Summer 2012

Goal programming approaches to deriving interval fuzzy preference relations

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Goal Programming Approaches to Deriving Interval Weights Based on Interval Fuzzy Preference Relations

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Abstract

This article investigates the consistency of interval fuzzy preference relations based on interval arithmetic, and new definitions are introduced for additive consistent, multiplicative consistent and weakly transitive interval fuzzy preference relations. Transformation functions are put forward to convert normalized interval weights into consistent interval fuzzy preference relations. By analyzing the relationship between interval weights and consistent interval fuzzy preference relations, goal-programming-based models are developed for deriving interval weights from interval fuzzy preference relations for both individual and group decision-making situations. The proposed models are illustrated by a numerical example and an international exchange doctoral student selection problem.

Keywords: Interval fuzzy preference relations, Additive transitivity, Multiplicative transitivity, Goal programming, Interval weights

1. Introduction

Since fuzzy logic was first introduced by Zadeh [63], it has become an alternative framework to tackle uncertainty and an indispensable tool in approximate reasoning and artificial intelligence [64, 65]. Along with other biology-inspired approaches such as artificial neural networks and evolutionary computing, fuzzy logic has greatly contributed to the flourishing development of soft computing technologies [25]. Recent years have witnessed numerous successful applications of soft computing tools in a host of areas ranging from intelligent systems design [25, 34], to environmental and water resources management [9, 29, 30, 33, 52, 53] as well as decision support [2, 36]. Among these applications, an important branch is to develop decision models within the fuzzy logic framework.

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Preference relations are among the most common ways to represent information for decision making problems. In multiple attribute decision making (MADM), the decision-maker (DM) generally needs to compare a set of n decision alternatives with respect to each attribute and construct a preference relation, then certain techniques are applied to derive aggregated weights based on individual preference relations. One widely used preference relation takes the multiplicative form, which was introduced by Saaty [38] to represent pairwise comparison data in the analytic hierarchy process (AHP). Since its inception, AHP has emerged as a key MADM approach and has been extensively and intensively studied [43]. The AHP has also been extended to the fuzzy environment [13, 15, 26, 27, 44] and group decision making with information granularity [37] and been applied to such areas as safety management [13], risk management [3], military personnel assignment [26]. Another commonly used preference relation takes the fuzzy form, and significant research [1, 10-12, 14, 16-18, 20-24, 31, 32, 35, 37, 42, 47, 48, 54, 59, 62] has been conducted to deal with fuzzy preference relations. One line of research on fuzzy preference relations is to investigate basic concepts and consistency properties and apply them to decision-making processes [10, 12, 18, 20-23, 35, 37, 42, 47, 59, 60]. Another active research topic is to examine the derivation of priority (weight) vectors based on fuzzy preference relations. For example, Xu and Da [62] propose a least deviation method to obtain a priority vector from a fuzzy preference relation; Wang and Fan [47] apply the logarithmic and geometric least squares methods to deal with the group decision analysis problems with fuzzy preference relations; Wang et al. [48] propose a chi-square method for obtaining a priority vector from multiplicative and fuzzy preference relations.

Due to the complexity and uncertainty involved in many real-world decision problems, it is sometimes unrealistic or impossible to acquire exact judgment data. As such, researchers have extended the MADM framework to accommodate decision situations where judgment data are expressed as intervals, fuzzy intervals [7], intuitionistic fuzzy numbers [8, 28, 58], or interval-valued intuitionistic fuzzy numbers [50, 51]. In the context of fuzzy preference relations, instead of demanding exact fuzzy numbers, a natural extension is to allow for interval fuzzy judgment. Researchers have started examining interval preference relations, such as interval multiplicative preference relations for pairwise comparison matrices [32, 39, 41, 46, 49] and interval fuzzy preference relations [1, 19, 20, 55, 57, 61].

For interval multiplicative preference relations where pairwise comparison matrices consist of interval values, a large body of literature has been developed over the years [4, 5, 32, 39, 41, 45, 46, 48, 49]. As Wang and Elhag [46] point out, an interval comparison matrix is expected to yield an interval weight. By following this guideline, Wang and Elhag [46] put forward a goal-programming approach to deriving interval weights based on a consistent or inconsistent interval comparison matrix. For an excellent overview of interval multiplicative preference relations, readers are referred to Wang and Elhag [46] and Xu [57].

For interval fuzzy preference relations where judgment data are expressed as interval fuzzy numbers, Xu [56] introduces the notion of compatibility degree and compatibility index for two interval fuzzy preference relations and analyzes the compatibility of interval fuzzy preference relations in group decision making. Herrera et al. [20] put forward an aggregation mechanism for group decision making that is able to handle hybrid information consisting of fuzzy binary preference relations, interval-valued preference relations and fuzzy linguistic relations. Xu and Chen [61] define additive and multiplicative consistent interval fuzzy preference relations based on crisp normalized weights, and establish some models for deriving priority weights from consistent or inconsistent interval fuzzy preference relations.

It is well known that the definitions of consistency play an important role in MADM with preference relations. When crisp preference relations are concerned, crisp arithmetic is employed to examine their consistency and crisp weights are derived. If preference relations are interval-valued, it is natural and logical to expect that interval arithmetic be used and interval weights be generated. As Wang and Elhag [46] and the literature review therein indicate, many existing approaches to handling interval data are only applicable to multiplicative preference relations. Although Xu and Chen's approach [61] is able to obtain interval weights from consistent or inconsistent interval fuzzy preference relations, their consistency definitions are based on crisp weights and the interval weight derivation process requires solving $2n+1$ linear programs (LPs). This paper focuses on interval fuzzy preference relations and employs interval arithmetic to define additive and multiplicative consistency of interval fuzzy preference relations. Based on the principle of minimizing deviations from additive and multiplicative consistency, two goal-programming approaches are developed to derive interval priority weights for decision problems for a single DM, where only one LP model has to be solved in each case. These two approaches are then extended to group decision making situations.

The rest of the paper is organized as follows. Section 2 provides preliminary background on fuzzy preference relations, comparisons and ranking of interval weights. Section 3 introduces new definitions of additive and multiplicative consistent interval fuzzy preference relations and their properties. In Section 4, goal-programming models are developed for deriving interval weights based on interval fuzzy preference relations for both individual and group decision making problems. An illustrative example is presented and the ranking result is compared with an existing approach in Section 5. Section 6 furnishes a case study on the international exchange doctoral student selection problem. The paper concludes with some remarks in Section 7.

2. Preliminaries

2.1 Consistent fuzzy preference relations

Consider an MADM problem with a finite set of n attributes or alternatives. Let $X=\{x_1,x_2,...,x_n\}$ be a finite set of attributes or alternatives. Without loss of generality, hereafter we refer to X as an alternative set. Fuzzy preference relations provide a DM with values between 0 and 1, representing the DM's varying degrees of preference for one alternative over another.

A fuzzy preference relation [35] R on the set X is a fuzzy subset of $X \times X$ characterized by a complementary matrix $R=(r_{ij})_{n\times n}$ with

$$0 \le r_{ij} \le 1, r_{ij}+r_{ji}=1, r_{ii}=0.5 \text{ for all } i,j=1,2,...,n \tag{2.1}$$

where r_{ij} represents the DM's preference ratio of alternative x_i over x_j. Especially, $r_{ij}=0.5$ means that the DM is indifferent between x_i and x_j, $r_{ij}=1$ indicates that x_i is definitely preferred to x_j and $r_{ij}=0$ signifies that x_j is definitely preferred to x_i, and $r_{ij}>0.5$ shows that x_i is preferred to x_j to a certain degree.

Tanino [42] proposes the definition of consistency for fuzzy preference relations and introduces additive and multiplicative transitivity conditions.

A fuzzy preference relation $R=(r_{ij})_{n\times n}$ is called additive consistent, if it satisfies [11, 23, 42, 57]:

$$r_{ij}=r_{ik}-r_{jk}+0.5 \qquad \text{for all } i,j,k=1,2,...,n \tag{2.2}$$

Since $r_{ij}=1-r_{ji}$ for all $i,j=1,2,...,n$, one can obtain

$$r_{ij}+r_{jk}+r_{ki}=r_{kj}+r_{ji}+r_{ik} \qquad \text{for all } i,j,k=1,2,...,n \tag{2.3}$$

It has been found that, for a fuzzy preference relation $R=(r_{ij})_{n\times n}$, if there exists a weight vector $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$, $\sum_{i=1}^{n} \omega_i = 1$ and $\omega_i \geq 0$ for $i = 1, 2, ..., n$, such that

$$r_{ij} = 0.5(\omega_i - \omega_j) + 0.5 \qquad \text{for all } i, j = 1, 2, ..., n \tag{2.4}$$

then R is additive consistent [11, 31, 55, 57].

A fuzzy preference relation $R = (r_{ij})_{n \times n}$ is called multiplicative consistent, if it satisfies [23, 42, 57]:

$$\frac{r_{ik}}{r_{ki}} \frac{r_{kj}}{r_{jk}} = \frac{r_{ij}}{r_{ji}} \qquad \text{for all } i, j, k = 1, 2, ..., n \tag{2.5}$$

Similarly, it has been pointed out that if there exists a weight vector $W = (\omega_1, \omega_2, ..., \omega_n)^T$, $\sum_{i=1}^{n} \omega_i = 1$ and $\omega_i \geq 0$ for $i = 1, 2, ..., n$, such that

$$r_{ij} = \frac{\omega_i}{\omega_i + \omega_j} \qquad \text{for all } i, j = 1, 2, ..., n \tag{2.6}$$

then R is multiplicative consistent [55, 57].

As $r_{ij} = 1 - r_{ji}$ for all $i, j = 1, 2, ..., n$, from (2.5), we have

$$\frac{r_{ji}}{r_{ij}} \frac{r_{kj}}{r_{jk}} \frac{r_{ik}}{r_{ki}} = \frac{r_{jk}}{r_{kj}} \frac{r_{ij}}{r_{ji}} \frac{r_{ki}}{r_{ik}} \qquad \text{for all } i, j, k = 1, 2, ..., n \tag{2.7}$$

A fuzzy preference relation $R = (r_{ij})_{n \times n}$ is called weakly transitive if $r_{ij} \geq 0.5$ and $r_{jk} \geq 0.5$ imply $r_{ik} \geq 0.5$ for all $i, j, k = 1, 2, ..., n$.

2.2 Comparison and ranking of interval weights

The commonly used comparison of interval weights is based on interval arithmetic. Given any two interval numbers $\overline{a} = [a^-, a^+]$ and $\overline{b} = [b^-, b^+]$, where $a^-, b^- \geq 0$, arithmetic operations of $\overline{a}$ and $\overline{b}$ can be summarized as follows:

(1) $\overline{a} + \overline{b} = [a^- + b^-, a^+ + b^+]$;

(2) $\overline{a} - \overline{b} = [a^- - b^+, a^+ - b^-]$;

(3) $\overline{a} \times \overline{b} = [a^- b^-, a^+ b^+]$

(4) $\frac{\overline{a}}{\overline{b}} = [\frac{a^-}{b^+}, \frac{a^+}{b^-}]$.

Let $\overline{\omega}_i = [\omega_i^-, \omega_i^+]$ be an interval weight, $i = 1, 2, ..., n$. To compare two interval weights, we refer to the notion of likelihood of one interval weight being greater than another. Denote $\overline{\omega}_i \geq \overline{\omega}_j$, indicating that $\overline{\omega}_i$ is no smaller than $\overline{\omega}_j$. The likelihood of $\overline{\omega}_i \geq \overline{\omega}_j$ is defined as [49]

$$p(\overline{\omega}_i \geq \overline{\omega}_j) = \frac{\max\{0, \omega_i^+ - \omega_j^-\} - \max\{0, \omega_i^- - \omega_j^+\}}{\omega_i^+ - \omega_i^- + \omega_j^+ - \omega_j^-} \tag{2.9}$$

It is obvious that $0 \leq p(\overline{\omega}_i \geq \overline{\omega}_j) \leq 1$ and $p(\overline{\omega}_i \geq \overline{\omega}_j) + p(\overline{\omega}_j \geq \overline{\omega}_i) = 1$. Especially, $p(\overline{\omega}_i \geq \overline{\omega}_i) = 0.5$.

The likelihood $p(\overline{\omega}_i \geq \overline{\omega}_j)$ possesses some useful properties as summarized below [49, 55, 61]:

(a) $p(\overline{\omega}_i \geq \overline{\omega}_j) = 1$ if and only if $\omega_i^- \geq \omega_j^+$;

(b) $p(\overline{\omega}_i \geq \overline{\omega}_j) = 0$ if and only if $\omega_i^+ \leq \omega_j^-$;

(c) $p(\overline{\omega}_i \geq \overline{\omega}_j) \geq 0.5$ if and only if $\frac{\omega_i^- + \omega_i^+}{2} \geq \frac{\omega_j^- + \omega_j^+}{2}$. Especially, $p(\overline{\omega}_i \geq \overline{\omega}_j) = 0.5$ if and only if $\frac{\omega_i^- + \omega_i^+}{2} = \frac{\omega_j^- + \omega_j^+}{2}$;

(d) Let $\overline{\omega}_i, \overline{\omega}_j$ and $\overline{\omega}_k$ be three interval weights, if $p(\overline{\omega}_i \geq \overline{\omega}_j) \geq 0.5$ and $p(\overline{\omega}_j \geq \overline{\omega}_k) \geq 0.5$, then $p(\overline{\omega}_i \geq \overline{\omega}_k) \geq 0.5$.

