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LETTER TO THE EDITOR

A rigorous treatment of $O(\alpha^6 mc^2)$ QED corrections to the fine structure splittings of helium

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Abstract. Relativistic formulae for the energy level shifts due to electron self-energy corrections are derived within the external potential Bethe–Salpeter formalism. A rigorous treatment of QED corrections to the fine structure splittings of helium is carried out. Although some individual self-energy diagrams give contributions of order $\alpha^6 mc^2$, they are shown to sum to zero. In addition, the $\alpha^6 mc^2$ correction from vertex modifications in the presence of Coulomb photons does not contribute. Therefore, a rigorous treatment of all QED corrections of order $\alpha^6 mc^2$ to the fine structure splittings of helium now appears to be complete.

Fine structure splittings in the helium 1s2p ^{3}P state have recently atracted considerable attention because of dramatic improvements in experimental accuracy to ± 3 kHz (1 part in 10^7) [1]. Also, the lowest order Breit interaction terms of $O(\alpha^4 mc^2)$ have now been calculated to better than 1 part in 10^9 [2]. However, a meaningful comparison of theory and experiment at the ± 3 kHz level of accuracy requires also terms of $O(\alpha^6 mc^2)$ and perhaps higher. Operators for the spin-dependent terms of $O(\alpha^6 mc^2)$ were obtained many years ago by Douglas and Kroll [3] (see also [4]) but their derivation was in part phenomenological in their treatment of electron self-energy and vertex corrections. We have therefore re-examined their derivation, and introduced a rigorous relativistic formalism based on the Bethe–Salpeter equation for all radiative processes. This paper reports the final results of this work for spin-dependent terms up to $O(\alpha^6 mc^2)$. It also lays the ground work for an extension to terms of $O(\alpha^7 \ln\alpha\ mc^2)$ and $O(\alpha^7 mc^2)$.

In a previous paper [5], we presented a relativistic formula for the energy level shift due to vertex modifications to transverse photon exchange. A new $\alpha^6 mc^2$ QED correction to the fine structure splittings of helium was found from the relativistic correction of the second order vertex modification to transverse photon exchange in the absence of Coulomb photons. In this paper, we present relativistic formulae for the energy level shifts due to electron self-energy corrections. Previously used phenomenological operators for electron self-energy corrections are verified. In addition, we show that the total spin-dependent contributions of order $\alpha^6 mc^2$ arising from electron self-energy corrections sum to zero. (We exclude thoughout spin-dependent contact terms which vanish for triplet states.) This result comes from a non-trivial cancellation of various terms. Furthermore, we calculate the QED corrections due to the vertex modifications in the presence of Coulomb photons, which were not included in the previous calculation [5].

The Feynman diagrams to be considered in this paper are shown in figure 1, in addition to the diagrams in figure 2 given previously [5]. As before, we choose a mixed gauge in which the Feynman gauge is used for the radiative photon propagator, and the Coulomb

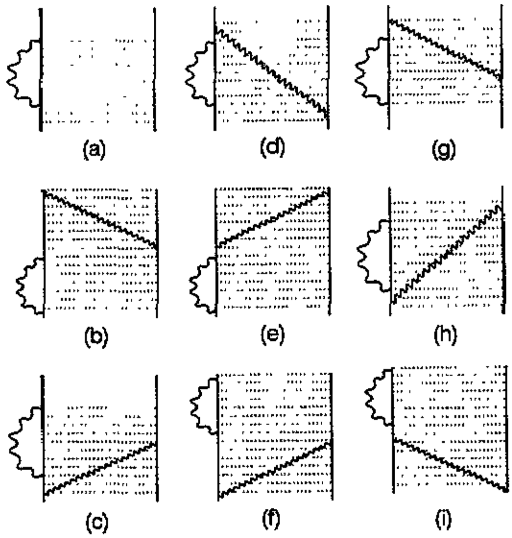

Figure 1. Feynman diagrams for electron self-energy corrections. The curved wavy lines denote covariant photons, the straight wavy lines transverse photons, and the broken lines instantaneous Coulomb photons.

