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An Approach to Multiattribute Decision Making with
Interval-Valued Intuitionistic Fuzzy Assessments and Incomplete Weights

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Abstract

This article proposes an approach to multiattribute decision making with incomplete attribute weight information where individual assessments are provided as interval-valued intuitionistic fuzzy numbers (IVIFNs). By employing a series of optimization models, the proposed approach derives a linear program for determining attribute weights. The weights are subsequently used to synthesize individual IVIFN assessments into an aggregated IVIFN value for each alternative. In order to rank alternatives based on their aggregated IVIFN values, a novel method is developed for comparing two IVIFNs by introducing two new functions: the membership uncertainty index and the hesitation uncertainty index. An illustrative investment decision problem is employed to demonstrate how to apply the proposed procedure and comparative studies are conducted to show its overall consistency with existing approaches.

Keywords: Multiattribute decision making, interval-valued intuitionistic fuzzy numbers (IVIFNs), uncertainty index, linear programming

1. Introduction

Since the seminal work of Zadeh [39], the traditional 0-1 logic has been extended to fuzzy logic, characterized by a membership function between 0 and 1. This extension has triggered significant theoretical developments and numerous successful industrial applications [17, 41], and provides a powerful alternative other than probability theory to characterize uncertainty, imprecision, and vagueness in many fields [40]. Intuitionistic fuzzy sets (IFSs), initiated by Atanassov [1], represent one of the key theoretical developments, which considers not only to what degree an element belongs to a particular

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set (membership function) but also to what degree this element does not belong to the set (nonmembership function). The notion of IFSs is further generalized [3] by allowing the membership and nonmembership functions to assume interval values, thereby introducing the concept of interval-valued intuitionistic fuzzy sets (IVIFSs).

From a voting perspective, the membership function of an IFS can be loosely regarded as the percentage of approval votes, the nonmembership function can be interpreted as the rejection percentage, and the remaining portion that is not included in either the membership or nonmembership function can be conveniently treated as abstention. Due to these distinct features in characterizing vagueness and uncertainty in human decision making processes, IFSs have been widely employed to develop diverse decision aid tools. For instance, the concept of score functions is introduced by Chen and Tan [6] to evaluate alternatives under multiple attributes where assessments of each alternative against the attributes are expressed as vague values, or equivalently, intuitionistic fuzzy numbers as pointed out by Deschrijver and Kerre [9]. Subsequently, Hong and Choi [14] indicate that the score function cannot discriminate some alternatives although they are apparently different and, hence, propose an accuracy function to measure how accurate are the membership and nonmembership (or negation in the vague set term) functions, thereby furnishing additional discrimination powers. Liu and Wang [22] extend this research by first introducing an evaluation function based on t -norm and t -conorm and, then defining an intuitionistic fuzzy point operator and developing several new score functions based on the evaluation function and point operator. If a score function is employed to rank alternatives, a higher score value means a more preferred alternative.

Another active research topic is the investigation of multiattribute decision making by introducing intuitionistic fuzzy aggregation operators. Xu and Yager [37] and Xu [32] examine geometric and arithmetic aggregation operators, respectively. Multiattribute decision making under IFSs is further investigated by Li [20], where a series of optimization models are introduced and manipulated to generate optimal attribute weights. The applications of IFSs are also extended to decision situations involving multiple decision-makers (DMs): Szmidt and Kacprzyk [27] put forward some solution concepts in group decision making with intuitionistic fuzzy preference relations, and

Szmidt and Kacprzyk [28] further investigate how to reach consensus with intuitionistic fuzzy preference relations. Atanassov et al. [4] also present an algorithm for multi-person multiattribute decision making with crisp weights and intuitionistic fuzzy attribute values. Xu [33] defines consistent, incomplete, and acceptable preference relations and develops another approach to group decision making under the intuitionistic fuzzy environment.

With the aforesaid extensive research on applying IFSs to decision analysis, it is natural to expect that IVIFSs play a significant role in enriching decision modeling. However, the extension from exact numbers to interval values for the membership and nonmembership functions of IFSs poses considerable challenges in working with IVIFSs. Current research mainly focuses on basic operations and relations of IVIFSs as well as their properties [2]. Correlation and coefficient of correlation are first introduced by Bustince and Burillo [5], and then generalized to a general probability space [13]. Subsequently, Hung and Wu [15] develop a so-called “centroid” approach to calculating the correlation coefficient of IVIFSs. Another method is proposed by Xu [31], which possesses a key property that the correlation coefficient of two IVIFSs is one if and only if the two IVIFSs are identical. Other aspects of IVIFSs are also investigated, such as topological properties [25], relationships between IFSs, L -fuzzy sets, interval-valued fuzzy sets and IVIFSs [7-9], and the entropy and subethood of IVIFSs [23]. It is still at an inception stage to apply IVIFSs to decision modeling and limited literature exists in this specialized area. Xu [34] proposes some aggregation operators for interval-valued intuitionistic fuzzy information and applies them to multiattribute decision analysis. Xu and Yager [38] further investigate dynamic intuitionistic fuzzy aggregation operators and devise two procedures for dynamic intuitionistic fuzzy multiattribute decision making with intuitionistic fuzzy numbers (IFNs) or interval-valued intuitionistic fuzzy numbers (IVIFNs).

Multiattribute decision approaches provide decision aid by examining tradeoffs among alternative performances over multiple attributes [16]. Key information required in a multiattribute decision model includes attribute values or performance measures (individual assessments on alternatives against each attribute), attribute weights (reflecting the importance of each attribute to the overall decision problem), and a mechanism to synthesize this information into an aggregated value or assessment for each

alternative. With ever increasing complexity in many decision situations in reality, it is often a challenge for a decision-maker (DM) to provide attribute values and weights in a precise manner. Therefore, a general trend in the literature is to investigate decision models with incomplete information. For instance, attribute values have been relaxed to be a range rather than an exact value [4, 6, 14, 18, 20, 22, 27-30, 33-35, 38], and incomplete attribute weight information has also been extensively investigated from different perspectives [18, 26, 36]. In addition, more and more research along this direction has been conducted within a fuzzy or intuitionistic fuzzy framework [14, 19-22, 27-30, 33-36, 38]. The purpose of this article is to propose a novel approach to multiattribute decision analysis in which attribute values are expressed as IVIFNs and incomplete attribute weights are identified as a set of linear constraints that may take any form as those in [18, 26, 36]. To rank alternatives based on their aggregated IVIFN values, a new method is devised to compare any two IVIFNs in Section 3. To obtain aggregated IVIFN values, this approach, motivated by the treatments in [20], starts with manipulating a series of linear and nonlinear programming models, and eventually derives a linear program to determine attribute weights for aggregating individual IVIFN assessments into a single IVIFN value for each alternative (Section 4).

Intuitively, extending from IFNs to IVIFNs furnishes additional capability to handle vague information because the membership and nonmembership degrees are only needed to be expressed as ranges of values rather than exact values. When the uncertainty in an IVIFN's membership and nonmembership degrees diminishes to zero, the IVIFN is reduced to an IFN. Therefore, compared to the multiattribute decision models in existing literature [14, 20, 22, 36, 38], the proposed approach makes a useful contribution by empowering a DM with more flexibility in tackling vagueness and uncertainty in its assessments, thereby providing an effective means to applying IVIFNs in multiattribute decision making with incomplete weights. Another key contribution of this article is the novel comparison method for IVIFNs in Section 3, which is able to differentiate any two IVIFNs.

An earlier version of this paper was presented at a conference and published in the proceedings [30]. The current manuscript significantly expands the conference paper by providing new theorems (Section 4) to validate the proposed approach and introducing a

new method (Section 3) to compare two IVIFNs rather than depending on a TOPSIS (technique for order performance by similarity to ideal solution [16]) based approach to ranking alternatives. Moreover, this paper has been thoroughly rewritten to explain the procedure more carefully and enhance its readability. The updated illustrative example in Section 5 demonstrates that two alternatives cannot be distinguished by using the TOPSIS approach in the conference paper, but a full ranking can be obtained by using the newly designed approach to comparing two IVIFNs in Section 3. The approach here also significantly differs from that reported in Wang and Wang [29], from the process of determining attribute weights (eigenvalue-based), to the aggregation operator (weighted arithmetic average) and ranking method (only score and accuracy functions are employed there).

The remainder of this paper is organized as follows: Section 2 reviews some basic concepts related to IFSs and IVIFSs. A novel method is introduced for comparing any two IVIFNs in Section 3. Section 4 establishes a linear programming approach to multiattribute decision making under interval-valued intuitionistic fuzzy environment. A numerical example is developed to demonstrate how to apply the proposed approach and some comparative studies are conducted in Section 5, followed by some concluding remarks in Section 6.

2. Preliminaries

Some basic concepts on IFSs and IVIFSs are introduced below to facilitate future discussions.

Definition 2.1 (Atanassov [1]). Let a set X be fixed, an intuitionistic fuzzy set (IFS) A in X is defined as

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$$

where the functions $\mu_A : X \rightarrow [0,1]$, $x \in X$, $\mu_A(x) \in [0,1]$ and $\nu_A : X \rightarrow [0,1]$, $x \in X$, $\nu_A(x) \in [0,1]$ satisfy the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, $\forall x \in X$.

$\mu_A(x)$ and $\nu_A(x)$ denote the degrees of membership and nonmembership of element $x \in X$ to set A , respectively. $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is usually called the intuitionistic fuzzy index of $x \in A$, representing the degree of indeterminacy or hesitation of x to A . It is

151 obvious that $0 \leq \pi_A(x) \leq 1$ for every $x \in X$.

152 Deschrijver and Kerre [9] have shown that IFSs are equivalent to interval-valued
 153 fuzzy sets (also called vague sets [10]) and both can be regarded as L -fuzzy sets in the
 154 sense of Goguen [11].

155 In reality, it may not be easy to identify exact values for the membership and
 156 nonmembership degrees of an element to a set. In this case, a range of values may be a
 157 more appropriate measurement to accommodate the vagueness. As such, Atanassov and
 158 Gargov [3] introduce the notion of interval-valued intuitionistic fuzzy set (IVIFS).