Properties (a) and (b) show that if two interval weights do not overlap, then the one on the upper end will 100 percent dominate the one on the lower end. Property (c) demonstrates how to compare two interval weights when the two intervals overlap. Property (d) indicates that the likelihood concept is transitive.

This likelihood makes it possible to compare any two interval weights, and the following steps are needed to rank a set of interval weights.

Step 1. Calculate the likelihood $p(\overline{\omega}_i \geq \overline{\omega}_j)$ for interval weights $\overline{\omega}_i$ and $\overline{\omega}_j$ ($i, j = 1, 2, ..., n$) by using (2.9), and construct the likelihood matrix $P = (p_{ij})_{n \times n}$, $p_{ij} = p(\overline{\omega}_i \geq \overline{\omega}_j)$.

Step 2. Determine the optimal degree θ_i of membership for interval weights $\overline{\omega}_i$ ($i = 1, 2, ..., n$) as per the following equation [45]:

$$\theta_i = \frac{1}{n(n-1)}\left(\sum_{j=1}^{n} p_{ij} + \frac{n}{2} - 1\right) \tag{2.10}$$

Step 3. Obtain a ranking for all interval weights $\overline{\omega}_i$ ($i = 1, 2, ..., n$) according to a decreasing order of θ_i, and “interval weight $\bar{\omega}_i$ being superior to $\bar{\omega}_j$” is denoted by $\bar{\omega}_i \overset{p(\bar{\omega}_i \ge \bar{\omega}_j)}{\succeq} \bar{\omega}_j$.

3. Consistency of interval fuzzy preference relations

This section puts forward the definitions of additive and multiplicative consistent interval fuzzy preference relations based on interval arithmetic and derives results to tell whether an interval fuzzy preference relation is additive or multiplicative consistent. The concept of weak transitivity is also defined for interval fuzzy preference relations and it is established that certain additive and multiplicative consistent preference relations are always weakly transitive.

Let I be the closed unit interval $I=[0,1]$, $D(I)=\{[a^-,a^+]: a^- \le a^+, a^-, a^+ \in I\}$. For any $x \in I$, define $x=[x,x]$.

Definition 3.1 [20, 56, 57] An interval fuzzy preference relation $\bar{R}$ on the set X is an interval-valued fuzzy subset of $X \times X$ characterized by a matrix $\bar{R}=(\bar{r}_{ij})_{n\times n}$ with

$$\bar{r}_{ij}=[r_{ij}^-, r_{ij}^+] \in D([0,1]), \bar{r}_{ji}=1-\bar{r}_{ij}=[1-r_{ij}^+, 1-r_{ij}^-], \bar{r}_{ii}=[0.5,0.5],\ i, j=1,2,...,n \tag{3.1}$$

where $\bar{r}_{ij}$ indicates the interval-valued fuzzy preference degree of alternative x_i over x_j, and r_{ij}^- and r_{ij}^+ are the lower and upper limits of $\bar{r}_{ij}$, respectively.

Based on the description of consistent fuzzy preference relations and interval arithmetic given in Section 2, we extend the concept of consistency to the situations where the preference values provided by the DM are interval fuzzy numbers.

Definition 3.2 An interval fuzzy preference relation $\bar{R}=(\bar{r}_{ij})_{n\times n}$ is called additive consistent, if the following additive transitivity is satisfied

$$\bar{r}_{ij}+\bar{r}_{jk}+\bar{r}_{ki}=\bar{r}_{kj}+\bar{r}_{ji}+\bar{r}_{ik} \qquad \text{for all } i, j, k=1,2,...,n \tag{3.2}$$

Definition 3.3 An interval fuzzy preference relation $\bar{R}=(\bar{r}_{ij})_{n\times n}$ is called multiplicative consistent, if the following multiplicative transitivity is satisfied

$$\left(\frac{\bar{r}_{ji}}{\bar{r}_{ij}}\right)\times\left(\frac{\bar{r}_{kj}}{\bar{r}_{jk}}\right)\times\left(\frac{\bar{r}_{ik}}{\bar{r}_{ki}}\right)=\left(\frac{\bar{r}_{jk}}{\bar{r}_{kj}}\right)\times\left(\frac{\bar{r}_{ij}}{\bar{r}_{ji}}\right)\times\left(\frac{\bar{r}_{ki}}{\bar{r}_{ik}}\right) \qquad \text{for all } i, j, k=1,2,...,n \tag{3.3}$$

Obviously, if all interval numbers $\bar{r}_{ij}$ ($i, j=1,2,...,n$) are reduced to exact real numbers, i.e., $r_{ij}^-=r_{ij}^+$, then the interval fuzzy preference relation becomes a regular fuzzy preference relation, and Eqs. (3.2) and (3.3) are reduced to Eqs. (2.3) and (2.7), respectively.

Note that interval arithmetic is very different from crisp arithmetic in terms of subtraction and division, and many properties of crisp arithmetic do not hold true any more. More specifically, for any interval $\bar{a}$, we often have $\bar{a}-\bar{a}\neq 0$ and $\frac{\bar{a}}{\bar{a}}\neq 1$. For instance, $[0.2,0.4]-[0.2,0.4]=[-0.2,0.2]\neq 0$ and $\frac{[0.2,0.4]}{[0.2,0.4]}=[\frac{2}{4},\frac{4}{2}]\neq 1$. Due to the fact that $\bar{a}-\bar{a}$ does not always yield 0, from (3.1), we cannot derive $\bar{r}_{ij}+\bar{r}_{ji}=1$ any more. For example, let $\bar{r}_{ij}=[0.1,0.2]$, as per (3.1), we have $\bar{r}_{ji}=1-\bar{r}_{ij}=1-[0.1,0.2]=[1,1]-[0.1,0.2]=[0.8,0.9]$, but $\bar{r}_{ij}+\bar{r}_{ji}=[0.1,0.2]+[0.8,0.9]=[0.9,1.1]\neq[1,1]=1$.

Moreover, due to the possibility of $\bar{a}-\bar{a}\neq 0$, which makes it impossible to manipulate an interval-valued equation by moving terms from one side to the other, (3.2) may not necessarily be able to produce equation $\bar{r}_{ij}=\bar{r}_{ik}-\bar{r}_{jk}+0.5$ in contrast to the case of regular fuzzy preference relations where these two expressions are equivalent. Consider, for example, the following interval fuzzy preference values: $\bar{r}_{12}=[0.4,0.5],\bar{r}_{13}=[0.35,0.45],\bar{r}_{23}=[0.4,0.5]$, based on (3.1), one can easily derive $\bar{r}_{21}=1-[0.4,0.5]=[0.5,0.6]$, $\bar{r}_{31}=[0.55,0.65],\bar{r}_{32}=[0.5,0.6]$. By applying the interval addition, one can verify that $\bar{r}_{12}+\bar{r}_{23}+\bar{r}_{31}=[1.35,1.65]=\bar{r}_{32}+\bar{r}_{21}+\bar{r}_{13}$, satisfying the additive transitivity condition (3.2). However, this condition does not lead to $\bar{r}_{13}-\bar{r}_{23}+0.5=[0.35,0.55]\neq[0.4,0.5]=\bar{r}_{12}$ any more .

Similarly, due to the possibility of $\frac{\bar{a}}{\bar{a}}\neq 1$ for intervals, (3.3) is not equivalent to $\left(\frac{\bar{r}_{ik}}{\bar{r}_{ki}}\right)\times\left(\frac{\bar{r}_{kj}}{\bar{r}_{jk}}\right)=\frac{\bar{r}_{ij}}{\bar{r}_{ji}}$ as in the case of regular fuzzy preference relations. For example, let $\bar{r}_{12}=[\frac{1}{4},\frac{1}{2}]$, $\bar{r}_{13}=[\frac{1}{5},\frac{2}{5}]$, $\bar{r}_{23}=[\frac{1}{3},\frac{1}{2}]$, as per (3.1), we have $\bar{r}_{21}=[\frac{1}{2},\frac{3}{4}],\bar{r}_{31}=[\frac{3}{5},\frac{4}{5}]$, $\bar{r}_{32}=[\frac{1}{2},\frac{2}{3}]$. It is easy to verify that the multiplicative transitivity condition (3.3) is satisfied, $\left(\frac{\bar{r}_{21}}{\bar{r}_{12}}\right)\times\left(\frac{\bar{r}_{32}}{\bar{r}_{23}}\right)\times\left(\frac{\bar{r}_{13}}{\bar{r}_{31}}\right)=[\frac{1}{4},4]=\left(\frac{\bar{r}_{23}}{\bar{r}_{32}}\right)\times\left(\frac{\bar{r}_{12}}{\bar{r}_{21}}\right)\times\left(\frac{\bar{r}_{31}}{\bar{r}_{13}}\right)$, but $\left(\frac{\bar{r}_{13}}{\bar{r}_{31}}\right)\times\left(\frac{\bar{r}_{32}}{\bar{r}_{23}}\right)=[\frac{1}{4},\frac{4}{3}]\neq[\frac{1}{3},1]=\left(\frac{\bar{r}_{12}}{\bar{r}_{21}}\right)$.

From Definition 3.1, we understand that $\bar{r}_{ij}$ gives the interval fuzzy preference degree of the alternative x_i over x_j, the greater $\bar{r}_{ij}$, the stronger the preference of alternative x_i over x_j; $\bar{r}_{ij} = [0.5, 0.5]$ denotes indifference between x_i and x_j. The preference information reflected in $\bar{R}$ is a result of pairwise comparisons among n alternatives. A mechanism is needed to aggregate this pairwise comparison matrix into a priority weight vector so that the DM can rank the alternatives based on the aggregated weights. As the input information in $\bar{R}$ is interval-valued, it is reasonable to expect that the aggregated weights be also interval-valued rather than real-valued [46].

Let $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n)^T = ([\omega_1^-, \omega_1^+], [\omega_2^-, \omega_2^+], ..., [\omega_n^-, \omega_n^+])^T$ be a normalized interval weight vector [41] with

$$0 \le \omega_i^- \le \omega_i^+ \le 1, \sum_{\substack{j=1 \\ j \ne i}}^{n} \omega_j^- + \omega_i^+ \le 1, \omega_i^- + \sum_{\substack{j=1 \\ j \ne i}}^{n} \omega_j^+ \ge 1 \qquad i = 1, 2, ..., n \qquad (3.4)$$

then the interval preference intensity of alternative x_i over alternative x_j, $\bar{p}_{ij}$, is given by the following transformation function

$$\bar{p}_{ij} = \varphi(\bar{\omega}_i, \bar{\omega}_j) = \begin{cases} [0.5, 0.5] & i = j \\ [0.5 + 0.5(\phi(\omega_i^-, \omega_j^+) - \phi(\omega_j^+, \omega_i^-)), & i \ne j \\ 0.5 + 0.5(\phi(\omega_i^+, \omega_j^-) - \phi(\omega_j^-, \omega_i^+))] & \end{cases} \qquad (3.5)$$

where $\phi : [0,1] \times [0,1] \to [0,1]$ satisfies (i) $\phi(x, x) = 0.5,\ \forall x \in [0,1]$, and (ii) $\phi(\cdot,\cdot)$ is nondecreasing in the first argument and nonincreasing in the second argument.

Theorem 3.1 Assume that the elements of the transformation matrix $\bar{P} = (\bar{p}_{ij})_{n \times n}$ are defined by (3.5), then $\bar{P}$ is an interval fuzzy preference relation.

Proof. As $0 \le \omega_i^- \le \omega_i^+ \le 1$, $0 \le \omega_j^- \le \omega_j^+ \le 1$ and $\phi(\cdot,\cdot)$ is nondecreasing in the first argument and nonincreasing in the second argument, it follows that $\phi(\omega_i^-, \omega_j^+) \le \phi(\omega_i^+, \omega_j^-)$ and $\phi(\omega_j^+, \omega_i^-) \ge \phi(\omega_j^-, \omega_i^+)$. Moreover, since $0 \le \phi(\cdot,\cdot) \le 1$, we have $\phi(\omega_i^-, \omega_j^+) - \phi(\omega_j^+, \omega_i^-) \ge -1$ and $\phi(\omega_i^+, \omega_j^-) - \phi(\omega_j^-, \omega_i^+) \le 1$. Therefore, it is ascertained that

$$0 \le 0.5 + 0.5(\phi(\omega_i^-, \omega_j^+) - \phi(\omega_j^+, \omega_i^-)) \le 0.5 + 0.5(\phi(\omega_i^+, \omega_j^-) - \phi(\omega_j^-, \omega_i^+)) \le 1.$$

So, we have $\bar{p}_{ij} \in D([0,1])$.

By applying the interval subtraction operation in Section 2, it is easy to verify that $\overline{p}_{ji} = 1 - \overline{p}_{ij}$.

