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The dialectical tier of mathematical proof

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ABSTRACT: Ralph Johnson argues that mathematical proofs lack a dialectical tier, and thereby do not qualify as arguments. This paper argues that, despite this disavowal, Johnson's account provides a compelling model of mathematical proof. The illative core of mathematical arguments is held to strict standards of rigour. However, compliance with these standards is itself a matter of argument, and susceptible to challenge. Hence much actual mathematical practice takes place in the dialectical tier.

KEYWORDS: argument, conclusive argument, dialectical tier, illative core, mathematical practice, mathematical reasoning, proof, Ralph Johnson

1. THE DANCE OF MATHEMATICAL PRACTICE

What is mathematics about? A standard answer has long been that mathematics is concerned with the derivation of formal proofs. And yet, as the mathematician David Ruelle points out, truly formal proof has little to do with actual mathematical practice:

> Human mathematics consists in fact in talking about formal proofs, and not actually performing them. One argues quite convincingly that certain formal texts exist, and it would in fact not be impossible to write them down. But it is not done: it would be hard work, and useless because the human brain is not good at checking that a formal text is error-free. Human mathematics is a sort of dance around an unwritten formal text, which if written would be unreadable. This may not seem very promising, but human mathematics has in fact been prodigiously successful. (Ruelle 2000: 254)

Explaining that success poses a problem for philosophy of mathematics as traditionally conceived. If mathematical practice were ultimately reducible to formal proof, which has been analysed in great detail in mathematical logic, then actual practice would differ only in degree from the elementary and/or foundational work upon which most philosophers of mathematics concentrate. But if mathematical practice cannot be understood solely in such terms, then philosophy of mathematics needs to pay it much closer attention.

In recent decades, some philosophers of mathematics have indeed begun to take a broader range of mathematical practice into account. Important milestones include (Pólya 1954), (Lakatos 1976), and (Kitcher 1984). In the last decade the pace has quickened. (Corfield 2002) is an explicit manifesto for a new, integrative field of research bringing together insights from philosophy of mathematics, history of mathematics, sociology of mathematics, mathematics education, and mathematics itself. Corfield's subsequent book, (Corfield 2003), makes good on some of this promise, which has been developed further by many authors, including contributors to (Hersh 2006), (Van Kerkhove and Van Bendegem 2007), (Van Kerkhove et al. 2010), and (Löwe and Müller 2010).

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2. JOHNSON'S TWO TIER MODEL OF ARGUMENT

In *Manifest Rationality* (2000), Ralph Johnson has provided a thoughtful and influential analysis of non-mathematical argument. He characterizes arguments as containing two levels—an 'illative core', in which the support that premisses provide for the conclusion is set out, and a 'dialectical tier', in which the proponent of the argument responds to potential or actual criticism. Hans Hansen summarizes Johnson's position as follows: The illative core comprises

> a thesis, T, supported by a set of reasons, R', whereas the 'dialectical tier must be a set of ordered pairs, with each pair consisting of an objection and one or more responses to the objection: thus:
>
> $\{\langle O_1, \{A_{1a}, ..., A_{1n}\}\rangle, \langle O_2, \{A_{2a}, ..., A_{2n}\}\rangle, ..., \langle O_N, \{A_{Na}, ..., A_{Nn}\}\rangle\}$
>
> Now, in advancing a Johnson-argument, a proponent has to do two things: (i) he must assert T because R, and (ii) for every objection, O_i, to R-T, he is obligated to respond with one or more answers, $A_{i1} - A_{ij}$. (Hansen 2002: 271 f.)

3. JOHNSON (AND HIS CRITICS) ON PROOF AND ARGUMENT

Johnson contends that mathematical proofs do not qualify as arguments.[1] This claim proceeds from his *Principle of Vulnerability*, that 'if the arguer claims to have insulated the argument against all possible criticism, then this is no arguer and no argument' (Johnson 2000: 224). It follows from this principle that there cannot be any conclusive arguments, and yet proofs would seem to be clear examples of conclusive arguments. So, if Johnson's principle is to survive, he must show that mathematical proofs are either not conclusive or not arguments. He defends the second of these alternatives, adducing four differences between proof and argument:

- (P1) Proofs require axioms; arguments do not have axioms.
- (P2) Proofs must be deductive; arguments need not be.
- (P3) Proofs have necessarily true conclusions; almost all arguments have contingent conclusions.
- (P4) "[A]n argument requires a dialectical tier, whereas no mathematical proof has or needs to have such" (Johnson 2000: 232)

I shall argue below that Johnson picked the wrong alternative. In his sense of 'conclusive argument', proofs are not conclusive, but they are arguments. Hence, suitably qualified, the Principle of Vulnerability may be preserved without jettisoning proof from the domain of argument. But first I should address some other criticism that Johnson's position has attracted.[2]

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[1] Or at least, in his subsequent clarification, not paradigmatically (Johnson 2002: 316).

