Moduli spaces for finite-order jets of Riemannian metrics

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March 5, 2009

Abstract

We construct the moduli space of $r-$jets of Riemannian metrics at a point on a smooth manifold. The construction is closely related to the problem of classification of jet metrics via differential invariants.

The moduli space is proved to be a differentiable space which admits a finite canonical stratification into smooth manifolds. A complete study on the stratification of moduli spaces is carried out for metrics in dimension $n = 2$.

Introduction

Let X be an $n-$dimensional smooth manifold. Fixed a point $x_0 \in X$ and an integer $r \geq 0$, we will denote by $J^r_{x_0}M$ the smooth manifold of $r-$jets at x_0 of Riemannian metrics on X. On the manifold $J^r_{x_0}M$, there exists a natural action of the group Diff_{x_0} of germs at x_0 of local diffeomorphisms leaving x_0 fixed, so it yields an equivalence relation on $J^r_{x_0}M$:

$$j^r_{x_0}g \equiv j^r_{x_0}\bar{g} \iff j^r_{x_0}(\tau^* g) = j^r_{x_0}\bar{g} \text{ , for some } \tau \in \mathrm{Diff}_{x_0} .$$

The quotient space $\mathbb{M}^r_n := J^r_{x_0}M/\mathrm{Diff}_{x_0}$ is called moduli space for $r-$jets of Riemannian metrics in dimension n. It depends neither on the point x_0 nor on the $n-$dimensional manifold X chosen.

The purpose of this paper is to study the structure of moduli spaces $\mathbb{M}^r_n$.

Moduli spaces $\mathbb{M}^r_n$ have been studied in the literature through their function algebras $\mathcal{C}^\infty(\mathbb{M}^r_n) := \mathcal{C}^\infty(J^r_{x_0}M)^{\mathrm{Diff}_{x_0}}$. This function algebra $\mathcal{C}^\infty(\mathbb{M}^r_n)$ is nothing but the algebra of differential invariants of order $\leq r$ of Riemannian metrics. Muñoz and Valdés ([8],[9]) prove that it is an essentially finitely-generated algebra and they determine the number of its functionally independent generators. In a more general setting, Vinogradov ([15]) has pointed out a simple and natural relationship between the algebra of differential invariants of homogeneous geometric structures and their characteristic classes. (See also [14].)

Let us also mention that in [4] García and Muñoz obtain a moduli space for linear frames, which has structure of smooth manifold.

However, apart from some trivial exceptions, moduli spaces $\mathbb{M}^r_n$ of jet metrics are not smooth manifolds, but they possess a differentiable structure in a more general sense: that of a differentiable space. (The typical example of differentiable space is a closed subset $Y \subseteq \mathbb{R}^m$ where a function $f : Y \to \mathbb{R}$ is said to be differentiable if it is the restriction to Y of a smooth function on $\mathbb{R}^m$, see [10].)

In addition, the differentiable structure of $\mathbb{M}_n^r$ is not too far from a smooth structure, since it admits a stratification by a finite number of smooth submanifolds. Our results can be summed up in the following

Theorem 0.1. *Every moduli space $\mathbb{M}_n^r$ is a differentiable space and it admits a finite canonical stratification*

$$\mathbb{M}_n^r = S^r_{[H_0]} \sqcup \ldots \sqcup S^r_{[H_s]} ,$$

for locally closed subspaces $S^r_{[H_i]}$ which are smooth manifolds. Moreover, one of them is an open connected dense subset of $\mathbb{M}_n^r$.

Each stratum of this decomposition of the space $\mathbb{M}_n^r$ consists of those jet metrics having essentially the same group of automorphisms. To be more precise, let us denote by $[H]$ the conjugacy class of a closed subgroup H of the orthogonal group $O(n)$. Then $S^r_{[H]}$ is the set of equivalence classes of jet metrics $j^r_{x_0}g$ whose group of automorphisms $\mathrm{Aut}(j^r_{x_0}g)$ is conjugate to H, viewing $\mathrm{Aut}(j^r_{x_0}g)$ as a subgroup of the orthogonal group $O(T_{x_0}X, g_{x_0}) \simeq O(n)$.

It is convenient to notice that Theorem 0.1 is not valid for semi-Riemannian metrics. For metrics of any signature, the problem lies on the existence of non-closed orbits for the action of Diff_{x_0} on the space $J^r_{x_0}M$ of $r-$jets of such metrics, which means that the corresponding moduli space $J^r_{x_0}M/\mathrm{Diff}_{x_0}$ is not a T_1 topological space, and consequently, it does not admit a structure of differentiable space either.

In dimension $n = 2$, we improve the above theorem by determining exactly all the strata which appear in the decomposition of each moduli space $\mathbb{M}^r_{n=2}$. Let us consider the only, up to conjugacy, closed subgroups of the orthogonal group $O(2)$: the finite group K_m of rotations of order m ($m \geq 1$), the dihedral group D_m of order $2m$ ($m \geq 1$), the special orthogonal group $SO(2)$ and $O(2)$ itself. The stratification of $\mathbb{M}_2^r$ is determined by the following

Theorem 0.2. *The strata in the moduli space $\mathbb{M}^r_{n=2}$ correspond exactly to the following conjugacy classes:* $[O(2)]$, $[D_1], \ldots, [D_{r-2}]$, $[K_1]$, $\ldots$, $[K_{r-4}]$. *(And also $[K_1]$, if $r = 4$.)*

Finally, we include two appendices. In the first one, we give a brief discussion of the notion of differential invariant. In the second one, we analyze the equivalence problem for infinite-order jets of Riemannian metrics.

1 Preliminaries

1.1 Quotient spaces

Throughout this paper, we are going to handle geometric objects of a more general nature than smooth manifolds, which appear when one considers the quotient of a smooth manifold by the action of a Lie group.

Definition 1.1. Let X be a topological space. A **sheaf of continuous functions** on X is a map $\mathcal{O}_X$ which assigns a subalgebra $\mathcal{O}_X(U) \subseteq \mathcal{C}(U, \mathbb{R})$ to every open subset $U \subseteq X$, with the following condition:

For every open subset $U \subseteq X$, every open cover $U = \bigcup U_i$ and every function $f : U \to \mathbb{R}$, it is verified

$$f \in \mathcal{O}_X(U) \iff f|_{U_i} \in \mathcal{O}_X(U_i)\,, \ \forall i\,.$$

In particular, if $V \subseteq U$ are open subsets in X, then it is verified

$$f \in \mathcal{O}_X(U) \implies f|_V \in \mathcal{O}_X(V)\,.$$

Definition 1.2. We will call **ringed space** the pair $(X, \mathcal{O}_X)$ formed by a topological space X and a sheaf of continuous functions $\mathcal{O}_X$ on X.

Although the concept of ringed space in the literature, specially in that concerning Algebraic Geometry, is much broader, the previous definition is good enough for our purposes.

Every open subset U of a ringed space $(X, \mathcal{O}_X)$ is itself, in a very natural way, a ringed space, if we define $\mathcal{O}_U(V) := \mathcal{O}_X(V)$ for every open subset $V \subseteq U$.

Hereinafter, a ringed space $(X, \mathcal{O}_X)$ will usually be denoted just by X, dropping the sheaf of functions.

Definition 1.3. Given two ringed spaces X and Y, a **morphism of ringed spaces** $\varphi : X \to Y$ is a continuous map such that, for every open subset $V \subseteq Y$, the following condition is held:

$$f \in \mathcal{O}_Y(V) \implies f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))\,.$$

A morphism of ringed spaces $\varphi : X \to Y$ is said to be an **isomorphism** if it has an inverse morphism, that is, there exists a morphism of ringed spaces $\phi : Y \to X$ verifying $\varphi \circ \phi = \mathrm{Id}_Y$, $\phi \circ \varphi = \mathrm{Id}_X$.

Example 1.4. (Smooth manifolds) The space $\mathbb{R}^n$, endowed with the sheaf $\mathcal{C}^\infty_{\mathbb{R}^n}$ of smooth functions, is an example of ringed space. An $n-$smooth manifold is precisely a ringed space in which every point has an open neighbourhood isomorphic to $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$. Smooth maps between smooth manifolds are nothing but morphisms of ringed spaces.

Example 1.5. (Quotients by the action of a Lie group) Let $G \times X \to X$ be a smooth action of a Lie group G on a smooth manifold X, and let $\pi : X \to X/G$ be the canonical quotient map.

We will consider on the quotient topological space X/G the following sheaf $\mathcal{C}^\infty_{X/G}$ of "differentiable" functions:

For every open subset $V \subseteq X/G$, $\mathcal{C}^\infty_{X/G}(V)$ is defined to be

$$\mathcal{C}^\infty_{X/G}(V) := \{f : V \longrightarrow \mathbb{R} : f \circ \pi \in \mathcal{C}^\infty(\pi^{-1}(V))\}\,.$$

Note that there exists a canonical $\mathbb{R}-$algebra isomorphism:

$$\begin{array}{ccc} \mathcal{C}^\infty_{X/G}(V) & = & \mathcal{C}^\infty(\pi^{-1}(V))^G \\ f & \longmapsto & f \circ \pi\,. \end{array}$$

The pair $(X/G, \mathcal{C}^\infty_{X/G})$ is an example of ringed space, which we will call **quotient ringed space** of the action of G on X.

As it would be expected, this space verifies the **universal quotient property**: Every morphism of ringed spaces $\varphi : X \to Y$, which is constant on every orbit of the action of G on X, factors uniquely through the quotient map $\pi : X \to X/G$, that is, there exists a unique morphism of ringed spaces $\tilde{\varphi} : X/G \to Y$ verifying $\varphi = \tilde{\varphi} \circ \pi$.

Example 1.6. (Inverse limit of smooth manifolds) Sometimes we will consider an inverse system

$$\cdots \longrightarrow X_{r+1} \longrightarrow X_r \longrightarrow \cdots \longrightarrow X_1$$

of smooth mappings between smooth manifolds (or, with some more generality, an inverse system of ringed spaces).