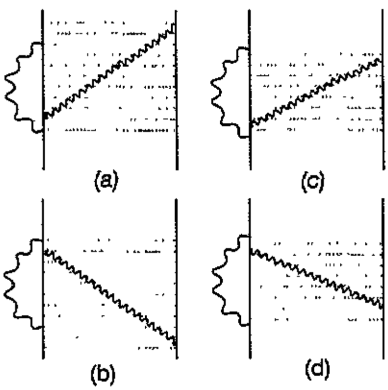

Figure 2. Feynman diagrams for vertex corrections to transverse photon exchange. The curved wavy lines denote covariant photons, the straight wavy lines transverse photons, and the broken lines instantaneous Coulomb photons.

gauge is used for the photon propagator between electrons. We adopt the same notation as used in reference [5]. The energy level shift from figure 1(a) is

$$\Delta E_{\mathrm{S}\sum_{n=0}^{\infty} C^n} = \frac{-\mathrm{i}e^2}{(2\pi)^4} \int \frac{\mathrm{d}^4 q}{q^2 + \mathrm{i}\delta} \langle \phi_{\mathrm{c}}(p_1, p_2) | \gamma_1^0 \gamma_1^\alpha \gamma_1^0 \\ \times \frac{1}{E - q_0 - H(p_1 - q) - \varepsilon(p_2) - \mathcal{L}_{2+}(p_2) I_{\mathrm{c}} \mathcal{L}_{2+}(p_2) + \mathrm{i}\delta} \gamma_{1\alpha} | \phi_{\mathrm{c}}(p_1, p_2) \rangle . \quad (1)$$

ϕ_{c} is the eigenfunction in the Coulomb ladder approximation defined in [6] and [3]. $\mathcal{L}_+$ is defined by

$$\mathcal{L}_+(p) = \frac{1}{2} \left(1 + \frac{H(p)}{\epsilon(p)} \right) .$$

Evaluating the corrections in equation (1), we reproduce the $\alpha^5 mc^2$ operators ($\Delta E_{\rm VR}$ and $\Delta E_{\rm CR}$) obtained by Douglas and Kroll [3], arising from electron self-energy modifications to the external potential and electron–electron Coulomb interaction. Together with the vertex correction of order $\alpha^5 mc^2$ in [5], we obtain the well known anomalous magnetic moment fine structure contributions [3] expressed in terms of Breit operators as

$$H^5 = \frac{\alpha}{2\pi}\left[2H^4_{\rm Z,so} + \frac{4}{3}H^4_{\rm e,so} + 2H^4_{\rm e,ss}\right]. \tag{2}$$

As expected, no $\alpha^6 mc^2$ QED correction to the fine structure splittings is found.

The diagrams shown in figures 1(*b*)–(*e*) give the following energy correction due to electron self-energy modification to transverse photon exchange

