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The Toric Approach to F-theory Model Building

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Abstract

We describe the theoretical motivation for F-theory as a non-perturbative generalization of string theory. The four complex-dimensional compactification spaces of F-theory, called elliptically-fibered Calabi-Yau manifolds, consist of the six compact dimensions of string theory, plus a two-dimensional fiber that describes the string coupling field as a function of position on the string theory manifold. The methods of toric geometry are developed and applied to construct examples of elliptically-fibered Calabi-Yau manifolds. We analyze in detail models in which the fiber is free of singularities as a test bed for a more general analysis.

Contents

1 Introduction

The description of physics is separated into two regimes: large scale physics governed by gravity and described by general relativity, and small scale quantum physics, described by the laws of the Standard Model. Over the past century, these theories have been put to the test in innumerable experiments and have stood the test of time. Nevertheless, while both these theories have been very successful phenomenologically, they are theoretically unsatisfying for a number of reasons.

Philosophically, it is disturbing that nature should be described by two different theories, depending on the length scale under consideration. More practically, because of the difficulties in marrying these two theories, answers to many deep questions about the nature of gravity on small distance scales remain elusive. A quantum theory of gravity is required to understand physical scenarios where this becomes important, such as black hole physics and the very early universe.

Both of these theories are also plagued by free parameters that cannot be determined within the context of the theory. In general relativity, this comes in the form of the cosmological constant. In the Standard Model, there are about two dozen dimensionless free parameters that describe various properties of the particles in the theory. Thus, it is desirable to find a more fundamental theory that can predict the value of these parameters without resorting to experiment.

1.1 The Stringy Solution

String theory is the leading candidate for a unified theory of all of the forces of nature. Originally developed as a theory of the strong force, intended to describe the "particle zoo" that began to emerge in the mid-20th century, string theory has evolved into a significant body of research. It provides a mechanism for solving many of the problems described above. The fundamental objects of string theory, called strings, can vibrate in many modes, much like a violin string (actually, higher dimensional generalizations of the string called branes – short for membrane – are also considered). Each of these vibrational modes gives rise to a distinct particle, one of which has all of the same properties of the graviton, the particle associated with the gravitational force. Thus, string theory has the potential to describe all of the fundamental particles of nature, and as a bonus, turns out to be a quantum theory of gravity as well! Furthermore, string theory has only one free parameter, called the string length, which describes the natural length scale of the theory [1]. This contrasts dramatically with the many dimensionless parameters in the Standard Model and general relativity.

Apparently, the success of string theory as a quantum theory of gravity is a result of the freedom obtained by allowing the fundamental objects of the theory be continuous with a non-zero length. The quantum states of the relativistic string give rise to gravitons, as expected for a quantum theory of gravity. However, the freedom that makes string theory so successful also leads to a significant challenge to the predictability of the theory. An unavoidable consequence of the use of strings is the requirement of additional spacetime dimensions. In the bosonic string theory, which is only capable of describing bosons, the dimension of spacetime can be shown to be 26! One may hope to reduce this number by adding additional symmetries that also make the theory useful for the description of fermions. This can be done by imposing supersymmetry, which associates to each boson a partner fermion. In this theory, sometimes called superstring theory, the dimension of spacetime drops to ten [1].

Clearly, if string theory is a correct description of nature, we must find a way to accommodate our experience of 3+1 dimensional spacetime with the mathematical requirement of ten spacetime dimensions. The traditional method of resolving this paradox is to posit the existence of a six-dimensional compact space that describes how the dimensions that we do not observe are curled up so small that they have avoided detection thus far. These spaces must have several properties in order to be consistent with our requirements for a sensible physical theory. The spaces which satisfy these requirements are called Calabi-Yau manifolds [2].

It turns out that important physics depends on the details of the geometry considered. In fact, the physics affected by the geometry of the extra dimensions ranges from very small scale to the largest scale. For example, the number and volume of the holes in a manifold describing the extra dimensions has consequences for the particle content of the theory. The volume of the manifold is important for applications to cosmology, such as the realization of cosmic inflation in the string theory context. Thus, understanding these geometries is essential to string theory.

1.2 Generalization to F-theory

The string coupling, determined by a field called the axion-dilaton, describes the strength of interactions between strings, much like Newton's constant in Newtonian gravity and general relativity. Much of the dynamics of string theory is understood using perturbative expansions in the string coupling when its value is small. However, because the value of the string coupling is related to a field, the coupling strength can fail to be constant and can even grow large. When this happens, perturbative expansions in the string coupling fail, and perturbative methods are no longer useful.

F-theory generalizes string theory by giving the string coupling field a geometric representation as a one complex-dimensional manifold with a volume that varies as a function of position on the six compact dimensions, described as a three complex-dimensional Calabi-Yau manifold. This construction gives rise to a four complex-dimensional elliptically-fibered Calabi-Yau manifold, also called just a Calabi-Yau fourfold. The name is descriptive of the geometrized axion-dilaton field (also called the elliptic fiber) being allowed to vary over the three dimensional Calabi-Yau base space. This representation has the advantage of being able to describe non-perturbative effects not apparent in the perturbative formulation of string theory. This motivates us to study the possible geometries that can be constructed in this way and the physics that arise from them.The purpose of this thesis is to describe the systematic construction and analysis of a large set of simple elliptically-fibered Calabi-Yau manifolds relevant to F-theory compactifications using the methods of toric geometry and analytical tools developed for the study of Calabi-Yau geometries.

