Theory of z-linear maps

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Introduction to the "Theory of z-linear maps"

The theory we develop in this memoir contemplates z-linear maps through three different points of view: as objects in a category, as homological tools, and functions.

The structure of the work follows, by and large, the previous guidelines as follows: the first point of view 1 is adopted in Chapters 1 and 2, the second in Chapters 3 and 4, and the third in Chapter 5. Although it is more true that the three perspectives are entwined in such a way that it could be said that the theory has a tridimensional character.

z-linear maps as objects of a category. In this work we introduce for the first time a category of z-linear maps (or of exact sequences of Banach spaces) which we shall denote $\mathfrak{Z}$. With it we intend to study categorical properties of z-linear maps; that is, those properties defined by the relationships established (via morphisms) between the objects.

Thus, if it is relevant to know, for instance, that l_1 is a projective object of the category **B** of Banach spaces (due to its lifting properties), we want to know which properties of z-linear maps come reflected in $\mathfrak{Z}$, and in which way they do. Put it otherwise, we want to be able to tell among objects of $\mathfrak{Z}$.

We identify three type of objects in $\mathfrak{Z}$: the object *zero*, the singular and cosingular objects, and some universal objects.

As one would expect, each of those objects is what it should naturally be. The zero object *are* the trivial maps (all of them isomorphic objects in $\mathfrak{Z}$). We define **singular** and **cosingulares** objects as those which have properties "opposite", in dual senses, to the zero object. The singular and cosingular objects turn out to be the objects induced by extensions having strictly singular quotient and strictly cosingular embedding, respectively.

The extensions inducing singular and cosingular objects have a certain importance in quasi-Banach space theory (they appear, for instance, in the solution of Klee's problem and the basic sequence problem). So, it is natural to ask for methods to detect them and construct them. Among the former, we show a technique to prove that the object induced by the quasi-linear map $\mathcal{Z}_1 : l_1 \curvearrowright l_1$ constructed by Kalton and Peck in [**45**] is a new singular object. About the latter, we consider the question of when is it possible, given Z and Y, to construct singular/cosingular objects $Z \curvearrowright Y$- we give some partial answers: on one hand we display a method to construct them, under certain restrictions on Z and Y (say, $Y = C[0,1]$ for the singular case, and $Z = l_1(X, \Gamma)$ for the cosingular case); on the other hand, we obtain the so-called **Proportion principles** [Sections 2.2 y 4.2 of Chapter 2] which establish the limitations for the general constructions: it is determined in terms of the cardinality of the spaces Z and Y, when no singular/cosingular objects $Z \curvearrowright Y$ exist.

Regarding universal properties, we establish the existence of finite products and coproducts which , naturally, correspond to finite "amalgams" of exact sequences $(0 \to l_p(Y_i) \to l_p(X_i) \to l_p(Z_i) \to 0)$. The approach we present to arbitrary products/coproducts forces us to introduce a new algebraic notion: **restricted products** and **coproducts**. They are universal properties analogous to the product/coproduct; they moreover quite clearly explain what happens in the categories such as **Q** (quasi-Banach spaces) and **B**. We obtain different restricted products/coproducts, which we call l_p-product, c_0-product and l_1-coproduct, and correspond to the different amalgamations of extensions. The existence of restricted products/coproducts provides the results we could call "of Ext-representation" because with them we identify the vector space $\operatorname{Ext}(X, l_\infty(Y_n))$ with a l_∞-amalgam of the spaces $\operatorname{Ext}(X, Y_n)$, the space $\operatorname{Ext}(X, c_0(Y_n))$ with a c_0-amalgam of $\operatorname{Ext}(X, Y_n)$, and $\operatorname{Ext}(l_1(X_n), Y)$ with a l_∞-amalgam of spaces $\operatorname{Ext}(X_n, Y)$. The most important of all those representations, and the less trivial indeed, is that of c_0, which we call, for very good reasons, **Super-Sobczyk Theorem**; we shall return to this at the end of the Introduction.

We also tackle the inductive limit of z-linear maps. We uncover two facts: it is possible tom complete certain diagrams of exact sequences; and every object of $\mathfrak{Z}$ defined on a separable space can be seen as an inductive limit. Shifting for a moment to consider z-linear maps as functions, that means that such maps admit inductive finite dimensional representations.

The category of z-linear maps is interesting for us for several reasons; one of them is that it allows to get answers: The question "Which difference, if any, exists between an extension $0 \to Y \to X \to Z \to 0$ and that obtained multiplying by the right (left) with another space E; $0 \to Y \to X \oplus E \to Z \oplus E \to 0$?" gets a natural answer: both objects are isomorphic. This does not mean that the objects can be identified in every aspect, because not all properties we shall consider are stable by isomorphisms (for instance, "to be singular" is not). Another question: Can we tell $F : Z \curvearrowright Y$ from $F : Z \curvearrowright \overline{< F(Z) >}$? We shall see [Section 4.1 in Chapter 2] that they can be distinguished if and only if F is not cosingular. Further questions such as "Which extensions are related via push-out or pull-back?" Or, "Does there exist an extension such that all of a given class $\mathcal{C}$ can be obtained from that via push-out/pull-back ?" should find an answer, in some cases, in terms of new "distinguished" objects of the category: the initial and final objects. Initial objects shall moreover be necessary to develop several techniques throughout the work.

z-linear maps as homological tools. We shall use z-linear maps as homological tools to treat classical problems of extension of operators. We shall study two types of problems: the extension of embeddings (and its dual problem of lifting of quotient maps); and the extension of $C(K)$-valued operators.

The interest on the problem of extension of embeddings stems from the Lindenstrauss-Rosenthal theorem [**57**] which establishes that every isomorphism between subspaces of c_0 can be extended to an automophism of c_0; and that every isomorphism between quotients of l_1 can be lifted to an automorphism of l_1. So, we introduce the notion of **automorphic** (resp. co-automorphic) space to describe those situations. After considering the (still open) conjectures the authors pose in that same paper; namely, that c_0 *and* l_2 *are the only automorphic Banach spaces; while* l_1 *and* l_2 *are the only co-automorphic Banach spaces*, and keeping in mind the properties of c_0 (to be separably injective) and l_1 (to be projective) underpinning those conjectures, we reach the the basic problem underlying the existence of "adequate" automorphisms: when , given two embeddings $i : Y \to X$, $j : Y \to X'$ such that each of them can be extended through the other, is it possible to find an automorphism $\tau : X \to X'$ such that $\tau i = j$? The z-linear approach provides an answer in the form of homological principles that we call **diagonal principles**. In "classical" terms, the first diagonal principle reads as: two embeddings i and j can be extended one through the other if and only if there exists an automorphism τ of the product $X \oplus X'$ so that $\tau(i \oplus 0) = 0 \oplus j$. As a categorical result this yields a characterization of all the isomorphisms of $\mathfrak{Z}$.

We also introduce the notions of partially automorphic and co-automorphic space, with the purpose of describing those spaces having the same behaviour as c_0 or l_1, but only with respect to a given class of subspaces or quotients. One example of the power of the diagonal principles is that they provide a unifying method of proof for all known results (all the results we knew) about partially (co) automorphic spaces; that unexpectedly reveals that all of them share a common homological nature. We also show a new homological technique [Section 4.1.1

of Chapter 3], using the natural transformations of the functor Ext, to obtain results about automorphic/co-automorphic spaces.

The concept of **automorphy class** emerges quite naturally, and with it we search to measure how close is a space to be automorphic. So, it is worth to notice how l_∞, which is the space which is closer to be automorphic (although it is not) by the Lindenstrauss-Rosenthal theorem, has a countable quantity of automorphy classes.

The diagonal principles also apply to the study of the Dunford-Pettis property.

The study of the extension problem for $C(K)$-valued operators was motivated by the Lindenstrauss-Pelczynski theorem [**55**]: "Every $C(K)$-valued operator defined on a subspace of c_0 can be extended to c_0"; and also by the Johnson-Zippin theorem [**38**]: "Every $\mathcal{L}_\infty$ -valued operator defined on a w^*-closed subspace of l_1 can be extended to l_1".

A fundamental tool in the extension problem for $C(K)$-valued operators is Zippin's lemma, that characterizes the subspaces $Y \hookrightarrow X$ such that every operator $Y \to C(K)$ extends to X (it is said that Y is almost complemented in X; or, in our terms, that the induced extension by $Y \hookrightarrow X$ is almost-trivial or $C(K)$-trivial). With this tool, and using a z-linear approach we uncover the existence of a "duality" between the two results, and that it lies in their homological nature: the Lindenstrauss-Pelczynski theorem essentially follows from the assertion "The c_0-product of uniformly almost trivial maps is almost trivial", while the Johnson-Zippin theorem appears as a consequence of the existence of the l_1-coproduct.

The Johnson-Zippin theorem is closely connected with the nature of the functor Ext; indeed, it can be stated as "$\mathrm{Ext}(H^*, C(K)) = 0$ for every subspace H of c_0". Since Kalton proves that $\mathrm{Ext}(X, C[0,1]) \neq 0$ for every separable Banach space X without the Schur property, it makes sense the question: Does there exists a subspace X of l_1 such that $\mathrm{Ext}(X, C(K)) \neq 0$? This question, which remains open in this memoir, has a special interest for us: it turns out to be equivalent to the so-called LP2 problem to which this memoir converges, and to which we shall speak below.

Trying to determine the range of validity of the Lindenstrauss-Pelczynski theorem we introduce a necessary definition [Section 2.2 of Chapter 4]: A space shall be said to be of type $\mathcal{LP}$ if every operator from a subspace of c_0 into that space extends to the whole c_0. All $\mathcal{LP}$ spaces are of type $\mathcal{L}_\infty$, although the converse fails. The complemented subspaces of $C(K)$-spaces, the separably injective spaces, the isometric preduals of L_1... all are $\mathcal{LP}$-spaces; the list is not exhaustive. By the way, we take the opportunity to *distinguish* between the two first classes by constructing a separably injective space which cannot be complemented in any $C(K)$-space. It remains open the problem of the characterization of $\mathcal{LP}$-spaces, nd also the question if every $\mathcal{LP}$-valued operator from a w^*-closed subspace of l_1 can be extended to the whole l_1.

One place where $C(K)$-valued extension problems and the problem of the existence of "adequate" automorphisms is [Section 5.1 of Chapter 4]: *If X does not contain l_1, the space $C[0,1]$ is X-automorphic if and only if X is almost complemented in $C[0,1]$.*

z-linear maps as functions. Throughout this memoir we shall contemplate different aspects of z-linear maps as functions.

At some places it shall be necessary to have a z-linear map in such form that its restriction to a finite dimensional subspace has finite dimensional range. That is possible after a process we call "convexification" applied to F. The resulting map shall be a version of F and shall be called a **convex version** of F. The existence of convex versions is precisely what allows us to represent z-linear maps as inductive limits of maps with finite dimensional range (this is what we called inductive finite dimensional representation of F).

The study of the convergence of sequences (F_n) of z-linear maps gives us an useful tool: the so-called **Change of convergence lemma**, which asserts that for canonical convex z-linear maps pointwise convergence of (F_n) on a finite dimensional space implies norm convergence.

The aspect of a z-linear map as a function most interesting for are its factorization properties. Precisely, we are interested in the extension problem for z-linear maps through embeddings; and, especially after the previous work for operators the case of $C(K)$-valued maps.

The extension problems for z-linear maps find their natural formulation in terms of the functor Ext^2, reason for which we call them "order 2 problems". The analogy which can be established with the extension problems for operators is surprising and is based on a fundamental fact: $\mathbb{R}$-valued z-linear maps play the role (this as been understood with some reservations, and put into the context) in the extension problems for z-linear maps of linear functionals in the extension problems for operators. In this way naturally emerge Banach space constructions and properties that we can interpret as "z-linear": we introduce the notions of **2-injective space**, **z-dual** of a Banach space, **envelope $\mathbf{co_z}$** and **z-metric projection**.

A space X shall be called 2-injective if every z-linear map with range X extends to any superspace. The z-dual X^z of a space X is the space of all scalar z-linear maps that vanish on a Hamel basis of X, endowed with the norm induced by the z-linearity constant. The space X^z is a dual. The $co_z(X)$ envelope is nothing but the natural predual of X^z, and has the property of representing the space through a map $\Omega_X : X \curvearrowright co_z(X)$ in such a way that any other map $X \curvearrowright Y$ factorizes through Ω_X. We say that X admits a z-metric projection if it exists a metric projection from the space of scalar z-linear maps to its algebraic dual.

2-injective spaces shall be characterized during the study of the general extension problem of z-linear maps [Section 2.2 of Chapter 5]; unlike injective spaces, 2-injective spaces need not by of type $\mathcal{L}_\infty$.

The key for the correspondence between the extension problem for $C(K)$-valued operators and the corresponding one for z-linear maps is given by the **order 2 version of Zippin's lemma**: Every map $Y \curvearrowright C(K)$ can be extended through $j : Y \hookrightarrow X$ if and only if there exists a w^*-continuous selection $B_{Y^z} \to \lambda B_{X^z}$ for $j^* : X^z \to Y^z$. Thus, we shall show [Theorem 5.2 of Chapter 5] in which different ways Zippin's lemma turns out to be equivalent to its order 2 version; namely, that to extend all $C(K)$-valued z-linear maps is equivalent to extend, from different spaces, $C(K)$-valued operators.

The problem to which this work converges is the one we call **Order 2 Lindenstrauss-Pelczynski** or (LP2): Does every $C(K)$-valued z-linear map defined on a subspace H of c_0 extend? By Sobczyk's theorem, to have an extension to c_0 and to any separable superspace are equivalent. And both equivalent to extend to $C(B_{H^*})$. The question remains open.

Where all perspectives intersect. The LP2 problem is one of the places in which all languages are interwoven, and where one more clearly realizes the possibility of a tridimensional aspect of the theory. For instance, one observes that the extension of z-linear maps $H \curvearrowright C[0,1]$ to the whole c_0 or, worse yet, the vanishing of the second derived functor in some long homology sequence, turn out to be equivalent to the fact that the space $l_\infty/C[0,1]$ is of type $\mathcal{LP}$ (say, complemented in some $C(K)$). And equivalent to the fact that the kernel $K(H)$ of the projective presentation of H is almost complemented in the corresponding kernel $K(c_0)$. And they follow from "$\mathrm{Ext}(M, C[0,1]) = 0$ for every subspace M of l_1". About this last assertion, we use the properties of z-metric projections to obtain a characterization of those spaces X such that $\mathrm{Ext}(X, C(K)) = 0$.

Another place where the three points of view are mixed is the Super-Sobczyk Theorem [Section 5.2 of Chapter 1] that we had already mentioned. Actually, the point is to identify a functor as $\mathrm{Ext}(\,\cdot\,, c_0(Y_n))$ or, what is the same, to represent the spaces $\mathrm{Ext}(X, c_0(Y_n))$. In this way, the theorem can be essentially stated as: *The c_0-product of uniformly trivial maps defined on a separable space is trivial.* Both the result and corollaries (including a new proof of Sobczyk's theorem) are obtained through a measured combination of three ingredients: the existence of the c_0-product [Section 3.1 of Chapter 1], the existence of finite dimensional representations for z-linear maps [Section 4.5 of Chapter 1] and the Change of convergence lemma [Section 4 of Chapter 4].

About which is original. About the originality of the work, I've tried to specify at each place, to the best of my knowledge, which results can be found in the literature. The remainder is, by exclusion, original.

Five papers have been published so far: [**13**], [**18**], [**19**], [**20**] and [**21**]; they are related with this memoir in the following form: Chapters 2 and 3 contain some of the results of [**18**], [**19**] and [**21**]; while Chapter 4 contains part of [**13**].

Nevertheless, as the reader can easily check by mere inspection, [**21**] versus Chapter 3 and [**13**] versus Chapter 4, the results which had been published are shown here from a different perspective; in my opinion, from *the* natural one.

Preliminaries

Categories and functors. A category $\mathfrak{C}$ is a collection of objects and morphisms between them, which we shall denote $f : A \to B$,satisfying the following conditions:

- There is composition of morphism, in such a way that if $f : A \to B$ and $g : B \to C$ then there exists a morphism $gf : A \to C$.
- The composition is associative.
- For each object A there is a morphism $1_A : A \to A$, called identity, such that for each morphism f one has $f = f1_A = 1_Af$.

Given two objects A, B the set of morphisms from A into B is denoted $\mathrm{Hom}_{\mathfrak{C}}(A, B)$.

A morphism $f : A \to B$ is an isomorphism if there is another morphism g such that $gf = 1_A$ y $fg = 1_B$.

A covariant functor $\mathcal{F} : \mathfrak{C} \to \mathfrak{A}$ between two categories is a correspondence assigning to each object of C an object of $F(C)$ and to each morphism $f : C \to D$ a morphism $F(f) : F(C) \to F(D)$ such that

- For each couple of morphisms f, g one has $\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f)$
- $\mathcal{F}(1_A) = 1_{\mathcal{F}A}$

Natural transformations. A natural transformation $\tau : \mathcal{F} \to \mathcal{G}$ between functors $\mathcal{F}, \mathcal{G} : \mathfrak{C} \to \mathfrak{A}$, defined between the same categories, is a correspondence assigning to each object C of $\mathfrak{C}$ a morphism $\tau_C : \mathcal{F}C \to \mathcal{G}C$ in such a way that if $f : C \to D$ is a morphism of $\mathfrak{C}$ then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{F}C & \xrightarrow{\tau_C} & \mathcal{G}C \\ {\scriptstyle \mathcal{F}(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{G}(f)} \\ \mathcal{F}D & \xrightarrow{\tau_D} & \mathcal{G}D. \end{array}$$

A natural transformation τ is called a natural equivalence if each of the morphisms τ_C is an isomorphism.

Adjoint functors. Given a functor $\mathcal{F} : \mathfrak{C} \to \mathfrak{A}$ it is said that another functor $\mathcal{G} : \mathfrak{A} \to \mathfrak{C}$ is its right adjoint (in which case one also says that $\mathcal{F}$ is left adjoint to $\mathcal{G}$) if for each pair of objects C of $\mathfrak{C}$ and A of $\mathfrak{A}$ there is a morphism

$$\mathrm{Hom}_{\mathfrak{C}}(C, \mathcal{G}A) \to \mathrm{Hom}_{\mathfrak{A}}(\mathcal{F}C, A)$$

That amounts the existence of a natural equivalence $\tau : GF \to 1$ and another $\eta : 1 \to FG$.

Two categories between which there exist two adjoint covariant functors are called [**22**] equivalent.

The categories Q y B. Given a vector space X, a quasi-norm on X i an application $\|\cdot\| : X \to \mathbb{R}^+$ verifying:

(1) $\|x\| = 0 \iff x = 0$.
(2) $\forall \lambda \in \mathbb{R}$, $\|\lambda x\| = |\lambda|\|x\|$.
(3) $\exists C \geq 1 : \forall x, y \in X$, $\|x + y\| \leq C(\|x\| + \|y\|)$.

We call quasi-normed space to a vector space endowed with a quasi-norm. If $C = 1$ it is said that $\|\cdot\|$ it is a norm.

A quasi-Banach space is a complete quasi-normed space. A Banach spaces is a complete normed space. A p-Banach space, $0 < p < 1$, is a quasi-Banach space whose quasi-norm verifies that for each pair of points x, y:

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p.$$

We denote by $\mathbf{Q}$ the category of quasi-Banach spaces and operators, $\mathbf{B}$ is the subcategory of Banach spaces and $\mathbf{Q_p}$ is the subcategory of p-Banach spaces.

In this memoir we shall call cokernel of an operator $T : Y \to X$ to the space $X/\overline{TY}$, quotient of X by the closure of the image of T.

Exact sequences or extensions. An extension (of Z by Y) or a short exact sequence in the category $\mathbf{QN}$ of quasi-normed spaces and operators is a diagram $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ in the category such that the image of each arrow coincides with the kernel of the previous one. We shall say that the sequence is topologically exact if, moreover, the inclusion j is an embedding (i.e., an into isomorphism) and the map q is open.

In $\mathbf{Q}$ all extensions are topologically exact, thanks to the open mapping theorem. This means that in an extension Y is a subspace of X and $X/Y \simeq Z$. We shall call the space X the **twisted sum** of Y and Z.

To each (topologically) exact sequence we can associate a parameter:

$$\rho = \sup\left\{\|j\|,\ \|j^{-1}\|,\ \|q\|,\ \|q^{*-1}\|\right\},$$

where $\|q^{*-1}\|$ establishes the openess of the map q; that is, $\|q^{*-1}\| = \inf\{\lambda > 0 : B_Z \subset \lambda q(B_X)\}$.

Definition 0.1 (Isomorphically equivalent extensions). *Two extensions $0 \to Y \to X \to Z \to 0$ and $0 \to Y' \to X' \to Z' \to 0$ in $\mathbf{Q}$ are called isomorphically equivalent if there exist isomorphisms α, β y γ making commutative the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & Z' & \longrightarrow & 0 \end{array} \tag{1}$$

In the particular case in which α and γ are the identity we have the usual equivalence relation $\equiv$ between exact sequences. Sometimes we shall speak of the $\equiv$-class of an extension; we shall also write $0 \to Y \to X \to Z \to 0 \ \in C$ to remark that the given extension is a representative of the class C.

Observe that two extensions between the same spaces are equivalent if and only there exists an operator β making commutative the diagram 1; it is so because the well-known 3-lemma and the open mapping theorem imply that β is an isomorphism.

We shall say that an extension is trivial if it is equivalent to the sequence $0 \to Y \to Y \oplus Z \to Z \to 0$ canonically induced by the product $Y \oplus Z$.

We shall denote by $\mathrm{Ext}_{\mathbf{Q}}(Z,Y)$ and $\mathrm{Ext}_{\mathbf{B}}(Z,Y)$ the vector spaces of $\equiv$-classes of extensions of Z by Y in $\mathbf{Q}$ and $\mathbf{B}$, respectively. For the Banach case we shall often simply write $\mathrm{Ext}(Z,Y)$. Those spaces have a vector space structure (see [**32**]).

Quasi- and z-linear maps. Kalton, [**40**], and Kalton and Peck, [**45**], introduced the so-called quasi-linear maps.

Definition 0.2 (Quasi-linear maps). *Let Z and Y be two quasi-Banach spaces, it is said that a map $F : Z \to Y$ is a quasi-linear map if it is homogeneous and verifies :*

$$\exists C > 0 :\ \forall z, z' \in Z,\ \|F(z+z') - F(z) - F(z')\| \leq C(\|z\| + \|z'\|)$$

We shall write $Q(F)$ for the infimum of those constants C satisfying the previous condition. We shall write $Q(Z,Y)$ for the space of all quasi-linear maps from Z into Y; it is a vector space with the standard operations.

A very special subclass of quasi-linear maps, which constitute the central notion of this work, are the z-linear maps, see [**9, 16**].

Definition 0.3 (z-linear maps). *Let Z and Y be two Banach spaces. A map $F : Z \to Y$ is called z-linear if it is homogeneous and verifies: $\exists C > 0 : \forall z_1, ...z_n \in Z$,*

$$\|F(\textstyle\sum_{i=1}^n z_i) - \sum_{i=1}^n F(z_i)\| \leq C\Big(\sum_{i=1}^n \|z_i\|\Big);$$

equivalently, there exists a constant $M > 0$ such that for each set $z_1, ...z_n \in Z$ such that $\sum_i z_i = 0$ one has

$$\|\textstyle\sum_{i=1}^n F(z_i)\| \leq M\left(\sum_{i=1}^n \|z_i\|\right).$$

To pass from one definition to the other one must take into account that the constants vary $C = 2M$. We shall denote by $Z(F)$ the infimum of the constants M satisfying the second definition (sometimes we shall write even $Z_X(F)$ to remark the space X on which F is defined). Although we shall use in each case the definition more convenient without explicitly say which one it is, let us remark that the second definition has the property that $Z(F) \leq \|F\|$ holds for a (homogeneous) bounded map; this estimate shall be useful at some place.

The space of all z-linear maps from Z into Y shall be denoted $Z(Z,Y)$; it is a vector space with the natural induced structure.

We shall use the notation $F : Z \curvearrowright Y$ to denote a given quasi- or z-linear map.

Equivalent quasi-linear maps. We define now the usual equivalence relation, also denoted $\equiv$, between quasi-linear maps from Z into Y. We shall say that F and G are equivalent if there exist maps $B, L : Z \to Y$, where B is homogeneous and bounded and L linear, such that $F - G = B + L$. We shall denote by $\mathbf{L}(Z,Y)$, or just $\mathbf{L}$ when the context is clear, the space of linear maps from Z into Y. Analogously, $\mathbf{B}(Z,Y)$ ($\mathbf{B}$ if the context is clear) denotes the space of all homogeneous bounded maps from Z into Y.

We shall say that F is trivial if it is equivalent to the map 0.

Given a quasi-linear map $F : Z \curvearrowright Y$ and a normalized Hamel basis (e_α) of Z it shall be helpful to get a version vanishing on the elements of the basis; to this end one defines the map $F - L_F$ where $L_F : Z \to Y$ is the linear map that in $z = \sum \lambda_\alpha e_\alpha \in Z$ takes the value $\sum \lambda_\alpha F(e_\alpha)$. We shall say that $F - L_F$ is a canonical form of F.

The space of all $\equiv$-classes of quasi-linear maps from Z into Y shall be called $\mathcal{Q}(Z,Y)$; and that of z-linear maps, $\mathcal{Z}(Z,Y)$. Each of them admits a natural semi-norm: $\mathcal{Q}(F) = \inf\{Q(F') : F' \equiv F\}$, and $\mathcal{Z}(F) = \inf\{Z(F') : F' \equiv F\}$ (see [**11**])

Correspondence between extensions and quasi-linear map. An exact sequence in $\mathbf{Q}$ comes described by a quasi-linear map and conversely; so, there is a correspondence between exact sequences and quasi-linear maps. Given an extension $0 \to Y \to X \xrightarrow{q} Z \to 0$, let us choose selections b, l, the former bounded and homogeneous and linear the latter, for q. The map $F = b - l$ takes values in Y and is quasi-linear:

$$\|F(x+y) - F(x) - F(y)\| = \|b(x+y) - b(x) - b(y)\| \leq 2\|b\|(\|x\| + \|y\|)$$

Conversely, given a quasi-linear map $F : Z \curvearrowright Y$, we can define the extension

$$0 \longrightarrow Y \longrightarrow Y \oplus_F Z \longrightarrow Z \longrightarrow 0,$$

where the twisted sum space is just the product $Y \times Z$ endowed with the quasi-norm

$$\|(y,z)\|_F = \|y - F(z)\|_Y + \|z\|_Z.$$

The embedding is $y \to (y,0)$ and the quotient map is $(y,z) \to z$. The correspondence is bijective between equivalence classes: two extensions $0 \to Y \to Y \oplus_F Z \to Z \to 0$ and $0 \to Y \to Y \oplus_G Z \to Z \to 0$ are equivalent if and only if F and G are equivalent.

It turns out that $\|\cdot\|_F$ is equivalent to a norm if and only if F is z-linear. this is the form in which extensions in **B** correspond with z-linear maps.

We shall often use the notation $0 \to Y \to X \to Z \to 0 \ \equiv F$ to indicate that F is a quasi-linear description of the extension, that is, F is an associated quasi-linear map.

The concept of isomorphically equivalent extensions has its analogue in quasi-linear terms: we shall say that two maps $F : Z \curvearrowright Y$ and $G : Z' \curvearrowright Y'$ are isomorphically equivalent if and only if there exist isomorphisms $\alpha : Z \to Z'$ and $\gamma : Y \to Y'$ such that $\alpha F \equiv G\gamma$. One has that two extensions $0 \to Y \to Y \oplus_F Z \to Z \to 0$ and $0 \to Y' \to Y' \oplus_G Z' \to Z' \to 0$ are isomorphically equivalent if and only if F and G are isomorphically equivalent.

Basic Homological tools.

Pull-back. Given a diagram

$$\begin{tikzcd} C & A \arrow[l, "f"'] \\ B, \arrow[u, "g"'] & \end{tikzcd}$$

in a category $\mathfrak{C}$, the pull-back of the pair $\{f, g\}$ is an object $PB(f, g)$, together with a couple of morphisms $u : PB(f, g) \to A$ and $v : PB(f, g) \to B$ such that $fu = gv$, and having the universal property that for each object X and morphisms $u' : X \to A$ and $v' : X \to B$ verifying $fu' = gv'$, there exists a unique morphism $t : X \to PB(f, g)$ tal que $v' = vt$ y $u' = ut$.

Pull-back exist both in **Q** and **B**: the pull-back of the pair $\{f, g\}$ shall be the space $PB(f, g) = \{(a, b) \in A \oplus B : fa = gb\}$ endowed with the topology induced by the product, together with the restrictions of the canonical projections of $A \oplus B$ onto the factors. In particular, given an extension $0 \to Y \to X \stackrel{q}{\to} Z \to 0$ and an operator $S : M \to Z$, we obtain by making pull-back a completed commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j"] & X \arrow[r, "q"] & Z \arrow[r] & 0 \equiv F \\ 0 \arrow[r] & Y \arrow[r, "i"] \arrow[u, equal] & PB \arrow[r] \arrow[u] & M \arrow[r] \arrow[u, "S"'] & 0 \equiv FS. \end{tikzcd} \tag{2}$$

The universal property of the pull-back ensures that all diagrams having this form are, modulus $\equiv$, pull-back diagrams. In the language of maps, making pull-back corresponds with right composition.