159 *Definition 2.2* (Atanassov and Gargov [3]). Let X be a non-empty set of the universe,
 160 and $D[0,1]$ be the set of all closed subintervals of $[0, 1]$, an interval-valued intuitionistic
 161 fuzzy set (IVIFS) \tilde{A} in X is defined by

$$162 \quad \tilde{A} = \{ \langle x, \tilde{\mu}_{\tilde{A}}(x), \tilde{\nu}_{\tilde{A}}(x) \rangle \mid x \in X \}$$

163 where $\tilde{\mu}_{\tilde{A}} : X \rightarrow D[0,1]$, $\tilde{\nu}_{\tilde{A}} : X \rightarrow D[0,1]$, with the condition $0 \leq \sup(\tilde{\mu}_{\tilde{A}}(x)) +$
 164 $\sup(\tilde{\nu}_{\tilde{A}}(x)) \leq 1$ for any $x \in X$.

165 Similarly, the intervals $\tilde{\mu}_{\tilde{A}}(x)$ and $\tilde{\nu}_{\tilde{A}}(x)$ denote the degree of membership and
 166 nonmembership of x to A , respectively. But, here, for each $x \in X$, $\tilde{\mu}_{\tilde{A}}(x)$ and $\tilde{\nu}_{\tilde{A}}(x)$ are
 167 closed intervals rather than real numbers and their lower and upper boundaries are
 168 denoted by $\tilde{\mu}_{\tilde{A}}^L(x), \tilde{\mu}_{\tilde{A}}^U(x), \tilde{\nu}_{\tilde{A}}^L(x), \tilde{\nu}_{\tilde{A}}^U(x)$, respectively. Therefore, another equivalent way
 169 to express an IVIFS \tilde{A} is

$$170 \quad \tilde{A} = \{ \langle x, [\tilde{\mu}_{\tilde{A}}^L(x), \tilde{\mu}_{\tilde{A}}^U(x)], [\tilde{\nu}_{\tilde{A}}^L(x), \tilde{\nu}_{\tilde{A}}^U(x)] \rangle \mid x \in X \} ,$$

171 where $\tilde{\mu}_{\tilde{A}}^U(x) + \tilde{\nu}_{\tilde{A}}^U(x) \leq 1, 0 \leq \tilde{\mu}_{\tilde{A}}^L(x) \leq \tilde{\mu}_{\tilde{A}}^U(x) \leq 1, 0 \leq \tilde{\nu}_{\tilde{A}}^L(x) \leq \tilde{\nu}_{\tilde{A}}^U(x) \leq 1$.

172 Similar to IFSs, for each element $x \in X$ we can compute its hesitation interval relative
 173 to \tilde{A} as:

$$174 \quad \tilde{\pi}_{\tilde{A}}(x) = [\tilde{\pi}_{\tilde{A}}^L(x), \tilde{\pi}_{\tilde{A}}^U(x)] = [1 - \tilde{\mu}_{\tilde{A}}^U(x) - \tilde{\nu}_{\tilde{A}}^U(x), 1 - \tilde{\mu}_{\tilde{A}}^L(x) - \tilde{\nu}_{\tilde{A}}^L(x)]$$

175 If each of the intervals $\tilde{\mu}_{\tilde{A}}(x)$ and $\tilde{\nu}_{\tilde{A}}(x)$ contains only one real value, i.e., if for every
 176 $x \in X$,

$$177 \quad \mu_A(x) = \tilde{\mu}_{\tilde{A}}^L(x) = \tilde{\mu}_{\tilde{A}}^U(x) , \nu_A(x) = \tilde{\nu}_{\tilde{A}}^L(x) = \tilde{\nu}_{\tilde{A}}^U(x)$$

then, the given IVIFS \tilde{A} is degraded to an ordinary IFS.

For any given x , the pair $(\tilde{\mu}_{\tilde{A}}(x), \tilde{\nu}_{\tilde{A}}(x))$ is called an interval-valued intuitionistic fuzzy number (IVIFN) [34,38]. For convenience, the pair $(\tilde{\mu}_{\tilde{A}}(x), \tilde{\nu}_{\tilde{A}}(x))$ is often denoted by $([a, b], [c, d])$, where $[a, b] \in D[0, 1]$, $[c, d] \in D[0, 1]$ and $b + d \leq 1$.

Remark 2.1

For IFSs, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ measures a DM's hesitation about the membership of x to A and also represents the DM's uncertainty. For IVIFSs, the uncertainty comes from three sources: membership uncertainty in $[\tilde{\mu}_{\tilde{A}}^L(x), \tilde{\mu}_{\tilde{A}}^U(x)]$, nonmembership uncertainty in $[\tilde{\nu}_{\tilde{A}}^L(x), \tilde{\nu}_{\tilde{A}}^U(x)]$, and hesitation uncertainty in $\tilde{\pi}_{\tilde{A}}(x) = [\tilde{\pi}_{\tilde{A}}^L(x), \tilde{\pi}_{\tilde{A}}^U(x)] = [1 - \tilde{\mu}_{\tilde{A}}^U(x) - \tilde{\nu}_{\tilde{A}}^U(x), 1 - \tilde{\mu}_{\tilde{A}}^L(x) - \tilde{\nu}_{\tilde{A}}^L(x)]$. This differentiation of uncertainty sources plays an instrumental role in devising a novel method for comparing two IVIFNs in Section 3.

3. A novel method for comparing two IVIFNs

In the proposed multiattribute decision approach in Section 4, the eventual evaluation of each alternative will be based on an aggregated IVIFN. In order to rank alternatives, it is necessary to consider how to compare two IVIFNs.

For intuitionistic fuzzy numbers (IFNs), Chen and Tan [6] introduce a score function, defined as the difference of membership and nonmembership function, to evaluate alternatives and, then, develop a multiattribute decision making approach under the IFS environment. Later, Hong and Choi [14] note that the score function alone cannot differentiate many IFNs even though they are obviously different. To make the comparison method more discriminatory, an accuracy function, defined as the sum of the membership and nonmembership function, is introduced to measure how accurate are the membership and nonmembership functions of an IFN. Subsequently, a procedure combining the score function and accuracy function is designed to handle multiattribute decision making problems with IFNs [14]. Built upon the concepts of score and accuracy functions, Xu [32] devises a new approach to comparing two IFNs.

When the comparison of two IFNs is extended to the interval-valued case, a similar line of thinking can be adopted. For instance, Xu [34] introduces the score and accuracy functions for IVIFNs and applies them to compare two IVIFNs. However, due to the

specific characteristics of intervals and the three different types of uncertainty (See Remark 2.1), the score and accuracy functions together sometimes cannot tell the difference between two IVIFNs. In this case, it is necessary to examine the difference between two IVIFNs using two additional functions as detailed below. The first two functions are proposed by Xu [34], but the last two are introduced in this research.

1. Score function: The difference between the membership and nonmembership functions, $\tilde{\mu}_{\tilde{A}} = [a, b]$ and $\tilde{\nu}_{\tilde{A}} = [c, d]$. As these functions are interval-valued, the means of the respective intervals are employed for the calculation. This difference is comparable to the score function in the IFN case and, hence, we have:

Definition 3.1 (Xu [34]) For an IVIFN $\tilde{\alpha} = ([a, b], [c, d])$, its score function is defined as $S(\tilde{\alpha}) = \frac{a + b - c - d}{2}$.

It is obvious that $-1 \leq S(\tilde{\alpha}) \leq 1$. The score function captures the overall degree of belonging to a certain set by deducting its nonmembership from its membership function and, hence, can be used as a basis to compare two IVIFNs. For two IVIFNs, the one with a smaller score function corresponds to a smaller IVIFN. However, two different IVIFNs may possess an identical score value as shown in the following example.

Example 3.1 Let $\tilde{\alpha}_1 = ([0.2, 0.3], [0.2, 0.3])$ and $\tilde{\alpha}_2 = ([0.4, 0.5], [0.4, 0.5])$. It is trivial to confirm that $S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2) = 0$, but these two IVIFNs are obviously different.

2. Accuracy function: When the score function alone cannot differentiate two IVIFNs as shown in Example 3.1, additional information, the sum of the membership and nonmembership functions, should now be considered. This idea is similar to the accuracy function in [14] except that the mean values of the intervals are employed here.

Definition 3.2 (Xu [34]) For an IVIFN $\tilde{\alpha} = ([a, b], [c, d])$, its accuracy function is defined as $H(\tilde{\alpha}) = \frac{a + b + c + d}{2}$.

Generally speaking, the accuracy function measures the amount of information captured by the membership and nonmembership functions, and the remaining portion characterizes the degree of hesitation. When the score function is the same for two IVIFNs, the smaller the accuracy function, the larger the hesitation and, hence, the

smaller the corresponding IVIFN. For the two IVIFNs in Example 3.1, since their score function value is identical but $H(\tilde{\alpha}_1) = 0.5 < H(\tilde{\alpha}_2) = 0.9$, we have $\tilde{\alpha}_1 < \tilde{\alpha}_2$.

It is clear that the introduction of the accuracy function increases the discriminatory power. Nevertheless, in some situations, the score and accuracy functions together still cannot tell the difference between two distinct IVIFNs. For instance,

Example 3.2 Let $\tilde{\alpha}_1 = ([0, 0.4], [0.3, 0.4])$, $\tilde{\alpha}_2 = ([0.1, 0.3], [0.3, 0.4])$, $\tilde{\alpha}_3 = ([0, 0.4], [0.18, 0.52])$, $\tilde{\alpha}_4 = ([0.05, 0.35], [0.2, 0.5])$, $\tilde{\alpha}_5 = ([0.2, 0.2], [0.3, 0.4])$. It is easy to verify that $S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2) = S(\tilde{\alpha}_3) = S(\tilde{\alpha}_4) = S(\tilde{\alpha}_5) = -0.15$ and $H(\tilde{\alpha}_1) = H(\tilde{\alpha}_2) = H(\tilde{\alpha}_3) = H(\tilde{\alpha}_4) = H(\tilde{\alpha}_5) = 0.55$. Therefore, these five IVIFNs are still indistinguishable.