$$\begin{aligned}
\Delta E_{\rm ST,1} = {} & \frac{\alpha}{2\pi^2}\frac{-{\rm i}e^2}{(2\pi)^4}\int\frac{{\rm d}^4q}{q^2+{\rm i}\delta}\left(\frac{d\omega}{-2\pi{\rm i}}\right)\frac{{\rm d}k}{\omega^2-k^2+{\rm i}\delta}\langle\phi_{\rm c}(p_1,p_2)|\Bigg[\alpha_1^i\mathcal{L}_{1+}(p_1-k)\\
& \times\frac{1}{E-\omega-\varepsilon(p_1-k)-\varepsilon(p_2)-\mathcal{L}_{++}(p_1-k,p_2)I_{\rm c}\mathcal{L}_{++}(p_1-k,p_2)+{\rm i}\delta}\\
& \times\gamma_1^0\gamma_1^\alpha\gamma_1^0\frac{1}{E-\omega-q_0-H(p_1-k-q)-\varepsilon(p_2)-\mathcal{L}_{2+}(p_2)I_{\rm c}\mathcal{L}_{2+}(p_2)+{\rm i}\delta}\gamma_{1\alpha}\\
& \times\frac{1}{E-\omega-\varepsilon(p_1-k)-\varepsilon(p_2)-\mathcal{L}_{++}(p_1-k,p_2)I_{\rm c}\mathcal{L}_{++}(p_1-k,p_2)+{\rm i}\delta}\alpha_2^i\Bigg]\\
& +\Bigg[\alpha_1^i\mathcal{L}_{1+}(p_1-k)\\
& \times\frac{1}{E-\omega-\varepsilon(p_1-k)-\varepsilon(p_2)-\mathcal{L}_{++}(p_1-k,p_2)I_{\rm c}\mathcal{L}_{++}(p_1-k,p_2)+{\rm i}\delta}\\
& \times\gamma_1^0\gamma_1^\alpha\gamma_1^0\frac{1}{E-\omega-q_0-H(p_1-k-q)-\varepsilon(p_2)-\mathcal{L}_{2+}(p_2)I_{\rm c}\mathcal{L}_{2+}(p_2)+{\rm i}\delta}\alpha_2^i\\
& \times\frac{1}{E-q_0-H(p_1-k-q)-\varepsilon(p_2+k)-\mathcal{L}_{2+}(p_2+k)I_{\rm c}\mathcal{L}_{2+}(p_2+k)+{\rm i}\delta}\gamma_{1\alpha}\Bigg]\\
& +\Bigg[\alpha_2^i\frac{1}{E+\omega-\varepsilon(p_1)-\varepsilon(p_2+k)-\mathcal{L}_{++}(p_1,p_2+k)I_{\rm c}\mathcal{L}_{++}(p_1,p_2+k)+{\rm i}\delta}\\
& \times\alpha_1^i\mathcal{L}_{1+}(p_1-k)\frac{1}{E-\varepsilon(p_1-k)-\varepsilon(p_2+k)}\gamma_1^0\gamma_1^\alpha\gamma_1^0\\
& \times\frac{1}{E-q_0-H(p_1-k-q)-\varepsilon(p_2+k)-\mathcal{L}_{2+}(p_2+k)I_{\rm c}\mathcal{L}_{2+}(p_2+k)+{\rm i}\delta}\gamma_{1\alpha}\Bigg]\\
& +\Bigg[\alpha_1^i\mathcal{L}_{1+}(p_1-k)\\
& \times\frac{1}{E-\omega-\varepsilon(p_1-k)-\varepsilon(p_2)-\mathcal{L}_{++}(p_1-k,p_2)I_{\rm c}\mathcal{L}_{++}(p_1-k,p_2)+{\rm i}\delta}
\end{aligned}$$

$$\times\alpha_2^i \frac{1}{E-\varepsilon(p_1-k)-\varepsilon(p_2+k)}\gamma_1^0\gamma_1^\alpha\gamma_1^0$$

$$\left.\times\frac{1}{E-q_0-H(p_1-k-q)-\varepsilon(p_2+k)-\mathcal{L}_{2+}(p_2+k)I_c\mathcal{L}_{2+}(p_2+k)+\mathrm{i}\delta}\gamma_{1\alpha}\right]$$

$$\times|\phi_c(p_1-k,p_2+k)\rangle. \tag{3}$$

The last two terms are associated with figures 1(*d*) and (*e*). They can be shown to cancel with another two terms arising from second order Brillouin–Wigner perturbation theory. To illustrated this cancellation, we start from the Brillouin–Wigner perturbation formula

$$\Delta E = E - E_c = \langle\phi_c|H_\Delta[1-\Gamma(E,\phi_c)H_\Delta]^{-1}|\phi_c\rangle \tag{4}$$

where

$$\Gamma(E,\phi_c) = (E-H_c)^{-1}[1-|\phi_c\rangle\langle\phi_c|] \tag{5}$$

is the Green function. The perturbative Hamiltonian H_Δ is defined by

$$H_\Delta = \mathcal{L}_{++}I_c(1-\mathcal{L}_{++}) - \mathcal{L}_{--}I_c + \int \frac{\mathrm{d}\epsilon}{-2\pi\mathrm{i}} D\mathcal{F}g_\Delta(\mathcal{F}-g_\Delta)^{-1}I_c \tag{6}$$

where

$$\mathcal{F} = (E/2+\epsilon-H_1)(E/2-\epsilon-H_2) \qquad D = E - H_1 - H_2$$

and g_Δ is the interaction integral operator. The energy level shift is expanded as

