LOCAL SYMMETRY OF HARMONIC SPACES AS DETERMINED BY THE SPECTRA OF SMALL GEODESIC SPHERES

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ABSTRACT. We show that in any harmonic space, the eigenvalue spectra of the Laplace operator on small geodesic spheres around a given point determine the norm $|\nabla R|$ of the covariant derivative of the Riemannian curvature tensor in that point. In particular, the spectra of small geodesic spheres in a harmonic space determine whether the space is locally symmetric. For the proof we use the first few heat invariants and consider certain coefficients in the radial power series expansions of the curvature invariants $|R|^2$ and $|\mathrm{Ric}|^2$ of the geodesic spheres. Moreover, we obtain analogous results for geodesic balls with either Dirichlet or Neumann boundary conditions. We also comment on the relevance of these results to constructions of Z.I. Szabó.

1. INTRODUCTION

For a compact closed Riemannian manifold S the *spectrum* of S is the eigenvalue spectrum, including multiplicities, of the associated (positive semi-definite) Laplace operator Δ acting on smooth functions. A central question of inverse spectral geometry asks to which extent the geometry of S is determined by its spectrum. The so-called *heat invariants* $a_k(S)$ of S are examples of geometric invariants which are determined by the spectrum of S; indeed, they are the coefficients in the famous asymptotic expansion by Minakshisundaram-Pleijel,

$$\mathrm{Tr}(\exp(-t\Delta)) \sim (4\pi t)^{-\dim(S)/2} \sum_{k=0}^{\infty} a_k(S) t^k$$

for $t \downarrow 0$. The first few of these coefficients are given by

$$a_0(S) = \mathrm{vol}(S), \quad a_1(S) = \frac{1}{6}\int_S \mathrm{scal}\, dvol_S, \quad a_2(S) = \frac{1}{360}\int_S (5\mathrm{scal}^2 - 2|\mathrm{Ric}|^2 + 2|R|^2)\, dvol_S,$$

where scal, Ric, and R denote the scalar curvature, the Ricci operator, and the Riemannian curvature operator of S, respectively. In general, each $a_k(S)$ is the integral over S of certain curvature invariants; see [9] for more information.

Nevertheless, there exist many examples of pairs or families of *isospectral* Riemannian manifolds (i.e., sharing the same spectrum) which are not isometric, sometimes not even locally isometric; see, for example, the survey article [10]. Still, many questions remain open; for example, it is not

known whether a locally symmetric compact closed Riemannian manifold can be isospectral to a locally nonsymmetric Riemannian manifold.

On the other hand, the geometry of *geodesic spheres* plays an interesting role in Riemannian geometry. Chen and Vanhecke [4] formulated the following general question: To what extent do the properties of small geodesic spheres determine the Riemannian geometry of the ambient space? For example, Gray and Vanhecke [12] studied the information contained in the volume function of small geodesic spheres and investigated the question whether a Riemannian manifold whose geodesic spheres have the same volumes as spheres in euclidean space must necessarily be flat (answering this question in the positive under various choices of additional assumptions).

In the context of inverse spectral geometry, an interesting special version of the above question is: To what extent do the spectra of small geodesic spheres in a (possibly noncompact) Riemannian manifold M determine the geometry of M? For example, Theorem 6.18 in [4] uses the information contained in the heat invariants a_0 and a_1 of small geodesic spheres (viewed as functions of the radius) and concludes local isometry of manifolds with adapted holonomy to certain model spaces under the assumption that all small geodesic spheres around each point are isospectral to the corresponding geodesic spheres in those model spaces.

In order to arrive at such and similar results, one uses radial power series expansions of curvature invariants, both of the ambient space and of the geodesic spheres. In general, even the first few coefficients of such expansions become very complicated; see, for example, the various formulas in [12] or [4]. One setting in which a quite restrictive geometric assumption on the ambient space makes the calculations considerably easier is the setting of *harmonic* ambient spaces.

A manifold is called harmonic if the volume density function of the geodesic exponential map is radial around each point. The notion of harmonicity was first introduced by Copson and Ruse [5] and intensively studied by Lichnerowicz [13]; see also [16]. Chapter 6 of the book by Besse [2] gives a useful survey of properties of harmonic spaces. One of the important facts about harmonic spaces is that they are Einstein [2] and hence analytic [7] (the latter result was not yet known when [2] was written). A locally symmetric manifold is harmonic if and only if it is flat or of rank one; the famous Lichnerowicz conjecture postulated that, conversely, each harmonic space is locally symmetric; i.e., satisfies $\nabla R = 0$ (this condition is classically known to be equivalent to the condition that the local geodesic symmetries around each point be isometries). For the case of compact manifolds with finite fundamental group the Lichnerowicz conjecture was proved by Szabó [18]; however, Damek and Ricci gave examples of noncompact homogeneous harmonic manifolds which are not locally symmetric in infinitely many dimensions greater or equal to seven [6]. These spaces are usually referred to as *Damek-Ricci spaces*; see [1] for more information.

Specializing the above question about the information contained in the spectra of small geodesic spheres to the setting of harmonic spaces, we are able to prove in the present paper that the spectra of small geodesic spheres in a harmonic space determine whether the space is locally symmetric (see Corollary 1.2 below). More precisely, we obtain:

Main Theorem 1.1. *Let M_1 and M_2 be harmonic spaces, and let $p_1 \in M_1$, $p_2 \in M_2$. If there exists $\varepsilon > 0$ such that for each $r \in (0, \varepsilon)$ the geodesic spheres $S_r(p_1)$ and $S_r(p_2)$ are isospectral, then $|\nabla R|^2_{p_1} = |\nabla R|^2_{p_2}$.*

Corollary 1.2. *Let M_1 and M_2 be harmonic spaces. Assume that the hypothesis of Theorem 1.1 is satisfied for* each *pair of points $p_1 \in M_1$, $p_2 \in M_2$. Then M_1 is locally symmetric if and only if M_2 is locally symmetric.*

In particular, note that in the case of *locally homogeneous* harmonic spaces M_1 and M_2, the hypothesis of Theorem 1.1 implies that M_1 is locally symmetric if and only if M_2 is locally symmetric. Actually, all known examples of harmonic spaces are locally homogeneous; it is an open question whether there exist harmonic spaces which are not locally homogeneous.

Interestingly, our result implies that certain pairs of geodesic spheres which were claimed to be isospectral by Szabó in [20], [21] are actually *not* isospectral. In fact, Szabó considered (as the featured examples in a more general construction) geodesic spheres in certain symmetric spaces M_1 (namely, quaternionic hyperbolic space of real dimension $4m \geq 12$) and in certain associated locally nonsymmetric Damek-Ricci spaces M_2 of the same dimension (see also Remark 2.4). He stated that every pair of geodesic spheres $S_r(p_1) \subset M_1$ and $S_r(p_2) \subset M_2$ of the same radius was isospectral. Since these ambient manifolds M_1 and M_2 are harmonic and homogeneous, and M_1 is locally symmetric while M_2 is not, Corollary 1.2 immediately implies that Szabó's result cannot be correct. Note that it was Fürstenau [8] who first discovered that actually there was a gap in Szabó's isospectrality argument. The question of whether that proof could be repaired or not had since remained open; our result settles this question in the negative.

The incorrect examples of geodesic spheres mentioned above had the notable property that one is homogeneous and the other not. While it remains unknown whether a homogeneous metric on a sphere can be isospectral to a non-homogeneous one, Szabó in an earlier article [19] did construct a pair of isospectral metrics, only one of which is homogeneous, on the product of a sphere and a torus (those results are not affected by the error in the later papers).

We obtain analogs of our above results for geodesic balls endowed with either Dirichlet or Neumann boundary conditions; see Theorem 5.1 and Corollary 5.2. Similarly as above, this implies that Szabó's examples in [20], [21] of isospectral geodesic balls (of any given radius) in quaternionic hyperbolic space of real dimension at least 12 and in certain associated locally nonsymmetric Damek-Ricci spaces were erroneous.

Note that nevertheless there do exist isospectral pairs and even continuous families of isospectral metrics on spheres and balls; the first such examples were due to Gordon [11].

In order to prove Theorem 1.1 we use the heat invariants $a_0(S_r(p))$ and $a_2(S_r(p))$ of geodesic spheres in harmonic spaces. In particular, we study the coefficients of r^2 in the radial power series expansions of $\frac{1}{\mathrm{vol}(S_r(p))} \int_{S_r(p)} |\mathrm{Ric}^S|^2 \, dvol_{S_r(p)}$ and $\frac{1}{\mathrm{vol}(S_r(p))} \int_{S_r(p)} |R^S|^2 \, dvol_{S_r(p)}$, where Ric^S and R^S denote the Ricci operator and the Riemannian curvature operator of $S_r(p)$. From the form of these coefficients (see Proposition 3.2 and its mean value version Proposition 4.2), we are able to conlude that the heat invariants a_0 and a_2 of $S_r(p)$, viewed as functions of r, together determine the value of $|\nabla R|^2$ at the midpoint p. Note that the same is not true for $a_0(S_r(p))$ alone; see Remark 2.4. Moreover, in the harmonic setting, the function $r \mapsto a_1(S_r(p)) = \frac{1}{6} \int_{S_r(p)} \mathrm{scal}^S \, dvol_{S_r(p)}$ does actually not contain more information than $r \mapsto a_0(S_r(p)) = \mathrm{vol}(S_r(p))$; see Remark 2.3. So it is indeed necessary for our purpose to consider $a_2(S_r(p))$.