[2] I shall restrict my attention to critics who address Johnson directly. However, there are many other commentators who have made similar points. For example, Michael Crowe lists as 'misconceptions' several theses which closely resemble Johnson's disanalogies, including 'The methodology of mathematics is deduction', 'Mathematics provides certain knowledge', 'Mathematical statements are invaria-

3.1 The Four Colour Theorem

The four colour theorem (4CT) states that four colours suffice to colour every planar map so that no neighbouring regions are the same colour. Johnson himself warns that the proof of 4CT 'creates a potential problem' for (P2), by undermining the position that proofs are necessarily deductive (Johnson 2000: 232). Presumably this problem arises because 4CT was the first and most widely discussed example of a theorem with a proof that can only be completed by computer. The proof involves a large set of configurations (633 in the most recent version) each of which has to be shown to possess a certain property (see Aberdein 2007: 140, for a more detailed discussion). Each of these demonstrations was arrived at by a computer which had been programmed with a general method for their construction. This makes the full proof far too long to be verified by hand by any one human mathematician, although individual passages have been checked. Moreover, the whole proof has been independently shown to be error-free by a different computer program (Gonthier 2008). Hence it is a rare exception to Ruelle's assertion that formal texts are not written down.

Of course, sceptics of computer-aided proof may very well ask why, if they did not trust the first computer program, they should be expected to trust the second. Nonetheless, the source of their anxiety is not, as Johnson would seem to imply, that the proof is non-deductive, but that it may be no proof at all. As Georges Gonthier, the architect of the computer-checked proof of 4CT, observes, "Coq [the proof assistant used] verifies that [the proof] strictly follows the rules of logic. Thus, our proof is more rigorous than a traditional one" (Gonthier 2008: 333). Gonthier is not begging the question against the sceptics when he insists that his proof is more rigorous; rather, he is identifying 'rigour' with deductive logic. It is still possible, if astronomically unlikely, that every program used either to prove 4CT or to check the proof has run into undetectable bugs that have caused it to misfire. But otherwise, the proof was conducted in strict adherence with deductive logic.

4CT is not the only candidate for a non-deductive mathematical proof (for others, see Baker 2009). However, although the existence of such a proof would contradict (P2), it is not clear why this should jeopardize Johnson's position. He is not arguing that no arguments can be deductive, but rather that the relative importance of deductive argumentation has been greatly overstated. So, since he concedes that there can be deductive arguments, the deductive nature of mathematical proofs may establish that they are an unusual sort of argument, but not that they are not arguments.

3.2 Finocchiaro

Maurice Finocchiaro observes of Johnson's position that treating 'geometrical proofs as not arguments but mere inferences or entailments ... would strike me as arbitrary insofar as Euclidean geometrical proofs are typically attempts to persuade oneself or others of the truth of the theorem in question by rational means' (Finocchiaro 2003: 32). This would seem to be related to a point Finocchiaro made to Johnson on some earlier occasion: 'Finocchiaro suggested that the difference between an argument and a proof is one of perspective. That is, a proof is an argument that has been found to have certain properties. I

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bly correct', 'Mathematical proof is unproblematic', and 'The methodology of mathematics is radically different from the methodology of science' (Crowe 1988: 260 ff.).

am not sure how to respond to this objection' (Johnson 2000: 232). Johnson's candour is striking and Finocchiaro's point is a highly pertinent challenge to (P4), but as it stands, frustratingly condensed. His central idea seems to be that proofs are articulated in different contexts, and often in a context in which the proof is intended to persuade. In this context, a dialectical tier is to be expected, but this need not be so in other contexts.

3.3 Dove

Ian Dove suggests a counterexample to (P1) and (P4): Cauchy's proof of the Euler Conjecture (Dove 2007: 348). The Euler Conjecture expresses a relationship between the number of vertices (V), edges (E), and faces (F) of a polyhedron: $V - E + F = 2$. Imre Lakatos's most celebrated work (Lakatos 1976) is a painstaking reconstruction of attempts made by several nineteenth-century mathematicians to prove this conjecture. Central to the story is Cauchy's proof of 1813, to which a series of counterexamples was advanced by later mathematicians, resulting in a succession of reworked proofs and a substantial clarification of the original concepts. As Dove shows, Cauchy's proof was not axiomatic, contrary to (P1), and in its subsequent history (as reconstructed by Lakatos) exhibited a sophisticated dialectical tier, contrary to (P4).

