# Dimensional curvature identities
# on pseudo-Riemannian geometry

Alberto Navarro [*]          José Navarro [†]

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### Abstract

For a fixed $n \in \mathbb{N}$, the curvature tensor of a pseudo-Riemannian metric, as well as its covariant derivatives, satisfy certain identities that hold on any manifold of dimension less or equal than $n$.

In this paper, we re-elaborate recent results by Gilkey-Park-Sekigawa regarding these $p$-covariant curvature identities, for $p = 0, 2$. To this end, we use the classical theory of natural operations, that allows us to simplify some arguments and to generalize the main results of Gilkey-Park-Sekigawa, both by dropping a symmetry hypothesis and by including $p$-covariant curvature identities, for any even $p$..

Thus, for any dimension $n$, our main result describes the first space (i.e., that of highest weight) of $p$-covariant dimensional curvature identities, for any even $p$.

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[*]ICMat, Madrid, Spain.

[†]Department of Mathematics, University of Extremadura, Avda. Elvas s/n, 06071, Badajoz, Spain.
*Email address:* navarrogarmendia@unex.es

# Introduction

The curvature tensor of a pseudo-Riemannian metric, as well as its covariant derivatives, satisfy certain identities, such as the linear and differential Bianchi identities, or the Ricci identities. These identities are *universal*, in the sense that they are satisfied by the curvature tensor of *any* non-singular metric, on *any* manifold. Indeed, it can be proved that, essentially, these are the only identities with these properties ([7]).

Nevertheless, there exists some other kind of identities satisfied by the curvature. As an example, recall that the Einstein tensor of a pseudo-Riemannian surface vanishes; that is, on any pseudo-Riemannian manifold of dimension 2, the following relation holds:

$$Ricc - \frac{r\,g}{2} \; = \; 0 \; , \tag{1}$$

where $Ricc$ denotes the Ricci tensor and $r$ the scalar curvature of $g$.

This 2-covariant identity is satisfied by *any* non-singular metric on *any* manifold *of dimension less or equal* than two. For this reason, these type of identities will be called dimensional curvature identities (Definition 1.8).

These dimensional identities already attracted attention very early in the development of General Relativity ([9]). Later on, they also proved relevant in mathematics: in 1973, P. Gilkey ([3]) characterized the vanishing of the Pfaffian as the only scalar dimensional curvature identity satisfying certain homogeneity condition, and used this result to simplify the heat equation proof of the index theorem. Let us also remark that, in the course of his investigations of the equivariant inverse problem of the calculus of variations, I. Anderson ([1]) was led to prove a similar statement for symmetric, 2-covariant curvature identities, extending, in a sense, the identity (1) written above.

Nevertheless, most of these results were established in the realm of Riemannian manifolds, and their proofs relied on lengthy calculations, involving large expressions of multi-indexes. Recently, these arguments have been dramatically simplified by Gilkey-Park-Sekigawa ([4]), who also have extended their validity to pseudo-Riemannian manifolds of any signature ([5]), and have started studying analogous results in different settings ([6]).

In this paper, we generalize the statements of Gilkey-Park-Sekigawa regarding scalar and symmetric, 2-covariant identities to the case of $p$-covariant identities, for any even $p$ and with no symmetry assumptions (Theorem 1.10). To this end, we use the machinery of the classical theory of natural operations ([7], [12]), that also allows us to simplify the exposition of certain arguments.

The paper is organized as follows: the first Section is devoted to present the statement of the main result, Theorem 1.10, whereas the proofs are all postponed until the second Section.

# 1 Statements

Fix a dimension $n \in \mathbb{N}$ and a signature $(n_+, n_-)$, where $n_+ + n_- = n$.

**Definition 1.1** A $p$-**covariant, natural tensor in dimension** $n$, **associated to metrics of signature** $(n_+, n_-)$, is an assignment $T$ that, for any metric $g$ on any smooth manifold $X$ of dimension $n$, produces a $p$-covariant tensor $T(g)$ on $X$, satisfying:

- It is *regular*: if $\{g_t\}_{t \in T}$ is a smooth family of metrics, parametrized by a smooth manifold $T$, then $\{T(g_t)\}_{t \in T}$ is also a smooth family of tensors.

- It is *natural*: for any local diffeomorphism $\tau \colon Y \to X$ between smooth manifolds of dimension $n$, it holds

$$T(\tau^* g) = \tau^* T(g) \ .$$

A natural tensor $T$ is **homogeneous of weight** $w \in \mathbb{R}$ if, for any metric $g$ on any smooth manifold of dimension $n$, and any positive real number $\lambda > 0$, it holds:

$$T(\lambda^2 g) = \lambda^w \, T(g) \ .$$

As it will be explained later (see Corollary 2.3), natural tensors do not depend on the fixed signature of the metrics: the vector space of homogeneous natural tensors associated to Riemannian metrics in dimension $n$ is canonically isomorphic to the space of homogeneous natural tensors associated to metrics of signature $(n_+, n_-)$, provided that $n_+ + n_- = n$.

Therefore, the $\mathbb{R}$-vector space of homogeneous natural tensors will be denoted:

$$\mathsf{T}_{p,w}[n] := \left[ \begin{array}{c} p\text{-Covariant, natural tensors } T \text{ in dimension } n \\ \text{homogeneous of weight } w \end{array} \right] \ .$$

**Example 1.2** For any $n$, the metric itself $g$ is a natural 2-tensor in dimension $n$, homogeneous of weight 2. The Riemann-Christoffel tensor, $R$, the Ricci tensor $Ricc$ and the scalar curvature $r$ are also natural tensors, homogeneous of weight 2, 0 and -2, respectively.

