\bar{Y}^{\Sigma z_i - 2B} \{(\sigma^2+n\sigma^2_a)nk\}^{B} \quad H_{p_1+2w_1, w_1} \; H_{p_2+2w_2, w_2} \cdots H_{p_k+2w_k, w_k}$$

$$\{(\sigma^2+\sigma^2_a)/(\sigma^2+n\sigma^2_a)\}^{\Sigma w_i} \sum_{c_1=1}^{n} \sum_{A_{1,1}=1}^{6} \sum_{A_{1,2}=1}^{6} \cdots \sum_{A_{1,c_1}=1}^{6} \sum_{c_2=1}^{n}$$

$$\sum^{6} A_{2,1}=1 \quad \sum^{6} A_{2,2}=1 \quad \ldots \quad \sum^{6} A_{2,c_2}=1 \quad \ldots \quad \sum^{n} c_k=1 \quad \sum^{6} A_{k,1}=1 \quad \sum^{6} A_{k,2}=1$$

$$\ldots \sum^{6} A_{k,c_k}=1 \qquad t_0^{kn-\Sigma c_i} \; t_{A_{1,1}} \, t_{A_{1,2}} \ldots t_{A_{1,c_1}}$$

$$t_{A_{2,1}} \, t_{A_{2,2}} \ldots t_{A_{2,c_2}} \ldots t_{A_{k,1}} \, t_{A_{k,2}} \ldots t_{A_{k,c_k}}$$

$$g_{1,A_{1,1}\ldots A_{1,c_1}} \left(\sum_{i=1}^{c_1} A_{1i} - P_1 - 2w_1 \right) \ldots g_{k,A_{k,1}\ldots A_{k,c_k}}$$

$$I(z_1)\, I(z_2) \ldots I(z_k) \quad \sum^{2} e_{1,1}=0 \quad \ldots \sum^{2} e_{1p_1}=0 \quad \ldots \sum^{2} e_{k,1}=0 \quad \ldots \sum^{2} e_{k,p_k}=0$$

$$b_{e_{11,1}} \ldots b_{e_{1,p_1},1} \ldots b_{e_{k_1},k} \ldots b_{e_{kp_k},k} , \left(\sum_{i=1}^{c_k} A_{k,i} - P_k - 2w_k \right) \tag{3.9}$$

and c is the normalizing constant.

3.3 Posterior Distribution of σ^2_a/σ^2

Using the same approach as we did in chapter 2, we make the following transformation from (3.8).

$$w = 1 + n\sigma^2_a/\sigma^2 , \qquad v = \sigma^2 .$$

Then the joint distribution

$$f(w,v|y) = cv^{-(k/2(n-1)+1)} (wv)^{-\frac{1}{2}(k+1)} v/n \quad \xi_2$$

$$e^{-S_1/2v - S_2/2wv}, \tag{3.10}$$

where $\xi_2 = \sum_{z_1=0}^{12n} \sum_{z_2=0}^{12n} \cdots \sum_{z_k=0}^{12n} \sum_{P_1=[(z_1+1)/2]}^{6n} \sum_{P_2=[(z_2+1)/2]}^{6n}$

$$\cdots \sum_{P_k=[(z_k+1)/2]}^{6n} \sum_{w_1}^{3n-[(P_1+1)/2]} \sum_{w_2=0}^{3n-[(P_2+1)/2]} \cdots$$

$$\sum_{w_k=0}^{3n-[(P_k+1)/2]} \sum_{B=0}^{[\Sigma z_i/2]} H_{\Sigma z_i, B} \; \bar{Y}^{\Sigma z_i - 2B} (wv/nk)^{B}$$

$$H_{p_{1+2w_1, w_1}} \quad H_{p_{2+2w_2, w_2}} \quad \dots \quad H_{p_{k+2w_k, w_k}}$$

$$\{(w-1)/nw^2\}^{\Sigma w_i} \sum_{c_1=1}^{n} \sum_{A_{1,1}=1}^{6} \sum_{A_{1,2}=1}^{6} \dots \sum_{A_{1,c_1}=1}^{6} \dots$$

$$\sum_{c_k=1}^{n} \sum_{A_{k,1}=1}^{6} \sum_{A_{k,2}=1}^{6} \dots \sum_{A_{k,c_k}=1}^{6} t_0^{kn-c_i} \; t'_{A_{1,1}} \; t'_{A_{1,2}} \dots$$

$$t'_{A_{1,c_1}} \dots t'_{A_{k_1}} \; t'_{A_{k_2}} \dots t'_{A_{k,c_k}} \; v^{-\sum_{i=1}^{k} \sum_{j=1}^{c_i} A_{ij}/2}$$

$$g_{1A_{11} \dots A_{1c_1}} \left(\sum_{i=1}^{c} A_{ij} - P_1 - 2w\right) \dots \; g_{k,A_{k,1} \dots A_{k,c_k}}$$

$$\left(\sum_{i=1}^{c_k} A_{ki} - P_k - 2w_k\right) \; I(z_1) \; I(z_2) \dots I(z_k)$$

$$\sum_{e_{11}=0}^{2} \cdots \sum_{e_{1p_1}=0}^{2} \cdots \sum_{e_{k_1}=0}^{2} \cdots \sum_{e_{kp_k}=0}^{2} b'^{\,e_{11}}_{1,1} \cdots b'^{\,e_{1p_1}}_{1} \cdots$$

$$b'^{\,e_{k,1}}_{k} \cdots b'^{\,e_{kp_k}}_{k} , \qquad (3.11)$$

where

$$b'_{2_i} = (w-1)/w ,$$

$$b'_{1_j} = 1-2\bar{Y}_i \; (w-1)/w ,$$

$$b'_{0_i} = \bar{Y}_i \; (w-1)/w ,$$

and $t'_1 = \gamma_1/2$,

$$t'_2 = -\gamma_2/4 + 45\gamma_1^{\,2}/72 ,$$

$$t'_3 = -\gamma_1/6 ,$$

$$t'_4 = \gamma_2/24 - 15\gamma_1^{\,2}/72 ,$$

$$t'_5 = 0 ,$$

$$t'_6 = \gamma_1^{\,2}/72 .$$

Integrating out v, we obtain

$$E(w|y) = cn^{-1}\, w^{-\frac{1}{2}(k+1)} \Big\{ \sum_{z_1=0}^{12n} \sum_{z_2=0}^{12n} \cdots \sum_{z_k=0}^{12n} \; \sum_{P_1=[(z_1+1)/2]}^{6n} \cdots$$

$$\sum_{P_2=[(z_2+1)/2]}^{6n} \cdots \sum_{P_k=[(z_k+1)/2]}^{6n} \; \sum_{w_1=0}^{3n-[(P_1+1)/2]} \; \sum_{w_2=0}^{3n-[(P_2+1)/2]} \cdots$$

$$\sum_{w_k=0}^{3n-[(P_k+1)/2]} \; \sum_{B=0}^{[\Sigma z_i/2]} H_{\Sigma z_i, B}\; \bar{Y}^{\Sigma z_i - 2B}$$