However, Lakatos's reconstruction is not without its critics. As the editors of his posthumously published *Proofs and Refutations* (1976) observe, some mathematicians see the struggle to prove the Euler Conjecture as uncharacteristic of mathematical practice: 'while the method of proof-analysis described by Lakatos may be applicable to the study of polyhedra, a subject which is "near empirical" and where the counterexamples are easily visualisable, it may be inapplicable to "real" mathematics' (Lakatos, 1976: *ix*). The editors do stress that Lakatos has other examples, and later writers have provided many more. Nonetheless, perhaps Johnson could preserve his characterization of proofs as not arguments by excluding these examples, and retreating to a statement about 'typical proofs', say.[3]

3.4 Dufour

Michel Dufour makes two criticisms of Johnson's position. Firstly, he notes that some proofs 'have been notoriously controversial, at least in their early days' (Dufour 2011). This challenge to (P4) is similar to Dove's, if much less explicit. Secondly, Dufour picks up on an important detail of Johnson's presentation:

> Johnson adds an interesting epistemic comment about the relationship between proof and argument. 'The proof that there is no greatest prime number is conclusive, meaning that anyone *who knows anything about such matters* sees that the conclusion must be true for the reasons given' (Johnson, 2000: 232, Dufour's emphasis). This is certainly true. But what happens when you just know some things, not any thing, in the mathematical field and you wonder if there is a greatest prime number? (Dufour 2011).

[3] Such a move would be doubly ironic: Johnson would be exhibiting a strategy which Lakatos stigmatizes as 'monster barring', redefining a concept to exclude anomalous cases, as well as coming close to violating his own Principle of Vulnerability.

Like Finocchiaro, Dufour draws attention to the different sorts of context in which proofs may arise. Where there is an epistemic asymmetry, as in a classroom context, the dialectical tier may be expected to play less of a role, since the student may take more on trust.

4. PROOFS AND CONCLUSIVE ARGUMENTS

In the last section we saw that there have been a number of piecemeal challenges to Johnson's contention that proofs are not arguments. More systematic criticism will require analysis of his characterization of 'conclusive argument'. Johnson states four properties that a conclusive argument must exhibit:

(C1) 'Its premises would have to be unimpeachable or uncriticizable.'
(C2) 'The connection between the premises and the conclusion would have to be unimpeachable—the strongest possible.'
(C3) 'A conclusive argument is one that can successfully (and rationally) resist every attempt at legitimate criticism.'
(C3) 'The argument would be regarded as a conclusive argument.' (Johnson 2000: 233 f.)

Johnson argues that no argument can satisfy all four criteria. He takes it that proofs, if they were arguments, would satisfy the criteria, but he denies that they are arguments. I shall argue that proofs cannot satisfy all of these criteria either, so there need be no objection to their being admitted as arguments in Johnson's system.

Actually, only two of these criteria are at issue: Johnson states that conclusive arguments are impossible 'principally because of the difficulty of satisfying (C1) but also because of (C3)' (Johnson 2000: 234). The other two criteria provide no such obstacle. Of (C2), which corresponds to (P2), deductive inference being an unimpeachable connection, Johnson notes that there is 'no problem with satisfying this requirement', since arguments may be deductive (Johnson 2000: 233). (C4) introduces a new point, but an uncontroversial one. In its defence, Johnson returns to mathematical proof, observing rightly that '[p]art of being a proof is being regarded as a proof' (Johnson 2000: 234).

So what of (C1) and (C3)? (C1) corresponds to (P1): the axioms of mathematics would be unimpeachable premisses. However, Johnson's demonstration that no argument satisfies (C1) also shows why (P1) is no obstacle to proofs being arguments. Considering the project of relativizing (C1) to a discourse community in which some premiss may be treated as unimpeachable, Johnson notes that 'that would not confer on that premise the status of being uncriticizable. Someone from outside that community of discourse might well have a legitimate criticism of the statement' (Johnson 2000: 233). But this is exactly the situation with axioms. By choosing to operate within a given axiomatic system, the mathematician undertakes to treat a set of axioms as uncriticizable. But other mathematicians (or the same mathematician in other moods) may still challenge these axioms from the perspective of other systems. While this would be quixotic for the most firmly entrenched axioms, it is commonplace for more controversial cases, such as the Axiom of Choice, or large cardinal axioms. (C3) shares with (P4) a focus on the dialectical tier: according to (P4) mathematical proofs have no dialectical tier; according to (C3) conclusive arguments would have an unbeatable dialectical tier. Johnson is right that (C3) sets a standard that no argument can meet, so it is no discredit to their rigour that proofs do not meet it either.

