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ABSTRACT: In argumentation studies, almost all theoretical proposals are applied, in general, to the analysis and evaluation of written argumentative texts. I will consider mathematics to illustrate some differences between argumentative practice and the products of it, to emphasize the need to address the different types of argumentative discourse and argumentative situation. Argumentative practice should be encouraged when teaching technical subjects to convey a better understanding and to improve thought and creativity.

KEYWORDS: Argument, Argumentation, Mathematical Practice, Learning, Proof

1. INTRODUCTION

Since Aristotle, logic has been considered a normative theory of reasoning and as such has been applied to the analysis and evaluation of arguments. In the last century, Toulmin (1958) noticed a gap between logic as a theory of argument and real practice; his work gave rise to new ways of conceiving and theorizing about arguments. The study of fallacies was pivotal in the development of the field of argumentation and emphasis on avoiding fallacious arguments when arguing in natural settings was the driving force behind the new theoretical proposals. In part because of these origins, almost all theoretical proposals in the field of argumentation apply to the analysis and evaluation of arguments, mostly in written texts.

Johnson (2000) states that a theory of argumentation, considered as a theory of the practice of argumentation, has to consider many aspects not included in the study of its products. He defines argumentation as "the socio-cultural activity of constructing, presenting, interpreting, criticizing and revising arguments" (p. 12). For Johnson, a theory of argument is a component of a theory of argumentation and he considers that proper work has to be done first towards a better theory of argument in order to have a balanced theory of argumentation.

Much work has been done to present actual examples of arguments as they appear in real practice but, nevertheless, it is still true that, as Hitchcock (2002, p. 288) remarks of Johnson's examples of argumentative interchanges, "[they] do not exhibit at first glance the features of one person interpreting and criticizing an argument and the argument's author revising it in response to this criticism",

features he considers constitutive of the practice of argumentation.

The relationship between the theory and practice of argumentation has been reconsidered again in several recent papers, for example, those by Pinto (2001), Johnson (2005) and Kvernbekk (2012). The positions of the authors differ, but, when talking of practice, they usually consider specific arguments as they appear in argumentative exchanges in order to analyze the distance a normative theory of argument should maintain to be of any value to evaluate practice. In general, they want to assess the arguments as part of the activity, but not the activity as a whole.

The ideal pragma-dialectical model of a critical discussion is "based on analytical considerations regarding the most pertinent presentation of the constitutive parts of a problem-valid procedure for carrying out a particular kind of discursive task" (van Eemeren & Houtlosser, 2005, p. 75). The emphasis is put in the activity, but here too, its application is devoted to the analysis and evaluation of the argumentative products. This is the case even in the latest attempts to look at the properties of what the above cited authors call activity types, defined as "conventionalized entities that can be distinguished by 'external' empirical observation of the communicative practices in the various domains […] of discourse" (p. 76).

In this paper, I consider some issues of recent papers associated with mathematical argumentation in an attempt to contribute to the discussion about the role of arguing in mathematical practice and in the evaluation of the products of this practice. I argue that, in mathematical practice, argumentation considered as a rational, social and communicative activity should be encouraged to improve collaborative, efficient and creative work, but this does not necessarily imply that direct application of the current theories of ordinary argumentation to evaluate its products should be undertaken. The particular constraints of mathematical activities, for example, the rigor required for mathematical definitions and proofs and their institutionalized forms, are sufficient for their evaluation in the different contexts in which they arise.

Application of problem-solving strategies has proved helpful for the successful accomplishment of mathematical tasks and the understanding of difficult mathematical concepts. In those cases, argumentation may be of help not only to raise and solve problems, formulate hypotheses, ask for justification of inferential steps, construct explanations and test one's understanding, but also to establish relationships between concepts and the application of methods in different situations. That is, argumentation may be of help if mathematics is considered as a critical and collaborative inquiry to look for a solution to a problem. Nevertheless, adaptation to the specific activity type and the actual context may have to be taken into account to design the tasks and argumentative practices that trigger collaborative and effective work.

2. A LOOK AT PROOFS, ARGUMENTS AND MATHEMATICAL PRACTICE

Discussion on the nature of mathematical proof has a long history that is beyond the scope of this paper. I refer only to some recent contributions that link the idea of proof, argument and the kind of processes that can be found in different contexts of

mathematical practice in order to support my claim that argumentative practice is important to promote understanding, and to improve thought and creativity, including in mathematics.