Our computations rely heavily on the harmonicity of the ambient space. Note that they are related to certain more general computations in [12] and [4]; for example, Theorem 8.1 of [4] actually includes a kind of analog to our Proposition 3.2, and this even for general, not only for harmonic manifolds; however, that theorem contains information only on the coefficients of r^j with $j \leq 0$, while we need the coefficients of r^2. In fact, in the harmonic case, the lower order coefficients turn out to be determined already by the function $r \mapsto a_0(S_r(p)) = \mathrm{vol}(S_r(p))$; see Proposition 3.2 and Remark 2.3(i).

This paper is organized as follows:

In Section 2 we gather the necessary background on harmonic spaces, mostly following [2]; in particular, we recall Ledger's recursion formula for the power series expansion of the second fundamental form of geodesic spheres, and the resulting curvature identities in harmonic spaces.

In Section 3, we study the coefficients of r^j for $j \leq 2$ in $|\mathrm{Ric}^{S_r(p)}|^2_{\exp(ru)}$ and $|R^{S_r(p)}|^2_{\exp(ru)}$ for unit tangent vectors u of harmonic spaces, using the power series expansion of the second fundamental form and its radial covariant derivative, as well as the Taylor series expansion of the Riemannian curvature tensor. Proposition 3.2 is the main result of this section.

Section 4 is devoted to the proof of the Main Theorem 1.1. In preparation for this, we first derive a mean value version of Proposition 3.2; see Proposition 4.2.

Finally, in Section 5, we prove the analog of Theorem 1.1 for geodesic balls. We consider the heat coefficients of geodesic balls in harmonic spaces and show that the functions $r \mapsto a_0(B_r(p))$ and $r \mapsto a_2(B_r(p))$ (either for Dirichlet or for Neumann boundary conditions) together determine the value of $|\nabla R|^2$ at the midpoint p of the balls. More precisely, we show that the coefficient of r^3 in the radial power series expansion of the quotient $a_2(B_r(p))/a_0(B_r(p))$ is a sum of a nonzero multiple of $|\nabla R|^2_p$ and of terms determined by the function $r \mapsto a_0(B_r(p))$.

2. Preliminaries

2.1. Volume density and the shape operator of geodesic spheres.

In the following, let M be a complete, connected, n-dimensional Riemannian manifold. For $p \in M$, let $\exp_p = \exp|_{T_pM} : T_pM \to M$ denote the associated geodesic exponential map. For a vector $v \in T_pM$ we denote by γ_v the geodesic with initial velocity v. Identifying $T_v(T_pM)$ with T_pM, we regard the differential $d(\exp_p)_v$ as a linear map from T_pM to $T_{\exp v}M$. We denote parallel translation along γ_v by $P^{s,t}_{\gamma_v} : T_{\gamma_v(s)}M \to T_{\gamma_v(t)}M$. Given any unit vector $u \in S_1(0_p) := \{u \in T_pM \mid |u| = 1\}$ and $r \in \mathbb{R}$, we consider the volume density

$$\theta_u(r) := \det\bigl(P^{r,0}_{\gamma_u} \circ d(\exp_p)_{ru}\bigr).$$

Note that $\theta_u(r)$ is the infinitesimal volume distortion of the map $\exp_p$ at the point $ru \in T_pM$. Recall the Gauss lemma: The vector $d(\exp_p)_{ru}u$ is a unit vector perpendicular to each $d(\exp_p)_{ru}w$ with $w \perp u$. Thus, for each $r \in (0, i(p))$, where $i(p)$ denotes the injectivity radius of M at p,

$$(1) \qquad\qquad v_u(r) := r^{n-1}\theta_u(r)$$

is the infinitesimal volume distortion at u of the map

$$S_1(0_p) \ni u \mapsto \gamma_u(r) = \exp(ru) \in S_r(p),$$

where $S_r(p) \subset M$ denotes the geodesic sphere of radius r around p. Let $\sigma_u(r)$ denote the shape operator of $S_r(p)$ at $\exp(ru)$; that is,

$$\sigma_u(r) := (\nabla \nu)|_{T_{\exp(ru)}M},$$

where ∇ is the Levi-Civita connection of M and ν denotes the outward pointing unit normal vector field on the geodesic ball $B_{i(p)}(p) \setminus \{p\}$. In particular, $\nu \circ \gamma_u = \dot\gamma_u$, $\sigma_u\nu = 0$, and the image of $\sigma_u(r)$ is contained in $T_{\gamma_u(r)}S_r(p)$. It is well-known that for all $r \in (0, i(p))$,

$$(2) \qquad\qquad v'_u(r)/v_u(r) = \mathrm{Tr}(\sigma_u(r)),$$

and that the covariant derivative σ'_u of the endomorphism field σ_u along $\gamma_u|_{(0,i(p))}$ satisfies the so-called Riccati equation

$$(3) \qquad\qquad \sigma'_u = -\sigma_u^2 - R_{\dot\gamma_u},$$

where R is the Riemannian curvature tensor of M, given by $R(x,y)z = -\nabla_x\nabla_y z + \nabla_y\nabla_x z + \nabla_{[x,y]}z$, and where $R_\nu := R(\nu, .)\nu$. (Note that here we use the same sign for R as Besse [2].) Let $C_u(r) := r\sigma_u(r)$. This endomorphism field along $\gamma_u|_{(0,i(p))}$ is smoothly extendable to $r = 0$ by $C_u(0) := I_u$, where I_u is defined by $I_u(u) = 0$ and $I_u|_{\{u\}^\perp} = \mathrm{Id}_{\{u\}^\perp}$. Moreover, from (3) one can derive Ledger's recursion formula for the covariant derivatives of C_u at $r = 0$ (see, e.g., [4]):

$$(k-1)C_u^{(k)}(0) = -k(k-1)R_u^{(k-2)} - \sum_{\ell=0}^{k}\binom{k}{\ell}C_u^{(\ell)}(0)C_u^{(k-\ell)}(0)$$

for all $k \in \mathbb{N}$, where $R_u^{(k)}$ is the k-th covariant derivative of the endomorphism field $R_{\dot\gamma_u}$ along γ_u at $r = 0$. This formula allows one to successively compute the $C_u^{(k)}(0)$ in terms of the endomorphisms $R_u^{(k)}$ of T_pM. Forming the Taylor series of C_u and dividing by r, one obtains (see, e.g., [2], [4]):

$$
\begin{aligned}
P_{\gamma_u}^{r,0} \circ \sigma_u(r) \circ P_{\gamma_u}^{0,r} = &\frac{1}{r}I_u - \frac{r}{3}R_u - \frac{r^2}{4}R_u' - \left(\frac{1}{10}R_u'' + \frac{1}{45}R_uR_u\right)r^3 \\
& - \left(\frac{1}{36}R_u''' + \frac{1}{72}R_uR_u' + \frac{1}{72}R_u'R_u\right)r^4 \\
& - \left(\frac{1}{168}R_u^{(4)} + \frac{1}{210}R_uR_u'' + \frac{1}{210}R_u''R_u + \frac{1}{112}R_u'R_u' + \frac{2}{945}R_uR_uR_u\right)r^5 \\
& + O(r^6).
\end{aligned}
$$

(4)

2.2. Curvature identities in harmonic spaces.

The manifold M is called a *harmonic space* if for every $p \in M$ the above function θ_u does not depend on $u \in S_1(0_p)$. An equivalent condition is that for all $r \in (0, i(p))$, the geodesic spheres $S_r(p)$ have constant mean curvature (recall equations (1), (2)). For more information on harmonic spaces see [16] or [2]. If M is harmonic then the function θ_u does in fact not even depend on p; that is, there exists $\theta : [0, \infty) \to \mathbb{R}$ such that

$$\theta_u(r) = \theta(r)$$

for all $u \in TM$ with $|u| = 1$. Moreover, even the local or infinitesimal versions of the above condition imply that the manifold is Einstein [2] and therefore analytic [7]. Hence, the local or infinitesimal versions of the above conditions are equivalent to the global versions. Since $\theta_u(r)$ depends only on r, so does $v_u(r)$ and hence $\mathrm{Tr}(\sigma_u(r))$. From this one can successively derive, using the expansion (4):

Proposition 2.1 (see [2], Chapter 6). *If M is harmonic then there exist constants $C, H, L \in \mathbb{R}$ such that for all $p \in M$ and all $u \in T_pM$ with $|u| = 1$:*

(i) $\mathrm{Tr}(R_u) = C$; *in particular:*
(ii) $\mathrm{Tr}(R_u^{(k)}) = 0$ *for all $k \in \mathbb{N}$.*
(iii) $\mathrm{Tr}(R_uR_u) = H$; *in particular:*
(iv) $\mathrm{Tr}(R_uR_u') = 0$ *and*
(v) $\mathrm{Tr}(R_uR_u'') = -\mathrm{Tr}(R_u'R_u')$.
(vi) $\mathrm{Tr}(32R_uR_uR_u - 9R_u'R_u') = L$.