First, we describe the physical picture underlying F-theory in greater detail in section 2. In section 3, we develop the methods of toric geometry used to construct Calabi-Yau manifolds using data from combinatorial geometry. Then, in section 4, we specialize to the F-theory case and apply these methods to the construction of elliptically-fibered Calabi-Yau fourfolds. In section 5, we describe the geometric analysis of these manifolds and the analytical tools developed to carry out this analysis for a general Calabi-Yau manifold, focusing on the computation of information most relevant to the physics that we observe. Finally, in section 6, we describe the outlook for extending this analysis to more complicated models than those studied here. In particular, studying models with singularities in the elliptic fiber is a promising direction, since such singularities give rise to gauge interactions analogous to those in the Standard Model.

2 The Physical Picture in F-theory

The discussion in this section is drawn from [3]. As described briefly above, F-theory is a generalization of string theory in 10 dimensions. More specifically, the starting point for F-theory is type IIB string theory compactified on six dimensions. The axion-dilaton in this theory takes the form

$$\tau = C_0 + ie^{-\phi} \tag{1}$$

where C_0 and ϕ are scalar fields called the axion and the dilaton, respectively. The value of the string coupling g_s at a given point in spacetime is related to the dilaton field by the expression

$$g_s = e^{\phi}. \tag{2}$$

It is important to note that the string coupling is a *function* of a dynamical field, the dilaton. Because of this relationship, the string coupling is not an independent parameter of the theory, like the universal constants familiar in classical physics, but is in fact a dynamical variable. This means that the string coupling can grow large compared to unity, making it impossible to perform perturbative expansions with the coupling as a small parameter. This is the motivation for seeking a non-perturbative theory that can describe the physics when the string coupling grows large.

The action for the type IIB string theory takes the form

$$
\begin{aligned}
S_{IIB} =& \frac{2\pi}{l_s^8}\Big[\int d^{10}x\sqrt{-g}R \\
& - \frac{1}{2}\int \frac{1}{Im(\tau)^2} d\tau \wedge *d\overline{\tau} + \frac{1}{Im(\tau)} G_3 \wedge *d\overline{G_3} + \frac{1}{2}\tilde{F_5} \wedge *\tilde{F_5} \\
& + C_4 \wedge H_3 \wedge F_3\Big],
\end{aligned} \tag{3}
$$

where g is the trace of the spacetime metric, R is the Ricci scalar, and F_3, G_3, H_3, C_4, $\tilde{F_5}$, and G_5 are tensor fields. This action is invariant under the $SL(2,\mathbb{Z})$ gauge symmetry

$$
\tau \to \frac{a\tau + b}{c\tau + d} \tag{4}
$$

$$
\begin{pmatrix} H \\ F \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} H \\ F \end{pmatrix} \tag{5}
$$

$$
\tilde{F_5} \to \tilde{F_5}. \tag{6}
$$

This action for the type IIB theory looks very much like the action for a twelve-dimensional string theory compactified on a two-dimensional torus, T^2, whose geometry is described by the axion-dilaton field, τ. In this picture, the $SL(2,\mathbb{Z})$ gauge transformation of τ simply becomes reparametrization invariance of the torus. It can be shown that these dimensions are not physical dimensions [3], in the sense that they do not constitute spatial directions orthogonal to our familiar four-dimensional spacetime and the six compact dimensions. Nevertheless, the idea of geometrizing the axion-dilaton field is fruitful because it gives rise to a non-perturbative theory, dubbed F-theory. The description of the axion-dilaton field as a T^2 in F-theory allows a geometric approach to describing both the string coupling and the six extra spatial dimensions. This approach gives a more direct understanding of strongly coupled physics because the theory does not fail when the string coupling grows large.

There is a freedom when constructing F-theory compactifications to choose how the torus varies over the six compact dimensions. For very simple models, the torus can be chosen to be constant over the compact dimensions, and for more complicated models, the torus is allowed to vary its geometry as a function of position. A torus that varies over the so-called base space (the six compact dimensions) is referred to as an elliptic fibration. Physically, this is equivalent to letting the axion-dilaton field vary and is necessary to find models with realistic physics. Within this background, the dynamical objects in string theory, strings and branes, also interact. Strings can be either closed, like a loop of twine, or open, with freely moving endpoints. In string theory, open strings are confined to end on objects called D-branes (short for Dirichlet branes). In the F-theory picture, models can be constructed where the elliptic fibration "pinches off" to a singularity. Physically, this is due to the presence of D7-branes (seven-dimensional D-branes) stacked atop one another, which deform the fibration. These singularities are associated to gauge symmetries, which give rise to charges. In this way, a D-brane can inherit charge from a singularity, and the strings that end on the D-brane inherit charge as well.

3 The Toric Approach to Model Building

In order to study F-theory models, we need a method of constructing suitable compactification spaces. One well-known method useful for constructing a large set of interesting models makes use of toric geometry. In toric geometry, geometric data about the Calabi-Yau space can be realized using data from combinatorial geometry. We will first describe the basic ideas of this method, then illustrate the method using several concrete examples. The development in this section is primarily drawn from [4, 5].

3.1 Projective Spaces and Fans

The goal of the toric approach is to use the geometry of simple combinatorial shapes to encode algebraic spaces. Specifically, the spaces that are interesting for the purpose of constructing Calabi-Yau manifolds are called complex projective spaces. These spaces are themselves simple examples of algebraic structures called toric varieties, from which the method derives its name. In this construction, Calabi-Yau manifolds are hypersurfaces embedded in a toric variety. For our purposes, it suffices to understand a few concrete examples to build intuition for the spaces we want to describe.