Type of Dialogue	**Initial Situation**	**Main Goal**	**Goal of Protagonist**	**Goal of Interlocutor**
Inquiry	Open-mindedness	Prove or disprove conjecture	Contribute to outcome	Obtain knowledge
Deliberation	Open-mindedness	Reach a provisional conclusion	Contribute to outcome	Obtain warranted belief
Persuasion	Difference of opinion	Resolve difference of opinion with rigour	Persuade interlocutor	Persuade protagonist
Negotiation	Difference of opinion	Exchange resources for a provisional conclusion	Contribute to outcome	Maximize value of exchange
Debate (Eristic)	Irreconcilable difference of opinion	Reveal deeper conflict	Clarify position	Clarify position
Information-Seeking (Pedagogical)	Interlocutor lacks information	Transfer of knowledge	Disseminate knowledge of results and methods	Obtain knowledge

Fig. 1. Some mathematical dialogue types

We saw in the last section that (P4) may be challenged by drawing attention to the context in which proofs are produced. I shall now make this challenge more precise. In Douglas Walton's account of argument an important role is played by the 'type of dialogue'. Dialogue types include persuasion, negotiation, inquiry, deliberation, information-seeking, and quarrel. They may be distinguished in terms of their initial situation, and the shared and individual aims of their participants (see, for example Walton and Krabbe 1995: 80). For Walton, different argumentational practices are legitimate in different types of dialogue, so the evaluation of arguments must have regard to the type of dialogue in which they are advanced. Elsewhere, I have argued that mathematical discourse also exhibits a diversity of dialogue types, similar to Walton's, and that the analysis of proofs should have regard to the type of dialogue in which the proof arises (Aberdein 2007: 144 ff.). Fig. 1 summarizes a variety of mathematical dialogue types, some of which are more appropriate for successful proof than others (cf. Walton and Krabbe 1995: 66). Finocchiaro's criticism can now be understood as the positive point that proofs are frequently advanced in persuasion dialogues, a context in which the interlocutor may be expected to raise objections to which the protagonist should have replies if the proof is to be accepted. Examples of persuasion dialogues of this sort include the presentation of new work and journal refereeing. Conversely, Dufour alludes to pedagogical, information-seeking dialogues, in which objections may also arise, but it may sometimes be admissible to ignore them (if, for example, a satisfactory answer would be unintelligible without a depth of knowledge that the student lacks).

Johnson, however, is sceptical about dialogue types, suspecting Walton of an unduly broad characterization of argument (Johnson 2000: 177). This has the effect of stranding him in a single type of dialogue. His conception of mathematical proof appears to be limited to an inquiry conducted by logically omniscient individuals. In such a context objections would never be raised, so the dialectical tier would indeed be empty, but this is, of course, an idealized fiction. However, if Johnson's account is augmented with Walton-style mathematical dialogue types, it becomes a supple and versatile instrument, that, as I shall argue in the next section, can contribute decisively to the understanding of mathematical practice.

5. TIERS OF MATHEMATICAL REASONING

Many mathematicians and philosophers of mathematics have observed the dual nature of mathematical proof: proofs must be both persuasive and rigorous. The passage from Ruelle quoted above is one example. Here is another from a more famous mathematician, G.H. Hardy:

> If we were to push it to its extreme we should be led to a rather paradoxical conclusion; that we can, in the last analysis, do nothing but *point*; that proofs are what Littlewood and I call *gas*, rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils. ... On the other hand it is not disputed that mathematics is full of proofs, of undeniable interest and importance, whose purpose is not in the least to secure conviction. Our interest in these proofs depends on their formal and aesthetic properties. Our object is *both* to exhibit the pattern and to obtain assent. (Hardy 1928: 18, his emphasis)

It follows from this account that 'proof' is ambiguous between two different activities: 'exhibiting the pattern' and 'obtaining assent'. In most circumstances both activities must be satisfactorily performed for the proof to be a success. There are some special cases, such as proofs that have been fully formalized, or have been reified as mathematical objects, where only the first activity is attempted. That sort of 'proof' may be harmlessly identified with its illative core. But in the more characteristic sense of 'proof' we need more than this; we need a dialectical interaction with the mathematical community. For Richard Epstein, proofs intended to obtain assent are arguments by means of which mathematicians convince each other that the corresponding inferences are valid. He represents this situation schematically (Fig. 2).