In fact, taking traces in (4), one has in the harmonic case:

$$\mathrm{Tr}(\sigma_u(r)) = (n-1)\frac{1}{r} - \frac{1}{3}Cr - \frac{1}{45}Hr^3 - \frac{1}{15120}Lr^5 + O(r^7)$$

(5)

for $r \downarrow 0$ and all $u \in TM$ with $|u| = 1$. Note that Proposition 2.1(i) just says that the Einstein constant of M is C; that is, $\mathrm{Ric} = C\mathrm{Id}$ on each $T_p M$. Recall that the Ricci operator is defined by $\langle \mathrm{Ric}(x), y \rangle = \mathrm{Tr}(R(x, .)y)$ for all $x, y \in T_p M$ and all $p \in M$. From Proposition 2.1 one can further derive:

Proposition 2.2 (see [2], Chapter 6). *If M is harmonic, then for the above constants C, H, L and each $p \in M$:*

(i) $\langle R(x, .)., R(y, .). \rangle = \frac{2}{3}((n+2)H - C^2)\langle x, y \rangle$ *for all $x, y \in T_p M$; in particular:*

(ii) $|R|_p^2 = \frac{2}{3}n((n+2)H - C^2)$.

(iii) $32\left(nC^3 + \frac{9}{2}C|R|_p^2 + \frac{7}{2}\hat{R}(p) - \mathring{R}(p)\right) - 27|\nabla R|_p^2 = n(n+2)(n+4)L$.

Here, the functions $\hat{R}, \mathring{R} \in C^\infty(M)$ are certain curvature invariants of order six which are defined as follows: If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_p M$ and $R_{ijk\ell} := \langle R(e_i, e_j)e_k, e_\ell \rangle$, then

$$\mathring{R}(p) := \sum_{i,j,k,\ell,a,b} R_{ijk\ell} R_{ja\ell b} R_{aibk}, \quad \hat{R}(p) := \sum_{i,j,k,\ell,a,b} R_{ijk\ell} R_{k\ell ab} R_{abij}.$$

Note that the term nC^3 in Proposition 2.2(iii) reads nC^2 in the corresponding equation 6.67 in [2], but this was obviously a misprint (note that curvature terms of different order cannot occur here); see also formula (3.1) in [22].

Proposition 2.2(iii) will be used in Section 4, together with the following formula which actually holds in any Einstein manifold; see formula (6-7) in [14] or formula (11.3) in [12]:

$$(6) \qquad\qquad -\frac{1}{2}\Delta(|R|^2) = 2C|R|^2 - \hat{R} - 4\mathring{R} + |\nabla R|^2,$$

where Δ denotes the Laplace operator on functions, that is, $\Delta f = -\sum_i (e_i(e_i f) - (\nabla_{e_i} e_i)f)$ for local orthonormal frames $\{e_1, \ldots, e_n\}$. (Again, there is a misprint in two of the coefficients in the corresponding formula 6.65 in [2].) If M is harmonic, then the left hand side of (6) is zero by Proposition 2.2(ii). Finally, we recall the following well-known observations which will be used in Section 4:

Remark 2.3. Let M be an n-dimensional harmonic space with volume density function θ as above.

(i) For any $p \in M$, the volume of the geodesic sphere $S_r(p)$ with $0 < r < i(p)$ equals the volume ω_{n-1} of the standard unit sphere S^{n-1} in $\mathbb{R}^n$ multiplied by the factor

$$v(r) := r^{n-1}\theta(r)$$

Note that $v(r) = v_u(r)$ for each unit vector $u \in TM$, where v_u is the function defined in (1). The function v determines the volume growth function v'/v of the geodesic spheres, and thus it determines, by (2), the function $\mathrm{Tr}(\sigma_u(r))$ (which is independent of u). By (5), the function which associates to small values of r the volume of geodesic spheres of radius r in a given harmonic space M determines the constants C, H, L (and of course n) associated with M.

(ii) Let $\mathrm{scal} = nC$ denote the scalar curvature of M. Let $p \in M$, fix some $r \in (0, i(p))$, and let scal^S denote the scalar curvature function of $S_r(p)$. A routine calculation using the Gauss equation shows that for each unit vector $u \in T_p M$ we have

$$\mathrm{scal}^S(\exp(ru)) = \mathrm{scal} - 2\langle \mathrm{Ric}(\dot{\gamma}_u(r)), \dot{\gamma}_u(r) \rangle + (\mathrm{Tr}(\sigma_u(r)))^2 - \mathrm{Tr}(\sigma_u(r)^2)$$

which by the Einstein condition and equations (2) and (3) implies

$$\mathrm{scal}^S(\exp(ru)) = (n-2)C + (v'(r)/v(r))^2 + \mathrm{Tr}(\sigma'_u(r)) + \mathrm{Tr}(R_{\dot{\gamma}_u(r)})$$
$$= (n-2)C + (v'(r)/v(r))^2 + (v'/v)'(r) + C = (n-1)C + v''(r)/v(r).$$

Therefore, geodesic spheres in M have constant scalar curvature, and the respective constant depends only on the radius, not on the midpoint. Finally, using (i) one concludes that the function which associates to small values of r the scalar curvature of geodesic spheres of radius r is determined already by the function which associates to small values of r the volume of geodesic spheres of radius r.

Remark 2.4. As mentioned in the Introduction, the aim of this paper is to show that in harmonic spaces, the heat invariants $a_0(S_r(p)) = \mathrm{vol}(S_r(p))$ and $a_2(S_r(p))$, viewed as functions of r, together determine $|\nabla R|_p^2$. This is not the case for a_0 alone, as manifested by certain pairs of Damek-Ricci spaces. A Damek-Ricci space AN is a certain type of solvable Lie groups with left invariant metric, namely, the standard 1-dimensional solvable extension of a simply connected Riemannian nilmanifold N of Heisenberg type. The volume density function of AN is radial and depends only on the dimensions of N and its center [6]; see also the book [1]. Within the class of Damek-Ricci spaces, there exist pairs of symmetric spaces AN and locally nonsymmetric spaces AN' where N and N' have the same dimension and so do their centers. (In fact, certain such pairs AN and AN' were the ambient manifolds used by Szabó in [20], [21]; recall the Introduction.) In particular, geodesic spheres of the same radius in AN and AN' have the same volume. This shows that in harmonic spaces, the function $r \mapsto \mathrm{vol}(S_r(p))$ alone does not determine $|\nabla R|_p^2$. In turn, Remark 2.3 shows that in harmonic spaces, the function $r \mapsto a_0(S_r(p)) = \mathrm{vol}(S_r(p))$ already determines the function $r \mapsto a_1(S_r(p)) = \frac{1}{6} \int_{S_r(p)} \mathrm{scal}^S \, dvol_{S_r(p)}$. Therefore, we need to consider $a_2(S_r(p))$. The next section gives some necessary preparations for this.

3. Radial expansions of $|\mathrm{Ric}|^2$ and $|R|^2$ for geodesic spheres in harmonic spaces

In this section we will describe a certain coefficient in the radial power series expansions of the curvature invariants $|\mathrm{Ric}|^2$ and $|R|^2$ of geodesic spheres in harmonic spaces. First we need the following lemma.

Lemma 3.1. *Let M be an n-dimensional harmonic space, and let C and H be the constants from Proposition 2.1. Let $p \in M$, and let $S := S_r(p)$ be a geodesic sphere around p with radius $r \in (0, i(p))$, endowed with the induced Riemannian metric. Let u be a unit vector in T_pM, let $\sigma := \sigma_u(r)$ be as in Section 2, and write $\sigma' := \sigma'_u(r)$. Let R^S and Ric^S denote the curvature tensor, resp. the Ricci operator, of S. Then in the point $q := \exp(ru) \in S$ we have:*

(i) $|\mathrm{Ric}^S|_q^2 = (n-1)C^2 + 2C(\mathrm{Tr}(\sigma))^2 + (\mathrm{Tr}(\sigma))^2 \mathrm{Tr}(\sigma^2) + 2C\mathrm{Tr}(\sigma') + 2\mathrm{Tr}(\sigma)\mathrm{Tr}(\sigma\sigma') + \mathrm{Tr}(\sigma'\sigma'),$

(ii) $|R^S|_q^2 = \frac{2}{3}(n-4)\big((n+2)H - C^2\big) + 4H + 2(\mathrm{Tr}(\sigma^2))^2 - 2\mathrm{Tr}(\sigma^4) + 4\sum_{i=1}^{n} \mathrm{Tr}\big(\sigma \circ R(e_i, \cdot)\sigma e_i\big),$

   where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_qM.