The simplest example of a complex projective space is the one-dimensional projective space,

$$\mathbb{P}^1 = (\mathbb{C}^2\backslash 0)/(\mathbb{C}\backslash 0). \tag{7}$$

Here, the division by $\mathbb{C}\backslash 0$ means that we identify points related by the equivalence relation

$$(x, y) \sim (\lambda x, \lambda y) \qquad \forall \lambda \in \mathbb{C}\backslash 0. \tag{8}$$

As we shall see, the exponents on the factors of λ, call them a_i, may be different from unity in general. For now, we note that the exponent on each factor of λ is 1. We can encode this equivalence relation in the language of vectors on a lattice in a straightforward way. For each coordinate x_i, assign a vector v_{x_i}, such that the vectors are related by the expression

$$\sum_i a_i v_{x_i} = 0. \tag{9}$$

In the case of $\mathbb{P}^1$, this equation is simply

$$v_x + v_y = 0. \tag{10}$$

Intuitively, the idea of "coming back" to the same point by the identification is encoded by the fact that when we add the vectors together, we "come back" to the origin.

As a slightly more complicated example, we consider a so-called weighted projective space, $\mathbb{P}^{(2,3,1)}$, an important space for the construction of F-theory compactifications. This space is defined similarly to $\mathbb{P}^1$:

$$\mathbb{P}^{(2,3,1)} = (\mathbb{C}^3\backslash 0)/(\mathbb{C}\backslash 0), \tag{11}$$

with the identification given by

$$(x, y, z) \sim (\lambda^2 x, \lambda^3 y, \lambda z) \qquad \forall \lambda \in \mathbb{C}\backslash 0. \tag{12}$$

In the same way as before, we encode this relation by defining vectors v_x, v_y, and v_z that satisfy

$$2v_x + 3v_y + 1v_z = 0. \tag{13}$$

The set of vectors that encodes the equivalence relation(s) of a projective space (or, more generally, a toric variety) is called the fan of the projective space. The two fans constructed above can be seen in Figure 3.1.

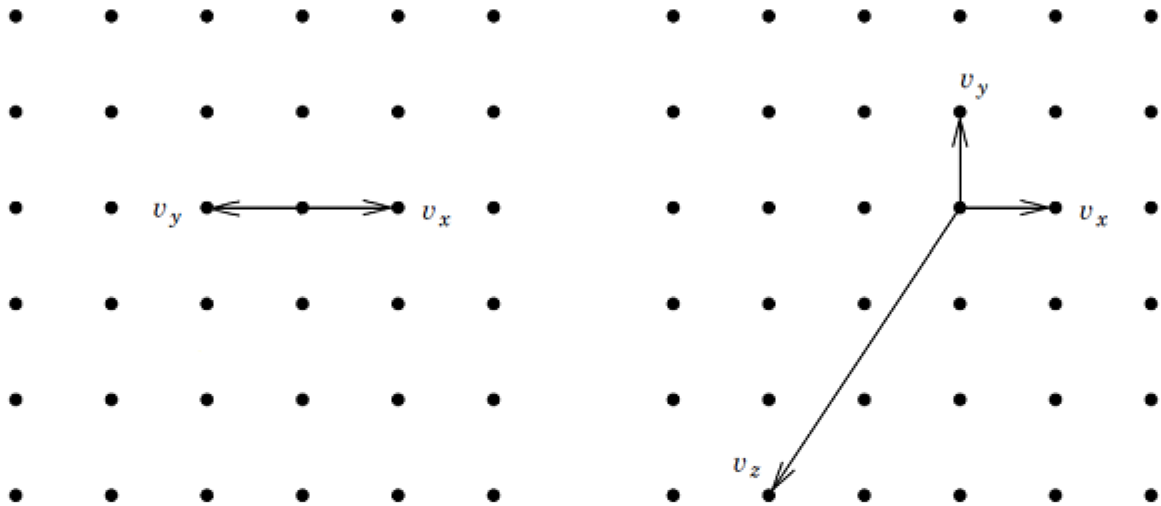

Figure 1: The fans of $\mathbb{P}^1$ and $\mathbb{P}^{(2,3,1)}$. Adapted from [4].

The final step in relating the projective space to the combinatorial data is to associate the vectors of the fan with their corresponding points, that is, associate the vector $v = a\hat{x} + b\hat{y} + c\hat{z}$ with the point (a, b, c). These points are taken to be the vertices of a polyhedron on the lattice, and the point at the origin is also added to the polyhedron. The polyhedra associated to the examples above are shown in Figure 3.1. In general, polyhedra constructed in this way have the additional properties of convexity (any two points of the polyhedron can be connected by a single line segment) and reflexivity (the only interior point is the origin), which is important for performing the reverse of the process described above, i.e. constructing projective spaces from polyhedra.