$$H_{p_1+2w_1,\,w_1}\; H_{p_2+2w_2,\,w_2} \cdots H_{p_k+2w_k,\,w_k}$$

$$\sum_{c_1=1}^{n} \sum_{A_{11}=1}^{6} \cdots \sum_{A_{1c_1}=1}^{6} \cdots \sum_{c_k=1}^{n} \sum_{A_{k1}=1}^{6} \cdots$$

$$\sum_{A_{kc_k}=1}^{6} t_0^{\,kn-\Sigma c_i}\; t'_{A_{11}} \cdots t'_{A_{1c_1}} \cdots t'_{A_{k1}} \cdots t'_{A_{kc_k}}$$

$$q_{1A_{11}\ldots A_{1c_1}} \left(\sum_{i=1}^{c_1} A_{1i} - P_1 - 2w_1\right) \cdots$$

$$q_{kA_{k1}\ldots A_{kc_k}} \left(\sum_{i=1}^{c_k} A_{ki} - P_k - 2w_k\right)$$

$$I(z_1)\, I(z_2) \ldots I(z_k) \sum_{e_{11}=0}^{2} \cdots \sum_{e_{1p_1}=0}^{2} \cdots \sum_{e_{k1}=0}^{2} \cdots \sum_{e_{kp_k}=0}^{2} \cdots$$

$$b_{e_{11},1} \ldots b_{e_{1p_1},1} \ldots b_{e_{k1},k} \ldots b_{e_{kp_k},k} \; (w/nk)^{B} \, (w-1/nw^2)^{\Sigma w_i}$$

$$\Gamma\left\{\tfrac{1}{2}(kn+1) + \sum_{i=1}^{k} \sum_{j=1}^{c_i} A_{ij}/2 - B - 2\right\}$$

$$\{S_1/2(1+\phi/w)^{-\frac{1}{2}(kn+1) - \sum_{i=1}^{k} \sum_{j=1}^{c_i} A_{ij}/2 + B + 2}\},\; 1 < w < \infty$$

(3.12)

We note that the form of $f(w|y)$ is basically the same as the one developed in chapter two, that is, a summation of truncated F distribution functions. Also, because of the presence of similar terms, we see that the tail area regions are approximately equal to the normal theory approach. The main difference is the increased complexity of the function reflected in the large number of summations.

3.4 Posterior Distribution of S_1/σ^2

From (3.9) we make the following transformations:

$$\delta = S_1/\sigma^2 , \qquad \rho = 2n\sigma_a^2/S_2 ,$$

and then $x = \tfrac{1}{2}(\phi\delta/1+\tfrac{1}{2}\phi\delta\rho)$,

and then obtain the joint posterior probability distribution of δ and ρ.

$$f(\delta,\rho) = c(S_1/\delta)^{-k/2(n-1)+1} (2x/S_2)^{\frac{1}{2}(k+1)} e^{-\delta/2} e^{-x}$$

$$(S_1 S_2/2n\delta^2)\,(1/x^2)\,\xi_2 , \qquad (3.13)$$

where $\xi_2 = \sum_{z_1=0}^{12n} \sum_{z_2=0}^{12n} \cdots \sum_{z_k=0}^{12n} \sum_{p_1=[(z_1+1)/2]}^{6n} \sum_{p_2=[(z_2+1)/2]}^{6n}$

$$\cdots \sum_{P_k=[(z_k+1)/2]}^{6n} \; \sum_{w_1=0}^{3n-[(P_1+1)/2]} \; \sum_{w_2=0}^{3n-[(P_2+1)/2]} \cdots$$

$$\sum_{w_k=0}^{3n-[(P_k+1)/2]} \Big) \; \sum_{B=0}^{[\Sigma z_i/2]} H_{\Sigma z_i, B} \; \bar{Y}^{\Sigma z_i - 2B}$$

$$H_{p_1+2w_1, w_1} \cdots H_{p_k+2w_k, w_k} \; \sum_{c_1=1}^{n} \; \sum_{A_{11}=1}^{6} \cdots \sum_{A_{1c_1}=1}^{6} \cdots$$

$$\sum_{c_k=1}^{n} \; \sum_{A_{k1}=1}^{6} \cdots \sum_{A_{kc_k}=1}^{6} \; t_0^{kn-\Sigma c_i} \; t'_{A_{11}} \cdots t'_{A_{1c_1}} \cdots$$

$$t'_{A_{k1}} \cdots t'_{A_{kc_k}} \; g_{1A_{11} \cdots A_{1c_1}} \Big(\sum_{i=1}^{c_1} A_{ij} - P_1 - 2w_1 \Big) \cdots$$

$$g_{kA_{k1} \cdots A_{kc_k}} \Big(\sum_{i=1}^{c_k} A_{ki} - P_k - 2w_k \Big)$$

$$I(z_1)\ldots I(z_k) \sum_{e_{11}=0}^{2} \ldots \sum_{e_{1p_1}=0}^{2} \ldots \sum_{e_{k_1}=0}^{2} \ldots \sum_{e_{kp_k}=0}^{2}$$

$$(S_2/2x)^{B} (1/nk)^{B} \{S_1/n\delta\ (1-2x/\phi\delta)\}^{\Sigma w_i} (S_1/\delta)^{\Sigma\Sigma A_{ij}/2}$$

$$b'_{e_{11},1} \ldots b'_{e_{1p_1},1} \ldots b'_{e_{k_1},k} \ldots b'_{e_{kp_k},k}, \tag{3.14}$$

where

$$b'_{2_i} = 1 - 2x/\phi\delta,$$

$$b'_{1_i} = 1 - 2\bar{Y}_i(1-2x/\phi\delta),$$

$$b'_{0_i} = \bar{Y}^2_i(1 - 2x/\phi\delta).$$

Now

$$b'_{e_{11},1} \ldots b'_{e_{1p_1},1} \ldots b'_{e_{k_1},k} \ldots b'_{e_{kp_k},k} \tag{3.15}$$

turns out to be quite a complex polynomial in $(1 - 2x/\phi\delta)$. Since we are most interested in the form of the distribution, we take (3.15) equal to $\sum_l c_l (1 - 2x/\phi\delta)^l$, where the c_l are functions of $\bar{y}_i$.