Proof. (i) Let ν be the outward pointing radial unit vector field as in Section 2. From the Gauss equation one easily derives the following formula whose analog is valid for submanifolds of codimension one in arbitrary Riemannian manifolds:

$$\mathrm{Ric}_q^S = (\mathrm{Ric} - R_{\nu_q} + \mathrm{Tr}(\sigma)\sigma - \sigma^2)|_{T_qS}$$

Using the Einstein condition and the Riccati equation (3), this formula becomes in our situation:

$$\operatorname{Ric}_q^S = (C\operatorname{Id} + \operatorname{Tr}(\sigma)\sigma + \sigma')|_{T_qS}$$

(see also [15], p. 67). Now one obtains the desired formula immediately, keeping in mind that both σ and σ' are symmetric and annihilate ν_q.

(ii) Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_qM such that $e_1 = \nu_q$. For all $i, j, k, \ell \in \{2, \ldots, n\}$ we have by the Gauss equation (recall our sign convention for R):

$$\langle R^S(e_i, e_j)e_k, e_\ell \rangle = \langle R(e_i, e_j)e_k, e_\ell \rangle + \langle \sigma e_i, e_k \rangle \langle \sigma e_j, e_\ell \rangle - \langle \sigma e_j, e_k \rangle \langle \sigma e_i, e_\ell \rangle.$$

Squaring both sides and forming the sum over i, j, k, ℓ, while recalling that σ is symmetric and annihilates e_1, we get

$$|R^S|_q^2 = \sum_{i,j,k,\ell=2}^{n} \langle R(e_i, e_j)e_k, e_\ell \rangle^2 + |\sigma|^2|\sigma|^2 + |\sigma|^2|\sigma|^2$$

$$- 2|\sigma^2|^2 + 2\sum_{i,j=1}^{n} \langle R(e_i, e_j)\sigma e_i, \sigma e_j \rangle - 2\sum_{i,j=1}^{n} \langle R(e_i, e_j)\sigma e_j, \sigma e_i \rangle$$

$$= \sum_{i,j,k,\ell=2}^{n} \langle R(e_i, e_j)e_k, e_\ell \rangle^2 + 2(\operatorname{Tr}(\sigma^2))^2 - 2\operatorname{Tr}(\sigma^4) + 4\sum_{i=1}^{n} \operatorname{Tr}(\sigma \circ R(e_i, .)\sigma e_i).$$

The desired formula now follows from the fact that the first sum on the right hand side is equal to $|R|_q^2 - 4|R(e_1, .).|^2 + 4|R_{e_1}|^2$ which by Proposition 2.2(i), (ii) and Proposition 2.1(iii) becomes $\frac{2}{3}(n-4)\big((n+2)H - C^2\big) + 4H$. $\qquad\square$

Using the radial power series expansion of σ together with the previous lemma, we will make conclusions concerning the first few coefficients of the radial expansions of $|\operatorname{Ric}^S|^2$ and $|R^S|^2$. The following proposition will be the key of the proof of the Main Theorem 1.1. Actually we will use only the statements about α_2 and β_2 in this proposition.

Proposition 3.2. *Let M be an n-dimensional harmonic space, and let C, H, and L be the constants from Proposition 2.1. Let $p \in M$, and let u be a unit vector in T_pM. Then*

$$|\operatorname{Ric}^{S_r(p)}|_{\exp(ru)}^2 = \alpha_{-4}r^{-4} + \alpha_{-2}r^{-2} + \alpha_0 + \alpha_2(u)r^2 + O(r^3) \ and$$

$$|R^{S_r(p)}|_{\exp(ru)}^2 = \beta_{-4}r^{-4} + \beta_{-2}r^{-2} + \beta_0 + \beta_2(u)r^2 + O(r^3)$$

for $r \downarrow 0$, where the coefficients α_i and β_i for $i \in \{-4, -2, 0\}$ are constants depending only on n, C, and H. Moreover,

$$\alpha_2(u) = \hat{\alpha}_2 + \frac{1}{16}\operatorname{Tr}(R_u' R_u') \ and$$

$$\beta_2(u) = \hat{\beta}_2 + \frac{4}{9}\sum_{i=1}^{n} \operatorname{Tr}\big(R_u \circ R(e_i, .)R_u e_i\big),$$

where $\hat{\alpha}_2$ and $\hat{\beta}_2$ are constants depending only on n, C, H, and L, and where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM.

Proof. We use Lemma 3.1 together with the power series expansions (4), (5) of $\sigma := \sigma_u(r)$ and $\operatorname{Tr}(\sigma)$. Let us first consider $|\operatorname{Ric}^{S_r(p)}|_{\exp(ru)}^2$ and the individual contributions of the nonconstant

terms in Lemma 3.1(i) to its expansion. By (5) we have

$$(\mathrm{Tr}(\sigma))^2 = \left((n-1)\frac{1}{r} - \frac{1}{3}Cr - \frac{1}{45}Hr^3 - \frac{1}{15120}Lr^5\right)^2 + O(r^6)$$

for $r \downarrow 0$. Moreover, from the expansion (4) and Proposition 2.1 one gets

$$(7) \qquad \mathrm{Tr}(\sigma^2) = (n-1)\frac{1}{r^2} - \frac{2}{3}C + \frac{1}{15}Hr^2 + \frac{1}{3024}Lr^4 + O(r^5).$$

Further,

$$\mathrm{Tr}(\sigma') = \frac{d}{dr}\mathrm{Tr}(\sigma) = -(n-1)\frac{1}{r^2} - \frac{1}{3}C - \frac{1}{15}Hr^2 + O(r^4)$$

by (5), and

$$2\mathrm{Tr}(\sigma\sigma') = \frac{d}{dr}\mathrm{Tr}(\sigma^2) = -2(n-1)\frac{1}{r^3} + \frac{2}{15}Hr + \frac{1}{756}Lr^3 + O(r^4)$$

by (7). Using these expansions and (5), one easily checks that each of the the four individual terms $2C(\mathrm{Tr}(\sigma))^2$, $(\mathrm{Tr}(\sigma))^2\mathrm{Tr}(\sigma^2)$, $2C\mathrm{Tr}(\sigma')$, and $2\mathrm{Tr}(\sigma)\mathrm{Tr}(\sigma\sigma')$ appearing on the right hand side of Lemma 3.1(i) has the property that the corresponding coefficients of r^{-4}, r^{-2}, r^0 depend only on n, C, H, the coefficient of r^2 depends only on n, C, H, L, and the coefficients of r^{-3}, r^{-1}, r vanish.

It remains to consider the term $\mathrm{Tr}(\sigma'\sigma')$ in Lemma 3.1(i). From (4) we get

$$\begin{aligned}
P_{\gamma_u}^{r,0} \circ \sigma' \circ P_{\gamma_u}^{0,r} = & -\frac{1}{r^2}I_u - \frac{1}{3}R_u - \frac{r}{2}R_u' - \left(\frac{3}{10}R_u'' + \frac{1}{15}R_uR_u\right)r^2 \\
& - \left(\frac{1}{9}R_u''' + \frac{1}{18}R_uR_u' + \frac{1}{18}R_u'R_u\right)r^3 \\
& - \left(\frac{5}{168}R_u^{(4)} + \frac{1}{42}R_uR_u'' + \frac{1}{42}R_u''R_u + \frac{5}{112}R_u'R_u' + \frac{2}{189}R_uR_uR_u\right)r^4 \\
& + O(r^5)
\end{aligned}$$

and thereby, using Proposition 2.1:

$$\begin{aligned}
\mathrm{Tr}(\sigma'\sigma') = & (n-1)\frac{1}{r^4} + \frac{2}{3}\frac{C}{r^2} + \frac{11}{45}H \\
& + \left(\left(-\frac{2}{21} + \frac{5}{56} - \frac{1}{5} + \frac{1}{4}\right)\mathrm{Tr}(R_u'R_u') + \left(\frac{4}{189} + \frac{2}{45}\right)\mathrm{Tr}(R_uR_uR_u)\right)r^2 + O(r^3).
\end{aligned}$$

The coefficient of r^2 in the latter expansion is

$$\frac{37}{840}\mathrm{Tr}(R_u'R_u') + \frac{62}{945}\mathrm{Tr}(R_uR_uR_u)$$

which by Proposition 2.1(vi) turns out to be

$$\frac{62}{32\cdot945}L + \left(\frac{37}{840} + \frac{9\cdot62}{32\cdot945}\right)\mathrm{Tr}(R_u'R_u') = \frac{31}{15120}L + \frac{1}{16}\mathrm{Tr}(R_u'R_u').$$

This concludes the proof of the statements concerning the expansion of $|\mathrm{Ric}^{S_r(p)}|^2_{\exp(ru)}$.

We now turn to $|R^{S_r(p)}|^2_{\exp(ru)}$ and study the individual contributions of the nonconstant terms in Lemma 3.1(ii) to its expansion. Squaring (7), we see that in the expansion of the term $2(\mathrm{Tr}(\sigma^2))^2$ the coefficients of r^{-4}, r^{-2}, r^0 depend only on n, C, H, the coefficient of r^2 depends only on n, C, H, L, and the coefficients of r^{-3}, r^{-1}, r vanish.