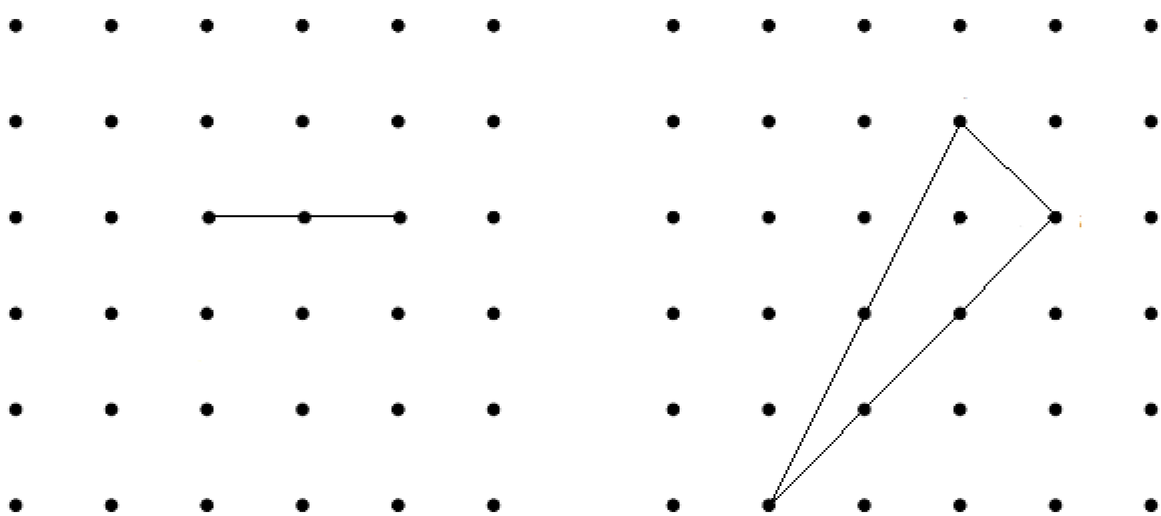

Figure 2: The convex reflexive polyhedra associated to $\mathbb{P}^1$ and $\mathbb{P}^{(2,3,1)}$.

3.2 Constructing Toric Varieties

In section 3.1, projective spaces were described and the correspondence to convex reflexive polyhedra was shown explicitly for simple examples. With this knowledge, it is natural to wonder how the process can be reversed. After all, convex reflexive polyhedra are well documented in the literature [6, 7, 8, 9], and there are many of these geometries to choose from. If we could construct toric varieties using the available data for convex reflexive polyhedra, we would be well on our way to constructing thousands of different spaces in which to construct Calabi-Yau manifolds! It turns out that this is possible, but we need a bit more than just the vertices of the polyhedron to carry out the procedure.

In dimensions higher than 2, the vertices of the polyhedron are not sufficient to describe a toric variety. In order to construct a one-to-one correspondence between toric varieties and convex reflexive polyhedra, we need a few more definitions that will allow the concept of a fan described above to be generalized to the higher-dimensional case. First, we define a strongly convex rational polyhedral cone σ to be a cone on the lattice with the origin as its apex and the following properties:

- 'Polyhedral': σ is bounded by finitely many hyperplanes,
- 'Rational': the edges of σ are spanned by lattice vectors,
- 'Strongly convex': σ contains no complete line.

In addition, a face of a cone σ is either σ itself or the intersection of σ with a bounding hyperplane.

Finally, this allows us to generalize and formalize the definition of a fan. A fan Σ is a collection of cones with the properties:

- The origin is a cone.
- For any cone σ in Σ, every face of σ is also in Σ.
- The intersection of any two cones in Σ is a face of both cones.

In this language, we can properly describe the fans of $\mathbb{P}^1$ and $\mathbb{P}^{(2,3,1)}$. The fan of $\mathbb{P}^1$ has a zero-dimensional cone at the origin, two one-dimensional cones given by the rays along the directions of v_x and v_y, and two two-dimensional cones given by the segments into which the rays cut the lattice. The fan of $\mathbb{P}^{(2,3,1)}$ similarly has a zero-dimensional cone at the origin, three one-dimensional cones given by the rays along the directions of v_x, v_y, and v_z, and three two-dimensional cones given by the segments into which the rays cut the lattice.

In the language of the polyhedron, a fan is constructed by making a choice of how to cut the polyhedron into simplices (which must all include the origin as a vertex). A particular choice is called a triangulation of the polyhedron. This explains why the naive picture developed in section 3.1 worked without need for the concept of cones. In the one- and two-dimensional cases, there is a unique way of triangulating a polytope. However, this is not necessarily true for higher-dimensional polyhedra.

Using our generalized definition of a fan, we can complete the correspondence between toric varieties and convex reflexive polyhedra. First, for every one-dimensional cone, assign a complex coordinate $z_i, i = 1, ..., k$. This assignment gives rise to $\mathbb{C}^k$. From this set, we remove the set

$$Z_\Sigma = \bigcup_{i\in I}\{(z_1, ..., z_k) : z_i = 0 \quad \forall i \in I\} \tag{14}$$

where the union is over all sets $I \subseteq \{1, ..., k\}$ for which $\{v_i : i \in I\}$ does not belong to the same cone in Σ. In other words, many z_i can vanish simultaneously only if the corresponding vectors v_i are in the same cone. We then quotient that space by $(\mathbb{C}\backslash 0)^{n-k}$ acting by the equivalence relations[1]

$$(z_1, ..., z_k) \sim (\lambda^{a^i_j} z_1, ..., \lambda^{a^k_j} z_k) \quad \text{if} \sum_{i=1}^{k} a^i_j v_i = 0. \tag{15}$$

This definition is somewhat obtuse, but it naturally generalizes the constructions studied in section 3.1. We will become more familiar with the definition when we construct toric varieties in which to embed F-theory compactifications.

[1] In the most general case, the divisor space is actually a product of the space defined by the equivalence relations and a finite abelian group that becomes important only if the v_i do not generate the entire lattice. However, this is not usually an issue for the cases we consider. For more details, see [4].