The inverse limit $\lim\limits_{\leftarrow} X_r$ is a ringed space in the following natural way. On $\lim\limits_{\leftarrow} X_r$ it is considered the inverse limit topology, that is, the initial topology induced by the evident projections $p_s : \lim\limits_{\leftarrow} X_r \to X_s$. A real function on an open subset of $\lim\limits_{\leftarrow} X_r$ is said to be "differentiable" if it locally coincides with the composition of a projection $p_s : \lim\limits_{\leftarrow} X_r \to X_s$ and a smooth function on X_s.

The topological space $\lim\limits_{\leftarrow} X_r$ endowed with the above sheaf of differentiable functions is a ringed space satisfying the suitable universal property:

For every ringed space Z, there exists the bijection

$$\begin{array}{ccc} \mathrm{Hom}\,(Z, \lim\limits_{\leftarrow} X_r) & = & \lim\limits_{\leftarrow} \mathrm{Hom}\,(Z, X_r) \\ \varphi & \longmapsto & (\ldots, p_r \circ \varphi, \ldots)\,. \end{array}$$

Example 1.7. Let Z be a locally closed subspace of $\mathbb{R}^n$. We define the sheaf $\mathcal{C}^\infty_Z$ of differentiable functions on Z to be the sheaf of functions locally coinciding with restrictions of smooth functions on $\mathbb{R}^n$. The pair $(Z, \mathcal{C}^\infty_Z)$ is another example of ringed space.

Definition 1.8. A (reduced) **differentiable space** is a ringed space in which every point has an open neighbourhood isomorphic to a certain locally closed subspace $(Z, \mathcal{C}^\infty_Z)$ in some $\mathbb{R}^n$.

A map between differentiable spaces is called **differentiable** if it is a morphism of ringed spaces.

Theorem 1.9. (Schwarz [11],[10] Th. 11.14) *Let $G \to Gl(V)$ be a finite-dimensional linear representation of a compact Lie group G. The quotient space V/G is a differentiable space.*

More precisely: Let $p_1, \ldots, p_s$ be a finite set of generators for the $\mathbb{R}-$algebra of $G-$invariant polynomials on V; these invariants define an isomorphism of ringed spaces

$$(p_1, \ldots, p_s) \,:\, V/G = Z \subseteq \mathbb{R}^s\,,$$

Z being a closed subspace of $\mathbb{R}^s$.

1.2 Normal tensors

Let X be an $n-$dimensional smooth manifold. Fix a point $x_0 \in X$ and a semi-Riemannian metric g on X of fixed signature (p,q), with $n = p+q$. Let us recall briefly some definitions and results:

Definition 1.10. A coordinate system $(z_1,\ldots,z_n)$ in a neighbourhood of x_0 is said to be a **normal coordinate** system for g at the point x_0 if the geodesics passing through x_0 at $t=0$ are precisely the "straight lines" $\{z_1(t)=\lambda_1 t,\ldots,z_n(t)=\lambda_n t\}$, where $\lambda_i \in \mathbb{R}$.

In particular, x_0 is the origin of any normal coordinate system for g at x_0.

Remark 1.11. Observe that we do not require $(\partial_{z_1},\ldots,\partial_{z_n})$ to be an orthonormal basis of $T_{x_0}X$.

As it is well known, via the exponential map $\exp_g : T_{x_0}X \to X$, normal coordinate systems on X correspond bijectively to linear coordinate systems on $T_{x_0}X$. Therefore, two normal systems differ in a linear coordinate transformation.

Proposition 1.12. *Let g, $\bar{g}$ be two semi-Riemannian metrics on X. Let us also consider their corresponding exponential maps* $\exp_g, \exp_{\bar{g}} : T_{x_0}X \to X$. *For every $r \geq 0$ it is verified:*

$$j^r_{x_0} g = j^r_{x_0}\bar{g} \implies j^{r+1}_0(\exp_g) = j^{r+1}_0(\exp_{\bar{g}}) .$$

As a consequence of Proposition 1.12, whose proof is routine, normal coordinate systems at x_0 for a metric g are determined up to the order $r+1$ by the jet $j^r_{x_0}g$. This fact will be used later on with no more explicit mention.

Definition 1.13. Let $r \geq 1$ be a fixed integer and let $x_0 \in X$. The space of **normal tensors** of order r at x_0, which we will denote by N_r, is the vector space of $(r+2)-$covariant tensors T at x_0 having the following symmetries:

- T is symmetric in the first two and last r indices:

$$T_{ijk_1\ldots k_r} = T_{jik_1\ldots k_r} \quad , \quad T_{ijk_1\ldots k_r} = T_{ijk_{\sigma(1)}\ldots k_{\sigma(r)}} \quad , \quad \forall\, \sigma \in S_r ;$$

- the cyclic sum over the last $r+1$ indices is zero:

$$T_{ijk_1\ldots k_r} + T_{ik_r jk_1\ldots k_{r-1}} + \ldots + T_{ik_1\ldots k_r j} = 0 .$$

If $r=0$, we will assume N_0 to be the set of semi-Riemannian metrics at x_0 of a fixed signature (p,q) (which is an open subset of $S^2T^*_{x_0}X$, but not a vector subspace).

A simple computation shows that, in general, $N_1 = 0$. Moreover, in [2] it is proved that N_r $(r \geq 2)$ is a linear irreducible representation of the linear group $\mathrm{Gl}\,(T_{x_0}X)$.

To show how a semi-riemannian metric g produces a sequence of normal tensors $g^r_{x_0}$ at x_0, let us recall this classical result:

Lemma 1.14. (Gauss Lemma) *Let $(z_1,\ldots,z_n)$ be germs of coordinates centred at $x_0 \in X$. These coordinates are normal for the germ of a semi-Riemannian metric g if and only if the metric coefficients g_{ij} verify the equations*

$$\sum_j g_{ij} z_j = \sum_j g_{ij}(x_0) z_j .$$

Let $(z_1, \ldots, z_n)$ be a normal coordinate system for g at $x_0 \in X$ and let us denote:

$$g_{ij,k_1\ldots k_r} := \frac{\partial^r g_{ij}}{\partial z_{k_1} \ldots \partial z_{k_r}}(x_0)\,.$$

If we differentiate $r+1$ times the identity of the Gauss Lemma, we obtain:

$$g_{ik_0,k_1\ldots k_r} + g_{ik_1,k_2\ldots k_r k_0} + \cdots + g_{ik_r,k_0\ldots k_{r-1}} = 0\,.$$

This property, together with the obvious fact that the coefficients $g_{ij,k_1\ldots k_r}$ are symmetric in the first two and in the last r indices, allows to prove that the tensor

$$g^r_{x_0} := \sum_{ijk_1\ldots k_r} g_{ij,k_1\ldots k_r}\, \mathrm{d}z_i \otimes \mathrm{d}z_j \otimes \mathrm{d}z_{k_1} \otimes \ldots \otimes \mathrm{d}z_{k_r}$$

is a normal tensor of order r at $x_0 \in X$. This construction does not depend on the choice of the normal coordinate system $(z_1, \ldots, z_n)$.

Definition 1.15. The tensor $g^r_{x_0}$ is called the $r-$th **normal tensor of the metric** g at the point x_0.

As a consequence of $N_1 = 0$, the first normal tensor of a metric g is always zero, $g^1_{x_0} = 0$.

The normal tensors associated to a metric were first introduced by Thomas [13]. The sequence $\{g_{x_0}, g^2_{x_0}, g^3_{x_0}, \ldots, g^r_{x_0}\}$ of normal tensors of the metric g at a point x_0 totally determines the sequence $\{g_{x_0}, R_{x_0}, \nabla_{x_0} R, \ldots, \nabla^{r-2}_{x_0} R\}$ of covariant derivatives at x_0 of the curvature tensor R of g and vice versa (see [13]). The main advantage of using normal tensors is the possibility of expressing the symmetries of each $g^s_{x_0}$ without using the other normal tensors, whereas the symmetries of $\nabla^s_{x_0} R$ depend on R (recall the Ricci identities).

Remark 1.16. Using the exact sequence

$$0 \longrightarrow N_r \longrightarrow S^2 T^*_{x_0} X \otimes S^r T^*_{x_0} X \stackrel{s}{\longrightarrow} T^*_{x_0} X \otimes S^{r+1} T^*_{x_0} X \longrightarrow 0\,,$$

where s stands for the symmetrization on the last $(r+1)-$indices, we obtain

$$\dim N_r = \binom{n+1}{2}\binom{n+r-1}{r} - n\binom{n+r}{r+1}\,.$$

2 Differential invariants of metrics

In the remainder of the paper, X will always be an $n-$dimensional smooth manifold.

Let us denote by $J^r M \to X$ the fiber bundle of $r-$jets of semi-Riemannian metrics on X of fixed signature $(p\,, q)$, with $n = p + q$. Its fiber over a point $x_0 \in X$ will be denoted $J^r_{x_0} M$.

Let Diff_{x_0} be the group of germs of local diffeomorphisms of X leaving x_0 fixed, and let $\mathrm{Diff}^r_{x_0}$ be the Lie group of $r-$jets at x_0 of local diffeomorphisms of X leaving x_0 fixed. We have the following exact group sequence:

$$0 \longrightarrow H^r_{x_0} \longrightarrow \mathrm{Diff}_{x_0} \longrightarrow \mathrm{Diff}^r_{x_0} \longrightarrow 0\,,$$

$H^r_{x_0}$ being the subgroup of Diff_{x_0} made up of those diffeomorphisms whose $r-$jet at x_0 coincides with that of the identity.

The group Diff_{x_0} acts in an obvious way on $J^r_{x_0}M$. Note that the subgroup $H^{r+1}_{x_0}$ acts trivially, so the action of Diff_{x_0} on $J^r_{x_0}M$ factors through an action of $\mathrm{Diff}^{r+1}_{x_0}$.

Definition 2.1. Two $r-$jets $j^r_{x_0}g\,, j^r_{x_0}\bar{g} \in J^r_{x_0}M$ are said to be **equivalent** if there exists a local diffeomorphism $\tau \in \mathrm{Diff}_{x_0}$ such that $j^r_{x_0}\bar{g} = j^r_{x_0}(\tau^* g)$.