As before we approximate

$$(1 - 2x/\phi\delta)^A \quad \text{by} \quad 1 - A\,(2x/\phi\delta) + A(A-1)/2\,(2x/\phi\delta)^2 + \ldots$$

$$(-1)^{[A]} \binom{A}{[A]} (2x/\phi\delta)^{[A]} . \qquad (3.16)$$

To obtain the posterior distribution of δ, we integrate out x. Thus

$$f(\delta|y) = c(S_1)^{-k/2(n-1)+1} \cdot (2/S_2)^{k+1/2} \; S_1 S_2/2n$$

$$\sum_{z_1=0}^{12n} \sum_{z_2=0}^{12n} \cdots \sum_{z_k=0}^{12n} \; \sum_{P_1=[(z_1+1)/2]}^{6n} \; \sum_{P_2=[(z_2+1)/2]}^{6n} \cdots$$

$$\sum_{P_k=[(z_k+1)/2]}^{6n} H_{\Sigma z_i, B} \; \bar{y}^{\Sigma z_i - 2B} \; H_{P_1+2w_1, w_1} \cdots H_{p_k+2w_k, w_k}$$

$$\sum_{w_1=0}^{3n-[(P_1+1)/2]} \; \sum_{w_2=0}^{3n-[(P_1+1)/2]} \cdots \sum_{w_k=0}^{3n-[(P_1+1)/2]} \; \sum_{B=0}^{[\Sigma z_i/2]}$$

$$\sum_{c_1=1}^{n} \sum_{A_{11}=1}^{6} \cdots \sum_{A_{1c_1}=1}^{6} \cdots \sum_{c_k=1}^{n} \sum_{A_{k1}=1}^{6} \cdots \sum_{A_{kc_k}=1}^{6}$$

$$t_0^{kn-c_i}\, t_{A_{11}} \cdots t'_{A_{1c_1}} \cdots t'_{A_{k1}} \cdots t'_{A_{kc_k}}\, g_{1A_{11}\ldots A_{1c_1}}\left(\sum_{i=1}^{c_1} A_{ij} - P_1 - 2w_1\right) \cdots$$

$$g_{kA_{k1}\ldots A_{kc_k}}\left(\sum_{i=1}^{c_k} A_{ki} - P_k - 2w_k\right) I(z_1)\, I(z_2) \cdots I(z_k)$$

$$\sum_{e_{11}=0}^{2} \cdots \sum_{e_{1p_1}=0}^{2} \cdots \sum_{e_{k1}=0}^{2} \cdots \sum_{e_{kp_k}=0}^{2} (S_1)^{\Sigma\Sigma A_{ij}/2} (S_2/2nk)^{B}$$

$$(S_1/n)^{\Sigma w_i} \sum_{e \in e}^{\Sigma c} \sum_{D=0}^{[\Sigma w_i + 1]} \binom{[\Sigma w_i + 1]}{D} (-\phi/2)^{-D}$$

$$f(kn/2 + k/2 - \Sigma w_i - \Sigma\Sigma A_{ij}/2 - 2)\; G_{\phi\delta/2}((k+1)/2 - B + D - 1), \qquad (3.17)$$

where $f(A)$ is a chi-square function with A degrees of freedom. We note that this distribution is basically in the same form as the one developed in chapter two. We can therefore apply the same techniques to get the approximate distribution. We would then obtain a scaled chi-square distribution function for the individual terms of $f(\delta|y)$. It also turns out that the results obtained for different values of ϕ hold here and the implications are the same.

3.5 The Posterior Distribution of $\rho = 2n\sigma^2_a/S_2$

If again we apply the same technique as in the previous chapter, we get

$$f(\rho|y) = \int_0^\infty cn^{-1} \sum_{z_1=0}^{12n} \sum_{z_2=0}^{12n} \cdots \sum_{z_k=0}^{12n} \sum_{P_1=[(z_1+1)/2]}^{6n}$$

$$\sum_{P_2=[(z_2+1)/2]}^{6n} \cdots \sum_{P_k=[(z_k+1)/2]}^{6n} \sum_{w_1=0}^{3n-[(P_1+1)/2]} \cdots \sum_{w_k=0}^{3n-[(P_K+1)/2]}$$

$$\sum_{B=0}^{[\Sigma z_i/2]} H_{\Sigma z_i, B} \, \bar{Y}^{\Sigma z_i - 2B} \, H_{P_1+2w_1, w_1} \cdots H_{P_k+2w_k, w_k}$$

$$\sum_{c_1=0}^{n} \sum_{A_{11}=1}^{6} \cdots \sum_{A_{1c_1}=1}^{6} \cdots \sum_{c_k=1}^{n} \sum_{A_{k1}=1}^{6} \cdots \sum_{A_{kc_k}=1}^{6} t^{kn-\Sigma c_i} \; 0$$

$$t'_{A_{11}} \ldots t'_{A_{1c_1}} \ldots t'_{A_{k1}} \ldots t'_{A_{kc_k}} \; g_{1A_{11} \ldots A_{1c_1}}\left(\sum_{i=1}^{c_1} A_{ij} - P_1 - 2w_1\right)$$

$$\ldots g_{kA_{k1} \ldots A_{kc_k}}\left(\sum_{i=1}^{c_k} A_{ki} - P_k - 2w_k\right) I(z_1) I(z_2) \ldots I(z_k)$$

$$\sum_{e_{11}=0}^{2} \ldots \sum_{e_{1p_1}=0}^{2} \ldots \sum_{e_{k1}=0}^{2} \ldots \sum_{e_{kp_k}=0}^{2} \; \Sigma c_{1e} \sum_{D=0}^{[\Sigma w_i+1]}$$

$$\binom{[\Sigma w_i+1]}{D} (-\phi/2)^{-D} S_1^{-k/2(n-1)+2+\Sigma\Sigma A_{ij}/2}$$

$$S_2^{-(k+1)/2+1} \; 2^{(k-1)/2} \; \delta^{k/2(n-1)-1-\Sigma\Sigma A_{ij}/2 + \Sigma w_i+1} \; e^{-\delta/2}$$

$$h_{(k+1)/2 - B + \Sigma w_i + 1 + D - 1}(\rho, T) \,, \qquad (3.18)$$

where $h_{A}(\rho,T)$ is defined above.

Again we note that the form of (3.18) is similar to the posterior distribution of ρ developed in chapter two. Hence we could proceed and get the same result.

3.6 Summary

The posterior probability distributions evolved in this chapter are extremely complex. However the terms of all three of the distributions were exactly the same as the ones in chapter two. The differences were only in the constants containing γ_1 and γ_2 and the Hermite polynomial coefficients. This fortunate event facilitated the subsequent analysis in that it was easy to see that the distributions behave the same way when the random variables take on extreme values.

Chapter IV

Bayesian Methods in Regression Analysis

4.1 Posterior Distribution of Regression Coefficients

The distribution of regression coefficients are well known results, studied by many authors - see for example, Scheffé (1959). The approach taken by most writers is the traditional sampling theory technique. However, recently Bayes' Theorem has been used to make inferences about these coefficients - see Tiao and Zellner (1964). The advantage, of course, is that prior knowledge may be combined with sample information in a mathematical way. In most of those works, the authors worked under the normal assumption. In this chapter we analyze the effect of departures from the underlying normal assumption using Bayesian methods.