As per Definition 3.1, $\overline{P} = (\overline{p}_{ij})_{n\times n}$ is an interval fuzzy preference relation. ■

Let $\phi(x,y) \triangleq 0.5 + 0.5(f(x) - f(y))$, where $f(\cdot)$ is a nondecreasing continuous function and $0 \le f(z) \le 1, \forall z \in [0,1]$. It is apparent that $\phi(x,x) = 0.5,\ \forall x \in [0,1]$, and $\phi(\cdot,\cdot)$ is nondecreasing in the first argument and nonincreasing in the second argument. By using this function, (3.5) can be expressed as:

$$\overline{p}_{ij} = \begin{cases} [0.5, 0.5] & i = j \\ [0.5 + 0.5(f(\omega_i^-) - f(\omega_j^+)), 0.5 + 0.5(f(\omega_i^+) - f(\omega_j^-)] & i \ne j \end{cases} \tag{3.6}$$

Theorem 3.2 Assume that the elements of the transformation matrix $\overline{P} = (\overline{p}_{ij})_{n\times n}$ are defined by (3.6), then $\overline{P}$ is an additive consistent interval fuzzy preference relation.

Proof. According to Theorem 3.1, it immediately follows that $\overline{P} = (\overline{p}_{ij})_{n\times n}$ is an interval fuzzy preference relation.

By applying the interval addition operation in Section 2, we have

$$\begin{aligned} &\overline{p}_{ij} + \overline{p}_{jk} + \overline{p}_{ki} = [1.5 + 0.5((f(\omega_i^-) - f(\omega_j^+) + f(\omega_j^-) - f(\omega_k^+) + f(\omega_k^-) - f(\omega_i^+)), \\ &1.5 + 0.5((f(\omega_i^+) - f(\omega_j^-) + f(\omega_j^+) - f(\omega_k^-) + f(\omega_k^+) - f(\omega_i^-))] \\ &= [1.5 + 0.5((f(\omega_i^-) + f(\omega_j^-) + f(\omega_k^-) - f(\omega_i^+) - f(\omega_j^+) - f(\omega_k^+)), \\ &1.5 + 0.5(f(\omega_i^+) + f(\omega_j^+) + f(\omega_k^+) - f(\omega_i^-) - f(\omega_j^-) - f(\omega_k^-))] \end{aligned}$$

Similarly,

$$\begin{aligned} &\overline{p}_{kj} + \overline{p}_{ji} + \overline{p}_{ik} = [1.5 + 0.5((f(\omega_k^-) - f(\omega_j^+) + f(\omega_j^-) - f(\omega_i^+) + f(\omega_i^-) - f(\omega_k^+)), \\ &1.5 + 0.5((f(\omega_k^+) - f(\omega_j^-) + f(\omega_j^+) - f(\omega_i^-) + f(\omega_i^+) - f(\omega_k^-))] \\ &= [1.5 + 0.5((f(\omega_i^-) + f(\omega_j^-) + f(\omega_k^-) - f(\omega_i^+) - f(\omega_j^+) - f(\omega_k^+)), \\ &1.5 + 0.5(f(\omega_i^+) + f(\omega_j^+) + f(\omega_k^+) - f(\omega_i^-) - f(\omega_j^-) - f(\omega_k^-))] \end{aligned}$$

As per Definition 3.2, it is verified that $\overline{P} = (\overline{p}_{ij})_{n\times n}$ is additive consistent. ■

On the other hand, if we let

$$\phi(x,y) \triangleq \begin{cases} 0.5 & x = 0, y = 0 \\ \dfrac{s(x)}{s(x) + s(y)} & \text{Otherwise} \end{cases}$$

where $s(\cdot)$ is a nondecreasing continuous function such that $s(0)=0$ and $0 \le s(z) \le 1, \forall z \in [0,1]$. Then, one can verify that $\phi(x,x)=0.5,\ \forall x \in (0,1]$, and $\phi(\cdot,\cdot)$ is nondecreasing in the first argument and nonincreasing in the second argument. In this case, (3.5) can be expressed as:

$$\overline{p}_{ij} = \begin{cases} [0.5, 0.5] & i = j \\ \left[\dfrac{s(\omega_i^-)}{s(\omega_i^-)+s(\omega_j^+)}, \dfrac{s(\omega_i^+)}{s(\omega_i^+)+s(\omega_j^-)} \right] & i \ne j \end{cases} \tag{3.7}$$

Theorem 3.3 Assume that the elements of the transformation matrix $\overline{P}=(\overline{p}_{ij})_{n\times n}$ are defined by (3.7), then $\overline{P}$ is a multiplicative consistent interval fuzzy preference relation.

Proof. By Theorem 3.1, we know that $\overline{P}=(\overline{p}_{ij})_{n\times n}$ is an interval fuzzy preference relation.

Since

$$\begin{aligned} &\left(\frac{\overline{p}_{ji}}{\overline{p}_{ij}}\right) \times \left(\frac{\overline{p}_{kj}}{\overline{p}_{jk}}\right) \times \left(\frac{\overline{p}_{ik}}{\overline{p}_{ki}}\right) = \left[\frac{s(\omega_j^-)}{s(\omega_i^+)}, \frac{s(\omega_j^+)}{s(\omega_i^-)}\right] \times \left[\frac{s(\omega_k^-)}{s(\omega_j^+)}, \frac{s(\omega_k^+)}{s(\omega_j^-)}\right] \times \left[\frac{s(\omega_i^-)}{s(\omega_k^+)}, \frac{s(\omega_i^+)}{s(\omega_k^-)}\right] \\ &= \left[\frac{s(\omega_i^-)s(\omega_j^-)s(\omega_k^-)}{s(\omega_i^+)s(\omega_j^+)s(\omega_k^+)}, \frac{s(\omega_i^+)s(\omega_j^+)s(\omega_k^+)}{s(\omega_i^-)s(\omega_j^-)s(\omega_k^-)}\right] \end{aligned},$$

On the other hand,

$$\begin{aligned} &\left(\frac{\overline{p}_{jk}}{\overline{p}_{kj}}\right) \times \left(\frac{\overline{p}_{ij}}{\overline{p}_{ji}}\right) \times \left(\frac{\overline{p}_{ki}}{\overline{p}_{ik}}\right) = \left[\frac{s(\omega_j^-)}{s(\omega_k^+)}, \frac{s(\omega_j^+)}{s(\omega_k^-)}\right] \times \left[\frac{s(\omega_i^-)}{s(\omega_j^+)}, \frac{s(\omega_i^+)}{s(\omega_j^-)}\right] \times \left[\frac{s(\omega_k^-)}{s(\omega_i^+)}, \frac{s(\omega_k^+)}{s(\omega_i^-)}\right] \\ &= \left[\frac{s(\omega_i^-)s(\omega_j^-)s(\omega_k^-)}{s(\omega_i^+)s(\omega_j^+)s(\omega_k^+)}, \frac{s(\omega_i^+)s(\omega_j^+)s(\omega_k^+)}{s(\omega_i^-)s(\omega_j^-)s(\omega_k^-)}\right] \end{aligned}.$$

By Definition 3.3, we know that $\overline{P}=(\overline{p}_{ij})_{n\times n}$ is multiplicative consistent. ■

Let $f(x) \triangleq x$ and $s(x) \triangleq x$, then $f(x)$ and $s(x)$ are apparently nondecreasing and continuous. Then, (3.6) and (3.7) can be rewritten as:

$$\overline{p}_{ij} = \begin{cases} [0.5, 0.5] & i = j \\ [0.5+0.5(\omega_i^- - \omega_j^+), 0.5+0.5(\omega_i^+ - \omega_j^-)] & i \ne j \end{cases} \tag{3.8}$$

$$\overline{p}_{ij} = \begin{cases} [0.5, 0.5] & i = j \\ \left[\dfrac{\omega_i^-}{\omega_i^- + \omega_j^+}, \dfrac{\omega_i^+}{\omega_i^+ + \omega_j^-} \right] & i \ne j \end{cases} \tag{3.9}$$

By Theorem 3.2, if the elements of $\overline{P}=(\overline{p}_{ij})_{n\times n}$ are defined by (3.8), then $\overline{P}$ is an additive consistent interval fuzzy preference relation. As per Theorem 3.3, if the elements of $\overline{P}=(\overline{p}_{ij})_{n\times n}$ are defined by (3.9), then $\overline{P}$ is a multiplicative consistent interval fuzzy preference relation.

It should be noted that if all interval weights $\overline{\omega}_i$ ($i=1,2,...,n$) are reduced to exact real values, i.e., $\omega_i^-=\omega_i^+$, the interval fuzzy preference relation becomes a regular fuzzy preference relation. In this case, (3.8) and (3.9) are simplified to (2.4) and (2.6), respectively, corresponding to additive and multiplicative consistent fuzzy preference relations.

Based on the aforesaid discussions, we are now ready to introduce the following corollaries.

Corollary 3.1 Let $\overline{R}=(\overline{r}_{ij})_{n\times n}$ be an interval fuzzy preference relation, if there exists a normalized interval weight vector $\overline{\omega}=(\overline{\omega}_1,\overline{\omega}_2,\cdots,\overline{\omega}_n)^T$ such that

$$\overline{r}_{ij}=[r_{ij}^-,r_{ij}^+]=\begin{cases}[0.5,0.5] & i=j\\ [0.5+0.5(\omega_i^--\omega_j^+),0.5+0.5(\omega_i^+-\omega_j^-)] & i\neq j\end{cases} \tag{3.10}$$

where $\overline{\omega}$ satisfies (3.4), then $\overline{R}$ is an additive consistent interval fuzzy preference relation.

Corollary 3.2 Let $\overline{R}=(\overline{r}_{ij})_{n\times n}$ be an interval fuzzy preference relation, if there exists a normalized interval weight vector $\overline{\omega}=(\overline{\omega}_1,\overline{\omega}_2,\cdots,\overline{\omega}_n)^T$ such that

$$\overline{r}_{ij}=[r_{ij}^-,r_{ij}^+]=\begin{cases}[0.5,0.5] & i=j\\ \left[\dfrac{\omega_i^-}{\omega_i^-+\omega_j^+},\dfrac{\omega_i^+}{\omega_i^++\omega_j^-}\right] & i\neq j\end{cases} \tag{3.11}$$

where $\overline{\omega}$ satisfies (3.4), then $\overline{R}$ is a multiplicative consistent interval fuzzy preference relation.

Definition 3.4 An interval fuzzy preference relation $\overline{R}=(\overline{r}_{ij})_{n\times n}$ is weakly transitive if $p(\overline{r}_{ij}\geq[0.5,0.5])\geq 0.5$ and $p(\overline{r}_{jk}\geq[0.5,0.5])\geq 0.5$ imply $p(\overline{r}_{ik}\geq[0.5,0.5])\geq 0.5$, for all $i,j,k=1,2,...,n$.

Theorem 3.4 If an interval fuzzy preference relation $\overline{R}=(\overline{r}_{ij})_{n\times n}$ can be expressed as (3.10), then $\overline{R}$ is weakly transitive.

Proof. If $k=i$ or $k=j$, it is obvious that $p(\overline{r}_{ik}\geq[0.5,0.5])\geq 0.5$.

Let $i\neq j\neq k$. According to property (c) of the likelihood concept in Section 2, if $p(\overline{r}_{ij}\geq[0.5,0.5])\geq 0.5$ and $p(\overline{r}_{jk}\geq[0.5,0.5])\geq 0.5$, we have $r_{ij}^-+r_{ij}^+\geq 1$ and $r_{jk}^-+r_{jk}^+\geq 1$. Since $\overline{R}$

can be expressed as (3.10), it follows that $\omega_i^- + \omega_i^+ \ge \omega_j^- + \omega_j^+$ and $\omega_j^- + \omega_j^+ \ge \omega_k^- + \omega_k^+$. Therefore, we have

$$1 + 0.5(\omega_i^- - \omega_k^+ + \omega_i^+ - \omega_k^-) \ge 1,$$

which is equivalent to $\frac{r_{ik}^- + r_{ik}^+}{2} \ge \frac{0.5+0.5}{2}$. By property (c) of the likelihood concept, the proof of Theorem 3.4 is completed. ■

Theorem 3.5 If an interval fuzzy preference relation $\bar{R} = (\bar{r}_{ij})_{n\times n}$ can be expressed as (3.11), then $\bar{R}$ is weakly transitive.

*Proof***.** If $k = i$ or $k = j$, it is obvious that $p(\bar{r}_{ik} \ge [0.5, 0.5]) \ge 0.5$.