In general, it can be proved that the local expression of a homogeneous natural tensor is a "universal" (i.e., valid on any chart) polynomial function on the curvature and its covariant derivatives, with coefficients smooth functions of the metric and its inverse ([7], [12]).

## 1.1 Dimensional reduction. Universal tensors

Let $(X, g)$ be an $(n-1)$-Riemannian manifold, an consider the cylinder $(X \times \mathbb{R}, g + \mathrm{d}t^2)$, which is an $n$-dimensional Riemannian manifold.

Let $i$ denote the embedding:

$$i\colon X \hookrightarrow X \times \mathbb{R} \qquad , \qquad x \;\mapsto\; (x,0)\;.$$

**Definition 1.3** The **dimensional reduction** of natural tensors is the linear map:

$$\mathsf{T}_{p,w}[n] \xrightarrow{\;r_n\;} \mathsf{T}_{p,w}[n-1] \quad , \quad r_n(T)(g) := i^*\left(T(g+\mathrm{d}t^2)\right)\;.$$

If $T \in \mathsf{T}_{p,w}[n]$, it is not difficult to check that $r_n(T)$ is a natural tensor in dimension $n-1$. Moreover, as $\lambda^2 g + \mathrm{d}t^2$ and $\lambda^2 g + \lambda^2 \mathrm{d}t^2$ are related by an isometry that is the identity on $X$, $r_n(T)$ is also homogeneous of weight $w$:

$$\begin{aligned}
r_n(T)(\lambda^2 g) = i^*\left(T(\lambda^2 g + \mathrm{d}t^2)\right) &= i^*\left(T(\lambda^2 g + \lambda^2 \mathrm{d}t^2)\right)\\
&= i^*\left(\lambda^w T(g+\mathrm{d}t^2)\right) = \lambda^w\, r_n(T)(g)\;.
\end{aligned}$$

Therefore, these linear maps establish a projective system:

$$\ldots \xrightarrow{\;r_{n+1}\;} \mathsf{T}_{p,w}[n] \xrightarrow{\;r_n\;} \mathsf{T}_{p,w}[n-1] \xrightarrow{\;r_{n-1}\;} \ldots \;,$$

and the linear maps $r_n$ can be proved to be surjective (Lemma 2.11).

**Definition 1.4** A **universal tensor**, homogeneous of weight $w$, is an element of the inverse limit

$$\mathsf{T}_{p,w} := \varprojlim \mathsf{T}_{p,w}[n]\;.$$

In other words, a universal tensor is a collection of natural tensors $\{T_n\}_{n\in\mathbb{N}}$, where each $T_n$ is a natural tensor in dimension $n$, satisfying that, for every manifold $X$ of dimension $n_1$ and every embedding into a cylinder $i\colon X \to X \times \mathbb{R}^{n_2}$, $x \mapsto (x,0)$, where $\mathbb{R}^{n_2}$ is endowed with the euclidean metric, it holds

$$i^*\left(T_{n_1+n_2}\Big(g+\sum_{i=1}^{n_2}\mathrm{d}t_i^2\Big)\right) = T_{n_1}(g)\;.$$

**Example 1.5** The metric $g$, the Riemann-Christoffel tensor $R$, the Ricci tensor $Ricc$, and the scalar curvature $r$ are all universal tensors.

For any fixed $\lambda \in \mathbb{R}$, the tensor $\lambda g$ is a universal tensor. However, $(\operatorname{tr}\operatorname{Id})g$, $(-1)^{\dim X}g$ or $(-1)^{n+}g$ are not universal tensors.

Let $(X, g)$ be a Riemannian manifold as above and let $\pi \colon X \times \mathbb{R} \to X$ be the first projection.

The curvature $R$ is a universal tensor that satisfies:

$$R(g + \mathrm{d}t^2) \;=\; \pi^*\left(R(g)\right) \;, \tag{2}$$

which is a stronger property than that of being universal.

The following Lemma can be checked in local coordinates, using property (2) above:

**Lemma 1.6** *Let $T$ be a universal tensor, and let $T' = c(R \otimes T)$ be the contraction of one index of the curvature with one index of $T$. Then $T'$ is a universal tensor.*

**Example 1.7** Consider the metric $g$ as a 1-form with values on 1-forms, so that

$$g^{2k+\bar{p}} := g \wedge \overset{2k+\bar{p}}{\cdots} \wedge g$$

is a $2k + \bar{p}$-form with values on $2k + \bar{p}$-forms, for any $k, \bar{p} \geq 0$.

Analogously, consider the curvature tensor $R$ as a 2-vector with values on 2-vectors, so that:

$$R^k := R \wedge \overset{k}{\cdots} \wedge R$$

is a $2k$-vector with values on $2k$-vectors.