Regarding the term $-2\mathrm{Tr}(\sigma^4)$ we obtain from (4):

$$
P_{\gamma_u}^{r,0} \circ \sigma^4 \circ P_{\gamma_u}^{0,r} = \frac{1}{r^4}I_u - \frac{4}{3r^2}R_u - \frac{1}{r}R_u' + \left(-\frac{2}{5}R_u'' + \frac{26}{45}R_u^2\right) + \left(-\frac{1}{9}R_u''' + \frac{4}{9}(R_u R_u' + R_u' R_u)\right)r
$$
$$
+ \left(-\frac{1}{42}R_u^{(4)} + \left(-\frac{2}{105} + \frac{1}{5}\right)(R_u'' R_u + R_u R_u'') + \left(-\frac{1}{28} + \frac{3}{8}\right)R_u' R_u' + \left(-\frac{8}{945} + \frac{4}{45} - \frac{4}{27}\right)R_u^3\right)r^2
$$
$$
+ O(r^3)
$$

for $r \downarrow 0$. Using Proposition 2.1 we get

$$
-2\mathrm{Tr}(\sigma^4) = -2(n-1)\frac{1}{r^4} + \frac{8}{3}\frac{C}{r^2} - \frac{52}{45}H
$$
$$
+ \left(\left(-\frac{8}{105} + \frac{4}{5} + \frac{1}{14} - \frac{3}{4}\right)\mathrm{Tr}(R_u' R_u') + \left(\frac{16}{945} - \frac{8}{45} + \frac{8}{27}\right)\mathrm{Tr}(R_u R_u R_u)\right)r^2 + O(r^3).
$$

The coefficient of r^2 in the latter expansion is

$$
\frac{19}{420}\mathrm{Tr}(R_u' R_u') + \frac{128}{945}\mathrm{Tr}(R_u R_u R_u)
$$

which by Proposition 2.1(vi) equals

$$
(8) \qquad \frac{128}{32 \cdot 945}L + \left(\frac{19}{420} + \frac{9 \cdot 128}{32 \cdot 945}\right)\mathrm{Tr}(R_u' R_u') = \frac{4}{945}L + \frac{1}{12}\mathrm{Tr}(R_u' R_u').
$$

It remains to consider the term $4\sum_{i=1}^{n}\mathrm{Tr}\bigl(\sigma \circ R(e_i, .)\sigma e_i\bigr)$ in Lemma 3.1(ii). We make some preliminary observations. For $k \in \mathbb{N}_0$, let $R^{(k)}$, resp. $\mathrm{Ric}^{(k)}$ denote the k-th covariant derivative of the curvature tensor, resp. the Ricci operator, along γ_u at $r = 0$. We will use the the Taylor series expansion of the Riemannian curvature tensor along γ_u (recall that M is analytic):

$$
(9) \qquad P_{\gamma_u}^{r,0} \circ R_{\gamma_u(r)} \circ P_{\gamma_u}^{0,r} = \sum_{k=0}^{\infty} \frac{r^k}{k!}R^{(k)}
$$

Moreover, $\mathrm{Ric}^{(k)} = 0$ for $k \geq 1$ since M is Einstein. Note that $\mathrm{Ric}|_{T_pM} = \sum_{i=1}^{n} R_{e_i}$ and similarly on each $T_{\gamma_u(r)}M$ if we extend $\{e_1, \ldots, e_n\}$ parallelly along γ_u. For any $k \in \mathbb{N}_0$ we have, using Proposition 2.1:

$$
(10) \qquad \sum_{i=1}^{n}\mathrm{Tr}\bigl(I_u \circ R^{(k)}(e_i, .)I_u e_i\bigr) = \mathrm{Tr}(I_u \circ \mathrm{Ric}^{(k)}) - \mathrm{Tr}(I_u \circ R_u^{(k)})
$$
$$
= \mathrm{Tr}(\mathrm{Ric}^{(k)}) - \langle \mathrm{Ric}^{(k)}u, u\rangle - \mathrm{Tr}(R_u^{(k)}) = \begin{cases}(n-2)C, & k = 0, \\ 0, & k \geq 1.\end{cases}
$$

Moreover,

$$
(11) \qquad \sum_{i=1}^{n}\mathrm{Tr}\bigl(R_u \circ R^{(k)}(e_i, .)I_u e_i\bigr) = \mathrm{Tr}(R_u \circ \mathrm{Ric}^{(k)}) - \mathrm{Tr}(R_u R_u^{(k)})
$$
$$
= \begin{cases}C^2 - H, & k = 0, \\ -\mathrm{Tr}(R_u R_u^{(k)}), & k \geq 1,\end{cases}
$$

$$(12) \quad \sum_{i=1}^{n} \mathrm{Tr}\big(R'_u \circ R^{(k)}(e_i, \,.\,)I_u e_i\big) = \mathrm{Tr}(R'_u \circ \mathrm{Ric}^{(k)}) - \mathrm{Tr}(R'_u R_u^{(k)})$$

$$= \begin{cases} 0, & k = 0, \\ -\mathrm{Tr}(R'_u R_u^{(k)}), & k \geq 1, \end{cases}$$

$$(13) \quad \sum_{i=1}^{n} \mathrm{Tr}\big(R''_u \circ R(e_i, \,.\,)I_u e_i\big) = \mathrm{Tr}\big(R''_u \circ \mathrm{Ric}\big) - \mathrm{Tr}(R''_u R_u) = 0 + \mathrm{Tr}(R'_u R'_u),$$

$$(14) \quad \sum_{i=1}^{n} \mathrm{Tr}\big(R_u R_u \circ R(e_i, \,.\,)I_u e_i\big) = \mathrm{Tr}(R_u R_u \circ \mathrm{Ric}) - \mathrm{Tr}(R_u R_u R_u) = CH - \mathrm{Tr}(R_u R_u R_u).$$

Note that for any pair of symmetric endomorphisms F, G of $T_p M$ we have

$$(15) \quad \sum_{i=1}^{n} \mathrm{Tr}\big(F \circ R^{(k)}(e_i, \,.\,)G e_i\big) = \sum_{i=1}^{n} \mathrm{Tr}\big(G \circ R^{(k)}(e_i, \,.\,)F e_i\big)$$

by the symmetries of the curvature operator. Keeping the expansions (4) and (9) in mind, we see that the expression in (10) contributes only to the coefficient of r^{-2} in the expansion of $\sum_{i=1}^{n} \mathrm{Tr}\big(\sigma \circ R(e_i, \,.\,)\sigma e_i\big)$, the expression in (11) contributes to the coefficients of r^0 and r^2 (and higher order), the expressions in (12), (13), (14) contribute to the coefficient of r^2 (and higher order). The only additional contribution to the coefficient of r^2 is given by the sum of $\mathrm{Tr}\big(R_u \circ R(e_i, \,.\,)R_u e_i\big)$. Recalling (15) (and multiplying $R^{(k)}$ by $1/k!$), we obtain from (4), 2.1(iv), (9), and the above observations:

$$4 \sum_{i=1}^{n} \mathrm{Tr}\big(\sigma \circ R(e_i, \,.\,)\sigma e_i\big) = 4\Big((n-2)\frac{C}{r^2} - \frac{2}{3}(C^2 - H)$$

$$+ \Big[\frac{2}{3 \cdot 2!}\mathrm{Tr}(R_u R''_u) + \Big(\frac{2}{4} - \frac{2}{10}\Big)\mathrm{Tr}(R'_u R'_u) - \frac{2}{45}CH + \frac{2}{45}\mathrm{Tr}(R_u R_u R_u)$$

$$+ \frac{1}{9}\sum_{i=1}^{n}\mathrm{Tr}\big(R_u \circ R(e_i, \,.\,)R_u e_i\big)\Big]r^2\Big) + O(r^3).$$

By Proposition 2.1(v), the coefficient of r^2 in the latter expansion is

$$-\frac{8}{45}CH - \frac{2}{15}\mathrm{Tr}(R'_u R'_u) + \frac{8}{45}\mathrm{Tr}(R_u R_u R_u) + \frac{4}{9}\sum_{i=1}^{n}\mathrm{Tr}\big(R_u \circ R(e_i, \,.\,)R_u e_i\big)$$

By Proposition 2.1(vi), the two terms involving $\mathrm{Tr}(R'_u R'_u)$ and $\mathrm{Tr}(R_u R_u R_u)$ become

$$\frac{8}{32 \cdot 45}L + \Big(-\frac{2}{15} + \frac{9 \cdot 8}{32 \cdot 45}\Big)\mathrm{Tr}(R'_u R'_u) = \frac{1}{180}L - \frac{1}{12}\mathrm{Tr}(R'_u R'_u).$$

Combining this with the result for the r^2-coefficient of $-2\mathrm{Tr}(\sigma^4)$ from (8), we conclude that the terms involving $\mathrm{Tr}(R'_u R'_u)$ in the coefficient of r^2 in the power series expansion of $|R^{S_r(p)}|^2_{\exp(ru)}$ cancel each other, and the only remaining term apart from those which depend solely on n, C, H, L is $\frac{4}{9}\sum_{i=1}^{n}\mathrm{Tr}\big(R_u \circ R(e_i, \,.\,)R_u e_i\big)$, as claimed. $\qquad\square$