Let $i \ne j \ne k$. According to the likelihood property (c), if $p(\bar{r}_{ij} \ge [0.5, 0.5]) \ge 0.5$ and $p(\bar{r}_{jk} \ge [0.5, 0.5]) \ge 0.5$, we have $r_{ij}^- + r_{ij}^+ \ge 1$ and $r_{jk}^- + r_{jk}^+ \ge 1$. Since $\bar{R}$ can be expressed as (3.11), from $r_{ij}^- + r_{ij}^+ \ge 1$, it follows that

$$\begin{aligned}
&\frac{\omega_i^-}{\omega_i^- + \omega_j^+} + \frac{\omega_i^+}{\omega_i^+ + \omega_j^-} \ge 1\\
&\frac{\omega_i^-}{\omega_i^- + \omega_j^+} \ge 1 - \frac{\omega_i^+}{\omega_i^+ + \omega_j^-} = \frac{\omega_j^-}{\omega_i^+ + \omega_j^-}\\
&\frac{1}{1 + \omega_j^+ / \omega_i^-} \ge \frac{1}{1 + \omega_i^+ / \omega_j^-}\\
&\frac{\omega_i^+}{\omega_j^-} \ge \frac{\omega_j^+}{\omega_i^-}
\end{aligned}$$

In the same way, from $r_{jk}^- + r_{jk}^+ \ge 1$, we have $\frac{\omega_j^+}{\omega_k^-} \ge \frac{\omega_k^+}{\omega_j^-}$. Multiplying these two inequalities, we have

$$\frac{\omega_i^+}{\omega_j^-}\frac{\omega_j^+}{\omega_k^-} \ge \frac{\omega_k^+}{\omega_j^-}\frac{\omega_j^+}{\omega_i^-}.$$

By cancelling $\frac{\omega_j^+}{\omega_j^-}$ on both sides, we get $\frac{\omega_i^+}{\omega_k^-} \ge \frac{\omega_k^+}{\omega_i^-}$. By reversing the aforesaid process of proving $\frac{\omega_i^+}{\omega_j^-} \ge \frac{\omega_j^+}{\omega_i^-}$, one can get

$$\frac{\omega_i^-}{\omega_i^- + \omega_k^+} + \frac{\omega_i^+}{\omega_i^+ + \omega_k^-} \ge 1,$$

implying $r_{ik}^- + r_{ik}^+ \ge 1$, or equivalently, $\frac{r_{ik}^- + r_{ik}^+}{2} \ge \frac{0.5+0.5}{2}$. As per the likelihood property (c), we have $P([r_{ik}^-, r_{ik}^+] \ge [0.5, 0.5]) \ge 0.5$, the proof of Theorem 3.5 is thus completed. ■

4. Goal programming models for generating interval weights

This section develops some goal programming models for deriving interval weights from interval fuzzy preference relations.

4.1 Goal programming models based on additive transitivity

As per Corollary 3.1, if there exists a normalized interval weight vector $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n)^T$, satisfying (3.4) , such that $\bar{R} = (\bar{r}_{ij})_{n \times n}$ can be expressed as (3.10), then $\bar{R}$ is an additive consistent interval fuzzy preference relation. By Theorem 3.4, $\bar{R}$ is also weakly transitive. However, in many real situations, preference relations provided by a DM are often not consistent and, hence, may not be expressed as (3.10). In this case, we turn to seek an interval weight vector $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n)^T$ such that the lower and upper bounds of $\bar{r}_{ij}$ ($i \ne j$) are as close to those of $[0.5 + 0.5(\omega_i^- - \omega_j^+), 0.5 + 0.5(\omega_i^+ - \omega_j^-)]$ as possible, or equivalently, we intend to find an interval weight vector $\bar{\omega}$ such that the deviation of $\bar{R}$ from an additive consistent interval fuzzy preference relation (3.10) is minimized. This modeling principle is consistent with the approaches for real-valued multiplicative and fuzzy preference relations [48, 62] as well as interval-valued multiplicative preference relations [46]. Consequently, the following multi-objective programming model is constructed:

$$\begin{array}{ll} \min & J_{ij} = \left|(0.5 + 0.5(\omega_i^- - \omega_j^+)) - r_{ij}^-\right| + \left|(0.5 + 0.5(\omega_i^+ - \omega_j^-)) - r_{ij}^+\right| \quad \begin{array}{l} i, j = 1, 2, ..., n, \\ i \ne j \end{array} \\ s.t. & 0 \le \omega_i^- \le \omega_i^+ \le 1, \sum_{\substack{j=1 \\ j \ne i}}^{n} \omega_j^- + \omega_i^+ \le 1, \omega_i^- + \sum_{\substack{j=1 \\ j \ne i}}^{n} \omega_j^+ \ge 1 \qquad i = 1, 2, ..., n \end{array} \tag{4.1}$$

Since $\bar{r}_{ji} = 1 - \bar{r}_{ij}$, i.e. $r_{ji}^- = 1 - r_{ij}^+$ and $r_{ji}^+ = 1 - r_{ij}^-$, one can obtain

$$\left|(0.5 + 0.5(\omega_i^- - \omega_j^+)) - r_{ij}^-\right| = \left|(0.5 + 0.5(\omega_j^+ - \omega_i^-)) - r_{ji}^+\right| \quad \text{for } i, j = 1, 2, ..., n, i \ne j.$$

Therefore, instead of examining the deviation from each off-diagonal interval element of $\bar{R}$ in the objective function, we can simplify (4.1) by considering only the upper diagonal elements as shown below:

$$\begin{aligned} \min \quad & J_{ij} = \left|(0.5+0.5(\omega_i^- - \omega_j^+)) - r_{ij}^-\right| + \left|(0.5+0.5(\omega_i^+ - \omega_j^-)) - r_{ij}^+\right| \qquad \begin{matrix} i=1,2,...,n-1, \\ j=i+1,...,n \end{matrix} \\ s.t. \quad & 0 \le \omega_i^- \le \omega_i^+ \le 1, \sum_{\substack{j=1 \\ j\ne i}}^{n} \omega_j^- + \omega_i^+ \le 1, \omega_i^- + \sum_{\substack{j=1 \\ j\ne i}}^{n} \omega_j^+ \ge 1 \qquad i=1,2,...,n \end{aligned} \tag{4.2}$$

Let

$$\xi_{ij} \triangleq (0.5+0.5(\omega_i^- - \omega_j^+)) - r_{ij}^-, \ \eta_{ij} \triangleq (0.5+0.5(\omega_i^+ - \omega_j^-)) - r_{ij}^+ \tag{4.3}$$

$$\xi_{ij}^+ \triangleq \frac{\left|\xi_{ij}\right| + \xi_{ij}}{2}, \ \xi_{ij}^- \triangleq \frac{\left|\xi_{ij}\right| - \xi_{ij}}{2}, \ \eta_{ij}^+ \triangleq \frac{\left|\eta_{ij}\right| + \eta_{ij}}{2}, \ \xi_{ij}^- \triangleq \frac{\left|\eta_{ij}\right| - \eta_{ij}}{2} \tag{4.4}$$

for $i=1,2,...,n-1,\ j=i+1,...,n$

Based on the definitions of ξ_{ij}^+ and ξ_{ij}^-, ξ_{ij} and $\left|\xi_{ij}\right|$ can be expressed as $\xi_{ij} = \xi_{ij}^+ - \xi_{ij}^-$ and $\left|\xi_{ij}\right| = \xi_{ij}^+ + \xi_{ij}^-$, respectively, where $\xi_{ij}^+ \cdot \xi_{ij}^- = 0$ for $i=1,2,...,n-1,\ j=i+1,...,n$. Similarly, η_{ij} and $\left|\eta_{ij}\right|$ can be expressed as $\eta_{ij} = \eta_{ij}^+ - \eta_{ij}^-$ and $\left|\eta_{ij}\right| = \eta_{ij}^+ + \eta_{ij}^-$, respectively, where $\eta_{ij}^+ \cdot \eta_{ij}^- = 0$ for $i=1,2,...,n-1,\ j=i+1,...,n$. Accordingly, the solution to the minimization problem (4.2) can be found by solving the following LP model:

$$\begin{aligned} &\min \quad J = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \lambda_{ij} (\xi_{ij}^+ + \xi_{ij}^- + \eta_{ij}^+ + \eta_{ij}^-) \\ &s.t. \begin{cases} (0.5+0.5(\omega_i^- - \omega_j^+)) - r_{ij}^- - \xi_{ij}^+ + \xi_{ij}^- = 0, & i=1,2,...,n-1,\ j=i+1,...,n \\ (0.5+0.5(\omega_i^+ - \omega_j^-)) - r_{ij}^+ - \eta_{ij}^+ + \eta_{ij}^- = 0, & i=1,2,...,n-1,\ j=i+1,...,n \\ 0 \le \omega_i^- \le \omega_i^+ \le 1, \sum_{\substack{j=1 \\ j\ne i}}^{n} \omega_j^- + \omega_i^+ \le 1, \omega_i^- + \sum_{\substack{j=1 \\ j\ne i}}^{n} \omega_j^+ \ge 1, & i=1,2,...,n \\ \xi_{ij}^+ \ge 0, \xi_{ij}^- \ge 0, \eta_{ij}^+ \ge 0, \eta_{ij}^- \ge 0 & i=1,2,...,n-1,\ j=i+1,...,n \end{cases} \end{aligned} \tag{4.5}$$

where λ_{ij} is the weighting factor corresponding to the goal function J_{ij} ($i=1,2,...,n-1,\ j=i+1,...,n$).

Assume that all individual goal functions (or deviation variables) are equally important, we can then set $\lambda_{ij}=1$, $i=1,2,...,n-1,\ j=i+1,...,n$, and the optimization model (4.5) can be rewritten as

$$
\begin{aligned}
&\min \quad J=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\xi_{ij}^{+}+\xi_{ij}^{-}+\eta_{ij}^{+}+\eta_{ij}^{-})\\
&s.t.\begin{cases}
(0.5+0.5(\omega_i^{-}-\omega_j^{+}))-r_{ij}^{-}-\xi_{ij}^{+}+\xi_{ij}^{-}=0, & i=1,2,\ldots,n-1,\ j=i+1,\ldots,n\\
(0.5+0.5(\omega_i^{+}-\omega_j^{-}))-r_{ij}^{+}-\eta_{ij}^{+}+\eta_{ij}^{-}=0, & i=1,2,\ldots,n-1,\ j=i+1,\ldots,n\\
0\le\omega_i^{-}\le\omega_i^{+}\le 1,\sum_{\substack{j=1\\ j\ne i}}^{n}\omega_j^{-}+\omega_i^{+}\le 1,\omega_i^{-}+\sum_{\substack{j=1\\ j\ne i}}^{n}\omega_j^{+}\ge 1, & i=1,2,\ldots,n\\
\xi_{ij}^{+}\ge 0,\xi_{ij}^{-}\ge 0,\eta_{ij}^{+}\ge 0,\eta_{ij}^{-}\ge 0 & i=1,2,\ldots,n-1,\ j=i+1,\ldots,n
\end{cases}
\end{aligned}
\tag{4.6}
$$

Solving (4.6), an optimal interval weight vector $\bar{\omega}^{*}=(\bar{\omega}_1^{*},\bar{\omega}_2^{*},\cdots,\bar{\omega}_n^{*})^{T}=([\omega_1^{-*},\omega_1^{+*}],[\omega_2^{-*},\omega_2^{+*}],\cdots,[\omega_n^{-*},\omega_n^{+*}])^{T}$ is obtained for the underlying interval fuzzy preference relation.

For an interval fuzzy preference relation $\bar{R}=(\bar{r}_{ij})_{n\times n}$ from which an optimal weight vector $\bar{\omega}$ is derived as given in (4.6), it is apparent that $\bar{R}$ is additive consistent if the objective function value of (4.6) $J^{*}=0$ in the optimal solution. This is natural because $J^{*}=0$ and the non-negativity of the deviation variables, ξ_{ij}^{+}, ξ_{ij}^{-}, η_{ij}^{+} and η_{ij}^{-}, imply that $\xi_{ij}^{+}=\xi_{ij}^{-}=\eta_{ij}^{+}=\eta_{ij}^{-}=0$. As such, the optimal weight vector obtained from (4.6) allows $\bar{R}$ to be expressed as (3.10). As per Corollary 3.1, $\bar{R}$ is additive consistent.