As a matter of fact, for any two IVIFNs, as long as the means of their membership and nonmembership intervals are respectively equal, the score and accuracy functions of the two IVIFNs will be identical and, hence, indistinguishable under these two functions.

3. Membership uncertainty index function: When both score and accuracy functions fail to distinguish two IVIFNs, the difference of the uncertainty in the membership and nonmembership functions is considered.

Intuitively, the uncertainty of a membership (nonmembership) function is measured by the width of the interval: the wider a membership (nonmembership) interval, the more uncertain an element's membership (nonmembership) is. When the width of the interval diminishes to zero, it is known exactly to what degree an element belongs (does not belong) to a particular set. In this case, no uncertainty exists about an element's membership (nonmembership) to the set.

Definition 3.3 For an IVIFN $\tilde{\alpha} = ([a, b], [c, d])$, its membership uncertainty index is defined as $T(\tilde{\alpha}) = b + c - a - d$.

It is easy to tell that $-1 \leq T(\tilde{\alpha}) \leq 1$. Understandably, when the score and accuracy functions are equal for two IVIFNs, the larger a $T(\cdot)$ value, the smaller the corresponding IVIFN is. For the five IVIFNs in Example 3.2, applying Definition 3.3 yields $T(\tilde{\alpha}_1) = 0.3$, $T(\tilde{\alpha}_2) = 0.1$, $T(\tilde{\alpha}_3) = 0.06$, $T(\tilde{\alpha}_4) = 0$, and $T(\tilde{\alpha}_5) = -0.1$. As $T(\tilde{\alpha}_1) > T(\tilde{\alpha}_2) > T(\tilde{\alpha}_3) > T(\tilde{\alpha}_4) > T(\tilde{\alpha}_5)$, one can have $\tilde{\alpha}_1 < \tilde{\alpha}_2 < \tilde{\alpha}_3 < \tilde{\alpha}_4 < \tilde{\alpha}_5$.

However, with the three functions, $S(\cdot)$, $H(\cdot)$, and $T(\cdot)$, some IVIFNs still cannot be

264 differentiated. For example,

265 *Example 3.3* Assume that $\tilde{\alpha}_1 = ([0.05, 0.35], [0.25, 0.55])$, $\tilde{\alpha}_2 = ([0.1, 0.3], [0.3, 0.5])$,

266 $\tilde{\alpha}_3 = ([0.15, 0.25], [0.35, 0.45])$, $\tilde{\alpha}_4 = ([0.2, 0.2], [0.4, 0.4])$, then,

267
$$S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2) = S(\tilde{\alpha}_3) = S(\tilde{\alpha}_4) = -0.2$$

268
$$H(\tilde{\alpha}_1) = H(\tilde{\alpha}_2) = H(\tilde{\alpha}_3) = H(\tilde{\alpha}_4) = 0.6$$

269
$$T(\tilde{\alpha}_1) = T(\tilde{\alpha}_2) = T(\tilde{\alpha}_3) = T(\tilde{\alpha}_4) = 0$$

270 Therefore, $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3$, and $\tilde{\alpha}_4$ cannot be differentiated by using $S(\cdot), H(\cdot)$, and $T(\cdot)$.

271 In general, for any two IVIFNs $\tilde{\alpha} = ([a_1, b_1], [c_1, d_1])$ and $\tilde{\beta} = ([a_2, b_2], [c_2, d_2])$, if $a_1 +$

272 $b_1 = a_2 + b_2$, $c_1 + d_1 = c_2 + d_2$, $b_1 + c_1 = b_2 + c_2$, and $a_1 + d_1 = a_2 + d_2$, then, $S(\tilde{\alpha}) = S(\tilde{\beta})$,

273 $H(\tilde{\alpha}) = H(\tilde{\beta})$, and $T(\tilde{\alpha}) = T(\tilde{\beta})$, hence, the three functions will not be able to distinguish

274 these two IVIFNs. In this case, the uncertainty contained in the hesitation interval has to
275 be examined.

276 4. Hesitation uncertainty index function: Once again, the uncertainty in the
277 hesitation interval, $\tilde{\pi}_{\tilde{A}}(x) = [\tilde{\pi}_{\tilde{A}}^L(x), \tilde{\pi}_{\tilde{A}}^U(x)] = [1 - b - d, 1 - a - c]$, is measured by its width.

278 *Definition 3.4* For an IVIFN $\tilde{\alpha} = ([a, b], [c, d])$, its hesitation uncertainty index is
279 defined as $G(\tilde{\alpha}) = b + d - a - c$.

280 When the other three functions are equal, a larger hesitation uncertainty corresponds
281 to a smaller IVIFN. By introducing $G(\cdot)$, the four IVIFNs in Example 3.3 can be ranked.
282 As $G(\tilde{\alpha}_1) = 0.6 > G(\tilde{\alpha}_2) = 0.4 > G(\tilde{\alpha}_3) = 0.2 > G(\tilde{\alpha}_4) = 0$, $\tilde{\alpha}_1 < \tilde{\alpha}_2 < \tilde{\alpha}_3 < \tilde{\alpha}_4$.

283 Given these analyses, we can now introduce a procedure to compare any two IVIFSs.

284 *Definition 3.5* For any two IVIFNs $\tilde{\alpha} = ([a_1, b_1], [c_1, d_1])$ and $\tilde{\beta} = ([a_2, b_2], [c_2, d_2])$,

285 If $S(\tilde{\alpha}) < S(\tilde{\beta})$, then $\tilde{\alpha}$ is smaller than $\tilde{\beta}$, denoted by $\tilde{\alpha} < \tilde{\beta}$;

286 If $S(\tilde{\alpha}) > S(\tilde{\beta})$, then $\tilde{\alpha}$ is greater than $\tilde{\beta}$, denoted by $\tilde{\alpha} > \tilde{\beta}$;

287 If $S(\tilde{\alpha}) = S(\tilde{\beta})$, then

288 1) If $H(\tilde{\alpha}) < H(\tilde{\beta})$, then $\tilde{\alpha}$ is smaller than $\tilde{\beta}$, denoted by $\tilde{\alpha} < \tilde{\beta}$;

289 2) If $H(\tilde{\alpha}) > H(\tilde{\beta})$, then $\tilde{\alpha}$ is greater than $\tilde{\beta}$, denoted by $\tilde{\alpha} > \tilde{\beta}$;

290 3) If $H(\tilde{\alpha}) = H(\tilde{\beta})$, then

- 291 i) If $T(\tilde{\alpha}) > T(\tilde{\beta})$, then $\tilde{\alpha}$ is smaller than $\tilde{\beta}$, denoted by $\tilde{\alpha} < \tilde{\beta}$;
- 292 ii) If $T(\tilde{\alpha}) < T(\tilde{\beta})$, then $\tilde{\alpha}$ is greater than $\tilde{\beta}$, denoted by $\tilde{\alpha} > \tilde{\beta}$;
- 293 iii) If $T(\tilde{\alpha}) = T(\tilde{\beta})$, then
- 294 a) If $G(\tilde{\alpha}) > G(\tilde{\beta})$, then $\tilde{\alpha}$ is smaller than $\tilde{\beta}$, denoted by $\tilde{\alpha} < \tilde{\beta}$;
- 295 b) If $G(\tilde{\alpha}) < G(\tilde{\beta})$, then $\tilde{\alpha}$ is greater than $\tilde{\beta}$, denoted by $\tilde{\alpha} > \tilde{\beta}$;
- 296 c) If $G(\tilde{\alpha}) = G(\tilde{\beta})$, then $\tilde{\alpha}$ and $\tilde{\beta}$ represent the same information, denoted by
- 297 $\tilde{\alpha} = \tilde{\beta}$

298 **Remark 3.1**

299 Definition 3.5 establishes a novel approach to comparing any two IVIFNs by taking a
 300 prioritized sequence of score, accuracy, membership uncertainty index, and hesitation
 301 uncertainty index functions. When two IVIFNs are compared, this sequence follows a
 302 logic order of examining the overall belonging degree, the level of accuracy or hesitation,
 303 the membership uncertainty index, and the hesitation uncertainty index. The comparison
 304 process continues until the two IVIFNs are distinguished by one of the four functions in
 305 Definition 3.5. Once these two IVIFNs are differentiated at a certain priority level, the
 306 calculation terminates and functions at lower priority levels will not be computed. This
 307 prioritized sequence of comparison method has many applications in reality. For instance,
 308 many Canadian research-intensive institutions recruit their tenure-track faculty members
 309 following a priority order of research first, teaching second, and service last. Theorem 3.1
 310 below confirms that any two different IVIFNs will always be distinguishable by
 311 Definition 3.5.

312 *Theorem 3.1* Let $\tilde{\alpha} = ([a_1, b_1], [c_1, d_1])$ and $\tilde{\beta} = ([a_2, b_2], [c_2, d_2])$ be two IVIFNs, then
 313 $\tilde{\alpha} = \tilde{\beta}$ iff $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$.

314 *Proof:* The sufficient condition obviously holds true. Next, if $\tilde{\alpha} = \tilde{\beta}$, then Definition
 315 3.5 implies that $S(\tilde{\alpha}) = S(\tilde{\beta})$, $H(\tilde{\alpha}) = H(\tilde{\beta})$, $T(\tilde{\alpha}) = T(\tilde{\beta})$, and $G(\tilde{\alpha}) = G(\tilde{\beta})$.
 316 From the definitions of $S(\cdot), H(\cdot), T(\cdot)$, and $G(\cdot)$, we have

317
$$a_1 + b_1 - c_1 - d_1 = a_2 + b_2 - c_2 - d_2, \quad a_1 + b_1 + c_1 + d_1 = a_2 + b_2 + c_2 + d_2$$

318
$$b_1 + c_1 - a_1 - d_1 = b_2 + c_2 - a_2 - d_2, \quad b_1 + d_1 - a_1 - c_1 = b_2 + d_2 - a_2 - c_2$$

319 By solving the four equations, we have $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$. Q.E.D.

Definition 3.6 Let $[a_1, b_1], [a_2, b_2]$ be two interval numbers over $[0, 1]$. A relation " \leq " in $D[0,1]$ is defined as: $[a_1, b_1] \leq [a_2, b_2]$ iff $a_1 \leq a_2$ and $b_1 \leq b_2$.