Given a pull-back diagram such as 2, there exists an exact sequence

$$0 \longrightarrow PB \longrightarrow M \oplus E \xrightarrow{q-S} Z \longrightarrow 0 \equiv iF,$$

which we shall call associated diagonal pull-back sequence, or simply **diagonal pull-back**.

Push-out. Given a diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"] & A \\ B, & \end{tikzcd}$$

in a category $\mathfrak{C}$, the push-out of the couple $\{f, g\}$ is an object $PO(f, g)$, together with a couple of morphisms $u : A \to PO(f, g)$ and $v : B \to PO(f, g)$ such that $uf = vg$, and having the universal property that given another object X and morphisms $u' : A \to X$ and $v' : B \to X$ such that $u'f = v'g$, there exists a unique morphism $t : PO(f, g) \to X$ such that $v = v't$ and $u = u't$.

Push-out exist in **Q** and **B**; it is the space $PO(f, g) = A \oplus B/\overline{\Delta}$, where $\Delta = \{(fc, -gc) : c \in C\}$, endowed with the quotient topology , together with the restrictions of the quotient map $A \oplus B \to PO$ to B and A.

In particular, given an extension $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ and an operator $T : Y \to E$, making push-out of (j, T) one gets a completed commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j} & X & \longrightarrow & Z & \longrightarrow & 0 \equiv F \\ & & {\scriptstyle T}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & E & \longrightarrow & PO & \xrightarrow{p} & Z & \longrightarrow & 0 \equiv TF \end{array} \tag{3}$$

The universal property of the push-out guarantees that all diagrams having this form are, modulus $\equiv$, push-out diagrams. In the language of quasi-linear maps the push-out corresponds with left composition.

Given a push-out diagram like 3, there exists an exact sequence

$$0 \longrightarrow Y \xrightarrow{T \oplus (-j)} E \oplus X \longrightarrow PO \longrightarrow 0 \equiv Fp,$$

which we shall call associated diagonal push-out sequence, or simply **diagonal push-out**.

It should be obvious now that, given F, making first push-out with T and then pull-back with S gives an equivalent extension to that obtained making first pull-back with S and then push-out with T. In both cases one obtains TFS.

We introduce a new notion that generalizes that of trivial extension:

Definition 0.4 ($\mathcal{A}$-trivial extensions). *Let $\mathcal{A}$ be a class of quasi-Banach spaces. We shall say that an extension or quasi-linear map $0 \to Y \to X \to Z \to 0 \equiv F$ is $\mathcal{A}$-trivial if every operator T from Y into a space A belonging to $\mathcal{A}$ can be extended to X (i.e., $TF \equiv 0$).*

The literature contains some types of extensions that corresponds with different types of $\mathcal{A}$-triviality: it is clear that trivial extensions coincide with $\mathcal{A}$-trivial extensions for $\mathcal{A}$ the class of all quasi-Banach spaces; the extensions that locally split correspond to $l_\infty(F_n)$-trivial extensions, where (F_n) denotes a dense sequence, in the Banach-Mazur distance, of finite dimensional spaces; and almost trivial extensions shall be the $C(K)$-trivial extensions.

Projective Presentation. We shall say that an object Z of a category $\mathfrak{C}$ is projective if every extension $0 \to Y \to X \to Z \to 0$ is trivial.

The spaces $l_1(\Gamma)$ are projective in **B**, and they are "enough" in the sense that, as it is well known, every Banach space can be written as a quotient of some $l_1(\Gamma)$ space for some set of indices Γ. In **Q** no (infinite dimensional) projective objects exist, while in $\mathbf{Q_p}$ the spaces $l_p(\Gamma)$ are projective and enough.

We shall call projective presentation of a space Z to an extension

$$0 \longrightarrow K(Z) \longrightarrow P \longrightarrow Z \longrightarrow 0 \equiv \mathcal{P}_Z,$$

in **B** ó $\mathbf{Q_p}$, with P projective.

We can represent extensions by operators using projective presentations: consider $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0 \equiv F$. Construct a push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(Z) & \longrightarrow & P & \xrightarrow{p} & Z & \longrightarrow & 0 \equiv \mathcal{P}_Z \\ & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{j} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \equiv F, \end{array}$$

since p lifts through q. Conversely, given an operator $\varphi : K(Z) \to Y$ one gets making push-out an extension of Z by Y, described by $\varphi\mathcal{P}_Z$. Let us consider now the equivalence relation $\asymp$ between operators $K(Z) \to Y$ defined

$$\varphi \asymp \varphi' \iff \varphi - \varphi' \text{ se extiende a P},$$

which gives a bijection between $\equiv$-classes of extensions and $\asymp$-classes of operators.

Injective Presentations. An object I of a category is said to be injective if every extension $0 \to I \to X \to Z \to 0$ is trivial. In $\mathbf{B}$ the Hahn-Banach theorem implies that $l_\infty(\Gamma)$-spaces are injective. Although those are not the only injective objects of $\mathbf{B}$, they are enough because, as it is well known, every Banach space X is a subspace of some $l_\infty(\Gamma)$-space.

We call injective presentation of X to an extension $0 \to X \to I \to I(X) \to 0$ with I an injective space .

Injective presentations in $\mathbf{B}$ allow us to represent exact sequences or z-linear maps. Let us consider $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0 \equiv F$. We construct a pull-back diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{i} & I & \longrightarrow & I(Y) & \longrightarrow & 0 \equiv \mathcal{I}_Y \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \xi} & & \\
0 & \longrightarrow & Y & \xrightarrow{j} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \equiv F,
\end{array}$$

since i can be extended to X thanks to the injectivity of I. Conversely, given an operator $\xi : Z \to I(Y)$ we obtain the extension $\mathcal{I}_Y \xi$, associated with the corresponding pull-back diagram.

Taking into account the equivalence relation between operators $Z \to I(Y)$

$$\xi \asymp \xi' \Longleftrightarrow \xi - \xi' \text{ se levanta a } I,$$

we will have a bijective correspondence between $\equiv$-classes of extensions and $\asymp$-classes of operators $Z \to I(Y)$.

All those representations of exact sequences are naturally equivalent, with the meaning that the different induced functors are naturally equivalent (the proofs can be seen in [**61**]).

A little bit more about notation. The symbols *comodines*: $\clubsuit$, $\diamondsuit$, $\heartsuit$, $\spadesuit$ shall appear quite often through the work. They mean that their place can be occupied by any quasi-Banach space, or Banach if the context says so. For instance, we could say "Every map $F : \heartsuit \curvearrowright Y$" to mean any Y-valued quasi-linear map.

References. General information about homological algebra can be found in [**32**] and [**59**]; about Banach spaces in [**58**], and about the combination of both in [**16**]and [**49**].

CHAPTER 1

The category $\mathfrak{Z}$ of z-linear maps

The purpose of this chapter is to set a context and find a natural language to perform a study of short exact sequences in the categories **Q** of quasi-Banach spaces and **B** of Banach spaces, as well as their corresponding representations as quasi-linear and z-linear maps, respectively. By "natural context" we refer to the setting that makes simpler the description of the objects one intends to study, their properties and the relationships among them. From this point of view, the best context one can expect is a category. So, we are looking for a good definition of the categories of exact sequences and of quasi-linear maps.

In what follows, when we talk of equivalent extensions or quasi-linear maps it is understood that we are referring to the usual notions of equivalence ($\equiv$) as can be seen in the Preliminaries section.

Since we shall scarcely, if ever, shall distinguish between equivalent sequences in **Q**, it is reasonable to choose as objects of the *category* $\mathfrak{S}$ *of extensions in* **Q** equivalence classes of short exact sequences. Let us see that the choice of morphisms comes now determined by the choice of the objects:

Taking as reference the category $\mathfrak{K}(\mathcal{A})$ of complexes in a category $\mathcal{A}$, see for instance [**59**] or [**32**], it would seem natural to choose as morphisms of $\mathfrak{S}$ triples of operators (α, β, γ) making commutative a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & Z' & \longrightarrow & 0. \end{array} \tag{4}$$

However, we wonder if those morphisms are well-defined between equivalence classes. For instances, what should it mean the triple (α, β, γ) when we change the extensions in the previous diagram by equivalent ones? Observe that $\beta : X \to X'$ it makes no sense since the twisted sum spaces X and X' have changed.

Precisely, the algebra contemplates an equivalence relation between certain chains of "operators" (those defining the morphisms of $\mathfrak{K}(\mathcal{A})$) called *homotopy*, [**59**] ó [**32**]; it allows us to define the morphisms between extensions of quasi-Banach spaces in a way which is compatible with the usual equivalence $\equiv$.

In this way, it is said that two triples of operators (α, β, γ) and $(\alpha', \beta', \gamma')$ making commutative a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{i} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \alpha,\, \alpha'} & & \big\downarrow{\scriptstyle \beta,\, \beta'} & & \big\downarrow{\scriptstyle \gamma, \gamma'} & & \\ 0 & \longrightarrow & Y' & \xrightarrow{i'} & X' & \xrightarrow{q'} & Z' & \longrightarrow & 0, \end{array}$$

are homotopic if and only if there exist operators $S : X \to Y'$ and $T : Z \to X'$ such that:

(1) $Si = \alpha - \alpha'$.
(2) $q'T = \gamma - \gamma'$.
(3) $i'S + Tq = \beta - \beta'$.

This definition, however, can be simplified and put in terms of the first and third coordinates since it is not hard to check that

$$(\alpha, \beta, \gamma) \ y \ (\alpha', \beta', \gamma') \text{ are homotopic } \iff (1) \ y \ (2)$$

On the other hand, the existence of a diagram such as 4 is equivalent, modulus homotopy, to the fact that the pull-back and push-out extensions in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\widehat{\alpha}} & & \| & & \\
0 & \longrightarrow & Y' & \longrightarrow & PO & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\tau} & & \| & & \\
0 & \longrightarrow & Y' & \longrightarrow & PB & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\widehat{\gamma}} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & Z' & \longrightarrow & 0
\end{array}$$

are equivalent. Thus, one has the equivalence

$$(1) \ y \ (2) \iff (1) \iff (2).$$

It is now clear which should be the definition of the morphisms of $\mathfrak{S}$: given two objects E y E' of $\mathfrak{S}$, a morphism $E \rightrightarrows E'$ from E to E' shall be a $\asymp$-class of operators (α, γ) being part of a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \in E \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & Z' & \longrightarrow & 0 \in E',
\end{array}$$

where the equivalence relation $\asymp$ comes defined as :

$$(\alpha, \gamma) \asymp (\alpha', \gamma') \iff \alpha - \alpha' \textit{ se extiende a } X \iff \gamma - \gamma' \textit{ se levanta a } X'.$$

We shall write $\mathrm{Hom}_{\mathfrak{S}}(E, E')$ to denote the set of all morphisms of $\mathfrak{S}$ from E to E'. To simplify notation we shall write (α, γ) to denote an element of $\mathrm{Hom}_{\mathfrak{S}}(E, E')$, although it has to be understood that it represents the $\asymp$ $-$equivalence class of the couple (α, γ). THe composition of morphisms comes defined in a natural way: $(\alpha, \gamma) \circ (\alpha', \gamma') = (\alpha \circ \alpha', \gamma \circ \gamma')$. The identity morphisms shall be (id, id). Therefore, (α, γ) is an isomorphism if and only if there exists another morphism (α', γ') such that $(\alpha', \gamma') \circ (\alpha, \gamma) \asymp (id, id) \asymp (\alpha, \gamma) \circ (\alpha', \gamma')$. It is obvious that if α y γ are isomorphisms of $\mathbf{Q}$ then (α, γ) is an isomorphism of $\mathfrak{S}$ and, in that case, $(\alpha, \gamma)^{-1} = (\alpha^{-1}, \gamma^{-1})$. Nevertheless, one has to keep in mind that those are not the only isomorphisms of the category; in fact, we shall characterize in Proposition **??** in Chapter 3 all the isomorphisms.

We shall say that an object is *trivial* if it is the element 0 of some $\mathrm{Ext}_{\mathbf{Q}}(Z, Y)$. It is clear that all trivial objects are isomorphic, and thus we shall denote all them by 0. When it is necessary to distinguish the object $0 \to Y \to Y \oplus Z \to Z \to 0$ from $0 \to Y' \to Y' \oplus Z' \to Z' \to 0$ we shall refer to them as 0_Y^Z and $0_{Y'}^{Z'}$, respectively. Trivial objects play a special role in the category; they turn out to be the *initial* and $final$ object of $\mathfrak{Q}$. Let us recall both definitions:

DEFINITION 1.1. *An object I in a category $\mathfrak{C}$ is said to be initial if for all objects C there is a unique morphism from I to C. Dually, an object F is said to be final if for all objects C there exists a unique morphism from C to F.*

The initial and final object of a category are isomorphic. It is now clear that for every object F of $\mathfrak{S}$ there exists a unique arrow in the sets $\mathrm{Hom}_{\mathfrak{S}}(F, 0)$ and $\mathrm{Hom}_{\mathfrak{S}}(0, F)$.

The definition of the **category $\mathfrak{Q}$ of quasi-linear maps** in $\mathbf{Q}$ is that induced in a natural form (recall the correspondence between quasi-linear maps and extensions; see the Preliminaries section) by that of $\mathfrak{S}$: the object of $\mathfrak{Q}$ shall be $\equiv$ $-$classes of quasi-linear maps. Give a pair of objects F, G, the set of morphisms $\mathrm{Hom}_{\mathfrak{Q}}(F,G)$ has as elements $\asymp$-classes of couples of operators (α,γ) such that $\alpha F\equiv G\gamma$, in such a way that

$$(\alpha',\gamma')\asymp(\alpha,\gamma)\Longleftrightarrow(\alpha-\alpha')F\equiv 0\Longleftrightarrow G(\gamma-\gamma')\equiv 0.$$

The composition is defined analogously to that of $\mathfrak{S}$, and the identity morphism is therefore (id,id). Trivial objects $Z\curvearrowright Y$ shall be denoted, when necessary, 0_Y^Z.

The categories $\mathfrak{S}$ and $\mathfrak{Q}$ are **equivalent** in the sense that there exist two covariant functors $\mathcal{S}:\mathfrak{Q}\to\mathfrak{S}$ and $\mathcal{F}:\mathfrak{S}\to\mathfrak{Q}$ adjoint one of the other (see [**22**, 5.6]). That is, for each pair of objects $\heartsuit$ of $\mathfrak{S}$ and $\spadesuit$ of $\mathfrak{Q}$ there is a bijection

$$\mathrm{Hom}_{\mathfrak{S}}(\heartsuit,\mathcal{S}(\spadesuit))\longleftrightarrow\mathrm{Hom}_{\mathfrak{Q}}(\mathcal{F}(\heartsuit),\spadesuit)$$

Precisely, $\mathcal{F}$ and $\mathcal{S}$ establish the biunivoque correspondence between classes of extensions and classes of quasi-linear maps: $\mathcal{F}$ assigns to the $\equiv$-class of $0\to Y\to X\xrightarrow{q}Z\to 0$ the $\equiv$-class of the quasi-linear map $b-l$, where b and l are, respectively, a bounded homogeneous and a linear selection for q. Reciprocally, $\mathcal{S}$ assigns to the $\equiv$-class of a quasi-linear map F the class of the extension $0\to Y\to Y\oplus_F Z\to Z\to 0$.

In what follows we shall identify the two categories; this fact, however, does not mean that all constructions are equally easy in the two categories. For instance, the l_p-amalgams 2.1, are easier in $\mathfrak{S}$ than in $\mathfrak{Q}$, while the construction of minimal versions **??** is quite clear in $\mathfrak{Q}$ but not in $\mathfrak{S}$.

0.1. The category $\mathfrak{Z}$ of z-linear maps. The categories $\mathfrak{Z}$ of z-linear maps and $\mathfrak{S}(\mathbf{B})$ of extensions of Banach spaces are subcategoríes of, respectively, $\mathfrak{Q}$ and $\mathfrak{S}$. As one might expect also $\mathfrak{Z}$ and $\mathfrak{S}(\mathbf{B})$ are equivalent categories, and so we shall identify them. Since this work is focused on the study of z-linear maps, most results are obtained in $\mathfrak{Z}$. However, when the results do not depend on the local convexity of the spaces and the generalization can be done without further difficulties, we shall state the results directly in $\mathfrak{Q}$.

0.2. $\mathfrak{Q}$ is an additive category. The following properties make of $\mathfrak{Q}$ an additive category:

(1) For each pair of objects F,G the set $\mathrm{Hom}_{\mathfrak{Q}}(F,G)$ admits a structure of abelian group defined by $(\alpha,\gamma)+(\alpha',\gamma')=(\alpha+\alpha',\gamma+\gamma')$. The *null* morphism of $\mathrm{Hom}_{\mathfrak{Q}}(F,G)$, which we shall denote 0, shall be the class of $(0,0)$. This addition is compatible with the *composition*, that is, the application $\mathrm{Hom}_{\mathfrak{Q}}(F,G)\times\mathrm{Hom}_{\mathfrak{Q}}(G,H)\to\mathrm{Hom}_{\mathfrak{Q}}(F,H)$, naturally defined as $(f,g)\circ(h,u)=(f\circ h,g\circ u)$ is bi-linear.
(2) There is an object whose identity is the null morphism; $\mathrm{Hom}_{\mathfrak{Q}}(0,0)=\{0\}=\{(id,id)\}$.

The property 2 defines the object *zero* of a category, which is essentially unique. For all objects F one has

$$\mathrm{Hom}_{\mathfrak{Q}}(F,0)=\{0\}=\mathrm{Hom}_{\mathfrak{Q}}(0,F).$$

The fact that the trivial objects of $\mathfrak{Q}$ coincide with the *zero* objects of the category can be interpreted as an indicator that the category $\mathfrak{Q}$ has been naturally defined.

Properties 1 and 2 jointly with the existence of finite products and coproducts in $\mathfrak{Q}$, that we shall show in the next section, ensure that $\mathfrak{Q}$ is an additive category.

0.3. Categories associated to $\mathfrak{Q}$. It can be checked that the basic homological tools can be interpreted inside $\mathfrak{Q}$ quite naturally. In particular, a push-out diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0\equiv F\\
 & & \big\downarrow{\scriptstyle\alpha} & & \big\downarrow & & \| & & \\
0 & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & Z & \longrightarrow & 0\equiv G,
\end{array}$$

corresponds to a morphism having the form (α, id), which we shall denote $F \xrightarrow{\alpha} G$; and a pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \equiv F \\ & & \| & & \uparrow & & \uparrow\gamma & & \\ 0 & \longrightarrow & Y & \longrightarrow & X' & \longrightarrow & Z' & \longrightarrow & 0 \equiv G, \end{array}$$

corresponds to a morphism (id, γ) which we shall denote $F \xleftarrow{\gamma} G$. Since those are specially interesting morphisms (they can be understood as the basic morphisms of $\mathfrak{Q}$) it shall be helpful to defined the associated categories to $\mathfrak{Q}$ in which the morphisms are only either $F \longrightarrow G$ or $F \longleftarrow G$. Let Z be a quasi-Banach space; the category $\mathfrak{Q}^Z$ shall have as objects $\equiv$-classes of quasi-linear maps $F : Z \curvearrowright \heartsuit$. Given a pair of objects F y G, a morphism $F \longrightarrow G$ shall be the class formed by all operators α such that $\alpha F \equiv G$. Analogously, given a quasi-Banach space Y, the category $\mathfrak{Q}_Y$ shall have as objects $\equiv$-classes of quasi-linear maps with range in Y. Given a pair of objects F, G, a morphism $G \longleftarrow F$ from F to G is the class of all operators γ such that $G\gamma \equiv F$.

A trivial object of $\mathfrak{Q}^Z$ shall be denoted 0, except when it is necessary to refer to the range space, in which case we shall write $0_E : Z \curvearrowright E$. Analogously, we shall write 0 to denote a trivial object of $\mathfrak{Q}_Y$, unless it is necessary to remark the domain space in which case we shall write $0^W : W \curvearrowright Y$.

0.3.1. **The categories $\mathfrak{Q}^Z$ y $\mathfrak{Q}_Y$ are not additive.** A simple remark shows that $\mathfrak{Q}^Z$ and $\mathfrak{Q}_Y$ are not additive categories:

$$\alpha F \equiv G, \quad \alpha' F \equiv G \implies (\alpha + \alpha')F \equiv 2G.$$

Certainly, this difficulty to provide an additive structure to the set of morphisms could be circumvented by defining the objects through a different equivalence relation $\equiv_\pi$ "to be projectively equivalent", defined as follows:

$$F \equiv_\pi F' \iff \exists \lambda \in \mathbb{R} : \ F \equiv \lambda F'.$$

However, we still do not obtain that the categories are additive. The reason is that when $F \neq 0$, the spaces $\mathrm{Hom}_{\mathfrak{Q}^Z}(0, F)$ and $\mathrm{Hom}_{\mathfrak{Q}_Y}(F, 0)$ still have not group structure. Indeed, although $\mathrm{Hom}_{\mathfrak{Q}^Z}(F, 0)$ has just one element, $\mathrm{Hom}_{\mathfrak{Q}^Z}(0, F) = \emptyset$ unless $F = 0$; this means that 0 is final but not initial in $\mathfrak{Q}^Z$. In $\mathfrak{Q}_Y$, one also has $\mathrm{Hom}_{\mathfrak{Q}_Y}(0, F) = \{0\}$ and, when $F \neq 0$, $\mathrm{Hom}_{\mathfrak{Q}_Y}(F, 0) = \emptyset$. Thus, 0 is initial and not final in $\mathfrak{Q}_Y$.

Further notation. We shall write $F \longleftrightarrow G$ when F and G are isomorphic in $\mathfrak{Q}$. Some distinguished isomorphisms in $\mathfrak{Q}$ deserve a special name: if either $F \longrightarrow G \longrightarrow F$ or $F \longleftarrow G \longleftarrow F$ we shall say that F and G are **semiequivalent**. If F and G are two extensions isomorphically equivalent then we shall say that the objects F and G are **strictly isomorphic**, and we shall denote it by $F \sim G$. It is simple to give examples in which $F \longleftrightarrow G$ for F and G two objects not strictly isomorphic. Just consider the following collection of objects of $\mathfrak{Q}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \quad \equiv F \\ 0 & \longrightarrow & E \oplus Y & \longrightarrow & E \oplus X & \longrightarrow & Z & \longrightarrow & 0 \equiv 0_E \oplus F \\ 0 & \longrightarrow & Y & \longrightarrow & X \oplus W & \longrightarrow & Z \oplus W & \longrightarrow & 0 \equiv F \oplus 0^W, \end{array}$$

in which it is clear that $F \longleftrightarrow 0_E \oplus F \longleftrightarrow F \oplus 0^W$. In this case we shall say that F and $0_E \oplus F$ are **elementarily isomorphic** and we shall denote it by $F \overset{e}{\sim} 0_E \oplus F$ (resp. $F \overset{e}{\sim} F \oplus 0^W$).

It is clear, following with the same notation , that if A and B are isomorphic spaces then $0_A \oplus F \sim 0_B \oplus F$ and $F \oplus 0^A \sim F \oplus 0^B$. In this context, when the space A is isomorphic to a twisted sum space $B \oplus_G C$ we shall write 0_G instead of the more appropriate $0_{B \oplus_G C}$.

1. Universal properties I: Finite Products and Coproducts. Pull-back and Push-out

It is important when studying a category to try to establish the existence of universal properties to identify which constructions are natural. A universal property is, depending on which language one prefers (see for instance [**32**]) either a representable functor or a right or left adjoint of a diagonal functor (that is, a limit or a colimit). To describe such properties we shall use the language more adequate in each situation. Thus, sometimes we shall speak of a universal object with respect to a certain diagram, sometimes we shall speak of a certain representable functor.

We shall first establish the existence of finite products and coproducts which, as we said in 0.2, make of $\mathfrak{Q}$ an additive category. At the same time as $\mathfrak{Q}$ we shall study the associated categories $\mathfrak{Q}^Z$ and $\mathfrak{Q}_Y$; in them, we shall study together products and pull-backs (the same limit-type properties, or right adjoint of a digonal functor); and then coproducts and push-outs (the same type of colimit-type property, or left adjoint of a digonal functor).

1.1. Finite products in $\mathfrak{Q}$. We begin recalling the definition of product of a family $(A_i)_i$ of object of a category $\mathfrak{C}$. The product in $\mathfrak{C}$ of $(A_i)_i$ is an object ΠA_i plus a family of morphisms $\pi_j : \Pi A_i \to A_j$ such that for each object B they induce a bijection

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{C}}(B, \prod A_i) & \longrightarrow & \prod_i \mathrm{Hom}_{\mathfrak{C}}(B, A_i) \\ T & \dashrightarrow & (\pi_i \circ T)_i. \end{array}$$

As one might guess, the product in $\mathfrak{Q}$ of finite families always exist. Let $F_i : Z_i \curvearrowright Y_i$ be a finite family of objects of $\mathfrak{Q}$. The product of (F_i) is the class generated by the quasi-linear map $\Pi F_i : \Pi Z_i \curvearrowright \Pi Y_i$ that takes at $(z_i) \in \Pi Z_i$ the value $(F_i z_i)$, together with the morphisms $(\pi_j, \eta_j) : \Pi F_i \rightrightarrows F_j$, where $\pi_j : \Pi Y_i \to Y_j$ and $\eta_j : \Pi Z_i \to Z_j$ are the canonical projections. For each object F the map induce by $(\pi_i, \eta_i)_i$ is indeed a bijection

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{Q}}(F, \prod F_i) & \longrightarrow & \prod \mathrm{Hom}_{\mathfrak{Q}}(F, F_i) \\ ((\alpha_i), (\gamma_i)) & \dashleftarrow & (\alpha_i, \gamma_i)_i. \end{array}$$

this bijection is moreover an isomorphism when the natural vector space structures are involved; this guarantees the quite convenient property that the finite product of trivial objects is trivial.

We have omitted on purpose when writing ΠZ_i and ΠY_i any reference to the topology set on the spaces. The reason is clear: no matter which quasi-norm compatible with the product is set, the objects ΠF_i are strictly isomorphic. The extensions defined by the product object ΠF_i are, depending on which quasi-norm $\|\cdot\|_p$, $0 < p \leq \infty$ is chosen, the so-called $l_p - amalgamas$:

$$0 \longrightarrow l_p(Y_i) \xrightarrow{(j_i)} l_p(X_i) \xrightarrow{(q_i)} l_p(Z_i) \longrightarrow 0 \equiv l_p(F_i).$$

Here, j_i and q_i are the operators defining the extension

$$0 \longrightarrow Y_i \xrightarrow{j_i} X_i \xrightarrow{q_i} Z_i \longrightarrow 0 \equiv F_i.$$

1.2. Finite product and pull-back in $\mathfrak{Q}^Z$. Let Z be a quasi-Banach space. The product in $\mathfrak{Q}^Z$ of a finite family $F_i : Z \curvearrowright Y_i$ is obtained by composing the product $\Pi F_i : \Pi_i Z \curvearrowright \Pi_i Y_i$ in $\mathfrak{Q}$ with the diagonal operator $D : Z \to \Pi_i Z$. The extension, described by $(\Pi F_i) \circ D$ is the one appearing in the pull-back diagram:

$$\begin{tikzcd} 0 \arrow[r] & \prod Y_i \arrow[r] & \prod X_i \arrow[r] & \prod Z \arrow[r] & 0 \equiv \prod F_i \\ 0 \arrow[r] & \prod Y_i \arrow[r] \arrow[u, equal] & PB \arrow[r] \arrow[u] & Z \arrow[r] \arrow[u, "\Delta"'] & 0 \equiv (\prod F_i)D. \end{tikzcd}$$

It is perhaps worth to remark that $(\Pi F_i)D$ is an equivalent extension to that obtained by making the *extended pull-back* of the collection of quotient maps q_i of the extensions:

$$0 \longrightarrow Y_i \xrightarrow{j_i} X_i \xrightarrow{q_i} Z \longrightarrow 0 \equiv F_i.$$

The pull-back of $(q_i)_i$ in $\mathbf{Q}$ is $P_\infty = \{(x_i) \in \Pi X_i : \forall j,k \;\; q_j x_j = q_k x_k\}$. The quotient operator $q : P_\infty \to Z$ shall be $q(x) = q_i x_i$ whose kernel is $\ker q = \{x \in P_\infty : \forall i \;\; q_i x_i = 0\} = \Pi Y_i$. It is easy to verify now that the extension

$$0 \longrightarrow \prod Y_i \xrightarrow{\ j\ } P_\infty \xrightarrow{\ q\ } Z \longrightarrow 0$$

comes also described by $(\Pi F_i)D$.