Let $c$ be the contraction operator:

$$\left(\Lambda^{2k} TX \otimes \Lambda^{2k+\bar{p}} T^*X\right) \otimes \left(\Lambda^{2k} TX \otimes \Lambda^{2k+\bar{p}} T^*X\right) \xrightarrow{\;c\;} \Lambda^{\bar{p}} T^*X \otimes \Lambda^{\bar{p}} T^*X \;.$$

For any $k, \bar{p} \geq 0$, let us define the $(2\bar{p})$-covariant universal tensors:

$$\mathsf{S}_{2\bar{p},k} := c\left(R^k \otimes g^{2k+\bar{p}}\right) \;.$$

Each $\mathsf{S}_{2\bar{p},k}$ is indeed a universal tensor because it is obtained contracting indices of the curvature with indices of a universal tensor (Lemma 1.6). Their local expression may be written as follows:

$$(\mathsf{S}_{2\bar{p},k})_{i_1 \ldots i_{2\bar{p}}} := R^{a_1 a_2, b_1 b_2} \ldots R^{a_{2k-1} a_{2k}, b_{2k-1} b_{2k}} \delta^{c_1 \ldots c_{2k} j_1 \ldots j_{\bar{p}}}_{b_1 \ldots b_{2k} i_1 \ldots i_{\bar{p}}} g_{a_1 c_1} \cdots g_{a_{2k} c_{2k}} g_{j_1 i_{\bar{p}+1}} \cdots g_{j_{\bar{p}} i_{2\bar{p}}}$$

where $\delta^{j_1 \ldots j_m}_{i_1 \ldots i_m}$ denotes the generalized Kronecker delta.

Moreover, these tensors $\mathsf{S}_{2\bar{p},k}$ are skew-symmetric in its first $\bar{p}$ indexes and in the last $\bar{p}$ indexes, and, due to the symmetries of $g$ and $R$, they are also symmetric under the interchange of these group of indexes.

If $\bar{p} = 0$, there is only one $\mathsf{S}_{2\bar{p},k}$, whereas, for $\bar{p} \geq 1$, any permutation $\sigma$ of $2\bar{p}$ elements produces the tensor

$$(\sigma \cdot \mathsf{S}_{2\bar{p},k})(D_1, \ldots, D_{2\bar{p}}) := \mathsf{S}_{2\bar{p},k}(D_{\sigma(1)}, \ldots, D_{\sigma(2\bar{p})}) \ .$$

Due to the symmetries of $\mathsf{S}_{2\bar{p},k}$, there may only be $\frac{1}{2}\left(p \cdot (p-1) \cdot \ldots \cdot (\frac{p}{2}+1)\right)$ different tensors among the $\{\sigma \cdot \mathsf{S}_{2\bar{p},k}\}_{\sigma \in S_{2\bar{p}}}$.

## 1.2 Dimensional curvature identities

Loosely speaking, any natural tensor is locally written in terms of the coefficients of the curvature and its covariant derivatives. Hence, if, for some $n$, a universal tensor lies in the kernel of the canonical projection $\mathsf{T}_{p,w} \longrightarrow \mathsf{T}_{p,w}[n]$, then it can be understood as an identity, satisfied by the coefficients of the curvature and its covariant derivatives, which is valid for any metric, of any signature, on any manifold of dimension less or equal than $n$.

Of course, if a universal tensor $\{T_n\}_{n \in \mathbb{N}}$ defines a curvature identity, then the universal tensor $\{\lambda T_n\}_{n \in \mathbb{N}}$ defines the same identity, for any $\lambda \in \mathbb{R} - \{0\}$.

This motivates the following definition:

**Definition 1.8** A **dimensional curvature identity** in dimension $n$ is an element of the projective space associated to the vector space:

$$\mathsf{K}_{p,w}[n] := \mathrm{Ker}\left[\mathsf{T}_{p,w} \longrightarrow \mathsf{T}_{p,w}[n]\right] \ .$$

**Example 1.9** For any fixed $\bar{p}, k \geq 0$ (apart from the cases $k = 0$ and $\bar{p} = 0, 1$) the tensor $\mathsf{S}_{2\bar{p},k}$ vanishes whenever $\dim X < 2k + \bar{p}$, because the form $g^{2k+\bar{p}}$ is identically zero.

Hence, this tensor $\mathsf{S}_{2\bar{p},k}$ defines a dimensional curvature identity

$$\mathsf{S}_{2\bar{p},k} \in \mathsf{K}_{2\bar{p},w}[2k + \bar{p} - 1] \ ,$$

with $w = 2(\bar{p} - k)$.

The following Theorem 1.10 establishes that it is, essentially, the only dimensional curvature identity of this kind.

Finally, let us observe that, if $T \in \mathsf{T}_{2\bar{p},w}$ is a $(2\bar{p})$-covariant, universal tensor, homogeneous of degree $w$, then $w$ has to be an even integer, lesser or equal than $p$ (see, v. gr., [13]), so that we may write, without loss of generality:

$$w = 2(\bar{p} - k)$$

for some $k \geq 0$.

**Theorem 1.10** *Consider covariant tensors with an even number $2\bar{p}$ of indices ($\bar{p} \geq 0$), associated to non-singular metrics.*

*For any[1] weight $w = 2(\bar{p} - k)$, with $k \geq 0$, the following holds:*

- *If $n \geq 2k + \bar{p}$, there are no dimensional curvature identities of weight $w$; that is,*

$$\mathsf{K}_{2\bar{p},w}[n] = 0 \quad , \quad for\ n \geq 2k + \bar{p} \ .$$

- *The vector space $\mathsf{K}_{2\bar{p},w}[2k + \bar{p} - 1]$ is generated by the tensors $\{\sigma \cdot \mathsf{S}_k\}_{\sigma \in S_{2\bar{p}}}$, and hence has dimension:*

$$\dim\ (\mathsf{K}_{2\bar{p},w}[2k + \bar{p} - 1]) = \bar{p}\ (\bar{p} + 1) \cdot \ldots \cdot (2\bar{p} - 2)(2\bar{p} - 1) \ ,$$

*or reduces to a single identity, when $\bar{p} = 0$.*

In the particular case $\bar{p} = 0$ and $n = 2k$, the smooth function $\mathsf{S}_{0,k}$ is proportional to the Pfaffian of the curvature. Hence, Theorem 1.10 implies the following result:

**Corollary 1.11 ([5])** *Consider tensors with $p = 0$ indices; i.e., scalar differential invariants associated to non-singular metrics.*

*For any weight $w = -2k$, with $k \geq 1$, the following holds:*

- *If $n \geq 2k$, there are no dimensional curvature identities of weight $w$; i.e.:*

$$\mathsf{K}_{0,w}[n] = 0 \quad , \quad for\ n \geq 2k \ .$$

- *If $n = 2k - 1$, the only dimensional curvature identity of weight $w = -2k$ is the vanishing of $\mathsf{S}_{0,k}$:*

$$\mathsf{K}_{0,w}[2k - 1] = \langle \mathsf{S}_{0,k} \rangle \ .$$

In the particular case $\bar{p} = 1$, the symmetric 2-tensors producing the identities are well-known: $\mathsf{S}_{2,1}$ is the Einstein tensor; $\mathsf{S}_{2,2}$ was first introduced by Lanczos ([9]) and the other $\mathsf{S}_{2,k}$ were independently introduced by Lovelock ([10], [11]) and Kuz'mina ([8]).

Theorem 1.10 then reads:

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[1]Apart from the exceptional cases $(p, k) = (0, 0)$ or $(1, 0)$

**Corollary 1.12** *Consider covariant tensors with $p = 2$ indices (not necessarily symmetric) associated to non-singular metrics.*

*For any weight $w = 2 - 2k$, with $k \geq 1$, the following holds:*

- *If $n \geq 2k + 1$, there are no dimensional curvature identities of weight $w$; i.e.:*

$$\mathsf{K}_{2,w}[n] = 0 \quad , \quad \text{for } n \geq 2k + 1 \ .$$

- *If $n = 2k$, the only dimensional curvature identity of weight $w = 2 - 2k$ is the vanishing of $\mathsf{S}_{2,k}$:*

$$\mathsf{K}_{2,w}[2k] = \langle \mathsf{S}_{2,k} \rangle \ .$$

This statement drops the symmetry hypothesis that is assumed in [5], as well as in the Riemanian statements of [1] and [4].

## 2 Proofs

The proof of Theorem 1.10 relies on the classical theory of natural constructions, that reduces the problem to a question regarding tensors (at a point) invariant under the action of the orthogonal group (see Theorem 2.2).

To state this result, let us firstly introduce the vector spaces of normal tensors:

**Definition 2.1** Let $X$ be a smooth manifold of dimension $n$, $x \in X$ be a point and $r \geq 2$ be an integer. The space $N_r \subset S^2 T_x^* X \otimes S^r T_x^* X$ of $r^{th}$-**order metric normal tensors** at $x$ is the kernel of the symmetrization in the last $(r + 1)$-indices:

$$0 \to N_r \to S^2 T_x^* X \otimes S^r T_x^* X \xrightarrow{s_{r+1}} T_x^* X \otimes S^{r+1} T_x^* X \to 0 \ .$$

Any germ of metric $g$ around the point $x$ defines a sequence of metric normal tensors $(g_x^2, \ldots, g_x^r, \ldots) \in N_2 \times \ldots \times N_r \times \ldots$; to construct this sequence, choose normal coordinates $x_1, \ldots, x_n$ for $g$ at $x$ and define

$$(g_x^k)_{abi_1 \ldots i_k} := \frac{\partial^k g_{ab}}{\partial x_{i_1} \ldots \partial x_{i_k}}(x) \quad , \qquad k = 2, \ldots$$

for the condition of the chart $(x_i)$ being normal guarantees that the symmetrization of the last $r + 1$ indices of $g_x^r$ is zero.

Let $\mathsf{T}_{p,w}[n_+, n_-]$ denote the vector space of homogeneous natural $p$-tensors (of weight $w$), associated to pseudo-Riemannian metrics of signature $(n_+, n_-)$.

**Theorem 2.2 ([12], [15])** *Let $X$ be a smooth manifold of dimension $n$, $x \in X$ be a point and $g_x$ be a pseudo-Riemannian metric at $x$ of signature $(n_+, n_-)$.*

*There exists an $\mathbb{R}$-linear isomorphism:*

$$\mathsf{T}_{p,w}[n_+, n_-] \simeq \bigoplus_{d \in D} \mathrm{Hom}_{O_{g_x}} \left( S^{d_2} N_2 \otimes \cdots \otimes S^{d_r} N_r \ , \ \otimes^p T_x^* X \ \right)$$

*where $D$ is the set of sequences of nonnegative integers $d = \{d_2, \ldots, d_r\}$ such that:*

$$2 d_2 + \ldots + r\, d_r = p - w \ . \tag{3}$$

*If such equation has no solutions the such vector space is zero.*

*If $\varphi \colon S^{d_2} N_2 \otimes \cdots \otimes S^{d_r} N_r \to \otimes^p T_x^* X$ is an $O_{g_x}$-equivariant linear map, then, on any metric $g$ with the prefixed value at $x$, the corresponding natural tensor $T$ is obtained by the formula:*

$$T(g)_x = \varphi \left( (g_x^2 \otimes \overset{d_2}{\cdots} \otimes g_x^2) \otimes \cdots \otimes (g_x^s \overset{d_s}{\cdots} \otimes g_x^s) \right)$$

*where $(g_x^2, g_x^3, \ldots)$ is the sequence of normal tensors of $g$ at the point $x \in X$.*