The usual regression model with coefficient vector $\beta' = (\beta^1, \beta^2, \ldots \beta^p)$ can be written $y = x\beta + e$ where y is a k x 1 vector of observations, x is a k x p matrix of fixed elements of rank p, and e is a k x 1 vector of random errors. We shall assume as we have before that the distribution of e is unknown, but can be approximated by the first four terms of the Edgeworth Series. Since we are primarily interested in the effect of non-normality, we shall for simplicity's sake, take $p = 2$. Using these assumptions, the joint likelihood function is

$$L(\beta,\sigma|y) = \{(1/\sqrt{2\pi})(1/\sigma)\}^k e^{-\sum_{i=1}^{k} e_i^2/2\sigma^2} \prod_{i=1}^{k} \{1+\lambda_1/3!$$

$$(e_i^3/\sigma^3 - e_i/\sigma) + \lambda^2/4!\ (e_i^4/\sigma^4 - 6e_1^2/\sigma^2 + 3) + 10\lambda_1^2/6!$$

$$(e_i^6/\sigma^6 - 15e_i^4/\sigma^4 + 45e_i^2/\sigma^2 - 15)\}, \tag{4.1}$$

where σ is the variance of e, $\lambda_1 = E(e^3/\sigma^3)$, and

$\lambda_2 = E(e^4)/\sigma^4 - 3$ (mean of course equals zero).

Let $E_i = b_0 + \gamma_i$,

where $\gamma_i = b_1 e_i + b_2 e_i^2 + \ldots + b_6 e_i^6$,

and $b_0 = 1 + 3\lambda_2/4! - 150\lambda_1^2/6!$,

$b_1 = -\lambda_1/3!\ (1/\sigma)$,

$$b_2 = (-6\lambda_2/4!)(1/\sigma^2) + 450\lambda_1^2/6!\,(1/\sigma^2),$$

$$b_3 = \lambda_1/3!(1/\sigma^3),$$

$$b_4 = (\lambda_2/4!)(1/\sigma^4) - (150\lambda_1^2/6!)(1/\sigma^4),$$

$$b_5 = 0, \text{ and}$$

$$b_6 = 10\lambda_1^2/6!(1/\sigma^6).$$

We also have

$$y_i - x_{1,i}\beta_1 - x_{2,i}\beta_2 = e_i.$$

Therefore

$$L(\beta_1,\beta_2,\sigma|y) = (1/\sqrt{2\pi})(1/\sigma)^k e^{-\sum_{i=1}^{k}(y_i - x_{1,i}\beta_1 - x_{2,i}\beta_2)^2/2\sigma^2}$$

$$E_1 E_2 \ldots E_k. \tag{4.2}$$

From a similar formula in chapter 2, we have

$$E_1 E_2 \ldots E_k = b_0^k + b_0^{k-1} \sum_{A_1=1}^{6} b_{A_1} \sum_{S_1=0}^{A_1} (-1)^{S_1} \sum_{h_1=1}^{k}$$

$$(x_{1,h_1}\beta_1 - x_{2,h_1}\beta_2)^{S_1} y_{h_1}^{A_1-S_1} \binom{A_1}{S_1} (\sigma^2)^{-A_1/2}$$

$$+ b_0^{k-2} \sum_{A_1=1}^{6} \sum_{A_2=1}^{6} b_{A_1} b_{A_2} \sum_{S_1=0}^{A} \sum_{S_2=0}^{A} (-1)^{\Sigma S_i} \sum_{h_1<h_2}^{k}$$

$$(x_{1,h_1}\beta_1 - x_{2,h_1}\beta_2)^{S_1} (x_{1,h_2}\beta_1 - x_{2,h_2}\beta_2)^{S_2} y_{h_1}^{A_1-S_1} y_{h_2}^{A_2-S_2}$$

$$(\sigma^2)^{-\Sigma A/2}$$

$$+$$

$$\vdots$$

$$b_0^{0} \sum_{A_1=1}^{6} \cdots \sum_{A_k=1}^{6} b_{A_1} \cdots b_{A_k} \sum_{S_1=0}^{A} \cdots \sum_{S_k=0}^{A_k} (-1)^{\Sigma S_i}$$

$$\sum_{h_1<h_2<\dots h_k}^{k} (x_{1,h_1}\beta_1 - x_{2,h_1}\beta_2)^{S_1}$$

$$(x_{1,h_2}\beta_1 - x_{2,h_2}\beta_2)^{S_2} \dots (x_{1,h_k}\beta_k - x_{2,h_k}\beta_2)^{S_k}$$

$$y_{h_1}^{A_1-S_1} y_{h_2}^{A_2-S_2} \dots y_{h_k}^{A_k-S_k} \binom{A_1}{S_1} \dots \binom{A_k}{S_k} \sigma^{2^{-\Sigma A_i/2}} . \quad (4.3)$$

Therefore $\xi = E_1 E_2 \ldots E_k =$

$$b_0^k + \sum_{t=1}^{k} b_0^{k-t} \sum_{A_1=1}^{6} \sum_{A_2=1}^{6} \cdots \sum_{A_t=1}^{6} \sum_{S_1=0}^{A_1} \sum_{S_2=0}^{A_2} \cdots \sum_{S_t}^{A_t}$$

$$\sum_{h_1 h_2 \ldots h_t}^{k} \binom{A_1}{S_1} \binom{A_2}{S_2} \cdots \binom{A_c}{S_c} y_{h_1}^{A_1-S_1} y_{h_2}^{A_2-S_2} \cdots y_{h_c}^{A_c-S_c}$$

$$(-1)^{\Sigma S_i} \cdot (x_{1,h_1}\beta_1 - x_{2,h_1}\beta_2)^{S_1} (x_{1,h_2}\beta_1 - x_{2,h_2}\beta_2)^{S_2}$$

$$\ldots (x_{1,h_t}\beta_1 - x_{2,h_t}\beta_2)^{S_t}. \qquad (4.4)$$

Therefore $L(\beta,\sigma^2 | y) = \{(1/\sqrt{2\pi})(1/\sigma)\}^k \ \xi \ e^{-\sum_{i=1}^{k} (y_i - x_{1,i}\beta_1 - x_{2,i}\beta_2)^2/2\sigma^2}.$

In situations where little is known about β and σ, Jeffreys (1961) and Savage (1962) suggest that the prior distributions should be

$$f'(\beta_1, \beta_2, \sigma) \propto 1/\sigma.$$

Therefore the joint posterior distribution is

$$f(\beta_1, \beta_2, \sigma|y) = c\ (\sigma)^{-k-1}\ \xi\ e^{-\sum_{i=1}^{k} (y_i - x_{1,i}\beta_1 - x_{2,i}\beta_2)^2/2\sigma^2}, \qquad (4.5)$$

where c is the normalizing constant.