A slightly surprising phenomenon is that given a diagram in $\mathfrak{Q}^Z$,

$$\begin{array}{ccc} F_1 & \xrightarrow{\alpha_1} & F \\ & & \uparrow{\scriptstyle \alpha_2} \\ & & F_2, \end{array}$$

the pull-back is the product of F_1 and F_2. thus, one has a completed diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{\alpha_1} & F \\ \uparrow{\scriptstyle \pi_1} & & \uparrow{\scriptstyle \alpha_2} \\ F_1 \Pi F_2 & \xrightarrow{\pi_2} & F_2. \end{array}$$

It is clear that the diagram is commutative since $(\alpha_1 F_1 \equiv \alpha_2 F_2)$. At the same time, the universal property of the product guarantees that given an object $P : Z \curvearrowright E$ and morphisms $a : P \longrightarrow F_1$, $b : P \longrightarrow F_2$, the morphism $P \longrightarrow F_1 \Pi F_2$ defined by the operator $(a,b) : E \to Y_1 \Pi Y_2$, $(a,b)(e) = (ae, be)$ is the only one that makes $\pi_1 \circ (a,b) \asymp a$ y $\pi_2 \circ (a,b) \asymp b$. This allows us to conclude that $F_1 \Pi F_2$ has the universal property of the pull-back since a and b verify the equation $\alpha_1 \circ a \asymp \alpha_2 \circ b$ in $\mathfrak{Q}^Z$.

1.3. Finite product and pull-back in $\mathfrak{Q}_Y$. We do not know if $\mathfrak{Q}_Y$ admits finite products; maybe the difficulties stem from the fact that $\mathrm{Hom}_{\mathfrak{Q}_Y}(0, F) = \emptyset$ when $F \neq 0$, which is the reason why $\mathfrak{Q}_Y$ is not an additive category.

So, the problem of the existence of pull-back is genuinely interesting in this case, as we shall see next. Let us observe the diagram $\mathfrak{Q}_Y$;

$$\begin{array}{ccc} F_0 & \xleftarrow{\gamma} & F \\ \uparrow{\scriptstyle \gamma'} & & \\ F', & & \end{array}$$

whose meaning in terms of extensions is:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \xrightarrow{q} & Z & \longrightarrow & 0 \equiv F \\ & & \| & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_0 & \xrightarrow{q_0} & Z_0 & \longrightarrow & 0 \equiv F_0 \\ & & \| & & \uparrow{\scriptstyle \beta'} & & \uparrow{\scriptstyle \gamma'} & & \\ 0 & \longrightarrow & Y & \longrightarrow & X' & \xrightarrow{q'} & Z' & \longrightarrow & 0 \equiv F'. \end{array}$$

It is not difficult to verify the existence of an exact sequence

$$0 \longrightarrow Y \xrightarrow{\ \varsigma\ } PB(\beta, \beta') \xrightarrow{\ \eta\ } PB(\gamma, \gamma') \longrightarrow 0 \equiv FB,$$

defined by the natural morphisms $\varsigma(y) = ((y,0),(y,0))$ and $\eta((x,z),(x',z')) = (z,z')$. The sequence FB describes an object of $\mathfrak{Q}_Y$ making commutative the diagram

$$\begin{tikzcd} F_0 & F \arrow[l, "\gamma"'] \\ F' \arrow[u, "\gamma'"] & FB, \arrow[l, "u_{\gamma'}"] \arrow[u, "u_\gamma"'] \end{tikzcd}$$

where u_γ, $u_{\gamma'}$ are the operators associated to the pull-back of (γ,γ') in $\mathbf{Q}$.

That FB has the universal property of the pull-back is much more tricky to prove. The difficulties begin when one wants to deduce that FB is universal with respect to the diagram 1.3 using the universal property of $PB(\gamma,\gamma')$ in $\mathbf{Q}$. That is, we know that given an object $P : W \curvearrowright Y$ and morphisms $a : P \to F'$, $b : P \to F$ in $\mathfrak{Q}_Y$ one has the equality $\gamma' a \asymp \gamma b$ in $\mathfrak{Q}$, which means $(\gamma' a - \gamma b)P \equiv 0$. However, in order to apply the universal property of the pull-back construction in quasi-Banach spaces one would need the equality $\gamma' a = \gamma b$ in $\mathbf{Q}$. Let us show that that is possible after choosing adequate representatives at each moment:

Proposition 1.1. *The object FB is the pull-back in $\mathfrak{Q}_Y$ of $F_0 \xleftarrow{\gamma} F$ and $F_0 \xleftarrow{\gamma'} F'$.*

Proof. Let $0 \to Y \to H \xrightarrow{Q} W \to 0 \equiv G$ be such that there exist morphisms such that $F \xleftarrow{\alpha} G$ and $F' \xleftarrow{\alpha'} G$ verifying $F_0\gamma\alpha \equiv F_0\gamma'\alpha'$. That is, that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] & X \arrow[r, "q"] & Z \arrow[r] & 0 \equiv F_0\gamma \equiv F \\ 0 \arrow[r] & Y \arrow[r] \arrow[u, equal] \arrow[d, equal] & H \arrow[r, "Q"] \arrow[u, "\xi"'] \arrow[d, "\xi'"] & W \arrow[r] \arrow[u, "\alpha"'] \arrow[d, "\alpha'"] & 0 \equiv G \\ 0 \arrow[r] & Y \arrow[r] & X' \arrow[r, "q'"'] & Z' \arrow[r] & 0 \equiv F_0\gamma' \equiv F'. \end{tikzcd}$$

Since $\gamma\alpha$ and $\gamma'\alpha'$ describe the same morphism in $\mathfrak{Q}_Y$, the triples $(id, \beta\xi, \gamma\alpha)$ and $(id, \beta'\xi', \gamma'\alpha')$ are homotopic, which means the existence of operators $\tau : H \to Y$ y $\theta : W \to X_0$ such as those of the diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j_0"] & X_0 \arrow[r, "q_0"] & Z_0 \arrow[r] & 0 \equiv F_0 \\ 0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & H \arrow[r, "Q"'] \arrow[u] \arrow[ul, "\tau"] & W \arrow[r] \arrow[u] \arrow[ul, "\theta"] & 0 \equiv G \end{tikzcd}$$

in such a way that

(1) $\gamma\alpha - \gamma'\alpha' = q_0\theta$
(2) $\beta\xi - \beta'\xi' = j_0\tau + \theta Q$.

Those equalities make the commutative the square

$$\begin{tikzcd} X_0 \arrow[r, "q_0"] & Z_0 \\ H \arrow[r, "\alpha Q"'] \arrow[u, "\theta Q + \beta'\xi'"] & Z \arrow[u, "\gamma"'] \end{tikzcd}$$

since $\gamma\alpha Q = q_0\beta\xi = q_0 j_0\tau + q_0\theta Q + q_0\beta'\xi' = q_0(\theta Q + \beta'\xi')$. Therefore, there will be an operator $h : H \to X$ such that $\beta h = \beta'\xi' - \theta Q$ and $qh = \alpha Q$.

Reasoning the same way with the analogous square 1.3 for q_0 and γ', we find an operator $h' : H \to X'$ such that $\beta' h' = \beta\xi - \theta Q$ y $q'h' = \alpha' Q$.

We have a new commutative square

$$\begin{array}{ccc} X & \xrightarrow{\beta} & X_0 \\ {\scriptstyle \xi - h}\uparrow & & \uparrow{\scriptstyle \beta'} \\ H & \xrightarrow[h'-\xi']{} & X'. \end{array}$$

On one hand $\beta(h-\xi) = \beta'\xi' - \theta Q - \beta\xi = j_0\tau + \theta Q - \theta Q = j_0\tau$ and on the other hand $\beta'(h'-\xi') = \beta\xi - \theta Q - \beta'\xi' = j_0\tau + \theta Q - \theta Q = j_0\tau$.

There exists therefore an operator $r : H \to PB(\beta, \beta')$ such that $u_\beta r = \xi - h$ y $u_{\beta'} r = h' - \xi'$. The operator r gives us a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\varsigma} & PB(\beta,\beta') & \xrightarrow{\eta} & PB(\gamma,\gamma') & \longrightarrow & 0 \\ & & \| & & {\scriptstyle r+\varsigma\tau}\uparrow & & \uparrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & H & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

as soon as we show that $r : Y \to Y$. Since $r(a) = ((\xi - h)(a), (h' - \xi')(a))$ one has $r(y) = ((y,0) - (\theta Q y - \beta'\xi' y, \alpha Q y), (\beta\xi y - \theta Q y, \alpha' Q y) - (y,0)) = ((y - \beta'\xi' y, 0), (\beta\xi y - y, 0))$. Therefore $ry = 0$, and $r + \varsigma\tau$ is the operator we were looking for . □

1.4. Finite coproduct in $\mathfrak{Q}$. Let us recall the definition of coproduct of an arbitrary family in a category $\mathfrak{C}$. The coproduct in $\mathfrak{C}$ of $(A_i)_i$ is an object $\oplus A_i$ and a family of morphisms $j_k : A_k \to \oplus A_i$ that, for each object B, induce a bijection

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{C}}(\bigoplus A_i, B) & \longrightarrow & \prod \mathrm{Hom}_{\mathfrak{C}}(A_i, B) \\ T & \leftrightsquigarrow & (T \circ j_i)_i. \end{array}$$

Let us see that the coproduct of finite families always exist in $\mathfrak{Q}$. Thus, the coproduct of $(F_i : Z_i \curvearrowright Y_i)_{i=1}^n$ is the object $\oplus F_i : \oplus Z_i \curvearrowright \oplus Y_i$ that acts $\oplus F_i(z_i)_i = (F_i z_i)_i$, endowed with the morphisms
$(\epsilon_j, \upsilon_j) : F_j \rightrightarrows \oplus F_i$ where $\epsilon_j : Y_j \to \oplus Y_i$ and $\upsilon_j : Z_j \to \oplus Z_i$ are the natural inclusion . For each object F, the natural correspondence induced by $(\epsilon_i, \upsilon_i)_i$ is a bijection;

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{Q}}(\bigoplus F_i, F) & \longrightarrow & \prod \mathrm{Hom}_{\mathfrak{Q}}(F_i, F) \\ ((\alpha_i),(\gamma_i)) & \dashleftarrow & (\alpha_i, \gamma_i)_i \end{array}$$

As we have already observed, $\oplus F_i$ and ΠF_i are the same object in $\mathfrak{Q}$, although they have different universal properties. Again, endowing the vector spaces $\oplus_i Y_i$ and $\oplus_i Z_i$ with different quasi-norms we obtain isomorphic coproducts. The extensions defined by $\oplus F_i$ are, when endowed with the quasi-norms $\|\cdot\|_p$ $(0 < p \le \infty)$, the corresponding l_p-amalgams:

$$0 \longrightarrow l_p(Y_i) \xrightarrow{(j_i)} l_p(X_i) \xrightarrow{(q_i)} l_p(Z_i) \longrightarrow 0 \equiv l_p(F_i),$$

in which (j_i), (q_i) are the operators appearing in the extensions

$$0 \longrightarrow Y_i \xrightarrow{j_i} X_i \xrightarrow{q_i} Z_i \longrightarrow 0 \equiv F_i.$$

1.5. Finite coproduct and push-out in $\mathfrak{Q}_Y$. Let Y be a quasi-Banach space. The coproducto in $\mathfrak{Q}_Y$ of a finite family $F_i : Z_i \curvearrowright Y$ can be obtained by composition of the coproduct $\oplus F_i : \oplus_i Z \curvearrowright \oplus_i Y_i$ in $\mathfrak{Q}$ with the operator sum $\sum : \oplus_i Y \to Y$ defined as $\sum(y_i) = \sum_i y_i$. The extension described by $\sum \circ (\oplus F_i) = \sum F_i$ is what appears in the push-out diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_i Y & \longrightarrow & \bigoplus_i X_i & \longrightarrow & \bigoplus_i Z_i & \longrightarrow & 0 \equiv \bigoplus F_i \\ & & \downarrow{\scriptstyle \Sigma} & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & PO & \longrightarrow & \bigoplus_i Z_i & \longrightarrow & 0 \equiv \sum F_i. \end{array}$$

Of course, the morphisms associated to the coproduct are those induced by the natural inclusions $Z_j \hookrightarrow \oplus_i Z_i$. It is interesting to observe that the extension $\sum F_i$ is equivalent to that obtained making the so-called *extended push-out* of the collection of embeddings j_i in the extensions

$$0 \longrightarrow Y_i \xrightarrow{j_i} X_i \xrightarrow{q_i} Z \longrightarrow 0 \equiv F_i.$$

The push-out space of (j_i) in $\mathbf{Q}$ is $P_1 = \oplus X_i/\mathcal{D}$ where $\mathcal{D} = \{(j_i(y))_i : y \in Y\}$. The embedding $j : Y \to P_1$ shall be $j(y) = (j_i y)$, and induces a quotient map $Q = \{(x_i + j_i(Y))_i : (x_i)_i \in P_1\} = \oplus_i(Z_i)$. It is easy to check that the extension

$$0 \longrightarrow Y \xrightarrow{j} P_1 \xrightarrow{q} \oplus(Z_i) \longrightarrow 0$$

also comes described by $\sum F_i$.

Analogously to the case of the product in $\mathfrak{Q}^Z$, we now have that given a diagram with the form

$$\begin{tikzcd} F_1 & F \arrow[l, "\alpha_1"'] \arrow[d, "\alpha_2"] \\ & F_2, \end{tikzcd}$$

the push-out coincides with the coproduct of $F_1 : Z_1 \curvearrowright Y$ and $F_2 : Z_2 \curvearrowright Y$. So, the diagram is completed with the coproduct $(F_1 \oplus F_2; j_1, j_2)$, and one has:

$$\begin{tikzcd} F_1 \arrow[d, "j_1"] & F \arrow[l, "\alpha_1"'] \arrow[d, "\alpha_2"] \\ F_1 \oplus F_2 & F_2. \arrow[l, "j_2"] \end{tikzcd}$$

It is evident that the diagram is commutative $(F_1\alpha_1 \equiv F_2\alpha_2)$. The universal property of the coproduct guarantees that given an object $C : Z \curvearrowright Y$ and morphisms $C \xleftarrow{a} F_1$, $C \xleftarrow{b} F_2$, the morphism $C \longleftarrow F_1 \oplus F_2$ defined by the operator $(a, b) : Z_1 \oplus Z_2 \to Z$, $(a, b)(z_1, z_2) = (az_1 + bz_2)$ is the only making $(a, b)j_1 \asymp a$ and $(a, b)j_2 \asymp b$. What allows us to conclude that $F_1 \oplus F_2$ has the universal property of the push-out is that a and b verify $a\alpha_1 \asymp b\alpha_2$.

1.6. Funite coproduct and push-out in $\mathfrak{Q}^Z$. We do not know if there exists coproduct of finite families in $\mathfrak{Q}^Z$. Thus, the existence of push-out must be treated as an independent problem. Let us observe the diagram in $\mathfrak{Q}^Z$;

$$\begin{tikzcd} F_0 \arrow[r, "\alpha"] \arrow[d, "\alpha'"] & F \\ F' & \end{tikzcd}$$

whose meaning in terms of extensions is:

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] & X \arrow[r] & Z \arrow[r] & 0 \equiv F \\ 0 \arrow[r] & Y_0 \arrow[r] \arrow[u, "\alpha"] \arrow[d, "\alpha'"] & X_0 \arrow[r] \arrow[u, "\widehat{\alpha}"] \arrow[d, "\widehat{\alpha'}"] & Z \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \equiv F_0 \\ 0 \arrow[r] & Y' \arrow[r] & X' \arrow[r] & Z \arrow[r] & 0 \equiv F'. \end{tikzcd}$$

It is easy to show that there is an exact sequence

$$0 \longrightarrow PO(\alpha, \alpha') \longrightarrow PO(\widehat{\alpha}, \widehat{\alpha'}) \longrightarrow Z \longrightarrow 0 \equiv FO$$

defining an object FO of $\mathfrak{Q}^Z$ that makes commutative the diagram

$$\begin{array}{ccc} F_0 & \xrightarrow{\alpha} & F \\ \Big\downarrow{\scriptstyle \alpha'} & & \Big\downarrow{\scriptstyle u_\alpha} \\ F' & \xrightarrow[u_{\alpha'}]{} & FO, \end{array}$$

where u_α, $u_{\alpha'}$ are the operators associated to the push-out of (α, α') in $\mathbf{Q}$.

To prove that the object FO has the universal property of the push-out is a little bit more complicated. One would like to deduce that FO is universal with respect to the preceding commutative diagram from the fact that FO is defined on $PO(\alpha, \alpha')$. That is, given an object $P : Z \curvearrowright W$ and morphisms $a : F' \longrightarrow P$, $b : F \longrightarrow P$ in $\mathfrak{Q}^Z$ one has $a\alpha' \asymp b\alpha$. This equality in the category means $(a\alpha' - b\alpha)P \equiv 0$. We need, however, the equality $a\alpha' = b\alpha$ in $\mathbf{Q}$ to be able to use the universal property of the push-out.

Dualizing the proof of Proposition 1.1 we obtain:

Proposition 1.2. *The object FO is the push-out in $\mathfrak{Q}^Z$ of $\alpha : F_0 \longrightarrow F$ y $\alpha' : F_0 \longrightarrow F'$.*

2. Universal properties II: An approach to arbitrary Products and Coproducts in $\mathfrak{Q}$.

In general, the existence of products and coproducts for finite families in a category $\mathfrak{C}$ does not present problems. Nevertheless, the product of infinite families often presents difficulties and even sometimes the "usual" definition of product is clearly inadequate. One may observe the following phenomenon in some categories: given an infinite family (A_i), there exists an object $\mathcal{P}$ and a collection of morphisms $\pi_i : \mathcal{P} \to A_i$ such that, for all objects C, the natural induced correspondence

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{C}}(C, \mathcal{P}) & \longrightarrow & \prod_i \mathrm{Hom}_{\mathfrak{C}}(C, A_i) \\ T & \dashrightarrow & (\pi_i \circ T)_i \end{array}$$

is not a bijection onto the whole range space, but only onto a region $\mathcal{D}(C) \subset \prod_i \mathrm{Hom}_{\mathfrak{C}}(C, A_i)$. We shall say that the couple $(\mathcal{P}, (\pi_i))$ is a **restricted product** or a $\mathcal{D}$-product of (A_i).

Dually, we can also speak of restricted coproducts or $\mathcal{D}$-coproducts $(\mathcal{C}, (j_i))$ of a family $(A_i)_i$ when (j_i) establishes, for each object E, a bijection between $\mathrm{Hom}_{\mathfrak{C}}(\mathcal{C}, E)$ and a proper subset $\mathcal{D}(E)$ of $\prod_i \mathrm{Hom}_{\mathfrak{C}}(A_i, E)$.

A type of categories in which the notion of restricted product is necessary are those termed *augmented categories.* It is said that $\mathfrak{C}$ is an augmented category, although we hasten to remark that this is just a vague notion (see for instance [**74**]), if there exists an additional structure on the sets $\mathrm{Hom}_{\mathfrak{C}}(A, B)$. Put it otherwise, one has that the functor $\mathrm{Hom}_{\mathfrak{C}}(\,\cdot\,, B)$ takes values in richer categories than Set. For instance, in the categories: **Vect** of vector spaces; **Top** of topological spaces; **B** of Banach spaces, etc... The more interesting examples for us of augmented categories are **Q** and **B**, since the spaces of morphisms $\mathcal{L}(A, B)$ have a Banach space structure.

It is important to observe that a restricted product can be as "good" as a standard product, in the sense that it can also reflect a universal property: let $(P, (\pi_i))$ a $\mathcal{D}$-product of a family (A_i) in a category $\mathfrak{C}$. Let us consider the map $\mathcal{D}(\cdot) : \mathfrak{C} \to \mathbf{Set}$ assigning to each object C the region $\mathcal{D}(C) \subset \prod_i \mathrm{Hom}_{\mathfrak{C}}(C, A_i)$ onto which the natural correspondence induced by (π_i) is a bijection. If $\mathcal{D}(\cdot)$ is a functor then the $\mathcal{D}$-product is a *universal property.* We can make this affirmation precisely because $\mathcal{D}(\cdot)$ is a *representable functor.*

Definition 1.2 (Representable functors). *A covariant functor $\mathbf{F} : \mathfrak{C} \longrightarrow \mathbf{Set}$ is said to be representable if there is some object M of $\mathfrak{C}$ such that F and $\mathrm{Hom}_{\mathfrak{C}}(M, \,\cdot\,)$ are isomorphic functors. Analogously, a contravariant functor $\mathbf{F} : \mathfrak{C} \longrightarrow \mathbf{Set}$ is said to be representable if there exists an object M de $\mathfrak{C}$ such that F and $\mathrm{Hom}_{\mathfrak{C}}(\,\cdot\,, M)$ are isomorphic functors.*

There is well-known algebraic result, see for instance [**62**], asserting that if F is representable, the corresponding isomorphism between functors comes defined in a canonical way by some element ξ of $F(M)$. So, it is said that the pair (M, ξ) represents the functor F. It is clear that the representative of a functor is unique.

Since each universal property is nothing else but the affirmation that a certain functor F is represented by a give pair (M, ξ), we shall have that a $\mathcal{D}$-product (-coproduct) is a universal property as soon as $\mathcal{D}(\cdot)$ defines a functor.

Restricted products and coproducts in Q. To gain some familiarity with these universal product-like properties, and to set a starting point for a further study in $\mathfrak{Q}$, let us see the different types of restricted products and coproducts that exist in the category of quasi-Banach spaces.

We know that $\mathbf{Q}$ does not necessarily admit products of infinite families $(E_i)_i$ because the vector space $\Pi_i E_i$, that should play that role, does not admit a quasi-normed structure. Moreover, it is clear that no family of operators $\pi_i : P \to A_i$ can establish a bijection

$$\begin{array}{ccc} \mathcal{L}(C, P) & \longrightarrow & \prod_i \mathcal{L}(C, A_i) \\ T & \dashrightarrow & (\pi_i \circ T)_i \end{array}$$

since we can always choose an element in the right space (T_i) so that the sequence $(\|T_i\|)$ is "faster" than $(\|\pi_i\|)$.

The natural notions behind the well-known c_0 and l_p-amalgams, $0 < p \leq \infty$, of quasi-Banach spaces turn out to be the restricted products. Let (A_i) be a family of q-Banach spaces.

$\mathbf{l_\infty}$-Product: The l_∞-product of (A_i) is the representative $[l_\infty(A_i), (\pi_i)]$ of the contravariant functor $l_\infty [\mathcal{L}(\,\cdot\,, A_i)] : \mathbf{Q} \to \mathbf{Set}$ that assigns to each object X of $\mathbf{Q}$ the set $l_\infty [\mathcal{L}(X, A_i)]$ of uniformly bounded families of operators $X \to A_i$. The applications $\pi_i : l_\infty(A_i) \to A_i$ are the natural projections. Therefore, there is, for each X, a bijection

$$\mathcal{L}(X, l_\infty(A_i)) \longleftrightarrow l_\infty [\mathcal{L}(X, A_i)] .$$

$\mathbf{l_p}$-Product, $\mathbf{0 < p < +\infty}$: The l_p-product of the family (A_i) is the representative $[l_p(A_i), (\pi_i)]$ of the functor $l_p^{SOT} [\mathcal{L}(\,\cdot\,, A_i)] : \mathbf{Q} \to \mathbf{Set}$ that assigns to each X the set $l_p^{SOT} [\mathcal{L}(X, A_i)]$ of pointwise p-summable families of operators $X \to A_i$. Again, $\pi_i : l_p(A_i) \to A_i$ are the natural projections. Therefore, there is, for each X, a bijection

$$\mathcal{L}(X, l_p(A_i)) \longleftrightarrow l_p^{SOT} [\mathcal{L}(X, A_i)] .$$

$\mathbf{c_0}$-Product: The c_0-product of (A_i) is the representative $[c_0(A_i), (\pi_i)_i]$ of the functor $c_0^{SOT} [\mathcal{L}(\,\cdot\,, A_i)] : \mathbf{Q} \to \mathbf{Set}$ that assigns to each object X the set $c_0^{SOT} [\mathcal{L}(X, A_i)]$ of pointwise convergent to 0 families of operators $X \to A_i$. The maps $\pi_i : l_p(A_i) \to A_i$ are the natural projections. Therefore, there is, for each X, a bijection

$$\mathcal{L}(X, c_0(A_i)) \longleftrightarrow c_0^{SOT} [\mathcal{L}(X, A_i)] .$$

$\mathbf{l_q}$- Coproduct in $\mathbf{Q_q}$, $\mathbf{0 < q \leq 1}$: The l_q-coproduct of (A_i) is the representative $[l_q(A_i), (j_i)_i]$ of the functor $l_\infty [\mathcal{L}(A_i, \,\cdot\,)] : \mathbf{Q_q} \to \mathbf{Set}$ that assigns to each object X the set $l_\infty [\mathcal{L}(A_i, X)]$ of uniformly bounded families of operators $A_i \to X$. The maps $j_i : A_i \to l_1(A_i)$ are the natural inclusions. Therefore, there is, for each X, a bijection induced by (j_i);

$$\mathcal{L}(l_1(A_i), X) \longleftrightarrow l_\infty [\mathcal{L}(A_i, X)] .$$

Except in the case of the l_p-product, the previous bijections are moreover isomorphisms of Banach spaces.

2.1. What amalgams of quasi-linear maps conceal. The existence of restricted products (resp. coproducts) in $\mathfrak{Q}$ analogous to those of $\mathbf{Q}$ does not appear to be impossible. We start observing that $\mathfrak{S}$ admits some constructions corresponding to the amalgams of $\mathbf{Q}$: given an adequate family $\{0 \to Y_i \to X_i \to Z_i \to 0 : i \in I\}$ we can construct the objects $0 \to l_\infty(Y_i) \to l_\infty(X_i) \to l_\infty(Z_i) \to 0$, $0 \to l_p(Y_i) \to l_p(X_i) \to l_p(Z_i) \to 0$, con $p > 0$, $0 \to c_0(Y_i) \to c_0(X_i) \to c_0(Z_i) \to 0$,.... Nevertheless, we are not quite sure of being able to tell one of the other, except by the quasi-norms with which they are made. Also, we do not know which properties they possess or whether they are universal with respect to them (a fact that will make them useful tools in $\mathfrak{S}$). For instance, Are those objects the corresponding l_∞, l_p, c_0-products in $\mathfrak{S}$? Do those objects depend and in which sense of the exact sequences chosen as representatives? Let us first examine the amalgamation processes for exact sequences (without equivalence). We consider first the l_∞-amalgam.

Intuition suggests that given a collection $0 \to Y_i \to X_i \to Z_i \to 0 \equiv F_i$, the l_∞-product extension should be the l_∞-amalgam

$$0 \longrightarrow l_\infty(Y_i) \longrightarrow l_\infty(X_i) \longrightarrow l_\infty(Z_i) \longrightarrow 0$$

in such a way that the l_∞-product of the quasi-linear maps F_i would be the quasi-linear map associated to that extension. Let us see when the l_∞-amalgam extension is well-defined and which is the associated quasi-linear map.

• Let us first observe that it is necessary that the family of extensions be "uniform" so that the l_∞-amalgam turn out to be an exact sequence in $\mathbf{Q}$; i.e., that the family of their parameters ρ_i, **??** must be bounded. That information can also be found in the associated quasi-linear maps F_i. Thus, given a family F_i, if we form the canonical extensions

$$0 \longrightarrow Y_i \longrightarrow Y_i \oplus_{F_i} Z_i \longrightarrow Z_i \longrightarrow 0 \equiv F_i,$$

we find that the constant of concavity C_i for the quasi-norm $\|\cdot\|_{F_i}$ can be a problem to make the amalgam. Since, luckily, that constant is $Q(F_i)$, to make $l_\infty(Y_i \oplus_{F_i} Z_i)$ a quasi-Banach space we'll have to ask $\sup_i Q(F_i) < +\infty$. If (F_i) are z-linear maps, to make the twisted sums space $l_\infty(Y_i \oplus_{F_i} Z_i)$ a Banach space we can replace each $Y_i \oplus_{F_i} Z_i$ by its Banach envelope $\mathrm{co}(Y_i \oplus_{F_i} Z_i)$; in this way we need not to worry about the concavity constants. We have to worry however about the distances $\mathrm{d}(Y_i \oplus_{F_i} Z_i, \mathrm{co}(Y_i \oplus_{F_i} Z_i))$, in order that

$$0 \longrightarrow Y_i \longrightarrow \mathrm{co}(Y_i \oplus_{F_i} Z_i) \longrightarrow Z_i \longrightarrow 0$$

be still a family "uniformly equivalent" to the original. Luckily once more, that distance is $Z(F_i)$, and thus the condition to amalgamate "canonical " extensions associated to families of z-linear maps (F_i) is that $\sup Z(F_i) < +\infty$.

Those remarks suggest that *to amalgamate a family of objects* (F_i) of $\mathfrak{Q}$ requires at least the existence of a family (F_i) of representatives (quasi-linear maps) $Q(\cdot)$-bounded. That condition exactly means to ask $\sup_i \mathcal{Q}(F_i) < +\infty$ recalling that for a given object F of $\mathfrak{Q}$ one had defined $\mathcal{Q}(F) = \inf_{\text{versiones F' de } F} Q(F')$. We shall say in that case that (F_i) is a $\mathcal{Q}$-bounded family of $\mathfrak{Q}$.