This definition can be treated as a special case of Definition 2.1 in [8] and, hence, $\langle D[0,1], "\leq" \rangle$ constitutes a complete lattice.

For any two IVIFNs, $\tilde{\alpha}$ and $\tilde{\beta}$, denote $\tilde{\alpha} \leq \tilde{\beta}$ iff $\tilde{\alpha} < \tilde{\beta}$ or $\tilde{\alpha} = \tilde{\beta}$.

Theorem 3.2 Let $\tilde{\alpha} = ([a_1, b_1], [c_1, d_1])$ and $\tilde{\beta} = ([a_2, b_2], [c_2, d_2])$ be two IVIFNs, if $[a_1, b_1] \leq [a_2, b_2]$ and $[c_1, d_1] \geq [c_2, d_2]$, then $\tilde{\alpha} \leq \tilde{\beta}$.

Proof: Since $[a_1, b_1] \leq [a_2, b_2]$ and $[c_1, d_1] \geq [c_2, d_2]$, Definition 3.6 implies that $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \geq c_2$, and $d_1 \geq d_2$.

By the definition of score functions, we have $S(\tilde{\alpha}) - S(\tilde{\beta}) = (a_1 + b_1 - c_1 - d_1)/2 - (a_2 + b_2 - c_2 - d_2)/2 = (a_1 + b_1 - a_2 - b_2)/2 + (c_2 + d_2 - c_1 - d_1)/2 \leq 0$. Two cases have to be considered:

1) if $S(\tilde{\alpha}) - S(\tilde{\beta}) < 0$, then $\tilde{\alpha} < \tilde{\beta}$ as per Definition 3.5. Otherwise,

2) if $S(\tilde{\alpha}) - S(\tilde{\beta}) = 0$ then

$$a_1 + b_1 - c_1 - d_1 = a_2 + b_2 - c_2 - d_2 \quad (3.1)$$

Rearranging the terms yields

$$c_1 + d_1 = a_1 + b_1 - a_2 - b_2 + c_2 + d_2 \quad (3.2)$$

According to the definition of accuracy functions,

$$H(\tilde{\alpha}) - H(\tilde{\beta}) = (a_1 + b_1 + c_1 + d_1)/2 - (a_2 + b_2 + c_2 + d_2)/2 \quad (3.3)$$

Plugging (3.2) into (3.3), one can have $H(\tilde{\alpha}) - H(\tilde{\beta}) = (a_1 - a_2) + (b_1 - b_2) \leq 0$. Once again, two cases may arise

a) if $H(\tilde{\alpha}) - H(\tilde{\beta}) < 0$ then $\tilde{\alpha} < \tilde{\beta}$ by Definition 3.5. Otherwise,

b) if $H(\tilde{\alpha}) - H(\tilde{\beta}) = 0$, i.e., $H(\tilde{\alpha}) - H(\tilde{\beta}) = (a_1 - a_2) + (b_1 - b_2) = 0$, then

$$a_1 + b_1 = a_2 + b_2 \quad (3.4)$$

(3.4) - (3.1) leads to $c_1 + d_1 = c_2 + d_2$. By rearranging these terms, we have

$$a_1 - a_2 = b_2 - b_1, \quad c_1 - c_2 = d_2 - d_1 \quad (3.5)$$

As $a_1 - a_2 \leq 0$ and $b_2 - b_1 \geq 0$, the first equation in (3.5) implies that $a_1 - a_2 = b_2 - b_1 = 0$. Similarly, as $c_1 - c_2 \leq 0$ and $d_2 - d_1 \geq 0$, the second part of (3.5) yields $c_1 - c_2 = d_2 - d_1 = 0$. Therefore, we have $a_1 = a_2, b_1 = b_2, c_1 = c_2$, and $d_1 = d_2$ and, hence, $\tilde{\alpha} = \tilde{\beta}$.

Q.E.D.

The proof also reveals that any two IVIFNs satisfying the conditions of Theorem 3.2 can be differentiated by the score and accuracy functions.

4. An approach to multiattribute decision making with interval-valued intuitionistic fuzzy assessments and incomplete weights

This section puts forward a framework for multiattribute decision making with incomplete weight information, where assessments of alternatives against attributes are given as interval-valued intuitionistic fuzzy numbers and incomplete attribute weight information is provided by the DM as a set of linear constraints.

4.1 Problem formulations

Given an alternative set $X = \{x_1, x_2, \dots, x_n\}$, consisting of n non-inferior decision alternatives, and an attribute set $A = (a_1, a_2, \dots, a_m)$. Each alternative is assessed on each of the m attributes and the assessment is expressed as an IVIFN, describing the satisfaction and dissatisfaction degree of the alternative to a fuzzy concept of “excellence” as per a particular attribute. The decision problem is to select a most preferred alternative from X or obtain a ranking of all alternatives based on the overall assessments of all alternatives on the m attributes.

More Specifically, let $\tilde{R} = (\tilde{r}_{ij})_{n \times m} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])_{n \times m}$ be the interval-valued intuitionistic fuzzy decision matrix, where $[a_{ij}, b_{ij}]$ and $[c_{ij}, d_{ij}]$ are the membership and nonmembership intervals of alternative x_i on attribute a_j as per a fuzzy concept “excellence” given by a decision-maker (DM), indicating to what degree x_i satisfies and does not satisfy the “excellence” requirement as per a_j , respectively. By Definition 2.2, $[a_{ij}, b_{ij}] \in D[0, 1]$, $[c_{ij}, d_{ij}] \in D[0, 1]$, and $b_{ij} + d_{ij} \leq 1$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$. It is clear that the lowest satisfaction degree of x_i with respect to a_j is $[a_{ij}, b_{ij}]$, as given in the

membership function, and the highest satisfaction degree of x_i with respect to a_j is $[1-d_{ij}, 1-c_{ij}]$, in the case that all hesitation is treated as membership or satisfaction.

In a multiattribute decision making problem, different weights on attributes reflect their varying importance in choosing the optimal alternative. Let $w = (w_1, w_2, \dots, w_m)^T$ be the attribute weight vector, where $w_j \geq 0, j = 1, 2, \dots, m$, and the weight vector is often

normalized to one, i.e. $\sum_{j=1}^m w_j = 1$. In reality, due to the increasing complexity of many

practical decision situations, the DM may not be confident in providing exact values for attribute weights. Instead, the DM may only possess partial knowledge about attribute weights [18]. This phenomenon has triggered significant research on developing decision models for handling incomplete attribute weights [18,26,36]. Generally speaking, the incomplete attribute weight information can be expressed as the following relationships among the weights:

- 1) A weak ranking: $\{w_{j_1} \geq w_{j_2}\}, j_1 \neq j_2$;
- 2) A strict ranking: $\{w_{j_1} - w_{j_2} \geq \varepsilon_{j_1 j_2} (> 0)\}, j_1 \neq j_2$;
- 3) A ranking with multiples: $\{w_{j_1} \geq \alpha_{j_1 j_2} w_{j_2}\}, 0 \leq \alpha_{j_1 j_2} \leq 1, j_1 \neq j_2$;
- 4) An interval form: $\{\beta_j \leq w_j \leq \beta_j + \varepsilon_j\}, 0 \leq \beta_j < \beta_j + \varepsilon_j \leq 1$;
- 5) A ranking of differences: $\{w_{j_1} - w_{j_2} \geq w_{j_3} - w_{j_4}\}, \text{ for } j_1 \neq j_2 \neq j_3 \neq j_4$.

In a particular decision problem, the partial knowledge about the attribute weights can be a subset of the aforementioned relationships, denoted by H .

As mentioned earlier, the satisfaction degree of x_i with respect to a_j , denoted by $[\xi_{ij}, \eta_{ij}]$, should lie between $[a_{ij}, b_{ij}]$ and $[1-d_{ij}, 1-c_{ij}]$. When all individual assessments of alternative x_i is aggregated by incorporating attribute weights, it is expected that the optimal satisfaction degree should also satisfy this condition, i.e., $[a_{ij}, b_{ij}] \leq [\xi_{ij}, \eta_{ij}] \leq [1-d_{ij}, 1-c_{ij}]$. According to Definition 3.6, ξ_{ij} and η_{ij} should satisfy $a_{ij} \leq \xi_{ij} \leq 1-d_{ij}$ and $b_{ij} \leq \eta_{ij} \leq 1-c_{ij}$.

Notice that as $a_{ij} \leq b_{ij}$, $c_{ij} \leq d_{ij}$ and $b_{ij} + d_{ij} \leq 1$, we have $a_{ij} \leq b_{ij} \leq 1-d_{ij} \leq 1-c_{ij}$.

4.2 An optimization model for deriving aggregated IVIFN values

Assume that the satisfaction degree interval of alternative x_i with respect to a_j is given as $[\xi_{ij}, \eta_{ij}]$, its aggregated interval value incorporating attribute weights can be expressed as

$$[z_i^L, z_i^U] = [\sum_{j=1}^m \xi_{ij} w_j, \sum_{j=1}^m \eta_{ij} w_j], i = 1, 2, \dots, n.$$

As the aggregated value $[z_i^L, z_i^U]$ reflects the overall satisfaction degree of alternative x_i to the fuzzy concept of “excellence”, the greater the $[z_i^L, z_i^U]$, the better the alternative x_i is. Therefore, a reasonable attribute weight vector $(w_1, w_2, \dots, w_m)^T$ is to maximize $[z_i^L, z_i^U]$. Motivated by the optimization models for multiattribute decision making under IFSs presented by Li [20], this article extends the idea and proposes a similar framework to handle multiattribute decision making problems with incomplete attribute weights under IVIFSs.

As per Definition 3.6, the following two optimization models can thus be established for each alternative:

$$\begin{aligned} & \max \left\{ z_i^L = \sum_{j=1}^m \xi_{ij} w_j \right\} \\ & \text{s.t.} \begin{cases} a_{ij} \leq \xi_{ij} \leq 1 - d_{ij} & (i = 1, 2, \dots, n; j = 1, 2, \dots, m), \\ w \in H, \\ \sum_{j=1}^m w_j = 1 \end{cases} \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} & \max \left\{ z_i^U = \sum_{j=1}^m \eta_{ij} w_j \right\} \\ & \text{s.t.} \begin{cases} b_{ij} \leq \eta_{ij} \leq 1 - c_{ij} & (i = 1, 2, \dots, n; j = 1, 2, \dots, m), \\ w \in H, \\ \sum_{j=1}^m w_j = 1 \end{cases} \end{aligned} \quad (4.2)$$

for each $i = 1, 2, \dots, n$.

