

# Constant Jacobi osculating rank of $U(3)/(U(1) \times U(1) \times U(1))$ - Appendix -

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*Dedicated to Salud Bartoll.*

## Abstract

This is the appendix of the paper [T. Arias-Marco, Constant Jacobi osculating rank of  $U(3)/(U(1) \times U(1) \times U(1))$ , Arch. Math. (Brno) 45 (2009), 241–254] where we obtain an interesting relation between the covariant derivatives of the Jacobi operator valid for all geodesic on the flag manifold  $M^6 = U(3)/(U(1) \times U(1) \times U(1))$ . As a consequence, an explicit expression of the Jacobi operator independent of the geodesic can be obtained on such a manifold. Moreover, we show the way to calculate the Jacobi vector fields on this manifold by a new formula valid on every g.o. space.

Mathematics Subject Classification (2000). 53C21, 53C25, 53C30, 53C20

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## Appendix

$\mathcal{J}_0^2 = (\mathcal{J}_{ij}^2(0))$ ,  $i, j = 1, \dots, 6$ , where

$$\begin{aligned}
\mathcal{J}_{11}^2(0) &= -\frac{3}{4} (x_4^2(x_2^2 + x_3^2 + x_5^2 + x_6^2) - 2(x_2x_6 + x_3x_5)^2), \\
\mathcal{J}_{12}^2(0) &= \frac{3}{8} (x_4x_5(x_1^2 + x_2^2 + x_4^2 + x_5^2) - 2(x_2x_6 + x_3x_5)(x_1x_6 + x_3x_4)), \\
\mathcal{J}_{13}^2(0) &= \frac{3}{8} (x_4x_6(x_1^2 + x_3^2 + x_4^2 + x_6^2) - 2(x_2x_6 + x_3x_5)(x_1x_5 + x_2x_4)), \\
\mathcal{J}_{14}^2(0) &= \frac{3}{4} (x_1x_4(x_2^2 + x_3^2 + x_5^2 + x_6^2) + 2(x_2x_6 + x_3x_5)(x_2x_3 - x_5x_6)), \\
\mathcal{J}_{15}^2(0) &= -\frac{3}{8} (x_2x_4(x_1^2 + x_2^2 + x_4^2 + x_5^2) + 2(x_2x_6 + x_3x_5)(x_1x_3 - x_4x_6)), \\
\mathcal{J}_{16}^2(0) &= -\frac{3}{8} (x_3x_4(x_1^2 + x_3^2 + x_4^2 + x_6^2) + 2(x_2x_6 + x_3x_5)(x_1x_2 - x_4x_5)), \\
\\
\mathcal{J}_{22}^2(0) &= -\frac{3}{4} (x_5^2(x_1^2 + x_3^2 + x_4^2 + x_6^2) - 2(x_1x_6 + x_3x_4)^2), \\
\mathcal{J}_{23}^2(0) &= \frac{3}{8} (x_5x_6(x_2^2 + x_3^2 + x_5^2 + x_6^2) - 2(x_1x_6 + x_3x_4)(x_1x_5 + x_2x_4)), \\
\mathcal{J}_{24}^2(0) &= -\frac{3}{8} (x_1x_5(x_1^2 + x_2^2 + x_4^2 + x_5^2) + 2(x_1x_6 + x_3x_4)(x_2x_3 - x_5x_6)), \\
\mathcal{J}_{25}^2(0) &= \frac{3}{4} (x_2x_5(x_1^2 + x_3^2 + x_4^2 + x_6^2) + 2(x_1x_6 + x_3x_4)(x_1x_3 - x_4x_6)), \\
\mathcal{J}_{26}^2(0) &= -\frac{3}{8} (x_3x_5(x_2^2 + x_3^2 + x_5^2 + x_6^2) + 2(x_1x_6 + x_3x_4)(x_1x_2 - x_4x_5)), \\
\\
\mathcal{J}_{33}^2(0) &= -\frac{3}{4} (x_6^2(x_1^2 + x_2^2 + x_4^2 + x_5^2) - 2(x_1x_5 + x_2x_4)^2), \\
\mathcal{J}_{34}^2(0) &= -\frac{3}{8} (x_1x_6(x_1^2 + x_3^2 + x_4^2 + x_6^2) + 2(x_1x_5 + x_2x_4)(x_2x_3 - x_5x_6)), \\
\mathcal{J}_{35}^2(0) &= -\frac{3}{8} (x_2x_6(x_2^2 + x_3^2 + x_5^2 + x_6^2) + 2(x_1x_5 + x_2x_4)(x_1x_3 - x_4x_6)), \\
\mathcal{J}_{36}^2(0) &= \frac{3}{4} (x_3x_6(x_1^2 + x_2^2 + x_4^2 + x_5^2) + 2(x_1x_5 + x_2x_4)(x_1x_2 - x_4x_5)), \\
\\
\mathcal{J}_{44}^2(0) &= -\frac{3}{4} (x_1^2(x_2^2 + x_3^2 + x_5^2 + x_6^2) - 2(x_2x_3 - x_5x_6)^2), \\
\mathcal{J}_{45}^2(0) &= \frac{3}{8} (x_1x_2(x_1^2 + x_2^2 + x_4^2 + x_5^2) - 2(x_2x_3 - x_5x_6)(x_1x_3 - x_4x_6)), \\
\mathcal{J}_{46}^2(0) &= \frac{3}{8} (x_1x_3(x_1^2 + x_3^2 + x_4^2 + x_6^2) - 2(x_2x_3 - x_5x_6)(x_1x_2 - x_4x_5)), \\
\\
\mathcal{J}_{55}^2(0) &= -\frac{3}{4} (x_2^2(x_1^2 + x_3^2 + x_4^2 + x_6^2) - 2(x_1x_3 - x_4x_6)^2), \\
\mathcal{J}_{56}^2(0) &= \frac{3}{8} (x_2x_3(x_2^2 + x_3^2 + x_5^2 + x_6^2) - 2(x_1x_3 - x_4x_6)(x_1x_2 - x_4x_5)), \\
\\
\mathcal{J}_{66}^2(0) &= -\frac{3}{4} (x_3^2(x_1^2 + x_2^2 + x_4^2 + x_5^2) - 2(x_1x_2 - x_4x_5)^2).
\end{aligned} \tag{1}$$