$$\Delta E = \Delta E^{(1)} + \Delta E^{(2)} + \dots \tag{7}$$

where

$$\Delta E^{(1)} = \langle\phi_c|H_\Delta|\phi_c\rangle \qquad \Delta E^{(2)} = \langle\phi_c|H_\Delta\Gamma H_\Delta|\phi_c\rangle$$

etc. Equation (3) is derived from $\Delta E^{(1)}$. A term arising from $\Delta E^{(2)}$ is

$$\Delta E^2_{\mathrm{ST},1} = \langle\phi_c|\left[D\int\frac{\mathrm{d}\epsilon}{-2\pi\mathrm{i}}\mathcal{F}^{-1}g_\mathrm{T}\mathcal{F}^{-1}I_c\right]\Gamma\left[D\int\frac{\mathrm{d}\epsilon'}{-2\pi\mathrm{i}}\mathcal{F}^{-1}g_\mathrm{S}\mathcal{F}^{-1}I_c\right]|\phi_c\rangle \tag{8}$$

where g_T and g_S are transverse photon and electron self-energy operators. Equation (8) expands into the corresponding second-order Rayleigh–Schrödinger perturbation term, which is of nominal order α^7mc^2, plus two additional terms which cancel with the last two terms in equation (3).

However, the first two terms in equation (3) do contribute to the fine splittings of order α^6mc^2. Without the presence of Coulomb photons, the first term is given by

$$\Delta E^1_{\mathrm{ST},1} = \frac{\alpha}{2\pi^2}\int\frac{\mathrm{d}k}{2k}\langle\phi_c(p_1,p_2)|\alpha_1^i\mathcal{L}_{1+}(p_1-k)\frac{1}{E-k-E(p_1-k)-E(p_2)}$$

$$\times\gamma_1^0\Sigma_\mathrm{R}(p)\frac{1}{E-k-E(p_1-k)-E(p_2)}\alpha_2^i|\phi_c(p_1-k,p_2+k)\rangle \tag{9}$$

where

$$p = (E - k - E(\boldsymbol{p}_1 - \boldsymbol{k}) - E(\boldsymbol{p}_2), \mathbf{0}) - p_1$$

and $E(\boldsymbol{p}) = \sqrt{m^2 + \boldsymbol{p}^2}$. The renormalized electron self-energy modification function is

$$\Sigma_{\mathrm{R}}(p) = -\frac{\alpha}{4\pi}\frac{1}{p^2}(\not{p} - m)^2 \not{p}\left[1 + 6\int_0^1 \frac{\mathrm{d}x}{x} - \frac{1}{p^2}(\not{p} - m)^2 \int_0^1 \frac{\mathrm{d}x}{x}\right]. \quad (10)$$

The second term in equation (3) is then

$$\Delta E^2_{\mathrm{ST},1} = \frac{\alpha}{2\pi^2}\int \frac{\mathrm{d}\boldsymbol{k}}{2k}\langle\phi_{\mathrm{c}}(\boldsymbol{p}_1, \boldsymbol{p}_2)|\alpha^i_1 \mathcal{L}_{1+}(\boldsymbol{p}_1 - \boldsymbol{k})\frac{1}{E - k - E(\boldsymbol{p}_1 - \boldsymbol{k}) - E(\boldsymbol{p}_2)}$$
$$\times\gamma^0_1 \alpha^i_2 \Lambda^0_{\mathrm{R}}(\boldsymbol{k})|\phi_{\mathrm{c}}(\boldsymbol{p}_1 - \boldsymbol{k}, \boldsymbol{p}_2 + \boldsymbol{k})\rangle \quad (11)$$

where

$$\Lambda_{\mathrm{R}}(\boldsymbol{k}) = -\frac{\alpha}{4\pi}\frac{k}{m}\left[(4 - 3\gamma^0_1) + 6\int_0^1 \frac{\mathrm{d}x}{x}\right]. \quad (12)$$

To order $\alpha^6 mc^2$, the two corrections given by equations (9) and (11) cancel with each other. *In the presence of Coulomb photons, the QED corrections from the first two terms in* equation (3) contribute to the fine structure splittings only to $O(\alpha^7 mc^2)$ and higher.