4.2 Goal programming models based on multiplicative transitivity

By Corollary 3.2, if there exists an interval weight vector $\bar{\omega}=(\bar{\omega}_1,\bar{\omega}_2,\cdots,\bar{\omega}_n)^{T}$, satisfying (3.4), such that $\bar{R}=(\bar{r}_{ij})_{n\times n}$ can be expressed as

$$r_{ij}^{-}(\omega_i^{-}+\omega_j^{+})=\omega_i^{-} \qquad i,j=1,2,\ldots,n, i\ne j \tag{4.7}$$

$$r_{ij}^{+}(\omega_i^{+}+\omega_j^{-})=\omega_i^{+} \qquad i,j=1,2,\ldots,n, i\ne j \tag{4.8}$$

Then, $\bar{R}$ is multiplicative consistent. Once again, the preference information provided by the DM may not always be consistent. As such, $\bar{R}$ may not be expressed as (4.7) and (4.8). In this case, (4.7) and (4.8) are relaxed by allowing some deviation, and the deviation from consistency is then minimized. To this end, the following multi-objective programming model is established, where the objectives are to minimize the sum of absolute deviations from the lower and upper bounds of each off-diagonal element in $\bar{R}$ and the constraints ensure that the weight vector satisfies (3.4):

$$
\begin{aligned}
\min \quad & \tilde{J}_{ij} = \left|\omega_i^- - r_{ij}^-(\omega_i^- + \omega_j^+)\right| + \left|\omega_i^+ - r_{ij}^+(\omega_i^+ + \omega_j^-))\right| \quad \begin{matrix} i, j = 1,2,\ldots,n, \\ i \neq j \end{matrix} \\
s.t. \quad & 0 \le \omega_i^- \le \omega_i^+ \le 1, \sum_{j\neq i}^{n} \omega_j^- + \omega_i^+ \le 1, \omega_i^- + \sum_{j \neq i}^{n} \omega_j^+ \ge 1 \quad i = 1,2,\ldots,n
\end{aligned}
\tag{4.9}
$$

As $\overline{r}_{ji} = 1 - \overline{r}_{ij}$, i.e. $r_{ji}^- = 1 - r_{ij}^+$ and $r_{ji}^+ = 1 - r_{ij}^-$, we have

$$\left|\omega_i^- - r_{ij}^-(\omega_i^- + \omega_j^+)\right| = \left|\omega_j^+ - r_{ji}^+(\omega_j^+ + \omega_i^-)\right| \qquad \text{for } i, j = 1,2,\ldots,n, i \neq j .$$

Similar to the treatment in Section 4.1, (4.9) can be simplified as

$$
\begin{aligned}
\min \quad & \tilde{J}_{ij} = \left|\omega_i^- - r_{ij}^-(\omega_i^- + \omega_j^+)\right| + \left|\omega_i^+ - r_{ij}^+(\omega_i^+ + \omega_j^-)\right| \quad \begin{matrix} i = 1,2,\ldots,n-1, \\ j = i+1,\ldots,n \end{matrix} \\
s.t. \quad & 0 \le \omega_i^- \le \omega_i^+ \le 1, \sum_{j\neq i}^{n} \omega_j^- + \omega_i^+ \le 1, \omega_i^- + \sum_{j \neq i}^{n} \omega_j^+ \ge 1 \quad i = 1,2,\ldots,n
\end{aligned}
\tag{4.10}
$$

Let

$$\tilde{\xi}_{ij} = \omega_i^- - r_{ij}^-(\omega_i^- + \omega_j^+),\ \tilde{\eta}_{ij} = \omega_i^+ - r_{ij}^+(\omega_i^+ + \omega_j^-) \tag{4.11}$$

$$\tilde{\xi}_{ij}^+ = \frac{\left|\tilde{\xi}_{ij}\right| + \tilde{\xi}_{ij}}{2}, \tilde{\xi}_{ij}^- = \frac{\left|\tilde{\xi}_{ij}\right| - \tilde{\xi}_{ij}}{2},\ \tilde{\eta}_{ij}^+ = \frac{\left|\tilde{\eta}_{ij}\right| + \tilde{\eta}_{ij}}{2},\ \tilde{\xi}_{ij}^- = \frac{\left|\tilde{\eta}_{ij}\right| - \tilde{\eta}_{ij}}{2} \tag{4.12}$$

for $i = 1,2,\ldots,n-1,\ j = i+1,\ldots,n$

Therefore, we have $\tilde{\xi}_{ij} = \tilde{\xi}_{ij}^+ - \tilde{\xi}_{ij}^-$ and $\left|\tilde{\xi}_{ij}\right| = \tilde{\xi}_{ij}^+ + \tilde{\xi}_{ij}^-$, where $\tilde{\xi}_{ij}^+ \cdot \tilde{\xi}_{ij}^- = 0$ for $i = 1,2,\ldots,n-1,\ j = i+1,\ldots,n$, and, $\tilde{\eta}_{ij} = \tilde{\eta}_{ij}^+ - \tilde{\eta}_{ij}^-$ and $\left|\tilde{\eta}_{ij}\right| = \tilde{\eta}_{ij}^+ + \tilde{\eta}_{ij}^-$. By applying the same process as model (4.2), (4.10) can be rewritten as a linear program:

$$
\begin{aligned}
\min \quad & \tilde{J} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\tilde{\xi}_{ij}^+ + \tilde{\xi}_{ij}^- + \tilde{\eta}_{ij}^+ + \tilde{\eta}_{ij}^-) \\
s.t. & \begin{cases}
\omega_i^- - r_{ij}^-(\omega_i^- + \omega_j^+) - \tilde{\xi}_{ij}^+ + \tilde{\xi}_{ij}^- = 0, \quad i = 1,2,\ldots,n-1,\ j = i+1,\ldots,n \\
\omega_i^+ - r_{ij}^+(\omega_i^+ + \omega_j^-) - \tilde{\eta}_{ij}^+ + \tilde{\eta}_{ij}^- = 0, \quad i = 1,2,\ldots,n-1,\ j = i+1,\ldots,n \\
0 \le \omega_i^- \le \omega_i^+ \le 1, \sum_{\substack{j=1 \\ j \neq i}}^{n} \omega_j^- + \omega_i^+ \le 1, \omega_i^- + \sum_{\substack{j=1 \\ j\neq i}}^{n} \omega_j^+ \ge 1, \quad i = 1,2,\ldots,n \\
\tilde{\xi}_{ij}^+ \ge 0, \tilde{\xi}_{ij}^- \ge 0, \tilde{\eta}_{ij}^+ \ge 0, \tilde{\eta}_{ij}^- \ge 0 \quad i = 1,2,\ldots,n-1,\ j = i+1,\ldots,n
\end{cases}
\end{aligned}
\tag{4.13}
$$

Solving this model, we can get the optimal interval weight vector $\overline{\omega}^{**} = (\overline{\omega}_1^{**}, \overline{\omega}_2^{**}, \cdots, \overline{\omega}_n^{**})^T$ $= ([\omega_1^{-**}, \omega_1^{+**}], [\omega_2^{-**}, \omega_2^{+**}], \cdots, [\omega_n^{-**}, \omega_n^{+**}])^T$ for the interval fuzzy preference relation $\overline{R}$.

Similar to the argument in the last paragraph in Section 4.1, if the objective function value $\tilde{J}^* = 0$, then $\overline{R} = (\overline{r}_{ij})_{n \times n}$ is a multiplicative consistent interval fuzzy preference relation.

4.3 Goal programming models for group interval fuzzy preference relations

Consider now a group decision-making situation, where an interval fuzzy preference relation $\overline{R}_k = (\overline{r}_{ijk})_{n\times n} = ([r_{ijk}^-, r_{ijk}^+])_{n\times n}$ is provided by DM k to express his/her preference on an alternative set $X = \{x_1, x_2, ..., x_n\}$, $k = 1, 2, ..., m$. Let $M = \{1, 2, ..., m\}$ be the set of DMs and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)^T$ be the normalized weight vector for the DMs such that $\sum_{k=1}^{m} \lambda_k = 1$ and $\lambda_k \ge 0$ for $k = 1, 2, ..., m$.

Due to the fact that different DMs usually have different preferences, it is nearly impossible to find a unified interval weight vector $\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2, \cdots, \overline{\omega}_n)^T$ that is able to characterize all DMs' preferences. As such, the following additive transitivity equations (4.14) and (4.15) or multiplicative consistency equations (4.16) and (4.17) may not hold true for all DMs.

$$r_{ijk}^- = (0.5 + 0.5(\omega_i^- - \omega_j^+)), \; i = 1, 2, ..., n-1, \; j = i+1, ..., n, k = 1, 2, ..., m \tag{4.14}$$

$$r_{ijk}^+ = (0.5 + 0.5(\omega_i^+ - \omega_j^-)), \; i = 1, 2, ..., n-1, \; j = i+1, ..., n, k = 1, 2, ..., m \tag{4.15}$$

$$r_{ijk}^- (\omega_i^- + \omega_j^+) = \omega_i^-, \; i = 1, 2, ..., n-1, \; j = i+1, ..., n, k = 1, 2, ..., m \tag{4.16}$$

$$r_{ijk}^+ (\omega_i^+ + \omega_j^-) = \omega_i^+, \; i = 1, 2, ..., n-1, \; j = i+1, ..., n, k = 1, 2, ..., m \tag{4.17}$$

In order to derive a unified interval weight vector from the collective interval fuzzy preference relations, the following two optimization models are established based on additive and multiplicative transitivity equations, respectively. The principle is, once again, to minimize the deviation from consistent relations. To differentiate the two goal programming models based on additive and multiplicative consistency, the following two objective functions are labeled with GA (goal-additive) and GM (goal-multiplicative) accordingly.

$$\begin{aligned} \min \quad & GA = \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \lambda_k \left(\left| (0.5 + 0.5(\omega_i^- - \omega_j^+)) - r_{ijk}^- \right| + \left| (0.5 + 0.5(\omega_i^+ - \omega_j^-)) - r_{ijk}^+ \right| \right) \\ s.t. \quad & 0 \le \omega_i^- \le \omega_i^+ \le 1, \sum_{\substack{j=1 \\ j\ne i}}^{n} \omega_j^- + \omega_i^+ \le 1, \omega_i^- + \sum_{\substack{j=1 \\ j\ne i}}^{n} \omega_j^+ \ge 1 \qquad i = 1, 2, ..., n \end{aligned} \tag{4.18}$$

$$\begin{aligned} \min \quad & GM = \sum_{k=1}^{m} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \lambda_k \left(\left| \omega_i^- - r_{ijk}^- (\omega_i^- + \omega_j^+) \right| + \left| \omega_i^+ - r_{ijk}^+ (\omega_i^+ + \omega_j^-) \right| \right) \\ s.t. \quad & 0 \le \omega_i^- \le \omega_i^+ \le 1, \sum_{\substack{j=1 \\ j\ne i}}^{n} \omega_j^- + \omega_i^+ \le 1, \omega_i^- + \sum_{\substack{j=1 \\ j\ne i}}^{n} \omega_j^+ \ge 1 \qquad i = 1, 2, ..., n \end{aligned} \tag{4.19}$$

For (4.18), let

$$\xi_{ijk} \triangleq (0.5+0.5(\omega_i^- - \omega_j^+)) - r_{ijk}^- \ , \ \eta_{ijk} \triangleq (0.5+0.5(\omega_i^+ - \omega_j^-)) - r_{ijk}^+ \tag{4.20}$$

$$\xi_{ijk}^+ \triangleq \frac{\left|\xi_{ijk}\right| + \xi_{ijk}}{2} \ , \ \xi_{ijk}^- \triangleq \frac{\left|\xi_{ijk}\right| - \xi_{ijk}}{2}, \ \eta_{ijk}^+ \triangleq \frac{\left|\eta_{ijk}\right| + \eta_{ijk}}{2} \ , \ \eta_{ijk}^- \triangleq \frac{\left|\eta_{ijk}\right| - \eta_{ijk}}{2} \tag{4.21}$$

for $i = 1,2,...,n-1,\ j = i+1,...,n, k = 1,2,...,m$.

Then, the solution to (4.18) can be found by solving the following linear program:

$$\begin{aligned}
&\min \quad GA = \sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \lambda_k (\xi_{ijk}^+ + \xi_{ijk}^- + \eta_{ijk}^+ + \eta_{ijk}^-) \\
&s.t. \begin{cases}
(0.5+0.5(\omega_i^- - \omega_j^+)) - r_{ijk}^- - \xi_{ijk}^+ + \xi_{ijk}^- = 0, & i = 1,2,...,n-1,\ j = i+1,...,n, k = 1,2,...,m \\
(0.5+0.5(\omega_i^+ - \omega_j^-)) - r_{ijk}^+ - \eta_{ijk}^+ + \eta_{ijk}^- = 0, & i = 1,2,...,n-1,\ j = i+1,...,n, k = 1,2,...,m \\
0 \le \omega_i^- \le \omega_i^+ \le 1, \sum_{\substack{j=1 \\ j \ne i}}^{n} \omega_j^- + \omega_i^+ \le 1, \omega_i^- + \sum_{\substack{j=1 \\ j \ne i}}^{n} \omega_j^+ \ge 1, & i = 1,2,...,n \\
\xi_{ijk}^+ \ge 0, \xi_{ijk}^- \ge 0, \eta_{ijk}^+ \ge 0, \eta_{ijk}^- \ge 0 & i = 1,2,...,n-1,\ j = i+1,...,n, k = 1,2,...,m
\end{cases}
\end{aligned} \tag{4.22}$$

As $(0.5+0.5(\omega_i^- - \omega_j^+)) - r_{ijk}^- - \xi_{ijk}^+ + \xi_{ijk}^- = 0$ ($i = 1,2,...,n-1,\ j = i+1,...,n, k = 1,2,...,m$) and $\sum_{k=1}^{m} \lambda_k = 1$, it is easy to verify that

$$(0.5+0.5(\omega_i^- - \omega_j^+)) - \sum_{k=1}^{m} \lambda_k r_{ijk}^- - \sum_{k=1}^{m} \lambda_k \xi_{ijk}^+ + \sum_{k=1}^{m} \lambda_k \xi_{ijk}^- = 0 \tag{4.23}$$

Similarly, from $(0.5+0.5(\omega_i^+ - \omega_j^-)) - r_{ijk}^+ - \eta_{ijk}^+ + \eta_{ijk}^- = 0$ and $\sum_{k=1}^{m} \lambda_k = 1$, one can obtain

$$(0.5+0.5(\omega_i^+ - \omega_j^-)) - \sum_{k=1}^{m} \lambda_k r_{ijk}^+ - \sum_{k=1}^{m} \lambda_k \eta_{ijk}^+ + \sum_{k=1}^{m} \lambda_k \eta_{ijk}^- = 0 \tag{4.24}$$