4 Application to F-theory

In the toric approach to constructing string theory models, toric varieties are the ambient spaces in which we embed Calabi-Yau manifolds. Now that we have developed a strategy for constructing these spaces using the abundant available data on convex reflexive polyhedra, our next step is to apply this knowledge to the construction of F-theory vacua. To do so, we must briefly review what we know about F-theory compactifications.

4.1 The Elliptic Fiber

The appropriate compactifications for F-theory are elliptically-fibered Calabi-Yau fourfolds. In these models, one dimension is needed to describe the elliptic fiber T^2, which is allowed to vary as a function of the remaining coordinates. The other dimensions serve as the base space over which the fiber varies. In order to realize these spaces using the toric approach, we require five-dimensional convex reflexive polyhedra (in general, an (n+1)-dimensional polyhedron is needed to construct a Calabi-Yau n-fold). However, we must also ensure that the spaces that arise from these polyhedra admit an elliptic fiber as part of the resulting Calabi-Yau manifold. This can be done by constructing five-dimensional polyhedra from two sub-polyhedra: one that describes the elliptic fiber, and another for the base space over which the elliptic fiber is defined [10].

The first sub-polyhedron is a two-dimensional polyhedron which we studied in section 3.1. The polyhedron Δ with vertices

$$\Delta = \{(1,0),(0,1),(-2,-3)\} \tag{16}$$

is of essential importance in our construction. As described above, Δ corresponds to the projective space $\mathbb{P}^(2,3,1)$. In this space, the torus is described as an elliptic fiber by the Weierstrass equation,

$$y^2 = x^3 + axz^4 + bz^6, \tag{17}$$

where x, y, and z are coordinates and a and b are complex coefficients that affect the skew of the torus[2].

For our construction of elliptically-fibered Calabi-Yau fourfolds, we seek a very closely related setup to this simple case. The F-theory torus is allowed to vary its geometry as a function of position on the six compact dimensions, so it is described by an equation of the form

$$y^2 = x^3 + a(z_i)xz^4 + b(z_i)z^6, \tag{18}$$

where the z_i are the coordinates on the six physical dimensions. Such a space is exactly what is obtained when we extend the two-dimensional space on which the elliptic fiber is defined to a higher dimensional toric variety, which means adding coordinates and additional points to the 2d polyhedron. To construct a four-dimensional elliptically-fibered Calabi-Yau manifold, we need a five dimensional toric variety that admits the torus as a fiber over some base space. This requires combining the two-dimensional polyhedron that describes the elliptic fiber with a three dimensional polyhedron that describes the base space. We will explore how this is done by studying some simple examples.

4.2 Example: $\mathbb{P}^1$ over $\mathbb{P}^1$

As a first step in understanding how to construct polyhedra for F-theory compactifications, we will focus on the simplest toric variety that can be constructed from two independent reflexive polyhedra. As illustrated in section 3.1, the polyhedron that encodes the simplest toric variety, the one-dimensional complex projective space $\mathbb{P}^1$, is

$$\Delta = \{(1),(-1)\}. \tag{19}$$

The simplest way to create a 2d polyhedron using this space is to "overlay" two copies of $\mathbb{P}^1$. This is called a direct product of the two spaces, and the polyhedron constructed by such a process is simply

$$\Delta = \{(1,0),(-1,0),(0,1),(0,-1)\}. \tag{20}$$

[2]This equation feels somewhat contrived at first sight. In fact, it is the most general polynomial that describes a torus. Some terms in the polynomial vanish by coordinate transformation, which is possible by using the equivalence relation and choosing λ cleverly.

It is easy to see that the two copies of $\mathbb{P}^1$ are independent. According to the dictionary from polyhedron to toric variety developed above, the equivalence relations determined by this polyhedron are given by sums of the vectors that sum to zero (see equation 15. In this case, the equivalence relations are

$$(x, y, u, v) \sim (\lambda x, \lambda y, u, v) \tag{21}$$

$$(x, y, u, v) \sim (x, y, \mu u, \mu v), \tag{22}$$

where x and y are coordinates on the first copy of $\mathbb{P}^1$, and u and v are coordinates on the second copy of $\mathbb{P}^1$. Since the pairs of coordinates scale independently, the toric variety is particularly simple. On the other hand, F-theory compactifications require that the fibration depend on the coordinates in the base space. This suggests that the polyhedra we seek will have more complicated relations among their vertices.

To generalize this example, we can consider forming convex reflexive polyhedra with more complicated relations among the vertices. One such model is called a $\mathbb{P}^1$ fibered over $\mathbb{P}^1$, or sometimes just $\mathbb{P}^1$ over $\mathbb{P}^1$. Instead of taking the direct product of two copies of projective space, we can allow the coordinates in one copy to scale in a non-trivial way with the coordinates in the other. The polyhedron that encodes such a model is of the form

$$\Delta = \{(1, 0), (0, 1), (0, -1), (-1, -n)\}, \tag{23}$$

where $n \in \mathbb{Z}$. Using the same notation as above, the scaling relations among the coordinates are

$$(x, y, u, v) \sim (\lambda x, \lambda^n y, u, \lambda v) \tag{24}$$

$$(x, y, u, v) \sim (x, \mu y, \mu u, v). \tag{25}$$

The interdependence between the fiber coordinates and the base coordinates is more representative of the models needed for F-theory.