Johnson (2000, p. 168) defines argument as "the distillate of the practice of argumentation". For him, in addition to the reasons to support a claim (the illative core of the argument), "an argument possesses a dialectical tier in which the arguer discharges his dialectical obligations". As he considers that mathematical proofs do not have this "dialectical tier", they are not (paradigmatic) arguments. As Tindale (2002) points out, "this concept of argument is hampered by an internal tension between the product an argument is and the process it captures" (p. 299) because mathematical proofs appear in many types of argumentative situation and the idea that mathematical proofs are more than chains of deductive inferences is nowadays supported in very different fields or disciplines.

From the field of argumentation, Aberdein (2009) presents an insightful recompilation of references in which authors appeal directly to studies on mathematical proof and mathematical practice. In this paper and in many others (see references), Aberdein considers that much of what mathematicians do, in particular proofs, may be understood as a "species of *argument*", considering it "an act of communication intended to lend support to a claim" (p. 1-2). Several authors (for example, Alcolea Banegas, 1998; Aberdein, 2010; Dove, 2009) consider that the way mathematicians analyze and assess mathematical reasoning is closer to the way informal logicians analyze and assess ordinary arguments than what the convention about mathematical proof asserts, namely, that the reconstruction of a mathematical proof should conform to a chain of valid deductive arguments. As a consequence, they think that elements of the new theories of argument(ation) may be of help to assess mathematical proofs as given in practice. Krabbe (2008) distinguishes different types of mathematical activity with various objectives and examines examples of strategic maneuvering in mathematical proofs.

From the field of the philosophy of mathematics, Pólya and Lakatos's pioneering work to present proofs based on their own heuristic experience did not resonate with the mainstream of the discipline, except perhaps for the application of their proposals to mathematical education. The deductivist and logicistic approaches to science in general, and to mathematics in particular, were the conventional approaches during the majority of the last century. The emphasis on the objects or results promoted losing sight of the processes by which they were obtained (Ferreirós, 2010). Nowadays, there is a widespread eagerness to overcome the foundational view on mathematics that considers mathematical theorems *a priori* truths that are there, in the void, waiting to be discovered. Instead, there is a new emphasis to understand how proofs are developed and built and many philosophers of mathematics are now concerned with the kinds of activities mathematicians perform, that is, how the practice of mathematics is actually carried out (Mancosu, 2008). There is a turn to the practical (Gabbay & Woods, 2005) and the dividing line between the pair product/process, the first conceived as an object of analysis subject to a normative evaluation, and the second seeking to accommodate the descriptive adequacy of real practices, has begun to blur. Aberdein (2011) provides us again with a good summary of references of works that

try to integrate insights from many fields into a new philosophy of mathematical practice.

In traditional mathematics education, mathematics consists of readymade perfect products transmitted directly by the teacher in an authoritative but also authoritarian way. All is set up to be accepted. The process of discovery or of construction that led to these products is hardly considered. Communication is oriented mainly to the explanation of difficult steps in proofs, the resolution of repetitive problems as applications of the theory and the assessment of the solutions. Argumentation (in the ordinary sense of the term) among students or with the teacher is not usual. The focus is on knowing that something is the case, not on how it can be constructed. This fact can be explained in part by the difficulty that many students encounter to assimilate abstract concepts and by the need to address very long curricula. At elementary levels, a constructivist approach to mathematics is nowadays more common, but as soon as the contents of the curriculum accumulate, the traditional way of doing mathematics is still prevalent in many countries. As a consequence, many students give up on understanding mathematics and apply the results or methods in a mechanical and rote way. If we think of education as a way of pursuing a method to construct knowledge in the mind of the student, the classical approach to mathematics is clearly not the ideal.

If we look at the practice of mathematics in any particular setting, we soon realize that, to establish the right path of valid inferences that lead to the solution of a problem, we first have to perform many different activities. For example, we have to conceptualize the new ideas that can be of help to solve the problem and to do so, we have, maybe, to translate it into another more familiar domain. We also have to find a proof strategy able to solve the problem and, to arrive at that, we have to identify a promising direction to find the solution and/or to dismiss other directions. We may have to explain to ourselves or to others the reasons to adopt or to reject this strategy, that is, to explain why we think this direction is appropriate or why it will not work. We have to confirm that the strategy works by being able to express the particular details that conform to it; to do so, we may have to express those technical or difficult details that lead to the solution to make it comprehensible to others and, at the same time, we may have to eliminate those details that at first seemed to be necessary, but that finally are not. In addition, we may have to go back to the first formulation of the problem to readdress its initial conditions by including additional preconditions to accommodate the solution found. We may also look for ways to adapt the problem and its solution to an actual problem in a specific field. We may want to refine the proof to make it clearer or more elegant. Finally, we may have to communicate the problem and its solution to different audiences.

The ideas involved in these tasks are in many cases tentative, incomplete or even incorrect and have to be developed or explained in order to be included in the final presentation of the solution or in the proof of the theorem. Not all this material is included in the final product, but all of this is part of the mathematical practice that leads to the solution. Those intermediate steps towards the solution are important to understand how mathematics works.