$\mathcal{J}_0^3 = (\mathcal{J}_{ij}^3(0))$ ,  $i, j = 1, \dots, 6$ , where

$$\begin{aligned}
\mathcal{J}_{11}^3(0) &= \frac{9}{8\sqrt{2}} x_4(x_2^2 - x_3^2 + x_5^2 - x_6^2)(x_2x_6 + x_3x_5), \\
\mathcal{J}_{12}^3(0) &= \frac{3}{16\sqrt{2}} (x_3(x_1^2(x_4^2 - 4x_5^2) + x_2^2(4x_4^2 - x_5^2) + (x_4^2 - x_5^2)(7x_3^2 + x_4^2 + x_5^2 + 7x_6^2)) \\
&\quad + x_1x_4x_6(x_1^2 + 4x_2^2 + 7x_3^2 + x_4^2 + 4x_5^2 + 7x_6^2) - x_2x_5x_6(4x_1^2 + x_2^2 + 7x_3^2 + 4x_4^2 + x_5^2 + 7x_6^2)), \\
\mathcal{J}_{13}^3(0) &= \frac{3}{16\sqrt{2}} (-x_4(x_1x_5 + x_2x_4)(x_1^2 + 7x_2^2 + 4x_3^2 + x_4^2 + 7x_5^2 + 4x_6^2)) \\
&\quad + x_6(x_2x_6 + x_3x_5)(4x_1^2 + 7x_2^2 + x_3^2 + 4x_4^2 + 7x_5^2 + x_6^2), \\
\mathcal{J}_{14}^3(0) &= \frac{9}{16\sqrt{2}} (x_2^2 - x_3^2 + x_5^2 - x_6^2)(x_1(x_2x_6 - x_3x_5) + x_4(x_2x_3 - x_5x_6)), \\
\mathcal{J}_{15}^3(0) &= \frac{3}{16\sqrt{2}} (x_4(x_1x_3 - x_4x_6)(x_1^2 + 4x_2^2 + 7x_3^2 + x_4^2 + 4x_5^2 + 7x_6^2)) \\
&\quad + x_2(x_2x_6 + x_3x_5)(4x_1^2 + x_2^2 + 7x_3^2 + 4x_4^2 + x_5^2 + 7x_6^2), \\
\mathcal{J}_{16}^3(0) &= \frac{3}{16\sqrt{2}} (x_5(x_1^2(x_4^2 - 4x_3^2) + x_6^2(4x_4^2 - x_3^2) + (x_4^2 - x_3^2)(7x_2^2 + x_3^2 + x_4^2 + 7x_5^2)) \\
&\quad - x_1x_2x_4(x_1^2 + 7x_2^2 + 4x_3^2 + x_4^2 + 7x_5^2 + 4x_6^2) - x_6x_2x_3(4x_1^2 + 7x_2^2 + x_3^2 + 4x_4^2 + 7x_5^2 + x_6^2)), \\
\mathcal{J}_{22}^3(0) &= \frac{9}{8\sqrt{2}} x_5(x_1^2 - x_3^2 + x_4^2 - x_6^2)(x_1x_6 + x_3x_4), \\
\mathcal{J}_{23}^3(0) &= \frac{3}{16\sqrt{2}} (x_5(x_1x_5 + x_2x_4)(7x_1^2 + x_2^2 + 4x_3^2 + 7x_4^2 + x_5^2 + 4x_6^2)) \\
&\quad - x_6(x_1x_6 + x_3x_4)(7x_1^2 + 4x_2^2 + x_3^2 + 7x_4^2 + 4x_5^2 + x_6^2), \\
\mathcal{J}_{24}^3(0) &= \frac{3}{16\sqrt{2}} (x_6(x_2^2(4x_1^2 - x_5^2) + x_4^2(x_1^2 - 4x_5^2) + (x_1^2 - x_5^2)(x_1^2 + 7x_3^2 + x_5^2 + 7x_6^2)) \\
&\quad - x_2x_3x_5(4x_1^2 + x_2^2 + 7x_3^2 + 4x_4^2 + x_5^2 + 7x_6^2) - x_4x_1x_3(x_1^2 + 4x_2^2 + 7x_3^2 + x_4^2 + 4x_5^2 + 7x_6^2)), \\
\mathcal{J}_{25}^3(0) &= \frac{9}{16\sqrt{2}} (x_1^2 - x_3^2 + x_4^2 - x_6^2)(x_2(x_1x_6 + x_3x_4) - x_5(x_1x_3 + x_4x_6)), \\