*The value at any other point, and for any other metric over another manifold, is computed adequately transforming with a diffeomorphism.*

Indeed, the $O_{g_x}$-equivariant linear maps in the theorem can, in certain cases, be explicitly computed applying the invariant theory of the orthogonal group explained in the next section. An interesting consequence of this Theorem 2.2 and Corollary 2.6 below is that the vector spaces $\mathsf{T}_{p,w}[n]$ do not depend on the signature of the metrics under consideration:

**Corollary 2.3** *If $n_+ + n_- = m_+ + m_-$, there is a canonical isomorphism:*

$$\mathsf{T}_{p,w}[n_+, n_-] = \mathsf{T}_{p,w}[m_+, m_-] \ .$$

## 2.1 Invariant theory for the orthogonal group

Let $(E, g)$ be a $\mathbb{R}$-vector space of dimension $n$ with a nonsingular metric $g$ of signature $(n_+, n_-)$ and let $O_g$ denote the Lie group of its linear isometries $(E, g) \to (E, g)$.

The main theorem of the invariant theory for the orthogonal group describes the polynomial functions on $m$ vectors

$$f \colon E \times \overset{m}{\cdots} \times E \longrightarrow \mathbb{R}$$

that are invariant under the action of $O_g$.

For any given $i, j = 1, \ldots, m$, the following functions $y_{ij}$ are examples of $O_g$-invariant polynomial functions:

$$y_{ij} \colon E \times \overset{m}{\ldots} \times E \to \mathbb{R} \quad , \quad y_{ij}(e_1, \ldots, e_m) := g(e_i, e_j) \ .$$

If $m > n := n_+ + n_-$, then such functions have relations: for any $1 \le i_0 < \ldots < i_n \le m$, $1 \le j_0 < \ldots < j_n \le m$, the following identities hold:

$$\begin{vmatrix} y_{i_0 j_0} & \cdots & y_{i_0 j_n} \\ \vdots & & \vdots \\ y_{i_n j_0} & \cdots & y_{i_n j_n} \end{vmatrix} (e_1, \ldots, e_m) = (e_{i_0} \wedge \ldots \wedge e_{i_n}) \cdot (e_{j_0} \wedge \ldots \wedge e_{j_n}) = 0 \cdot 0 = 0,$$

where $\cdot$ denotes the metric induced by $g$ on the corresponding tensor algebra.

The so-called Main Theorem then states that these are, essentially, the only invariant functions and the only relations among them:

**Theorem 2.4 ([14], [16])** *The algebra $\mathsf{A}_m^g$ of $O_g$-invariant polynomial functions on $E \times \overset{m}{\ldots} \times E$ is generated by the functions $y_{ij}$.*

*Moreover, let $Y_{ij}$ be free symmetric variables. The map $Y_{ij} \mapsto y_{ij}$ induces a canonical isomorphism*

$$\mathbb{R}[Y_{ij}]/M_{n+1} \xlongequal{\quad} \mathsf{A}_m^g \ ,$$

*where $M_{n+1}$ is the ideal generated by the functions:*

$$M_{i_0 \ldots i_n}^{j_0 \ldots j_n} = \begin{vmatrix} Y_{i_0 j_0} & \cdots & Y_{i_0 j_n} \\ \vdots & & \vdots \\ Y_{i_n j_0} & \cdots & Y_{i_n j_n} \end{vmatrix} \ , \quad \text{for any} \quad \begin{matrix} 1 \le i_0 < \ldots < i_n \le m \\ 1 \le j_0 < \ldots < j_n \le m \end{matrix} \ .$$

*In particular, if $m \le n$, these functions $y_{ij}$ are algebraically independent.*

As a consequence, if $(E', g')$ is another $n$-dimensional $\mathbb{R}$-vector space with a nonsingular metric of signature $(n'_+, n'_-)$, then there are canonical isomorphisms:

$$\mathsf{A}_m^{g'} \xlongequal{\quad} \mathbb{R}[Y_{ij}]/M_{n+1} \xlongequal{\quad} \mathsf{A}_m^g \ ,$$

This classical statement is usually proved in the realm of algebraic varieties ([14], [16]); that is, for the *affine algebraic $\mathbb{R}$-group $O_g$*. The corresponding version for the (non-compact) Lie group $O_g$ requires some argument, to reduce the proof to the algebraic case (see, for example, [2], [5], or [12]).

As a consequence, it readily follows a useful description of the space of linear forms

$$E \otimes \overset{m}{\ldots} \otimes E \longrightarrow \mathbb{R}$$

which are invariant under the action of $O_g$:

**Corollary 2.5** *The vector space* $\mathrm{Hom}_{O_g}(E \otimes \overset{m}{\ldots} \otimes E, \mathbb{R})$ *of invariant linear forms is zero if* $m$ *is odd and, if* $m = 2k$ *is even, it is spanned by total contractions:*

$$\omega_\sigma \colon e_1 \otimes \ldots \otimes e_{2k} \;\mapsto\; g\big(e_{\sigma(1)}, e_{\sigma(2)}\big) \cdot \ldots \cdot g\big(e_{\sigma(2k-1)}, e_{\sigma(2k)}\big)$$

*where* $\sigma \in S_{2k}$ *is a permutation.*

*Moreover, if* $m \leq 2n$, *the only relations among these generators are the obvious ones due to the symmetry of* $g$.

*Proof:* Among the polynomials on $y_{ij}$, observe that $m$-multilinear maps are precisely the linear combinations of

$$y_{\sigma(1)\sigma(2)} \cdot \ldots \cdot y_{\sigma(2k-1)\sigma(2k)}$$

where $\sigma$ is a permutation of $1, \ldots, 2k$.