• Keeping in mind the previous observations, it is not difficult to show that

$$0 \longrightarrow l_\infty(Y_i) \longrightarrow l_\infty(Y_i \oplus_{F_i} Z_i) \longrightarrow l_\infty(Z_i) \longrightarrow 0$$

has as associated quasi-linear map something very close to the product $\prod F_i$, understood as the simple product of map. Let $(F_i : Z_i \curvearrowright Y_i)$ be a Q-bounded family of quasi-linear maps. Let us consider the map

$$\Pi F_i : l_\infty(Z_i) \curvearrowright \Pi Y_i$$

taking the value $(F_i(z_i))$ at $(z_i) \in l_\infty(Z_i)$. The point is to bring $(F_i(z_i))_i$ from the product $\Pi_i Y_i$ down to $l_\infty(Y_i)$. To that end, let us fix a normalized Hamel basis $(e_\alpha)_\alpha$ of $l_\infty(Z_i)$ and then construct the linearization

$$L_{\prod F_i} : l_\infty(Z_i) \longrightarrow \prod Y_i$$

of $\Pi_i F_i$ with respect to (e_α). Since the family (F_i) is Q-bounded, given a pair of points (z_i), $(z_i') \in l_\infty(Z_i)$ the family $(F_i(z_i + z_i') - F_i(z_i) - F_i(z_i'))_i$ of Cauchy differences belongs to $l_\infty(Y_i)$. Reasoning inductively, one gets that given a finite collection $\{(z_i^j)_i : j = 1, ..., n\}$ of elements of $l_\infty(Z_i)$, the collection $(F_i(\sum_{j=1}^n z_i^j) - \sum_{j=1}^n F_i(z_i^j))_i$ also belongs to $l_\infty(Y_i)$. We thus deduce that $\Pi_i F_i - L_{\Pi_i F_i}$ has its range in $l_\infty(Z_i)$; it is enough to write each point z of $l_\infty(Z_i)$ as a linear combination $z = \sum_\alpha \lambda_\alpha e_\alpha$ with respect to $(e_\alpha)_\alpha$. It is also immediate that $G := \Pi_i F_i - L_{\Pi_i F_i}$ is quasi-linear.

Finally, it is not difficult to show the existence of a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & l_\infty(Y_i) & \longrightarrow & l_\infty(Y_i \oplus_{F_i} Z_i) & \longrightarrow & l_\infty(Z_i) & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle T}\uparrow & & \| & & \\
0 & \longrightarrow & l_\infty(Y_i) & \longrightarrow & l_\infty(Y_i) \oplus_G l_\infty(Z_i) & \longrightarrow & l_\infty(Z_i) & \longrightarrow & 0,
\end{array}$$

in which T is the operator sending $((y_i), (z_i))$ to $(y_i, z_i)_i$.

It is immediate that the same type of construction can be made for the c_0-product and the l_p-product, that is, that an analogous construction shall provide the associated quasi-linear map that defines the c_0-amalgam

$$0 \longrightarrow c_0(Y_i) \longrightarrow c_0(Y_i \oplus_{F_i} Z_i) \longrightarrow c_0(Z_i) \longrightarrow 0$$

y la l_p-amalgam, $p > 0$,

$$0 \longrightarrow l_p(Y_i) \longrightarrow l_p(Y_i \oplus_{F_i} Z_i) \longrightarrow l_p(Z_i) \longrightarrow 0,$$

; the only condition is to ask that (F_i) be a Q-bounded family. We shall denote by $l_\infty(F_i)$, $l_p(F_i)$, $c_0(F_i)$, respectively, the quasi-linear maps that define the l_∞, l_p and c_0-amalgams, respectively.

2.1.1. **The derived space.** Behind the constructions $l_p(F_i)$, $0 < p \leq \infty$ and $c_0(F_i)$ one can see the simple fact that if we have a map $F : Z \to \Pi\mathbb{R}$ such that the Cauchy differences $F(z + z') - F(z) - F(z')$ belong to a certain quasi-Banach space Y, and verify an estimate $\|F(z + z') - F(z) - F(z')\|_Y \leq K(\|z\| + \|z'\|)$ then each canonical version $\Omega = F - L_F$ of F is a quasi-linear map $Z \curvearrowright Y$. the z-linear version is analogous. If we construct now the space d_Ω, often known as the "derived space"

$$d_\Omega = \{(v, z) \in \prod \mathbb{R} \times Z : v - Fz \in Y\}$$

endowed with the quasi-norm $\|(v, z)\|_\Omega = \|v - Fz\|_Y + \|z\|_Z$, oner obtains an exact sequence in $\mathbf{Q}$

$$0 \to Y \to d_\Omega \to Z \to 0.$$

It is immediate now to verify that this extension is equivalent to $0 \to Y \to Y \oplus_\Omega Z \to Z \to 0$.

Observation: It is not immediate to realize that, with an adequate choice of the Hamel basis involved (and after making completion to a basis of the bigger superspace) one has for $p < q$ morphisms $l_p(F_i) \rightrightarrows l_q(F_i)$ in $\mathfrak{Q}$; more yet, there exist commutative diagrams with the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & l_\infty(Y_i) & \longrightarrow & l_\infty(Y_i \oplus_{F_i} Z_i) & \longrightarrow & l_\infty(Z_i) & \longrightarrow & 0 \equiv l_\infty(F_i) \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & l_p(Y_i) & \longrightarrow & l_p(Y_i \oplus_{F_i} Z_i) & \longrightarrow & l_p(Z_i) & \longrightarrow & 0 \equiv l_p(F_i).
\end{array}$$

All this means that with a proper choice of the Hamel bases one can make that the restriction of $l_\infty(F_i)$ to $l_p(Z_i)$ be $l_p(Y_i)$-valued.

3. Universal properties III: Restricted products and coproducts of quasi-linear maps.

It is now clear that the product of an infinite family $(F_i : Z_i \curvearrowright Y_i)$ of objects of $\mathfrak{Q}$ needs not to exist; and it is so because even the existence of the product in $\mathbf{Q}$ of the families (Z_i) and (Y_i) is not guaranteed. The question we plan to consider now is whether or not the amalgams of quasi-linear maps provide the adequate notion of l_∞-product, l_p-product or c_0-product of the objects F_i in the category $\mathfrak{Q}$.

Everything seems to indicate that the role of the l_p, $0 < p \leq \infty$, and c_0-products of a $\mathcal{Q}$-bounded family (F_i) of objects of $\mathfrak{Q}$ should be played by the objects $l_p(F_i)$ and $c_0(F_i)$ constructed with a Q-bounded family of representatives. Nevertheless, if we accept that the desired product should have the property that the product of trivial objects is trivial, we are forced to discard the previous constructions. The reason is that, as we shall see, that it is perfectly possible that $l_\infty(F_i)$ is not trivial when (F_i) is a $\mathcal{Q}$-bounded family of trivial objects: to see this, let us choose $B_n : Z_n \to Y_n$ bounded maps such that $\operatorname{dist}(B_n, \mathbf{L}) \to \infty$ and $Q(B_n) \leq 1$ for all $n \in \mathbb{N}$. Let us form the (trivial) extensions $0 \to Y_n \to Y_n \oplus_{B_n} Z_n \to Z_n \to 0$ and then construct their l_∞-amalgam

$$0 \longrightarrow l_\infty(Y_n) \longrightarrow l_\infty(Y_n \oplus_{B_n} Z_n) \longrightarrow l_\infty(Z_n) \longrightarrow 0.$$

This extension is quite clearly not trivial. However, if we choose as representative of each object B_n the map $0 : Z_n \curvearrowright Y_n$ then we shall get as l_∞-amalgam a trivial extension. We have thus just shown that given a $\mathcal{Q}$-bounded family (F_i) of objects, it is possible to choose Q-bounded families of representatives (F_i') and (F_i'') such that the objects $l_\infty(F_i')$ and $l_\infty(F_i'')$, which should both be representatives of the l_∞-product in $\mathfrak{Q}$ of (F_i), are not even isomorphic.

All this makes it impossible to continue with the idea of making products of quasi-linear maps in terms of their equivalence classes. We shall look for a different approach. We define the category $\mathcal{Q}$ of quasi-linear maps in which the morphisms $F \rightrightarrows G$ are couples of operators (α, γ) such that $\alpha F \equiv G\gamma$. We shall denote by $\mathcal{Z}$ the subcategory of $\mathcal{Q}$ whose objects are z-linear maps. The corresponding category of short exact sequences of quasi-Banach spaces (without equivalence) shall be denoted $\mathcal{S}$; the morphisms are here triples (α, β, γ) of operators making commutative the diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, "\alpha"] & X \arrow[r] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & Y' \arrow[r] & X' \arrow[r] & Z' \arrow[r] & 0.
\end{tikzcd}$$

We shall write $\mathcal{S}(\mathbf{B})$ to denote the subcategory of $\mathcal{S}$ whose objects are extensions in $\mathbf{B}$.

Although it is not true that $\mathcal{Q}$ and $\mathcal{S}$ (or else $\mathcal{Z}$ and $\mathcal{S}(\mathbf{B})$) are equivalent categories, the fact is that what what one does in one of them can be reproduced in the other. Nevertheless, in order to translate results from one to the other, is good to keep in mind the following:

- The fact that a family (F_i) of quasi-linear maps is Q-bounded corresponds with the condition on a family (E_i) of exact sequences of being ρ-bounded.
- We define below at $(*)$ the notion of uniformly representable family of morphisms $(\alpha_i, \gamma_i) : F_i \rightrightarrows G_i$. This notion corresponds to the idea of having a family of morphisms $(\alpha_i, \beta_i, \gamma_i)$ in $\mathcal{S}$ so that (α_i), (β_i) and (γ_i) are uniformly bounded collections of operators.

► **l_∞-Product in $\mathcal{Q}$.** Our plan is to define a representable functor whose representative in $\mathcal{Q}$ is the l_∞-amalgam of quasi-linear maps.

We shall denote by $l_\infty(Q(X_i, Y_i))$ the space formed by all uniformly Q-bounded collections $(F_i : X_i \curvearrowright Y_i)$ of quasi-linear maps. We introduce an equivalence relation $\bowtie$ in $l_\infty(Q(X_i, Y_i))$:

$$(F_i) \bowtie (G_i) \iff \forall i, \;\; F_i - G_i = B_i + L_i, \;\; \text{where } \sup_i \|B_i\| < +\infty,$$

where each $B_i : X_i \to Y_i$ denotes a bounded homogeneous map and $L_i : X_i \to Y_i$ a linear map.

($*$) We shall say that a collection of morphisms $(\alpha_i, \gamma_i) : F_i \rightrightarrows G_i$ of $\mathcal{Q}$ is **uniformly representable** if (α_i), (γ_i) are uniformly bounded families of operators such that

$$(\alpha_i F_i) \bowtie (G_i \gamma_i).$$

We define now the new category $\mathcal{Q}^{\bowtie}$ whose objects are $\bowtie$-equivalence classes of uniformly Q-bounded families of quasi-linear maps. A morphism $(F_i) \rightrightarrows (G_i)$ in $\mathcal{Q}^{\bowtie}$ shall be a uniformly representable family of morphisms $(\alpha_i, \gamma_i) : F_i \rightrightarrows G_i$ of $\mathcal{Q}$

Finally, we shall consider the diagonal functor $\boldsymbol{\Delta} : \mathcal{Q} \to \mathcal{Q}^{\bowtie}$ defined by $\Delta(F) = (F)$.

The representative $[\Pi_\infty F_i, (\pi_i, \eta_i)]$, if it exists, of the functor

$$\mathcal{Q} \xrightarrow{\ \Delta\ } \mathcal{Q}^{\bowtie} \xrightarrow{\mathrm{Hom}_{\mathcal{Q}^{\bowtie}}(\cdot,(F_i))} \mathbf{Set}$$

shall be called the l_∞-product of the family (F_i) of $\mathcal{Q}$.

Proposition 3.1. *Let $(F_i : Z_i \curvearrowright Y_i)$ be a Q-bounded family of $\mathcal{Q}$. The functor $\mathrm{Hom}_{\mathcal{Q}^{\bowtie}}(\Delta(\cdot), (F_i))$ is representable and its representative is $[l_\infty(F_i), (\pi_i, \eta_i)]$, where $\pi_j : l_\infty(Y_i) \to Y_j$ and $\eta_j : l_\infty(Z_i) \to Z_j$ are the canonical projections .*

Proof. Let us consider the quasi-linear l_∞-amalgam $l_\infty(F_i) = \Pi F_i - L_{\Pi F_i}$, where $L_{\Pi F_i}$ is the linearization of ΠF_i with respect to a normalized Hamel basis (e_α) of $l_\infty(Z_i)$. Let us see that the canonical projections (π_i), (η_i) define a family of morphisms $(\pi_i, \eta_i) : l_\infty(F_i) \rightrightarrows F_i$ uniformly representable: let $(z_i)_i = (\sum_\alpha \lambda_\alpha e_\alpha(i))_i$ be an element of $l_\infty(Z_i)$; the difference

$$\pi_j \circ (\Pi F_i - L_{\Pi F_i})((z_i)_i) - F_j \circ \eta_j((z_i)_i) = F_j(z_j) - \textstyle\sum_\alpha \lambda_\alpha F_j(e_\alpha(j)) - F_j(z_j),$$

is linear. We check now that, given an object $F : Z \curvearrowright Y$ of $\mathcal{Q}$, the natural correspondence induce by (π_i, η_i),

$$\mathrm{Hom}_{\mathcal{Q}}(F, l_\infty(F_i)) \longmapsto \mathrm{Hom}_{\mathcal{Q}^{\bowtie}}(\Delta F, (F_i))$$

$$(T, S) \longrightarrow (\pi_i T, \eta_i S)_i,$$

is a bijection: given a uniformly representable collection of morphisms $(\alpha_i, \gamma_i) : F \rightrightarrows F_i$, it turns out that $((\alpha_i), (\gamma_i)) : F \rightrightarrows l_\infty(F_i)$ defines a morphism in $\mathcal{Q}$ since

$$l_\infty(F_i)(\gamma_i) = (F_i \gamma_i) - L_{\Pi F_i} \circ (\gamma_i) = (\alpha_i F) + B_i + L_i - L_{\Pi F_i}(\gamma_i),$$

because of $(F_i \gamma_i) \bowtie (\alpha_i F)$. We can conclude thanks to the fact that the family of maps $(B_i : Z \to Y_i)$ is bounded and therefore it defines a bounded map $Z \to l_\infty(Y_i)$. Thus, since $(\alpha_i F)$ obviously takes values in $l_\infty(Y_i)$, it is clear that the linear map $(L_i - L_{\Pi F_i}(\gamma_i))$ also has its range in $l_\infty(Y_i)$. So, $l_\infty(F_i)(\gamma_i) \equiv (\alpha_i)F$, as we wanted. Finally, it is clear that, given a morphism $(\alpha, \gamma) : F \rightrightarrows l_\infty(F_i)$, the composition $(\alpha, \gamma) \to (\pi_i \alpha, \eta_i \gamma)_i \to ((\pi_i \alpha), (\eta_i \gamma))$ is the identity. $\square$

Observe that if $(F_i') \bowtie (F_i)$ then $[l_\infty(F_i'), (\pi_i, \eta_i)]$ also represents the functor $\mathrm{Hom}_{\mathcal{Q}^{\bowtie}}(\Delta(\cdot), (F_i))$. It is easy to verify that (see also Proposition 5.1), $l_\infty(F_i') \equiv l_\infty(F_i)$, and thus we shall say that the l_∞-product of (F_i) is unique modulus equivalence. With some abuse of notation, we shall sometimes speak of the l_∞-product of uniformly equivalent families.

► **$\mathbf{l_p}$-Product, $\mathbf{0 < p < \infty}$, in $\mathcal{Q}$.** Let us consider an element (F_i) of $l_\infty^{\bowtie}(Q(Z_i, Y_i))$. Let $F : Z \curvearrowright Y$ be a quasi-linear map and ΔF its image by the diagonal functor. We shall define the space
p-$\mathrm{Hom}_{\mathcal{Q}^{\bowtie}}(\Delta F, (F_i))$ whose elements are families of morphisms $(\alpha_i, \gamma_i) : F \rightrightarrows F_i$ such that:

(1) (α_i) and (γ_i) are pointwise p-summable.
(2) There exists a $\|\cdot\|$-bounded family $(B_i : Z \to Y_i)$ of homogeneous maps, pointwise p-summable, such that $\alpha_i F - F_i \gamma_i = B_i + L_i$, being $L_i : Z \to Y_i$ linear.

We shall call l_p-product of a family (F_i) of objects of $\mathcal{Q}$ to the representative $(\Pi_p F_i, (\pi_i, \eta_i))$, if it exists, of the functor

$$p - \mathrm{Hom}_{\mathcal{Q}^{\bowtie}}(\Delta(\cdot), (F_i)) : \mathcal{Q} \longrightarrow \mathbf{Set}\,.$$

PROPOSITION 3.2. *Let $(F_i : Z_i \curvearrowright Y_i)$ be a Q-bounded family of $\mathcal{Q}$. The functor $p - \mathrm{Hom}_{\mathcal{Q}^{\bowtie}}(\Delta(\cdot), (F_i))$ is representable and its representative is $[l_p(F_i), (\pi_i, \eta_i)]$, where $\pi_j : l_p(Y_i) \to Y_j$ and $\eta_j : l_p(Z_i) \to Z_j$ are the canonical projections .*

PROOF. The proof is exactly the same as that of Proposición 3.1. We just make a couple of observations: *a*) Given a quasi-linear map $F : Z \curvearrowright Y$, each element $(\alpha_i, \gamma_i)_i$ of p-$\mathrm{Hom}_{\mathcal{Q}^{\bowtie}}(\Delta F, (F_i))$ defines a morphism $((\alpha_i), (\gamma_i)) : F \rightrightarrows l_p(F_i)$; *b*) Each $\|\cdot\|$-bounded family $(B_i : Z \to Y_i)$ of homogeneous pointwise p-summable maps defines a homogeneous bounded map $(B_i) : Z \to l_p(Y_i)$. □

We shall say that the l_p-product of (F_i) is unique modulus equivalence with the same meaning as in the case of the l_∞-product. Occasionally we shall speak of the l_p-product of uniformly equivalent families of quasi-linear maps.

► **$\mathbf{c_0}$-product in $\mathcal{Q}$.** Let us consider an element (F_i) of $l_\infty^{\bowtie}(Q(Z_i, Y_i))$. Let $F : Z \curvearrowright Y$ be a quasi-linear map and ΔF its image by the diagonal functor. We define the space 0-$\mathrm{Hom}_{\mathcal{Q}^{\bowtie}}(\Delta F, (F_i))$ of families of morphisms $(\alpha_i, \gamma_i) : F \rightrightarrows F_i$ such that:

(1) (α_i) y (γ_i) are pointwise convergent to 0.
(2) There is a $\|\cdot\|$-bounded family $(B_i : Z \to Y_i)$ of homogeneous maps pointwise convergent to 0 such that $\alpha_i F - F_i \gamma_i = B_i + L_i$, where $L_i : Z \to Y_i$ are linear.

We shall term c_0-product of a family (F_i) of objects of $\mathcal{Q}$ to the representative $[\Pi_0 F_i, (\pi_i, \eta_i)]$, if it exists, of the functor

$$0 - \mathrm{Hom}_{\mathcal{Q}^{\bowtie}}(\Delta(\cdot), (F_i)) : \mathcal{Q} \longrightarrow \mathbf{Set}\,.$$

PROPOSITION 3.3. *Let $(F_i : Z_i \curvearrowright Y_i)$ be a Q-bounded family of objects of $\mathcal{Q}$. The functor $0 - \mathrm{Hom}_{\mathcal{Q}^{\bowtie}}(\Delta(\cdot), (F_i))$ is representable and its representative is $[c_0(F_i), (\pi_i, \eta_i)]$, where $\pi_j : c_0(Y_i) \to Y_j$ and $\eta_j : c_0(Z_i) \to Z_j$ are the canonical projections.*

PROOF. Once again the proof is the dame as in Propositions 3.1 and 3.2. Only a couple of observations are perhaps necessary: *a*) Given an object $F : Z \curvearrowright Y$ of $\mathcal{Q}$, each element $(\alpha_i, \gamma_i)_i$ of 0-$\mathrm{Hom}_{\mathcal{Q}^{\bowtie}}(\Delta F, (F_i))$ defines a morphism $((\alpha_i), (\gamma_i)) : F \rightrightarrows c_0(F_i)$; *b*) Every $\|\cdot\|$-bounded family $(B_i : Z \to Y_i)$ of homogenous maps pointwise convergent to 0 defines a homogeneous bounded map $(B_i) : Z \to c_0(Y_i)$. □

The c_0-product is unique modulus equivalence with the same meaning as in the case of the l_∞-product. Occasionally we shall speak of the c_0-product of uniformly equivalent families of quasi-linear maps.

► **$\mathbf{l_q}$-coproduct in $\mathcal{Q}_q$, $\mathbf{0 < q \leq 1}$.** We shall call l_q-coproduct of a family (F_i) of objects of $\mathcal{Q}_q$ (the category of quasi-linear maps in $\mathbf{Q_q}$) to the representative $[\bigoplus_i F_i, (\xi_i, \upsilon_i)]$, if it exists, of the functor

$$\mathrm{Hom}_{\mathcal{Q}^{\bowtie}}((F_i), \Delta(\cdot)) : \mathcal{Q}_q \longrightarrow \mathbf{Set}\,.$$

PROPOSITION 3.4. *Let $(F_i : Z_i \curvearrowright Y_i)$ be a Q-bounded family of objects of $\mathcal{Q}$. The functor $\mathrm{Hom}_{\mathcal{Q}^{\bowtie}}((F_i), \Delta(\cdot))$ is representable and is represented by $[l_q(F_i), (\varsigma_j, \upsilon_j)]$, where $\varsigma_j : Y_j \to l_1(Y_i)$ y $\upsilon_j : Z_j \to l_1(Z_i)$ are the canonical inclusions .*

PROOF. Taking into account the previous proofs and the fact the covariant character of the functor we want to represent, we only need a couple of observations: *a*) Each element (α_i, γ_i) of $\mathrm{Hom}_{\mathcal{Q}^{\bowtie}}((F_i), \Delta F)$ clearly defines a morphism $(\sum_i \alpha_i, \sum_i \gamma_i) : l_q(F_i) \rightrightarrows F$; *b*) Every $\|\cdot\|$-bounded family (B_i) of homogeneous maps defines a homogeneous bounded map $\sum_i B_i : l_q(X_i) \to Y$. □

The l_q-coproduct shall be unique modulus equivalence in the standard sense. Sometimes we shall speak of the l_q-coproduct of uniformly equivalent families.

We do not know about the existence of the l_q-coproduct, for $q > 1$. An interesting remark here is that an l_p-amalgam $l_p(F_i)$ when $p > 1$ cannot represent the functor $\mathrm{Hom}_{\mathcal{Q}^{\bowtie}}((F_i), \Delta(\cdot))$ since it is not possible, in general, to amalgamate a family of operators $\alpha_i : Y_i \to A$ uniformly bounded in a single operator $l_p(Y_i) \to A$.

3.1. Restricted products in $\mathcal{Q}^Z$. We shall study the existence of restricted products in the category $\mathcal{Q}^Z$ of quasi-linear maps defined on a fixed space Z, in which morphisms $F \longrightarrow G$ are operators α such that $\alpha F \equiv G$. The corresponding category $\mathcal{Q}^{Z\bowtie}$ has as objects $\bowtie$-equivalence classes of Q-bounded families (F_i) of objects of $\mathcal{Q}^Z$; and as morphisms $(F_i) \longrightarrow (G_i)$ uniformly representable families of morphisms $\alpha_i : F_i \longrightarrow G_i$ of $\mathcal{Q}^Z$; that is, (α_i) is uniformly bounded and $(\alpha_i F_i) \bowtie (G_i)$.

▶ **$\mathbf{l_\infty}$-Product.** We shall call l_∞-product of a family (F_i) of objects of $\mathcal{Q}^Z$ to the representative $(\Pi_\infty(F_i), (\pi_i))$, if it exists, of the functor

$$\mathrm{Hom}_{\mathcal{Q}^{Z\bowtie}}(\Delta(\cdot), (F_i)) : \mathcal{Q}^Z \longrightarrow \mathbf{Set}\,.$$

PROPOSITION 3.5. *Let $(F_i : Z \curvearrowright Y_i)$ be a $\mathcal{Q}$-bounded family of objects of $\mathcal{Q}^Z$. The functor $\mathrm{Hom}_{\mathcal{Q}^{Z\bowtie}}(\Delta(\cdot), (F_i))$ is representable and comes represented by $[l_\infty(F_i)D, (\pi_i)]$, where $\pi_j : l_\infty(Y_i) \to Y_j$ are the canonical projections.*

PROOF. Following Proposition 3.1 we know the existence of the l_∞-product $(l_\infty(F_i), (\pi_i, \eta_i))$ of (F_i) in $\mathcal{Q}$. Making the pull-back diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & l_\infty(Y_i) & \longrightarrow & l_\infty(Y_i) \oplus_{l_\infty(F_i)} l_\infty(Z) & \longrightarrow & l_\infty(Z) & \longrightarrow & 0 \equiv l_\infty(F_i) \\
 & & \| & & \uparrow & & \uparrow D & & \\
0 & \longrightarrow & l_\infty(Y_i) & \longrightarrow & PB & \longrightarrow & Z & \longrightarrow & 0 \equiv l_\infty(F_i)D,
\end{array}$$

with D the diagonal operator $Dz = (z)$, we shall obtain the quasi-linear map $l_\infty(F_i)D$. This together with the family $(\pi_j : l_\infty(F_i)D \longrightarrow F_j)$ is the product in $\mathcal{Q}^Z$ of (F_i). That means that for each object F there is a natural correspondence

$$\mathrm{Hom}_{\mathcal{Q}^Z}(F, l_\infty(F_i)D) \longleftrightarrow \mathrm{Hom}_{\mathcal{Q}^{Z\bowtie}}(\Delta F, (F_i)),$$

which is a bijection. □

As it was the case with the finite product, one can check that $l_\infty(F_i)D$ defines an equivalent extension to that obtained making the *extended pull-back* of the family of quotient maps q_i in the extensions:

$$0 \longrightarrow Y_i \xrightarrow{j_i} X_i \xrightarrow{q_i} Z \longrightarrow 0 \equiv F_i.$$

The l_∞-product in $\mathcal{Q}^Z$ is unique modulus equivalence, in the sense that if $(F_i') \bowtie (F_i)$ then $[l_\infty(F_i')D, (\pi_i)]$ also represents the functor $\mathrm{Hom}_{\mathcal{Q}^{Z\bowtie}}(\Delta(\cdot), (F_i))$; it is also clear that $l_\infty(F_i')D \equiv l_\infty(F_i)D$. We shall sometimes speak of the l_∞-product of uniformly equivalent families.

Although the existence of the l_∞-product in $\mathcal{Q}$ guarantees the existence of the l_∞-product in $\mathcal{Q}^Z$, we cannot say the same of the l_p, $0 < p < \infty$ and c_0-products, since no diagonal operator D exists in these cases. Nevertheless, ate the cost of imposing some extra conditions on the family (F_i) it is still possible to obtain such products .

► $\mathbf{l_p}$**-Product,** $\mathbf{0 < p < \infty}$**.** We define the set $l_p\left(\mathrm{Hom}_{\mathcal{Q}^Z}(F, F_i)\right)$ of families of morphisms $\alpha_i : F \longrightarrow F_i$ such that:

(1) The family of operators (α_i) is pointwise p-summable.
(2) There exists a $\|\cdot\|$-bounded family $(B_i : Z \to Y_i)$ of homogeneous maps, pointwise p-summable such that $\alpha_i F - F_i = B_i + L_i$, where $L_i : Z \to Y_i$ are linear.

We shall denote l_p-product in $\mathcal{Q}^Z$ of a Q-bounded family F_i of quasi-linear maps to a pair $(\Pi_p(F_i), (\pi_i))$, if it exists, that represents the functor

$$l_p\left[\mathrm{Hom}_{\mathcal{Q}^Z}(\,\cdot\,, F_i)\right] : \mathcal{Q}^Z \longrightarrow \mathbf{Set}\,.$$

Let us see that the l_p-product in $\mathcal{Q}^Z$ exists for certain Q-bounded families of quasi-linear maps.

Proposition 3.6. *Let (F_i) be a family of objects of $\mathcal{Q}^Z$ pointwise p-summable and such that $(\sum_i Q(F_i)^p)^{1/p} < \infty$. The functor $l_p\left[\mathrm{Hom}_{\mathcal{Q}^Z}(\,\cdot\,, F_i)\right]$ is representable.*

Proof. It is probably enough to make a couple of remarks: $a)$ It is clear that $\Pi F_i : Z \curvearrowright l_p(Y_i)$, $\Pi F_i(z) = (F_i(z))$ is a well-defined quasi-linear map; $b)$ The morphisms $\pi_j : \Pi F_i \longrightarrow F_j$ in $\mathcal{Q}^Z$ induced by the natural projections $\pi_j : l_p(Y_i) \to Y_j$ define, for each quasi-linear map F, a bijection

$$\mathrm{Hom}_{\mathcal{Q}^Z}(F, \Pi F_i) \longleftrightarrow l_p\left[\mathrm{Hom}_{\mathcal{Q}^Z}(F, F_i)\right],$$

since for each element $(\alpha_i : F \longrightarrow F_i)$ de $l_p\left[\mathrm{Hom}_{\mathcal{Q}^Z}(F, F_i)\right]$ one has a well defined morphism $(\alpha_i) : F \longrightarrow \Pi F_i$ in $\mathcal{Q}^Z$. □

► $\mathbf{c_0}$**-Product.** We define the set $c_0\left[\mathrm{Hom}_{\mathcal{Q}^Z}(F, F_i)\right]$ of families of morphisms $\alpha_i : F \longrightarrow F_i$ such that:

(1) The family of operators (α_i) is pointwise convergent to 0.
(2) There is a $\|\cdot\|$-bounded family $(B_i : X \to Y_i)$ of homogeneous maps, pointwise convergent to 0 such that $\alpha_i F - F_i = B_i + L_i$, where $L_i : X \to Y_i$ are linear.