\mathcal{J}_{26}^3(0) &= \frac{3}{16\sqrt{2}} (x_4(x_2^2(4x_3^2 - x_5^2) + x_6^2(x_3^2 - 4x_5^2) + (x_3^2 - x_5^2)(7x_1^2 + x_3^2 + 7x_4^2 + x_5^2)) \\
&\quad + x_2x_1x_5(7x_1^2 + x_2^2 + 4x_3^2 + 7x_4^2 + x_5^2 + 4x_6^2) + x_6x_1x_3(7x_1^2 + 4x_2^2 + x_3^2 + 7x_4^2 + 4x_5^2 + x_6^2)), \\
\mathcal{J}_{33}^3(0) &= \frac{9}{8\sqrt{2}} x_6(x_1^2 - x_2^2 + x_4^2 - x_5^2)(x_1x_5 + x_2x_4), \\
\mathcal{J}_{34}^3(0) &= \frac{3}{16\sqrt{2}} (x_6(x_2x_3 - x_5x_6)(4x_1^2 + 7x_2^2 + x_3^2 + 4x_4^2 + 7x_5^2 + x_6^2)) \\
&\quad + x_1(x_1x_5 + x_2x_4)(x_1^2 + 7x_2^2 + 4x_3^2 + x_4^2 + 7x_5^2 + 4x_6^2), \\
\mathcal{J}_{35}^3(0) &= \frac{3}{16\sqrt{2}} (x_6(-x_1x_3 + x_4x_6)(7x_1^2 + 4x_2^2 + x_3^2 + 7x_4^2 + 4x_5^2 + x_6^2)) \\
&\quad - x_2(x_1x_5 + x_2x_4)(7x_1^2 + x_2^2 + 4x_3^2 + 7x_4^2 + x_5^2 + 4x_6^2), \\
\mathcal{J}_{36}^3(0) &= \frac{9}{16\sqrt{2}} (x_1^2 - x_2^2 + x_4^2 - x_5^2)(-x_3(x_1x_5 + x_2x_4) + x_6(x_1x_2 - x_4x_5)), \\
\mathcal{J}_{44}^3(0) &= \frac{9}{8\sqrt{2}} x_1(x_2^2 - x_3^2 + x_5^2 - x_6^2)(-x_2x_3 + x_5x_6), \\
\mathcal{J}_{45}^3(0) &= \frac{3}{16\sqrt{2}} (x_3(-x_1^2(x_4^2 + 4x_5^2) + x_2^2(4x_4^2 + x_5^2) - (x_1^2 - x_2^2)(x_1^2 + x_2^2 + 7x_3^2 + 7x_6^2)) \\
&\quad + x_4x_1x_6(x_1^2 + 4x_2^2 + 7x_3^2 + x_4^2 + 4x_5^2 + 7x_6^2) + x_5x_2x_6(4x_1^2 + x_2^2 + 7x_3^2 + 4x_4^2 + x_5^2 + 7x_6^2)), \\
\mathcal{J}_{46}^3(0) &= \frac{3}{16\sqrt{2}} (x_1(x_1x_2 - x_4x_5)(x_1^2 + 7x_2^2 + 4x_3^2 + x_4^2 + 7x_5^2 + 4x_6^2)) \\
&\quad - x_3(x_2x_3 - x_5x_6)(4x_1^2 + 7x_2^2 + x_3^2 + 4x_4^2 + 7x_5^2 + x_6^2), \\
\mathcal{J}_{55}^3(0) &= \frac{9}{8\sqrt{2}} x_2(x_1^2 - x_3^2 + x_4^2 - x_6^2)(x_1x_3 - x_4x_6), \\
\mathcal{J}_{56}^3(0) &= \frac{3}{16\sqrt{2}} (x_1(-x_2^2(x_5^2 + 4x_6^2) + x_3^2(4x_5^2 + x_6^2) - (x_2^2 - x_3^2)(7x_1^2 + x_2^2 + x_3^2 + 7x_4^2)) \\
&\quad + x_5x_2x_4(7x_1^2 + x_2^2 + 4x_3^2 + 7x_4^2 + x_5^2 + 4x_6^2) - x_6x_3x_4(7x_1^2 + 4x_2^2 + x_3^2 + 7x_4^2 + 4x_5^2 + x_6^2)), \\
\mathcal{J}_{66}^3(0) &= \frac{9}{8\sqrt{2}} x_3(x_1^2 - x_2^2 + x_4^2 - x_5^2)(-x_1x_2 + x_4x_5),
\end{aligned}$$