$\square$

The isomorphisms $\mathsf{A}_m^{g'} = \mathbb{R}[Y_{ij}]/M_{n+1} = \mathsf{A}_m^g$ takes $m$-multilinear maps into $m$-multilinear maps. Hence:

**Corollary 2.6** *Let* $(E, g)$, $(E', g')$ *be vector spaces of the same dimension, endowed with non-singular metrics. There exists a canonical isomorphism:*

$$\mathrm{Hom}_{O_g}(E \otimes \overset{m}{\ldots} \otimes E, \mathbb{R}) \;=\; \mathrm{Hom}_{O_{g'}}(E' \otimes \overset{m}{\ldots} \otimes E', \mathbb{R}) \;.$$

From now on, we put
$$\mathsf{T}_m[n] := \mathrm{Hom}_{O_g}(E \otimes \overset{m}{\ldots} \otimes E, \mathbb{R})$$

where $g$ is any non-singular metric; for example a scalar product.

**Definition 2.7** For all $n > 1$, there exist **dimensional reduction** linear maps

$$r_n \colon \mathsf{T}_m[n] \longrightarrow \mathsf{T}_m[n-1] \;,$$

defined as follows: if $\omega \in \mathsf{T}_m[n]$ is an invariant linear form on $n$-dimensional euclidean spaces,

let $r_n(\omega)$ be the invariant form on $(n-1)$-dimensional euclidean spaces obtained as:

$$E \otimes \ldots \otimes E \hookrightarrow (E \perp \mathbb{R}) \otimes \ldots \otimes (E \perp \mathbb{R}) \xrightarrow{\omega} \mathbb{R} \ .$$

The following statement is a consequence of Theorem 2.4:

**Proposition 2.8** *The dimensional reduction maps* $r_n \colon \mathtt{T}_m[n] \longrightarrow \mathtt{T}_m[n-1]$ *are surjective for all* $n > 1$ *and, for* $n > m-1$*, they are linear isomorphisms.*

### 2.1.1 Invariant forms on the subspace of normal tensors

Let $X$ be a smooth manifold of dimension $n$, $g_x$ a Riemannian metric at a point $x \in X$, and let $D = (d_2, \ldots, d_r)$ be a multi-index.

As the orthogonal group is semisimple, restriction to the spaces of normal tensors induce surjective linear maps, for each $n \in \mathbb{N}$:

$$\mathtt{T}_D[n] := \mathrm{Hom}_{O_{g_x}} \left( T_x^* X \otimes \overset{4d_2 + \ldots + (2+r)d_r + p}{\cdots} \otimes T_x^* X \ , \ \mathbb{R} \right)$$

$$i_n^* \Big\downarrow$$

$$\mathtt{N}_D[n] := \mathrm{Hom}_{O_{g_x}} \left( S^{d_2} N_2 \otimes \cdots \otimes S^{d_r} N_r \otimes^p T_x^* X \ , \ \mathbb{R} \right) \ .$$

That is to say,

**Lemma 2.9** *Any* $O_{g_x}$*-invariant linear map:*

$$S^{d_2} N_2 \otimes \cdots \otimes S^{d_r} N_r \otimes^p T_x^* X \ \longrightarrow \ \mathbb{R}$$

*is the restriction of a* $O_{g_x}$*-invariant linear map:*

$$T_x^* X \otimes \overset{4d_2 + \ldots + (2+r)d_r + p}{\cdots} \otimes T_x^* X \ \longrightarrow \ \mathbb{R} \ .$$

In terms of the generators $\omega_\sigma$ introduced in Corollary 2.5, the dimensional reduction maps $r_n$ have a simple expression:

$$\mathtt{T}_D[n] \xrightarrow{\ r_n\ } \mathtt{T}_D[n-1] \qquad , \qquad \omega_\sigma \longmapsto \omega_\sigma \ .$$

Therefore, they specialize to the subspaces of normal tensors, defining maps $\bar{r}_n$:

$$
\begin{array}{ccc}
\mathsf{T}_D[n] & \xrightarrow{\ r_n\ } & \mathsf{T}_D[n-1] \\
{\scriptstyle i_n^*}\downarrow & & \downarrow{\scriptstyle i_{n-1}^*} \\
\mathsf{N}_D[n] & \dashrightarrow{\ \bar{r}_n\ } & \mathsf{N}_D[n-1]
\end{array}
$$

The following Lemma states that any relation satisfied by the dimensional reduction $\bar{r}_n$ in the subspace is indeed the restriction of a relation satisfied by the dimensional reduction $r_n$ in the ambient space:

**Lemma 2.10** *Restriction to the space of normal tensors induce surjective linear maps:*

$$
\mathsf{K}_D[n] := \operatorname{Ker}\ (r_n \colon \mathsf{T}_D[n] \to \mathsf{T}_D[n-1])
$$

$$
{\scriptstyle i_n^*}\downarrow
$$

$$
\bar{\mathsf{K}}_D[n] := \operatorname{Ker}\ \big(\bar{r}_n \colon \mathsf{N}_D[n] \to \mathsf{N}_D[n-1]\big)
$$

*Proof:* The kernel of the restriction maps $i_n^*$ is generated by those total contractions $\omega_\sigma$ that become zero when restricted to $S^{d_2} N_2 \otimes \ldots \otimes S^{d_r} N_r \otimes^p T_x^* X$. Due to the particular ("geometrical") definition of these subspaces, the vanishing of these contractions does not depend on the dimension $n$, but only on the symmetries defining the normal tensors and the symmetric powers. Hence, the dimensional reductions $r_n$ induce a surjective maps

$$
\operatorname{Ker} i_n^* \ \xrightarrow{\ r_n\ } \ \operatorname{Ker} i_{n-1}^* \ .
$$

Now, the thesis follows using the Snake's Lemma.