If we think of practice in mathematics as a set of complex activities and tasks to be performed in order to solve a mathematical problem, not only logical reasoning but also good arguing is a very valuable tool to improve understanding and creativity, as I try to show in the following sections.

3. MATHEMATICS AND ARGUMENTATION

The influence of Johnson's definition of argument and the reaction to it from some of the researchers coming from the field of mathematical argumentation can be easily seen in many of the papers cited in the last section. For example, Dove (2009) presents many mathematical examples to try to show that the method by which mathematicians assess mathematical reasoning resembles the practice of informal logic or argumentation theory. Alcolea Banegas (1998), Aberdein (2005) and many others (see Aberdein, 2009 for references) try to adapt Toulmin's layout to mathematics. Aberdein (2010) and Dove (2009) consider how some of the argumentation schemes in the work of Walton, Reed and Macagno (2008) may be of use to evaluate mathematics. Epstein (2012) proposes an account of mathematical reasoning by means of two parallel structures: an inferential structure of formal derivations and an argumentation structure by which mathematicians attempt to convince each other. Aberdein (2012), following this proposal, discusses theoretical conceptions of mathematical practice by analyzing the nature of steps that should be admitted to the argumentational structure in a mathematical proof.

In many of these papers, the main concern is to show that there is argumentation in mathematics. To do so, the authors often discuss examples of problems, proofs or different kinds of mathematical error, either to show how mathematicians evaluate them in practice or to show that the way in which the neat final results were reached was by refining previous faulty results, using to that end ordinary communicative forms in a language that was not totally formalized. Either way, the authors justify, can have a parallel in ordinary theories of argumentation.

To give an example, Dove (2009, pp. 140-141) comments on one of the faulty *proofs* in the work of Maxwell (1959) in which the proposed task is to prove that any given triangle is isosceles[1].

First of all, maybe there could be someone, somewhere, who considers this a true problem and is trying to find a *proof* for it, but it is difficult to imagine such a situation. Moreover, the proposed *proof* begins with a diagram of a triangle that clearly is not isosceles. After that, a very detailed notation is used; there is an appeal to two mathematical theorems (the angle bisection theorem and the sine rule) and a series of careful mathematical steps are given. That is, the method used is mathematical. Then, the supposed author of the *proof* makes the error of considering that, from the equality of sines, the equality of angles follows.

The error is trickier to find in the original example because of many of the factors already cited (use of graphics, notational details, the appeal to mathematical

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[1] Examples from Maxwell can also be found in Aberdein (2010). A very similar example to this one from Wikipedia can be found in Krabbe (2008).

theorems, etc.). Dove uses this example to illustrate how people proceed when confronted with an evidently faulty result and to defend that "the process one uses to discover the mistake is analogous to the process one might use to criticize an unpalatable argument in non-mathematical settings" (p. 6). Nevertheless, in my opinion, the example lacks a real context. For example, in a classroom setting, a teacher would not need much time to discover the error. It is a common error made time and again by students of elementary trigonometry. What is not typical of a classroom context is the careful notation, the use of two theorems and the attentive and mathematical style of the proof. All of this makes natural for many to attempt to find a more sophisticated kind of error. I think that, in order to learn more about the nature of mathematical practice and how its products are evaluated, we should be looking at real examples of this practice, including the contextual elements of the situations in which they were produced.

Other lines of defense on the argumentative nature of mathematics are the appeal to the axiom of choice (Dove, 2009; Alcolea Banegas, 1998, among others), the surveyability of long proofs (Coleman, 2009) or even the use of mathematical diagrams (Larvor, 2012). All of them try to underline the "challengeable" nature of at least some mathematical proofs and to defend the idea that some results can be accepted not only because of their validity, but also because they are useful for mathematics. This being true, it is also true that many proofs in mathematics are deductive. In my opinion, there is no need to appeal to special cases to defend the assertion that, in mathematical practice, there is a place for argumentation. We only have to distinguish between mathematical products and mathematical practice. As Kuhn (1992) states, thinking as argument "arises every time a significant decision must be made" (p. 157).

Pólya (1945, 1954) and Lakatos (1976) are cited in every work that tries to emphasize the plausible and heuristic nature of mathematical practice but, then, examples tend to show the argument in the products instead of the practice. As noticed by many authors, mathematical practice is not always successful and in many cases it creates "knowledge" that is neither precise, rigorous nor certain (Chazan, 1990). It is, I think, in this process that argumentation has a natural place.