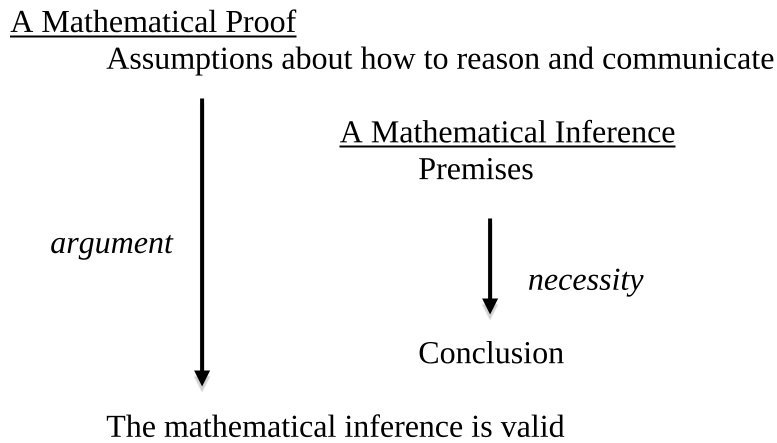

Fig. 2. Epstein's picture of mathematical proof (Epstein 2008: 419)

Proofs are typically made up of many steps, not all of which are necessarily developed with the same rigour. So closer examination of proofs will represent them not as single arguments but as structures of arguments (technically trees, or directed acyclic graphs). Hence the construction of proofs requires the articulation of two parallel structures: an inferential structure of formal derivations linking formal statement to formal statement, and an argumentational structure of arguments by which mathematicians attempt to convince each other of the soundness of the inferential structure. Fig. 3 summarizes this picture.

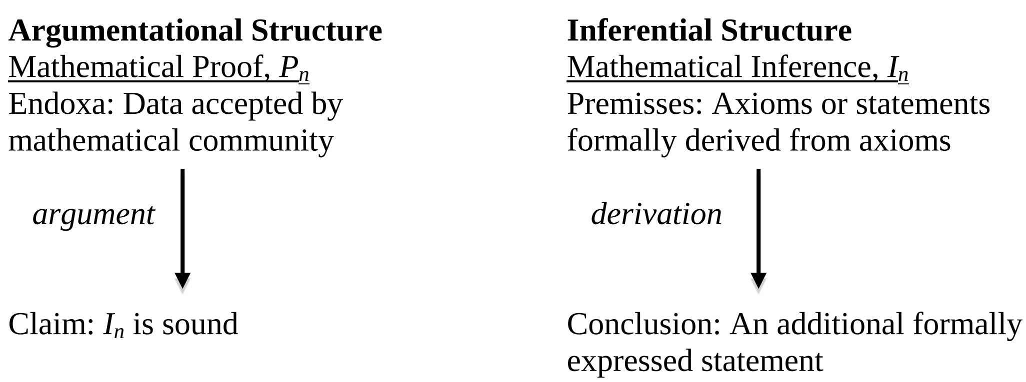

Fig. 3. The parallel structure of mathematical proof

The relationship between the corresponding steps in the inferential and argumentational structures is broadly that between illative core and dialectical tier. One might object that the argumentational structure contains more than just objections and replies: it has its own theses and reasons. There is a close overlap of content between the nodes of the two structures, since the nodes of the argumentational structure assert that the corresponding nodes of the inferential structure have been soundly derived. Nonetheless, the argumentational structure must contain additional data, namely facts about the acceptability of various inferential moves within the mathematical community. However, this could in principle be couched in the form of answers to objections. Hence the presence of a dialectical tier should be seen as characteristic of mathematical proof, at least in the sense in which it is concerned with obtaining assent. Where the steps in the inferential structure are unproblematic—since they are fully worked out formal derivations, or more typically, it is clear how they could be—the argumentational structure can be very light; it need do no more than point (as Hardy puts it) at the steps of the inferential structure. But where the derivation is more complex or contested, much more of the burden of the proof rests on the argumentational structure. In those circumstances it becomes critical to track and provide responses to the objections that may be raised to the gaps in the inferential structure.