Remark 3.3. For the purpose of the proof of the Main Theorem 1.1 in Section 4, which we will perform using the heat invariants $a_0(S_r(p)) = \mathrm{vol}(S_r(p))$ and $a_2(S_r(p)) = \frac{1}{360}\int_{S_r(p)}(5(\mathrm{scal}^S)^2 - 2|\mathrm{Ric}^S|^2 + 2|R^S|^2)\,dvol_{S_r(p)}$, we would actually not have needed the exact statement of the previous proposition – which might, however, be interesting in its own right. Rather, we could have restricted our attention to the term $\mathrm{Tr}(\sigma'\sigma')$ in the expression of $|\mathrm{Ric}^S|^2_{\gamma_u(r)}$ in Lemma 3.1(i), and to the last two terms in the expression of $|R^S|^2_{\gamma_u(r)}$ in Lemma 3.1(ii). In fact, even without the explicit calculation of the expansion of the other terms, one easily sees that those are determined by the volume function $r \mapsto a_0(S_r(p)) = \mathrm{vol}(S_r(p))$ of the geodesic spheres (which is just the function v multiplied by the volume of the standard unit sphere, see Remark 4.1 below). More precisely, in the spirit of Remark 2.3 we obtain

$$2C(\mathrm{Tr}(\sigma))^2 = 2C(v'/v)^2,$$
$$(\mathrm{Tr}(\sigma))^2\mathrm{Tr}(\sigma^2) = (v'/v)^2(-(v'/v)' - C),$$
$$2C\mathrm{Tr}(\sigma') = 2C\mathrm{Tr}(\sigma)' = 2C(v'/v)',$$
$$2\mathrm{Tr}(\sigma)\mathrm{Tr}(\sigma\sigma') = 2v'/v \cdot \tfrac{1}{2}(\mathrm{Tr}(\sigma^2))' = v'/v \cdot (-(v'/v)''),$$
$$2(\mathrm{Tr}(\sigma^2))^2 = 2(-(v'/v)' - C)^2.$$

4. Proof of the Main Theorem

In this section we will first derive an integrated version of Proposition 3.2. Using this and the heat invariants a_0, a_1, a_2 of geodesic spheres in harmonic spaces we will then prove our Main Theorem 1.1. We need the following general remark on mean values.

Remark 4.1. In any harmonic space M, the average (or mean value) of a smooth function f on a geodesic sphere $S_r(p)$ (with $0 < r < i(p)$) is the same as the average of $f(\exp(r\,.))$ over the unit sphere $S_1(0_p)$ in T_pM. More explicitly: Let ω_{n-1} denote the volume of the $(n-1)$-dimensional standard sphere. In particular, ω_{n-1} is the volume of $S_1(0_p)$. Recall from Section 2 that $\theta(r) = \theta_u(r)$ is independent of u (and even of p) by harmonicity. We have

$$\mathrm{vol}(S_r(p)) = r^{n-1}\theta(r)\omega_{n-1} = v(r)\omega_{n-1}\,,$$

and for any smooth function f on $S_r(p)$,

$$\frac{1}{\mathrm{vol}(S_r(p))}\int_{S_r(p)} f\,dvol_{S_r(p)} = \frac{1}{v(r)\omega_{n-1}}\int_{S_1(0_p)} f(\exp(ru))v(r)\,du$$
$$= \frac{1}{\omega_{n-1}}\int_{S_1(0_p)} f(\exp(ru))\,du.$$

Now we can give an "integrated" version of Proposition 3.2.

Proposition 4.2. *Let M be an n-dimensional harmonic space, and let C, H, and L be the constants from Proposition 2.1. Let $p \in M$. Then*

$$\frac{1}{\mathrm{vol}(S_r(p))}\int_{S_r(p)} |\mathrm{Ric}^{S_r(p)}|^2\,dvol_{S_r(p)} = \alpha_{-4}r^{-4} + \alpha_{-2}r^{-2} + \alpha_0 + \overline{\alpha}_2 r^2 + O(r^3) \text{ and}$$
$$\frac{1}{\mathrm{vol}(S_r(p))}\int_{S_r(p)} |R^{S_r(p)}|^2\,dvol_{S_r(p)} = \beta_{-4}r^{-4} + \beta_{-2}r^{-2} + \beta_0 + \overline{\beta}_2 r^2 + O(r^3)$$

for $r \downarrow 0$, where the coefficients α_i and β_i for $i \in \{-4, -2, 0\}$ are the constants from Proposition 3.2 depending only on n, C, and H. Moreover,

$$\overline{\alpha}_2 = \tilde{\alpha}_2 + \frac{3}{16n(n+2)(n+4)}|\nabla R|_p^2 \quad and$$

$$\overline{\beta}_2 = \tilde{\beta}_2 + \frac{1}{8n(n+2)}|\nabla R|_p^2,$$

where $\tilde{\alpha}_2$ and $\tilde{\beta}_2$ are constants depending only on n, C, H, and L.

Proof. For any unit vector u in $T_p M$, let $\alpha_2(u)$ and $\beta_2(u)$ be the coefficients from Proposition 3.2. Using that proposition and Remark 4.1, we only need to show that

$$\overline{\alpha}_2 := \frac{1}{\omega_{n-1}} \int_{S_1(0_p)} \alpha_2(u) \, du = \hat{\alpha}_2 + \frac{1}{\omega_{n-1}} \int_{S_1(0_p)} \frac{1}{16} \mathrm{Tr}(R'_u R'_u) \, du \quad and$$

$$\overline{\beta}_2 := \frac{1}{\omega_{n-1}} \int_{S_1(0_p)} \beta_2(u) \, du = \hat{\beta}_2 + \frac{1}{\omega_{n-1}} \int_{S_1(0_p)} \frac{4}{9} \sum_{i=1}^{n} \mathrm{Tr}\big(R_u \circ R(e_i, \, . \,)R_u e_i\big) \, du$$

are of the claimed form, where $\hat{\alpha}_2, \hat{\beta}_2$ are as in Proposition 3.2. For $\overline{\alpha}_2$ this follows immediately (with $\tilde{\alpha}_2 := \hat{\alpha}_2$) from the following formula (see the proof of Theorem 5.7 of [15]; details of the computation can be found on p. 170 of [12]):

$$(16) \qquad \int_{S_1(0_p)} \mathrm{Tr}(R'_u R'_u) \, du = \frac{3\omega_{n-1}}{n(n+2)(n+4)}|\nabla R|_p^2$$

This confirms the statement concerning $\overline{\alpha}_2$.

We now consider $\overline{\beta}_2$. Writing $u = \sum_{i=1}^{n} u_i e_i$ and $R_{ijk\ell} = \langle R(e_i, e_j)e_k, e_\ell \rangle$ we have

$$(17) \qquad \begin{aligned} \sum_{i=1}^{n} \mathrm{Tr}\big(R_u \circ R(e_i, \, . \,)R_u e_i\big) &= \sum_{i,j,k,\ell=1}^{n} \langle R(e_i, e_j)e_k, e_\ell \rangle \langle R_u e_i, e_k \rangle \langle R_u e_j, e_\ell \rangle \\ &= \sum_{a,b,c,d=1}^{n} \Big[\sum_{i,j,k,\ell=1}^{n} R_{ijk\ell} R_{aibk} R_{cjd\ell} \Big] u_a u_b u_c u_d. \end{aligned}$$

Note that the integral of $u_a u_b u_c u_d$ over $S_1(0_p)$ is zero whenever $\{a, b, c, d\}$ contains at least three different elements. Abbreviating $A_{abcd} := \sum_{i,j,k,\ell=1}^{n} R_{ijk\ell} R_{aibk} R_{cjd\ell}$ we have, using the Einstein condition and recalling the definition of $\hat{R}$ and $\mathring{R}$ from Section 2:

$$\sum_{a,b=1}^{n} A_{aabb} = \sum_{a,b,i,j,k,\ell=1}^{n} R_{ijk\ell} R_{aiak} R_{bjb\ell} = C^2 \sum_{i,j,k,\ell=1}^{n} R_{ijk\ell}\delta_{ik}\delta_{j\ell} = C^2 \sum_{i,j=1}^{n} R_{ijij} = nC^3,$$

$$\sum_{a,b=1}^{n} A_{abab} = \sum_{a,b,i,j,k,\ell=1}^{n} R_{ijk\ell} R_{aibk} R_{ajb\ell} = \sum_{a,b,i,j,k,\ell=1}^{n} R_{ijk\ell} R_{aibk} R_{ja\ell b} = \mathring{R}(p),$$

$$\sum_{a,b=1}^{n} A_{abba} = \sum_{a,b,i,j,k,\ell=1}^{n} R_{ijk\ell} R_{aibk} R_{bja\ell} = \sum_{a,b,i,j,k,\ell=1}^{n} R_{ijk\ell} R_{aibk} R_{jb\ell a} = \mathring{R}(p) - \frac{1}{4}\hat{R}(p),$$

where for the last equality we have used formula (2.7)(vi) of [17]; see also formula (2.15) of [12]. Let $S^{n-1} \subset R^n$ denote the $(n-1)$-dimensional standard sphere. Note that $\int_{S^{n-1}} u_1^2 u_2^2 \, du = \frac{\omega_{n-1}}{n(n+2)}$

and $\int_{S^{n-1}} u_1^4 \, du = \frac{3\omega_{n-1}}{n(n+2)}$. From the above equations and (17) we thus obtain

$$\int_{S_1(0_p)} \sum_{i=1}^n \mathrm{Tr}\big(R_u \circ R(e_i, \, . \,) R_u e_i\big) \, du$$

$$= \sum_{\substack{a,b\in\{1,\dots,n\} \\ a\neq b}} [A_{aabb} + A_{abab} + A_{abba}] \int_{S^{n-1}} u_1^2 u_2^2 \, du + \sum_{a=1}^n A_{aaaa} \int_{S^{n-1}} u_1^4 \, du$$

$$= \sum_{a,b=1}^n [A_{aabb} + A_{abab} + A_{abba}] \frac{\omega_{n-1}}{n(n+2)} = \big(nC^3 + 2\mathring{R}(p) - \tfrac{1}{4}\hat{R}(p)\big)\frac{\omega_{n-1}}{n(n+2)}$$