We shall call c_0-product in $\mathcal{Q}^Z$ of a Q-bounded family (F_i) of quasi-linear maps to a pair $(\Pi_0(F_i), (\pi_i))$, if it exists, that represents the functor

$$c_0\left[\mathrm{Hom}_{\mathcal{Q}^Z}(\,\cdot\,, F_i)\right] : \mathcal{Q}^Z \longrightarrow \mathbf{Set}\,.$$

Let us see that the c_0-product in $\mathcal{Q}^Z$ exists for certain Q-bounded families of quasi-linear maps.

Proposition 3.7. *Let $(F_i : Z \curvearrowright Y_i)$ be a Q-bounded family of objects of $\mathcal{Q}^Z$ pointwise convergent to 0. The functor $c_0\left[\mathrm{Hom}_{\mathcal{Q}^Z}(\,\cdot\,, F_i)\right]$ is representable.*

Proof. It is enough to make a couple of remarks: $a)$ It is clear that $\Pi F_i : Z \curvearrowright c_0(Y_i)$, $\Pi F_i(z) = (F_i(z))$, is a well-defined quasi-linear map; $b)$ The morphisms $\pi_j : \Pi F_i \longrightarrow F_j$ in $\mathcal{Q}^Z$ induced by the natural projections $\pi_j : c_0(Y_i) \to Y_j$, define, for eeach quasi-linear map F, a bijection

$$\mathrm{Hom}_{\mathcal{Q}^Z}(F, \Pi F_i) \longleftrightarrow c_0\left[\mathrm{Hom}_{\mathcal{Q}^Z}(\,\cdot\,, F_i)\right],$$

since for each element $(\alpha_i : F \longrightarrow F_i)$ of $c_0\left[\mathrm{Hom}_{\mathcal{Q}^Z}(F, F_i)\right]$ one has a well-defined morphism $(\alpha_i) : F \longrightarrow \Pi F_i$. □

The c_0-product is not, in principle, unique modulus equivalence; we have to wait until the next section where we shall prove what we have called “super-Sobczyk” theorem.

3.2. l_q**-Coproduct in** $\mathcal{Q}_{q,Y}$**,** $\mathbf{0 < q \leq 1}$**.** We study now the l_q-coproduct of arbitrary families in the category of quasi-linear maps in $\mathbf{Q_q}$. Let Y be a fixed q-Banach space. We define the category $\mathcal{Q}_{q,Y}$ of quasi-linear maps defined on q-Banach spaces with range Y, in which the morphisms $F \longleftarrow G$ are operators γ such that $F \equiv G\gamma$. The corresponding category $\mathcal{Q}_{q,Y}^{\bowtie}$ has as objects $\bowtie$-equivalence classes of Q-bounded families (F_i) of objects of $\mathcal{Q}_{q,Y}$, and as morphisms $(F_i) \longleftarrow (G_i)$ uniformly representable families of morphisms $F_i \xleftarrow{\gamma_i} G_i$ of $\mathcal{Q}_{q,Y}$.

we shall call *l_q-coproduct in $\mathcal{Q}_{q,Y}$* of a Q-bounded family $F_i : Z_i \curvearrowright Y$ to a pair $(\bigoplus_i(F_i),(\upsilon_i))$, when it exists, which represents the functor

$$\mathrm{Hom}_{\mathcal{Q}^{\bowtie}_{q,Y}}((F_i),\Delta(\cdot)) : \mathcal{Q}_{q,Y} \longrightarrow \mathbf{Set}\,.$$

Let us see that the l_q-coproduct in $\mathcal{Q}_{q,Y}$ of Q-bounded families (F_i) follows from the existence of the l_q-coproduct in $\mathcal{Q}_q$.

PROPOSITION 3.8. *Let (F_i) be a Q-bounded family of objects of $\mathcal{Q}_{q,Y}$. the functor $\mathrm{Hom}_{\mathcal{Q}^{\bowtie}_{Y}}((F_i),\Delta(\cdot))$ is representable and comes represented by $[\sum_i F_i,(\upsilon_i)]$, where $\upsilon_j : Z_j \to l_q(Z_i)$ are the canonical inclusions.*

PROOF. Let us consider the l_q-coproduct $(l_q(F_i),(\epsilon_i,\upsilon_i))$ de (F_i) in $\mathcal{Q}_q$. Constructing the push-out diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & l_1(Y) & \longrightarrow & l_1(Y)\oplus_{l_1(F_i)} l_1(Z_i) & \longrightarrow & l_1(Z_i) & \longrightarrow & 0\equiv l_1(F_i)\\
 & & \Big\downarrow{\scriptstyle\Sigma} & & \Big\downarrow & & \| & & \\
0 & \longrightarrow & Y & \longrightarrow & PO & \longrightarrow & l_1(Z_i) & \longrightarrow & 0\equiv \sum F_i,
\end{array}$$

in which $\sum$ is the sum operator $\sum(y_i)=\sum_i y_i$, we obtain the quasi-linear map $\sum_i F_i$ which, together with the family (υ_i), forms the l_q-coproduct in $\mathcal{Q}_{q,Y}$ of (F_i); that is, for each F, there is a natural bijection

$$\mathrm{Hom}_{\mathcal{Q}_{q,Y}}(\textstyle\sum_i F_i,F) \longleftrightarrow \mathrm{Hom}_{\mathcal{Q}^{\bowtie}_{q,Y}}((F_i),\Delta F).$$

□

It can be easily checked that $\sum_i F_i$ defines an equivalent extension to that obtained making the *extended push-out* of the collection of embeddings j_i in the extensions:

$$0 \longrightarrow Y \xrightarrow{j_i} X_i \xrightarrow{q_i} Z_i \longrightarrow 0 \equiv F_i.$$

Finally, we have that the l_q-coproduct in $\mathcal{Q}_{q,Y}$ is unique modulus equivalence, because that was the case of the l_q-coproduct in $\mathcal{Q}_q$. Therefore, we can speak of the l_q-coproduct of uniformly equivalent families.

4. Inductive representations

It is useful to know that a given quasi-Banach space E can be represented in the form $E=\overline{\cup_n En}$ where each E_n has some additional properties. In this section we wonder about an analogous situation in $\mathfrak{Q}$. Thus, in terms of exact sequences we ask:

(1) When a diagram such as

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \\
0 & \longrightarrow & Y_2 & \longrightarrow & X_2 & \longrightarrow & Z_2 & \longrightarrow & 0\\
 & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \\
0 & \longrightarrow & Y_1 & \longrightarrow & X_1 & \longrightarrow & Z_1 & \longrightarrow & 0
\end{array}$$

can be completed?

(2) Conversely, given an exact sequence $0\to Y\to X\to Z\to 0$; ¿is it possible to represent $0\to Y\to X\to Z\to 0$ as the "completion" (in the same sense as before) of a diagram as the preceding?

The same questions can be formulate in terms of quasi-linear maps

(1) Does there exist the limit, in some sense, of a system

$$(*)\quad F_1 \rightrightarrows F_2 \rightrightarrows .... \rightrightarrows F_n \rightrightarrows ...?$$

(2) Conversely, given an object F, is it possible to write F as the limit (in the same sense as before) of a system like the preceding one?

Since we have already observed the difficulty that working with equivalent classes presents to study the universal properties (recall the case of restricted products and coproducts), we shall not consider equivalence classes regarding the previous questions. We shall wait a little to define the right category to formulate the problems.

The considerations (1) and (2) make us wonder about the existence of the *límit* of a system of exact sequences. We begin by recalling the definition of **inductive límit**. Let I be a partially ordered set. An inductive system $\langle A_i, \phi_{ij}\rangle$, with indices in I, of objects of a category $\mathfrak{C}$ is a family of objects (A_i) and morphisms $\phi_{ij} : A_i \to A_j$, $i \leq j$ such that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ for each ordered triple $i \leq j \leq k$. We shall call *inductive límit* of the system $\langle A_i, \phi_{ij}\rangle$ in $\mathfrak{C}$ to an object $\lim_{\rightarrow} A_i$ endowed with a collection of morphisms $\psi_j : A_j \to \lim_{\rightarrow} A_i$ such that for all $i < j$ one has $\psi_j \phi_{ij} = \psi_i$; and such that it is universal with respect to that property, that is, if there is another object G and another collection of morphisms $\psi'_j : A_j \to G$ such that for each $i < j$ one has $\psi'_j \phi_{ij} = \psi'_i$, then there is a unique morphism $\Phi : \lim_{\rightarrow} A_i \to G$ verifying that for each $i \in$ I, $\Phi\psi_i = \psi'_i$. Obviously, the inductive límit of an inductive system is unique in the category, modulus isomorphisms.

4.1. Inductive limits of p-Banach spaces. Let us recall which types of inductive systems of $\mathbf{Q}$ admit limits: Let $\langle E_n, i_n\rangle$ be an inductive system formed by a sequence of p-Banach spaces and isometries $i_n : E_n \hookrightarrow E_{n+1}$. Let us consider the vector space

$$X = \{(x_n)_n \in \textstyle\prod_{n=1} E_n : \exists \mu \in \mathbb{N} : i_n(x_n) = x_{n+1},\ \forall n > \mu\},$$

endowed with the semi-p-norm $\|(x_n)_n\| = \lim \|x_n\|$. Let $K = \ker \|\cdot\|$. The limit $\mathfrak{X}$ of $\langle E_n, i_n\rangle$ is the completion of the quotient X/K together with the family of isometries $I_n : E_n \to \mathfrak{X}$ defined as $I_n(x) = [(0, 0, \ldots, x, x, \ldots)]$. In this way, the space inductive limit can be described in the form $\mathfrak{X} = \overline{\cup I_n(E_n)}$.

It is important to observe that the object $\mathfrak{X}$ scan only be universal with respect to the property "$\forall n \in \mathbb{N},\ I_{n+1} i_n = I_n$" for uniformly bounded families of operators $f_n : E_n \to X$. Thus, if for all $n \in \mathbb{N}$ it occurs that $f_{n+1} i_n = f_n$ then there is an operator $T : \mathfrak{X} \to X$ defined by $T((x_n) + K) = f_N x_N$, where N is the least index for which $i_j x_j = x_{j+1}$ for all $j \geq N$.That operator verifies $TI_n = f_n$ an is unique. We then have that $(\mathfrak{X}, I_n)$ would only be the "restricted inductive limit" of $\langle E_n, i_n\rangle$, (in the same sense as the products and coproducts of the previous section), because they are representatives of a certain functor which we do not need to describe now. All in all, we shall refer to it as the inductive limit, or just the limit. It is also important to observe that if each f_n is an injective isometry then also T is an injective isometry

$$\|T((x_n) + K)\| = \|f_N x_N\| = \|x_N\| = \lim \|x_n\| = \|(x_n) + K)\|.$$

Another interesting observation is that given an increasing sequence $E_1 \hookrightarrow E_2 \hookrightarrow \cdots \hookrightarrow E_n \hookrightarrow \cdots$ of subspaces of a quasi-Banach space X, we can identify the limit of the system with $\overline{\cup_n E_n}$.

4.2. The category of extensions $\mathcal{S}_1$. To obtain the limit of a system of extensions

$$(*) \qquad [0 \to Y_i \to X_i \to Z_i \to 0, (\alpha_{ij}, \beta_{ij}, \gamma_{ij})]$$

it seems to be necessary that the systems $\langle Y_i, \alpha_{ij}\rangle$, $\langle X_i, \beta_{ij}\rangle$ and $\langle Z_i, \gamma_{ij}\rangle$ have limits in $\mathbf{Q}$. That takes us to define a new category of extensions $\mathcal{S}_1$ in which we shall work. An object $0 \to Y \to X \to Z \to 0$ of $\mathcal{S}_1$ is an extension in $\mathbf{Q}$ with parameter $\rho = 1$ (that is, $Y \to X$ is an into isometry and $X \to Z$ the corresponding quotient map). A morphisms between two extensions shall be a triple (α, β, γ) of isometries making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_2 & \longrightarrow & X_2 & \longrightarrow & Z_2 & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\uparrow & & {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & Y_1 & \longrightarrow & X_1 & \longrightarrow & Z_1 & \longrightarrow & 0. \end{array}$$

4.3. Inductive limits in $\mathcal{S}_1$.

PROPOSITION 4.1. *Let $(*)$ be an inductive system in $\mathcal{S}_1$ formed by exact sequences of q-Banach spaces, $0 < q \leq 1$. The extension*

$$0 \longrightarrow \varinjlim Y_n \longrightarrow \varinjlim X_n \longrightarrow \varinjlim Z_n \longrightarrow 0$$

is the inductive limit in $\mathcal{S}_1$ of the system.

PROOF. Maintaining the notation we for the limit of quasi-Banach spaces we set $\lim_{\rightarrow} Y_n = [\overline{\cup I_n Y_n}, (I_n)]$, $\lim_{\rightarrow} X_n = [\overline{\cup S_n X_n}, (S_n)]$, $\lim_{\rightarrow} Z_n = [\overline{\cup J_n Z_n}, (J_n)]$; it is not difficult to check that we have a commutative diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \cup I_n Y_n & \longrightarrow & \cup S_n(X_n) & \longrightarrow & \cup J_n Z_n & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \vdots & & \\ & & \uparrow{\scriptstyle\alpha_2} & & \uparrow{\scriptstyle\beta_2} & & \uparrow{\scriptstyle\gamma_2} & & \\ 0 & \longrightarrow & Y_2 & \longrightarrow & X_2 & \longrightarrow & Z_2 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle\alpha_1} & & \uparrow{\scriptstyle\beta_1} & & \uparrow{\scriptstyle\gamma_1} & & \\ 0 & \longrightarrow & Y_1 & \longrightarrow & X_1 & \longrightarrow & Z_1 & \longrightarrow & 0 \end{array}$$

in which the upper extension is topologically exact with parameter $\rho = 1$. Then, it is easy to verify that the sequence of their completions

$$0 \longrightarrow \lim_{\rightarrow} Y_n \longrightarrow \lim_{\rightarrow} X_n \longrightarrow \lim_{\rightarrow} Z_n \longrightarrow 0$$

is still an exact sequence. The universal property of this last extension is quite clear, although we prove it next. Given an extension $0 \to A \to B \to C \to 0$ with $\rho = 1$ and a collection of morphisms $(\Upsilon_n, \Xi_n, \Psi_n)$ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle\Upsilon_n}\uparrow & & {\scriptstyle\Xi_n}\uparrow & & \uparrow{\scriptstyle\Psi_n} & & \\ 0 & \longrightarrow & Y_n & \longrightarrow & X_n & \longrightarrow & Z_n & \longrightarrow & 0, \end{array}$$

in such a way that $\Upsilon_{n+1}\alpha_n = I_n$, $\Xi_{n+1}\beta_n = S_n$ and $\Psi_{n+1}\gamma_n = J_n$ it is clear that one has isometries $\Upsilon : \lim_{\rightarrow} Y_n \to A$, $\Xi : \lim_{\rightarrow} X_n \to B$, $\Psi : \lim_{\rightarrow} Z_n \to C$ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle\Upsilon}\uparrow & & {\scriptstyle\Xi}\uparrow & & \uparrow{\scriptstyle\Psi} & & \\ 0 & \longrightarrow & \lim_{\rightarrow} Y_n & \longrightarrow & \lim_{\rightarrow} X_n & \longrightarrow & \lim_{\rightarrow} Z_n & \longrightarrow & 0. \end{array}$$

□

In the last proposition we have completed isometrically a topologically exact sequence. Returning for a while to classical terms, the completion of topologically exact sequences is feasible:

LEMMA 4.1. *Let $0 \to Y_0 \xrightarrow{i} X_0 \xrightarrow{q} Z_0 \to 0 \equiv F_0$ be a topologically exact sequence of quasi-normed spaces. The extension of their completions $0 \to \widehat{Y_0} \xrightarrow{\widehat{i}} \widehat{X_0} \xrightarrow{\widehat{q}} \widehat{Z_0} \to 0 \equiv F$, is an exact sequence in* $\mathbf{Q}$.

PROOF. Let us consider $Y_0 \overset{j}{\hookrightarrow} \widehat{Y_0}$ the embedding of Y_0 in its completion and the push-out diagram they generate:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_0 & \longrightarrow & X_0 & \longrightarrow & Z_0 & \longrightarrow & 0 \equiv F_0 \\ & & \big\downarrow j & & \big\downarrow & & \| & & \\ 0 & \longrightarrow & \widehat{Y_0} & \longrightarrow & PO & \longrightarrow & Z_0 & \longrightarrow & 0 \equiv jF_0. \end{array}$$

The extension jF_0 obtained is topologically exact. Now, the classical extension lemma (see [**45**]) yields an extension of jF_0 to $\widehat{Z_0}$ through the natural inclusion $u : Z_0 \hookrightarrow \widehat{Z_0}$; in that way we obtain the extension F we were looking for. □

Moreover, there is a *exact extension* of F_0 (see, for instance, Lemma 1.1 in Chapter 5); that is, there exists a version F' of F such that $F'u = jF_0$.

4.4. Inductive limit of quasi-linear maps. Let us form the *corresponding* category to $\mathcal{S}_1$ in terms of quasi-linear maps. The category $\mathcal{F}_1$ that we define here has as objects quasi-linear maps $F : Z \curvearrowright Y$ with $Q(F) = 1$. If $G : Z' \curvearrowright Y'$ then a morphism $(\alpha, \gamma) : F \rightrightarrows G$ are two injective isometries $\alpha : Y \to Y'$ y $\gamma : Z \to Z'$ such that $\alpha F = G\beta$. Let us remark that this last equality is as applications.

A functor $\mathcal{E} : \mathcal{F}_1 \to \mathcal{S}_1$ comes cleanly defined assigning to an F the extension $\mathcal{E}(F) = 0 \to Y \to Y \oplus_F Z \to Z \to 0$. The morphism $\mathcal{E}(\alpha, \gamma)$ of $\mathcal{S}_1$ associated to a morphism (α, γ) of $\mathcal{F}_1$ shall be (α, β, γ) with $\beta(y, z) = (\alpha y, \gamma z)$. This observation is a particular case of the study about *isometrically equivalent* extensions made in [**20**]. Going the other direction, it is not possible to define a functor such that $\mathcal{F}_1$ and $\mathcal{S}_1$ be made equivalent categories. For that reason we shall need to explicitly perform the construction of the inductive limit in $\mathcal{F}_1$, although we shall use the existence of limits in $\mathcal{S}_1$.

PROPOSITION 4.2. *Let $\langle F_n, (\alpha_n, \gamma_n)\rangle$ be an inductive system in $\mathcal{F}_1$ in which $(F_n : Z_n \curvearrowright Y_n)$ and (Z_n), (Y_n) are families of q-Banach spaces. The system has inductive limit.*

PROOF. It is obvious that our problem is to define $F : \lim_{\to} Z_n \curvearrowright \lim_{\to} Y_n$. With that purpose we define first the quasi-linear map $F_0 : \cup J_n Z_n \curvearrowright \cup I_n Y_n$; in each point $z \in \cup_n J_n Z_n$ it takes the value $F(z) = I_N F_N(z_N)$ for $N = \min\{n \in \mathbb{N} : z \in J_n Z_n\}$ and $J_N(z_N) = z$. One has:

(1) F_0 is quasi-linear: $\forall z, z' \in \cup J_n Z_n$, there exist natural numbers N and N' such that $z = I_N(z_N)$ and $z' = I_{N'}(z_{N'})$. Let us assume that $M = \max\{N, N'\} = N'$:

$$\|F_0(z + z') - F_0 z - F_0 z'\| = \|I_M F_M(\gamma_M \cdots \gamma_N(z_N) + z_{N'}) - I_N F_N(z_N) - I_{N'} F_{N'}(z_{N'})\| =$$

$$\|I_M F_M(\gamma_M \cdots \gamma_N(z_N) + z_{N'}) - I_M \alpha_{M-1} \cdots \alpha_N F_N(z_N) - I_M F_M(z_{N'})\| =$$

$$\|I_M F_M(\gamma_M \cdots \gamma_N(z_N) + z_M) - I_M F_M(\gamma_M \cdots \gamma_N z) - I_M F_M(z_M)\| \leq Z(F_M)(\|z\| + \|z'\|).$$

.

(2) F_0 verifies that for all $n \in \mathbb{N}$, $F_0 J_n(z_n) = I_n F_n(z_n)$.

We shall prove now that that map is precisely the inductive limit of the system $\langle F_n, (\alpha_n, \gamma_n)\rangle$. Benefiting from the work already done in $\mathcal{S}_1$, we just have to define the following isometry:

$$T((y_n, z_n) + K_0) = ((y_n) + K_1, (z_n) + K_2),$$

(where $K_0 = \ker(\lim \|\cdot\|_{F_n})$, $K_1 = \ker(\lim \|\cdot\|_{Y_n})$ and $K_2 = \ker(\lim \|\cdot\|_{Z_n})$ as we have already seen in Section 4.1), that makes commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \lim_{\to} Y_n & \longrightarrow & \lim_{\to} Y_n \oplus_{F_0} \lim_{\to} Z_n & \longrightarrow & \lim_{\to} Z_n & \longrightarrow & 0. \\ & & \| & & {\scriptstyle T}\uparrow & & \| & & \\ 0 & \longrightarrow & \lim_{\to} Y_n & \longrightarrow & \lim_{\to} [Y_n \oplus_{F_n} Z_n] & \longrightarrow & \lim_{\to} Z_n & \longrightarrow & 0. \end{array}$$

□

Finally, we remark that one should not expect that the quasi-linear "límit" map F behaves as a restricted inductive limit in $\mathfrak{Q}$ since, in this case, the identities $(\Upsilon_{n+1}, \Psi_{n+1}) \circ (\alpha_n, \gamma_n) \asymp (\Upsilon_n, \Psi_n)$ would be as morphisms in $\mathfrak{Q}$, and thus it would not be possible to apply the universal properties of the restricted inductive limits in $\mathbf{Q}$. This problem remains open.

4.5. Finite dimensional inductive representations in $\mathcal{Z}$. Let us address now to the problem (2). We are going to show a result to represent certain objects $\mathcal{Z}$ as limits of a system $\langle F_i, (\alpha_i, \gamma_i)\rangle$ in which each F_i is defined between finite dimensional spaces.

With that purpose we need the concept of **convexification** of a z-linear map $F : Z \curvearrowright Y$. Let $A = (a_i)_{i=0,1,...}$ with $a_0 = 0$ a subset of the ball of radius $1 + \varepsilon$ of Z such that the unit ball of Z is contained in the convex hull of A. We define an order for finite subsets of A: $\{a_{i_1}, ..., a_{i_N}\} \leq \{a_{j_1}, ..., a_{j_M}\}$ if either $N < M$ or $N = M$ and $\{i_1, ..., i_N\}$ is lesser than or equal to $\{j_1, ..., j_N\}$ in the lexicographical order. We define the homogeneous map $F_c : Z \curvearrowright Y$ which on a point p of the unit sphere of Z takes the value $F_c(p) = \sum_i \theta_i F a_i$ where $(a_i)_i$ il the minimal set for which p is a convex combination $\sum_i \theta_i a_i$. It is clear that $\|F - F_c\| < Z(F)(1 + \varepsilon)$ since if $\|p\| = 1$ then

$$\|F(p) - F_c(p)\| \leq Z(F) \sum |\theta_i|\|a_i\|.$$

Therefore F_c is a z-linear map equivalent to F an with constant $Z(F_c) \leq 2Z(F)(1 + \varepsilon)$. We shall say that F_c is a **convex version** of F. It is still possible to obtain a convex version of F with a better estimate for its constant, at the cost of making more difficult to estimate the distance between F y and its convexification. That variation can be found in [**13**].

Occasionally it is useful to have simultaneously a convex canonical version of a given z-linea map $F : Z \curvearrowright Y$; to do that, we take first a canonical version $F - L_F$ of F with respect to a Hamel basis (e_α) formed by elements with norm $1 + \varepsilon$ of Z and then one only has to include the collection (e_α) in the set A with respect to which one convexifies.

The next results show the usefulness of the convex versions.

Lemma 4.2. *Let $F : Z \curvearrowright Y$ be a z-linear map defined on a finite dimensional space. There is a version of F with finite dimensional range at a distance $Z(F)(1 + \varepsilon)$ from F.*

Proof. It is clear that the convexification of F has finite dimensional range. □

Proposition 4.3. *Finite dimensional representation of a z-linear mapLet $F : E \curvearrowright Y$ be a z-linear map defined on a separable space. We set $E = \overline{\bigcup E_n}$ with each E_n finite dimensional spaces. There is a canonical version F^c of F such that $F^c|_{E_n}$ has finite dimensional range for each n.*

Proof. One only has to choose carefully the sets $A_n \subset E_n$ with respect to which the convexification takes place: let first A_1 be a subset of $(1 + \varepsilon)B_{E_1}$ such that $B_{E_1} \subset \mathrm{conv}(A_1)$. We add a $(1+\varepsilon)$-normalized basis (e_α) of E_1, and continue denoting the set as A_1. The next set $A_2 \subset E_2$ contain A_1, sufficient vectors to complete the basis of E_1 up to a basis of E_2 and the points with norm at most $1+\varepsilon$ necessary so that $\mathrm{conv}(A_2) \supset B_{E_2}$. The process continues in this way. We define at each step n an order on the finite subsets of A_n compatible with the previous order for the $n - 1$ step. Let us consider now the canonical version $G = F - L_F$ of F with respect to the Hamel basis (e_γ) of $\bigcup E_n$ we have just constructed (i.e., $(e_\gamma) \cap E_k$ is the Hamel bases we had chosen for E_k). Finally, we convexify the restrictions $G_{|_{E_n}}$ with respect to the sets A_n. It is clear we have a map defined on $\bigcup E_n$ that, when extended to the whole E turns out to be a z-linear map in canonical form at finite distance from F, and whose restrictions to each E_n are in convex form. $\square$

We shall say that the final map is in convex form with respect to the sequence (E_n).

Proposition 4.4 (Inductive finite dimensional representation of a z-linear map). *Each object $F : Z \curvearrowright Y$ of $\mathcal{Z}$ defined on a separable space admits a version which is the inductive limit of a system $\langle F_n, (\alpha_n, \gamma_n)\rangle$ in which each F_n is defined between finite dimensional spaces.*

Proof. Let $Z = \overline{\cup Z_n}$ be an inductive finite dimensional representation of Z. After Proposition 4.3, there is a convex version F^c of F such that the following commutative diagram is possible

$$\begin{tikzcd}
0 \arrow[r] & \bigcup_n [F_n^c(Z_n)] \arrow[r] & \bigcup_n X_n \arrow[r] & \bigcup_n Z_n \arrow[r] & 0 \equiv F_0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
0 \arrow[r] & {[F_2^c(Z_2)]} \arrow[r] \arrow[u] & X_2 \arrow[r] \arrow[u] & Z_2 \arrow[r] \arrow[u] & 0 \equiv F_2^c \\
0 \arrow[r] & {[F_1^c(Z_1)]} \arrow[r] \arrow[u] & X_1 \arrow[r] \arrow[u] & Z_1 \arrow[r] \arrow[u] & 0 \equiv F_1^c
\end{tikzcd}$$

Since the sequence $(Z(F_i^c))_i$ is bounded and the operators $Z_n \hookrightarrow Z_{n+1}$ and $[F_n^c Z_n] \hookrightarrow [F_{n+1}^c Z_{n+1}]$ in the diagram are isometries, the Proposition 4.2 provides the limit L of the system. That L and F are the same object follows from the fact that L takes, by definition, at the point $z \in \cup_{n=1}^{\infty} Z_n$ the value $F_0^c(z) = F_N^c(z)$ for $N = \min\{n \in \mathbb{N} : z \in Z_n\}$; so, on $\cup_{n=1}^{\infty} Z_n$ one has $\|L - F\| < Z(F)(1 + \varepsilon)$. An exact extension to the whole space is what we need to conclude.. $\square$

It is clear from the previous proposition that the inductive finite dimensional representation of each exact sequence $0 \to Y \to X \to Z \to 0$ depends on the inductive representation $\overline{\cup Z_n}$ chosen for Z.

4.6. Inductive representations in $\mathcal{Q}_Y$. We can also give an inductive presentation of a quasi-linear map $F : Z \curvearrowright Y$ defined on a separable space without need to use the convexification process.