The marginal posterior distribution of (β_1, β_2) is obtained by integrating out σ from the joint posterior distribution. Therefore

$$f(\beta_1, \beta_2 | y) = c \int_0^{\infty} (\sigma)^{-(k+1)}\ \xi\ e^{-\sum_{i=1}^{k} (y_i - x_{1,i}\beta_1 - x_{2,i}\beta_2)^2/2\sigma^2}\ d\sigma \qquad (4.6)$$

Since each of the terms in the above expression is in the form of an inverted gamma function and

$$\int_0^{\infty} v^{-A} e^{-B/2v}\, dv = \Gamma(A-1)(2/B)^{A-1}.$$

We have

$$f(\beta_1, \beta_2 | y) = c/2\ b_0^{k}\ \Gamma(k/2)\ (2/S_\beta)^{k/2} +$$

$$\sum_{t=1}^{k} b_0^{k-t} \sum_{A_1=1}^{6} \sum_{A_2=1}^{6} \cdots \sum_{A_t=1}^{6} \sum_{S_1=0}^{A_1} \sum_{S_2=0}^{A_2} \cdots \sum_{S_t=0}^{A_t} \Gamma(k/2+\Sigma A_i/2)$$

$$(2S\beta)^{k/2 + \Sigma A_i/2} \sum_{h_1<h_2<\dots<h_t}^{k} \binom{A_1}{S_1} \binom{A_2}{S_2} \cdots \binom{A_t}{S_t}\ b_{A_1}\ b_{A_2} \cdots b_{A_t}$$

$$(y_{h_1})^{A_1-S_1}(y_{h_2})^{A_2-S_2}\ldots(y_{h_t})^{A_t-S_t}(x_{1,h_1}\beta_1-x_{2,h_1}\beta_2)^{S_1}$$

$$(x_{1,h_2}\beta_1-x_{2,h_2}\beta_2)^{S_2}\ldots(x_{1,h_t}\beta_1-x_{2,h_t}\beta_2)^{S_t}, \qquad (4.7)$$

where $S_\beta = \sum_{i=1}^{k}(y_i - x_{1,i}\beta_1 - x_{2,i}\beta_2)^2$.

From Tiao and Zellner (1964) we have

$$f(\beta|y) = \text{constant}\ \{1 + \Sigma x_i^2(\beta-\hat\beta)^2/\Sigma(y_i-x_i\hat\beta)^2\}^{-k/2}$$

where $\hat\beta = \Sigma x_i y_i/\Sigma x_i^2$, which is an equivalent form of the first term of (4.7).

4.2 Comparison with Normal-Theory Results

Several examples were generated in order to examine the effects of non-normality (see Appendix II). In a great deal of them the graphs had the following appearance (β_1' and β_2' represent the true value of β_1 and β_2 respectively).

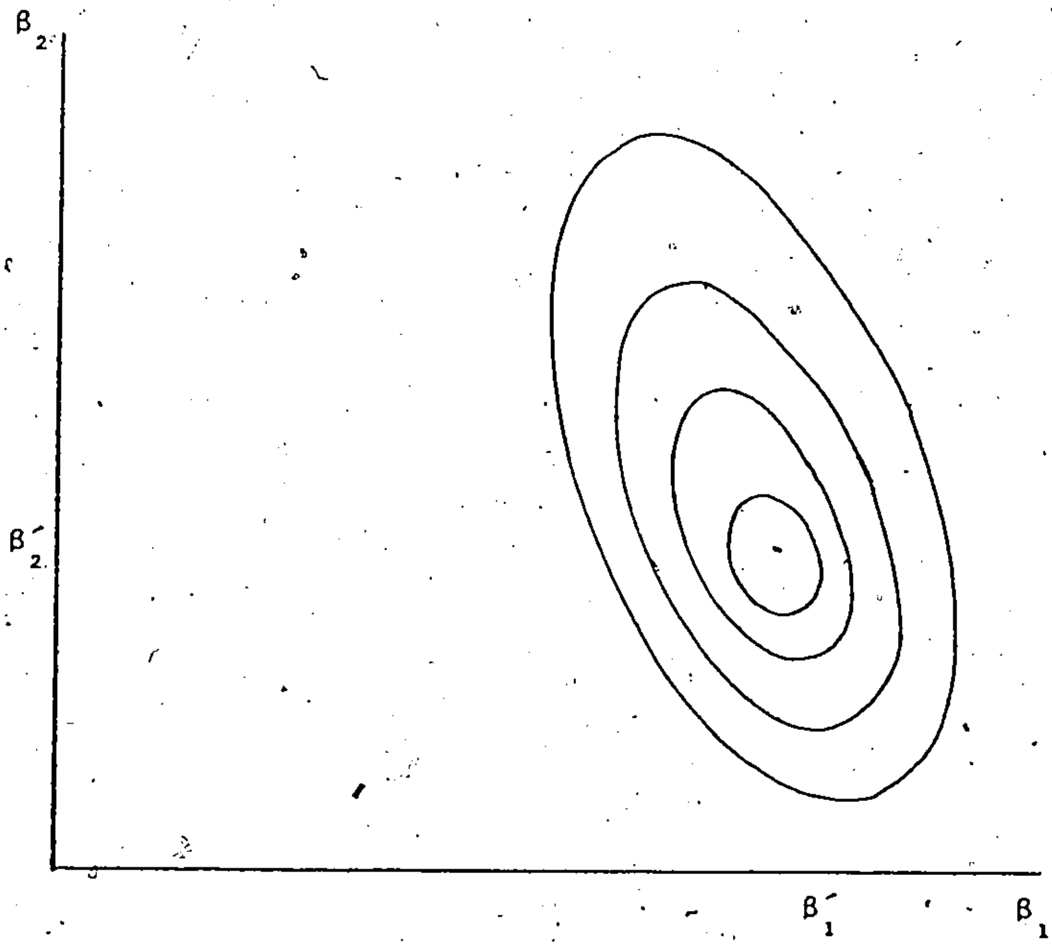

Fig. 2 Contours of the Posterior Probability of Regression Coefficients

It is interesting to note that in general, the posterior distributions for the normal-theory approach were quite sharp reaching their maximum on or near the true value. However, the non-normal technique produced even sharper peaks. The result is certainly not unexpected. In general, a great deal of information using Bayes' Theorem is reflected by an extremely sharp posterior distribution. In our examples, the information supplied by the third and fourth moments results in that extra information and hence, a sharp curve. There were several examples where the normal-theory result was better. These probably arose because the values of λ_1 and λ_2 did not adequately reflect the true values of λ_1 and λ_2.

4.3 An Illustrative Example

A problem was generated with the following information:

$$k = 5, \qquad \lambda_1 = .6, \qquad \lambda_2 = 3.0 .$$

The posterior probability distribution was computed for both techniques. The results are depicted below.