Let $\dot{\xi}_{ij}^+ \triangleq \sum_{k=1}^{m} \lambda_k \xi_{ijk}^+, \dot{\xi}_{ij}^- \triangleq \sum_{k=1}^{m} \lambda_k \xi_{ijk}^-, \dot{\eta}_{ij}^+ \triangleq \sum_{k=1}^{m} \lambda_k \eta_{ijk}^+$ and $\dot{\eta}_{ij}^- \triangleq \sum_{k=1}^{m} \lambda_k \eta_{ijk}^-$, then (4.22) can be converted to the following linear program.

$$
\begin{aligned}
&\min\quad GA=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\dot{\xi}_{ij}^{+}+\dot{\xi}_{ij}^{-}+\dot{\eta}_{ij}^{+}+\dot{\eta}_{ij}^{-})\\
&s.t.\begin{cases}
(0.5+0.5(\omega_i^- -\omega_j^+))-\sum_{k=1}^{m}\lambda_k r_{ijk}^- -\dot{\xi}_{ij}^{+}+\dot{\xi}_{ij}^{-}=0, & i=1,2,...,n-1,\ j=i+1,...,n\\
(0.5+0.5(\omega_i^+ -\omega_j^-))-\sum_{k=1}^{m}\lambda_k r_{ijk}^+ -\dot{\eta}_{ij}^{+}+\dot{\eta}_{ij}^{-}=0, & i=1,2,...,n-1,\ j=i+1,...,n\\
0\le\omega_i^-\le\omega_i^+\le 1,\sum_{\substack{j=1\\ j\ne i}}^{n}\omega_j^- +\omega_i^+\le 1,\omega_i^- +\sum_{\substack{j=1\\ j\ne i}}^{n}\omega_j^+\ge 1, & i=1,2,...,n\\
\dot{\xi}_{ij}^{+}\ge 0,\dot{\xi}_{ij}^{-}\ge 0,\dot{\eta}_{ij}^{+}\ge 0,\dot{\eta}_{ij}^{-}\ge 0 & i=1,2,...,n-1,\ j=i+1,...,n
\end{cases}
\end{aligned}
\tag{4.25}
$$

By solving (4.25), we can obtain a unified interval weight vector $\dot{\varpi}=(\dot{\varpi}_1,\dot{\varpi}_2,\cdots,\dot{\varpi}_n)^T=([\dot{\omega}_1^-,\dot{\omega}_1^+],\ [\dot{\omega}_2^-,\dot{\omega}_2^+],\cdots,[\dot{\omega}_n^-,\dot{\omega}_n^+])^T$ for the collective interval fuzzy preference relations $\overline{R}_k$ ($k=1,2,...,m$).

In a similar way, for (4.19), let

$$\tilde{\xi}_{ijk}\triangleq\omega_i^- -r_{ijk}^-(\omega_i^- +\omega_j^+)\,,\quad \tilde{\eta}_{ijk}\triangleq\omega_i^+ -r_{ijk}^+(\omega_i^+ +\omega_j^-) \tag{4.26}$$

$$\tilde{\xi}_{ijk}^{+}\triangleq\frac{\left|\tilde{\xi}_{ijk}\right|+\tilde{\xi}_{ijk}}{2}\ ,\ \tilde{\xi}_{ijk}^{-}\triangleq\frac{\left|\tilde{\xi}_{ijk}\right|-\tilde{\xi}_{ijk}}{2},\ \tilde{\eta}_{ijk}^{+}\triangleq\frac{\left|\tilde{\eta}_{ijk}\right|+\tilde{\eta}_{ijk}}{2}\ ,\ \tilde{\eta}_{ijk}^{-}\triangleq\frac{\left|\tilde{\eta}_{ijk}\right|-\tilde{\eta}_{ijk}}{2} \tag{4.27}$$

for $i=1,2,...,n-1,\ j=i+1,...,n,k=1,2,...,m$

Then, the solution to (4.19) can be found by solving the following linear program:

$$
\begin{aligned}
&\min\quad GM=\sum_{k=1}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\lambda_k(\tilde{\xi}_{ijk}^{+}+\tilde{\xi}_{ijk}^{-}+\tilde{\eta}_{ijk}^{+}+\tilde{\eta}_{ijk}^{-})\\
&s.t.\begin{cases}
\omega_i^- -r_{ijk}^-(\omega_i^- +\omega_j^+)-\tilde{\xi}_{ijk}^{+}+\tilde{\xi}_{ijk}^{-}=0, & i=1,2,...,n-1,\ j=i+1,...,n,k=1,2,...,m\\
\omega_i^+ -r_{ijk}^+(\omega_i^+ +\omega_j^-)-\tilde{\eta}_{ijk}^{+}+\tilde{\eta}_{ijk}^{-}=0, & i=1,2,...,n-1,\ j=i+1,...,n,k=1,2,...,m\\
0\le\omega_i^-\le\omega_i^+\le 1,\sum_{\substack{j=1\\ j\ne i}}^{n}\omega_j^- +\omega_i^+\le 1,\omega_i^- +\sum_{\substack{j=1\\ j\ne i}}^{n}\omega_j^+\ge 1, & i=1,2,...,n\\
\tilde{\xi}_{ijk}^{+}\ge 0,\tilde{\xi}_{ijk}^{-}\ge 0,\tilde{\eta}_{ijk}^{+}\ge 0,\tilde{\eta}_{ijk}^{-}\ge 0 & i=1,2,...,n-1,\ j=i+1,...,n,k=1,2,...,m
\end{cases}
\end{aligned}
\tag{4.28}
$$

From $\omega_i^- -r_{ijk}^-(\omega_i^- +\omega_j^+)-\tilde{\xi}_{ijk}^{+}+\tilde{\xi}_{ijk}^{-}=0$ ($i=1,2,...,n-1,\ j=i+1,...,n,k=1,2,...,m$) and $\sum_{k=1}^{m}\lambda_k=1$, it is easy to confirm that

$$\omega_i^- -\sum_{k=1}^{m}\lambda_k r_{ijk}^-(\omega_i^- +\omega_j^+)-\sum_{k=1}^{m}\lambda_k\tilde{\xi}_{ijk}^{+}+\sum_{k=1}^{m}\lambda_k\tilde{\xi}_{ijk}^{-}=0 \tag{4.29}$$

Similarly, as $\omega_i^+ - r_{ijk}^+(\omega_i^+ + \omega_j^-) - \tilde{\eta}_{ijk}^+ + \tilde{\eta}_{ijk}^- = 0$ and $\sum_{k=1}^{m}\lambda_k = 1$, we have

$$\omega_i^+ - \sum_{k=1}^{m}\lambda_k r_{ijk}^+(\omega_i^+ + \omega_j^-) - \sum_{k=1}^{m}\lambda_k\tilde{\eta}_{ijk}^+ + \sum_{k=1}^{m}\lambda_k\tilde{\eta}_{ijk}^- = 0 \tag{4.30}$$

Let $\ddot{\xi}_{ij}^+ \triangleq \sum_{k=1}^{m}\lambda_k\tilde{\xi}_{ijk}^+, \ddot{\xi}_{ij}^- \triangleq \sum_{k=1}^{m}\lambda_k\tilde{\xi}_{ijk}^-, \ddot{\eta}_{ij}^+ \triangleq \sum_{k=1}^{m}\lambda_k\tilde{\eta}_{ijk}^+$ and $\ddot{\eta}_{ij}^- \triangleq \sum_{k=1}^{m}\lambda_k\tilde{\eta}_{ijk}^-$, then (4.28) can be rewritten as

$$\min \;\; GM = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\ddot{\xi}_{ij}^+ + \ddot{\xi}_{ij}^- + \ddot{\eta}_{ij}^+ + \ddot{\eta}_{ij}^-)$$
$$s.t.\begin{cases} \omega_i^- - \sum_{k=1}^{m}\lambda_k r_{ijk}^-(\omega_i^- + \omega_j^+) - \ddot{\xi}_{ij}^+ + \ddot{\xi}_{ij}^- = 0, & i=1,2,...,n-1,\ j=i+1,...,n \\ \omega_i^+ - \sum_{k=1}^{m}\lambda_k r_{ijk}^+(\omega_i^+ + \omega_j^-) - \ddot{\eta}_{ij}^+ + \ddot{\eta}_{ij}^- = 0, & i=1,2,...,n-1,\ j=i+1,...,n \\ 0 \le \omega_i^- \le \omega_i^+ \le 1, \sum_{\substack{j=1\\ j\ne i}}^{n}\omega_j^- + \omega_i^+ \le 1, \omega_i^- + \sum_{\substack{j=1\\ j\ne i}}^{n}\omega_j^+ \ge 1, & i=1,2,...,n \\ \ddot{\xi}_{ij}^+ \ge 0, \ddot{\xi}_{ij}^- \ge 0, \ddot{\eta}_{ij}^+ \ge 0, \ddot{\eta}_{ij}^- \ge 0 & i=1,2,...,n-1,\ j=i+1,...,n \end{cases} \tag{4.31}$$

Solving this model, the optimal solution yields a unified interval weight vector $\ddot{\bar{\omega}} = (\ddot{\bar{\omega}}_1, \ddot{\bar{\omega}}_2, \cdots, \ddot{\bar{\omega}}_n)^T = ([\ddot{\omega}_1^-, \ddot{\omega}_1^+], [\ddot{\omega}_2^-, \ddot{\omega}_2^+], \cdots, [\ddot{\omega}_n^-, \ddot{\omega}_n^+])^T$ for the collective interval fuzzy preference relations $\bar{R}_k$ ($k = 1,2,...,m$).

5 A numerical example and comparative analysis

This section presents a multiple criteria decision making problem to demonstrate how to apply the proposed models in Sections 4.1 and 4.2.

Consider a multiple criteria decision making problem, consisting of four criteria x_i $(i = 1,2,3,4)$. Assume that a DM conducts an exhaustive pairwise comparison of criteria x_i and x_j, and the result is given as the following interval fuzzy preference relation:

$$\bar{R} = (\bar{r}_{ij})_{4\times 4} = \begin{bmatrix} [0.50, 0.50] & [0.35, 0.50] & [0.50, 0.60] & [0.45, 0.60] \\ [0.50, 0.65] & [0.50, 0.50] & [0.55, 0.70] & [0.50, 0.70] \\ [0.40, 0.50] & [0.30, 0.45] & [0.50, 0.50] & [0.40, 0.55] \\ [0.40, 0.55] & [0.30, 0.50] & [0.45, 0.60] & [0.50, 0.50] \end{bmatrix}$$

This interval fuzzy preference relation matrix $\bar{R}$ reflects the DM's judgment of the importance between each pair of criteria. The cells along the diagonal are always [0.50, 0.50], implying the DM's indifference between any criterion and itself. The elements off the diagonal give the DM's pairwise comparison result between two criteria and any two elements symmetric about the diagonal are complementary in the sense of $\overline{r}_{ji} = 1 - \overline{r}_{ij}$ as defined in Definition 3.1. For instance, $\overline{r}_{12} = [0.35, 0.50]$ indicates that the DM's preference of x_1 over x_2 is between 0.35 and 0.50. The element symmetric about the diagonal, $\overline{r}_{21}$, is given as $\overline{r}_{21} = 1 - \overline{r}_{12} = [1-0.50, 1-0.35] = [0.50, 0.65]$, signifying the DM's preference of x_2 over x_1 is between 0.50 and 0.65. Remaining elements in $\overline{R}$ can be interpreted similarly.

Plugging the interval fuzzy preference relation $\overline{R}$ into (4.6), and solving this model, one can obtain its optimal solution $J^* = 0$, and the optimal interval weight vector as:

$$\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3, \overline{\omega}_4)^T = ([0.175, 0.275], [0.275, 0.475], [0.075, 0.175], [0.075, 0.275])^T$$

As $J^* = 0$, we know that $\overline{R}$ is additive consistent. Based on the procedure of ranking interval weights described in the Section 2.2, the following likelihood matrix is derived.

$$P = \begin{bmatrix} 0.5 & 0 & 1 & 0.6667 \\ 1 & 0.5 & 1 & 1 \\ 0 & 0 & 0.5 & 0.3333 \\ 0.3333 & 0 & 0.6667 & 0.5 \end{bmatrix}$$

As per (2.10), we get $\theta_1 = 0.2639, \theta_2 = 0.375, \theta_3 = 0.1528$ and $\theta_4 = 0.2083$. Then, we have $\overline{\omega}_2 \overset{1}{\succeq} \overline{\omega}_1 \overset{0.6667}{\succeq} \overline{\omega}_4 \overset{0.6667}{\succeq} \overline{\omega}_3$, which indicates that $\overline{\omega}_2$ is superior to $\overline{\omega}_1$ to the degree of 100%, $\overline{\omega}_1$ is superior to $\overline{\omega}_4$ to the degree of 66.67%, and $\overline{\omega}_4$ is superior to $\overline{\omega}_3$ to the degree of 66.67%.