Proposition 4.5. *Let $F : Z \curvearrowright Y$ be a quasi-linear map defined on a separable space Z. Then F is the inductive limit of a system $\langle F_n, \gamma_n\rangle$.*

Proof. Let $Z = \overline{\cup Z_n}$ be an inductive finite dimensional presentation Z. Let us consider the sequence (F_n) of of quasi-linear maps obtained as follows: for each n, we make pull-back with the natural inclusion $Z_n \hookrightarrow \cup Z_n$, that is, $F_n = F_{|_{Z_n}}$. It is clear that

$F_0 := F_{|\cup Z_n}$ makes the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X_0 & \longrightarrow & \cup_{n=1} Z_n & \longrightarrow & 0 \equiv F_0 \\
 & & \| & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \longrightarrow & PB_n & \longrightarrow & Z_n & \longrightarrow & 0 \equiv F_n \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \longrightarrow & PB_2 & \longrightarrow & Z_2 & \longrightarrow & 0 \equiv F_2 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \longrightarrow & PB_1 & \longrightarrow & Z_1 & \longrightarrow & 0 \equiv F_1
\end{array}$$

commutative. Let now $0 \to Y \to B \to C \to 0 \equiv E$ be an object such that there exists a family of morphisms $E \xleftarrow{\phi_n} F_n$ of $\mathcal{Q}_Y$ such that ϕ_n are uniformly bounded operators and, for all n, one has $\phi_n \circ \gamma_{n-1} = \phi_{n-1}$. Since Z is the (restricted) inductive limit of the system (Z_n), there exists an operator $\Phi : Z \to C$ such that for each n one has $\Phi_{|Z_n} = \phi_n$. Thus, we have a morphisms $E \xleftarrow{\Phi} F$. Indeed: let us consider Φ on $\cup Z_n$; when $z \in \cup Z_n$, one has $E\Phi(z) = E\phi_N(z) = F_N z$, being $N = \min\{n \in \mathbb{N} : z \in Z_n\}$. Thus, the inductive limit of the system $F_1 \longrightarrow F_2 \longrightarrow .... \longrightarrow F_n \longrightarrow \cdots$ is F. $\square$

5. z-linear maps on vector amalgams

We focus now our attention on the question: to which extent the l_∞-product in $\mathcal{Q}^Z$ allows us to represent the space $\mathcal{Q}(Z, l_\infty(Y_i))$? And, analogously, to which extent the c_0-product in $\mathcal{Z}^Z$ allows us to represent the space $\mathcal{Z}(Z, c_0(Y_i))$? In particular, we want to know if it is possible to identify the spaces $\mathcal{Q}(Z, l_\infty(Y_i))$ and $\mathcal{Z}(Z, c_0(Y_i))$ with amalgams, in a sense to be determined, of quasi-linear maps.

5.1. Representation of spaces $\mathcal{Q}(\clubsuit, l_\infty(Y_i))$. We study first what occurs with the simplest of the restricted products: the l_∞-product.

Proposition 5.1. *Let (Y_i) be a collection of quasi-Banach spaces. For each quasi-Banach space X, the existence of the l_∞-product in $\mathfrak{Q}^Z$ establishes an isomorphism of vector spaces*

$$\mathcal{Q}\left(X, l_\infty(Y_i)\right) \longleftrightarrow l_\infty^{\bowtie}\left(Q(X, Y_i)\right).$$

Proof. Let us consider the family $\pi_j : l_\infty(Y_i) \to Y_j$ of natural projections. The map $F \to (\pi_i F)$ is well defined because if $G \equiv F$ then $(\pi_i F) \bowtie (\pi_i G)$. Its inverse is $(F_i) \to l_\infty(F_i)$, and is equally well defined: we observe that when $(F_i) \bowtie 0$ (i.e., for each i there exist maps $B_i, L_i : X \to Y_i$ which are, respectively, bounded-homogeneous and linear, so that
$F_i = B_i + L_i$ with $\sup_i \|B_i\| < \infty$) then $l_\infty(F_i) = (B_i) + (L_i - \pi_i L_{\Pi F_i})_i$, where $(B_i) : Z \to l_\infty(Y_i)$ is a well-defined bounded map; therefore, also the linear map $(L_i - \pi_i L_{\Pi F_i}) : Z \to l_\infty(Y_i)$ is well defined. So, $(F_i) \bowtie (G_i)$ implies $l_\infty(F_i) \equiv l_\infty(G_i)$. $\square$

The fact that the previous isomorphism sends 0 to 0 admits a nice equivalent formulation: *The l_∞-product of a family (F_i) of quasi-linear maps uniformly trivial is trivial.* In particular, one has the following corollary:

COROLLARY 5.1. *The l_∞-product $l_\infty(Y_i)$ in $\mathbf{B}$ of family (Y_i) of λ-injective spaces is λ-injective. If (Y_i) is a family of spaces λ-separably injective then $l_\infty(Y_i)$ is λ-separably injective.*

The identification in the Proposition 5.1 corresponds in quasi-Banach spaces to the classical equality

$$\mathrm{Ext}(X, \Pi Y_i) = \Pi\,\mathrm{Ext}(X, Y_i)$$

for groups (see [**32**]). However, it is still possible a more deep and precise algebraic formulation for the result in terms of the representability of a functor:

DEFINITION 1.3 (Functor Ext-representable). *Let $\mathfrak{C}$ be a category that admits exact sequences. We say that a covariant functor $F : \mathfrak{C} \to \mathbf{Set}$ is* Ext*-representable if there exists an object M of $\mathfrak{C}$ and an element ξ of $F(M)$ so that ξ establishes an isomorphism between the functors F and* $\mathrm{Ext}(\,\cdot\,, M)$*. Analogously, a contravariant functor*

$$F : \mathfrak{C} \to \mathbf{Set}$$

is Ext*-representable if there exists an object M and an element $\xi \in F(M)$ such that ξ establishes an isomorphism between the functors F y* $\mathrm{Ext}(M,\,\cdot\,)$*. In both cases, we say that the couple (M, ξ) is an* Ext*-representative of F in $\mathfrak{C}$.*

In these terms, Proposition 5.1 ensures that $(l_\infty(Y_i), \pi_i)$ is the $l_\infty - \mathrm{Ext} -$*producto* in $\mathbf{Q}$ of the family (Y_i).

5.2. Representation of the spaces $\mathcal{Z}\,(\clubsuit, \mathbf{c_0(Y_n)})$. We consider now the c_0-product. Let $c_0(Z(X, Y_n))$ be the space of families of z-linear maps $F_n : X \curvearrowright Y_n$ uniformly Z-bounded and pointwise convergent to 0. We denote by $c_0^{\bowtie}(Z(X, Y_n))$ the space whose elements are $\bowtie$-equivalence classes of elements of $c_0(Z(X, Y_n))$.

Let us recall from [**34**] that a Banach space E is said to be a Π_λ-space if there exists a constant $\lambda > 0$ such that every finite dimensional subspace $A_1 \subset E$ is contained in a finite dimensional superspace which is moreover λ-complemented in E.

THEOREM 5.1 (**super-Sobczyk**). *Let (Y_n) be a family of Banach spaces and let X be a separable Π_λ-space. There is an isomorphism of vector spaces*

$$\mathcal{Z}\,(X, c_0(Y_n)) \longleftrightarrow c_0^{\bowtie}\,(Z(X, Y_n))\,.$$

PROOF. Let us consider the family $\pi_j : c_0(Y_i) \to Y_j$ of natural projections. The map $F \to (\pi_i F)$ induced by (π_i) is well defined, because it is clear that if $F \equiv G$ then $(\pi_i F) \bowtie (\pi_i G)$. Its inverse map $(F_n) \to c_0(F_n)$ comes induced by the c_0-product: it associates to each Z-bounded sequence (F_n) of z-linear maps pointwise convergent to 0 its c_0-product. What remains to be seen is that thes map is compatible with the $\bowtie$- relation, what amounts to prove that if $(F_n) \bowtie 0$ then $c_0(F_n) \equiv 0$.

By hypothesis, there exists a representation $X = \overline{\cup_n E_n}$ of X, with (E_n) an increasing sequence of finite dimensional λ-complemented in X. Let us denote by $\eta_j : X \to E_j$ the corresponding associated projections such that $\|\eta_j\| \le \lambda$ for each $j \in \mathbb{N}$. Given $(F_n) \in c_0^{\bowtie}(Z(X, Y_i))$, after Proposition 4.2, we can select a canonical convex version $c_0(F_n)^c$ of $c_0(F_n)$ with respect to the family (E_j). In particular, for each $j \in \mathbb{N}$, the space generated in $c_0(Y_n)$ by $c_0(F_n)^c(E_j)$ is finite dimensional; moreover, $c_0(F_n)^c$ vanishes on a normalized Hamel basis $(e_\alpha)_\alpha$ of X compatible with the structure $\overline{\cup E_n}$. Also, $c_0(F_i) \equiv c_0(F_i)^c$ since $\|c_0(F_i) - c_0(F_i)^c\| \le \sup_i Z(F_i)(1+\varepsilon)$. Observe that each $\pi_i c_0(F_n)^c$ is in canonical convex form with respect to each E_j.

Assume now that (F_n) is a uniformly trivial family, $(F_n) \bowtie 0$, which means that there exists a constant μ and linear maps $L_n : X \to Y_n$ such that $\|F_n - L_n\| \le \mu$, for all $n \in \mathbb{N}$. To simplify notation we shall write $G_i = \pi_i c_0(F_n)^c$. We thus have a z-linear map $c_0(G_n) \equiv c_0(F_n)$ verifying:

(1) (G_n) is uniformly trivial.
(2) (G_n) is pointwise convergent to 0.
(3) Each G_n is in canonical convex form with respect to E_n
(4) $c_0(G_n)$ is in canonical convex form with respect to (E_i).

Our aim is to show that $c_0(G_n) \equiv 0$. We need now the "change of convergence" lemma that shall prove in Chapter 4 (Lemma 4.2): *For a sequence of z-linear maps* (F_n) *in canonical convex form defined over the same finite dimensional space* E *and such that* $\sup dim[F_nE] < +\infty$, *pointwise convergence implies norm convergence.* From the lema it follows that for each $j \in \mathbb{N}$, $\|\cdot\| - \lim_n G_{n|_{E_j}} = 0$, since obviously (G_n) is pointwise convergent to 0 on each E_j. That allows us to choose for each j a natural number $N(j)$ such that for each $n \geq N(j)$ one has $\|G_{n|_{E_j}}\| \leq 2^{-j}$. We are ready to define a linear map $L : X \to c_0(Y_n)$ at finite distance from $c_0(G_n)$. We define L as follows: if $x \in X$

$$L(x)(n) = (L_n - L_n \circ \eta_j)(x) \quad para \quad N(j) \leq n < N(j+1).$$

We can give the value $L_n(x)$ for the first $N(1) - 1$ coordinates.

a) The map L is pointwise convergent to 0. Indeed, it is enough to check that on the dense part $\cup_{n=1}^{\infty} E_n$ of X; if $x \in \cup_{n=1}^{\infty} E_n$, there exists j such that $x \in E_j$; thus, $L(x)(n) = 0$ for all $n \geq N(j)$ since in this case $L_n(x) = L_n \eta_s(x)$ for all $s \geq j$.

b) Taking into account that $\|L_{n|_{E_j}}\| \leq \|G_{n|_{E_j}} - L_{n|_{E_j}}\| + \|G_{n|_{E_j}}\| \leq \mu + 2^{-j}$, one gets

$$\|c_0(G_n) - L\| = \sup_n \|G_n - \pi_n L\| \leq \sup_n \|G_n - L_n\| + \|L_{n|_{E_j}}\| \leq \mu + (\mu + 2^{-j})\lambda$$

□

COROLLARY 5.2. *Let* X *be a separable* Π_λ*-space. If* $\mathcal{Z}(X, Y_i) = 0$ *uniformly in* i *then* $\mathcal{Z}(X, c_0(Y_i)) = 0$.

The reason to call Theorem 5.1 "super-Sobczyk" is because its proof contains a proof for Sobczyk's theorem. In our terms:

THEOREM 5.2 (**Sobczyk**). *Each z-linear* $F : S \curvearrowright c_0$ *map defined on a separable Banach space* S *is trivial.*

PROOF. The proof goes as before. In this case it is not necessary to use the the convexification procedure to apply the change of convergence lemma since all the maps have range $\mathbb{R}$. It only remains to see how to define the linear map $L : S \to c_0$ at finite distance from the c_0-product $c_0(F_n)$ of the canonical versions of each F_n. To do that we make a slight modification in the prosecution device for the linear maps L_n: instead of composing each L_n with good enough projections η_j that the space S may have (as it was the case of Π_λ-spaces), we simply take Hahn-Banach extensions $\widehat{L_{n|_{E_j}}}$ of each $L_{n|_{E_j}}$. We thus define L as follows:

$$L(x)(n) = \left(L_n - \widehat{L_{n|_{E_j}}}\right)(x) \quad si \quad N(j) \leq n < N(j+1)$$

(we can set $L_n(x)$ for the first $N(1) - 1$ coordinates).

Let us check that L is well defined and at finite distance from $c_0(G_n)$;

a) The application L pointwise converges to 0. Indeed, it is enough to verify that on the dense part $\cup_{n=1}^{\infty} E_n$ of X. If $x \in \cup_{n=1}^{\infty} E_n$, there exists j such that $x \in E_j$; thus, $L(x)(n) = 0$ for all $n \geq N(j)$ because in that case $L_n(x) = \widehat{L_{n|_{E_s}}}(x)$ for all $s \geq j$. Taking into account that $\|L_{n|_{E_j}}\| \leq \|G_{n|_{E_j}} - L_{n|_{E_j}}\| + \|G_{n|_{E_j}}\| \leq \mu + 2^{-j}$, one gets

$$\|c_0(G_n) - L\| = \sup_n \|G_n - \pi_n L\| \leq \sup_n \|G_n - L_n\| + \|L_{n|_{E_j}}\| \leq 2\mu + 2^{-j}.$$

□

A generalization of the device yields (see also Rosenthal [**72**], Johnson y Oikberg [**33**]):

THEOREM 5.3. *If* (Y_n) *is a sequence of* λ*-separably injective spaces the amalgam* $c_0(Y_n)$ *is a* λ*-separably injective space.*

With some variation, the previous devices provide:

THEOREM 5.4. *Let (Y_n) be a sequence of spaces λ-complemented in its bidual and let Z be a separable $\mathcal{L}_1$-space. Let $F_n : Z \curvearrowright Y_n$ be a $Z(\cdot)$-bounded sequence of z-linear maps. The c_0-product $c_0(F_n)$ is trivial.*

As a corollary one gets.

COROLLARY 5.3. *Let (Y_n) be a sequence of spaces λ-complemented in its bidual. Then for each separable $\mathcal{L}_1$ space one has $\mathcal{Z}(\mathcal{L}_1, c_0(Y_n)) = 0$.*

5.3. Representation of the spaces $\mathcal{Z}\left(\mathbf{l_1(X_n)}, \diamondsuit\right)$. We are looking now a representation of the spaces $\mathcal{Z}(l_1(X_n), Y)$. We shall use the existence of l_1-coproducts in $\mathcal{Z}_Y$.

PROPOSITION 5.2. *Let (Y_n) be a sequence of Banach spaces. For each Banach space Y, the existence of the l_1-coproduct in $\mathcal{Z}_Y$ establishes an isomorphism of vector spaces*

$$\mathcal{Z}(l_1(X_n), Y) \longleftrightarrow l_\infty^{\bowtie}\left(Z(X_n, Y)\right).$$

PROOF. Let $\upsilon_j : X_j \to l_1(X_n)$ be the natural inclusions. Going in one direction we have that $F \to (F\upsilon_n)$ is well defined because if $G \equiv F$ then $(F\upsilon_n) \bowtie (G\upsilon_n)$. It is not difficult to realize that $(F_n) \to l_1(F_n)$ is its inverse map since $l_1(F\upsilon_n)$ is equivalent to F on the dense subspace $l_1(Z_n)$ generated by the finitely supported elements (z_n);

$$\| \textstyle\sum_{i=1}^n Fz_i - F(\sum_{i=1}^N z_i)\| \leq Z(F) \sum_{i=1}^N \|z_i\|.$$

Moreover, it is also well defined because $(F_i) \bowtie (G_i)$ implies $l_1(F_i) \equiv l_1(G_i)$; it is enough to see that if $(F_i) \bowtie 0$ then $l_1(F_i) = \sum_i F_i - \sum L_{\Pi F_i} = \sum_i B_i + \sum_i L_i - \sum L_{\pi F_i}$, where $L_i : Z \to Y_i$ are linear maps and $B_i : Z \to Y_i$ are homogeneous bounded maps such that $\sup_i \|B_i\| < +\infty$. Therefore, $\sum_i B_i : l_1(Z_i) \to Y$ is a bounded well defined map, which makes also $\sum_i L_i - \sum L_{\pi F_i} : l_1(Z_i) \to Y$ well defined. In conclusion, $l_1(F_n) \equiv 0$. $\square$

In particular:

COROLLARY 5.4. *Let Y be a Banach space. If (X_n) is a family of Banach spaces such that $\mathcal{Z}(X_n, Y) = 0$ uniformly in n, then $\mathcal{Z}(l_1(X_n), Y) = 0$.*

In algebraic terms, the previous proposition asserts that the functor $l_1^{\bowtie}(Z(X_n, \cdot))$ is Ext-representable. We shall call its representative $(l_1(X_n), \upsilon_n)$ the l_1-Ext-coproduct in $\mathbf{B}$ of the family (Z_n).

CHAPTER 2

Singular and cosingular objects

The main goal of this chapter is to examine the *singular* objects of $\mathfrak{Q}$ from different points of view. We do not attempt to elaborate a list of nontrivial objects; to that end the reader can consult, for instance, the monograph [**16**], in which an exhaustive study of the main nontrivial sequences of Banach spaces in connection with the so-called *3-space* problems.

Our purpose will rather be to examine those objects of $\mathfrak{Q}$ which, one way or another, have a behaviour *completely* different from that of a trivial object; we shall refer to such objects as *singular objects*. The motivation comes from the existence of certain extensions whose twisted sum spaces have very peculiar topological properties, solving often classical problems: such is the case of Kalton's construction [**43**] solving negatively the basic sequence problem for quasi-Banach spaces, the Kalton-Peck solution [**45**] to Palais problem, so different in their properties from the original one of Enflo, Lindenstrauss and Pisier [**26**], etc (see below Section 2.1).

1. The trivial object

Obviously, an object of $\mathfrak{Q}$ can be "very different" from a trivial object in several forms. Let us start characterizing trivial objects, namely the *zero* object of the category from different points of view to establish, by opposition, the criteria to determine if an object is or is not *singular*. From now on, given an object F of $\mathfrak{Q}$, we shall write j_F y q_F to denote the operators of the extension

$$0 \longrightarrow Y \xrightarrow{j_F} Y \oplus_F Z \xrightarrow{q_F} Z \longrightarrow 0.$$

Lemma 1.1. *Let $F : Z \curvearrowright Y$ be an object of $\mathfrak{Q}$. Let $0 \to Y \to I \to I(Y) \to 0 \equiv \mathcal{I}$ be an injective presentation of Y and let $0 \to K(Z) \to P \to Z \to 0 \equiv \mathcal{P}$ be a projective presentation of Z. The following properties are equivalent.*

(1) *F is the trivial, or zero object, of $\mathfrak{Q}$.*
(2) *Every embedding $\heartsuit \to Z$ lifts to $Y \oplus_F Z$ throughq_F.*
(3) *Every quotient map $Y \to \diamondsuit$ extends to $Y \oplus_F Z$ through j_F.*
(4) *For every embedding i one has $F \circ i \equiv 0$.*
(5) *For every quotient map q one has $q \circ F \equiv 0$.*
(6) *The canonical morphism $\phi_F : \mathcal{P} \longrightarrow F$ is zero.*
(7) *The canonical morphism $\psi_F : \mathcal{I} \longleftarrow F$ is zero.*
(8) *F is an initial, not final, object of $\mathfrak{Q}_Y$.*
(9) *F is a final, not initial, object of $\mathfrak{Q}^Z$.*

One can observe from the lemma 1.1 that the properties characterizing a trivial object F can be divided into two groups: those that either implicit or explicitly involve the quotient map q_F (2, 4, 6 and 8) and those that depend on the embedding j_F (3, 5, 7 and 9). We'll see that each group of properties determine, by opposition, a unique type of objects. Those determined from the point of view of of the quotient shall be termed *singular object* and those determined from the point of view of the embedding shall be termed *cosingular objects*.

2. Singular objects

An object $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0 \equiv F$ is the opposite of the trivial object with respect to condition (2) if no embedding $i : M \to Z$ can be lifted to X through q; equivalently, q is a

strictly singular operator. The basic properties of strictly singular operators can be seen in [**24, 67**].

Conditions (4) and (6) do not present difficulties. About property (8), we observe that the opposite statement "For every object G there exists a morphism $F \longleftarrow G$ but it does not always exist a morphism $G \longleftarrow F$" is just impossible: it is enough to take as $G = \mathcal{I}$ an injective presentation of Y and the canonical morphism $\mathcal{I} \longleftarrow F$ always exist. Thus, regarding (8), an object F shall be less like a trivial one as less morphisms $G \longleftarrow F$ admit.

It shall be helpful one more definition. Let Y and X be quasi Banach spaces and let $i : Y \to X$ be an embedding. We'll say that a quasi Banach space E is a i-superspace (sometimes we just say a superspace when the operator i is clear from the context) if there exist embeddings i' and i'' making a commutative diagram

$$\begin{array}{ccc} & E & \\ i' \nearrow & & \searrow i'' \\ Y & \xrightarrow{\;i\;} & Z. \end{array}$$

An object F shall be called **singular** if there are satisfied any of the equivalent conditions of the following Proposition 2.1; more or less the opposites to (2), (4), (6) and (8) which characterize in Lemma 1.1 the trivial objects in terms of the quotient map. It is clear that the conditions do not depend on the chosen representatives (extension, operators or quasi linear map). The extensions $0 \to Y \to X \to Z \to 0 \equiv F$ representing a singular object F shall be called **singular extensions**. Sometimes we shall call the space X *singular twisted sum* and even, if the way in which X is a twisted sum of Y and Z is clear, *singular space.*

PROPOSITION 2.1. *Let $F : Z \curvearrowright Y$ be an object of $\mathfrak{Q}$ and let $0 \to K(Z) \xrightarrow{i} P \xrightarrow{\pi} Z \to 0 \equiv \mathcal{P}$ be a projective presentation of Z. The following conditions are equivalent.*

(2) *The quotient operator q_F is strictly singular.*
(4) *If $j : M \to Z$ is an embedding and M is infinite dimensional then Fj is not trivial.*
(6) *No representative of the canonical morphism $\mathcal{P} \longrightarrow F$ can be extended to a i-superspace in which $K(Z)$ has infinite codimension.*
(8) *Given an object G, it does not exist a quotient map p with infinite dimensional kernel such that $G \circ p \equiv F$.*

PROOF. **2 $\Longrightarrow$ 4** : If $j : M \to Z$ is a closed infinite dimensional of Z such that $Fj \equiv 0$ then one has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_F Z & \xrightarrow{q_F} & Z & \longrightarrow & 0 \equiv F \\ & & \| & & \uparrow & & \uparrow j & & \\ 0 & \longrightarrow & Y & \longrightarrow & PB & \xrightarrow{q} & M & \longrightarrow & 0 \equiv Fj. \end{array}$$

There must exist then a section s for q that is an into isomorphism. Thus, $s(M)$ is a closed subspace of PB and $q : s(M) \to M$ is an isomorphism. Therefore q_F cannot be a strictly singular operator.

4 $\Longrightarrow$ 2 : Let us assume that q_F is an isomorphism on an infinite dimensional subspace M of $Y \oplus_F Z$. Then $q_F(M)$ is a closed subspace of Z and the corresponding pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_F Z & \xrightarrow{q_F} & Z & \longrightarrow & 0 \equiv F \\ & & \| & & \uparrow & & \uparrow j & & \\ 0 & \longrightarrow & Y & \longrightarrow & Y \oplus M & \longrightarrow & q_F(M) & \longrightarrow & 0 \equiv Fj. \end{array}$$

yields a trivial extension Fj.

6 $\Longrightarrow$ **2** : Let us consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K(Z) \arrow[r, "i"] \arrow[d, "\phi_F"] & P \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \equiv \mathcal{P} \\
0 \arrow[r] & Y \arrow[r] & Y \oplus_F Z \arrow[r, "q_F"] & Z \arrow[r] & 0 \equiv F
\end{tikzcd}$$

Assume that q_F is not strictly singular. Then there exist a closed infinite dimensional subspace $j : M \to Z$ such that the extension Fj in the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_Z \arrow[r, "i"] \arrow[d, "\phi_F"] & P \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \equiv \mathcal{P} \\
0 \arrow[r] & Y \arrow[r] & Y \oplus_F Z \arrow[r, "q_F"] & Z \arrow[r] & 0 \equiv F \\
0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & W \arrow[r] \arrow[u] & M \arrow[r] \arrow[u, "j"] & 0 \equiv Fj
\end{tikzcd}$$

is trivial. Which means, as can be seen in the equivalent diagram

$$\begin{tikzcd}
0 \arrow[r] & K_Z \arrow[r, "i"] & P \arrow[r] & Z \arrow[r] & 0 \equiv \mathcal{P} \\
0 \arrow[r] & Y \arrow[r] \arrow[u, equal] \arrow[d, "\phi_F"] & PB \arrow[r] \arrow[u] \arrow[d] & M \arrow[r] \arrow[u, "j"] \arrow[d, equal] & 0 \equiv \mathcal{P}j \\
0 \arrow[r] & Y \arrow[r] & W \arrow[r] & M \arrow[r] & 0 \equiv Fj,
\end{tikzcd}$$

that ϕ_F extends to PB.

2 $\Longrightarrow$ **6** : Let us assume that ϕ_F extends to some i-superspace E of $K(Z)$ such that $E/K(Z)$ is infinite dimensional. Then $j : E/K(Z) \to Z$ is an embedding and $Fj \equiv \phi\mathcal{P}j \equiv 0$.

2 $\Longleftrightarrow$ **8** : Everything one has to do is to look at the diagram

$$\begin{tikzcd}
 & & & 0 & \\
 & & & B \arrow[u] & \\
0 \arrow[r] & Y \arrow[r, "j_F"] & X \arrow[r, "q_F"] & Z \arrow[r] \arrow[u, "p"] & 0 \equiv F \\
 & & & A \arrow[u, "j"] & \\
 & & & 0, \arrow[u] &
\end{tikzcd}$$

in which A is infinite dimensional. If $Fj \equiv 0$ then the exactness of the long homology sequence (see [**11**]) yields the existence of a quasi-linear map $G : B \curvearrowright Y$ such that $F \equiv Gp$. Reciprocally, it is clear that if there exists $G : B \curvearrowright Y$ such that $Gp \equiv F$, then $Fj \equiv 0$. $\square$

2.1. Singular objects in the classical theory. Not many singular extensions are known, and they are not obtained without effort. The simplest example of singular extension is that of a projective presentation of a Banach space containing no copies of l_1; since $l_1(\Gamma)$ is hereditarily l_1, the quotient operator in

$$0 \longrightarrow K(X) \longrightarrow l_1(\Gamma) \longrightarrow X \longrightarrow 0$$

must be strictly singular.

As we said at the beginning of the section, one of the motivations to observe the singular objects of $\mathfrak{Q}$ was the founding of "exotic" topological properties in their associated twisted sum spaces. Some singular extensions even offer solution to classical problems. Let us briefly describe the basic constructions of this kind appearing in the literature.

2.1.1. **Kalton and Peck Z_p extensions.** In [**45**] Kalton and Peck reinvent Ribe's method of construction of a nontrivial quasi-linear map $\mathcal{R} : l_1 \curvearrowright \mathbb{R}$ (see [**68**]) and obtain nontrivial quasi-linear maps between l_p spaces, for $p < +\infty$. The process is as follows. Let E be any of the spaces l_p for $0 < p < \infty$. Given a lipschitz map $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi(t) = 0$ for $t \leq 0$, one defines a quasi-linear map F_Φ on the subspace of all finitely supported sequences as follows:

$$F_\Phi(x)(n) = x_n \Phi\Big(\log \tfrac{\|x\|}{|x_n|}\Big)$$

In this way one obtains exact sequences

$$0 \longrightarrow l_p \longrightarrow l_p \oplus_{F_\Phi} l_p \longrightarrow l_p \longrightarrow 0 \equiv F_\Phi.$$

Kalton and Peck show that F_Φ is trivial if and only if Φ is bounded (hence, two functions F_ϕ and F_θ are equivalent if and only if $\phi - \theta$ is bounded). The spaces $l_p \oplus_{F_\Phi} l_p$ are usually denoted $Z_p(\Phi)$. With the choice $\Phi(t) = t$ we shall simply put Z_p.

Using Banach space techniques it is proved in [**45**] that Z_p is singular for $p > 1$. This fact yields exotic properties to Z_p: they are spaces with basis, although not unconditional; they are Z_p-hereditarily complemented, hence they do not have complemented copies of l_p, etc. The case of Z_2 is especially interesting since, in addition to being a different and, in a certain sense, extremal solution to the 3-space problem for Hilbert spaces, is the only concrete (and, in some sense, natural) possible counterexample to Banach's hyperplane problem (negatively solved by Gowers [**30**]): it seems to be still unknown if Z_2 is isomorphic to its hyperplanes.