(2)

$\mathcal{J}_0^4 = (\mathcal{J}_{ij}^4(0))$ ,  $i, j = 1, \dots, 6$ , where

$$\begin{aligned}
\mathcal{J}_{11}^4(0) &= \frac{3}{32} (-8(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_2x_6 + x_3x_5)^2 \\
&\quad + x_4^2((x_1^2 + 7x_2^2 + 7x_3^2 + x_4^2 + 7x_5^2 + 7x_6^2)(x_2^2 + x_3^2 + x_5^2 + x_6^2) - 6(x_2^2 + x_5^2)(x_3^2 + x_6^2))) \\
\mathcal{J}_{12}^4(0) &= \frac{3}{64} (8x_6(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1x_2x_6 + x_1x_3x_5 + x_2x_3x_4) \\
&\quad - x_4x_5((x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(3x_3^2 - 5x_6^2) \\
&\quad + 12((x_1^2 + x_4^2)(x_2^2 + x_5^2) - (x_3^2 + x_6^2)^2))) \\
\mathcal{J}_{13}^4(0) &= \frac{3}{64} (8x_5(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1x_2x_6 + x_1x_3x_5 + x_2x_3x_4) \\
&\quad - x_4x_6((x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 + (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(-3x_2^2 + 5x_5^2) \\
&\quad + 12((x_1^2 + x_4^2)(x_3^2 + x_6^2) - (x_2^2 + x_5^2)^2))) \\
\mathcal{J}_{14}^4(0) &= \frac{3}{32} (8(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(-x_2x_3 + x_5x_6)(x_2x_6 + x_3x_5) \\
&\quad - x_1x_4((x_1^2 + x_4^2)(x_2^2 + x_3^2 + x_5^2 + x_6^2) - 6(x_2^2 + x_5^2)(x_3^2 + x_6^2) + 7(x_2^2 + x_3^2 + x_5^2 + x_6^2)^2)) \\
\mathcal{J}_{15}^4(0) &= \frac{3}{64} (8x_3(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1x_2x_6 + x_1x_3x_5 - x_4x_5x_6) \\
&\quad + x_2x_4((x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 + (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(5x_3^2 - 3x_6^2) \\
&\quad + 12((x_1^2 + x_4^2)(x_2^2 + x_5^2) - (x_3^2 + x_6^2)^2))) \\
\mathcal{J}_{16}^4(0) &= \frac{3}{64} (8x_2(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1x_2x_6 + x_1x_3x_5 - x_4x_5x_6) \\
&\quad + x_3x_4((x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 + (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(5x_2^2 - 3x_5^2) \\
&\quad + 12((x_1^2 + x_4^2)(x_3^2 + x_6^2) - (x_2^2 + x_5^2)^2))) \\
\mathcal{J}_{22}^4(0) &= \frac{3}{32} (-8(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1x_6 + x_3x_4)^2 \\
&\quad + x_5^2((7x_1^2 + x_2^2 + 7x_3^2 + 7x_4^2 + x_5^2 + 7x_6^2)(x_1^2 + x_3^2 + x_4^2 + x_6^2) - 6(x_1^2 + x_4^2)(x_3^2 + x_6^2))) \\
\mathcal{J}_{23}^4(0) &= \frac{3}{64} (8x_4(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1x_2x_6 + x_1x_3x_5 + x_2x_3x_4) \\
&\quad - x_5x_6((x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(3x_1^2 - 5x_4^2) \\
&\quad + 12((x_2^2 + x_5^2)(x_3^2 + x_6^2) - (x_1^2 + x_4^2)^2))) \\
\mathcal{J}_{24}^4(0) &= \frac{3}{64} (8x_3(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1x_2x_6 + x_2x_3x_4 - x_4x_5x_6) \\
&\quad + x_1x_5((x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 + (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(5x_3^2 - 3x_6^2) \\
&\quad + 12((x_1^2 + x_4^2)(x_2^2 + x_5^2) - (x_3^2 + x_6^2)^2))) \\
\mathcal{J}_{25}^4(0) &= \frac{3}{32} (8(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(-x_1x_3 + x_4x_6)(x_1x_6 + x_3x_4) \\
&\quad - x_2x_5((x_2^2 + x_5^2)(x_1^2 + x_3^2 + x_4^2 + x_6^2) - 6(x_1^2 + x_4^2)(x_3^2 + x_6^2) + 7(x_1^2 + x_3^2 + x_4^2 + x_6^2)^2)) \\
\mathcal{J}_{26}^4(0) &= \frac{3}{64} (8x_1(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1x_2x_6 + x_2x_3x_4 - x_4x_5x_6) \\