Equivalence classes of $r-$jets of metrics constitute a ringed space. To be precise:

Definition 2.2. We call **moduli space** of $r-$jets of semi-Riemannian metrics of signature $(p\,, q)$ the quotient ringed space

$$\mathbb{M}^r_{p,q} := J^r_{x_0}M/\mathrm{Diff}_{x_0} = J^r_{x_0}M/\mathrm{Diff}^{r+1}_{x_0}\,.$$

In the case of Riemannian metrics, that is $p = n\,,\ q = 0$, the moduli space will be denoted $\mathbb{M}^r_n$.

It is important to observe that the moduli space depends neither on the point x_0 nor on the chosen $n-$dimensional manifold:

Given a point $\bar{x}_0$ in another $n-$dimensional manifold $\bar{X}$, let us consider an arbitrary diffeomorphism

$$X \supset U_{x_0} \xrightarrow{\varphi} U_{\bar{x}_0} \subset \bar{X}$$

between corresponding neighbourhoods of x_0 and $\bar{x}_0$, verifying $\varphi(x_0) = \bar{x}_0$. Such a diffeomorphism induces an isomorphism of ringed spaces between the corresponding moduli spaces,

$$\begin{array}{ccc} J^r_{\bar{x}_0}\bar{M}/\mathrm{Diff}_{\bar{x}_0} & \overline{\overline{\qquad}} & J^r_{x_0}M/\mathrm{Diff}_{x_0} \\ [j^r_{\bar{x}_0}\bar{g}] & \longmapsto & [j^r_{x_0}\varphi^*\bar{g}]\,, \end{array}$$

which is independent of the choice of the diffeomorphism φ. So both moduli spaces are canonically identified.

Let us now consider the quotient morphism

$$J^r_{x_0}M \xrightarrow{\pi} J^r_{x_0}M/\mathrm{Diff}_{x_0} = \mathbb{M}^r_{p,q}\,.$$

Recall that a function f defined on an open subset $U \subseteq \mathbb{M}^r_{p,q}$ is said to be **differentiable** if $f \circ \pi$ is a smooth function on $\pi^{-1}(U)$, that is,

$$\mathcal{C}^\infty(U) = \mathcal{C}^\infty(\pi^{-1}(U))^{\mathrm{Diff}_{x_0}}\,.$$

Every semi-Riemannian metric g on X of signature (p, q) defines a map

$$\begin{array}{ccc} X & \xrightarrow{m_g} & \mathbb{M}^r_{p,q} \\ x & \longmapsto & [j^r_x g]\,, \end{array}$$

which is “differentiable”, that is, it is a morphism of ringed spaces.

Definition 2.3. A **differential invariant** of order $\leq r$ of semi-Riemannian metrics of signature (p, q) is defined to be a global differentiable function on $\mathbb{M}^r_{p,q}$.

Taking into account the ringed space structure of $\mathbb{M}^r_{p,q}$, we can simply write:

$$\{\text{Differential invariants of order } \leq r\} = \mathcal{C}^\infty(\mathbb{M}^r_{p,q}) = \mathcal{C}^\infty(J^r_{x_0}M)^{\mathrm{Diff}_{x_0}}\,.$$

A differential invariant $h : \mathbb{M}^r_{p,q} \to \mathbb{R}$ associates with every semi-Riemannian metric g on X a smooth function on X, denoted by $h(g)$, through the formula $h(g) := h \circ m_g$, that is,

$$h(g)(x) = h([j^r_x g]) .$$

In any local coordinates, $h(g)$ is a function smoothly depending on the coefficients of the metric and their subsequent partial derivatives up to the order r,

$$h(g)(x) = h\left(g_{ij}(x), \frac{\partial g_{ij}}{\partial x_k}(x), \dots, \frac{\partial^r g_{ij}}{\partial x_{k_1} \dots \partial x_{k_r}}(x) \right) ,$$

which is equivariant with respect to the action of local diffeomorphisms,

$$h(\tau^* g) = \tau^*(h(g)) .$$

For a discussion on the concept of differential invariant, see Section 6.

3 A fundamental lemma

The aim of this section is to prove that there exist a certain linear finite-dimensional representation V^r of the orthogonal group $O(p,q)$ and an isomorphism of ringed spaces

$$\mathbb{M}^r_{p,q} \;=\; V^r / O(p,q) .$$

This bijection is already known at a set-theoretic level (see [2] and also [7] for $G-$structures which posses a linear connection). We just add the fact that this bijection is an isomorphism of ringed spaces.

Let us fix for this entire section a local coordinate system $(z_1, \dots, z_n)$ centred at x_0.

We will denote by $\mathcal{N}^r_{x_0}$ the smooth submanifold of $J^r_{x_0} M$ formed by $r-$jets at x_0 of metrics of signature (p,q) for which $(z_1, \dots, z_n)$ is a normal coordinate system (that is, Taylor expansions of the coefficients of such metrics with respect to coordinates $(z_1, \dots, z_n)$ satisfy the equations of the Gauss Lemma up to the order r).

Consider the subgroup of Diff_{x_0}

$$H^1_{x_0} := \{ \tau \in \mathrm{Diff}_{x_0} \,:\, j^1_{x_0}\tau \;=\; j^1_{x_0}(\mathrm{Id}) \} .$$

Note the following exact group sequence:

$$0 \longrightarrow H^1_{x_0} \longrightarrow \mathrm{Diff}_{x_0} \longrightarrow \mathrm{Gl}\,(T_{x_0} X) \longrightarrow 0 ,$$

where the epimorphism $\mathrm{Diff}_{x_0} \to \mathrm{Gl}\,(T_{x_0} X)$ takes every diffeomorphism to its linear tangent map at x_0.

Lemma 3.1. *There exists an isomorphism of ringed spaces*

$$\mathcal{N}^r_{x_0} =\!=\!= J^r_{x_0} M / H^1_{x_0} .$$

Proof. Let us start by constructing a smooth section of the natural inclusion

$$\mathcal{N}^r_{x_0} \hookrightarrow J^r_{x_0} M \,.$$

Given a jet metric $j^r_{x_0} g \in J^r_{x_0} M$, consider a metric g representing it. Let $(\bar{z}_1, \ldots, \bar{z}_n)$ be the only normal coordinate system centred at x_0 with respect to g which satisfies $\mathrm{d}_{x_0} \bar{z}_i = \mathrm{d}_{x_0} z_i$.

Let τ be the local diffeomorphism which transforms one coordinate system into another: $\tau^*(\bar{z}_i) = z_i$. The condition $\mathrm{d}_{x_0} \bar{z}_i = \mathrm{d}_{x_0} z_i$ implies that the linear tangent map of τ at x_0 is the identity, i.e. $\tau \in H^1_{x_0}$.

As $(\bar{z}_1, \ldots, \bar{z}_n)$ is a normal coordinate system for g, $(z_1 = \tau^*(\bar{z}_1), \ldots, z_n = \tau^*(\bar{z}_n))$ is a normal coordinate system for $\tau^* g$; that is, $j^r_{x_0}(\tau^* g) \in \mathcal{N}^r_{x_0}$.

Therefore, the section we were looking for is the following map:

$$\begin{array}{ccc} J^r_{x_0} M & \xrightarrow{\varphi} & \mathcal{N}^r_{x_0} \\ j^r_{x_0} g & \longmapsto & j^r_{x_0}(\tau^* g) \,, \end{array}$$

with τ depending on g.

Let us now see that φ is constant on each orbit of the action of $H^1_{x_0}$. Let $j^r_{x_0} g'$ be another point in the same orbit as $j^r_{x_0} g$, so we can write $g' = \sigma^* g$ for some $\sigma \in H^1_{x_0}$.

Since $(\bar{z}_1, \ldots, \bar{z}_n)$ are normal coordinates for g, $(z'_1 = \sigma^*(\bar{z}_1), \ldots, z'_n = \sigma^*(\bar{z}_n))$ is a normal coordinate system for $g' = \sigma^* g$. Then $z_i = \tau^*(\bar{z}_i) = \tau^*({\sigma^*}^{-1}(z'_i))$, and, if we apply the definition of φ, we get

$$\varphi(j^r_{x_0} g') = j^r_{x_0}(\tau^* {\sigma^*}^{-1} g') = j^r_{x_0}(\tau^* g) = \varphi(j^r_{x_0} g) \,.$$

As φ is constant on each orbit of the action of $H^1_{x_0}$, it induces, according to the universal quotient property, a morphism of ringed spaces:

$$J^r_{x_0} M / H^1_{x_0} \longrightarrow \mathcal{N}^r_{x_0} \,.$$

This map is indeed an isomorphism of ringed spaces, because it has an obvious inverse morphism, which is the following composition:

$$\mathcal{N}^r_{x_0} \hookrightarrow J^r_{x_0} M \to J^r_{x_0} M / H^1_{x_0} \,.$$

□

Let us denote by Gl_n the general linear group in dimension n:

$$\mathrm{Gl}_n := \{\, n \times n \text{ invertible matrices with coefficients in } \mathbb{R} \,\} \,.$$

Considering every matrix in Gl_n as a linear transformation of the coordinate system $(z_1, \ldots, z_n)$, we can think of Gl_n as a subgroup of Diff_{x_0}.

Via the action of the group Diff_{x_0} on $J^r_{x_0} M$, the subgroup Gl_n, for its part, acts leaving the submanifold $\mathcal{N}^r_{x_0}$ stable, and then we can state the following

Lemma 3.2. *There exists an isomorphism of ringed spaces*

$$\mathcal{N}^r_{x_0} / \,\mathrm{Gl}_n = J^r_{x_0} M / \mathrm{Diff}_{x_0} = \mathbb{M}^r_{p,q} \,.$$

Proof. Via the epimorphism

$$\mathrm{Diff}_{x_0} \longrightarrow \mathrm{Diff}_{x_0}/\, H^1_{x_0} \;=\; \mathrm{Gl}\,(T_{x_0}X)\,,$$

the subgroup Gl_n gets identified with $\mathrm{Gl}\,(T_{x_0}X)$. Consequently, the subgroups $H^1_{x_0}$ and Gl_n generate Diff_{x_0}.