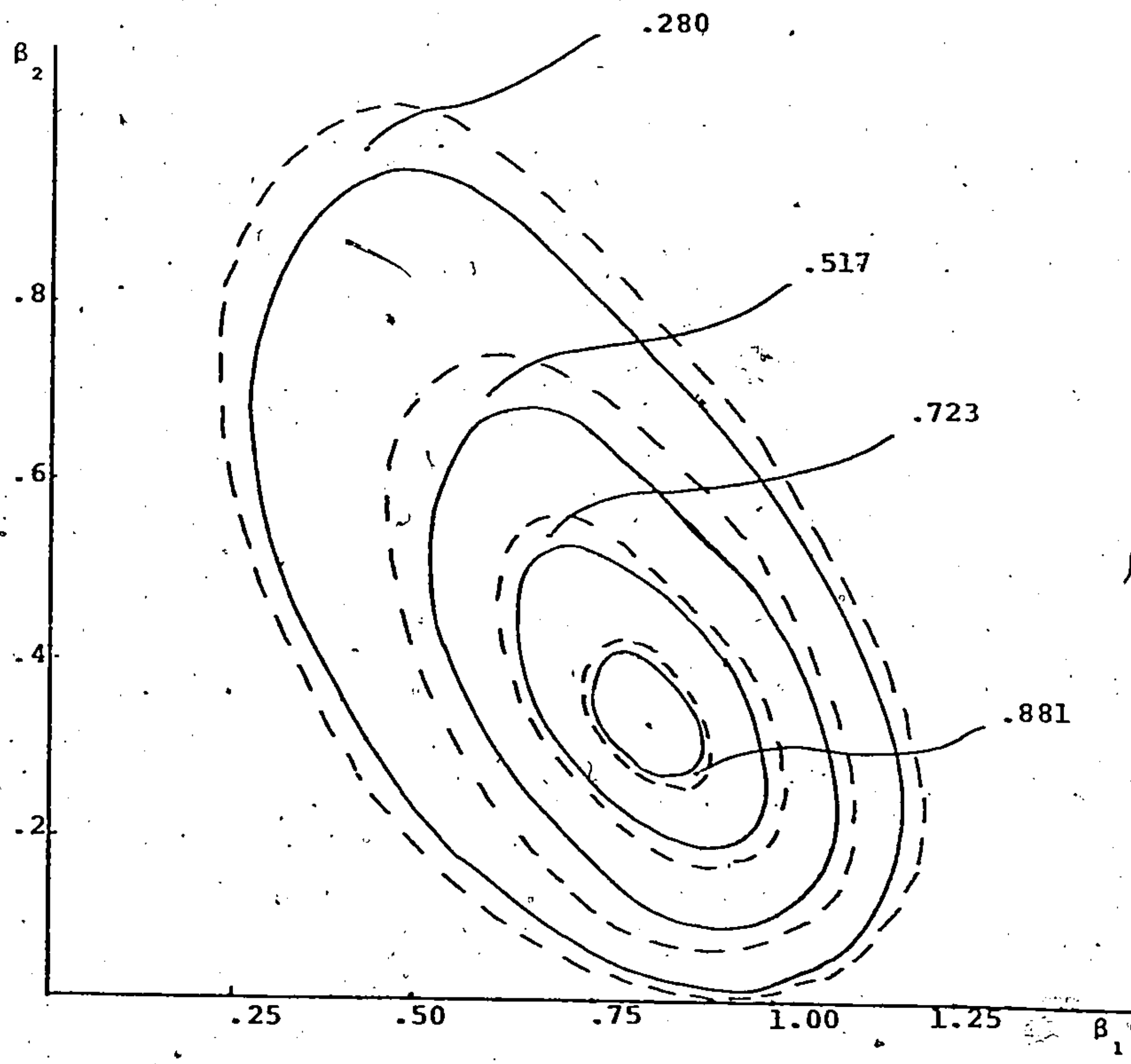

Fig. 3 Contours of (β_1, β_2) for the above example. Normal ---, Non-Normal ———.

Peak for normal theory approach is .913.

Peak for Non-Normal approach is .976.

The true values of β_1 and β_2 were .8 and .35 respectively. An examination of Figure 3 reveals that the non-normal and normal-theory approaches result in a posterior probability distribution whose peak is approximately centered over the true value of (β_1, β_2). However the non-normal approach results in a distinctly sharper narrower distribution, producing more certain estimates.

4.4 Summary

As in chapter 2, the posterior distributions are quite complex and somewhat difficult to analyze. However, we did have a computer work out several examples with different values of λ_1 and λ_2. Almost all of these showed differences between the two methods with the non-normal technique providing the better distributions. The largest differences occurred when λ_1 was non-zero. Therefore once again we find λ_1 being more important than λ_2.

Chapter V

Conclusion

5.1 An Overview

In all of the distribution functions developed over the last four chapters, one feature is most prominent. The form of those functions was basically the same. The first term in the complex summation was equal to the distribution developed under normal theory assumptions. The remaining terms which had the same distribution-type form, acted as correction terms. They were functions of λ_1 and λ_2 and provided information as to the impact these moments had on the overall function.

Another feature was the necessity of the use of electronic computers to calculate practical examples. One of our original goals was to use a computer to create a table of tail-area probabilities. However, this idea had to be dropped because of the large number of variables affecting the distributions. Therefore an integral part of this study is the computer programs that we have included in the appendices.

A philosophical difficulty arose in some of the assumptions we made. In all of our work we assumed that the third and fourth moments were known and that the higher moments were negligible. Was this a reasonable assumption? Our answer is that from previous examples, we could get such knowledge. For example, R. C. Geary (1947) suggested the use of Fisher's k-statistics for such a use. In addition, the only alternative, the normal assumption, seems to be a worse offender. Under the normality conditions, we are assuming that all higher moments are zero. We at least are providing a more general form. As such, our assumptions seem to provide a measure of the knowledge garnered from previous such examples. Also we may claim that one of the purposes of this study was to examine the effects of the departures from normality and therefore our knowledge of λ_1 and λ_2 may be considered hypothetical.

Finally, a problem usually arising in such a work is ignored. The problem, of course, is the use of Bayes' Law and the prior probability distribution. It seemed to us that this procedure is well established and that any comment on our part would be irrelevant and unimportant. In any case our work was merely meant as an attempt to answer some of the questions arising from this technique and not to criticize it.

5.2 Opportunities for Future Research

The chances for finding future topics of study in this field seem to be quite good. Both major topics, Bayes' Law and the Edgeworth Series, seem to have hardly been touched. We list some of the ideas that have occurred to us.

1. The application of the Edgeworth series to study higher way designs in the analysis of variance - Bayesian and classical methods. In addition, the inclusion of the more general case of unequal observations is a possibility. A superficial study indicates that the calculations would only be moderately more complex.

2. The extension of the Edgeworth series to more than the first four terms - again in Bayesian and classical approaches. It is obvious that the work would grow quite complicated. Another approach might be required.

3. The uses of the Edgeworth series as applied to Bayesian and classical methods in the analysis of variance and regression theory with auto correlated errors. Our study seems to preclude the possibility of attempting the study from a "first principles" basis as we have in our study. Such a work would involve a complex transformation that expands rapidly when the Edgeworth series is used.

4. A philosophical discussion of the impact of the use of the Edgeworth series to estimate the prior probability distribution.