417 Similar to the treatment in Li [20], (4.1) can be converted to the following two linear
 418 programs:

$$\begin{aligned} & \min \left\{ z_i^{LL} = \sum_{j=1}^m a_{ij} w_j \right\} \\ & \text{s.t.} \begin{cases} w \in H, \\ \sum_{j=1}^m w_j = 1 \end{cases} \end{aligned} \quad (4.3)$$

420 and

$$\begin{aligned} & \max \left\{ z_i^{LU} = \sum_{j=1}^m (1 - d_{ij}) w_j \right\} \\ & \text{s.t.} \begin{cases} w \in H, \\ \sum_{j=1}^m w_j = 1 \end{cases} \end{aligned} \quad (4.4)$$

422 for each $i=1,2,\dots,n$.

423 By following the same manner, (4.2) is transformed to the following two linear
 424 programs:

$$\begin{aligned} & \min \left\{ z_i^{UL} = \sum_{j=1}^m b_{ij} w_j \right\} \\ & \text{s.t.} \begin{cases} w \in H, \\ \sum_{j=1}^m w_j = 1 \end{cases} \end{aligned} \quad (4.5)$$

426 and

$$\begin{aligned} & \max \left\{ z_i^{UU} = \sum_{j=1}^m (1 - c_{ij}) w_j \right\} \\ & \text{s.t.} \begin{cases} w \in H, \\ \sum_{j=1}^m w_j = 1 \end{cases} \end{aligned} \quad (4.6)$$

428 for each $i=1,2,\dots,n$.

429 Models (4.3)-(4.6) are standard linear programs that can be conveniently solved.

430 Denote their optimal solutions by $\tilde{W}_i^{LL} = (\tilde{w}_{i1}^{LL}, \tilde{w}_{i2}^{LL}, \dots, \tilde{w}_{im}^{LL})^T$, $\tilde{W}_i^{LU} = (\tilde{w}_{i1}^{LU}, \tilde{w}_{i2}^{LU}, \dots, \tilde{w}_{im}^{LU})^T$,

431 $\tilde{W}_i^{UL} = (\tilde{w}_{i1}^{UL}, \tilde{w}_{i2}^{UL}, \dots, \tilde{w}_{im}^{UL})^T$ and $\tilde{W}_i^{UU} = (\tilde{w}_{i1}^{UU}, \tilde{w}_{i2}^{UU}, \dots, \tilde{w}_{im}^{UU})^T$ ($i = 1, 2, \dots, n$), respectively,

432 and let

$$\begin{aligned}
\tilde{z}_i^{LL} &\triangleq \sum_{j=1}^m a_{ij} \tilde{w}_{ij}^{LL} \\
\tilde{z}_i^{UL} &\triangleq \sum_{j=1}^m b_{ij} \tilde{w}_{ij}^{UL} \\
\tilde{z}_i^{LU} &\triangleq \sum_{j=1}^m (1-d_{ij}) \tilde{w}_{ij}^{LU} \\
\tilde{z}_i^{UU} &\triangleq \sum_{j=1}^m (1-c_{ij}) \tilde{w}_{ij}^{UU}
\end{aligned} \tag{4.7}$$

for each $i=1,2,\dots,n$. Then Theorem 4.1 follows.

Theorem 4.1 Assume that \tilde{z}_i^{LL} , \tilde{z}_i^{UL} , \tilde{z}_i^{LU} , and \tilde{z}_i^{UU} are respectively defined by (4.7), then $\tilde{z}_i^{LL} \leq \tilde{z}_i^{UL}$, $\tilde{z}_i^{LU} \leq \tilde{z}_i^{UU}$, and $\tilde{z}_i^{UL} \leq \tilde{z}_i^{LU}$, $i=1,2,\dots,n$.

Proof. Note that $\tilde{W}_i^{LL} = (\tilde{w}_{i1}^{LL}, \tilde{w}_{i2}^{LL}, \dots, \tilde{w}_{im}^{LL})^T$, $\tilde{W}_i^{LU} = (\tilde{w}_{i1}^{LU}, \tilde{w}_{i2}^{LU}, \dots, \tilde{w}_{im}^{LU})^T$, $\tilde{W}_i^{UL} = (\tilde{w}_{i1}^{UL}, \tilde{w}_{i2}^{UL}, \dots, \tilde{w}_{im}^{UL})^T$, and $\tilde{W}_i^{UU} = (\tilde{w}_{i1}^{UU}, \tilde{w}_{i2}^{UU}, \dots, \tilde{w}_{im}^{UU})^T$ are optimal solutions of (4.3), (4.4), (4.5), and (4.6), respectively, and $a_{ij} \leq b_{ij}$ and $c_{ij} \leq d_{ij}$. According to (4.3), we have

$$\tilde{z}_i^{LL} \triangleq \sum_{j=1}^m a_{ij} \tilde{w}_{ij}^{LL} \leq \sum_{j=1}^m a_{ij} \tilde{w}_{ij}^{UL} \leq \sum_{j=1}^m b_{ij} \tilde{w}_{ij}^{UL} \triangleq \tilde{z}_i^{UL}$$

where the first inequality is due to the fact that \tilde{w}_{ij}^{LU} is an optimal solution of (4.3) and \tilde{w}_{ij}^{UL} is a feasible solution of this minimization problem, and the second inequality holds true as $a_{ij} \leq b_{ij}$.

Similarly, from (4.6), one can obtain

$$\tilde{z}_i^{LU} \triangleq \sum_{j=1}^m (1-d_{ij}) \tilde{w}_{ij}^{LU} \leq \sum_{j=1}^m (1-c_{ij}) \tilde{w}_{ij}^{LU} \leq \sum_{j=1}^m (1-c_{ij}) \tilde{w}_{ij}^{UU} \triangleq \tilde{z}_i^{UU}$$

where the first inequality is confirmed since $1-d_{ij} \leq 1-c_{ij}$ or equivalently, $0 \leq c_{ij} \leq d_{ij} \leq 1$, and the second inequality is derived because \tilde{w}_{ij}^{UU} is an optimal solution of (4.6) and \tilde{w}_{ij}^{LU} is a feasible solution of this maximization problem.

Furthermore, since $b_{ij} + d_{ij} \leq 1$, or equivalently, $b_{ij} \leq 1 - d_{ij}$, as per (4.4), we have

$$\tilde{z}_i^{UL} \triangleq \sum_{j=1}^m b_{ij} \tilde{w}_{ij}^{UL} \leq \sum_{j=1}^m (1-d_{ij}) \tilde{w}_{ij}^{UL} \leq \sum_{j=1}^m (1-d_{ij}) \tilde{w}_{ij}^{LU} \triangleq \tilde{z}_i^{LU}$$

451 Once again, the first inequality holds as $b_{ij} \leq 1 - d_{ij}$, and the second inequality comes

452 from the fact that \tilde{w}_{ij}^{LU} is an optimal solution of the maximization problem in (4.4) and

453 \tilde{w}_{ij}^{UL} is a feasible solution. The proof is thus completed. Q.E.D.

454 Theorem 4.1 indicates that the optimal aggregated value of $x_i \in X$ can be

455 characterized by a pair of intervals: $[\tilde{z}_i^{LL}, \tilde{z}_i^{UL}]$ and $[\tilde{z}_i^{LU}, \tilde{z}_i^{UU}]$. As $\tilde{z}_i^{LL} \leq \tilde{z}_i^{UL}, \tilde{z}_i^{LU} \leq \tilde{z}_i^{UU}$,

456 one can have $\tilde{z}_i^{LL} \leq \tilde{z}_i^{UL}, 1 - \tilde{z}_i^{UU} \leq 1 - \tilde{z}_i^{LU}$. Furthermore, since $\tilde{z}_i^{UL} \leq \tilde{z}_i^{LU}$, it is implied that

457 $\tilde{z}_i^{UL} + 1 - \tilde{z}_i^{LU} \leq 1$. Therefore, written in an IVIFN format, the optimal aggregated value of

458 the alternative $x_i \in X$ can be given as

$$\begin{aligned} \tilde{\alpha}_i &= \left([\tilde{z}_i^{LL}, \tilde{z}_i^{UL}], [1 - \tilde{z}_i^{UU}, 1 - \tilde{z}_i^{LU}] \right) \\ &= \left(\left[\sum_{j=1}^m a_{ij} \tilde{w}_{ij}^{LL}, \sum_{j=1}^m b_{ij} \tilde{w}_{ij}^{UL} \right], \left[\sum_{j=1}^m c_{ij} \tilde{w}_{ij}^{UU}, \sum_{j=1}^m d_{ij} \tilde{w}_{ij}^{LU} \right] \right) \end{aligned} \quad (4.8)$$

460 As the weight vectors $\tilde{W}_i^{LL}, \tilde{W}_i^{LU}, \tilde{W}_i^{UL}$, and \tilde{W}_i^{UU} are independently determined by the

461 four linear programs (4.3), (4.4), (4.5) and (4.6), respectively, it is understandable that

462 they are generally different, i.e., $\tilde{W}_i^{LL} \neq \tilde{W}_i^{LU} \neq \tilde{W}_i^{UL} \neq \tilde{W}_i^{UU}$ for $x_i \in X$, or $\tilde{w}_{ij}^{LL} \neq \tilde{w}_{ij}^{LU}$

463 $\neq \tilde{w}_{ij}^{UL} \neq \tilde{w}_{ij}^{UU}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Therefore, it is not fair to compare the

464 aggregated values of all alternatives based on the different weight vectors. A more

465 reasonable common ground for comparing the aggregated values is to derive a unified

466 weight vector and apply this same weight vector to all alternatives. The following

467 discussions aim to obtain such a weight vector. The general procedure is similar to that

468 reported in [20], but it has been adapted to accommodate the specific structure of IVIFNs.