A similar analysis applies to figures 1(f)–(i). Here the energy correction is

$$\Delta E_{\mathrm{ST},2} = \frac{\alpha}{2\pi^2}\frac{-\mathrm{i}e^2}{(2\pi)^4}\int \frac{\mathrm{d}^4 q}{q^2 + \mathrm{i}\delta}\left(\frac{\mathrm{d}\omega}{-2\pi\mathrm{i}}\right)\frac{\mathrm{d}\boldsymbol{k}}{\omega^2 - k^2 + \mathrm{i}\delta}\langle\phi_{\mathrm{c}}(\boldsymbol{p}_1, \boldsymbol{p}_2)|$$
$$\times\left[\alpha^i_2 \frac{1}{E + \omega - \varepsilon(\boldsymbol{p}_1) - \varepsilon(\boldsymbol{p}_2 + \boldsymbol{k}) - \mathcal{L}_{++}(\boldsymbol{p}_1, \boldsymbol{p}_2 + \boldsymbol{k}) I_{\mathrm{c}} \mathcal{L}_{++}(\boldsymbol{p}_1, \boldsymbol{p}_2 + \boldsymbol{k}) + \mathrm{i}\delta}\right.$$
$$\times\gamma^0_1 \gamma^\alpha_1 \gamma^0_1$$
$$\times\frac{1}{E + \omega - q_0 - H(\boldsymbol{p}_1 - \boldsymbol{q}) - \varepsilon(\boldsymbol{p}_2 + \boldsymbol{k}) - \mathcal{L}_{2+}(\boldsymbol{p}_2 + \boldsymbol{k}) I_{\mathrm{c}} \mathcal{L}_{2+}(\boldsymbol{p}_2 + \boldsymbol{k}) + \mathrm{i}\delta}$$
$$\times\gamma_{1\alpha} \mathcal{L}_{1+}(\boldsymbol{p}_1)$$
$$\left.\times\frac{1}{E + \omega - \varepsilon(\boldsymbol{p}_1) - \varepsilon(\boldsymbol{p}_2 + \boldsymbol{k}) - \mathcal{L}_{++}(\boldsymbol{p}_1, \boldsymbol{p}_2 + \boldsymbol{k}) I_{\mathrm{c}} \mathcal{L}_{++}(\boldsymbol{p}_1, \boldsymbol{p}_2 + \boldsymbol{k}) + \mathrm{i}\delta}\alpha^i_1\right]$$
$$+\left[\gamma^0_1 \gamma^\alpha_1 \gamma^0_1 \frac{1}{E - q_0 - H(\boldsymbol{p}_1 - \boldsymbol{q}) - \varepsilon(\boldsymbol{p}_2) - \mathcal{L}_{2+}(\boldsymbol{p}_2) I_{\mathrm{c}} \mathcal{L}_{2+}(\boldsymbol{p}_2) + \mathrm{i}\delta}\alpha^i_2\right.$$
$$\times\frac{1}{E + \omega - q_0 - H(\boldsymbol{p}_1 - \boldsymbol{q}) - \varepsilon(\boldsymbol{p}_2 + \boldsymbol{k}) - \mathcal{L}_{2+}(\boldsymbol{p}_2 + \boldsymbol{k}) I_{\mathrm{c}} \mathcal{L}_{2+}(\boldsymbol{p}_2 + \boldsymbol{k}) + \mathrm{i}\delta}$$
$$\times\gamma_{1\alpha} \mathcal{L}_{1+}(\boldsymbol{p}_1)$$
$$\left.\times\frac{1}{E + \omega - \varepsilon(\boldsymbol{p}_1) - \varepsilon(\boldsymbol{p}_2 + \boldsymbol{k}) - \mathcal{L}_{++}(\boldsymbol{p}_1, \boldsymbol{p}_2 + \boldsymbol{k}) I_{\mathrm{c}} \mathcal{L}_{++}(\boldsymbol{p}_1, \boldsymbol{p}_2 + \boldsymbol{k}) + \mathrm{i}\delta}\alpha^i_1\right]$$