&\quad + x_3x_5((x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 + (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(5x_1^2 - 3x_4^2) \\
&\quad + 12((x_2^2 + x_5^2)(x_3^2 + x_6^2) - (x_1^2 + x_4^2)^2))) \\
\mathcal{J}_{33}^4(0) &= \frac{3}{32} (8(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1x_5 + x_2x_4)^2 \\
&\quad - x_6^2((7x_1^2 + 7x_2^2 + x_3^2 + 7x_4^2 + 7x_5^2 + x_6^2)(x_1^2 + x_2^2 + x_4^2 + x_5^2) - 6(x_1^2 + x_4^2)(x_2^2 + x_5^2))) \\
\mathcal{J}_{34}^4(0) &= \frac{3}{64} (8x_2(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1x_3x_5 + x_2x_3x_4 - x_4x_5x_6) \\
&\quad + x_1x_6((x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 + (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(5x_2^2 - 3x_5^2) \\
&\quad + 12((x_1^2 + x_4^2)(x_3^2 + x_6^2) - (x_2^2 + x_5^2)^2))) \\
\mathcal{J}_{35}^4(0) &= \frac{3}{64} (8x_1(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1x_3x_5 + x_2x_3x_4 - x_4x_5x_6) \\
&\quad + x_2x_6((x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 + (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(5x_1^2 - 3x_4^2) \\
&\quad + 12((x_2^2 + x_5^2)(x_3^2 + x_6^2) - (x_1^2 + x_4^2)^2))) \\
\mathcal{J}_{36}^4(0) &= \frac{3}{32} (8(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(-x_1x_2 + x_4x_5)(x_1x_5 + x_2x_4) \\
&\quad - x_3x_6((x_3^2 + x_6^2)(x_1^2 + x_2^2 + x_4^2 + x_5^2) - 6(x_1^2 + x_4^2)(x_2^2 + x_5^2) + 7(x_1^2 + x_2^2 + x_4^2 + x_5^2)^2))
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}_{44}^4(0) &= \frac{3}{32} (-8(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_2x_3 - x_5x_6)^2 \\
&\quad + x_1^2((x_1^2 + 7x_2^2 + 7x_3^2 + x_4^2 + 7x_5^2 + 7x_6^2)(x_2^2 + x_3^2 + x_5^2 + x_6^2) - 6(x_2^2 + x_5^2)(x_3^2 + x_6^2))) \\
\mathcal{J}_{45}^4(0) &= \frac{3}{64} (8x_6(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(-x_1x_3x_5 - x_2x_3x_4 + x_4x_5x_6) \\
&\quad - x_1x_2((x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 + (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(3x_3^2 - 5x_6^2) \\
&\quad + 12((x_1^2 + x_4^2)(x_2^2 + x_5^2) - (x_3^2 + x_6^2)^2))) \\
\mathcal{J}_{46}^4(0) &= \frac{3}{64} (8x_5(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(-x_1x_2x_6 - x_2x_3x_4 + x_4x_5x_6) \\
&\quad - x_1x_3((x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(3x_2^2 - 5x_5^2) \\
&\quad + 12((x_1^2 + x_4^2)(x_3^2 + x_6^2) - (x_2^2 + x_5^2)^2))) \\
\mathcal{J}_{55}^4(0) &= \frac{3}{32} (-8(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1x_3 - x_4x_6)^2 \\
&\quad + x_2^2((7x_1^2 + x_2^2 + 7x_3^2 + 7x_4^2 + x_5^2 + 7x_6^2)(x_1^2 + x_3^2 + x_4^2 + x_6^2) - 6(x_1^2 + x_4^2)(x_3^2 + x_6^2))) \\
\mathcal{J}_{56}^4(0) &= \frac{3}{64} (8x_4(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(-x_1x_2x_6 - x_1x_3x_5 + x_4x_5x_6) \\
&\quad - x_2x_3((x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(3x_1^2 - 5x_4^2) \\
&\quad + 12((x_2^2 + x_5^2)(x_3^2 + x_6^2) - (x_1^2 + x_4^2)^2))) \\
\mathcal{J}_{66}^4(0) &= \frac{3}{32} (-8(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1x_2 - x_4x_5)^2 \\
&\quad + x_3^2((7x_1^2 + 7x_2^2 + x_3^2 + 7x_4^2 + 7x_5^2 + x_6^2)(x_1^2 + x_2^2 + x_4^2 + x_5^2) - 6(x_1^2 + x_4^2)(x_2^2 + x_5^2))) \\
\end{aligned} \tag{3}$$