469 Since X is a non-inferior alternative set, no alternative dominates or is dominated by

470 any other alternative. Hence, when all alternatives, rather than a single alternative in (4.3)

471 and (4.4), have to be considered, the contribution to the objective function from each

472 alternative should be treated with an equal weight of $1/n$. Therefore, parallel to (4.3) and

473 (4.4), we have the following two aggregated linear programs.

$$\min \left\{ z_0^{LL} = \frac{\sum_{i=1}^n \sum_{j=1}^m a_{ij} w_j}{n} \right\} \quad (4.9)$$

$$s.t. \begin{cases} w \in H, \\ \sum_{j=1}^m w_j = 1 \end{cases}$$

and

$$\max \left\{ z_0^{LU} = \frac{\sum_{i=1}^n \sum_{j=1}^m (1 - d_{ij}) w_j}{n} \right\} \quad (4.10)$$

$$s.t. \begin{cases} w \in H, \\ \sum_{j=1}^m w_j = 1 \end{cases}$$

Note that (4.9) can be converted to an equivalent maximization linear programming model in (4.11) by multiplying its objective function with -1.

$$\max \left\{ \bar{z}_0^{LL} = -\frac{\sum_{i=1}^n \sum_{j=1}^m a_{ij} w_j}{n} \right\} \quad (4.11)$$

$$s.t. \begin{cases} w \in H, \\ \sum_{j=1}^m w_j = 1 \end{cases}$$

Since (4.10) and (4.11) are both maximization problems and share the same constraints, if we treat the two objective functions as equally important, a typical way to translate the bi-objective linear programs into a single linear program is given below:

483

$$\begin{aligned} \max \left\{ z^L = \frac{\sum_{i=1}^n \sum_{j=1}^m (1 - a_{ij} - d_{ij}) w_j}{n} \right\} \\ \text{s.t.} \begin{cases} w \in H, \\ \sum_{j=1}^m w_j = 1 \end{cases} \end{aligned} \quad (4.12)$$

484 By applying the same procedure, (4.5) and (4.6) can be transformed to the following
485 linear program:

486

$$\begin{aligned} \max \left\{ z^U = \frac{\sum_{i=1}^n \sum_{j=1}^m (1 - b_{ij} - c_{ij}) w_j}{n} \right\} \\ \text{s.t.} \begin{cases} w \in H, \\ \sum_{j=1}^m w_j = 1 \end{cases} \end{aligned} \quad (4.13)$$

487 Once again, as (4.12) and (4.13) are both maximization problems and have the same
488 constraints, they can be combined to formulate the following linear program:

489

$$\begin{aligned} \max \left\{ z = \frac{\sum_{i=1}^n \sum_{j=1}^m (2 - a_{ij} - b_{ij} - c_{ij} - d_{ij}) w_j}{n} \right\} \\ \text{s.t.} \begin{cases} w \in H, \\ \sum_{j=1}^m w_j = 1 \end{cases} \end{aligned} \quad (4.14)$$

490 **Remark 4.1**

491 If $a_{ij} = b_{ij}$ and $c_{ij} = d_{ij}, i = 1, 2, \dots, n; j = 1, 2, \dots, m$, the IVIFNs in the decision matrix are
492 reduced to IFNs, and (4.14) is equivalent to Eq. (15) in [20] if the weight constraint
493 $w \in H$ in (4.14) takes the same form of being bounded on the lower and upper sides as
494 that in [20]. From this perspective, the proposed approach can be treated as a natural
495 extension of the work reported in [20] from the IFN to IVIFN environment.

496 Similarly, the linear programming model (4.14) can be easily solved by using the
 497 simplex method or an appropriate optimization computer package. Denote its optimal
 498 solution by $w^0 = (w_1^0, w_2^0, \dots, w_m^0)^T$, and follow the similar notation as (4.7) to define:

$$\begin{aligned}
 \tilde{z}_i^{0LL} &\triangleq \sum_{j=1}^m a_{ij} \tilde{w}_j^0 \\
 \tilde{z}_i^{0UL} &\triangleq \sum_{j=1}^m b_{ij} \tilde{w}_j^0 \\
 \tilde{z}_i^{0LU} &\triangleq \sum_{j=1}^m (1-d_{ij}) \tilde{w}_j^0 \\
 \tilde{z}_i^{0UU} &\triangleq \sum_{j=1}^m (1-c_{ij}) \tilde{w}_j^0
 \end{aligned}
 \tag{4.15}$$

500 As $a_{ij} \leq b_{ij}$, $c_{ij} \leq d_{ij}$ and $b_{ij} + d_{ij} \leq 1$, it follows that $\tilde{z}_i^{0LL} \leq \tilde{z}_i^{0UL}$, $\tilde{z}_i^{0LU} \leq \tilde{z}_i^{0UU}$ and
 501 $\tilde{z}_i^{0UL} \leq \tilde{z}_i^{0LU}$. Therefore, the optimal aggregated value of alternative x_i using a unified
 502 weight vector w^0 can be determined by a pair of closed intervals, $[\tilde{z}_i^{0LL}, \tilde{z}_i^{0UL}]$ and
 503 $[\tilde{z}_i^{0LU}, \tilde{z}_i^{0UU}]$. Equivalently, this aggregated value can be expressed as an IVIFN:

$$\begin{aligned}
 \tilde{\alpha}_i^0 &= \left([\tilde{z}_i^{0LL}, \tilde{z}_i^{0UL}], [1 - \tilde{z}_i^{0UU}, 1 - \tilde{z}_i^{0LU}] \right) \\
 &= \left(\left[\sum_{j=1}^m a_{ij} w_j^0, \sum_{j=1}^m b_{ij} w_j^0 \right], \left[1 - \sum_{j=1}^m (1-c_{ij}) w_j^0, 1 - \sum_{j=1}^m (1-d_{ij}) w_j^0 \right] \right) \\
 &= \left(\left[\sum_{j=1}^m a_{ij} w_j^0, \sum_{j=1}^m b_{ij} w_j^0 \right], \left[\sum_{j=1}^m c_{ij} w_j^0, \sum_{j=1}^m d_{ij} w_j^0 \right] \right)
 \end{aligned}
 \tag{4.16}$$

505 for each $i = 1, 2, \dots, n$. Note that $1 - \sum_{j=1}^m (1-c_{ij}) w_j^0 = \sum_{j=1}^m c_{ij} w_j^0$ and $1 - \sum_{j=1}^m (1-d_{ij}) w_j^0 =$
 506 $\sum_{j=1}^m d_{ij} w_j^0$ are due to the fact that $\sum_{j=1}^m w_j^0 = 1$. Now Theorem 4.2 can be established.

507 *Theorem 4.2* For $x_i \in X, i=1, 2, \dots, n$, assume that IVIFNs $\tilde{\alpha}_i$ and $\tilde{\alpha}_i^0$ are defined by
 508 (4.8) and (4.16), respectively, then

$$[\tilde{z}_i^{LL}, \tilde{z}_i^{UL}] \leq [\tilde{z}_i^{0LL}, \tilde{z}_i^{0UL}] \leq [\tilde{z}_i^{0LU}, \tilde{z}_i^{0UU}] \leq [\tilde{z}_i^{LU}, \tilde{z}_i^{UU}]$$

510 *Proof.* As $w^0 = (w_1^0, w_2^0, \dots, w_m^0)^T$ is an optimal solution of (4.14), it is also a feasible
 511 solution of (4.3), (4.4), (4.5) and (4.6) as these linear programs share the same constraints.

512 Note that $\tilde{W}_i^{LL} = (\tilde{w}_{i1}^{LL}, \tilde{w}_{i2}^{LL}, \dots, \tilde{w}_{im}^{LL})^T$ and $\tilde{W}_i^{LU} = (\tilde{w}_{i1}^{LU}, \tilde{w}_{i2}^{LU}, \dots, \tilde{w}_{im}^{LU})^T$ are an optimal
 513 solution of (4.3) and (4.4), respectively, and $a_{ij} \leq b_{ij}$ and $b_{ij} + d_{ij} \leq 1$, it follows that

$$514 \quad \tilde{z}_i^{LL} \triangleq \sum_{j=1}^m a_{ij} \tilde{w}_{ij}^{LL} \leq \sum_{j=1}^m a_{ij} w_j^0 \triangleq z_i^{0LL} \leq \sum_{j=1}^m b_{ij} w_j^0 \leq \sum_{j=1}^m (1-d_{ij}) w_j^0 \triangleq z_i^{0LU} \leq \sum_{j=1}^m (1-d_{ij}) \tilde{w}_{ij}^{LU} \triangleq \tilde{z}_i^{LU}$$

515 Here the first inequality holds true because \tilde{w}_{ij}^{LL} is an optimal solution of (4.3) and w_j^0
 516 is a feasible solution of this minimization problem. The 2nd and 3rd inequalities are due to
 517 $a_{ij} \leq b_{ij} \leq 1-d_{ij}$. The last inequality is confirmed because the objective function value of a
 518 feasible solution w_j^0 is always no more than that of an optimal solution \tilde{w}_{ij}^{LU} for the
 519 maximization problem (4.4). Therefore, we have $\tilde{z}_i^{LL} \leq z_i^{0LL} \leq z_i^{0LU} \leq \tilde{z}_i^{LU}$.

520 Similarly, as $\tilde{W}_i^{UL} = (\tilde{w}_{i1}^{UL}, \tilde{w}_{i2}^{UL}, \dots, \tilde{w}_{im}^{UL})^T$ and $\tilde{W}_i^{UU} = (\tilde{w}_{i1}^{UU}, \tilde{w}_{i2}^{UU}, \dots, \tilde{w}_{im}^{UU})^T$ are an
 521 optimal solution of (4.5) and (4.6), respectively, and $c_{ij} \leq d_{ij}$ and $b_{ij} + d_{ij} \leq 1$, following
 522 similar arguments, one can have

$$523 \quad \tilde{z}_i^{UL} \triangleq \sum_{j=1}^m b_{ij} \tilde{w}_{ij}^{UL} \leq \sum_{j=1}^m b_{ij} w_j^0 \triangleq z_i^{0UL} \leq \sum_{j=1}^m (1-d_{ij}) w_j^0 \leq \sum_{j=1}^m (1-c_{ij}) w_j^0 \triangleq z_i^{0UU} \leq \sum_{j=1}^m (1-c_{ij}) \tilde{w}_{ij}^{UU} \triangleq \tilde{z}_i^{UU}$$

524 i.e., $\tilde{z}_i^{UL} \leq z_i^{0UL} \leq z_i^{0UU} \leq \tilde{z}_i^{UU}$.