APPENDIX I

Computer program to compute the posterior probabilities for the variance ratio

```
      Dimension Fact (8),F(8,8),Z(20,20),H(20,20),G(42),Y(2)
C     Calculation of Combinatorial Function
      I1=6
      Fact(1)=1
      DO 1 I2=2,I1
      I3=I2-1
      AI2=I2
1     Fact(I2)=AI2*Fact(13)
      DO 2 I4=2,I1
      DO 2 J1=2,I4
      F(I4,1)=1.
      F(I4,I2+1)=1.
      F(I4,I4+1)=1.
2     F(I4,J1)=Fact(I4)/(Fact(J1-1)*Fact(I4-J1+1))
      F(1,1)=1.
      F(1,2)=1.
C     Calculation of Hermite Polynomial Coefficients
      Z(2,2)=3.
      N=18
      DO 3 I7=1,N
      18=I7+1
      Z(I7,1)=(-1.)**I8
3     Z(I7,I8)=0.
      DO 4 I9=3,N
      DO 4 J3=2,I9
      AI9=I9
      AJ3=J3
      I10=I9-1
      J4=J3-1
4     Z(I9,J3)=(-Z(I10,J3))+(AI9-2.*(AJ3-2.))*Z(I10,J4)
      DO 5 I11=4,N
      I11=(I11+1)/2
      I13=I11-2
      DO 5 I14=1,I12
5     H(I11,I14)=ABS(Z(I13,I14))
      H(1,1)=1.
      H(2,1)=1.
      H(3,1)=1.
      H(3,2)=1.
C     Calculation of Gamma Function
      DO 105 I16=1,40
```

```
      BI16=I16
      AI16=BI16/2.
 105  G(I16)=Gamma(AI16)
C     Main Program
      Print 96
      DO 9991 I9991=1,100
      Read 901,W
 901  Format(F5.3)
C     Calculated Data
      Read AN,AK,TH,YBAR,S1,S2,PHI,X1,X2
      DO 905 I=1,K
      905 Read, Y(I)
      X2=-1.0
      DO 1=.125*X2*((W-1)/W)**2
      DO2=.2083*(X1**2)*((W-1.)/W)**3
      DO=1.+DO1-DO2
      D(1)=.5*X1*(AN**.5)*((W-1.)**1.5)*(W**(-2))
      D21=.25*X2*AN*((W-1.)**2)*(W**(-3))
      D22=.625*(X1**2)8AN*((W-1.)**3)*(W88(-4))
      D(2)=D22-D21
      D(3)=(-.1667)*X1*(AN**1.5)*((W-1.)**1.5)*(W**(-3))
      D41=.0417*X2*(AN**2)*((W-1.)**2)*(W**(-4))
      D42=.2083*(X1**2)*(AN**2)*((W-1.)**3)*(W**(-5))
      D(4)=D41-D42
      D(5)=0.
      D(6)=.0139*(X1**2)*(AN**3)*((W-1.)**3)*(W**(-6))
      C1=(S1**((AK/2.)*(AN-1.)))*(S2**((AK-1.)/2.))
      K1=AK-1.
      K2=AK*(AN-1.)
      K3=AK*AN-1.
      C=C1/(G(K1)*G(K2)*TH*2.**((AK*AN-1.)/2.))
      C2=(DO**AK)*2.**((AK*AN-1.)/2)*G(K3)
      HA=(W**((AK/2.)*(AN-1.)-1.))/((1.+W/PHI)**((AK*AN-1.)/2.))
      ANRM=(C2/(S2**((AK*AN-1.)/2.)))*HA*C
      GNRMAL=ANRM/(DO**AK)
      ANS1=0
      DO 6 IR=1,13
      R=IR
      IRR=(IR+1)/2
      DO 6 IP=1,IRR
      P=IP
      DO 6 IA1=1,6
      A1=IA1
      JIA1=IA1+1
      DO 6 IS=1,JIA1
      S11=IS
      DO 6I18=1,2
      I19=IA1-IR+1
      IF(A1-R.)7,8,8
    8 IF(S11-A1+R-2.)7,9,7
    7 AAK=0.
```

```
      BAK=0.
      CAK=0.
      DAK=0.
      IAK=2
      GAK=0.
      HAK=0.
      GO TO 6
 9    AAK=H(IR,IP)*((YBAR)**(R-2.*P+1.))*((W/(AN*AK))**(P-1.))
                                                    *D(IA1)
      BAK=((-1.)**I19)*DO
      CAK=F(IA1,IS)*(Y(I18)**(S11-1.))
      DAK=(2./(S1*PHI))**(((AK*AN-1.)/2.)+A1/2.-P+1.)
      IAK=((AK*AN-1.)+A1-2.*P+2.)
      GAK=(1.+W/PHI)**(P-1.-((AK*AN-1.)/2.)-(A1/2.))
      HAK=W**((AK*AN-AK-2.)/2.)+(A1/2.)-P+1.)
 6    ANS1=ANS1+AAK*BAK*CAK*DAK*GAK*HAK*G(IAK)
      PANS=C*ANS1
      ANS2=0
      DO 16 IR=1,13
      R=IR
      IRR=(IR+1)/2
      DO 16 IP=1,IRR
      P=IP
      DO 16 IA1=1,6
      A1=IA1
      JIA1=IA1+1
      DO 16 IS=1,JIA1
      S11=IS
      DO 16 IA2=1,6
      A2=IA2
      JIA2=IA2+1
      DO 16 JS=1,JIA2
      S22=JS
      I19=IA1+IA2-IR+1
      IF(A1+A2-R+1.)17,18,18
18    IF(S11+S22-A1-A2+R-3.)17,19,17
17    AAK=0.
      BAK=0.
      CAK=0.
      DAK=0.
      IAK=2
      GAK=0.
      HAK=0.
      GO TO 16
19    AAK=H(IR,IP)*((YBAR)**(R-2.*P+1.))*((W/(AN*AK))**
                                         (P-1.))*D(IA1)*D(IA2)
      BAK=((-1.)**I19)
      CAK=F(IA1,IS)*(Y(1)**(S11-1.))*F(IA2,JS)*(Y(2)**(S22-1.))
```

```
      DAK=(2./(S1*PHI))**(((AK*AN-1.)/2.)+A1/2.+A2/2.-P+1.)
      IAK=((AK*AN-1.)+A1+A2-2.*P+2.)
      GAK=(1.+W/PHI)**(P-1.-((AK*AN-1.)/2.)-(A1/2.)-(A2/2.))
      HAK=W**(((AK*AN-AK-2.)/2.)+(A1/2.)+(A2/2.)-P+1.)
   16 ANS2=ANS2+AAK*BAK*CAK*DAK*GAK*HAK*G(IAK)
      PANT=C*ANS2
      FFANS=ANRM+PANS+PANT
 9991 PRINT 999,W,GNRMAL,ANRM,PANS,PANT,FFANS
  999 FORMAT('0',6F10.6)
      PRINT 97,X1,X2
   97 FORMAT('0','X1=',F10.4,'X2=',F10.4)
      STOP
      END
```