$\mathcal{J}_0^5 = (\mathcal{J}_{ij}^5(0))$ ,  $i, j = 1, \dots, 6$ , where

$$\begin{aligned}
\mathcal{J}_{11}^5(0) &= -\frac{45}{64\sqrt{2}} x_4(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_2^2 - x_3^2 + x_5^2 - x_6^2)(x_2x_6 + x_3x_5), \\
\mathcal{J}_{12}^5(0) &= \frac{3}{32\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 (x_3(x_4^2 - x_5^2) + x_6(x_1x_4 - x_2x_5)) \\
&\quad - \frac{15}{128\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) (x_3(x_1^2(x_4^2 - 4x_5^2) + x_2^2(4x_4^2 - x_5^2)) \\
&\quad + (x_4^2 - x_5^2)(7x_3^2 + x_4^2 + x_5^2 + 7x_6^2)) + x_1x_4x_6(x_1^2 + 4x_2^2 + 7x_3^2 + x_4^2 + 4x_5^2 + 7x_6^2) \\
&\quad - x_2x_5x_6(4x_1^2 + x_2^2 + 7x_3^2 + 4x_4^2 + x_5^2 + 7x_6^2)), \\
\mathcal{J}_{13}^5(0) &= -\frac{3}{32\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 (x_2(x_4^2 - x_6^2) + x_5(x_1x_4 - x_3x_6)) \\
&\quad - \frac{15}{128\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) (-x_4(x_1x_5 + x_2x_4)(x_1^2 + 7x_2^2 + 4x_3^2 \\
&\quad + x_4^2 + 7x_5^2 + 4x_6^2) + x_6(x_2x_6 + x_3x_5)(4x_1^2 + 7x_2^2 + x_3^2 + 4x_4^2 + 7x_5^2 + x_6^2)), \\
\mathcal{J}_{14}^5(0) &= -\frac{45}{128\sqrt{2}} (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_2^2 - x_3^2 + x_5^2 - x_6^2)(x_1(x_2x_6 - x_3x_5) \\
&\quad + x_4(x_2x_3 - x_5x_6)), \\
\mathcal{J}_{15}^5(0) &= \frac{3}{32\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 (x_6(x_2^2 - x_4^2) + x_3(x_1x_4 + x_2x_5)) \\
&\quad - \frac{15}{128\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) (x_4(x_1x_3 - x_4x_6)(x_1^2 + 4x_2^2 + 7x_3^2 \\
&\quad + x_4^2 + 4x_5^2 + 7x_6^2)) + x_2(x_2x_6 + x_3x_5)(4x_1^2 + x_2^2 + 7x_3^2 + 4x_4^2 + x_5^2 + 7x_6^2)), \\
\mathcal{J}_{16}^5(0) &= -\frac{3}{32\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 (x_5(x_3^2 - x_4^2) + x_2(x_1x_4 + x_3x_6)) \\
&\quad - \frac{15}{128\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) (x_5(x_1^2(x_4^2 - 4x_3^2) + x_6^2(4x_4^2 - x_3^2)) \\
&\quad + (x_4^2 - x_3^2)(7x_2^2 + x_3^2 + x_4^2 + 7x_5^2)) - x_1x_2x_4(x_1^2 + 7x_2^2 + 4x_3^2 + x_4^2 + 7x_5^2 + 4x_6^2) \\
&\quad - x_6x_2x_3(4x_1^2 + 7x_2^2 + x_3^2 + 4x_4^2 + 7x_5^2 + x_6^2)), \\
\mathcal{J}_{22}^5(0) &= -\frac{45}{64\sqrt{2}} x_5(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1^2 - x_3^2 + x_4^2 - x_6^2)(x_1x_6 + x_3x_4), \\