From a review of many of the papers cited above we can extract two main ideas. First, Johnson's influential definition placed a burden on many of their authors to justify the claim that mathematical products are argumentative. Second, there is a manifest tension in these works between the examples of mathematical products considered as arguments and the process that leads to them.

At first glance, Krabbe (2008) seems to avoid this problem because he takes note of the "various contexts in which proofs occur and of the various objectives they may serve" (p. 453). He also proposes a list of contexts of proof and the supposed functions of reasoning in them by their association to different types of dialogue:

1. thinking up a proof to convince oneself of the truth of some theorem;
2. thinking up a proof in dialogue with other people (inquiry dialogue; probative functions of reasoning);

3. presenting a proof to one's fellow discussants in an inquiry dialogue (persuasion dialogue embedded in inquiry dialogue; persuasive and probative functions of reasoning);
4. presenting a proof to other mathematicians, e.g. by publishing it in a journal (persuasion dialogue; persuasive and probative functions of reasoning);
5. presenting a proof when teaching (information-seeking and persuasion dialogue; explanatory, persuasive, and probative functions of reasoning) (Krabbe, 2008, p. 457).

The first two types of activity are not considered argumentative owing to the characteristics associated by Walton & Krabbe (1995) with the type of dialog in which they occur. For Krabbe, probative functions are intended to extend knowledge and are not argumentative. In order to have persuasive functions, the aim should be to convince another by overcoming her doubts (p. 457).

Nevertheless, it is in the situation of thinking up a proof or of looking for the solution to a problem that there is doubt and dialectic situations can appear (clearly with respect to others, but even with respect to oneself). When trying to establish the inferential structure that glues together the initial conditions and the solution or the claim, there are many situations in which a choice has to be made in conditions of uncertainty. There may be many methods to try that could be of use. You may need to persuade the other (or yourself) that a particular path of inquiry is better that another or that some conjecture is adequate to solve a problem. There may also be situations in which, although the inference seems valid, you may ask yourself or other people to look for possible counterexamples to the claim. There may be cases in which the solution to a problem is already known, but, nevertheless, you may want to try it with specific examples before thinking up how to prove it, and so on.

Another difference between the first two situations and the other three in the list that could, maybe, be invoked is that, when thinking up a proof, there is no need for language interaction. Language is an important tool for thinking in mathematics (Thurston, 1994). The need to formulate careful definitions and the use of specific notation seem fundamental to advance in the construction of a proof. Thinking up a proof with other people without linguistic interaction seems impossible and, if communication is needed, it is difficult to distinguish this case from that of a presentation of a proof to other people (case 4) except for the fact that, here, there is doubt involved and dialectical and rhetorical elements have, in my opinion, a role to play.

The two first activities in the list are just those that correspond to the discovery process in mathematics (Lakatos, 1976) or in rhetorical terms to the *inventio* part before the deliverance of a speech or the production of a written text. In those situations, the dialog types are always complex. It could be of use to separate the five types of activity for theoretical purposes, but at least the first two appear in general to be mixed up with one of the others.

A problem is always proposed or considered in a specific contextual situation (be it a classroom context, an academic situation or even a proposal to solve a problem through the internet) and, although the solution to it may be unique, if we consider the definition of argumentation given by van Eemeren et al. (1996) that considers argumentation as a verbal and social activity of reason aimed at

increasing (or decreasing) the acceptability of a standpoint, the activity of looking up for a solution stands in accordance with this definition.

Krabbe considers the dialectical component of a proof to lie in the number of inferential steps it contains. However, I think that inferential steps correspond to the logical part of the proof and dialectical components correspond to the communicative situations in which mathematical practice is undertaken. All the situations in the list are communicative events in which an audience is involved (in the first one, only oneself). When communicating mathematical work, abstract notation, uses of previous works and deductive inferences are always involved. In those cases, explanation of some inferential steps or definitions may be needed to enhance understanding and to accept the proof. As Mancosu (2011) states, demands for explanation in proofs do not always come with a new proof, but they may contribute to reinforce it, so I consider that they have their place in the dialectical tier of the argumentative exchange.

Besides, there could be some (easy) cases where persuasion (or conviction) could be reached only by understanding the inferential process in the proof. In those cases, when presenting the proof to others, rhetorical and communicational elements would surely be present, but not necessarily dialectical moves.

To finish with, I think that what is behind this list is the necessity of a final proof to assign persuasive functions to a mathematical situation. There is again the tension between the practice and the products of this practice. There is also the legacy of the deductivist approach to mathematics and, perhaps, the added difficulty of observing practice in different real situations.