If we consider the isomorphism

$$\mathcal{N}^r_{x_0} =\!=\!= J^r_{x_0}M/H^1_{x_0}$$

of Lemma 3.1 and take quotient with respect to the action of Gl_n, we get the desired isomorphism:

$$\mathcal{N}^r_{x_0}/\,\mathrm{Gl}_n =\!=\!= (J^r_{x_0}M/\,H^1_{x_0})/\mathrm{Gl}_n \;=\; J^r_{x_0}M/\mathrm{Diff}_{x_0}\,.$$

□

Let us express the previous result in terms of normal tensors by using the following

Lemma 3.3. *The map*

$$\mathcal{N}^r_{x_0} =\!=\!= N_0 \,\times\, N_2 \,\times\, \ldots \,\times\, N_r \quad , \quad j^r_{x_0}g \longmapsto (g_{x_0}, g^2_{x_0}, \ldots, g^r_{x_0})$$

is a diffeomorphism.

Proof. The inverse map is defined in the obvious way:

Given $(T^0, T^2, \ldots, T^r) \in N_0 \times N_2 \times \ldots \times N_r$, consider the jet metric $j^r_{x_0}g$ which in coordinates $(z_1, \ldots, z_n)$ is determined by the identities

$$g_{ij,k_1\ldots k_s} := \frac{\partial^s g_{ij}}{\partial z_{k_1}\cdots\partial z_{k_s}}(x_0) \;=\; T^s_{ijk_1\ldots k_s} \;\;,\;\; s = 0, \ldots, r\,.$$

The symmetries of tensors T^s guarantee that the coefficients g_{ij} of the metric g verify the equations of the Gauss Lemma up to the order r, that is, $j^r_{x_0}g \in \mathcal{N}^r_{x_0}$. □

Combining Lemma 3.2 and Lemma 3.3, we obtain an isomorphism of ringed spaces:

$$\begin{array}{ccc} \mathbb{M}^r_{p,q} \;=\; J^r_{x_0}M/\,\mathrm{Diff}_{x_0} & =\!=\!= & (N_0 \times N_2 \times \ldots \times N_r)/\,\mathrm{Gl}\,(T_{x_0}X) \\ [j^r_{x_0}g] & \longmapsto & [(g_{x_0}, g^2_{x_0}, \ldots, g^r_{x_0})]\,. \end{array}$$

Let us now fix a metric $g_{x_0} \in N_0$ at x_0 and let us consider the orthogonal group $O(p,q) := O(\,T_{x_0}X, g_{x_0})$. As the linear group $\mathrm{Gl}(\,T_{x_0}X)$ acts transitively on the space of metrics N_0, and $O(p,q)$ is the stabilizer subgroup of $g_{x_0} \in N_0$, we obtain the following isomorphism:

$$(N_0 \times N_2 \times \ldots \times N_r)/\,\mathrm{Gl}(\,T_{x_0}X) =\!=\!= (N_2 \times \ldots \times N_r)/\,O(p,q)\,.$$

To sum up, we can state the main result of this section:

Lemma 3.4. (Fundamental Lemma) *The moduli space $\mathbb{M}^r_{p,q}$ is isomorphic to the quotient space of a linear representation of the orthogonal group $O(p,q)$, through the following isomorphism of ringed spaces:*

$$\mathbb{M}^r_{p,q} \;=\!=\!=\; (N_2 \times \ldots \times N_r)/\, O(p,q)\,.$$

This isomorphism takes every class $[j^r_{x_0}\bar{g}] \in \mathbb{M}^r_{p,q}$, with $\bar{g}_{x_0} = g_{x_0}$, to the sequence of normal tensors $[(\bar{g}^2_{x_0}, \ldots, \bar{g}^r_{x_0})] \in (N_2 \times \ldots \times N_r)/\, O(p,q)$.

4 Structure of the moduli spaces

Let V be a finite-dimensional linear representation of a reductive Lie group G. The $\mathbb{R}-$algebra of $G-$invariant polynomials on V is finitely generated (Hilbert-Nagata theorem, see [3]). Let $p_1, \ldots, p_s$ be a finite set of generators for that algebra; by a result of Luna [6], every smooth $G-$invariant function f on V can be written as $f = F(p_1, \ldots, p_s)$, for some smooth function $F \in \mathcal{C}^\infty(\mathbb{R}^s)$.

Theorem 4.1. (Finiteness of differential invariants, [8]) *There exists a finite number $p_1, \ldots, p_s \in \mathcal{C}^\infty(\mathbb{M}^r_{p,q})$ of differential invariants of order $\leq r$ such that any other differential invariant f of order $\leq r$ is a smooth function of the former ones, i.e. $f = F(p_1, \ldots, p_s)$, for a certain $F \in \mathcal{C}^\infty(\mathbb{R}^s)$.*

Proof. By the Fundamental Lemma (3.4),

$$\mathcal{C}^\infty(\mathbb{M}^r_{p,q}) \;=\; \mathcal{C}^\infty(N_2 \times \ldots \times N_r)^{O(p,q)}\,,$$

and we can conclude by applying the above theorem by Luna to the linear representation $N_2 \times \ldots \times N_r$ of the orthogonal group $O(p,q)$. □

Remark 4.2. Using the theory of invariants for the orthogonal group and the fact that the sequence of normal tensors $\{g_{x_0}, g^2_{x_0}, g^3_{x_0}, \ldots, g^r_{x_0}\}$ is equivalent to the sequence $\{g_{x_0}, R_{x_0}, \nabla_{x_0} R, \ldots, \nabla^{r-2}_{x_0} R\}$, it can be proved that the generators $p_1, \ldots, p_s$ of Theorem 4.1 can be chosen to be **Weyl invariants**, that is, scalar quantities constructed from the sequence $\{g_{x_0}, R_{x_0}, \nabla_{x_0} R, \ldots, \nabla^{r-2}_{x_0} R\}$ by reiteration of the following operations: tensor products, raising and lowering indices, and contractions.

Theorem 4.3. *In the Riemannian case, differential invariants of order $\leq r$ separate points in the moduli space $\mathbb{M}^r_n$.*

Consequently, differential invariants of order $\leq r$ classify $r-$jets of Riemannian metrics (at a point).

Proof. For positive definite metrics, the orthogonal group $O(n)$ is compact. It is a well-known fact that, if V is a linear representation of a compact Lie group G, then smooth $G-$invariant functions on V separate the orbits of the action of G, or, in other words, the algebra $\mathcal{C}^\infty(V/G)$ separates the points in V/G.

Using this, together with the Fundamental Lemma, we conclude our proof. □

Neither assertion in Theorem 4.3 is valid for semi-Riemannian metrics. See Note in Subsection 5.2 for a counterexample. For such metrics, moduli spaces $\mathbb{M}^r_{p,q}$ are generally pathological in a topological sense, since they have non-closed points (they are not T_1 topological spaces).

In the Riemannian case, Schwarz Theorem 1.9 and the Fundamental Lemma directly provide the following

Theorem 4.4. *In the Riemannian case, moduli spaces* $\mathbb{M}^r_n$ *are differentiable spaces.*

More precisely: Let $p_1, \ldots, p_s$ be the basis of differential invariants of order $\leq r$ mentioned in Theorem 4.1. These invariants induce an isomorphism of differentiable spaces

$$(p_1, \ldots, p_s) \, : \, \mathbb{M}^r_n = \!= \!= Z \subseteq \mathbb{R}^s \,,$$

Z being a closed subspace of $\mathbb{R}^s$.

Although the differentiable space $\mathbb{M}^r_n$ is not in general a smooth manifold, its structure is not so deficient as it could seem at first sight, since we are going to prove that it admits a finite stratification by certain smooth submanifolds.

Definition 4.5. Let us consider $V_n = \mathbb{R}^n$ endowed with its standard inner product δ, and the corresponding orthogonal group $O(n) := O(V_n, \delta)$. We will denote by $\mathcal{T}$ the set of conjugacy classes of closed subgroups in $O(n)$.

Given another $n-$dimensional vector space $\bar{V}_n$ with an inner product $\bar{\delta}$, we can also consider the set $\bar{\mathcal{T}}$ of conjugacy classes of closed subgroups in $O(\bar{V}_n, \bar{\delta})$.

Observe that there exists a canonical identification

$$\mathcal{T} \longrightarrow \bar{\mathcal{T}} \ , \ [H] \longmapsto [\varphi \circ H \circ \varphi^{-1}] \,,$$

where φ stands for any isometry $\varphi : V_n \to \bar{V}_n$.

As the identification is canonical (i.e. it does not depend on the choice of the isometry φ), from now on we will suppose that the set $\mathcal{T}$ is just "the same" for every pair $(\bar{V}_n, \bar{\delta})$.

Note that $\mathcal{T}$ possesses a partial order relation: $[H] \leq [H']$, if there exist some representatives H and H' of $[H]$ and $[H']$ respectively, such that $H \subseteq H'$.

Definition 4.6. The **group of automorphisms** of a Riemannian jet metric $j^r_{x_0} g$ is defined to be the stabilizer subgroup $\mathrm{Aut}(j^r_{x_0} g) \subseteq \mathrm{Diff}^{r+1}_{x_0}$ of $j^r_{x_0} g$:

$$\mathrm{Aut}(j^r_{x_0} g) := \{ j^{r+1}_{x_0} \tau \in \mathrm{Diff}^{r+1}_{x_0} \ : \ j^r_{x_0}(\tau^* g) = j^r_{x_0} g \} \,.$$

Given $\tau \in \mathrm{Diff}_{x_0}$, let us denote by $\tau_{*,x_0} : T_{x_0} X \to T_{x_0} X$ the linear tangent map of τ at x_0.

Lemma 4.7. *The group morphism*

$$\begin{array}{ccc} \mathrm{Aut}(j^r_{x_0} g) & \longrightarrow & O(T_{x_0} X, g_{x_0}) \simeq O(n) \\ j^{r+1}_{x_0} \tau & \longmapsto & \tau_{*,x_0} \end{array}$$

is injective.