4.3 Example: Elliptically-Fibered K3

Now, with an improved intuition for how to construct composite polyhedra, we turn to constructing an elliptically-fibered Calabi-Yau manifold. The simplest elliptically-fibered manifold that we might be interested in is the elliptically-fibered K3. This is an elliptically-fibered Calabi-Yau twofold with the T^2 as the fiber. From section 4.1, recall that the natural ambient space for the T^2 is $\mathbb{P}^{(2,3,1)}$, encoded by the polyhedron

$$\Delta = \{(1, 0), (0, 1), (-2, -3)\}. \tag{26}$$

To obtain the trivial direct product of $\mathbb{P}^{(2,3,1)}$ over $\mathbb{P}^1$, we would simply add one additional coordinate and additional points orthogonal to the fiber polyhedron to represent the base $\mathbb{P}^1$. This would give the polyhedron

$$\Delta_{dp} = \{(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, 0, 1), (0, -2, -3)\}. \tag{27}$$

However, as pointed out above, the direct product does not give the desired interdependence between the fiber and base coordinates. We seek a non-trivial scaling relation between the fiber and the base.

There are many polyhedra that give rise to such scaling relations – one simple example is

$$\Delta_{dp} = \{(0, 1, 0), (0, 0, 1), (0, -2, -3), (1, -2, -3), (-1, -2, -3)\}. \tag{28}$$

This polyhedron gives rise to the scaling relations

$$(x, y, z, s, t) \sim (\lambda^2 x, \lambda^3 y, \lambda z, s, t) \tag{29}$$

$$(x, y, z, s, t) \sim (x, y, \mu^{-2} z, \mu s, \mu t). \tag{30}$$

In general, polyhedra representing an elliptic fibration of T^2 take a form similar to that of the above example. For constructing a Calabi-Yau d-fold, we require a $(d+1)$-dimensional polyhedron. The polyhedron is constructed by adding a $(d-1)$-dimensional reflexive polyhedron representing the base space to the two-dimensional reflexive polyhedron representing $\mathbb{P}^{(2,3,1)}$ as follows. The first three points of the space are of the form

$$\Delta_{P^{(2,3,1)}} = \{(0^{d-1}, 1, 0), (0^{d-1}, 0, 1), (0^{d-1}, -2, -3)\}. \tag{31}$$

This is simply an embedding of $\mathbb{P}^{(2,3,1)}$ as a subspace of the larger toric variety to be constructed. The remaining points are simply those points of the base polyhedron placed 'above' the point $(-2,-3)$ in the plane of the $\mathbb{P}^{(2,3,1)}$ polyhedron. In the above example, $\mathbb{P}^1$ is encoded by $\{(1),(-1)\}$, which appear as the first coordinates of the points of the full polyhedron, $\{(1,-2,-3),(-1,-2,-3)\}$. We can construct more complicated relations between the coordinates by adding more vertices to the polyhedron, but these simple models are useful as a starting point for understanding some key features of elliptically-fibered Calabi-Yau manifolds and their applications to F-theory.

4.4 Example: Elliptic Fibration over F_3

A more complicated example, called the elliptic fibration over F_3, incorporates parts of both of the previously encountered examples. Here we will consider an elliptic fibration over a base space that is itself a fibration of two spaces. The F-theory base space we will use, called F_3, is the fibration of $\mathbb{P}^1$ over a base of $\mathbb{P}^1$. As we saw in section 4.2, such a fibration takes the form

$$\Delta_{F_3} = \{(1,0),(0,1),(0,-1),(-1,-n)\}. \tag{32}$$

In this case, we will fix $n = 3$ (hence the name F_3) and take this space as our F-theory base space.

As discussed in section 4.3, we can obtain a toric description of the elliptic fiber over this base space by adding points of the two polyhedra. The resulting polyhedron is

$$\begin{aligned}\Delta = \{&(0,0,1,0),(0,0,0,1),(0,0,-2,-3),\\ &(1,0,-2,-3),(0,1,-2,-3),(0,-1,-2,-3),(-1,-3,-2,-3)\}.\end{aligned} \tag{33}$$

By inspection, the associated toric variety obeys the scaling relations

$$\begin{aligned}(t,u,v,w,x,y,z) &\sim (\lambda^2 t, \lambda^3 u, \lambda v, w, x, y, z)\\ (t,u,v,w,x,y,z) &\sim (t, u, \mu^{-2} v, w, \mu x, \mu y, z)\\ (t,u,v,w,x,y,z) &\sim (t, u, \nu^{-5} v, \nu w, \nu^3 x, y, \nu z),\end{aligned} \tag{34}$$

where t, u, and v are associated to the vertices of the elliptic fiber polyhedron, w, x, y, and z are associated to the base space vertices, and λ, μ, and ν are non-zero complex numbers. This geometry is very closely related to the types of geometries that are studied for F-theory applications.

5 Geometric Analysis of Calabi-Yau Fourfolds

With a method of constructing elliptically-fibered Calabi-Yau fourfolds in hand, we now turn to calculating some of the geometric quantities relevant for understanding the physics associated to the geometry of the extra dimensions. In particular, we will focus on the Hodge numbers and volumes of the geometries. Hodge numbers are quantities that describe the numbers of holes of a given dimension. These are important for understanding what fields arise in the theory, and are thus essential for understanding the particle content for a given model. The volumes of these holes and of the whole manifold are important in many physical processes and are particularly relevant for understanding cosmological evolution in the theory, since they appear in the calculation of the vacuum potential for the theory.