Much more work is needed to observe and understand the relationship between actual mathematical practices and argumentation in different contexts in order to design protocols that can help in the development of a better understanding of, and to improve thought and creativity in, mathematics. Empirical research from the field of mathematical education could be of help to understand better how mathematics and argumentation are handled in the classroom. The analysis of Pease & Martin (2012) of the third Mini-Polymath project (Tao, 2011), involving online collaborative work to solve a mathematical problem, also represents a good step to explore another type of context. Finally, it is worth considering the first exploration of van Bendengem & van Kerkhove (2009) to situate mathematical arguments in context, by looking at their commentaries on the organization of a large research program to prove a difficult theorem and on the mode of presentation of a paper by Pólya.

4. ARGUING, PROVING AND LEARNING IN MATHEMATICS

Learning has traditionally been defined as the integration of new information with existing knowledge (Andriessen, 2009). However, a learner's previous knowledge can inhibit the integration of new information because this knowledge may have proved efficient in different situations in the past (Balacheff, 2010). A good way to overcome this problem is by argumentation.

Mathematical ideas may not be a matter of opinion or belief, but arguing is important as a type of communication to fix a (shared) understanding of

mathematical concepts, to improve inferential steps and to question the solution to a problem. Although some researchers in mathematical education consider that "the key role of proof is the promotion of mathematical understanding" (Hanna, 2000, p. 5), a more careful look at what those authors mean by proof shows that, for them too, it means much more than mere syntax or a chain of valid deductive steps and includes ordinary argumentative elements to promote understanding.

Schwarz (2009) presents an outline of the complex relationships between argumentation and learning. To begin with, there are different approaches to the definition of what constitutes learning and each conception determines the role of the argumentation in the classroom.

For some psychologists, learning is a psychological change in the individual that can be observed indirectly between successive activities. For others, learning emerges through interactions. These two views may not be incompatible, but they have been considered as if they were so from a theoretical point of view and they can be representative of the way mathematical education has been undertaken throughout history.

While the traditional view maintains that the acquisition of mathematical skills is individually undertaken, a more accurate view considers that the interaction with peers and a teacher is essential. When adopting the second view, many researchers think that the role of argumentation is central, even in mathematics (Muller Mirza & Perret-Cremont, 2009). Nevertheless, the relationship between argumentation and learning in mathematics is complex because, in learning, there are multiple processes involved and different forms of undertaking them.

One of the processes involved in mathematical learning is the conceptualization of mathematical ideas. Empirical research proves that argumentation may represent an important tool to intervene in the progressive construction of basic mathematical concepts and in the development of consciousness and systematic links when it is guided by careful mediation of the teacher and a good design of the task, which has to take into account the appropriateness of it with respect to the class (Douek & Scali, 2000). As these authors have shown for elementary education, the relationship between doubt and communication through argumentation can serve as the basis to encourage questioning, expression and evolution of conceptualization in mathematics.

Another process in mathematical education is the acquisition of reasoning skills. Several researchers have stressed the psychological gap that separates arguing and proving in the classroom (Schwarz, 2009). In many cases, the presentation of a proof is not persuasive enough to convince a student of its validity (Duval, 1991; Healy & Hoyles, 2000). Argumentative dialogs between students and/or with the teacher may help to bridge this gap because proofs are then conceived as constructions built up through an interactive process that looks for the understanding and the acknowledgment of the student who has to explain all the steps of the inferential process. Nevertheless, careful guidance of the process may be needed to transform spontaneous or even authoritarian interchanges into argumentative situations that are of help to understand it (Atzmon, Hershkowitz & Schwarz, 2006).

The educational system does not facilitate the development of good argumentative practices in higher levels of education. The pressure to cover all the material leaves insufficient time for arguing in mathematics classes and thus students merely assume the value of proof in mathematics, even if they do not fully understand how it works. They assume that the lack of understanding is due to a lack of knowledge. Discussion to promote understanding is not common in mathematics classes in higher education. As soon as the curriculum becomes more advanced, this lack of understanding presents an obstacle to many students and the gap between the application of the theory and the practical work they need to do to solve problems may widen. Repetition of techniques is a typical way to acquire mathematical knowledge. As a consequence, many students do not fully understand what they are doing and fail when a different kind of problem is proposed or integration of different concepts is needed.

Argumentative dialogs can be used to attain different goals depending on the context in which they arise. For example, a mathematical problem in the classroom can be presented as a kind of collaborative task in which two or more parties work together to resolve it. Another goal of an argumentative dialog can be that of developing competences related to critical reasoning. However, how could we relate that to mathematics? For instance, a way of promoting mathematical understanding and avoiding mistakes could be to look at the different solutions proposed to solve a task in order to compare, to relate and to evaluate them. A simple first-order equation can be solved, for instance, by algebraic or by geometric means. The solution may be unique, but comparison of methods may help to improve the critical assessment of mathematical methods.