This account both conserves and transcends the conventional view of mathematical proof. The illative core of mathematical arguments is held to strict standards of rigour, without which the proof would not qualify as mathematical. However, the step-by-step compliance of the proof with these standards is itself a matter of argument, and susceptible to challenge. Hence much actual mathematical practice takes place in the dialectical tier. Careful demarcation of these two levels is essential to the proper understanding of mathematics; a virtue of Johnson's account is that attention may be directed to the dialectical tier without undermining the rigour of the illative core. If this account is correct, important concepts in the philosophy of mathematics, such as mathematical rigour and mathematical explanation, can only properly be addressed when both of the parallel structures are accounted for. Mathematicians have a sophisticated grasp of the inferential structure. But we still need a system for analysis and appraisal of the argumentational structure. Despite Johnson's disavowal, his account may contribute significantly to this pursuit.

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Commentary on "THE DIALECTICAL TIER OF MATHEMATICAL PROOF" by Andrew Aberdein

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I am honoured to be asked to comment on the paper by Andrew Aberdein on the Dialectical Tier of Mathematical Proof, for although I have been interested in this area for many years, I have not been active in it lately.

The paper addresses the question of what mathematics is about, and it does so in an interesting way. It is generally agreed that part of mathematical proof is to make sure that "everything is correct," i.e. that all logical inferences, etc., are correct. But part of mathematical proof is to convince others—mathematicians, students and assorted others—of the correctness of the proof, either just in its own right, or in defence against objections.

Aberdein mentions, *inter alia*, the famous four colour theorem. This is one of the most famous theorems that, so far at least, can only be proved with the aid of a computer, since a number of cases have to be dealt with, and these are too many and too complex to be done by hand. As Aberdein points out, this may be seen by sceptics as a problem—why trust one computer program when another may have the same flaw?—but to computer science people like myself, who can verify by hand that the computer codes of the two or more programs are truly independent, this does not seem like a serious objection.[1]

Some of us may remember Euclidean geometry, and proofs of such theorems as that of Pythagoras, or the theorem that asserts that, in any triangle, the lines from each angle to the mid-point of the opposite side, meet in one point. Aberdein comments on geometrical proofs, but I think he is right when he says that (geometrical) proofs are often articulated in a context in which the proof is intended to persuade: to make a geometric proof fully 'watertight' would require a lot of additional machinery. But that, surely, is part of the issue: persuading one's audience is part of the task, as well as making sure that the proof is correct.

Aberdein also mentions, and briefly discusses the Euler conjecture relating the number of vertices, edges and faces of a polyhedron.

A brief mention of the proof that there is no greatest prime number is of some interest. The proof is conclusive to those who know at least some things about number theory, but Aberdein wonders, with Dufour, what happens when "you just know some things..." Here it is a case of whom you're trying to convince.[2] Well, in the end I'm tempted to say, "tough."

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[1] When I was a graduate student at the University of Waterloo in the late 60s, the 4CT was still the four colour conjecture, and I and my fellow graduate students spent much time trying to prove the 4CT. Several of us independently re-discovered the five colour theorem.

[2] The proof is one by contradiction: assume there is a greatest prime number, P. Then multiply all the prime numbers between 2 and P, and then add 1. The resulting number is clearly greater than P, and is prime, thereby contradicting the assumption that P is the greatest prime number. This proof, though

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But now we really get to the crux of the matter: is a dialectical tier necessary and/or useful? And it strikes me that it is.

There is, clearly, a big difference between talking about mathematics and *doing* mathematics, and also between talking about mathematics and *teaching* mathematics. When a mathematician encounters a problem, he or she uses all the tools at his or her disposal to prove the conjecture[3], and this almost certainly involves the dialectical tier.

But moreover, one needs to convince oneself and others of the correctness of the proof, and here's where the dialectical tier really "kicks in." Once one is satisfied that the proof is correct, one can start worrying about the niceties of making sure all the details have been filled in.

As Aberdein points out, " 'proof' is ambiguous between two different activities: 'exhibiting the pattern' and 'obtaining assent'. In most cases we need a "dialectical interaction with the mathematical community."

The dialectical tier is important.

simple, is somewhat subtle and may not convince the mathematically uninitiated: it may need a lot of further explanation to convince the uninitiated.

3 Or to disprove it, as the case may be: one of my fellow graduate students in the late 60s tried for months to prove a conjecture by his supervisor. When he did not succeed, he tried to find a counter example, and he did find one in a matter of hours.