5.1 Hodge Numbers

The Hodge numbers of a given compactification determine many of the fields, called axions and moduli, that arise in the theory. In turn, these fields are important in understanding the four-dimensional effective field theory that such a compactification yields. Thus, the Hodge numbers for a given compactification are among the most important quantities for understanding particle physics in that theory.

Because of the duality between polyhedra and toric varieties explored above, there are two ways of calculating Hodge numbers. The direct method, which involves studying the cohomology classes of the manifold, turns out to be more complicated than necessary. Instead, we can make use of the duality between polyhedra and toric varieties to turn the calculation of the Hodge numbers into a simpler problem in combinatorial geometry.

In the most general d-dimensional manifold, there are $(d+1)^2$ Hodge numbers, $h^{p,q}$ $(p, q = 0, 1, ..., d)$. However, many of these numbers are trivially related, and for a Calabi-Yau manifold some take simple values of either 1 or 0. Using these relations (see [11] for details), it can be shown that there are four independent Hodge numbers: $h^{1,1}$, $h^{3,1}$, $h^{2,1}$, and $h^{2,2}$.

It was first shown in [12] that the Hodge numbers can be computed combinatorially using data for the polyhedron that corresponds to the toric variety in which the Calabi-Yau surface is embedded. Actually, we need a bit more than this. Reflexive polyhedra have a built-in duality so that they occur in dual pairs. Given a lattice polyhedron Δ, define the dual polyhedron Δ^* to be

$$\Delta^* = \{v \in N_{\mathbb{R}} : \langle v, w \rangle \geq -1 \quad \forall w \in \Delta\}, \tag{35}$$

where $N_{\mathbb{R}}$ is the lattice defined over the reals and $\langle \cdot, \cdot \rangle$ is the inner product. Δ is called reflexive is Δ^* is a lattice polytope[3]. Because this is a symmetric definition, it is easy to see that if Δ is reflexive, so is the dual polyhedron Δ^*. Just swap the roles of Δ and Δ^*, and note that Δ is, by hypothesis, a lattice polytope. We call such polyhedra as Δ and Δ^* dual pairs.

Using the above definition, we now make use of identities proven in [12] and generalized in [11]. Three of the four Hodge numbers can be computed easily by the identities

$$\begin{aligned} h^{1,1}(X) &= l(\Delta^*) - (d+2) - \sum_{\dim\Theta^*=d} l'(\Theta^*) + \sum_{\mathrm{codim}\Theta_i^*=2} l'(\Theta_i^*)l'(\Theta_i) \\ h^{3,1}(X) &= l(\Delta) - (d+2) - \sum_{\dim\Theta^*=d} l'(\Theta) + \sum_{\mathrm{codim}\Theta_i^{=}2} l'(\Theta_i)l'(\Theta_i^*), \\ h^{2,1}(X) &= \sum_{\mathrm{codim}\Theta_i=3} l'(\Theta_i)l'(\Theta_i^*) \end{aligned} \tag{36}$$

where X denotes the Calabi-Yau hypersurface, Θ (Θ^*) is a face of Δ (Δ^*), $l(\Theta)$ is the number of all points of a face Θ, and $l'(\Theta)$ is the number of all points inside Θ. The remaining Hodge number, $h^{2,2}$, can be shown to be dependent on the above Hodge numbers using considerations from cohomology[4]. Carrying out the analysis results in the simple relation

$$h^{2,2} = 2(22 + 2h^{1,1} + 2h^{3,1} - h^{2,1}). \tag{37}$$

Thus, all of the Hodge numbers can be calculated from the polyhedral data.

5.2 Volumes

As mentioned above, volumes are essential to understanding the effective potential of a given theory. Unlike the problem of computing Hodge numbers, calculating volumes is more involved and cannot be reduced to a problem in combinatorial geometry. We will need some techniques in differential geometry to proceed. The discussion here is drawn primarily from [3].

A fundamental concept in differential geometry is that of a differential form. These objects are used to generalize integration from Euclidean space to a general differentiable manifold. They behave as coordinate-free integrands that allow integration over a curve, a surface, a volume, or any manifold. As an example consider the volume element dV in Euclidean space. dV is a three-dimensional volume element (or a three-form), which when integrated gives the volume of the total space of integration. We can re-write dV in terms of one-dimensional volume elements (or one-forms):

$$dV = dx \wedge dy \wedge dz, \tag{38}$$

where dx, dy, and dz are one-forms, and $\wedge$, called the "wedge product", is the product between forms.

[3]Note that this more rigorous definition implies all of the properties used as a working definition above.

[4]Actually, this is only true for Calabi-Yau fourfolds with $SU(4)$ holonomy. Luckily, these are precisely the manifolds in which we are interested!

This suggests that we seek a differential form that can be used to represent our manifold. This is done as follows. Define a divisors D_i of the Calabi-Yau d-fold to be the k $(d-1)$-dimensional surfaces obtained by setting the coordinates $z_i = 0 (i = 1, 1, ..., k)$. Among these divisors, we choose a basis of d divisors, denoted K_α, to simplify our calculations. Next, define the Kähler form, denoted J, as a sum of these basis divisors which describes the entire manifold,

$$J = \xi^\alpha K_\alpha \qquad (\xi^\alpha \geq 0). \tag{39}$$

These surfaces act as forms in the language of differential geometry. To find volumes, we integrate products of the Kähler form with itself. In general, the volume of a closed m-dimensional hypersurface C (called a m-cycle) takes the form[5]

$$\int_C \frac{J^m}{m!}. \tag{40}$$

In particular, the volume of the entire Calabi-Yau manifold X is

$$\int_X \frac{J^d}{d!}. \tag{41}$$

5.3 Flop Transitions

Constructing Calabi-Yau manifolds using the toric method involves the use of triangulations of a single polyhedron. This results in one Calabi-Yau manifold for each triangulation of the polyhedron. However, these manifolds are all related to one another geometrically. These relationship are called flop transitions, and they describe how moving from one region of the toric variety to another often corresponds to moving from one Calabi-Yau manifold to another.