To sum up, in order to achieve good cognitive development, it is important that the student learns to argue, but also that she argues to learn in different contexts with different goals (Andriessen, Baker & Suthers, 2003).

Nevertheless, not all the dialogical attempts to use argumentation as a collaborative method to solve a problem and to improve understanding are successful (Andriessen, 2009). As empirical research shows (Douek, 2005), mediation by an instructor may be needed to trigger productive argumentative practices. For example, it could be necessary to question statements that are not really helpful in understanding a problem, to integrate discussions or arguments provided by different students, and to generate and integrate new statements, among others. As a consequence, it is important that teachers have pedagogical and theoretical skills to foster argumentation in the classroom. Moreover, the activities need to be well designed and the implementation of a good design for an argumentative mathematical task may need to include some of the sociological characteristics of the group for which the task is designed. Only in this way can argumentation serve as a tool to promote understanding, to reinforce reasoning skills and as an efficient method to achieve good results in mathematics.

The discovery part of a proof is possibly the most difficult phase of any mathematical work. As Kerber & Pollet (2007, p. 87) state, deduction systems may be suitable as proof checkers in many cases, but lack the capacity to act as proof assistants for the exploration and construction of new mathematical knowledge. Argumentation theory and direct observation of real mathematical practice may be

of help to design protocols to facilitate mathematical work, but as Pease & Martin (2012) remark, we are still a long way from a system that could contribute in a human-like manner to a mathematical discussion that has the goal of solving a problem.

5. CONCLUSIONS

In this paper, I have presented several considerations from mathematics to try to show some of the differences between the product of a practice and the practice itself that are not reflected by careful analysis and evaluation of its products.

Much of what is done in mathematics is informal in the sense that it is not done in a pure formal system. In practice, in mathematical proofs, there are gaps and appeals to intuition (by the use of diagrams, for example), and proofs are not fully formalized. Controversies occur and are in practice dealt with without fully formalizing them. However, standards of rigor are specific and additional requirements of mathematical practice and proofs are always achieved and checked by the mathematical community according to those standards.

Proofs arise in dialogical contexts (even when thinking up a proof to convince oneself). Doubt is always present in the period of discovery of a proof or while looking for the solution to a problem. As a consequence, in the process of proving, argumentation, as in ordinary contexts, is always present. As Pólya (1954) stated, "we secure our mathematical knowledge by demonstrative reasoning, but we support our conjectures by plausible reasoning" (p. vi).

Presentation of mathematical products is a communicative act and, as usual in such acts, not all the communicational elements are made explicit. In mathematical proofs, gaps are intentionally left, but those gaps need not correspond to faulty inferential steps (Fallis, 2003). In many cases, several steps of the inferential process are left out to facilitate communication and to adapt to the context. For example, a long proof with all the small inferential steps made explicit may be boring for working mathematicians. An outline of the proof or of the problem may be sufficient and more informative than a complete proof in a classroom or in a scientific meeting. We can express the difference by saying that, in this case, we are making someone see the proof versus letting someone know the proof (Vega, 1999).

Rhetorical elements to persuade may and should be part of the process of communicating a mathematical result, but the dialectical component may or may not be present. For example, in a presentation of a mathematical result, there can be (or not) demands for a better explanation of it, and there can be (or not) requests for a better, clearer or more detailed display of the steps in the proof or some of the concepts involved in it. The appeal to diagrams, images, analogies or rhetorical figures can be not only of help but even a must in order to make the result understandable and, as a consequence, acceptable for the (mathematical) audience. As a result doubts and even a display of counterexamples or a rebuttal of the proof can occur, that is, an argumentative dialog may begin, but does not have to.

Mathematical practice is complex and, in many cases, collaborative work can be helpful to advance towards comprehension and solution of a problem. This is

particularly clear in classroom settings, but it can also be seen in contexts involving more advanced mathematics. The Mini-Polymath projects are a good example of collaborative work over the internet to solve difficult conjectures and open problems in mathematics (Pease & Martin, 2011). When collaboration is undertaken, argumentation is always present and may help to accomplish many mathematical tasks that go beyond those of analyzing and evaluating a proof. The use of argumentative diagrams may also be useful to organize the process towards the proof. As Pease and Martin state, careful consideration should be given to mathematical practice in order to design protocols that help in a human-like manner to improve mathematical thinking. To advance in this direction, more attention should be paid to the different contexts in which practice is undertaken in order to look for special requirements that apply in those contexts. Social dimensions of practice should be considered if we want to construct better ways of arguing and, as a consequence, of thinking, including in mathematics.

REFERENCES

Aberdein, A. (2005). The Uses of Argument in Mathematics. *Argumentation, 19,* 287-301.