\mathcal{J}_{23}^5(0) &= \frac{3}{32\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 (x_1(x_5^2 - x_6^2) + x_4(x_2x_5 - x_3x_6)) \\
&\quad - \frac{15}{128\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) (x_5(x_1x_5 + x_2x_4)(7x_1^2 + x_2^2 + 4x_3^2 \\
&\quad + 7x_4^2 + x_5^2 + 4x_6^2)) - x_6(x_1x_6 + x_3x_4)(7x_1^2 + 4x_2^2 + x_3^2 + 7x_4^2 + 4x_5^2 + x_6^2)), \\
\mathcal{J}_{24}^5(0) &= -\frac{3}{32\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 (x_6(x_1^2 - x_5^2) + x_3(x_1x_4 + x_2x_5)) \\
&\quad - \frac{15}{128\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) (x_6(x_2^2(4x_1^2 - x_5^2) + x_4^2(x_1^2 - 4x_5^2)) \\
&\quad + (x_1^2 - x_5^2)(x_1^2 + 7x_3^2 + x_5^2 + 7x_6^2)) - x_2x_3x_5(4x_1^2 + x_2^2 + 7x_3^2 + 4x_4^2 + x_5^2 + 7x_6^2) \\
&\quad - x_4x_1x_3(x_1^2 + 4x_2^2 + 7x_3^2 + x_4^2 + 4x_5^2 + 7x_6^2)), \\
\mathcal{J}_{25}^5(0) &= -\frac{45}{128\sqrt{2}} (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1^2 - x_3^2 + x_4^2 - x_6^2)(x_2(x_1x_6 + x_3x_4) \\
&\quad - x_5(x_1x_3 + x_4x_6)), \\
\mathcal{J}_{26}^5(0) &= \frac{3}{32\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 (x_4(x_3^2 - x_5^2) + x_1(x_2x_5 + x_3x_6)) \\
&\quad - \frac{15}{128\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) (x_4(x_2^2(4x_3^2 - x_5^2) + x_6^2(x_3^2 - 4x_5^2)) \\
&\quad + (x_3^2 - x_5^2)(7x_1^2 + x_3^2 + 7x_4^2 + x_5^2)) + x_2x_1x_5(7x_1^2 + x_2^2 + 4x_3^2 + 7x_4^2 + x_5^2 + 4x_6^2) \\
&\quad + x_6x_1x_3(7x_1^2 + 4x_2^2 + x_3^2 + 7x_4^2 + 4x_5^2 + x_6^2)), \\
\mathcal{J}_{33}^5(0) &= -\frac{45}{64\sqrt{2}} x_6(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1^2 - x_2^2 + x_4^2 - x_5^2)(x_1x_5 + x_2x_4), \\
\mathcal{J}_{34}^5(0) &= \frac{3}{32\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 (x_5(x_1^2 - x_6^2) + x_2(x_1x_4 + x_3x_6)) \\
&\quad - \frac{15}{128\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) (x_6(x_2x_3 - x_5x_6)(4x_1^2 + 7x_2^2 + x_3^2 \\
&\quad + 4x_4^2 + 7x_5^2 + x_6^2)) + x_1(x_1x_5 + x_2x_4)(x_1^2 + 7x_2^2 + 4x_3^2 + x_4^2 + 7x_5^2 + 4x_6^2)), \\
\mathcal{J}_{35}^5(0) &= -\frac{3}{32\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 (x_4(x_2^2 - x_6^2) + x_1(x_2x_5 + x_3x_6)) \\