Proof. For any $\tau \in \mathrm{Diff}_{x_0}$ and any metric g on X we have the following commutative diagram of local diffeomorphisms:

$$\begin{array}{ccc} T_{x_0}X & \xrightarrow{\exp_{\tau^* g}} & X \\ {\scriptstyle \tau_*}\downarrow & & \downarrow{\scriptstyle \tau} \\ T_{x_0}X & \xrightarrow{\exp_g} & X \end{array}$$

If $j^{r+1}_{x_0}\tau \in \mathrm{Aut}(j^r_{x_0}g)$, that is, $j^r_{x_0}(\tau^* g) = j^r_{x_0} g$, then $j^{r+1}_0(\exp_{\tau^* g}) = j^{r+1}_0(\exp_g)$ because of Proposition 1.12.

Now, taking $(r+1)-$jets in the above diagram, we obtain:

$$j^{r+1}_{x_0}\tau \;=\; j^{r+1}_0(\exp_g) \circ j^{r+1}_0\tau_* \circ j^{r+1}_{x_0}(\exp_g^{-1})\,,$$

hence $j^{r+1}_{x_0}\tau$ is determined by its linear part τ_*. □

By the previous lemma, the group $\mathrm{Aut}(j^r_{x_0}g)$ can be viewed as a subgroup (determined up to conjugacy) of the orthogonal group $O(n)$.

Definition 4.8. The **type map** is defined to be the map

$$t : \mathbb{M}^r_n \longrightarrow \mathcal{T} \;\;,\;\; [j^r_{x_0}g] \longmapsto [\mathrm{Aut}(j^r_{x_0}g)]\,.$$

For each $[H] \in \mathcal{T}$, the **stratum of type** $[H]$ is said to be the subset $S_{[H]} \subseteq \mathbb{M}^r_n$ of those points of type $[H]$.

Theorem 4.9. (Stratification of the moduli space) *The type map* $t : \mathbb{M}^r_n \to \mathcal{T}$ *verifies the following properties:*

1. *t takes a finite number of values $[H_0], \ldots, [H_k]$, one of which, say $[H_0]$, is minimum.*
2. *Semicontinuity: For every type $[H] \in \mathcal{T}$, the set of points in $\mathbb{M}^r_n$ of type $\leq [H]$ is an open subset of $\mathbb{M}^r_n$. In particular, every stratum $S_{[H_i]}$ is a locally closed subspace of $\mathbb{M}^r_n$.*
3. *Every stratum $S_{[H_i]}$ is a smooth submanifold of $\mathbb{M}^r_n$.*
4. *The (also called generic) stratum $S_{[H_0]}$ of minimum type is a dense connected open subset of $\mathbb{M}^r_n$.*

Proof. Fix a positive definite metric g_{x_0} on $T_{x_0}X$ and denote by $O(n)$ its orthogonal group. The Fundamental Lemma 3.4 tells us that there exists an isomorphism

$$\mathbb{M}^r_n = (N_2 \times \ldots \times N_r)/\, O(n)\,.$$

This isomorphism takes every class $[j^r_{x_0}\bar{g}] \in \mathbb{M}^r_n$, with $\bar{g}_{x_0} = g_{x_0}$, to the sequence of normal tensors $[\bar{g}^2_{x_0}, \ldots, \bar{g}^r_{x_0}] \in (N_2 \times \ldots \times N_r)/\, O(n)$.

Let us check that the subgroup $\mathrm{Aut}(j^r_{x_0}\bar{g}) \hookrightarrow O(n)$, $j^{r+1}_{x_0}\tau \mapsto \tau_*$, coincides with the subgroup

$$\mathrm{Aut}(\bar{g}^2_{x_0}, \ldots, \bar{g}^r_{x_0}) := \{\sigma \in O(n) : \sigma^*(\bar{g}^k_{x_0}) \;=\; \bar{g}^k_{x_0}\,,\, \forall\, k \leq r\}\,.$$

It is clear that if an automorphism $j^{r+1}_{x_0}\tau$ leaves $j^r_{x_0}\bar g$ fixed, then the sequence of its normal tensors must also remain fixed by the automorphism: $\tau^*(\bar g^k_{x_0}) = \bar g^k_{x_0}$.

Reciprocally, given an automorphism $\sigma : T_{x_0}X \to T_{x_0}X$ of the sequence of normal tensors $(\bar g^2_{x_0}, \dots, \bar g^r_{x_0})$, let us consider a normal coordinate system $z_1, \dots, z_n$ for $\bar g$ at x_0 .

Via the identification provided by the exponential map $\exp_g : T_{x_0}X \to X$, the map σ can be viewed as a diffeomorphism of X (a linear transformation of normal coordinates).

In normal coordinates, the expression of the normal tensor $\bar g^k_{x_0}$ corresponds to the expression of the homogeneous part of degree k of the jet metric $j^r_{x_0}\bar g$. Hence it is an immediate consequence that the linear transformation σ leaves $j^r_{x_0}\bar g$ fixed, i.e. $j^{r+1}_{x_0}\sigma \in \mathrm{Aut}(j^r_{x_0}\bar g)$.

The identity $\mathrm{Aut}(j^r_{x_0}\bar g) = \mathrm{Aut}(\bar g^2_{x_0}, \dots, \bar g^r_{x_0})$ implies that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{M}^r_n & \xrightarrow{\ t\ } & \mathcal{T} \\ \| & & \| \\ (N_2 \times \ldots \times N_r)/\,O(n) & \xrightarrow{\ t\ } & \mathcal{T} \\ [\bar g^2_{x_0}, \dots, \bar g^r_{x_0}] & \longmapsto & [\mathrm{Aut}(\bar g^2_{x_0}, \dots, \bar g^r_{x_0})]\,. \end{array}$$

Therefore, our theorem has come down to the case of a linear representation $V (= N_2 \times \ldots \times N_r)$ of a compact Lie group $G (= O(n))$ and the corresponding type map:

$$\begin{array}{ccl} V/G & \xrightarrow{\ t\ } & \mathcal{T} = \{\text{conjugacy classes of closed subgroups of } G\} \\ [v] & \longmapsto & [\text{Stabilizer subgroup of } v]\,. \end{array}$$

For this type map, the analogous properties to $1-4$ in the statement are well known (see [1], Chap. IX, § 9, Th. 2 and Exer. 9). □

Remark 4.10. Except for trivial cases, the generic stratum has type $H_0 = \{0\}$.

Remark 4.11. The dimension of the moduli space $\mathbb{M}^r_n$ (or rather that of its generic stratum) can be deduced directly from the Fundamental Lemma and the formulae giving the dimensions of spaces N_r of normal tensors which were presented in Section 1.

The result (due, in a different language, to J. Muñoz and A. Valdés, [9]) is as follows:

$$\dim \mathbb{M}^0_n = \dim \mathbb{M}^1_n = 0 \ , \ \forall n \geq 1\,;$$

$$\dim \mathbb{M}^r_1 = 0 \ , \ \forall r \geq 0\,;$$

$$\dim \mathbb{M}^2_2 = 1 \quad , \quad \dim \mathbb{M}^r_2 = \frac{1}{2}(r+1)(r-2) \ , \ \forall r \geq 3\,;$$

$$\dim \mathbb{M}^r_n = n + \frac{(r-1)n^2 - (r+1)n}{2(r+1)} \binom{n+r}{r} \ , \ \forall n \geq 3\,,\ r \geq 2\,.$$

5 Moduli spaces in dimension $n = 2$

5.1 Stratification

We are going to determine the stratification of moduli spaces $\mathbb{M}_2^r$ of $r-$jets of Riemannian metrics in dimension $n = 2$.

Let us consider the vector space $\mathbb{R}^2 = \mathbb{C}$, endowed with the standard Euclidean metric, and its corresponding orthogonal group $O(2)$. We will denote by (x, y) the Cartesian coordinates and by $z = x + iy$ the complex coordinate.

Let us denote by $\sigma_m : \mathbb{C} \to \mathbb{C}$ the rotation of angle $2\pi/m$ (that is, $\sigma_m(z) = \varepsilon_m z$, with $\varepsilon_m = \cos(2\pi/m) + i\sin(2\pi/m)$ a primitive mth root of unity) and by $\tau : \mathbb{C} \to \mathbb{C}$, $\tau(z) = \bar{z}$ the complex conjugation.

The only (up to conjucagy) closed subgroups of $O(2)$ are the following ones:

$$SO(2) := \{\varphi \in O(2) : \det\varphi = 1\} \quad \text{(special orthogonal group)},$$

$$K_m := <\sigma_m> \qquad \text{(group of rotations of order } m\text{)} \quad (m \geq 1),$$

$$D_m := <\sigma_m, \tau> \qquad \text{(dihedral group of order } 2m\text{)} \quad (m \geq 1),$$

and $O(2)$ itself. All these subgroups are normal but the dihedral D_m.

The subgroup $SO(2)$ of rotations is identified with the multiplicative group $S_1 \subset \mathbb{C}$ of complex numbers of modulus 1,

$$\begin{array}{ccc} S_1 & = & SO(2) \\ \alpha & \longmapsto & \rho_\alpha \quad , \quad \rho_\alpha(z) := \alpha z\,. \end{array}$$

Besides, every element in $O(2)$ is either ρ_α or $\tau\rho_\alpha$, for some $\alpha \in S_1$.

The action of $O(2)$ on $\mathbb{R}^2$ induces an action on the algebra $\mathbb{R}[x, y]$ of the polynomials on $\mathbb{R}^2$, to be more specific: $\varphi \cdot P(x, y) := P(\varphi^{-1}(x, y))$.

The following lemma provides us with the list of all invariant polynomials with respect to each of the subgroups of $O(2)$ above mentioned:

Lemma 5.1. *The following identities hold:*

1. $\mathbb{R}[x, y]^{K_m} = \mathbb{R}[x^2 + y^2, p_m(x, y), q_m(x, y)]$,
2. $\mathbb{R}[x, y]^{D_m} = \mathbb{R}[x^2 + y^2, p_m(x, y)]$,
3. $\mathbb{R}[x, y]^{O(2)} = \mathbb{R}[x, y]^{SO(2)} = \mathbb{R}[x^2 + y^2]$,

with $p_m(x, y) = \mathrm{Re}(z^m)$ *and* $q_m(x, y) = \mathrm{Im}(z^m)$.