$\square$

## 2.2   Final computations

A system of generators for $\mathsf{T}_{2\bar{p},w}[n]$ is given, via Theorem 2.2, by total contraction maps

$$
\omega_\sigma \colon S^{d_2} N_2 \otimes \ldots \otimes S^{d_r} N_r \otimes^{2\bar{p}} T_x^* X \longrightarrow \mathbb{R}
$$

where $\sigma \in S_m$ is a permutation of $m = 4d_2 + \ldots + (2+r)d_r + 2\bar{p}$ indices, and the multi-index $D = (d_2, \ldots, d_r)$ satisfies:

$$
2d_2 + \ldots + r d_r \ = \ 2\bar{p} - w \ .
$$

In terms of these generators, the dimensional reduction maps are:

$$\mathsf{T}_{2\bar{p},w}[n] \xrightarrow{\ r_n\ } \mathsf{T}_{2\bar{p},w}[n-1] \quad , \quad \omega_\sigma \longmapsto \omega_\sigma \ .$$

**Lemma 2.11** *For any weight* $w = 2\bar{p} - 2k$, *with* $k \geq 0$, *the dimensional reduction maps*

$$r_n \colon \mathsf{T}_{2\bar{p},w}[n] \longrightarrow \mathsf{T}_{2\bar{p},w}[n-1]$$

*satisfy:*

- *They are surjective, for all* $n$.

- *If* $n > 2k + \bar{p}$, *they are linear isomorphisms:*

- *If* $n = 2k + \bar{p}$, *then any tensor on the kernel of* $r_n$ *is second-order.*

*Proof:* Theorem 2.2, in conjunction with Proposition 2.8, imply that $r_n$ is surjective, for all $n$.

To study the kernels, first observe that the number $m = 4d_2 + \ldots + (2 + r)d_r + 2\bar{p}$ of indices to contract on a generator $\omega_\sigma$ is bounded by:

$$\begin{aligned} m = \ & 4d_2 + \ldots + (2+r)d_r + 2\bar{p} \ = \ 2(d_2 + \ldots + d_r) + 2d_2 + \ldots + rd_r + 2\bar{p} \\ \leq \ & 2\bar{p} - w + 2\bar{p} - w + 2\bar{p} \ = \ 4k + 2\bar{p} \ = \ 2\,(2k + \bar{p}) \ . \end{aligned}$$

Therefore, if the dimension is big enough, $n > 2k + \bar{p}$, then there are no relations among the generators $\omega_\sigma$ due to dimensional considerations (Corollary 2.5).

Nevertheless, on a manifold of dimension $n = 2k + \bar{p} - 1$, it may happen that the (even) number of indices $m$ is strictly greater than twice the dimension; in that case,

$$2\,(2k + \bar{p}) \ \leq \ m = 4d_2 + \ldots + (2 + r)d_r + 2\bar{p} \ \leq \ 2\,(2k + \bar{p}) \ ,$$

so both inequalities are indeed equalities, and hence $d_3 = \ldots = d_r = 0$.

This amounts to saying that the corresponding universal tensor is second-order.

$\square$

**Lemma 2.12** *For any weight* $w = 2\bar{p} - 2k$, *with* $k \geq 0$, *it holds:*

$$\dim \mathsf{K}_{2\bar{p},w}\,[2k + \bar{p} - 1\,] \ = \ \bar{p}\,(\bar{p} + 1) \cdot \ldots \cdot (2\bar{p} - 2)(2\bar{p} - 1) \ .$$

*If* $p = 0$, *then* $\dim \mathsf{K}_{0,w}[2k - 1] = 1$, *for all* $k \geq 0$.

*Proof:* By the previous Lemma, any tensor in the kernel of $r_n$, for $n = 2k + \bar{p}$, is second-order, so that it is defined by a linear combination of invariant linear maps:

$$\omega : \; S^{\frac{n}{2}} N_2 \otimes^{2\bar{p}} T_x^* X \longrightarrow \mathbb{R} \; ,$$

which are non-zero if $\dim X = n + \bar{p}$, but vanish when $\dim X = n + \bar{p} - 1$.

Theorem 2.4, together with Corollary 2.5 and Lema 2.10, imply that any such $\omega$ is a linear combination of elements of the form

$$e_1 \otimes \ldots \otimes e_{2(n+\bar{p})} \; \longmapsto \; (e_{r_1} \wedge \ldots \wedge e_{r_{n+\bar{p}}}) \cdot (e_{s_1} \wedge \ldots \wedge e_{s_{n+\bar{p}}}) \qquad (4)$$

where the indexes $r_1, \ldots, r_{n+\bar{p}}, s_1, \ldots, s_{n+\bar{p}}$ run from 1 to $2(n + \bar{p})$.