2.1.2. **The space Z_1.** The techniques that work for proving that Z_p is singular when $p > 1$ seem to fail when $p = 1$. In [**18**] we proved that Z_1 is still singular; to do that we need different tools from those of [**45**]: we shall use a variation of Ribe's argument [**68**] that shows that the map $\mathcal{R} : l_1 \curvearrowright \mathbb{R}$ is not trivial plus an added principle that we call "transfer principle":

LEMMA 2.1 (Transfer principle). *If the restriction of a quasi-linear map $F : l_1 \curvearrowright Y$ to some closed infinite dimensional subspace H is trivial then there exists a sequence $(u_n)_n$ formed by blocks of the canonical basis of l_1 such that the restriction of F to $[(u_n)_n]$ is trivial.*

PROOF. Let $(v_n)_n$ be a basic sequence in H, and let $(u)_n$ be a sequence of blocks of the canonical basis of l_1 such that $\|v_n - u_n\|_1 \leq 2^{-n}/n$. Let us see that F is trivial when restricted to $< (u_n)_n >$. Let $\sum \lambda_n u_n$ be a finite combination in $< (u_n)_n >$. The difference

$$F\left(\sum \lambda_n u_n\right) - F\left(\sum \lambda_n (u_n - v_n)\right) - F\left(\sum \lambda_n v_n\right)$$

is a bounded map due to the quasi-linearity of F. Moreover, since the restriction of F to H is trivial, $F(\sum \lambda_n v_n)$ is the sum of bounded plus lineal. On the other hand, the estimate (see [**42**]),

$$\|F(\sum_{i=1}^n x_i) - \sum_{i=1}^n F(x_i)\| \leq C \sum_{i=1}^n i\|x_i\|$$

and the fact that $(v_n - u_n)_n$ is absolutely summable, and the fact that we can assume that $F(u_n - v_n) = 0$ for all n, imply that $F(\sum \lambda_n u_n)$ is trivial. □

We are ready to show that Z_1 is singular.

PROPOSITION 2.2. *If $\theta : \mathbb{R} \to \mathbb{R}$ is a lipschitz increasing map, then $F_\theta : l_1 \curvearrowright l_1$ is strictly singular.*

PROOF. The proof is based on Ribe's arguments [**68**]. Let $(x_n)_n$ be a normalized sequence of blocks of the canonical basis of l_1. We fix $n \in \mathbb{N}$ and define the following collection of n-points:

$$y_{n,k} = \frac{1}{\theta\left(\log(n-1)\right)} x_k - \sum_{i \neq k}^{n} \frac{1}{n-1} \cdot \frac{1}{\theta\left(\log(n-1)\right)} x_i.$$

One has $\sum_{k=1}^{n} y_{n,k} = 0$ and $\|y_{n,k}\|_1 = \frac{2}{\theta(\log(n-1))}$. We take for fixed k

$$-u_{n,k} = \sum_{\substack{i=1 \\ i \neq k}}^{n} \sum_{j \in Sop(x_i)} \frac{x_i(j)}{(n-1)\theta(\log(n-1))} \theta\left(\log\left(\frac{2(n-1)}{|x_i(j)|}\right)\right) e_j.$$

The convex combination

$$(z_n, 0) = \frac{1}{n} \sum_{k=1}^{n} (u_{n,k}, y_{n,k}).$$

has norm greater than or equal to 1.
For fixed k, $\lim_{n\to\infty} \|(u_{n,k}, y_{n,k})\|_{F_\theta} = 0$:

$$\|u_{n,k} - F_\theta(y_{n,k})\|_1 = \sum_{j \in Sop(x_k)} \frac{|x_k(j)|}{\theta(\log(n-1))} \theta\left(\log\left(\frac{2}{|x_k(j)|}\right)\right)$$

from where it follows that Z_1 is not locally convex (this was proved in [**45**, Thm. 4.2], although their proof is perhaps more complicated).

However, the previous process guarantees that the space $\mathbb{R} \oplus_{F_\theta} M$ is not locally convex for every subspace M generated by a sequence of blocks of the canonical basis of l_1. The transfer principle then ensures that F_θ is strictly singular. $\square$

2.1.3. **Solution to a problem of Klee.** In [**52**], Klee asked if every vector topology on a real vector space X is the supremum of other two topologies τ_1 and τ_2 (not necessarily Hausdorff) such that $(X, \tau_1)^* = 0$ - one then says that τ_1 is dual-less - and $(X, \tau_2)^*$ has enough elements to separate points ($x \in X - \overline{\{0\}}^{\tau_2}$, there exists $f \in (X, \tau_2)^*$ such that $f(x) \neq 0$) - τ_2 is then called quasi-convex -. A real topological vector space (X, τ) has Klee's property if $\tau = sup\{\tau_1, \tau_2\}$, with τ_1 is dual-less and τ_2 is quasi-convex. In [**63**], Peck gave a sufficient condition for X to have Klee's property. In [**43**] Kalton gave the first example of a space without Klee's property (about which we shall talk in the next paragraph). In a later paper, [**48**], Kalton and Peck characterize those quasi-Banach separable space having Klee's property; at the same time, they give new and simpler examples. In essence, the result shows that for a quasi-Banach space X Klee's property depends on the position of the *kernel* subspace of X, namely $N = \cap_{x^* \in X^*} \ker x^*$, inside X; in such a way that as more the extension $0 \to N \to X \to X/N \to 0$ is singular more X lacks Klee's property. More precisely, the theorem asserts: *A quasi-Banach separable space has Klee's property if and only if N has infinite codimension in X and the extension $0 \to N \to X \to X/N \to 0$ is not singular.* Using this characterization they construct for each $0 < p < 1$ a push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & l_2 & \longrightarrow & Z_2 & \longrightarrow & l_2 & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \| & & \\ 0 & \longrightarrow & L_p(0,1) & \longrightarrow & PO & \longrightarrow & l_2 & \longrightarrow & 0 \end{array}$$

and obtain a space PO lacking Klee's property.

2.1.4. **The basic sequence problem.** Using as starting point the Gowers-Maurey space [**31**], Kalton constructed in [**43**] a singular extension

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{K} \longrightarrow l_1 \longrightarrow 0,$$

that provides a solution to the basic sequence problem in quasi-Banach spaces: the space $\mathfrak{K}$ actually contains no basic sequences. It is especially interesting the characterization lemma of singular extensions of l_1 by $\mathbb{R}$ that there appears:

LEMMA 2.2 (Kalton, [**?**]). *Let $F : l_1 \curvearrowright \mathbb{R}$ be a quasi-linear map. They are equivalent:*

(1) *$X = \mathbb{R} \oplus_F l_1$ does not contain a basic sequence.*
(2) *If Y is a closed infinite dimensional subspace of X then the kernel $N = \cap_{x^* \in X^*} \ker x^*$ of X is contained in Y.*
(3) *The quotient map of $0 \to \mathbb{R} \to \mathbb{R} \oplus_F l_1 \to l_1 \to 0$ is singular.*
(4) *$\mathbb{R} \oplus_F l_1$ has not Klee's property.*
(5) *Every bounded operator from l_1 into $\mathbb{R} \oplus_F l_1$ is compact.*
(6) *Every bounded operator $T : \mathbb{R} \oplus_F l_1 \to \mathbb{R} \oplus_F l_1$ has the form $T = \lambda I + S$ for some scalar λ and some compact operator S.*

2.1.5. **The Figiel, Ghoussoub and Johnson extensions.** Figiel, Ghoussoub y Johnson construct in [**27**] singular extensions having the form

$$0 \longrightarrow \ker T_p \longrightarrow X_p \xrightarrow{T_p} c_0 \longrightarrow 0,$$

with $1 < p < +\infty$, in order to solve a question about factorization of operators through c_0. In [**16**] they were used as counterexamples to the 3-space problem for properties (u) and V of Pelczynski.

2.1.6. **Singular presentations with kernel $C(K)$.** In [**12**] it is given a method to obtain singular presentations

$$0 \longrightarrow C[0,1] \longrightarrow E(X) \longrightarrow X \longrightarrow 0 \equiv \mathcal{E}_X$$

of any Banach space X not containing l_1. In the particular case $X = c_0$, the space $E(c_0)$ cannot be a quotient of a $C(K)$ space by [**64**]; so, the construction solves several questions of the classical theory of $\mathcal{L}_\infty$-spaces: to be isomorphic to a $C(K)$-space is not a 3-space property (posed in [**16**]); to be isomorphic to $C[0,1]$ is not a 3-space property (implicity posed in [**55**]). The space $E(c_0)$ cannot be a predual of L_1; that is, $E(c_0)$ cannot be renormed so that its dual is a L_1-space, although its dual is clearly isomorphic to $C[0,1]^* \oplus l_1$.

Under some additional hypothesis on X, see [**12**], it is possible to obtain a singular presentation

$$0 \longrightarrow C(\omega^\omega) \longrightarrow E_\omega(X) \longrightarrow X \longrightarrow 0.$$

The conditions on X allow the choice c_0. The space $E_\omega(c_0)$ is such that $E_\omega(c_0)^*$ is isomorphic to l_1 although $E_\omega(c_0)$ is not a predual of l_1.

2.1.7. **Singular presentations with hereditarily indecomposable spaces.** A Banach space is called *indecomposable* if it cannot be decomposed as the direct sum of of any two of its (infinite dimensional) subspaces. It is called *hereditarily indecomposable* (H.I.)if all its infinite dimensional closed subspaces are indecomposable. The concept is due to Johnson and the proof that they exist to Gowers and Maurey [**31**]. Once the existence H.I. is known, it is clear that every extension

$$0 \longrightarrow Y \longrightarrow H.I. \longrightarrow X \longrightarrow 0$$

is singular. The question is is reasonable sequences like that do exist. this has been considered by Argyros and Felouzis [**1**], who obtain presentations

$$0 \longrightarrow Y \longrightarrow HI(X) \longrightarrow X \longrightarrow 0.$$

for 'many' Banach spaces X. In particular, c_0 and every reflexive space having unconditional basis satisfying a certain property (P); this gives a list that includes L_p spaces and Tsirelson space as well. In those cases $HI(X)$ can be chosen reflexive. Moreover, every Banach space not containing l_1 contains a subspace which is a quotient of some H.I. space.

2.1.8. **Singular presentations that locally split.** Starting with James's example, Lindenstrauss showed that any WCG space X admits a presentation

$$0 \longrightarrow Z \longrightarrow Z^{**} \longrightarrow X \longrightarrow 0,$$

being Z a space with basis. In particular, for $X = c_0$ the extension obtained is singular: Z shall be separable since it has basis, so Z^{**} is separable by a 3-space argument. Thus, Z^{**} cannot contain c_0 (or else it would have it complemented). And so the quotient is strictly singular.

2.2. Construction of singular objects. A basic problem in the theory is to know if given two spaces A, B there must necessarily exist a singular object $F : B \curvearrowright A$. Thus, we wonder about the existence of methods to construct singular objects.

The theorem we prove next shows that under certain general assumptions on the space Z it is possible to obtain singular objects $F : Z \curvearrowright \diamondsuit$ just choosing $\diamondsuit$ "big enough", and provided, of course, that some nontrivial object on Z exists.

Theorem 2.1. *Let us assume that Z satisfies the following condition: there is a constant λ such that every closed infinite dimensional subspace H of Z contains a subspace λ-isomorphic to Z and λ-complemented in Z. If there exists a nontrivial object $F : Z \curvearrowright Y$ then there exists singular object $F_s : Z \curvearrowright l_\infty(\Gamma, Y)$.*

Proof. . Let Γ be the set of all λ-complemented subspaces of Z which are λ-isomorphic to Z. Every subspace of Z contains an element of Γ by hypothesis. For each $E \in \Gamma$ let $\gamma_E : E \to Z$ be a λ-isomorphism and let $\pi_E : Z \to E$ be a λ-projection. Let finally $F : Z \curvearrowright Y$ be a nontrivial map. Since $Z(F\gamma_E\pi_E) \le \lambda^2 Z(F)$ the l_∞-product

$$F_s = l_\infty(F\gamma_E\pi_E)$$

is well defined and is a z-linear map $Z \curvearrowright l_\infty(\Gamma, Y)$. It is, moreover, singular since every subspace of Z shall contain some $E_0 \in \Gamma$ on which the product map cannot be trivial because if $\pi_0 : l_\infty(\Gamma, Y) \to Y$ is the projection onto the E_0-coordinate then

$$\pi_0 l_\infty(F\gamma_E\pi_E)_{|E_0} = F\gamma_{E_0} + \pi_0 L$$

for a certain linear map $L : E_0 \to Y$. And $F\gamma_{E_0}$ cannot be trivial since it is isomorphically equivalent to F. $\square$

More difficult is the question of the reduction of the range $l_\infty(\Gamma, Y)$ of F_s to become, hopefully, Y; this would give singular objects in $\mathcal{Q}(Z, Y)$. Let us see that some reduction is possible.

Proposition 2.3. *With the hypothesis for Z as in theorem 2.1, if Z is separable and admits a nontrivial extension by $\mathbb{R}$ (i.e., Z is not a K-space) then $\mathcal{Q}(Z, C[0,1])$ admits a singular element.*

Proof. . It is enough to set $Y = \mathbb{R}$ in the previous theorem and take into account that every quasi-linear map defined on a separable space has a version having separable range. We can assume without loss of generality that the singular map F_s has separable range in $l_\infty(\Gamma)$. Moreover, the algebra generated by a separable space in $l_\infty(\Gamma)$ is a $C(K)$-space with K a metric compact. If is countable we'll have a countable ordinal interval; if not, Milutin's theorem makes $C(K)$ isomorphic with $C[0,1]$. In any case, $\mathrm{Im}(F_s) \subseteq C[0,1]$. $\square$

Hence, the existence of nontrivial extensions of l_1 by $\mathbb{R}$ implies that $\mathcal{Q}(l_1, C[0,1])$ admits singular elements. In [**12**] it is shown, as we have already said, that for every separable space Z

not containing l_1, $\mathcal{Z}(Z, C[0,1])$ contains a singular element. The result remains valid replacing $\mathbb{R}$ by any $\mathcal{C}^*$-commutative algebra..

The separability hypothesis on Z, instead, is not as superfluous as it seems. The following result is inspired by the paper [**8**] of Cabello. In it, a question posed in [**18**] was solved: show that if the cardinal of Γ is big enough there are not singular objects in $\mathcal{Q}(l_1(\Gamma), \mathbb{R})$. Remarkably, the technique used by Cabello can not be used for the general case we present; and conversely, the technique we use next cannot be used when $Z = l_1(\Gamma), Y = \mathbb{R}$. In what follows, $|A|$ shall denote the cardinal of a set A.

THEOREM 2.2 (Proportionality principle). *Let Y and Z be two Banach spaces. If $|Z| > (2^{\aleph_0})^{densY}$ then no singular elements exist in $\mathcal{Z}(Z, Y)$.*

PROOF. Let $0 \to Y \to X \to Z \to 0 \equiv F$ be an extension and let $0 \to Y \to l_\infty(|Y|) \to Q \to 0 \equiv \mathcal{I}$ be an injective presentation of Y. If ψ denotes a representative of the canonical morphism $\mathcal{I} \longleftarrow F$, there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & l_\infty(|Y|) & \longrightarrow & Q & \longrightarrow & 0 \equiv \mathcal{I} \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & Y & \longrightarrow & \square & \longrightarrow & Z & \longrightarrow & 0 \equiv F \\
 & & & & & & \uparrow{\scriptstyle i_\psi} & & \\
 & & & & & & \ker\psi. & &
\end{array}$$

For an infinite dimensional Banach space B one has $\dim B = |B|$. Moreover, if I is a set then $|l_\infty(I)| = |[-1,1]^I| = (2^{\aleph_0})^{|I|}$. So, if $|Z| > (2^{\aleph_0})^{|I|}$ then $\dim \ker\psi = \infty$. It is therefore clear that $Fi_\psi \equiv \mathcal{I}\psi i_\psi \equiv 0$ and thus F cannot be singular. Finally, one can choose $I = B_{Y^*}$ and thus a set of indices with cardinal the density character of Y. $\square$

2.2.1. **Some questions about singular objects.** We know that no singular extension $0 \to Y \to X \to Z \to 0$ exists when Z contains l_1 or $Y = c_0$. The basic question is:
Problem. ¿Under which conditions on Y and Z one has that $\mathcal{Q}(Z, Y) \neq 0$ implies that $\mathcal{Q}(Z, Y)$ contains a singular element?

More specifically, we are interested in knowing when a nontrivial extension $0 \to Y \to X \to Z \to 0$ in which Z and Y are incomparable is singular. The answer to this last question is not "always" because if Z is stable, $Z \sim Z \oplus Z$, then starting with any extension of Z by Y and multiplying by the right with Z we get a nonsingular extension. For instance, starting with a projective presentation $\mathcal{P}_0$ of c_0 we can construct a sequence

$$0 \longrightarrow K(c_0) \longrightarrow l_1 \oplus c_0 \longrightarrow c_0 \oplus c_0 \longrightarrow 0 \equiv \mathcal{P}_0 \oplus c_0,$$

which is obviously not singular .

Let us remark that the "desingularization" process (multiply by the right) we have just described yields isomorphic objects in$\mathfrak{Q}$ to the original ones (in the previous case, $\mathcal{P}_0 \leftrightarrow \mathcal{P}_0 \oplus c_0$), which shows that *to be singular* is not a property stable by isomorphisms. However, it is stable by strict isomorphisms: every extension isomorphically equivalent to a singular extension is singular.

A problem with especial interest for us is to know if there exists a singular object $\mathcal{F}: c_0 \curvearrowright l_1$. As a partial answer we recall that in [**10**] they were constructed singular objects $jF_2 : l_2 \curvearrowright L_1$ by the method of making push-out from the extension $\mathcal{Z}_2$ of Kalton-Peck with an inclusion $j : l_2 \to L_1(0,1)$. A little more complicated is making then pull-back with a quotient $q : l_\infty \to l_2$ and show that the resulting extension $jF_2q : l_\infty \curvearrowright L_1$ is not trivial. This last extension cannot be singular. The point would be if the "localization procedure" (see [**10**]) that allows one to obtain a nontrivial object $F : c_0 \curvearrowright l_1$ adds the singular character.

3. Initial objects

Another basic problem which can be traced back to the origins of the theory of exact sequences of quasi-Banach spaces is to find out which sequences are obtained as push-out from others; or which sequences are such that all the others can be obtained from them by push-out.

To study these questions we consider the category $\mathfrak{Q}^Z$ as an ordered set with the order $>$ induced:

$$F > G \iff F \longrightarrow G.$$

It is evident that the notion of equality coincides with the notion of isomorphism in the category. It is not hard to give examples showing that $>$ is not a total order.

With this new interpretation of $\mathfrak{Q}^Z$ the question at the beginning can be stated as: ¿Which objects of $\mathfrak{Q}^Z$ are comparable? ¿Which subclasses $\mathcal{C}$ of objects of $\mathfrak{Q}^Z$ admit maximal or minimal element?

It is evident that every projective presentation of Z is a maximum of all $\mathfrak{Q}^Z$, and that every trivial object is a minimum. This information however is not quite relevant now. What is interesting is to study the existence of minima and maxima of proper subclasses of objects of $\mathfrak{Q}^Z$.

Given a collection $\mathcal{C}$ of objects of $\mathfrak{Q}^Z$ we'll say that an object $\curlywedge$ **is initial** for $\mathcal{C}$ it is an upper bound for $\mathcal{C}$; that is, $\curlywedge \to F$ for every object $F \in \mathcal{C}$. Observe that $\curlywedge$ does not necessarily belong to $\mathcal{C}$. When it does, $\curlywedge \in \mathcal{C}$, we'll say that $\mathcal{C}$ **admits an initial element**. More often than not the class $\mathcal{C}$ will have a special form: given a class $\mathcal{B}$ of quasi-Banach spaces we shall denote $\mathfrak{Q}_{\mathcal{B}}$, resp. $\mathfrak{Z}_{\mathcal{B}}$, the collection of all objects $0 \to B \to X \to Z \to 0 \equiv F$ of $\mathfrak{Q}$ (resp. $\mathfrak{Z}$)with $B \in \mathcal{B}$.

It is possible to characterize the initial elements with respect to a class $\mathcal{C}$. The proof can be seen in its natural context in the next chapter, in Lemma [**?**].

LEMMA 3.1. *An object F is initial for $\mathcal{C}$ if and only if $Gq_F \equiv 0$ for all $G \in \mathcal{C}$.*

3.1. Construction of initial objects. We show now examples of extension representing initial objects.

► The l_p-product $l_p(F_i)$, $0 < p \leq \infty$, of a family of quasi-linear maps (F_i) that have it is initial for the family $\mathcal{C} = (F_i)$ de $\mathfrak{Q}^Z$. In particular, the l_∞-product of all quasi-linear maps F defined on Z with $Q(F) \leq 1$ is an initial object of $\mathfrak{Q}^Z$; it is isomorphic to the projective presentations of Z.

► Following the idea in [**12**], given a projective presentation $0 \to K(Z) \to P \to Z \to 0 \equiv \mathcal{P}$ of Z we form the set of indices $\Gamma = \mathcal{L}(K(Z), Y)$ and define an operator

$$\delta : K(Z) \to l_\infty(\Gamma, Y)$$

by $\delta(k)(\gamma) = \gamma(k)$. The object $\delta\mathcal{P}$ is initial for $\mathcal{Z}(Z, Y)$, since given F, if $\phi_F : \mathcal{P} \longrightarrow F$ represents the canonical morphism and $\delta_F : l_\infty(\Gamma, Y) \to Y$ is the evaluation at ϕ_F then

$$\delta_F \delta\mathcal{P} \equiv \phi_F \mathcal{P} \equiv F$$

► When $Y = C[0,1]$ y Z is separable, making a reduction of the range of $l_\infty(\Gamma, Y)$ as we did in Proposición 2.3 one obtain an initial element **in** $\mathcal{Z}(Z, C[0,1])$. Without asking Z separable, at the cost of using more general $C(K)$-spaces we can obtain an initial element **for** $\mathcal{Z}(Z, C[0,1])$. Nevertheless, it is possible to make a cleaner construction of an initial object in $\mathfrak{Z}_{C(K)}$, where $C(K)$ denotes the class of all spaces of continuous functions over Hausdorff compact sets: see **??** in Chapter 4.

► Again in Chapter 4, **??**, we shall define the z-dual of a Banach space. The natural predual of the z-dual turns out to be an initial object in $\mathfrak{Z}^Z$.

► Let Z be a quasi-Banach space. It has some interest to construct an initial object for $\mathfrak{Q}^Z_{Banach}$. The first example that comes to on'e mind is the l_∞-product of all quasi-linear maps $F : Z \curvearrowright$ Banach (which means maps with range in a Banach space) and such that $Q(F) \leq 1$.

The second one will be to take a projective presentation $\mathcal{P}$ of Z in Q_p (being Z a $p-Banach$-space) and making push-out with the canonical operator $\delta : K(Z) \to co(K(Z))$. It is clear that $\delta\mathcal{P}$ is initial for $\mathfrak{Q}^Z_{Banach}$.

Initial object have a certain inclination to be singular; this means that if there is a singular object in $\mathcal{Q}(Z,Y)$ then an initial object for $\mathcal{Q}(Z,Y)$ shall be singular.

PROPOSITION 3.1. *The class $\mathcal{Q}(l_1,\mathbb{R})$ does not admit initial objects.*

PROOF. This follows from the existence in $\mathcal{Q}(l_1,\mathbb{R})$ of both singular and non-singular and from the fact that a push-out of an extension by $\mathbb{R}$ is either trivial or isomorphically equivalent to the starting sequence. □

For the next proposition we adopt the notation $\mathcal{L}_\infty, \mathcal{L}_1$ to represent, respectively, unspecified infinite dimensional spaces of type $\mathcal{L}_\infty, \mathcal{L}_1$.

PROPOSITION 3.2. *Let $\mathfrak{U}$ be the class of all Banach space ultrasummands. An extension $0 \to \mathcal{L}_1 \to X \to \mathcal{L}_\infty \to 0$ is not an initial element for $\mathfrak{Z}_\mathfrak{U}$.*

PROOF. . Let $0 \to \mathcal{L}_1 \xrightarrow{j} X \xrightarrow{q} \mathcal{L}_\infty \to 0 \equiv F$ be an initial element for $\mathfrak{Z}_\mathfrak{U}$. Making homology taking valued in a space $\diamondsuit$ we'll get an exact sequence

$$\cdots \longrightarrow L(\mathcal{L}_1,\diamondsuit) \xrightarrow{F^*} \mathcal{Z}(\mathcal{L}_\infty,\diamondsuit) \xrightarrow{q^*} \mathcal{Z}(X,\diamondsuit) \xrightarrow{j^*} \mathcal{Z}(\mathcal{L}_1,\diamondsuit) \longrightarrow \cdots$$

The characterization in Lemma 4.1 means that F^* is surjective; thus $q^* = 0$ and therefore j^* is injective. This means that if $G : X \to U$ verifies $Gj \equiv 0$ then $G \equiv 0$. But that implies that $\mathcal{Z}(X,U) = 0$ for all ultrasummands U, what makes X an $\mathcal{L}_1$-space. The proof concludes applying the following lemma:

LEMMA 3.2. *In an exact sequence $0 \to \mathcal{L}_1 \xrightarrow{j} X \xrightarrow{q} \mathcal{L}_\infty \to 0 \equiv F$ the space X cannot be neither an $\mathcal{L}_1$ nor an $\mathcal{L}_\infty$-space*

PROOF. Let us see first that X cannot be of type $\mathcal{L}_1$. Choose a nontrivial extension $0 \to l_2 \to T \to \mathcal{L}_\infty \to 0 \equiv G$ (they do always exist, as can be seen with local arguments; see [**10**]). One has the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}_1 & \xrightarrow{j} & X & \xrightarrow{q} & \mathcal{L}_\infty & \longrightarrow & 0 \equiv F \\ & & & & & & \| & & \\ 0 & \longrightarrow & l_2 & \longrightarrow & T & \longrightarrow & \mathcal{L}_\infty & \longrightarrow & 0 \equiv G \end{array}$$

If X is of type $\mathcal{L}_1$ then $Gq \equiv 0$, what means the existence of a push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}_1 & \xrightarrow{j} & X & \xrightarrow{q} & \mathcal{L}_\infty & \longrightarrow & 0 \equiv F \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \| & & \\ 0 & \longrightarrow & l_2 & \longrightarrow & T & \longrightarrow & \mathcal{L}_\infty & \longrightarrow & 0 \equiv G \end{array}$$

Now, the operator $\alpha : \mathcal{L}_1 \to l_2$ must be 2-summing and this it extends to all X, what makes G trivial, against the hypothesis. The space X cannot be of type $\mathcal{L}_\infty$ by duality: the dual sequence shall have the form $0 \to L_1(\mu) \to L_1(\nu) \to L_\infty(\eta) \to 0$ which we know is impossible. □

□

3.1.1. **Open questions.**

- ¿Does $\mathcal{Q}(l_1, l_1)$ admit an initial element? Let us observe that if there is a projective presentation of l_1 in Q_p for some $p < 1$ with kernel $K_p(l_1)$ has Banach envelope $co(K_p(l_1)) = l_1$ then the answer is affirmative.
- ¿Does $\mathcal{Z}(l_2, l_2)$ admit an initial element? In this case the previous attack fails since no subspace of l_p, $0 < p < 1$, can have as Banach envelope a B-convex space, such as l_2 (see [**40**])

4. Cosingular objects

Let us review once more the properties that characterize n Lemma 1.1 trivial objects Θ from the point of view of the embedding j_Θ.

(3) Every quotient map $Y \to \diamondsuit$ extends to $Y \oplus_\Theta Z$ through j_Θ.
(5) Every quotient map q makes $q \circ \Theta \equiv 0$.
(7) The canonical morphism $\mathcal{I} \longleftarrow \Theta$ is zero.
(9) F is a final, but not initial, object in $\mathfrak{Q}^Z$.

By duality with the singular case, we shall study in this section the objects of $\mathfrak{Q}$ having *opposite* properties to the previous ones. Now, an object $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0 \equiv F$ is opposite to a trivial object in the sense of (3) when no infinite dimensional quotient of Y extends to X through j; that is, j is a **strictly cosingular** operator. There is not much to say about (5) and (7). About (9) we observe that there is no opposite statement since every object $F : Z \curvearrowright Y$ admits a morphism $\mathcal{P} \longrightarrow F$ when $\mathcal{P}$ is the projective presentation of Z. Thus, with regards to (9), an object F is increasingly far from being trivial when it is reluctant to admit morphisms $G \longrightarrow F$. Equivalently, F is not easily obtained as a push-out.

Again, one more definition shall be useful. Let $q : X \to Z$ be a quotient operator. We'll say that a quasi-Banach W is a q-subquotient (or just a subquotient, when the context is clear) of Z if there exist quotient maps q' and q'' making a commutative diagram

$$\begin{tikzcd} & W \arrow[dr, "q''"] & \\ X \arrow[ur, "q'"] \arrow[rr, "q"] & & Z. \end{tikzcd}$$

An object shall be called **cosingular** if it verifies any of the equivalent conditions of the following Proposition 4.1.

Proposition 4.1. *Let $F : Z \curvearrowright Y$ be an object of $\mathfrak{Q}$ and let $0 \to Y \xrightarrow{i} I \xrightarrow{q} I(Y) \to 0 \equiv \mathcal{I}$ be an injective presentation of Y. The following conditions are equivalent.*

(3) *The embedding j_F is strictly cosingular.*
(5) *If $p : Y \to E$ is a quotient map with infinite dimensional range then $pF \not\equiv 0$.*
(7) *No representative ψ_F of the canonical morphism $\mathcal{I} \longleftarrow F$ can be lifted to a subquotient of $I(Y)$ with infinite dimensional kernel.*
(9) *No embedding j with infinite dimensional cokernel can induce a morphism $G \longrightarrow F$.*

Proof. **3** $\Longleftrightarrow$ **5** : The proof of this equivalence is just a simple dualization of the argument used to prove $1 \Leftrightarrow 2$ in 2.1.