Proof. 1. Let us consider the algebra of polynomials on $\mathbb{R}^2$ with complex coefficients,

$$\mathbb{C}[x, y] = \mathbb{C}[z, \bar{z}] = \bigoplus_{ab} \mathbb{C} z^a \bar{z}^b\,.$$

Every summand is stable under the action of K_m, since

$$\sigma_m \cdot (z^a \bar{z}^b) = \frac{1}{\varepsilon_m^a \bar{\varepsilon}_m^b} z^a \bar{z}^b = \varepsilon_m^{b-a} z^a \bar{z}^b\,.$$

This formula also tells us that the monomial $z^a\bar{z}^b$ is invariant by K_m if and only if $b-a \equiv 0 \bmod m$, that is, $b-a = \pm km$ for some $k \in \mathbb{N}$. Then invariant monomials are of the form

$$z^a\bar{z}^b = (z\bar{z})^a\bar{z}^{km} \quad \text{or} \quad z^a\bar{z}^b = (z\bar{z})^b z^{km} ,$$

whence

$$\mathbb{C}[x,y]^{K_m} = \mathbb{C}[z\bar{z}, z^m, \bar{z}^m] .$$

As $z\bar{z} = x^2+y^2$, $z^m+\bar{z}^m = 2p_m(x,y)$ and $z^m-\bar{z}^m = 2iq_m(x,y)$, we can conclude that

$$\mathbb{C}[x,y]^{K_m} = \mathbb{C}[x^2+y^2, p_m(x,y), q_m(x,y)] ,$$

and particularly,

$$\mathbb{R}[x,y]^{K_m} = \mathbb{R}[x^2+y^2, p_m(x,y), q_m(x,y)] .$$

2. As $D_m =< K_m, \tau >$, we get

$$\mathbb{C}[x,y]^{D_m} = (\mathbb{C}[x,y]^{K_m})^{<\tau>} = \mathbb{C}[z\bar{z}, z^m, \bar{z}^m]^{<\tau>}$$

$$= \left[\left(\bigoplus_k \mathbb{C}[z\bar{z}]z^{km}\right) \oplus \left(\bigoplus_k \mathbb{C}[z\bar{z}]\bar{z}^{km}\right)\right]^{<\tau>}$$

(as $\tau \cdot z = \bar{z}$ and $\tau \cdot \bar{z} = z$)

$$= \bigoplus_k \mathbb{C}[z\bar{z}](z^{km}+\bar{z}^{km}) = \mathbb{C}[z\bar{z}, z^m+\bar{z}^m] = \mathbb{C}[x^2+y^2, p_m(x,y)] ,$$

and, in particular,

$$\mathbb{R}[x,y]^{D_m} = \mathbb{R}[x^2+y^2, p_m(x,y)] .$$

3. Every summand in the decomposition

$$\mathbb{C}[z,\bar{z}] = \bigoplus_{ab} \mathbb{C}z^a\bar{z}^b$$

is stable under the action of $SO(2)$, since for every $\rho_\alpha \in SO(2)$ it is satisfied:

$$\rho_\alpha \cdot (z^a\bar{z}^b) = \frac{1}{\alpha^a\bar{\alpha}^b} z^a\bar{z}^b .$$

Moreover, this formula assures us that the only monomials $z^a\bar{z}^b$ which are $SO(2)-$invariant are those verifying $a=b$. Then,

$$\mathbb{C}[x,y]^{SO(2)} = \mathbb{C}[z,\bar{z}]^{SO(2)} = \mathbb{C}[z\bar{z}] = \mathbb{C}[x^2+y^2] ,$$

whence

$$\mathbb{R}[x,y]^{SO(2)} = \mathbb{R}[x^2+y^2] .$$

Finally, this identity tells us that $SO(2)-$invariant polynomials are $O(2)-$invariant too, so the obvious inclusion $\mathbb{R}[x,y]^{O(2)} \subseteq \mathbb{R}[x,y]^{SO(2)}$ is indeed an equality. □

Corollary 5.2. *With the same notations used in the previous lemma, it is verified:*

1. D_m *is the stabilizer subgroup of the polynomial* $p_m(x,y)$ *, and there exists no polynomial in* $\mathbb{R}[x,y]$ *of degree* $< m$ *whose stabilizer subgroup is* D_m *.*

2. K_m *(*$m \geq 2$*) is the stabilizer subgroup of the polynomial* $p_m(x,y)+(x^2+y^2)q_m(x,y)$ *, and there exists no polynomial in* $\mathbb{R}[x,y]$ *of degree* $< m+2$ *whose stabilizer subgroup is* K_m *.*

3. $K_1 = \{\mathrm{Id}\}$ *is the stabilizer subgroup of the polynomial* $x + xy$ *, and there exists no polynomial in* $\mathbb{R}[x,y]$ *of degree* < 2 *whose stabilizer subgroup is* K_1 *.*

Proof. 1. Using that every element in $O(2)$ is either of the form ρ_α or of the form $\rho_\alpha \circ \tau$, it is a matter of routine to check that the stabilizer subgroup of the polynomial $p_m(x,y) = \mathrm{Re}(z^m)$ is D_m.

If there were another polynomial $\bar{p}(x,y)$ of degree $< m$ with the same property, $\bar{p}(x,y)$ should be a power of x^2+y^2, because of Lemma 5.1(2), and in that case its stabilizer subgroup would be the whole $O(2)$, against our hipothesis.

2. According to Lemma 5.1 (1), every K_m-invariant polynomial of degree $\leq m$ is of the form $\lambda p_m(x,y) + \mu q_m(x,y)$ (up to addition of a power of x^2+y^2). However, a polynomial of such a form does not have K_m as its stabilizer subgroup, but a larger dihedral group: after multiplying by a scalar, we can indeed assume $\lambda^2 + \mu^2 = 1$; if $\alpha = \lambda - i\mu$, then

$$\lambda p_m(x,y) + \mu q_m(x,y) \; = \; \mathrm{Re}(\alpha z^m) \; = \; \mathrm{Re}((\beta z)^m)$$

(with $\beta^m = \alpha$)

$$= \rho_{\beta^{-1}} \cdot \mathrm{Re}(z^m) \; = \; \rho_{\beta^{-1}} \cdot p_m(x,y)\,,$$

whose stabilizer subgroup is the dihedral group $\rho_{\beta^{-1}} \cdot D_m \cdot \rho_\beta$, which is conjugate to the stabilizer subgroup D_m of $p_m(x,y)$. (In particular, taking $\lambda = 0$, $\mu = -1$, we get that the stabilizer subgroup of $q_m(x,y)$ is $\rho_{\beta^{-1}} \cdot D_m \cdot \rho_\beta$, for $\beta^m = i$).

As no polynomial of degree $\leq m$ has the desired stabilizer subgroup K_m, and there are not any K_m-invariant polynomials of degree $m+1$ (up to a power of x^2+y^2), the following degree to be considered is $m+2$. The stabilizer subgroup of the polynomial $p_m(x,y)+(x^2+y^2)q_m(x,y)$, of degree $m+2$, is the intersection of the stabilizer subgroups of its two homogeneous components, $p_m(x,y)$ and $(x^2+y^2)q_m(x,y)$, that is,

$$D_m \cap (\rho_{\beta^{-1}} \cdot D_m \cdot \rho_\beta) \; = \; K_m \quad (\beta^m = i)\,.$$

3. This case is trivial. □

Theorem 5.3. *The strata in the moduli space* $\mathbb{M}_2^r$ *correspond exactly to the following types:* $[O(2)]$, $[D_1], \ldots, [D_{r-2}]$, $[K_1]$, $\ldots$, $[K_{r-4}]$*. (And also* $[K_1]$*, if* $r = 4$*.)*

Proof. It is a classical result (see [2]) that in dimension 2 every Riemannian metric can be written in normal coordinates (x,y) (in a unique way up to an orthogonal transformation) as follows:

$$g \; = \; \mathrm{d}x^2 + \mathrm{d}y^2 + h(x,y)(y\mathrm{d}x - x\mathrm{d}y)^2\,,$$

for some smooth function $h(x,y)$.

Observe that the stabilizer subgroup of $O(2)$ for the jet $j_0^k h$ is the same as that for $j_0^{k+2} g$.

If we take $h(x,y)=0$, we get a metric (the Euclidean one, i.e. $g=\mathrm{d}x^2+\mathrm{d}y^2$) whose group of automorphisms (for any jet order) is $O(2)$.

Choosing $h(x,y)=p_m(x,y)$, we obtain an $r-$jet metric (with $r\geq m+2$) whose stabilizer subgroup is D_m, because of Corollary 5.2 (1).

If we choose $h(x,y)=p_m(x,y)+(x^2+y^2)q_m(x,y)$, we get an $r-$jet metric (with $r\geq m+4$) whose stabilizer subgroup is K_m, by Corollary 5.2 (2).

If we make $h(x,y)=x+xy$, then we get an $r-$jet metric (with $r\geq 4$) whose stabilizer subgroup is K_1, according to Corollary 5.2 (3).

Finally, let us note that no $r-$jet metric can have $SO(2)$ as its stabilizer subgroup, since such a metric would correspond to a jet function $j_0^{r-2}h$ whose stabilizer subgroup should be $SO(2)$, which is impossible, because, by Lemma 5.1 (3), every $SO(2)-$invariant polynomial is also $O(2)-$invariant. $\square$

Corollary 5.4. *Every closed subgroup of $O(2)$, except for $SO(2)$, is the group of automorphisms of a jet metric $j_0^r g$ on $\mathbb{R}^2$ for some order r.*

Corollary 5.5. *The number of strata in $\mathbb{M}_2^r$ is:*

$$\text{Number of strata in } \mathbb{M}_2^r = \begin{cases} 1 & \text{for } r=0,1,2 \\ 2 & \text{for } r=3 \\ 4 & \text{for } r=4 \\ 2r-5 & \text{for } r\geq 5 \end{cases}$$

5.2 Examples

Now we describe, without proofs, low order jets in dimension $n=2$.