Let us now prove that, due to the symmetries of the space $S^{\frac{n}{2}} N_2 \otimes^{2\bar{p}} T_x^* X$, we can extract, among these generators, the following basis:

$$\omega_{i_1 \ldots i_{\bar{p}}}^{j_1 \ldots j_{\bar{p}}} : \; S^{\frac{n}{2}} N_2 \otimes^{2\bar{p}} T_x^* X \longrightarrow \mathbb{R}$$

$$e_1 \otimes \ldots \otimes e_{2(n+\bar{p})} \; \longmapsto \; (e_1 \wedge \ldots \wedge e_{2n-1} \wedge e_{i_1} \wedge \ldots \wedge e_{i_{\bar{p}}}) \cdot (e_2 \wedge \ldots \wedge e_{2n} \wedge e_{j_1} \wedge \ldots \wedge e_{j_{\bar{p}}})$$

where $i_1, \ldots, i_{\bar{p}}, j_1, \ldots, j_{\bar{p}}$ are different indexes, running from $2n + 1$ to $2(n + \bar{p})$.

So let $\omega$ be a non-zero linear map as in (4). Up to a sign, we can assume that $e_{r_1} = e_1$. Since normal tensors in $N_2$ are symmetric in the first two indexes, we may also assume that $e_{s_1} = e_2$. Analogously, normal tensors of order two are symmetric on the third and four indexes, so that we can also write $e_{r_2} = e_3$ and $e_{s_2} = e_4$.

A similar argument easily proves that $\omega$ is proportional to the linear map that sends $e_1 \otimes \ldots \otimes e_{2(n+\bar{p})}$ into

$$(e_1 \wedge e_3 \wedge \ldots \wedge e_{2n-1} \wedge e_{i_1} \wedge \ldots \wedge e_{i_{\bar{p}}}) \cdot (e_2 \wedge e_4 \wedge \ldots \wedge e_{2n} \wedge e_{j_1} \wedge \ldots \wedge e_{j_{\bar{p}}})$$

where $i_1, \ldots, i_{\bar{p}}, j_1, \ldots, j_{\bar{p}}$ are different indexes, running from $2n + 1$ to $2(n + \bar{p})$; that is to say, $\omega$ is proportional to the $\omega_{i_1 \ldots i_{\bar{p}}}^{j_1 \ldots j_{\bar{p}}}$ defined above.

If $\bar{p} = 0$, we are done. Otherwise, the dimension of the kernel of $r_n$ is bounded by the number of possible unordered choices of $\bar{p}$ elements over a set $2\bar{p}$ elements, and divided by 2 (because the metric in $\Lambda^k E$ is symmetric). In other words, it is bounded by

$$\bar{p} \, (\bar{p} + 1) \cdot \ldots \cdot (2\bar{p} - 2)(2\bar{p} - 1) \; .$$

As the linear maps $\omega_{i_1 \ldots i_{\bar{p}}}^{j_1 \ldots j_{\bar{p}}}$ are linearly independent, the statement follows.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

*Proof of Theorem 1.10:* As it was explained in Example 1.9, the universal tensors $\sigma \cdot \mathsf{S}_{2\bar{p},k}$ define elements in $\mathsf{K}_{2\bar{p},2(\bar{p}-k)}\left[2k+\bar{p}-1\right]$, for any permutation $\sigma$ of $2\bar{p}$ elements.

The tensors $\sigma \cdot \mathsf{S}_{2\bar{p},k}$ are all $\mathbb{R}$-linearly independent, so Theorem 1.10 readily follows from Lemma 2.11 and Lemma 2.12

$\square$

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## References

[1] Anderson, I. M.: *Natural variational principles on Riemannian manifolds*, Ann. of Math., (2) 120 (1984) 329-370.

[2] Castrillón, M., Muñoz, J.: *Gauge-invariant characterization of Yang-Mills-Higgs lagrangians*, Ann. Henri Poincaré **8**, 203–217 (2007)

[3] Gilkey, P.: *Curvature and the eigenvalues of the Laplacian for elliptic complexes*, Adv. in Math, **10** (1973) 344-382.

[4] Gilkey, P.; Park, J.H.; Sekigawa, K.: *Universal curvature identities*, Diff. Geom. App., **62** (2011) 814-825.

[5] Gilkey, P.; Park, J.H.; Sekigawa, K.: *Universal curvature identities II*, J. Geom. Phys., **62** (2012) 814-825.

[6] Gilkey, P.; Park, J.H.; Sekigawa, K.: *Universal curvature identities and Euler Lagrange Formulas for Kaehler manifolds*, ArXiv: 1311.2622

[7] Kolár, I., Michor, P.W., Slovák, J.: *Natural operations in differential geometry*, Springer-Verlag, 1993.

[8] Kuz'mina, G. M.: *Some generalizations of the Riemann spaces of Einstein*, Math. Notes **16** (1974) 961–963

[9] Lanczos, C.: *A remarkable property of the Riemann-Christoffel tensor in four dimensions*, Ann. Math. **38** (1938) 842–850

[10] Lovelock, D.: *The Einstein tensor and its generalizations*, J. Math. Phys. **12** (1971) 498–501.

[11] Navarro, A., Navarro, J.: *Lovelock's theorem revisited*, J. Geom. Phys., **61** (2011) 1950–1956.

[12] Navarro, J.: *Divergence-free tensors associated to a metric*, Ph. D. Thesis, Universidad de Extremadura, (2013)

[13] Navarro, J., Sancho, J.B. *On the naturalness of Einstein equation*, J. Geom. Phys. **58** (2008) 1007-1014.

[14] Sancho, C.: *Grupos algebraicos y teoría de invariantes*, Soc. Mat. Mexicana, (2001)

[15] Stredder, P.: *Natural differential operators on riemannian manifolds and representations of the orthogonal and special orthogonal groups*, J. Diff. Geom. **10** (1975) 647–660.

[16] Weyl, H.: *The Classical Groups. Their Invariants and Representations*, Princeton University Press, (1939)