$$
+\Bigg[\gamma_1^0\gamma_1^\alpha\gamma_1^0\frac{1}{E-q_0-H(\boldsymbol{p}_1-\boldsymbol{q})-\varepsilon(\boldsymbol{p}_2)-\mathcal{L}_{2+}(\boldsymbol{p}_2)I_c\mathcal{L}_{2+}(\boldsymbol{p}_2)+\mathrm{i}\delta}
$$

$$
\times\gamma_{1\alpha}\mathcal{L}_{1+}(\boldsymbol{p}_1)\frac{1}{E-\varepsilon(\boldsymbol{p}_1)-\varepsilon(\boldsymbol{p}_2)}\alpha_1^i
$$

$$
\times\frac{1}{E-\omega-\varepsilon(\boldsymbol{p}_1-\boldsymbol{k})-\varepsilon(\boldsymbol{p}_2)-\mathcal{L}_{++}(\boldsymbol{p}_1-\boldsymbol{k},\boldsymbol{p}_2)I_c\mathcal{L}_{++}(\boldsymbol{p}_1-\boldsymbol{k},\boldsymbol{p}_2)+\mathrm{i}\delta}\alpha_2^i\Bigg]
$$

$$
+\Bigg[\gamma_1^0\gamma_1^\alpha\gamma_1^0\frac{1}{E-q_0-H(\boldsymbol{p}_1-\boldsymbol{q})-\varepsilon(\boldsymbol{p}_2)-\mathcal{L}_{2+}(\boldsymbol{p}_2)I_c\mathcal{L}_{2+}(\boldsymbol{p}_2)+\mathrm{i}\delta}
$$

$$
\times\gamma_{1\alpha}\mathcal{L}_{1+}(\boldsymbol{p}_1)\frac{1}{E-\varepsilon(\boldsymbol{p}_1)-\varepsilon(\boldsymbol{p}_2)}\alpha_2^i
$$

$$
\times\frac{1}{E+\omega-\varepsilon(\boldsymbol{p}_1)-\varepsilon(\boldsymbol{p}_2+\boldsymbol{k})-\mathcal{L}_{++}(\boldsymbol{p}_1,\boldsymbol{p}_2+\boldsymbol{k})I_c\mathcal{L}_{++}(\boldsymbol{p}_1,\boldsymbol{p}_2+\boldsymbol{k})+\mathrm{i}\delta}\alpha_1^i\Bigg]
$$

$$
\times|\phi_c(\boldsymbol{p}_1-\boldsymbol{k},\boldsymbol{p}_2+\boldsymbol{k})\rangle. \tag{13}
$$

As before, the last two terms cancel with two terms derived from second order Brillouin–Wigner perturbation theory, and the spin-dependent parts of the first two terms cancel with each other to order α^6mc^2. Therefore, the total spin-dependent contribution from electron self-energy modifications of order α^6mc^2 is zero.

In our previous paper [5], we considered the additional diagrams shown in figure 2. In the approximation that Coulomb photons are neglected, we found a previously unsuspected contribution of

$$
\Delta E_6=-2\frac{\alpha^4}{12\pi^2}\langle\phi_0(\boldsymbol{r}_1,\boldsymbol{r}_2)|\sigma_2\cdot\left(\frac{\boldsymbol{r}}{r^4}\times\nabla_1/i\right)|\phi_0(\boldsymbol{r}_1,\boldsymbol{r}_2)\rangle \tag{14}
$$

to the fine structure splittings of helium. We have now verified that this result is correct to order α^6mc^2 even if additional Coulomb photons are included.

In summary, we have presented fully relativistic formulae for the energy level shift due to electron self-energy modifications. Using these formulae, the previously known QED corrections from electron self-energy are put on a rigorous footing. No new spin-dependent contribution of order α^6mc^2 is found beyond that recently reported [5]. We believe that the now known terms from Douglas and Kroll [3] and Zhang and Drake [5] are likely to be the total QED corrections of order α^6mc^2 to the fine structure splittings of helium. The relativistic formulae presented here and in [5] are necessary for an extension of this work to terms of order α^7mc^2.

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