For order $r=0,1$ (and in any dimension n) moduli spaces $\mathbb{M}_n^r$ come down to a single point.

Case $r=2$.

The moduli space is a line:

$$\mathbb{M}_2^2 = \mathbb{R} \quad , \quad [j_{x_0}^2 g] \longmapsto K_g(x_0) \ .$$

In other words, the curvature classifies $2-$jets of Riemannian metrics in dimension $n=2$.

In this case there is just one stratum, the generic one, whose type is $[O(2)]$.

Case $r=3$.

The moduli space is a closed semiplane:

$$\mathbb{M}_2^3 = \mathbb{R}\times[0,+\infty) \quad , \quad [j_{x_0}^3 g] \longmapsto (K_g(x_0)\,, |\mathrm{grad}_{x_0} K_g|^2) \ .$$

That is to say, the curvature and the square of the modulus of the gradient of the curvature classify $3-$jet metrics in dimension $n=2$.

Now we have two different strata:

The generic stratum $S_{[D_1]} = \mathbb{R}\times(0,+\infty)$, with type $[D_1]$. This stratum is the set of all classes of jets $j^3_{x_0}g$ verifying $\mathrm{grad}_{x_0}K_g \neq 0$ (in this case, the group of automorphisms is the group of order 2 generated by the reflection across the vector $\mathrm{grad}_{x_0}K_g$).

The non-generic stratum $S_{[O(2)]} = \mathbb{R}\times\{0\}$, with type $[O(2)]$, is the set of all classes of jets $j^3_{x_0}g$ verifying $\mathrm{grad}_{x_0}K_g = 0$ (which are invariant with respect to every orthogonal transformation of normal coordinates).

Note: If we consider metrics of signature $(+,-)$, instead of Riemannian metrics, then the map

$$\mathbb{M}^3_2 \longrightarrow \mathbb{R}\times[0,+\infty) \quad , \quad [j^3_{x_0}g] \longmapsto (K_g(x_0)\,, |\mathrm{grad}_{x_0}K_g|^2)\ .$$

is not injective, that is, differential invariants do not classify $3-$jet metrics of signature $(+,-)$. To illustrate this, consider two metrics $g\,,\bar{g}$ of signature $(+,-)$, such that $K_g(x_0) = K_{\bar{g}}(x_0)$, $\mathrm{grad}_{x_0}K_g = 0$ and $\mathrm{grad}_{x_0}K_{\bar{g}}$ is a non-zero isotropic vector with respect to $\bar{g}_{x_0}$. Both jets $j^3_{x_0}g$, $j^3_{x_0}\bar{g}$ cannot be equivalent (because the gradient of the curvature at x_0 equals zero for the first metric, whereas it is non-zero for the other one), but its differential invariants coincide: $K_g(x_0) = K_{\bar{g}}(x_0)$ and $|\mathrm{grad}_{x_0}K_g|^2 = |\mathrm{grad}_{x_0}K_{\bar{g}}|^2 = 0$.

Case $r = 4$.

A set of generators for differential invariants of order 4 is given by the following five functions:

$$\begin{aligned}
p_1(j^4_{x_0}g) &= K_g(x_0)\,,\\
p_2(j^4_{x_0}g) &= |\mathrm{grad}_{x_0}K_g|^2\,,\\
p_3(j^4_{x_0}g) &= \mathrm{trace}\,(\mathrm{Hess}_{x_0}K_g)\,,\\
p_4(j^4_{x_0}g) &= \det\,(\mathrm{Hess}_{x_0}K_g)\,,\\
p_5(j^4_{x_0}g) &= \mathrm{Hess}_{x_0}K_g(\mathrm{grad}_{x_0}K_g\,, \mathrm{grad}_{x_0}K_g)\,,
\end{aligned}$$

where $\mathrm{Hess}_{x_0}K_g := (\nabla \mathrm{d}K_g)_{x_0}$ stands for the hessian of the curvature function at x_0.

These above functions satisfy the following inequalities:

$$p_2 \geq 0 \quad , \quad p_3^2 - 4p_4 \geq 0 \quad , \quad (2p_5 - p_2p_3)^2 \leq p_2^2(p_3^2 - 4p_4)\,.$$

To say it in other words, these five differential invariants define an isomorphism of differentiable spaces

$$(p_1,\ldots,p_5) : \mathbb{M}^4_2 =\!=\!=\!= Y \subset \mathbb{R}^5$$

Y being the closed subset in $\mathbb{R}^5$ determined by the inequalities

$$x_2 \geq 0 \quad , \quad x_3^2 - 4x_4 \geq 0 \quad , \quad (2x_5 - x_2x_3)^2 \leq x_2^2(x_3^2 - 4x_4)\,.$$

In this case, the moduli space $\mathbb{M}^4_2$ has the following four strata:

- The generic stratum of all classes of jets $j^4_{x_0}g$ verifying that $\mathrm{grad}_{x_0}K_g$ is not an eigenvector of $\mathrm{Hess}_{x_0}K_g$ (therefore, the eigenvalues of $\mathrm{Hess}_{x_0}K_g$ are different). The type of this stratum (group of automorphisms of its jets) is $[K_1 = \{\mathrm{Id}\}]$.

- The stratum of those classes of jet metrics $j^4_{x_0}g$ verifying that $\text{grad}_{x_0}K_g$ is a non-zero eigenvector of $\text{Hess}_{x_0}K_g$. Its type is $[D_1]$: the group of automorphisms of each jet metric is generated by the reflection across the vector $\text{grad}_{x_0}K_g$.

- The stratum composed of those classes of jet metrics $j^4_{x_0}g$ with $\text{grad}_{x_0}K_g = 0$ and verifying that the eigenvectors of $\text{Hess}_{x_0}K_g$ are different. The type of this stratum is $[D_2]$: the group of automorphisms of each jet metric is generated by the reflections across either eigenvector of $\text{Hess}_{x_0}K_g$.

- The stratum of all classes of jets $j^4_{x_0}g$ with $\text{grad}_{x_0}K_g = 0$ and verifying that the eigenvectors of $\text{Hess}_{x_0}K_g$ are both equal. The type of the stratum is $[O(2)]$.

6 Appendix A: On the notion of differential invariant of metrics

The aim of this Appendix A is to discuss the notion of differential invariant and to back up the Definition 2.3 given in Section 2.

The notion of differential invariant must be understood as a particular case of the concept of regular and natural operator between natural bundles (see [5] for an exposition of the theory of natural bundles). What follows is an adaptation of this point of view, getting around, though, the concept of natural bundle.

Let X be an $n-$dimensional smooth manifold. Let $M \to X$ be the bundle of semi-Riemannian metrics of a fixed signature (p,q) and let $\mathcal{M}_X$ denote its sheaf of smooth sections.

Loosely speaking, the concept of differential invariant refers to a function "intrinsically, locally and smoothly constructed from a metric". Rigorously, as it is a *local* construction, a differential invariant is a morphism of sheaves:

$$f : \mathcal{M}_X \longrightarrow \mathcal{C}^\infty_X \,,$$

where $\mathcal{C}^\infty_X$ stands for the sheaf of smooth functions on X.

The intuition of "intrinsic and smooth construction" can be encoded by saying that the morphism f also satisfies the following two properties:

1.- **Regularity:** If $\{g_s\}_{s\in S}$ is a family of metrics depending smoothly on certain parameters, the family of functions $\{f(g_s)\}_{s\in S}$ also depends smoothly on those parameters.

To be exact, let S be a smooth manifold (the space of parameters) and let $U \subseteq X\times S$ be an open set. For each $s \in S$, consider the open set in X defined as $U_s := \{x \in X : (x,s) \in U\}$. A family of metrics $\{g_s \in \mathcal{M}(U_s)\}_{s\in S}$ is said to be *smooth* if the fibre map $U \to S^2T^*X$, $(x,s) \mapsto (g_s)_x$, is smooth. In the same way, a family of functions $\{f_s \in \mathcal{C}^\infty(U_s)\}_{s\in S}$ is said to be smooth if the function $U \to \mathbb{R}$, $(x,s) \mapsto (f_s)(x)$, is smooth.

In these terms, the regularity condition expresses that for each smooth manifold S, each open set $U \subseteq X \times S$ and each smooth family of metrics $\{g_s \in \mathcal{M}(U_s)\}_{s\in S}$, the family of functions $\{f(g_s) \in \mathcal{C}^\infty(U_s)\}_{s\in S}$ is smooth.

2.- **Naturalness:** The morphism of sheaves f is equivariant with respect to the action of local diffeomorphisms of X.

That is, for each diffeomorphism $\tau : U \to V$ between open sets of X and for each metric g on V, the following condition must be satisfied:

$$f(\tau^* g) \,=\, \tau^*(f(g))\,.$$

Taking into account the previous comments, the suitability of the following definition is now clear:

Definition 6.1. A **differential invariant** associated to semi-Riemannian metrics (of the fixed signature) is a regular and natural morphism of sheaves $f : \mathcal{M}_X \to \mathcal{C}^\infty_X$.

Note that this definition of differential invariant seems to be far too general, since a differential invariant $f(g)$ is not assumed *a priori* to be constructed from the coefficients of the metric g and their subsequent partial derivatives. As we are going to show below, this question is clarified by a beautiful result by J. Slovák.

For every integer $r \geq 0$, we denote by $J^r M \to X$ the fiber bundle of $r-$jets of semi-Riemannian metrics on X (of the prefixed signature). The fiber bundle $J^\infty M \to X$ of $\infty-$jets of semi-Riemannian metrics is not a smooth manifold, but it can be endowed with the structure of a ringed space as follows. On $J^\infty M \to X$ we consider the inverse limit topology: $J^\infty M = \lim\limits_{\leftarrow} J^r M$; a function on an open set $U \subseteq J^\infty M$ is said to be differentiable if it is locally the composition of one of the natural projections $U \subseteq J^\infty M \to J^r M$ with a smooth function on $J^r M$. This way, $J^\infty M$ is a ringed space, with its sheaf of differentiable functions.