&\quad - \frac{15}{128\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) (x_6(-x_1x_3 + x_4x_6)(7x_1^2 + 4x_2^2 + x_3^2 + 7x_4^2 \\
&\quad + 4x_5^2 + x_6^2)) - x_2(x_1x_5 + x_2x_4)(7x_1^2 + x_2^2 + 4x_3^2 + 7x_4^2 + x_5^2 + 4x_6^2)), \\
\mathcal{J}_{36}^5(0) &= -\frac{45}{128\sqrt{2}} (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1^2 - x_2^2 + x_4^2 - x_5^2)(-x_3(x_1x_5 + x_2x_4) \\
&\quad + x_6(x_1x_2 - x_4x_5)),
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}_{44}^5(0) &= -\frac{45}{64\sqrt{2}} x_1(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_2^2 - x_3^2 + x_5^2 - x_6^2)(-x_2x_3 + x_5x_6), \\
\mathcal{J}_{45}^5(0) &= \frac{3}{32\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 (x_3(x_2^2 - x_1^2) + x_6(x_1x_4 - x_2x_5)) \\
&\quad - \frac{15}{128\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) (x_3(-x_1^2(x_4^2 + 4x_5^2) + x_2^2(4x_4^2 + x_5^2)) \\
&\quad - (x_1^2 - x_2^2)(x_1^2 + x_2^2 + 7x_3^2 + 7x_6^2)) + x_4x_1x_6(x_1^2 + 4x_2^2 + 7x_3^2 + x_4^2 + 4x_5^2 + 7x_6^2) \\
&\quad + x_5x_2x_6(4x_1^2 + x_2^2 + 7x_3^2 + 4x_4^2 + x_5^2 + 7x_6^2)), \\
\mathcal{J}_{46}^5(0) &= -\frac{3}{32\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 (x_2(x_3^2 - x_1^2) + x_5(x_1x_4 - x_3x_6)) \\
&\quad - \frac{15}{128\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) (x_1(x_1x_2 - x_4x_5)(x_1^2 + 7x_2^2 + 4x_3^2 + x_4^2 \\
&\quad + 7x_5^2 + 4x_6^2)) - x_3(x_2x_3 - x_5x_6)(4x_1^2 + 7x_2^2 + x_3^2 + 4x_4^2 + 7x_5^2 + x_6^2), \\
\mathcal{J}_{55}^5(0) &= -\frac{45}{64\sqrt{2}} x_2(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1^2 - x_3^2 + x_4^2 - x_6^2)(x_1x_3 - x_4x_6), \\
\mathcal{J}_{56}^5(0) &= \frac{3}{32\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 (x_1(x_3^2 - x_2^2) + x_4(x_2x_5 - x_3x_6)) \\
&\quad - \frac{15}{128\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) (x_1(-x_2^2(x_5^2 + 4x_6^2) + x_3^2(4x_5^2 + x_6^2)) \\
&\quad - (x_2^2 - x_3^2)(7x_1^2 + x_2^2 + x_3^2 + 7x_4^2)) + x_5x_2x_4(7x_1^2 + x_2^2 + 4x_3^2 + 7x_4^2 + x_5^2 + 4x_6^2) \\
&\quad - x_6x_3x_4(7x_1^2 + 4x_2^2 + x_3^2 + 7x_4^2 + 4x_5^2 + x_6^2)), \\
\mathcal{J}_{66}^5(0) &= -\frac{45}{64\sqrt{2}} x_3(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)(x_1^2 - x_2^2 + x_4^2 - x_5^2)(-x_1x_2 + x_4x_5),
\end{aligned} \tag{4}$$