If we plug the interval fuzzy preference relation $\overline{R}$ into (4.13) and solve this model, then it follows that $\tilde{J}^* = 0.0037$ and the interval weight vector as:

$$\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3, \overline{\omega}_4)^T = ([0.2143, 0.2619], [0.2619, 0.4074], [0.1746, 0.2143], [0.1746, 0.2619])^T$$

Once again, by following the procedure of ranking interval weights, the following likelihood matrix is obtained:

$$P = \begin{bmatrix} 0.5 & 0 & 1 & 0.6471 \\ 1 & 0.5 & 1 & 1 \\ 0 & 0 & 0.5 & 0.3126 \\ 0.3529 & 0 & 0.6874 & 0.5 \end{bmatrix}$$

According to (2.10), one can have $\theta_1 = 0.2623, \theta_2 = 0.375, \theta_3 = 0.1511$ and $\theta_4 = 0.2117$. As such, the ranking of the four interval weights is $\bar{\omega}_2 \overset{1}{\succeq} \bar{\omega}_1 \overset{0.6471}{\succeq} \bar{\omega}_4 \overset{0.6874}{\succeq} \bar{\omega}_3$, meaning that $\bar{\omega}_2$ is superior to $\bar{\omega}_1$ to the degree of 100%, $\bar{\omega}_1$ is superior to $\bar{\omega}_4$ to the degree of 64.71%, and $\bar{\omega}_4$ is superior to $\bar{\omega}_3$ to the degree of 68.74%.

The aforesaid analyses indicate that the rankings of interval weights obtained by (4.6) and (4.13) are consistent with slightly different likelihood.

Next, models (M-3, M-4, M-5) and (M-11, M-12, M-13) in Xu and Chen [61] will be employed to derive interval priority weights based on the same interval fuzzy preference relation $\bar{R}$, and the ranking results will be compared with those obtained using our proposed models.

Using Xu and Chen's model (M-3) [61], one can obtain an optimal objective function value of 0 with all deviation values being zero. Solving (M-4) and (M-5) with the deviation values being set at zero, we derive an interval weight vector as:

$$\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_4)^T = ([0.150, 0.350], [0.275, 0.525], [0.050, 0.250], [0.075, 0.325])^T.$$

Based on the ranking procedure of interval weights, the following likelihood matrix is obtained:

$$P = \begin{bmatrix} 0.5 & 0.1667 & 0.75 & 0.6111 \\ 0.8333 & 0.5 & 1 & 0.9 \\ 0.25 & 0 & 0.5 & 0.3889 \\ 0.3889 & 0.1 & 0.6111 & 0.5 \end{bmatrix}$$

Thus, $\bar{\omega}_2 \overset{0.8333}{\succeq} \bar{\omega}_1 \overset{0.6111}{\succeq} \bar{\omega}_4 \overset{0.6111}{\succeq} \bar{\omega}_3$.

Similarly, using Xu and Chen's (M-11) [61], one can confirm an objective function value of 0 with all deviation values being zero in the optimal solution. Solving (M-12) and (M-13) with all deviation values being set at zero leads to an interval weight vector

$$\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_4)^T = ([[0.1969, 0.3000]], [0.2619, 0.4174], [0.1579, 0.2475], [0.1651, 0.2870])^T.$$

The ranking procedure of interval weights results in the following likelihood matrix:

$$P = \begin{bmatrix} 0.5 & 0.1473 & 0.7374 & 0.6 \\ 0.8527 & 0.5 & 1 & 0.9095 \\ 0.2626 & 0 & 0.5 & 0.3896 \\ 0.4 & 0.0905 & 0.6104 & 0.5 \end{bmatrix}$$

Thus, $\bar{\omega}_2 \overset{0.8527}{\succeq} \bar{\omega}_1 \overset{0.6}{\succeq} \bar{\omega}_4 \overset{0.6104}{\succeq} \bar{\omega}_3$.

The ranking results based on the models in [61] and our proposed approaches are summarized in Table 1.

Table 1. A comparative study for the interval fuzzy preference relation $\bar{R}$

Decision model	Reference	# of LP models to solve	Ranking result
M-3, M-4, M-5	Xu and Chen [61]	9	$\overline{\omega}_2 \overset{0.8333}{\succeq} \overline{\omega}_1 \overset{0.6111}{\succeq} \overline{\omega}_4 \overset{0.6111}{\succeq} \overline{\omega}_3$
M-11, M-12, M-13	Xu and Chen [61]	9	$\overline{\omega}_2 \overset{0.8527}{\succeq} \overline{\omega}_1 \overset{0.6}{\succeq} \overline{\omega}_4 \overset{0.6104}{\succeq} \overline{\omega}_3$
(4,6)	This article	1	$\overline{\omega}_2 \overset{1}{\succeq} \overline{\omega}_1 \overset{0.6667}{\succeq} \overline{\omega}_4 \overset{0.6667}{\succeq} \overline{\omega}_3$
(4.13)	This article	1	$\overline{\omega}_2 \overset{1}{\succeq} \overline{\omega}_1 \overset{0.6471}{\succeq} \overline{\omega}_4 \overset{0.6874}{\succeq} \overline{\omega}_3$

Table 1 demonstrates the overall consistency of the ranking results between the two different approaches, but our proposed framework significantly reduces the computation burden: our approach only requires solving one LP model, while the method reported in Xu and Chen [61] has to entertain $2n+1$ LP models.

To further verify the effectiveness of the proposed approaches in this article, substantial numerical experiments have been carried out by varying the pairwise comparison values in the interval fuzzy preference relation $\bar{R}$. Our approaches generally produce ranking results that are consistent with those generated from Xu and Chen's models [61].

6 An application to the international exchange doctoral student selection problem

In this section, the proposed models in Section 4.3 are applied to examine a two-level group decision making problem with a hierarchical structure. The purpose is to recommend highly competitive doctoral students for publicly-funded international exchange opportunities at the first author's university, and both faculty-level and institution-level panels are convened to rank applicants for final recommendations.

With the continuing internationalization of the Chinese higher education system, numerous universities and research institutions in China have established international partnerships for jointly training their postgraduate students with a focus at the doctoral level. Under this framework, a small proportion of these students, presumably of exceptional quality and potentials, are selected and sent to foreign institutions to work on joint research projects for one to two years. These students are expected to return to their home institutions in China after the visit to complete their theses and defense. Although the number of scholarships to support Chinese students and scholars to conduct research abroad has dramatically increased over the past decade, the competition of getting such an award remains fierce given the size of the applicant pool.

The first author of this article has been actively involved in a faculty-wide selection committee to rank their applicants and make recommendations to the university. A general practice at this university is to call for an institution-wide committee to come up with a criteria weighting scheme for assessing applications. This scheme has to follow the published guidelines from the granting agency but also reflects the committee members' personal judgment on the importance of different criteria. Once the committee reaches a consensus on criteria weights, this information will be distributed to all faculties and schools on the campus for their evaluation process at the faculty level. Each faculty and school then strikes their selection committee to assess applications from their graduate students based on the weighting scheme provided by the university. This decision process involves two levels and each level can be treated as a group decision making problem.

At the upper level, the institution-wide committee considers a well-defined list of criteria based on the guidelines from the granting agency. The criteria consist of the following four aspects:

c_1: Academic capability, achievements, and potentials as reflected in refereed publications and other research output.

c_2: Academic profile and prestige of the proposed foreign host institution.

c_3: Communication skills and foreign language proficiency.

c_4: Academic background in the proposed area of study.

The deliberation of this university committee is expected to generate a weighting scheme for these four criteria. At the lower level, the faculty selection committee is responsible for assessing applicants based on the weights determined by the committee at the university level. The hierarchical structure of this decision process is illustrated in Fig. 1.

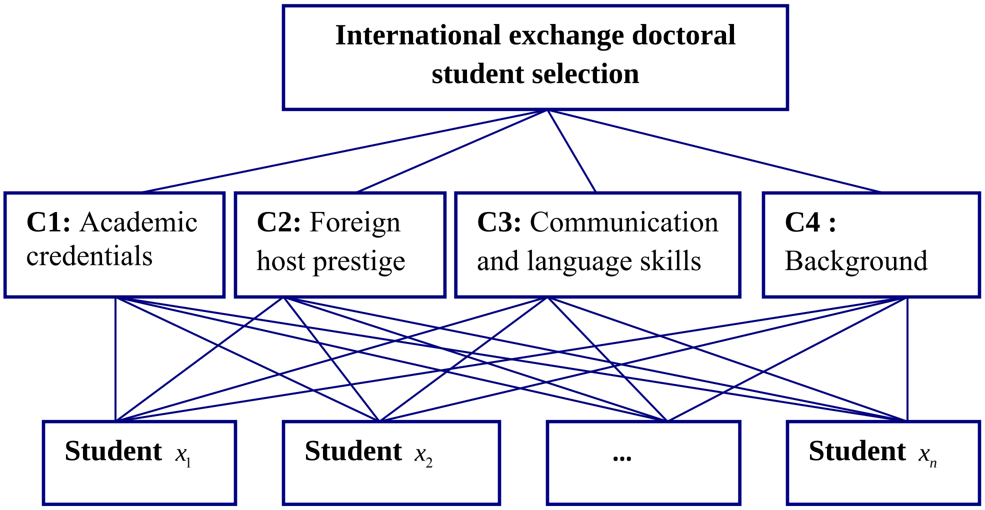

Fig. 1 A hierarchical structure of the decision process

For the sake of tractability and illustration, it is assumed that the university-level committee consists of four members and each member has an equal weight in determining the final criteria weights i.e., $\lambda_k = 0.25, (k = 1,2,3,4)$. However, the approaches proposed in this article can conveniently handle any practical number of committee members as well as the case that certain committee members have more influence powers than others in determining criteria weights. Each committee member is asked to furnish his/her pairwise comparison results among the four criteria. In reality, it tends to be easier for a DM to provide an assessment falling within a range rather than an exact value. In this case, it is sensible to assume that each committee member's assessments can be converted into an interval fuzzy preference relation as follows, where the subscript, k = 1, 2, 3, 4, indicates a specific committee member:

$$\overline{R}_1 = \begin{bmatrix} [0.50, 0.50] & [0.35, 0.45] & [0.40, 0.55] & [0.52, 0.65] \\ [0.55, 0.65] & [0.50, 0.50] & [0.70, 0.90] & [0.65, 0.75] \\ [0.45, 0.60] & [0.10, 0.30] & [0.50, 0.50] & [0.55, 0.65] \\ [0.35, 0.48] & [0.25, 0.35] & [0.35, 0.45] & [0.50, 0.50] \end{bmatrix}$$

$$\overline{R}_2 = \begin{bmatrix} [0.50, 0.50] & [0.75, 0.85] & [0.65, 0.75] & [0.35, 0.45] \\ [0.15, 0.25] & [0.50, 0.50] & [0.50, 0.65] & [0.50, 0.65] \\ [0.25, 0.35] & [0.35, 0.50] & [0.50, 0.50] & [0.62, 0.75] \\ [0.55, 0.65] & [0.35, 0.50] & [0.25, 0.38] & [0.50, 0.50] \end{bmatrix}$$

$$\bar{R}_3 = \begin{bmatrix} [0.50, 0.50] & [0.60, 0.70] & [0.75, 0.85] & [0.60, 0.72] \\ [0.30, 0.40] & [0.50, 0.50] & [0.50, 0.70] & [0.55, 0.70] \\ [0.15, 0.25] & [0.30, 0.50] & [0.50, 0.50] & [0.45, 0.55] \\ [0.28, 0.40] & [0.30, 0.45] & [0.45, 0.55] & [0.50, 0.50] \end{bmatrix}$$

$$\bar{R}_4 = \begin{bmatrix} [0.50, 0.50] & [0.30, 0.40] & [0.45, 0.65] & [0.63, 0.75] \\ [0.60, 0.70] & [0.50, 0.50] & [0.50, 0.70] & [0.68, 0.76] \\ [0.35, 0.55] & [0.30, 0.50] & [0.50, 0.50] & [0.65, 0.74] \\ [0.25, 0.37] & [0.24, 0.32] & [0.26, 0.35] & [0.50, 0.50] \end{bmatrix}$$

If the additive-transitivity based goal programming model (4.25) is employed, these four interval fuzzy preference relations, $\bar{R}_1, \bar{R}_2, \bar{R}_3, \bar{R}_4$, would lead to the following normalized interval weight vector for the four criteria:

$$\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \bar{\omega}_4)^T = ([\omega_1^-, \omega_1^+], [\omega_2^-, \omega_2^+], [\omega_3^-, \omega_3^+], [\omega_4^-, \omega_4^+])^T = \\ ([0.3583, 0.5333], [0.3333, 0.4301], [0.1333, 0.2333], [0.0001, 0.1433])^T$$

Based on the weighting scheme for the four criteria, a lower level committee is struck to evaluate applications from their individual faculty. Assume, once again, that the committee consists of four members and each member is equally important in evaluating the candidates. Without loss of generality and for the sake of tractability, consider the deliberation of four applicants. The application packages are distributed to the committee members, and each member is expected to provide his/her independent assessment of each candidate against the four criteria in terms of interval fuzzy preference relations to accommodate potential uncertainty in the judgment. These assumptions are reasonable representations of the first author's experience while he serves on the selection committee in his school.