525 By Definition 3.6, the proof of Theorem 4.2 is completed. Q.E.D.

526 **Remark 4.2**

527 Theorem 4.2 confirms that the aggregated value of x_i obtained by (4.14) is always
 528 bounded by that obtained by (4.3) – (4.6) in terms of Definition 3.6.

529 Based on the aforesaid analyses, we are now in a position to formulate an interval-
 530 valued intuitionistic fuzzy approach to multiattribute decision making with incomplete
 531 attribute weight information as described in the following steps.

532 *Step 1.* Obtain an optimal weight vector $w^0 = (w_1^0, w_2^0, \dots, w_m^0)^T$ as per (4.14).

533 *Step 2.* Determine the optimal aggregated value $\tilde{\alpha}_i^0$ for all alternatives $x_i \in X$,
 534 $i = 1, 2, \dots, n$ by plugging w^0 into (4.16).

535 *Step 3.* Calculate the values of the score function $S(\tilde{\alpha}_i^0)$, accuracy function $H(\tilde{\alpha}_i^0)$,
 536 membership uncertainty index $T(\tilde{\alpha}_i^0)$, and hesitation uncertainty index $G(\tilde{\alpha}_i^0)$ for each
 537 alternative in a sequential order, and rank all alternatives as per Definition 3.5 and/or

choose the best alternative(s).

Remark 4.3

In an actual decision process, it is often unnecessary to calculate the values for all four functions. For instance, if the purpose of the decision problem is to choose the best alternative(s) and the sequential order in Definition 3.5 is followed to compute the function values, whenever no tie is found for the best value of a function (largest for $S(\cdot)$ and $H(\cdot)$, but smallest for $T(\cdot)$ and $G(\cdot)$), the best choice is ascertained and it is not necessary to calculate remaining function values in any lower hierarchy as detailed in Definition 3.5. Even if the decision problem is to obtain a full ranking of all alternatives, calculations may terminate before all four functions are entertained. For an example, see Section 5.

Remark 4.4

From the modeling process, one can understand that the proposed framework here is able to handle incomplete weight information characterized by a subset of linear relationships given in Section 4.1. In addition, the aggregation process is achieved through a series of optimization models that take the individual IVIFN assessments as input, and the conversion from IVIFNs to real values is delayed until the last step when different alternatives' aggregated IVIFN values are compared. This treatment avoids loss of information due to conversions at early stages. Another advantage of this framework is its novel comparison method that is able to distinguish any two different IVIFNs as shown in Section 3. In terms of limitations of the proposed approach, an inherent assumption of the aggregation process is that the attributes are independent and the individual membership and nonmembership functions are linearly additive. If other forms of information fusion schemes are required, this model would not be applicable. In addition, the proposed approach requires that all individual assessment information must be provided as IVIFNs in full and no missing data are allowed in the decision matrix. Further research is necessary to expand this approach to accommodate these needs for different fusion mechanisms and missing assessment data.

5 An illustrative example

This section adapts an investment decision problem in [12] to demonstrate how to apply the proposed approach. Although this example is provided in the context of selecting an optimal investment opportunity from a list of four choices in respect to four attributes against which each choice is assessed, it should be noted that, as suggested and illustrated by Merigo and Gil-Lafuente [24] and Xu and Yager [38], the proposed approach can be easily applied to a host of practical decision problems that involve choosing an optimal alternative from a list of alternatives when multiple attributes must be considered. For instance, selecting the best candidate to fill a tenure-track faculty position at a Canadian university typically requires each recruitment committee member to rank short-listed applicants based on different criteria such as research achievements/potentials, teaching/presentation skills, ability to attract funding from government agencies and industries, and service to the profession and academic community. From each committee member's perspective, this is a typical multiattribute decision making situation and the weights among different attributes can be conveniently captured by a list of constraints as shown in Section 4.1 and individual assessments may well be expressed as IVIFNs.

For the following example, assume that a fund manager in a wealth management firm is assessing four potential investment opportunities, $X = \{x_1, x_2, x_3, x_4\}$. The firm mandates that the fund manager has to evaluate each investment against four attributes: risk (a_1), growth (a_2), socio-political issues (a_3), and environmental impacts (a_4). In addition, the fund manager is only comfortable with providing his/her assessment of each alternative on each attribute as an IVIFN and the decision matrix is

$$\tilde{R} = \begin{bmatrix} ([0.42, 0.48], [0.4, 0.5]) & ([0.6, 0.7], [0.05, 0.25]) & ([0.4, 0.5], [0.2, 0.5]) & ([0.55, 0.75], [0.15, 0.25]) \\ ([0.4, 0.5], [0.4, 0.5]) & ([0.5, 0.8], [0.1, 0.2]) & ([0.3, 0.6], [0.3, 0.4]) & ([0.6, 0.7], [0.1, 0.3]) \\ ([0.3, 0.5], [0.4, 0.5]) & ([0.1, 0.3], [0.2, 0.4]) & ([0.7, 0.8], [0.1, 0.2]) & ([0.5, 0.7], [0.1, 0.2]) \\ ([0.2, 0.4], [0.4, 0.5]) & ([0.6, 0.7], [0.2, 0.3]) & ([0.5, 0.6], [0.2, 0.3]) & ([0.7, 0.8], [0.1, 0.2]) \end{bmatrix}$$

Each element of this matrix is an IVIFN, representing the fund manager's assessment as to what degree an alternative is and is not an excellent investment as per an attribute. For instance, the top-left cell, $([0.42, 0.48], [0.4, 0.5])$, reflects the fund manager's belief that alternative x_1 is an excellent investment from a risk perspective (a_1) with a margin

of 42% to 48% and x_1 is not an excellent choice given its risk profile (a_1) with a chance between 40% and 50%.

If the fund manager is able to provide the following attribute weight information: $w_1 = 0.13$ (risk), $w_2 = 0.17$ (growth), $w_3 = 0.39$ (socio-political issues), and $w_4 = 0.31$ (environmental impacts), calculations for our proposed approach start with Step 2 and determine as follows the aggregated IVIFN values for the four alternatives by plugging the given weights into (4.16):

$$\begin{aligned}\tilde{\alpha}_1 &= ([0.4831, 0.6089], [0.1850, 0.3800]), \\ \tilde{\alpha}_2 &= ([0.4400, 0.6520], [0.2170, 0.3480]), \\ \tilde{\alpha}_3 &= ([0.4840, 0.6450], [0.1560, 0.2730]), \\ \tilde{\alpha}_4 &= ([0.5400, 0.6530], [0.1950, 0.2950]).\end{aligned}$$

Next, Step 3 applies Definition 3.5 to compare the four alternatives based on their aggregated IVIFNs. As $S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2) = 0.2635$, $S(\tilde{\alpha}_3) = 0.35$, $S(\tilde{\alpha}_4) = 0.3515$, one can tell that $x_4 \succ x_3 \succ \{x_1, x_2\}$, but the score function cannot distinguish x_1 and x_2 as they have the same score function value. Therefore, we move on to calculate the accuracy function values for x_1 and x_2 , $H(\tilde{\alpha}_1) = H(\tilde{\alpha}_2) = 0.8285$. Note that we do not need to compute $S(\tilde{\alpha}_3)$ and $S(\tilde{\alpha}_4)$ as x_3 and x_4 are differentiated by the score function at a higher priority level. Since the accuracy function values are also identical for x_1 and x_2 , it is necessary to move to the next priority level and calculate the membership uncertainty index function values, $T(\tilde{\alpha}_1) = -0.0692$, $T(\tilde{\alpha}_2) = 0.081$. Now a full ranking of the four alternatives is obtained as: $x_4 \succ x_3 \succ x_1 \succ x_2$.

This assumption of complete knowledge on attribute weights allows a comparative study with other approaches in the current literature that require complete weight information. The comparative study will utilize the decision matrix \tilde{R} and the aforesaid weights to compare the ranking result of our proposed approach with those obtained from Procedure II ($p = 1$) in Xu and Yager [38] and Xu [34] (both weighted arithmetic and weighted geometric average aggregation operators).

To begin, the same decision matrix \tilde{R} and weights are fed into the approach, Procedure II, developed by Xu and Yager [38] (Note that Procedure I therein handles the

case with IFN assessments rather than IVIFNs and, hence, is omitted here for the comparative study). Let $p = 1$ and \tilde{R} be the resulting decision matrix from Step 1 therein. Then, the closeness coefficient of each alternative (See Eq (73) on p258 in [38]) can be rewritten as follows by using the notation in this article:

$$c(x_i) = \frac{\sum_{j=1}^m w_j (2 - (c_{ij} + d_{ij}))}{\sum_{j=1}^m w_j (4 - (a_{ij} + b_{ij}) - (c_{ij} + d_{ij}))} \quad (5.1)$$

Plugging the decision matrix and weights into (5.1) yields $c(x_1) = 0.6125$, $c(x_2) = 0.6125$, $c(x_3) = 0.6433$, $c(x_4) = 0.6517$. Based on the decision rule in Xu and Yager [38], the larger a closeness coefficient of an alternative, the better the alternative. Therefore, the ranking result from this approach is $x_4 \succ x_3 \succ \{x_1 ? x_2\}$, where the question mark indicates that this approach cannot differentiate x_1 from x_2 .