In a similar manner, the structure of a ringed space is defined for the fiber of the bundle $J^\infty M \to X$ over a given point $x_0 \in X$: $J^\infty_{x_0} M = \lim\limits_{\leftarrow} J^r_{x_0} M$.

Theorem 6.2. (Slovák) *There exists the following bijective correspondence:*

$$
\begin{array}{ccc}
\{\text{differentiable functions } \tilde{f} : J^\infty M \to \mathbb{R}\} & & \tilde{f} \\
\Vert & & \downarrow \\
\{\text{regular morphisms of sheaves } f : \mathcal{M}_X \to \mathcal{C}^\infty_X\} & & f
\end{array}
$$

with $f(g)(x) := \tilde{f}(j^\infty_x g)$.

The result by Slovák [12] refers, with a bit more of generality, to regular morphisms between sheaves of sections of fiber bundles.

If a regular morphism $\mathcal{M}_X \to \mathcal{C}^\infty_X$ is, furthermore, natural (that is, a differential invariant), then the corresponding smooth function $\tilde{f} : J^\infty M \to \mathbb{R}$ is determined by its restriction to the fiber $J^\infty_{x_0} M$ of an arbitrary point $x_0 \in X$. This assertion can be expressed more precisely in the following way.

Corollary 6.3. *Fixed a point* $x_0 \in X$ *, the set of differential invariants* $f : \mathcal{M}_X \to \mathcal{C}^\infty_X$ *is in bijection with the set of differentiable* $\mathrm{Diff}_{x_0}-$*invariant functions* $\tilde{f} : J^\infty_{x_0} M \to \mathbb{R}$.

Definition 6.4. A differential invariant $f : \mathcal{M}_X \to \mathcal{C}^\infty_X$ is said to be **of order** $\leq r$ if the corresponding differentiable function $\tilde{f} : J^\infty M \to \mathbb{R}$ factors through the projection $J^\infty M \to J^r M$.

Reformulating Corollary 6.3 for invariants of order r , we obtain that Definition 6.4 coincides with that originally given in Section 2 (Definition 2.3):

Corollary 6.5. *Fixed a point* $x_0 \in X$ *, the set of all differential invariants*

$$f : \mathcal{M}_X \to \mathcal{C}^\infty_X$$

of order $\leq r$ *is in bijection with the set of all smooth* $\mathrm{Diff}_{x_0}-$*invariant functions*

$$\tilde{f} : J^r_{x_0} M \to \mathbb{R}\,.$$

7 Appendix B: Classification of $\infty-$jets of metrics

In Section 4 we have seen that differential invariants of order $\leq r$ classify $r-$jets of Riemannian metrics at a point (Theorem 4.3). We are now going to generalize this result for infinite-order jets.

In the proof of next lemma we will use the following well-known fact ([1], Chap. IX, § 9, Lemma 6):

Let G be a compact Lie group. Every decreasing sequence of closed subgroups $H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$ stabilizes, that is, there exists an integer s such that $H_s = H_{s+1} = H_{s+2} = \cdots$

Lemma 7.1. *Let* G *a compact Lie group and let*

$$\cdots \longrightarrow X_{r+1} \longrightarrow X_r \longrightarrow \cdots \longrightarrow X_1$$

be an inverse system of smooth $G-$*equivariant maps between smooth manifolds endowed with a smooth action of* G *. There exists an isomorphism of ringed spaces:*

$$\begin{array}{ccc} (\lim\limits_{\leftarrow} X_r)/G & = & \lim\limits_{\leftarrow} (X_r/G) \\ [(\ldots, x_2, x_1)] & \longmapsto & (\ldots, [x_2], [x_1])\,. \end{array}$$

Proof. Because of the universal quotient property, compositions of morphisms

$$\begin{array}{ccccc} \lim\limits_{\leftarrow} X_r & \longrightarrow & X_r & \longrightarrow & X_r/G \\ (\ldots, x_2, x_1) & \longmapsto & x_r & \longmapsto & [x_r] \end{array}$$

induce morphisms of ringed spaces

$$\begin{array}{ccc} (\lim\limits_{\leftarrow} X_r)/G & \longrightarrow & (X_r/G) \\ [(\ldots, x_2, x_1)] & \longmapsto & [x_r]\,, \end{array}$$

which, for their part, because of the universal inverse limit property, define a morphism of ringed spaces

$$\begin{array}{ccc} (\lim\limits_{\leftarrow} X_r)/G & \xrightarrow{\varphi} & \lim\limits_{\leftarrow} (X_r/G) \\ [(\ldots, x_2, x_1)] & \longmapsto & (\ldots, [x_2], [x_1])\,. \end{array}$$

It is easy to check that this morphism is surjective. Let us see that it is also injective.

First note that, given a point $(\ldots, x_2, x_1) \in \lim\limits_{\leftarrow} X_r$, we can get the decreasing sequence $H_{x_1} \supseteq H_{x_2} \supseteq H_{x_3} \supseteq \cdots$ of closed subgroups of G, where H_{x_k} stands for the stabilizer subgroup of x_k. This chain stabilizes, since G is compact, so for a certain s it is verified $H_{x_s} = H_{x_{s+1}} = H_{x_{s+2}} = \cdots$

Let now $[(\ldots, x_2, x_1)]$ and $[(\ldots, x'_2, x'_1)]$ be two points in $(\lim\limits_{\leftarrow} X_r)/G$ having the same image through φ, i.e. $[x_k] = [x'_k]$, for each $k \geq 0$. Write $x'_s = g \cdot x_s$ for some $g \in G$. As the morphisms $X_s \to X_k$ (with $s \geq k$) are $G-$equivariant, it is verified that $x'_k = g \cdot x_k$ for every $k \leq s$.

Let us show that the same happens when $k > s$. As $[x_k] = [x'_k]$, we have $x'_k = g_k \cdot x_k$ for a certain $g_k \in G$; applying that $X_k \to X_s$ is equivariant yields $x'_s = g_k \cdot x_s$, and then (comparing with $x'_s = g \cdot x_s$) $g^{-1} g_k \in H_{x_s}$; since $H_{x_s} = H_{x_k}$, it follows that $g^{-1} g_k \in H_{x_k}$, and hence the condition $x'_k = g_k \cdot x_k$ is equivalent to $x'_k = g \cdot x_k$. In conclusion, $x'_k = g \cdot x_k$ for every $k > 0$, and therefore $[(\ldots, x_2, x_1)]$ and $[(\ldots, x'_2, x'_1)]$ are the same point in $(\lim\limits_{\leftarrow} X_r)/G$.

Once we have proved that φ is bijective, it is routine to check that φ is an isomorphism of ringed spaces. □

Definition 7.2. Let $x_0 \in X$ and let

$$J^\infty_{x_0} M := \lim_{\leftarrow} J^r_{x_0} M$$

be the ringed space of $\infty-$jets of Riemannian metrics at x_0 on X. The quotient ringed space

$$\mathbb{M}^\infty_n := J^\infty_{x_0} M / \mathrm{Diff}_{x_0}$$

is called **moduli space** of $\infty-$jets of Riemannian metrics in dimension n.

In the same fashion as for finite-order jets, the moduli space $\mathbb{M}^\infty_n$ depends neither on the choice of the point x_0 nor on that of the $n-$dimensional manifold X.

For every integer $r > 0$ we have an evident morphism of ringed spaces

$$\begin{array}{ccc} \mathbb{M}^\infty_n & \longrightarrow & \mathbb{M}^r_n \\ [j^\infty_{x_0} g] & \longmapsto & [j^r_{x_0} g], \end{array}$$

and these morphisms allow us to define another morphism of ringed spaces:

$$\begin{array}{ccc} \mathbb{M}^\infty_n & \longrightarrow & \lim\limits_{\leftarrow} \mathbb{M}^r_n \\ [j^\infty_{x_0} g] & \longmapsto & (\ldots, [j^r_{x_0} g], \ldots), . \end{array}$$

Theorem 7.3. *There exists an isomorphism of ringed spaces*

$$\begin{array}{ccc} \mathbb{M}^\infty_n & = & \lim\limits_{\leftarrow} \mathbb{M}^r_n \\ [j^\infty_{x_0} g] & \longmapsto & (\ldots, [j^r_{x_0} g], \ldots). \end{array}$$

Proof. Fix a local coordinate system $(z_1, \ldots, z_n)$ centered at x_0. With the same notations as in Section 3, let us define

$$\mathcal{N}^\infty := \lim_{\leftarrow} \mathcal{N}^r .$$

In other words, $\mathcal{N}^\infty$ is the subspace of $J^\infty_{x_0} M$ formed by all those $\infty-$jets at x_0 of Riemannian metrics having $(z_1, \ldots, z_n)$ as a normal coordinate system. All lemmas in

Section 3, with their corresponding proofs, remain valid when substituting the integer ∞ for r. In particular, our Fundamental Lemma 3.4, when $r=\infty$, gives us the desired isomorphism of ringed spaces:

$$\mathbb{M}_n^\infty = \left(\prod_{k\geq 2} N_k\right)/O(n) = \left(\lim_{\leftarrow}(N_2\times\cdots\times N_r)\right)/O(n)$$

(by Lemma 7.1)

$$= \lim_{\leftarrow}((N_2\times\cdots\times N_r)/O(n)) = \lim_{\leftarrow}\mathbb{M}_n^r\,.$$

□

Corollary 7.4. *Differential invariants of finite order classify $\infty-$jets of Riemannian metrics: Two jet metrics $j^\infty_{x_0}g$ and $j^\infty_{x_0}\bar{g}$ are equivalent if and only if for each finite-order differential invariant h it is satisfied $h(g)(x_0)=h(\bar{g})(x_0)$.*

Proof. According to Theorem 7.3, we get:

$$j^\infty_{x_0}g\equiv j^\infty_{x_0}\bar{g} \iff j^r_{x_0}g\equiv j^r_{x_0}\bar{g}\ ,\ \forall r\geq 0\,.$$

To complete our proof, it is sufficient to use the fact that differential invariants of order $\leq r$ classify $r-$jet metrics (Theorem 4.3). □

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