The geometry of the Calabi-Yau is often described in terms of divisors as described above. Curves, or two-cycles, in the Calabi-Yau manifold can be described by intersections of $d-1$ of these divisors. In fact, there is a description of the geometry of Calabi-Yau manifolds dual to that of the divisors.

Recall that divisors are codimension one surfaces in the Calabi-Yau manifold obtained by setting one coordinate to zero. The Calabi-Yau manifold can be described by the geometry of the divisors. Similarly, a basis of two-cycles can be constructed such that the Calabi-Yau manifold can be described fully in terms of the cycles. The full set of these curves is called the Mori cone. The dual set of all divisors of the Calabi-Yau manifold is called the Kähler cone. Everywhere within the parameter space that describes the Kähler cone, the areas of the two-cycles are, by definition, positive.

Because of the definition of the Kähler cone, as we move out of the defining parameter space, the area of a two-cycle becomes formally negative. This process is called a flop transition. However, moving outside of the parameter space of the Kähler cone of one Calabi-Yau manifold often takes one into the parameter space of the Kähler cone of another Calabi-Yau manifold. Such a flop transition is called a true flop because it shows the two apparently different manifolds to be different phases of a single smooth geometry. In other cases, called ineffective flops, the two Calabi-Yau manifolds are not smoothly related. This occurs when the curve being shrunken to zero size is excluded from both of the Calabi-Yau manifolds.

6 Results

The analysis above is a general procedure for computing important geometric data for F-theory compactifications. Indeed, with few changes, the procedures can be adapted to work for any Calabi-Yau manifold. The purpose of this project was to construct a toolkit that could be used to rapidly analyze any F-theory compactification. This toolkit was adapted from tools developed by the author [13] for the study of compactifications in type IIB string theory, which involve Calabi-Yau threefolds. It was developed in Python using the open-source mathematics software Sage [14] and consists of two primary parts.

The first tool uses the methods described in sections 3 and 4 to construct polyhedra that correspond to F-theory compactifications. This tool is perhaps the most limited at this time because we have restricted ourselves in this project to the study of simple models where no additional points are added as vertices to

[5]For more information on how to carry out these integrals by hand, see any textbook on differential geometry.

the polyhedron. As mentioned in section 4, adding vertices may result in new relations among them, which correspond to new identifications among the points of the toric variety. This results in changes to the toric variety and thus the Calabi-Yau manifold constructed. Such manifolds would be of more phenomenological interest because of the physics associated to singularities that often occur in the elliptic fiber. This presents an opportunity for generalization in future work.

The second tool performs the geometric analysis described in section 5. The independent Hodge numbers are calculated directly from the polyhedral data using equation 36. Then, the corresponding toric variety is constructed and we restrict to the Calabi-Yau hypersurface. At this point, the geometry of the manifold can be studied in a variety of ways. As discussed here, the volumes of the manifold and the holes in the geometry are computed in order to understand the vacuum energy and the fields that arise in the theory. With this data easily available, we can classify models and proceed to analyzing the physics associated to each model by hand using the data.

These tools have been tested on models with a variety of properties. They are general enough to be applied to five- and lower-dimensional reflexive polyhedra, which allows their use in studying any string/M-/F-theory compactification to four dimensions, as well as compactifications to more than four dimensions. The code is fast enough to perform the described calculations for models with Hodge number $h^{1,1} = 2, 3, 4$, which is a rough measure of the complexity, in seconds on a laptop. More optimization would likely be needed to allow an efficient analysis of models with $h^{1,1} > 4$. The ability to quickly carry out the geometric analysis of these manifolds opens up a variety of options for applications to physics.

7 Discussion and Future Work

Future work may lay in either of two complementary directions. The first approach is to extend existing tools to carry out a more efficient and detailed analysis of models currently under study. There are many geometric quantities that can be computed using the data describing the Calabi-Yau that aid in classification of models. Furthermore, as mentioned above, greater efficiency is needed for the analysis of models with $h^{1,1} > 4$. By adding a few of these calculations, and optimizing existing tools, the scope of the possible classifications could be greatly improved.

The second direction for continued work is the generalization of these mathematical tools to study more complicated models. The simple models to which we have limited ourselves during this project are of limited physical interest because they do not possess singularities of the elliptic fiber. Such singularities are important because they give rise to generalizations of the gauge interactions in the Standard Model of particle physics. Early looks at these more realistic models have already turned up surprising results [15]. Namely, some geometries with identical Hodge numbers appear to exhibit different gauge interactions.

Furthermore, the ability to easily carry out geometric analyses of compactifications allows the models to be classified. By studying and comparing the resulting physics of these models, we can hope to find patterns that may improve our understanding of the kind of physics allowed in string theory compactifications. Such a comparative analysis is exactly what led to the discovery of mirror symmetry by string theorists [16]. Such classifications may help provide more order in what is now a rather chaotic picture of a vast landscape of possible models of compactification.

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