Aberdein, A. (2009). Mathematics and Argumentation. *Foundations of Science, 14,* 1-8.

Aberdein, A. (2010). Observations on sick mathematics. In B. van Kerkhove, J. P. van Bendegem, & J. de Vuyst (Eds.), *Philosophical Perspectives on Mathematical Practices* (pp. 269–300). London: College Publications.

Aberdein, A. (2011). The Dialectical Tier of Mathematical Proof. In F. Zenker (Ed.), *Argumentation: Cognition & Community. Proceedings of the 9th International Conference of the Ontario Society for the Study of Argumentation (OSSA), May 18-21*. Windsor: OSSA.

Aberdein, A. (2012). The Parallel Structure of Mathematical Reasoning. In A. Pease, & B. Larvor (Eds.), *Proceedings of the Symposium on Mathematical Practice and Cognition II: A symposium at the AISB/IACAP World Congress 2012* (pp. 7–14). Birmingham: Society for the Study of Artificial Intelligence and the Simulation of Behaviour.

Alcolea Banegas, J. (1998). L'Argumentació en Matemàtiques. In E. Casaban i Moya (Ed.), *XIIè Congrés Valencià de Filosofia* (pp. 135–147). Valencià. Translation online at http://my.fit.edu/%E7aberdein/Alcolea.pdf

Andriessen J. (2009). Argumentation in Higher Education: Examples of Actual Practices with Argumentation Tools. In N. Muller Mirza, & A.-N. Perret-Clermont (Eds.), *Argumentation and Education: Theoretical Foundations and Practice* (pp. 195-213). Dordrecht: Springer.

Andriessen, J., Baker, M., & Suthers D. (Eds.). (2003). *Arguing to Learn: Confronting Cognitions in Computer-Supported Collaborative Learning Environments.* Dordrecht: Kluwer.

Atzmon, S., Hershkowitz R., & Schwarz, B. B. (2006). The role of teachers in turning claims to arguments. In J. Novotna (Ed.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education, Vol. 5* (pp. 65-72). Prague.

Balacheff, N. (2010). Bridging Knowing and Proving in Mathematics: A Didactical Perspective**.** In G. Hanna, H. N. Jahnke, & H. Pulte. (Eds.), *Explanation and Proof in Mathematics: Philosophical and Educational Perspectives* (pp. 115-135). Dordrecht: Springer Science

Bendegem, J. P van, & Kerkhove, B. van (2009). Mathematical Arguments in Context. *Foundations of Science, 14,* 45-57.

Chazan, D. (1990). Quasi-empirical views of mathematics and mathematics teaching. *Interchange, 21(1)*, 14-23.

Coleman, E. (2009). The Surveyablity of Long Proofs. *Foundations of Science, 14,* 27-43.

Douek, N., & Scali, E. (2000). About argumentation and conceptualisation. *Proceedings of PME-XXIV*, *Vol. 2*: 249–256. Hiroshima.

Douek, N. (2005). Communication in the Mathematics Classroom. Argumentation and Development of Mathematical Knowledge. In A. Chronaki, & I. M. Christiansen (Eds.), *Challenging Perspectives on Mathematics Classroom Communication* (pp. 145-172). Charlotte, NC: Information Age Pb.

Dove, I. J. (2009). Towards a Theory of Mathematical Argument. *Foundations of Science, 14,* 137-152.

Duval, R. (1991). Structure du raisonnement déductif et apprentissage de la démonstration. *Educational Studies in Mathematics, 22(3)*, 233-261

Eemeren, F.H. van, Grootendorst, R., Snoeck Henkemans, F., Blair, J. A., Johnson, R. H., Krabbe, E. C. W., ... Zarefsky, D. (1996). *Fundamentals of Argumentation Theory. A Handbook of Historical Backgrounds and Contemporary Developments.* Mahwah, NJ: Erlbaum.

Eemeren, F.H. van, & Houtlosser, P. (2005). Theoretical construction and argumentative reality: an analytic model of critical discussion and conventionalised types of argumentative activity. In D. Hitchcock, & D. Farr (Eds.), *The Uses of Argument. Proceedings of a Conference at McMaster University 18–21 May 2005* (pp. 75-84). Hamilton: Ontario Society for the Study of Argumentation.

Epstein, R.L. (2012). Mathematics as the art of abstraction. In A. Aberdein, & I. J. Dove (Eds.), *The argument of mathematics* (pp. 249-279). Dordrecht: Springer.

Fallis, D. (2003). Intentional Gaps in Mathematical Proofs. *Synthese, 134,* 45-69.