Hence,

$$(18) \quad \overline{\beta}_2 = \hat{\beta}_2 + \frac{4}{9}\big(nC^3 + 2\mathring{R}(p) - \tfrac{1}{4}\hat{R}(p)\big)\frac{1}{n(n+2)} = \hat{\beta}_2 + \frac{4C^3}{9(n+2)} + \big(\tfrac{8}{9}\mathring{R}(p) - \tfrac{1}{9}\hat{R}(p)\big)\frac{1}{n(n+2)}$$

Recall from Proposition 2.2(ii), (iii) and equation (6) that

$$112\hat{R}(p) - 32\mathring{R}(p) = 27|\nabla R|_p^2 + \text{some constant depending only on } n, C, H, L \text{ and}$$

$$\hat{R}(p) + \; 4\mathring{R}(p) = \quad |\nabla R|_p^2 + \text{some constant depending only on } n, C, H,$$

using which one easily computes that

$$-\frac{1}{9}\hat{R}(p) + \frac{8}{9}\mathring{R}(p) = \frac{1}{8}|\nabla R|_p^2 + \text{some constant depending only on } n, C, H, L.$$

Thus we conclude from (18):

$$\overline{\beta}_2 = \tilde{\beta}_2 + \frac{1}{8n(n+2)}|\nabla R|_p^2,$$

where $\tilde{\beta}_2$ is a constant depending only on n, C, H, L. $\square$

Proof of the Main Theorem 1.1:
Let M_1, M_2 be harmonic spaces, $p_1 \in M_1$, $p_2 \in M_2$, and assume there exists ε in the interval $(0, \min\{i(p_1), i(p_2)\})$ such that for each $0 < r < \varepsilon$ the geodesic spheres $S_r(p_1)$ and $S_r(p_2)$ are isospectral. Then $\dim M_1 =: n = \dim M_2$, and the heat invariants of the geodesic spheres coincide:

$$a_k(S_r(p_1)) = a_k(S_r(p_2))$$

for each $r \in (0, \varepsilon)$ and all $k \in \mathbb{N}_0$. We want to deduce that $|\nabla R|_{p_1}^2 = |\nabla R|_{p_2}^2$. Actually this will follow using just a_0 and a_2.

Reformulating the problem, let M be an n-dimensional harmonic space and $p \in M$. We want to show that for any $\varepsilon \in (0, i(p))$, the two functions

$$\varphi_k : (0, \varepsilon) \ni r \mapsto a_k(S_r(p)) \in \mathbb{R}$$

with $k \in \{0, 2\}$ together determine the value of $|\nabla R|_p^2$. By Remark 2.3(i), the function

$$\varphi_0 : r \mapsto a_0(S_r(p)) = \mathrm{vol}(S_r(p)) = v(r)\omega_{n-1}$$

determines the constants C, H, L associated with M (see Section 2). Recall that the scalar curvature $\mathrm{scal}^S =: \mathrm{scal}^{S_r}$ of $S_r(p)$ is constant on the manifold $S_r(p)$, and that the function $v : (0, \varepsilon) \to \mathbb{R}$ determines, by Remark 2.3(ii), the function $(0, \varepsilon) \ni r \mapsto \mathrm{scal}^{S_r} \in \mathbb{R}$. In particular, the function

$\varphi_0 = v\omega_{n-1}$ also determines the function $(0, \varepsilon) \ni r \mapsto \int_{S_r(p)} (\text{scal}^S)^2 \, dvol_{S_r(p)} = \varphi_0(r) \cdot (\text{scal}^{S_r})^2 \in \mathbb{R}$. By

$$\varphi_2(r) = a_2(S_r(p)) = \frac{1}{360} \int_{S_r(p)} \left(5(\text{scal}^S)^2 - 2|\text{Ric}^S|^2 + 2|R^S|^2 \right) dvol_{S_r(p)}$$

it follows that the functions φ_0 and φ_2 together determine the function

$$(0, \varepsilon) \ni r \mapsto \frac{1}{\text{vol}(S_r(p))} \int_{S_r(p)} \left(|R^S|^2 - |\text{Ric}^S|^2 \right) dvol_{S_r(p)} \in \mathbb{R}.$$

By Proposition 4.2, the r^2-coefficient in the power series expansion of this function is the sum of the term

$$\left(\frac{1}{8n(n+2)} - \frac{3}{16n(n+2)(n+4)} \right) |\nabla R|_p^2 = \frac{2n+5}{16n(n+2)(n+4)} |\nabla R|_p^2$$

and $\tilde{\beta}_2 - \tilde{\alpha}_2$. Recall that the latter is a constant depending only on n, C, H, L, and is thus determined by φ_0. We conclude that the functions φ_0 and φ_2 together determine $|\nabla R|_p^2$, as claimed. $\qquad \square$

5. Geodesic balls

In this section we will prove the following version of the Main Theorem 1.1 for geodesic balls:

Theorem 5.1. *Let M_1 and M_2 be harmonic spaces, and let $p_1 \in M_1$, $p_2 \in M_2$. If there exists $\varepsilon > 0$ such that for each $r \in (0, \varepsilon)$ the geodesic balls $B_r(p_1)$ and $B_r(p_2)$ are Dirichlet isospectral, then $|\nabla R|_{p_1}^2 = |\nabla R|_{p_2}^2$. The same holds if the assumption of Dirichlet isospectrality is replaced by the assumption of Neumann isospectrality.*

This theorem implies the corresponding analog of our Main Corollary 1.2:

Corollary 5.2. *Let M_1 and M_2 be harmonic spaces. Assume that the Dirichlet isospectrality hypothesis of Theorem 5.1 is satisfied for each pair of points $p_1 \in M_1$, $p_2 \in M_2$. Then M_1 is locally symmetric if and only if M_2 is locally symmetric. The same holds if the assumption of Dirichlet isospectrality is replaced by the assumption of Neumann isospectrality.*

For the proof of Theorem 5.1 we will use the heat invariants for manifolds with boundary. Let M be an n-dimensional Riemannian manifold, and let $B \subset M$ be a compact domain with smooth boundary. If Δ denotes the Laplace operator on B with Dirichlet boundary conditions then there is an asymptotic expansion

$$\text{Tr}(\exp(-t\Delta)) \sim (4\pi t)^{-n/2} \sum_k a_k^D(B) t^k$$

for $t \downarrow 0$, where $k = 0, 0.5, 1, 1.5, \ldots$ ranges over the nonnegative half integers (see [3]). For the Laplace operator on B with Neumann boundary conditions the analog of this formula holds with certain coefficients $a_k^N(B)$. The coefficients $a_k^D(B)$ (resp. $a_k^N(B)$) are given by certain curvature integrals over B and ∂B. One has $a_0^D(B) = a_0^N(B) = \text{vol}(B)$ and $a_{0.5}^D(B) = -a_{0.5}^N(B) = -\frac{\sqrt{\pi}}{2}\text{vol}(\partial B)$ (see [3]). In the proof of Theorem 5.1 we will use the explicit formulas for $a_2^D(B)$ and $a_2^N(B)$ from [3]. Let ν denote the outward pointing unit vector field on the boundary ∂B of B, and let $\sigma = \nabla \nu$ be the associated shape operator. Let scal, Ric, R always refer to the usual objects on M