APPENDIX II

Computer program to compute the posterior probabilities for the regression coefficients

```
      DIMENSION X1(9),X2(9),B1(9),B2(9),Y(9),D(9),FACT(8),
                                       F(8,8),G(99)
C     CALCULATION OF COMBINATORIAL FUNCTION
      I1=6
      FACT(1)=1.
      DO 1 I2=2,I1
      I3=I2-1
      AI2=I2
    1 FACT(I2)=AI2*FACT(I3)
      DO 2 I4=2,I1
      DO 2 J1=2,I4
      F(I4,1)=1.
      F(I4,I2+1)=1.
      F(I4,I4+1)=1.
    2 F(I4,J1)=FACT(I4)/(FACT(J1-1)*FACT(I4-J1+1))
      F(1,1)=1.
      F(1,2)=1.
C     CALCULATION OF GAMMA FUNCTION
      DO 105 I16=1,40
      BI16=I16
      AI16=BI16/2.
  105 G(I16)=GAMMA(AI16)
      K=2
      AK=K
      DO 11 I=1,K
   11 READ 12,X1(I),X2(I),Y(I)
   12 FORMAT(3F6.3)
      DO 97 II1=1,25
   97 READ ;5.B1(II1)
   15 FORMAT(F6.3)
      DO 98 II2=1,8
   98 READ 15,B2(II2)
      DO 99 II1=1,8
      DO99 II2=1,8
      SUM=0.0
      DO 96 I=1,K
   96 SUM=SUM+(Y(I)-X1(I)*B1(II1)-X2(I)*B2(II2))**2
      SB=SUM
      AL1=.6
      AL2=3.0
```

```
      DO=1.0+(0.125*AL2)-(0.2083*AL1**2)
      D(1)=(-.1667*AL1)
      D(2)=(-.25*AL2)+(.625*AL1**2)
      D(3)=(.1667*AL1)
      D(4)=(.0417*AL2)-(.2083*AL1**2)
      D(5)=0.0
      D(6)=(.0139*AL1**2)
      ANRM1=(D0**K)*G(K)*(2.0/SB)**(AK/2.0)
      ANRM2=ANRM1/(D0**K)
      TOT1=0.0
      DO 100 IA1=1,6
      JA1=IA1+1
      DO 100 IS1=1,JA1
      A1=IA1
      S1=IS1-1
      DO 100 L1=1,K
      E1=(D0**(K-1))*((X1(L1)*B1(II1)-X2(L1)*B2(II2))**
                                (IS1-1))*((-1)**(1IS1-1))
      E2=D(IA1)*G(K+IA1)*(Y(L1)**(A1-S1))*F(IA1,IS1)*((2./SB)
                                **((AK+A1)/12.))
100   TOT1=TOT1+E1*E2
      TOT2=0.0
      DO 101 IA1=1,6
      DO 101 IA2=1,6
      JA1=IA1+1
      JA2=IA2+1
      DO 101 IS1=1,JA1
      DO 101 IS2=1,JA2
      S1=IS1-1
      S2=IS2-1
      KK=K-1
      DO 101 L1=1,KK
      LL1=L1+1
      DO 101 L2=LL1,K
      A1=IA1
      A2=IA2
      E1=(D0**(K-2))*(X1(L1)*B1(II1)+X2(L1)*B2(II2)**IS1-1)
      E2=D(IA1)*D(IA2)*(YL1)**(A1-S1))*(Y(L2)**(A2-S2))
      E3=F(IA1,IS1)*F(IA2,IS2)*G(K+IA1+IA2)*((2./SB)**((AK
                                +A1+A2)/2.))
      E4=(X1(L2)*B1(II1)+X2(L2)*B2(II2))**(IS2-1)
101   TOT2=TOT2+E1*E2*E3*E4
      TOTAL=ANRM1+TOT1+TOT2
      PRINT 93,B1(II1),B2(II2),TOTAL
 93   FORMAT('',F6.3,F6.3,F30.4)
      PRINT75,ANRM2,ANRM1.TOT1,TOT2
 75   FORMAT('',4F18.4)
 99   CONTINUE
      STOP
      END
```

APPENDIX III

Computer program to find the values of λ_1 and λ_2 which produce positive Edgeworth Series

```
      DO 1 ID=1,81
      JD=ID-21
      D=JD/10.
      DO 1 IA=1,51
      JA=IA-26
      A=JA/10.
      DO 7 I=1,2000
      J=I-1000
      X=J/100.
      B=(1./72.)*(A**2)*(X**6-15.*X**4+45.*X**2-15.)
      C=(A/6.)*(X**3-3.*X)
      E=(D/24.)*(X**4-6.*X**2+3.)
      Y=B+C+E+1.
      IF(Y.LT.0.0)GO TO 1
 7    CONTINUE
      PRINT 2,A,D
 2    FORMAT('', F10.4,F20.5)
 1    CONTINUE
      STOP
      END
```

APPENDIX IV

Computer program to compute the cumulative probability distribution of an Edgeworth Series (using the Trapezoid Rule)

```
      TSUM=0
      READ 9,A,C,S,AL1,AL2
 9    FORMAT(5F6.3)
 2    CONTINUE
      ANRM=.4*(1./S)*EXP(-(A**2.)/(2.*S**2.)
      A1=(AL1/6.)*((A**3.)/(S**3.)-(3.*A)/S)
      A2=(AL2/24.)*((A**4.)/(S**4.)-(6.*A**2)/(S**2.)+3.)
      A3=(AL1**2.)/72.
      A4=(A**6.)/(S**6.)-(15.*A**4.)/(S**4.)(45.*A**2.)/(S**2.)-15.
      F1=ANRM*(1.+A1+A2+A3*A4)
      B=A+.05
      BNRM=.4*(1./S)*EXP(-(B**2.)/(2.*S**2.)
      B1=(AL1/6.)*((B**3.)/(S**3.)-(3.*B)/S)
      B2=(AL2/24.)*((B**4.)/(S**4.)-(6.*B**2)/(S**2.)+3.)
      B4=(B**6.)/(S**6.)-(15.*B**4.)/(S**4.)(45.*B**2.)/(S**2.)-15.
      F2=BNRM*(1.+B1+B2+A3*B4)
      SUM=.0005*(F1+F2)
      DO 1 I=1,49
      A=A+.001
      ANRM=.4*(1./S)*EXP(-(A**2.)/(2.*S**2.)
      A1=(AL1/6.)*((A**3.)/(S**3.)-(3.*A)/S)
      A2=(AL2/24.)*((A**4.)/(S**4.)-(6.*A**2)/(S**2.)+3.)
      A4=(A**6.)/(S**6.)-(15.*A**4.)/(S**4.)(45.*A**2.)/(S**2.)-15.
      F=ANRM*(1.+A1+A2+A3*A4)
 1    SUM=SUM+(.001*F)
      TSUM=TSUM+SUM
      PRINT 10, TSUM,B,SUM
10    FORMAT('0','INTEGRAL=',F10.7,'UPPER LI IT.',F8.4,' ',F10.7)
      A=B
      IF(A-C)2,2,3
 3    STOP
      END
```

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