Ferreirós, J. (2010). Mathematical Knowledge and the Interplay of Practices. In M. Suárez, M. Dorato, & M. Rédei (Eds.), *EPSA Philosophical Issues in Science* (pp. 55-64). Dordrecht: Springer.

Gabbay, D., & Woods J. (2005). The practical turn in Logic. In D. Gabbay & G. Guenthner (Eds.), *Handbook of Philosophical Logic. Vol. 13* (pp.15-122). The Netherlands: Springer.

Hanna, G. (2000). Proof, explanation and exploration: an overview. *Educational Studies in Mathematics, 44*, 5–23.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31(4),* 396–428

Hitchcock, D. (2002). The Practice of Argumentative Discussion. *Argumentation, 16,* 287-298.

Johnson, R. H. (2000). *Manifest rationality. A pragmatic theory of argument.* Mahwah, NJ: Lawrence Erlbaum Associates.

Johnson, R. H. (2005). Theory and practice again: Challenges from Pinto and Toulmin. In D. Hitchcock, & D. Farr (Eds.), *The uses of argument: Proceedings of a conference at McMaster University 18–21 May 2005* (pp. 222-231). Hamilton: Ontario Society for the Study of Argumentation.

Kvernbekk, T. (2012). Argumentation in Theory and Practice: Gap or Equilibrium? *Informal Logic, 32(3)*, 288-305

Kerber, M., & Pollet, M. (2007). Informal and Formal Representations in Mathematics. *Studies in Logic, Grammar and Rhetoric, 10 (23),* 75-94.

Krabbe, E. C. W. (2008). Strategic maneuvering in Mathematical Proofs. *Argumentation,* 22, 453-468.

Kuhn, D. (1992). Thinking as argument. *Harvard Educational Review, 62(2)*, 155-178.

Lakatos, I. (1976). *Proofs and Refutations.* Cambridge, MA: Cambridge University Press.

Larvor, B. (2012). What Philosophy of Mathematical Practice Can Teach Argumentation Theory about Diagrams and Pictures. In A. Aberdein, & I. J. Dove (Eds.), *The argument of mathematics* (pp. 209-222). Dordrecht: Springer.

Mancosu, P. (2011). Explanation in Mathematics. In E. N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy*. *(Summer 2011 Edition),* <http://plato.stanford.edu/archives/sum2011/entries/mathematics-explanation/>

Mancosu, P. (Ed.). (2008). *The Philosophy of Mathematical Practice.* Oxford: Oxford University Press.

Maxwell, E. A. (1959). *Fallacies in Mathematics*. Cambridge: Cambridge University Press.

Muller Mirza N., & Perret-Clermont A.-N (Eds.). (2009). *Argumentation and Education: Theoretical foundations and Practices.* Dordrecht: Springer.

Pease, A., & Martin U. (2012). Seventy four minutes of mathematics: An analysis of the third Mini-Polymath project. *Symposium on Mathematical Practice and Cognition II: A symposium at the*

AISB/IACAP World Congress 2012. Birmingham: Society for the Study of Artificial Intelligence and the Simulation of Behaviour.

Pinto, R.C. (2001). *Argument, Inference and Dialectic.* Dordrecht: Kluwer Academic Publishers.

Pólya, G. (1945). *How to solve it.* Princeton, NJ: Princeton University Press.

Pólya, G. (1954). *Mathematics and Plausible Reasoning*. Two Volumes. Princeton, NJ: Princeton University Press.

Schwarz, B. B. (2009). Argumentation and Learning. In N. Muller Mirza, & A.-N. Perret-Clermont (Eds.), *Argumentation and Education: Theoretical Foundations and Practice* (pp. 91-126). Dordrecht: Springer.

Tao, T. (2011). *The mini-polymath3 project* <http://polymath_projects.org/2011/07/19/mini-polymath3-project-2011-imo/>

Tindale, Ch. W. (2002). A Concept Divided: Ralph Johnson's Definition of Argument. *Argumentation, 16,* 209-309.

Thurston, W. P. (1994). On Proof and Progress in Mathematics. *Bulletin of the American Mathematical Society, 30,* 161-177.

Toulmin, S. E. (1958). *The Uses of Argument*. Cambridge: Cambridge University Press.

Vega Reñón, L. (1999). Hacer ver, hacer saber (El rigor informal de las pruebas matemáticas clásicas). *X Congreso Nacional de filosofía de la Asociación Filosófica Argentina.* Universidad nacional de Córdoba.

Walton, D.N., Reed, Ch., & Macagno, F. (2008). *Argumentation Schemes*. Cambridge: Cambridge University Press.

Walton, D.N., & Krabbe, E.C.W. (1995). *Commitment in dialogue: Basic concepts of interpersonal reasoning.* Albany, NY: State University of New York Press.