To calibrate the models, each committee member's assessments on the students against each criterion have to be obtained. This important decision information can be garnered by sitting in an official deliberation meeting as the first author has experienced. For the illustration purpose and without loss of generality, assume that committee member k's assessment of the four candidates x_1, x_2, x_3, x_4 with respect to criterion c_1 is given as an interval fuzzy preference relation $\bar{R}_k^{c_1}$, k = 1, 2, 3, 4:

$$\overline{R}_1^{c_1} = \begin{bmatrix} [0.50, 0.50] & [0.56, 0.65] & [0.45, 0.55] & [0.45, 0.55] \\ [0.35, 0.44] & [0.50, 0.50] & [0.35, 0.45] & [0.35, 0.46] \\ [0.45, 0.55] & [0.55, 0.65] & [0.50, 0.50] & [0.43, 0.57] \\ [0.45, 0.55] & [0.54, 0.65] & [0.43, 0.57] & [0.50, 0.50] \end{bmatrix}$$

$$\overline{R}_2^{c_1} = \begin{bmatrix} [0.50, 0.50] & [0.65, 0.75] & [0.55, 0.65] & [0.53, 0.66] \\ [0.25, 0.34] & [0.50, 0.50] & [0.24, 0.36] & [0.25, 0.35] \\ [0.35, 0.45] & [0.64, 0.76] & [0.50, 0.50] & [0.53, 0.66] \\ [0.34, 0.47] & [0.65, 0.75] & [0.34, 0.47] & [0.50, 0.50] \end{bmatrix}$$

$$\overline{R}_3^{c_1} = \begin{bmatrix} [0.50, 0.50] & [0.62, 0.78] & [0.45, 0.60] & [0.46, 0.58] \\ [0.22, 0.38] & [0.50, 0.50] & [0.26, 0.38] & [0.38, 0.45] \\ [0.40, 0.55] & [0.62, 0.74] & [0.50, 0.50] & [0.46, 0.54] \\ [0.42, 0.55] & [0.55, 0.62] & [0.46, 0.54] & [0.50, 0.50] \end{bmatrix}$$

$$\overline{R}_4^{c_1} = \begin{bmatrix} [0.50, 0.50] & [0.60, 0.76] & [0.60, 0.70] & [0.58, 0.72] \\ [0.24, 0.40] & [0.50, 0.50] & [0.36, 0.44] & [0.35, 0.45] \\ [0.30, 0.40] & [0.56, 0.64] & [0.50, 0.50] & [0.57, 0.71] \\ [0.28, 0.42] & [0.55, 0.65] & [0.29, 0.43] & [0.50, 0.50] \end{bmatrix}$$

Committee member k's assessment of the four candidates x_1, x_2, x_3, x_4 with respect to criterion c_2 is given as an interval fuzzy preference relation $\overline{R}_k^{c_2}$, k = 1, 2, 3, 4:

$$\overline{R}_1^{c_2} = \overline{R}_2^{c_2} = \begin{bmatrix} [0.50, 0.50] & [0.25, 0.45] & [0.25, 0.45] & [0.10, 0.30] \\ [0.55, 0.75] & [0.50, 0.50] & [0.35, 0.50] & [0.35, 0.50] \\ [0.55, 0.75] & [0.50, 0.65] & [0.50, 0.50] & [0.45, 0.65] \\ [0.70, 0.90] & [0.50, 0.65] & [0.35, 0.55] & [0.50, 0.50] \end{bmatrix}$$

$$\overline{R}_3^{c_2} = \overline{R}_4^{c_2} = \begin{bmatrix} [0.50, 0.50] & [0.28, 0.40] & [0.29, 0.39] & [0.12, 0.22] \\ [0.60, 0.72] & [0.50, 0.50] & [0.45, 0.55] & [0.30, 0.40] \\ [0.61, 0.71] & [0.45, 0.55] & [0.50, 0.50] & [0.28, 0.42] \\ [0.78, 0.88] & [0.60, 0.70] & [0.58, 0.72] & [0.50, 0.50] \end{bmatrix}$$

Committee member k's assessment of the four candidates x_1, x_2, x_3, x_4 with respect to criterion c_3 is given as an interval fuzzy preference relation $\bar{R}_k^{c_3}$, k = 1, 2, 3, 4:

$$\bar{R}_1^{c_3} = \bar{R}_2^{c_3} = \begin{bmatrix} [0.50, 0.50] & [0.35, 0.55] & [0.25, 0.45] & [0.15, 0.30] \\ [0.45, 0.65] & [0.50, 0.50] & [0.35, 0.50] & [0.25, 0.40] \\ [0.55, 0.75] & [0.50, 0.65] & [0.50, 0.50] & [0.25, 0.55] \\ [0.70, 0.85] & [0.60, 0.75] & [0.45, 0.75] & [0.50, 0.50] \end{bmatrix}$$

$$\bar{R}_3^{c_3} = \bar{R}_4^{c_3} = \begin{bmatrix} [0.50, 0.50] & [0.36, 0.47] & [0.27, 0.39] & [0.20, 0.30] \\ [0.53, 0.64] & [0.50, 0.50] & [0.38, 0.50] & [0.28, 0.38] \\ [0.61, 0.73] & [0.50, 0.62] & [0.50, 0.50] & [0.35, 0.46] \\ [0.70, 0.80] & [0.62, 0.72] & [0.54, 0.65] & [0.50, 0.50] \end{bmatrix}$$

Committee member k's assessment of the four candidates x_1, x_2, x_3, x_4 with respect to criterion c_4 is given as an interval fuzzy preference relation $\bar{R}_k^{c_4}$, k = 1, 2, 3, 4:

$$\bar{R}_1^{c_4} = \bar{R}_2^{c_4} = \begin{bmatrix} [0.50, 0.50] & [0.45, 0.65] & [0.50, 0.60] & [0.55, 0.65] \\ [0.35, 0.55] & [0.50, 0.50] & [0.50, 0.60] & [0.55, 0.65] \\ [0.40, 0.50] & [0.40, 0.50] & [0.50, 0.50] & [0.50, 0.75] \\ [0.35, 0.45] & [0.35, 0.45] & [0.25, 0.50] & [0.50, 0.50] \end{bmatrix}$$

$$\bar{R}_3^{c_4} = \bar{R}_4^{c_4} = \begin{bmatrix} [0.50, 0.50] & [0.45, 0.70] & [0.50, 0.75] & [0.55, 0.65] \\ [0.30, 0.55] & [0.50, 0.50] & [0.50, 0.60] & [0.53, 0.66] \\ [0.25, 0.50] & [0.40, 0.50] & [0.50, 0.50] & [0.50, 0.60] \\ [0.35, 0.45] & [0.34, 0.47] & [0.40, 0.50] & [0.50, 0.50] \end{bmatrix}$$

Similarly, if the additive-transitivity based goal programming model (4.25) is entertained, a normalized interval assessment of each alternative x_i with respect to each criterion c_j , $i, j = 1, 2, 3, 4$, denoted by $\bar{\omega}_{ij} = \left[\omega_{ij}^-, \omega_{ij}^+\right]$, can be obtained as shown in columns 1-4 in Table 2, where the first row lists the upper level criteria weights obtained earlier.

Table 2. Interval weights for alternatives under each criterion based on (4.25) and the aggregated interval assessments

	c_1 [0.3583, 0.5333]	c_2 [0.3333, 0.4301]	c_3 [0.1333, 0.2333]	c_4 [0.0001, 0.1433]	Aggregated interval weights
x_1	[0.3450, 0.4717]	[0.0000, 0.0417]	[0.0000, 0.0900]	[0.2750, 0.5250]	[0.1236, 0.2851]
x_2	[0.0000, 0.1267]	[0.1917, 0.3167]	[0.1000, 0.2500]	[0.1750, 0.3750]	[0.0772, 0.2481]
x_3	[0.3117, 0.3950]	[0.2667, 0.3917]	[0.2500, 0.4300]	[0.1750, 0.1750]	[0.2691, 0.4021]
x_4	[0.2167, 0.3350]	[0.4167, 0.5417]	[0.4700, 0.6500]	[0.0650, 0.1750]	[0.2954, 0.4929]

If criteria weights and the assessment of each candidate against each criterion are real-valued, the aggregation process is simply a sumproduct function. But in the interval-valued case, the interval arithmetic cannot be applied directly [40]. As such, LP models are proposed by Bryson and Mobolurin [6] to handle the aggregation process. This same procedure is also adopted by Wang and Elhag [46] in their research. The basic idea is to treat criteria weights as decision variables and obtain the lower and upper bounds of the aggregated assessment for each alternative x_i, i = 1, 2, 3, 4, by constructing a pair of LP models.

$$\min \quad \omega_{x_i}^{-} = \sum_{j=1}^{4} \omega_{ij}^{-}\omega_j \qquad s.t. \begin{cases} \omega_j^{-} \le \omega_j \le \omega_j^{+}, & j = 1,2,3,4 \\ \sum_{j=1}^{4} \omega_j = 1 \end{cases} \tag{6.1}$$

$$\max \quad \omega_{x_i}^{+} = \sum_{j=1}^{4} \omega_{ij}^{+}\omega_j \qquad s.t. \begin{cases} \omega_j^{-} \le \omega_j \le \omega_j^{+}, & j = 1,2,3,4 \\ \sum_{j=1}^{4} \omega_j = 1 \end{cases} \tag{6.2}$$

By applying (6.1) and (6.2), one can obtain the aggregated interval assessment for each alternative x_i ($i = 1,2,3,4$) as shown in the last column of Table 2.

As per the interval ranking procedure in Section 2.2, the aggregated interval assessments can be translated to a final ranking of $x_4 \overset{0.6772}{\succeq} x_3 \overset{0.9457}{\succeq} x_1 \overset{0.5442}{\succeq} x_2$, signifying that candidate x_4 is superior to x_3 to the degree of 67.72%, x_3 is superior to x_1 to the degree of 94.57%, and x_1 is superior to x_2 to the degree of 54.42%.

On the other hand, if the multiplicative-transitivity based goal programming model (4.31) is employed, the interval criteria weights and assessment of each candidate against each criterion are presented in Table 3 in a similar structure.

Table 3. Interval weights for alternatives under each criterion based on (4.31) and aggregated interval assessments

	c_1 [0.2992, 0.3723]	c_2 [0.2482, 0.2992]	c_3 [0.1596, 0.2327]	c_4 [0.1193, 0.1689]	Aggregated interval weights
x_1	[0.2698, 0.3654]	[0.0920, 0.1429]	[0.1134, 0.1492]	[0.2478, 0.4192]	[0.1765, 0.2724]
x_2	[0.1317, 0.1620]	[0.1934, 0.2552]	[0.1733, 0.2060]	[0.2195, 0.3028]	[0.1689, 0.2538]
x_3	[0.2355, 0.3038]	[0.2309, 0.2901]	[0.2060, 0.3227]	[0.2018, 0.2478]	[0.2216, 0.2977]
x_4	[0.2169, 0.2645]	[0.3119, 0.4016]	[0.3222, 0.4806]	[0.1595, 0.2018]	[0.2524, 0.3483]

Similarly, (6.1) and (6.2) are adopted to aggregate individual interval weights into overall interval assessments as shown in the last column of Table 3. Once again, the interval ranking process in Section 2.2 yields a final ranking of the four candidates as $x_4 \overset{0.7366}{\succeq} x_3 \overset{0.7047}{\succeq} x_1 \overset{0.5723}{\succeq} x_2$, meaning that candidate x_4 is superior to x_3 to the degree of 73.66%, x_3 is superior to x_1 to the degree of 70.47%, and x_1 is superior to x_2 to the degree of 57.23%.

This case study demonstrates the robustness of the ranking results based on additive and multiplicative transitivity approaches: the final ranking is basically the same, except slightly different degrees of possibility.

7 CONCLUSIONS

Based on interval arithmetic, this article introduces new definitions of additive and multiplicative consistency for interval fuzzy preference relations. Transformation functions are established to convert interval weights into additive and multiplicative consistent interval fuzzy preference relations. This inherent link allows us to develop goal-programming based models for deriving interval weights from both consistent and inconsistent interval fuzzy preference relations for individual and group decision making situations. The basic modeling principle is that the derived interval weight vector minimizes the deviation between the converted consistent fuzzy preference relation and the given interval fuzzy preference relation. Numerical examples demonstrate how the proposed framework can be applied in practice.

Significant future work remains open. For instance, the proposed approaches assume that the preference relation provided by the DM is complete. In a real decision process, it is possible that some pairwise comparison preference values are missing [1]. In this case, it becomes important to examine how the proposed models should be modified to accommodate incomplete interval-valued fuzzy preference relations. Another worthy topic is to extend these approaches to the case that the judgment matrix is given as complete or incomplete interval-valued intuitionistic fuzzy preference relations.

ACKNOWLEDGMENTS

The authors appreciate the constructive comments from the Editor-in-Chief, Dr. Witold Pedrycz, and the three anonymous referees that have helped improve the presentation and quality of this article. Zhoujing Wang would like to acknowledge the financial support from the Natural Science Foundation of Fujian Province of China (Grant No. 2010J01362) and Kevin W. Li is grateful for the financial support from the Natural Sciences and Engineering Research Council of Canada (NSERC) under its Discovery Grant program.

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