Xu [34] also develops weighted arithmetic and weighted geometric average aggregation operators for IVIFN information fusions. Both operators are employed to obtain rankings for the four alternatives here. As per the weighted arithmetic average aggregation operator, the aggregated IVIFN value of an alternative is determined by [34, Eqs. (14) and (16)]:

$$\tilde{\alpha}_i = \sum_{j=1}^m \omega_j \tilde{r}_{ij} = \left(\left[1 - \prod_{j=1}^m (1 - a_{ij})^{w_j}, 1 - \prod_{j=1}^m (1 - b_{ij})^{w_j} \right], \left[\prod_{j=1}^m c_{ij}^{w_j}, \prod_{j=1}^m d_{ij}^{w_j} \right] \right) \quad (5.2)$$

Based on (5.2), the aggregated IVIFNs for the four alternatives are derived as

$$\begin{aligned} \tilde{\alpha}_1 &= ([0.4904, 0.6283], [0.1581, 0.3585]), \\ \tilde{\alpha}_2 &= ([0.4553, 0.6653], [0.1838, 0.3347]), \\ \tilde{\alpha}_3 &= ([0.5271, 0.6839], [0.1347, 0.2535]), \\ \tilde{\alpha}_4 &= ([0.5632, 0.6761], [0.1765, 0.2827]). \end{aligned}$$

According to the score and accuracy functions developed by Xu [34] and given in Section 3 here, one can determine that $s(\tilde{\alpha}_1) = 0.30105$, $s(\tilde{\alpha}_2) = 0.30105$, $s(\tilde{\alpha}_3) = 0.4114$, $s(\tilde{\alpha}_4) = 0.39005$. It is clear that the score function ranks the four alternatives as $x_3 \succ x_4 \succ \{x_1 ? x_2\}$ and it cannot differentiate x_1 from x_2 . Then, it is necessary to calculate

the accuracy functions for the aggregated IVIFN values for x_1 and x_2 , $H(\tilde{\alpha}_1) = 0.81765$, $H(\tilde{\alpha}_2) = 0.81955$. Therefore, this approach generates a full ranking: $x_3 \succ x_4 \succ x_2 \succ x_1$.

Similarly, the weighted geometric average aggregation operator given in Eqs. (15) and (17) by Xu [34] is reproduced below for self-containment.

$$\tilde{\alpha}_i = \prod_{j=1}^m \tilde{r}_{ij}^{w_j} = \left(\left[\prod_{j=1}^m a_{ij}^{w_j}, \prod_{j=1}^m b_{ij}^{w_j} \right], \left[1 - \prod_{j=1}^m (1 - c_{ij})^{w_j}, 1 - \prod_{j=1}^m (1 - d_{ij})^{w_j} \right] \right) \quad (5.3)$$

Plugging \tilde{R} and the weights into (5.3) yields the following aggregated IVIFNs:

$$\begin{aligned} \tilde{\alpha}_1 &= ([0.4760, 0.5972], [0.1915, 0.3926]), \\ \tilde{\alpha}_2 &= ([0.4211, 0.6454], [0.2260, 0.3546]), \\ \tilde{\alpha}_3 &= ([0.4057, 0.6112], [0.1632, 0.2833]), \\ \tilde{\alpha}_4 &= ([0.5081, 0.6389], [0.2007, 0.3016]). \end{aligned}$$

The corresponding score function values are $s(\tilde{\alpha}_1) = 0.24455$, $s(\tilde{\alpha}_2) = 0.24295$, $s(\tilde{\alpha}_3) = 0.2852$, $s(\tilde{\alpha}_4) = 0.32235$, resulting in a full ranking $x_4 \succ x_3 \succ x_1 \succ x_2$.

In summary, the results of this comparison study can be shown in Table 1.

Table 1. A comparative study when attribute weight information is complete

Decision approach	Reference	Ranking result
Procedure II, $p = 1$	Xu and Yager [38]	$x_4 \succ x_3 \succ \{x_1 ? x_2\}$
Arithmetic operator	Xu [34]	$x_3 \succ x_4 \succ x_2 \succ x_1$
Geometric operator	Xu [34]	$x_4 \succ x_3 \succ x_1 \succ x_2$
This approach	This article	$x_4 \succ x_3 \succ x_1 \succ x_2$

Table 1 demonstrates the overall consistency of the ranking results based on the proposed approach in this article and other approaches. All of the four approaches rank x_3 and x_4 as the first two alternatives, with x_4 being identified as the most preferred investment opportunity by three approaches except the weighted arithmetic average aggregation operator in Xu [34]. For the remaining two investment opportunities, x_1 and x_2 , the weighted geometric average aggregation operator in Xu [34] and our approach rank x_1 first, but the weighted arithmetic average aggregation operator in Xu [34] ranks x_2 in front of x_1 , and the Xu and Yager [38] approach cannot distinguish these two

alternatives. The subtle differences in ranking are simply due to the distinct information fusion mechanisms in these approaches.

It should be noted that the two approaches in Xu [34] do not always provide a full ranking for the alternatives under consideration because the comparison mechanism there utilizes only score and accuracy functions. As indicated in Section 3, it is possible that certain alternatives cannot be distinguished by these two functions only. Similarly, the first approach in Xu and Yager [38] sometimes cannot differentiate all distinct alternatives, either. Furthermore, to make the comparative study possible, it is assumed that the attribute weight information is completely known as the other three approaches cannot handle the case when attribute weights are incomplete.

In reality, however, complete weight information is not always readily available. Instead only partial knowledge of attribute weights may be obtained as a group of linear constraints such as those given in Section 4.1. For instance, assume that the fund manager can only provide his/her incomplete knowledge about the weights as follows:

$$H = \{0.15 \leq w_1 \leq 0.3, 0.15 \leq w_2 \leq 0.25, \\ 0.25 \leq w_3 \leq 0.4, 0.3 \leq w_4 \leq 0.45, 2.5w_1 \leq w_3\}$$

In this case, the other three approaches in the previous comparative study would not be applicable, but the proposed approach in this article will be able to solve the problem. According to (4.14), the following linear program is established.

$$\begin{aligned} & \max \{z = (1.2w_1 + 2w_2 + 1.4w_3 + 1.3w_4) / 4\} \\ & s.t. \begin{cases} 0.15 \leq w_1 \leq 0.3, \\ 0.15 \leq w_2 \leq 0.25, \\ 0.25 \leq w_3 \leq 0.4, \\ 0.3 \leq w_4 \leq 0.45, \\ 2.5w_1 \leq w_3, \\ \sum_{j=1}^4 w_j = 1 \end{cases} \end{aligned} \quad (5.4)$$

Solving this linear programming, one can obtain its optimal solution as:

$$w^0 = (w_1^0, w_2^0, w_3^0, w_4^0)^T = (0.1500, 0.1750, 0.3750, 0.3000)^T$$

Note that the derived weight vector slightly differs from that given in the comparative study. Plugging the weight vector w^0 and individual assessments in the decision matrix \tilde{R} into (4.16), the optimal aggregated values for the four alternatives are determined.

$$\tilde{\alpha}_1^0 = ([0.48300, 0.60700], [0.18875, 0.38125]),$$

$$\tilde{\alpha}_2^0 = ([0.44000, 0.65000], [0.22000, 0.35000]),$$

$$\tilde{\alpha}_3^0 = ([0.4750, 0.6375], [0.1625, 0.2800]),$$

$$\tilde{\alpha}_4^0 = ([0.5325, 0.6475], [0.2000, 0.3000]).$$

Next, the score function is calculated for each aggregated value as

$$S(\tilde{\alpha}_1^0) = 0.2600, S(\tilde{\alpha}_2^0) = 0.2600, S(\tilde{\alpha}_3^0) = 0.3350, S(\tilde{\alpha}_4^0) = 0.3400$$

Obviously, $S(\tilde{\alpha}_4^0) > S(\tilde{\alpha}_3^0) > S(\tilde{\alpha}_1^0) = S(\tilde{\alpha}_2^0)$ and, hence, $x_4 \succ x_3 \succ \{x_1 ? x_2\}$. The score function values indicate that the most preferred alternative is x_4 , followed by x_3 , and then x_1 and x_2 . As $\tilde{\alpha}_1^0 \neq \tilde{\alpha}_2^0$, the question mark between x_1 and x_2 indicates that their ranking cannot be determined by the score function as both have the same score of 0.2600. If the purpose is to choose the best investment alternative only, the problem is completed now. On the other hand, if the fund manager is interested in a full ranking of the four investments, it is necessary to calculate the accuracy function values of $\tilde{\alpha}_1^0$ and $\tilde{\alpha}_2^0$ for the first two investment opportunities.

By Definition 3.2, it is easy to verify that

$$H(\tilde{\alpha}_1^0) = H(\tilde{\alpha}_2^0) = 0.8300$$

Once again, the ranking between x_1 and x_2 still cannot be determined. Therefore, we proceed with the membership uncertainty index $T(\tilde{\alpha}_i^0)$ ($i = 1, 2$)

$$T(\tilde{\alpha}_1^0) = -0.0685, T(\tilde{\alpha}_2^0) = 0.08$$

As $S(\tilde{\alpha}_1^0) = S(\tilde{\alpha}_2^0), H(\tilde{\alpha}_1^0) = H(\tilde{\alpha}_2^0), T(\tilde{\alpha}_1^0) < T(\tilde{\alpha}_2^0)$, by Definition 3.5, we have $x_1 \succ x_2$. Therefore, a full ranking of all four alternatives is obtained as

$$x_4 \succ x_3 \succ x_1 \succ x_2.$$

6 CONCLUSIONS

This article puts forward a framework to tackle multiattribute decision making problems with interval-valued intuitionistic fuzzy assessments and incomplete attribute weight information. The proposed approach employs a series of optimization models to derive a unified weight vector and this weight vector is then applied to synthesize individual IVIFN assessments into an aggregated IVIFN value for each alternative. To rank alternatives based on aggregated IVIFNs, a novel method is devised to compare any two IVIFNs.

An illustrative example is developed to demonstrate how to apply the proposed procedure and comparative studies show its overall ranking consistency with existing research. Numerical experiments illustrate the benefit of this proposed framework: it is capable for handling incomplete weights and a full ranking can always be obtained as long as the alternatives' aggregated IVIFN values are not identical. On the other hand, this approach is not without limitations as the decision matrix must be provided without any missing assessments and the information fusion mechanism is essentially linearly additive. Further research is required to extend the proposed approach to accommodate the cases when the decision matrix contains missing data and different aggregation schemes have to be entertained.

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