(not to the ones associated with the induced metric on ∂B). Then

$$a_2^D(B) = \frac{1}{360}\Bigg[\int_B\big(-12\Delta(\mathrm{scal}) + 5\mathrm{scal}^2 - 2|\mathrm{Ric}|^2 + 2|R|^2\big)\,dvol_B$$

$$+ \int_{\partial B}\Big(18\nu(\mathrm{scal}) + 20\mathrm{scal}\cdot\mathrm{Tr}(\sigma) - 4\mathrm{Tr}(R_\nu)\mathrm{Tr}(\sigma) + 12\mathrm{Tr}(R_\nu\circ\sigma)$$

$$- 4\mathrm{Tr}((\mathrm{Ric} - R_\nu)\circ\sigma) + \frac{40}{21}(\mathrm{Tr}(\sigma))^3 - \frac{88}{7}\mathrm{Tr}(\sigma)\mathrm{Tr}(\sigma^2) + \frac{320}{21}\mathrm{Tr}(\sigma^3)\Big)\,dvol_{\partial B}\Bigg],$$

$$a_2^N(B) = \frac{1}{360}\Bigg[\int_B\big(-12\Delta(\mathrm{scal}) + 5\mathrm{scal}^2 - 2|\mathrm{Ric}|^2 + 2|R|^2\big)\,dvol_B$$

$$+ \int_{\partial B}\Big(-42\nu(\mathrm{scal}) + 20\mathrm{scal}\cdot\mathrm{Tr}(\sigma) - 4\mathrm{Tr}(R_\nu)\mathrm{Tr}(\sigma) + 12\mathrm{Tr}(R_\nu\circ\sigma)$$

$$- 4\mathrm{Tr}((\mathrm{Ric} - R_\nu)\circ\sigma) + \frac{40}{3}(\mathrm{Tr}(\sigma))^3 + 8\mathrm{Tr}(\sigma)\mathrm{Tr}(\sigma^2) + \frac{32}{3}\mathrm{Tr}(\sigma^3)\Big)\,dvol_{\partial B}\Bigg].$$

If M is harmonic then, by the results of Section 2, the previous formulas simplify to

$$a_2^D(B) = \frac{1}{360}\Bigg[\mathrm{vol}(B)\cdot\big(5(nC)^2 - 2nC^2 + \frac{4}{3}n((n+2)H - C^2)\big)$$

$$(19) \qquad\qquad + \int_{\partial B}\Big(20nC\mathrm{Tr}(\sigma) - 8C\mathrm{Tr}(\sigma) + 16\mathrm{Tr}(R_\nu\circ\sigma)$$

$$+ \frac{40}{21}(\mathrm{Tr}(\sigma))^3 - \frac{88}{7}\mathrm{Tr}(\sigma)\mathrm{Tr}(\sigma^2) + \frac{320}{21}\mathrm{Tr}(\sigma^3)\Big)\,dvol_{\partial B}\Bigg],$$

$$a_2^N(B) = \frac{1}{360}\Bigg[\mathrm{vol}(B)\cdot\big(5(nC)^2 - 2nC^2 + \frac{4}{3}n((n+2)H - C^2)\big)$$

$$(20) \qquad\qquad + \int_{\partial B}\Big(20nC\mathrm{Tr}(\sigma) - 8C\mathrm{Tr}(\sigma) + 16\mathrm{Tr}(R_\nu\circ\sigma)$$

$$+ \frac{40}{3}(\mathrm{Tr}(\sigma))^3 + 8\mathrm{Tr}(\sigma)\mathrm{Tr}(\sigma^2) + \frac{32}{3}\mathrm{Tr}(\sigma^3)\Big)\,dvol_{\partial B}\Bigg].$$

In the proof of Theorem 5.1 we will follow a similar strategy as in the proof of Theorem 1.1. To this end, we need some preliminary results in the special case that $B = B_r(p)$ with $r \in (0, i(p))$ and M is harmonic. We remark – without going into details this time – that one can compute in this case, using equation (4), Proposition 2.1, and the Taylor series expansion

$$P_{\gamma_u}^{r,0}\circ(R_\nu)_{\gamma_u(r)}\circ P_{\gamma_u}^{0,r} = \sum_{k=0}^{\infty}\frac{r^k}{k!}R_u^{(k)},$$

that the r^3-coefficient in the power series expansion of $\mathrm{Tr}(R_\nu\circ\sigma)$ equals $-\frac{1}{1440}L + \frac{1}{96}\mathrm{Tr}(R_u'R_u')$, and that the r^3-coefficient in the power series expansion of $\mathrm{Tr}(\sigma^3)$ equals $\frac{1}{30240}L - \frac{1}{96}\mathrm{Tr}(R_u'R_u')$. (That the contributions of $\mathrm{Tr}(R_u'R_u')$ in these terms are negatives of each other can also be checked as follows: Using (3) twice, we have $\mathrm{Tr}(R_\nu\circ\sigma) + \mathrm{Tr}(\sigma^3) = -\mathrm{Tr}(\sigma'\sigma) = -\frac{1}{2}\mathrm{Tr}(\sigma^2)' = \frac{1}{2}\mathrm{Tr}(R_\nu + \sigma')' = \frac{1}{2}\mathrm{Tr}(\sigma)''$ whose r^3-coefficient indeed depends only on L by (5).) Using (16), we conclude that the r^3-coefficient in the power series expansion of $r \mapsto \frac{1}{vol(S_r(p))}\int_{S_r(p)}\mathrm{Tr}(R_\nu\circ\sigma)\,dvol_{S_r(p)}$ is

$$(21) \qquad\qquad -\frac{1}{1440}L + \frac{1}{32n(n+2)(n+4)}|\nabla R|_p^2,$$

Similarly, the r^3-coefficient in the power series expansion of $r \mapsto \frac{1}{vol(S_r(p))} \int_{S_r(p)} \mathrm{Tr}(\sigma^3) dvol_{S_r(p)}$ is

$$(22) \qquad \frac{1}{30240} L - \frac{1}{32n(n+2)(n+4)} |\nabla R|_p^2.$$

Proof of Theorem 5.1:
Let M_1, M_2 be harmonic spaces, $p_1 \in M_1$, $p_2 \in M_2$, and assume there exists ε in the interval $(0, \min\{i(p_1), i(p_2)\})$ such that for each $0 < r < \varepsilon$ the geodesic spheres $B_r(p_1)$ and $B_r(p_2)$ are Dirichlet isospectral (resp. Neumann isospectral). Then $\dim M_1 =: n = \dim M_2$, and the heat invariants of the geodesic spheres coincide:

$$a_k^D(B_r(p_1)) = a_k^D(B_r(p_2)), \quad \text{resp.} \quad a_k^N(B_r(p_1)) = a_k^N(B_r(p_2))$$

for each $r \in (0, \varepsilon)$ and all $k \in \mathbb{N}_0$. We want to deduce that $|\nabla R|_{p_1}^2 = |\nabla R|_{p_2}^2$. Actually this will follow using just a_0 and a_2. (We remark without proof here that, viewed as functions of r, the heat coefficients $a_{0.5}$, a_1, and $a_{1.5}$ do actually not contain more information than a_0 in our situation.)

We first consider the case of Dirichlet conditions. Similarly as in the proof of Theorem 1.1, we reformulate the problem as follows: Let M be an n-dimensional harmonic space and $p \in M$. We want to show that for any $\varepsilon \in (0, i(p))$, the two functions

$$\psi_k^D : (0, \varepsilon) \ni r \mapsto a_k^D(B_r(p)) \in \mathbb{R}$$

with $k \in \{0, 2\}$ together determine the value of $|\nabla R|_p^2$. Note that the function

$$\psi_0^D : r \mapsto a_0^D(B_r(p)) = \mathrm{vol}(B_r(p))$$

determines its own derivative which is just

$$r \mapsto \mathrm{vol}(S_r(p)) = v(r)\omega_{n-1}$$

(see the previous section). By Remark 2.3(i), we conclude that ψ_0^D again determines the constants C, H, L associated with M. Moreover, the function $v : (0, \varepsilon) \to \mathbb{R}$ determines the radial functions $\mathrm{Tr}(\sigma) = v'/v$ and $\mathrm{Tr}(\sigma^2) = -(v'/v)' - C$ (compare Remark 3.3). By (19) it now follows that ψ_0^D and ψ_2^D together determine the function

$$(0, \varepsilon) \ni r \mapsto \frac{1}{\mathrm{vol}(S_r(p))} \int_{S_r(p)} \left(16\mathrm{Tr}(R_\nu \circ \sigma) + \frac{320}{21}\mathrm{Tr}(\sigma^3) \right) dvol_{S_r(p)} \in \mathbb{R}.$$

Recalling (21) and (22), we see that the r^3-coefficient in the power series expansion of the latter function is the sum of

$$\frac{1}{32n(n+2)(n+4)} \left(16 - \frac{320}{21} \right) |\nabla R|_p^2 = \frac{1}{42n(n+2)(n+4)} |\nabla R|_p^2$$

and a term depending only on L. Since L is determined by ψ_0^D, we conclude that ψ_0^D and ψ_2^D together determine $|\nabla R|_p^2$, as claimed.

In the Neumann case, letting $\psi_k^N(r) := a_k^N(B_r(p))$, we again have $\psi_0^N(r) = \mathrm{vol}(B_r(p)) = \psi_0^D(r)$. Proceeding exactly as in the Dirichlet case, using (20) this time, we see that ψ_0^N and ψ_2^N together determine the sum of

$$\frac{1}{32n(n+2)(n+4)} \left(16 - \frac{32}{3} \right) |\nabla R|_p^2 = \frac{1}{6n(n+2)(n+4)} |\nabla R|_p^2$$

and a term depending only on L. Hence they determine $|\nabla R|_p^2$, as claimed. $\qquad\square$

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