7 $\Longrightarrow$ **3** : Assume that there exist an infinite dimensional quasi-Banach space E and a quotient map $p : Y \to E$ that extends to $Y \oplus_F Z$ in such a way that $pF \equiv 0$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, "p"] & I \arrow[r, "q"] \arrow[d] & I(Y) \arrow[r] \arrow[d, equal] & 0 \equiv \mathcal{I} \\
0 \arrow[r] & E \arrow[r] & PO \arrow[r] & I(Y) \arrow[r] & 0 \equiv p\mathcal{I} \\
0 \arrow[r] & E \arrow[r] \arrow[u, equal] & PB \arrow[r] \arrow[u] & Z \arrow[r] \arrow[u, "\psi_F"] & 0 \equiv T\mathcal{I}\psi_F
\end{tikzcd}$$

in which $p\mathcal{I}\psi_F \equiv pF$ is trivial and thus ψ_F lifts to PO, which is a subquotient of $q : I \to I(Y)$ with infinite dimensional kernel E.

$\mathbf{3} \Longrightarrow \mathbf{7}$: Assume that ψ_F lifts to a q-subquotient W of $I(Y)$ in such a way that the kernel of $q'' : W \to I/Y$ is infinite dimensional. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "i"] \arrow[d, "q'_{|Y}"] & I \arrow[r, "q"] \arrow[d, "q'"] & I(Y) \arrow[r] \arrow[d, equal] & 0 \equiv \mathcal{I} \\
0 \arrow[r] & \ker q'' \arrow[r] & W \arrow[r, "q''"] & I(Y) \arrow[r] & 0 \equiv q'_{|Y}\mathcal{I} \\
0 \arrow[r] & \ker q'' \arrow[r] \arrow[u, equal] & PB \arrow[r] \arrow[u] & Z \arrow[r] \arrow[u, "\psi_F"] & 0 \equiv q'_{|Y}\mathcal{I}\psi_F
\end{tikzcd}$$

Is clear that $q'_{|Y} : Y \to \ker q''$ is also a quotient map. Since $q'_{|Y}\mathcal{I}\psi \equiv 0$ by hypothesis, it can be observed in the equivalent diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & I \arrow[r] & I(Y) \arrow[r] & 0 \equiv \mathcal{I} \\
0 \arrow[r] & Y \arrow[r, "j_F"] \arrow[u, equal] \arrow[d, "q'_{|Y}"] & Y \oplus_F Z \arrow[r] \arrow[u] \arrow[d] & Z \arrow[r] \arrow[u, "\psi_F"] \arrow[d, equal] & 0 \equiv F \\
0 \arrow[r] & \ker q'' \arrow[r] & PO \arrow[r] & Z \arrow[r] & 0 \equiv q'_{|Y}F
\end{tikzcd}$$

the contradiction with pour initial assumption.

$\mathbf{3} \Longrightarrow \mathbf{9}$: Look at the diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & A \arrow[d, "i"'] & & & \\
0 \arrow[r] & Y \arrow[r, "j_F"] \arrow[d, "p"'] & Y \oplus_F Z \arrow[r, "q_F"] & Z \arrow[r] & 0 \equiv F \\
 & B \arrow[d] & & & \\
 & 0. & & &
\end{tikzcd}$$

If there exists a quasi-linear map $G : Z \curvearrowright A$ such that $iG \equiv F$ then $pF \equiv piG \equiv 0$ y p can be extended to $Y \oplus_F Z$. If B was an infinitely dimensional space the hypothesis that j_F is strictly cosingular would be contradicted .

9 $\Longrightarrow$ **3** : If there exists a quotient map $p : Y \to B$ with infinite dimensional range B that extends to $Y \oplus_F Z$ then the exactness of the long homology sequence yields a quasi-linear map $G : Z \curvearrowright \ker p$ such that $F \equiv iG$ where $i : \ker p \to Y$. □

4.1. Minimal versions. Cosingular objects, besides being in connection with the idea of being opposites to a trivial object, seems also to correspond to the idea of *being minimal* in the following sense: let $F : Z \curvearrowright Y$ be a quasi-linear map . It is clear that the map

$$F_m : Z \curvearrowright [FZ]$$

defined by $F_m z = Fz$ is different from F. Since F_m has, in a certain sense, minimal range, we'll say that F_m is a **minimal versión** of F. We hasten to remark that the word "version" has not the habitual meaning of "representative" of a quasi-linear map. Of course that an object F admits infinitely many minimal versions. The relationships among them are not clear further of the existence of morphisms $F_m \longrightarrow F$.

We focus now our attention on the existence nontrivial objects isomorphic to (the objects generated by) their minimal versions: if H is a Hilbert space all object of $\mathfrak{Q}_H$ are isomorphic. This affirmation is consequence of:

Lemma 4.1. *Let $F : Z \curvearrowright Y$ be a quasi-linear map and let F_m be its minimal version. If $[F_m(Z)]$ is complemented in Y then F_m and F are isomorphic objects.*

Proof. It is clear that a projection $Y \to [F_m(Z)]$ induces a morphism $F \longrightarrow F_m$. □

Still more interesting shall be those objects that coincide (excepts for a finite dimensional space) with all their minimal versions. They can be characterized.

Proposition 4.2. *An object F is cosingular if and only if every version F' of F is such that* $\mathrm{span}(F'(Z))$ *is dense on a finitely codimensional subspace.*

Proof. One only needs to go to the characterization theorem 4.1, put $A = [F'(Z)]$ in the proof of (1) $\Longrightarrow$ (4) and conclude that B is infinite dimensional. □

4.2. Construction of cosingular objects. A simple example of cosingular object is that represented by a sequence

$$0 \longrightarrow Y \longrightarrow l_\infty \longrightarrow l_\infty/Y \longrightarrow 0$$

with Y a separable Banach space without reflexive quotients. We know not many more cosingular extensions The extensions $\mathcal{Z}_p$, $p > 1$, of Kalton and Peck are cosingular.

We ask now about methods to construct cosingular objects. The following result says that we can obtain cosingular objects $\heartsuit \curvearrowright Y$ by choosing $\heartsuit$ "big enough".

Proposition 4.3. *Let Y be a p-Banach space with the following property: there is a positive constant λ such that every infinite dimensional quotient $q : Y \to E_0$ of Y admits an infinite dimensional quotient $p : E_0 \to E$ which is λ-isomorphic to Y, and such that $p \circ q$ admits a section with norm at most λ. If there is a nontrivial map $F : Z \curvearrowright Y$, then there is a cosingular element $F_{cs} : l_1(Z, \Gamma) \curvearrowright Y$, for a certain index set Γ.*

Proof. Let Γ be the set formed by all quotients of Y which are λ-isomorphic to Y and whose quotient maps admit a section with norm at most λ. By hypothesis, every quotient of Y has a quotient in Γ. Given $E \in \Gamma$ let $\alpha_E : Y \to E$ be a λ-isomorphism and let $s_E : E \to Y$ be a λ-section for the quotient map. Let finally $F : Z \to Y$ be a nontrivial map. Since $Z(s_E \alpha_E F) \leq \lambda^2 Z(F)$ the l_p-coproduct

$$F_{cs} = \bigoplus_{E \in \Gamma} s_E \alpha_E F$$

is well defined and is a quasi-linear map $l_p(Z,\Gamma) \to Y$. It is, moreover, cosingular since given any quotient of Y there will be an element $\nu in \Gamma$ and a quotient map $q_\nu : Y \to \nu$ such that $q_\nu F_{cs}$ cannot be trivial: the composition $F_{sc} j_\nu$ with the natural inclusion $j_\nu : Z \to l_p(Z,\Gamma)$ shall give

$$\left(\bigoplus_{E\in\Gamma} s_E \alpha_E F\right) j_\nu = s_\nu \alpha_\nu F + L j_\nu$$

for a certain linear map L. Thus, $q_\nu s_\nu \alpha_\nu F \equiv \alpha_\nu F$, which is isomorphically equivalent to F. $\square$

We also pose now the question if given two spaces Z and Y it is possible to find cosingular elements in $\mathcal{Q}(Z,Y)$. In terms of the previous proof, the question is if the domain of F_{cs} can be reduced to Z.

Let us see now that the size of the spaces is important.

Proposition 4.4 (Proportionality principle). *If* dens$Z <$ dens$Y$ *then* $\mathcal{Q}(Z,Y)$ *does not contain cosingular elements.*

Proof. Let $(z_\gamma)_{\gamma\in\Gamma}$ be a dense subset of Z and let us consider $(Fz_\gamma)_\gamma$. There is a certain version F' of F with range contained in $[Fz_\gamma]$; this space has the same order of density that Z. So, if $densY > densZ$ the image of F' cannot be dense on a finitely codimensional subspace of Y. $\square$

5. Final objects

In complete duality with the case of initial objects we are now interested in knowing which extensions can be obtained from others via pull-back or when, given a class $\mathcal{C}$ of extensions there is one extension such that all the elements of $\mathcal{C}$ can be obtained as pull-back from it.

Again, the natural way to study this problem, in our opinion, is to interpret the category $\mathfrak{Q}_Y$ as an ordered set with the order $<$ induced by its morphisms; that is,

$$F < G \iff F \longleftarrow G.$$

It is not hard to give examples showing that $<$ is not total.

Given a fixed quasi-Banach space Y the question interesting for us can be formulated in terms of $<$ as follows: Which elements of $\mathfrak{Q}_Y$ are comparable? ¿Which classes $\mathcal{C}$ of objects of $\mathfrak{Q}_Y$ admit maximal or minimal elements ? We know that the injective presentation of Y is a minimum while the object 0 is a maximum. However, the relevant information will be to know the maximal and minimal elements of proper subsets of objects of $\mathfrak{Q}_Y$.

Given a collection $\mathcal{C}$ of objects of $\mathfrak{Q}_Y$ we shall say that an object Υ **is final** for $\mathcal{C}$ if $\Upsilon \leftarrow F$ for all objects $F \in \mathcal{C}$. Observe that Υ does not necessarily belong to $\mathcal{C}$. When it does and $\Upsilon \in \mathcal{C}$ then we shall say that $\mathcal{C}$ **admits a final element**. Quite often the class $\mathcal{C}$ shall have a special form. Thus, given a class $\mathcal{B}$ of quasi-Banach spaces (resp. Banach spaces), we shall denote by $\mathfrak{Q}^{\mathcal{B}}$ (resp. $\mathfrak{Z}^{\mathcal{B}}$) the collection of all objects $0 \to Y \to X \to B \to 0 \equiv F$ with $B \in \mathcal{B}$. It is possible to characterize the initial elements for a class $\mathcal{C}$: the proof can be seen in its proper context in the next chapter.

Lemma 5.1. *An object F is final for $\mathcal{C}$ if and only if $j_F G \equiv 0$ for all elements G of $\mathcal{C}$.*

It is clear that the l_p-coproduct $l_p(F_i)$, $p \le 1$ of a family of quasi-linear maps (F_i), when it exists, is final for the class $\mathcal{C} = (F_i)$ itself. Thus, the l_1-coproduct of all z-linear maps F with range in Y and $Z(F) \le 1$ is a final element in $\mathfrak{Z}_Y$; therefore, it is isomorphic to the injective presentation of Y.

A reason why it is more complicated to construct final objects than initial objects is that, although it is relatively easy to manipulate the range of a map, there is not much one can do with its domain.

Final objects tend to be cosingular, in the sense that the existence of a cosingular object in $\mathcal{Q}(Z,Y)$ implies that every final object must also be cosingular. Let us write **Sep** to denote the class of all separable Banach spaces.

PROPOSITION 5.1. *Let Y be separable space that is not a subspace of c_0. There are not final elements for $\mathfrak{Z}^{\mathbf{S}ep}$ de la forma* $0 \to Y \xrightarrow{j} X \to c_0 \to 0 \equiv \mathcal{F}$.

PROOF. Let S be separable. If $G : S \curvearrowright X$ is a z-linear map, the exactness of the homology sequence yields the existence of some $F : S \curvearrowright Y$ such that $jF \equiv G$. Since $\mathcal{F} \longleftarrow F$ then one has $jF \equiv 0$. Thus, X is separably injective. It follows from Zippin's theorem [**78**] that $X = c_0$. □

A more general version of this result would say:

COROLLARY 5.1. *If* $0 \to Y \to X \to Z \to 0 \equiv F$ *is final for* $\mathfrak{Z}^{\mathrm{Sep}}$ *and Z is separably injective, then X is separably injective .*

5.0.1. **Open questions.**

- Does $\mathcal{Q}(l_1, \mathbb{R})$ admit a final element?
- Does $\mathcal{Z}(l_2, l_2)$ admits a final element?
- Does $\mathcal{Q}(l_1, l_1)$ admit a final element?
- Let us fix $Y = C[0,1]$. Does there exist a final element for $\mathfrak{Z}^{\mathbf{S}ep}$ having the form $0 \to C[0,1] \to X \to C(K) \to 0$? Proposition 5.1 says that the answer is no for $C(K) = c_0$.

6. Minimal extensions of l_1

To remark the difficulty of disentangling the order structure of either $\mathfrak{Q}^Z$ or $\mathfrak{Q}_Y$ let us make a brief exposition of what we know about the apparently simplest case: $\mathcal{Q}(l_1, \mathbb{R})$. To begin with, there are known three types of nontrivial objects: Ribe's type, constructed from a lipschitz unbounded map θ and which shall be denoted R_θ; Kalton's map constructed in [**40**], which we shall denote K; and Kalton's singular extension constructed in [**43**], which we shall denote $\mathfrak{K}$. There is one more example in the literature, Roberts's example [**69**], but we know nothing about it in this context. If we study $\mathcal{Q}(l_1, \mathbb{R})$ as an ordered subset of either $\mathfrak{Q}^{l_1}$ or $\mathfrak{Q}_{\mathbb{R}}$ we want to determine which ones of the previous elements are comparable. The general situation is :

- All nontrivial elements are cosingular.
- There exist singulares elements.
- There are not initial elements.
- There is a final element (the coproduct) which is not singular.

Let us see now that Ribe's type elements can be obtained from $\mathcal{Z}_1$ type extensions (see [**18**]); precisely, one has that $\mathcal{Z}_1(\theta) \longrightarrow \mathcal{R}_\theta$ through the morphism induced by the functional $1 \in l_\infty$. Moreover,

PROPOSITION 6.1. *Let $f \in l_\infty$. The quasi-linear map $fF_\theta : l_1 \curvearrowright \mathbb{R}$ shall be:*
a) Trivial if $f \in l_1$.
b) Not trivial and not singular when $f \notin c_0$.

PROOF. If $\delta_n : l_1 \to \mathbb{R}$ represents the sequence of coordinate functionals, $\delta_n(x) = x_n$, then the composition $\delta_n \circ F_\theta$ is bounded:

$$|\delta_n F_\theta x| = |x_n \theta\Big(\log \frac{\|x\|}{|x_n|}\Big)| \leq Lip(\theta)|x_n| \log \frac{\|x\|}{x_n} \leq Lip(\theta)\|x\|.$$

Therefore, the functionals $\delta_n : l_1 \to \mathbb{R}$ can be extended to functionals $D_n : Z_1 \to \mathbb{R}$ with uniformly bounded norms. We observe now that if a functional $f : l_1 \to \mathbb{R}$ comes defined by a sequence $f \in l_1$ than it factorizes as $f = f \circ j$ where $f : l_\infty \to \mathbb{R}$ y $j : l_1 \to l_\infty$ is the natural inclusion. Since j can be extended to Z_1, obtaining $(D_n)_n : Z_1 \to l_\infty$, the same occurs to the composition; hence, the quasi-linear map $f \circ F_\theta$ is trivial.

On the contrary, if the functional f is not in c_0 then we can reduce the case to $f = 1$ (in some subspace of l_1 both are essentially equal). In this way we obtain a map not essentially different from (at least in some subspace) to Ribe's map

$$R_\theta(x) = \sum_n x_n \theta \left(\log \frac{\|x\|}{|x_n|} \right).$$

Let us see that it is not trivial. Let 1 be the sum functional; we'll show that the composition $1 \circ F_\theta$ is not trivial when θ is not bounded. Indeed, if it was trivial then 1 would extend as $f : Z_1 \to \mathbb{R}$ with norm, say, C. Consider the following collection of points $(z_{n,k})_{k=1}^n$

$$\begin{aligned} z_{n,k}(i) &= \frac{1}{\theta(\log(n-1))}, i = k \\ z_{n,k}(i) &= -\frac{1}{n-1}\left(\frac{1}{\theta(\log(n-1))}\right), 1 \le i \le n, i \ne k \\ z_{n,k}(n) &= 0,\ i > n. \end{aligned}$$

Taking the points $(F_\theta(z_{n,k}), z_{n,k})$ we have

$$|f\left(F_\theta(z_{n,k}), z_{n,k}\right)| = |R_\theta\left(z_{n,k}\right) + f(0, z_{n,k})| \le C\|z_{n,k}\|_1 = \frac{2C}{\theta(\log(n-1))}$$

on the other hand

$$R_\theta(z_{n,k}) = \frac{1}{\theta(\log(n-1))}\left(\theta(\log 2) - \theta(\log 2(n-1))\right)$$

adding the n coordinates in k and recalling that $\sum_k z_{n,k} = 0$ we get

$$-2C - \theta(\log 2) \le -\theta(\log 2(n-1)) \le 2C + \theta(\log 2),$$

which implies that θ must be bounded.

Finally, the map R_θ cannot be singular since it is invariant by permutations of l_1's canonical basis; if σ is a permutation such that $E = \{x - \sigma x : x \in l_1\}$ is infinite dimensional then $R_\theta|_E$ is trivial. □

We have just given a method to obtain infinitely many non-equivalent nontrivial minimal extensions of l_1. Indeed, let θ and ϕ two lipschitz maps verifying the conditions of Theorem 2.2 and such that $\theta - \phi$ is increasing and unbounded. Let $f \in l_\infty$ be such that $f \notin c_0$; since $f \circ F_\theta - f \circ F_\phi = f \circ F_{\theta-\phi}$ the maps $f \circ F_\theta$ y $f \circ F_\phi$ are not equivalent.

In [**42**] it is shown that R and K are not projectively equivalent. In [**18**] we asked if they could at least be isomorphically equivalent. Cabello gives in [**8**] a negative answer showing that there do not exist morphisms $R \longleftarrow K$. He also shows that if θ and ϕ are lipschitz maps such that

$$\liminf_{t\to\infty} \frac{\theta(t)}{\phi(t)} = 0$$

then it is not possible a morphism $R_\theta \longleftarrow R_\phi$; which in particular produces a continuum of non-mutually isomorphically equivalent elements in $\mathcal{Q}(l_1, \mathbb{R})$. This obviously implies that the corresponding extensions in $Z_1(\theta)$ are not isomorphically equivalent, although this can be directly shown using the technique of the previous result.

We have already seen in the characterizations in Lemma 2.1 that singular objects in $\mathcal{Q}(l_1, \mathbb{R})$ are clearly different from non-singular objects. Let $\mathcal{S}$ be a singular object. Let us see that every morphism $\mathcal{S} \longleftarrow R$ comes defined by a strictly singular operator.

PROPOSITION 6.2. *Let $\mathcal{S} : l_1 \curvearrowright \mathbb{R}$ be a singular object and let R be Ribe's map. Every representative γ of a morphism $\mathcal{S} \longleftarrow R$ is a strictly singular operator.*

PROOF. Assume that there exists a morphism $\mathcal{S} \longleftarrow R$ with a representative γ which is not strictly singular. Let W be a closed infinite dimensional subspace l_1 on which γ is an isomorphism. Kalton, [**?**], showed that Ribe's twisted sum space R is a subspace of L_p, for $p < 1$. Now, since every subspace of L_p, $p < 1$, contains a basic sequence ([**?**]), L_p cannot contain $\mathcal{S}$ as subspace by **??**. So, there exists a closed infinite dimensional subspace M of W such that $ij\gamma\mathcal{S}$ is trivial;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R} \oplus_{\mathcal{S}} l_1 & \longrightarrow & l_1 & \longrightarrow & 0 \equiv \mathcal{S} \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R} \oplus_R l_1 & \longrightarrow & l_1 & \longrightarrow & 0 \equiv R \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle j} & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & PB & \longrightarrow & W & \longrightarrow & 0 \equiv Rj \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle i} & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R} \oplus M & \longrightarrow & M & \longrightarrow & 0 \equiv Rji
\end{array}$$

This contradicts the strict singularity of $\mathcal{S}$. $\square$

Using again Kalton's Lemma 2.1 we obtain the following 3-space type result:

PROPOSITION 6.3. *The property $\mathcal{L}(\,\cdot\,, \spadesuit) = \mathcal{K}(\,\cdot\,, \spadesuit)$ is not a 3-space property in the domain of quasi-Banach spaces.*

PROOF. If $0 \to \mathbb{R} \to S \to l_1 \to 0$ is a singular extension, then $\mathcal{L}(\mathbb{R}, S) = \mathcal{K}(\mathbb{R}, S)$ and $\mathcal{L}(l_1, S) = \mathcal{K}(l_1, S)$, although $id_S \notin \mathcal{K}(S, S)$. $\square$

The result is curious if we realize that the previous one actually is a 3-space property in the domain of Banach spaces:

PROPOSITION 6.4. *The property $\mathcal{L}(\,\cdot\,, \spadesuit) = \mathcal{K}(\,\cdot\,, \spadesuit)$ is a 3-space property in the domain of Banach spaces.*

PROOF. Let $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ be an extension of Banach spaces in which $\mathcal{L}(Y, S) = \mathcal{K}(Y, S)$ and $\mathcal{L}(Z, S) = \mathcal{K}(Z, S)$. Let $T : X \to S$ be an operator and let $\delta : S \to C(B_{S^*})$ be the canonical inclusion. Since $\delta T j$ is compact by hypothesis , it admits a compact extension $\widehat{T} : X \to C(B_{S^*})$. Since $\delta T j = \widehat{T} j$ there is an operator $\nu : Z \to C(B_{S^*})$ such that $\delta T - \widehat{T} = \nu q$. This operator shall be compact by hypothesis, and thus $\delta T = \widehat{T} + \nu q$ hall be compact and therefore T itself shall be compact. $\square$

CHAPTER 3

Extension and lifting of isomorphisms

1. Formulation of the problem in the context of the category

Among classical problems of Banach space theory they stand the extension problems for operators and, of course, the dual problems of lifting. An extension problem means to know if given an operator $t : Y \to M$ from a subspace $j : Y \to X$ of X there exists an operator $T : X \to M$ making commutative the diagram

$$\begin{array}{ccc} Y & \stackrel{j}{\to} & X \\ t\downarrow & \swarrow & T \\ M, & & \end{array}$$

that is, such that $Tj = t$. In this chapter we focus on a particular although especially interesting case: the extension of embeddings to automophisms.

We shall carry our study in the category $\mathfrak{Q}$. To do that, we shall establish the relationships between the extension of embeddings and the existence of morphisms of $\mathfrak{Q}$; and, although they are more subtle, the relationships between the extension of embeddings and the existence of isomorphisms of $\mathfrak{Q}$.

Let therefore $i : Y \to X$ be an embedding. We say that an object F_i of $\mathfrak{Q}$ comes generated by i if it is the equivalence class of the quasi-linear map $F_i : X/i(Y) \curvearrowright Y$ that describes the extension $0 \to Y \stackrel{i}{\to} X \to X/i(Y) \to 0$. Analogously, given a quotient map $q : X \to Z$, the object F_q of $\mathfrak{Q}$ induced by q shall be the equivalence class of the quasi-linear map $F_q : Z \curvearrowright \ker q$ that describes the extension $0 \to \ker q \to X \to Z \to 0$. We shall occasionally use the notation F_i to refer to an object of $\mathfrak{Q}$ to mean that there exists an exact sequence $0 \to Y \to X \to Z \to 0 \equiv F$ in such a way that $F \equiv F_i$. Obviously, an object F of $\mathfrak{Q}$ has as many representations F_i as isomorphisms i lo generate it.

Let us observe now that an extension $\beta : X' \to X$ of an embedding $j : Y \to X$ from a subspace $i : Y \to X'$ originates a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \stackrel{j}{\longrightarrow} & X & \longrightarrow & Z & \longrightarrow & 0 \equiv F_j \\ & & \| & & \uparrow\beta & & \uparrow\gamma & & \\ 0 & \longrightarrow & Y & \stackrel{i}{\longrightarrow} & X' & \longrightarrow & Z' & \longrightarrow & 0 \equiv F_i. \end{array}$$

The existence of such diagram corresponds to the existence of a morphism $F_j \longleftarrow F_i$ in the category. In this way, if i extends through j and viceversa we 'll have $F_i \longleftarrow F_j \longleftarrow F_i$ and the objects F_i and F_j are isomorphic in $\mathfrak{Q}$.

We are especially interested in obtaining an extension operator which is in turn an isomorphism, that is, that the"applications" F_i and F_j be isomorphically equivalent. In terms of $\mathfrak{Q}$ we ask:

Question 1. Under which conditions two objects F_i, F_j isomorphic in $\mathfrak{Q}_Y$ are strictly isomorphic?

That is precisely what occurs in the case of embeddings with range c_0, after the Lindenstrauss-Rosenthal theorem [**57**] which we shall carefully study a few sections later. Although it is obvious that the same does not occur in general, as the following example shows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0 & \xrightarrow{i} & l_1 & \longrightarrow & c_0 & \longrightarrow & 0 \\ & & \| & & \uparrow\downarrow & & \uparrow\downarrow & & \\ 0 & \longrightarrow & K_0 & \xrightarrow{i\oplus 0} & l_1 \oplus l_1 & \longrightarrow & c_0 \oplus l_1 & \longrightarrow & 0. \end{array}$$

Here the embeddings i and $i \oplus 0$ extends to l_1 but the sequences are not isomorphically equivalent since c_0 is not isomorphic to $c_0 \oplus l_1$.

The dual problem to the extension of isomorphisms is the lifting of a quotient operator $q : X \to Z$ through another quotient map $p : X' \to Z$; that means to know if there exists an operator $\beta : X \to X'$ such that $p\beta = q$. The existence of that operator corresponds with the existence of a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \xrightarrow{q} & Z & \longrightarrow & 0 \equiv F_q \\ & & \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X' & \xrightarrow{p} & Z & \longrightarrow & 0 \equiv F_p, \end{array}$$

which in turn means the existence of a morphism $F_q \longrightarrow F_p$. In this way, if q lifts through p and viceversa, we'll have that $F_q \longrightarrow F_p \longrightarrow F_q$, and the objects F_q and F_p are isomorphic in $\mathfrak{Q}$.

If we moreover want that the lifted operator be an isomorphism then what we want is that the maps F_q and F_p be isomorphically equivalent. So, the question now is:

Question 2. Under which conditions two isomorphic objects F_q and F_p are strictly isomorphic?

An answer to this question we once more find in [**57**], where it is shown that F_q and F_p are strictly isomorphic when the quotients q and p are defined on l_1. It is of course easy to find examples to show that the answer to Question 2 cannot be "always".

2. Diagonal Principles

Let $F : Z \curvearrowright Y$ be a quasi-linear map. We define the **isomorphy class of F in the category $\mathfrak{Q}^Z$** as the set of all objects of that category strictly isomorphic to F; i.e.,:

$$\lceil F \;=\; \{\alpha F : \quad \alpha \text{ isomorfismo de } \mathbf{Q}\}$$

The **isomorphy class of F in the category $\mathfrak{Q}_Y$** will be the set

$$F\rfloor \;=\; \{F\gamma : \quad \gamma \text{ isomorfismo de } \mathbf{Q}\}.$$

Naturally then, the *isomorphy class $\lceil F\rfloor$ of an object F in* $\mathfrak{Q}$ shall be the set of all objects strictly isomorphic to F; i.e.,

$$\lceil F\rfloor = \{\alpha F\gamma : \quad \alpha, \gamma \text{ isomorfismos en } \mathbf{Q}\}.$$

2.1. First diagonal principle. Given a quasi-linear map F we shall denote by i_F and q_F the operators appearing in the extension

$$0 \longrightarrow Y \xrightarrow{i_F} Y \oplus_F Z \xrightarrow{q_F} Z \longrightarrow 0 \equiv F.$$

We observe that given two quasi-linear maps $F : Z \curvearrowright Y$ and $G : Z' \curvearrowright Y$ one can construct a complete push-out diagram :

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & Y \arrow[r, "i_G"] \arrow[d, "i_F"'] & Y \oplus_G Z' \arrow[r, "q_G"] \arrow[d, "i_a"] & Z' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \oplus_F Z \arrow[r, "i_b"'] \arrow[d, "q_F"'] & PO \arrow[r, "b"'] \arrow[d, "a"] & Z' \arrow[r] & 0 \\
 & Z \arrow[r, equal] \arrow[d] & Z \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
\tag{5}
$$

▲ **First observation.** The extensions in the cross $i_G F$ and $i_F G$ associated with the objects F and G are well-defined, in the following sense: if F and G are replaced by equivalent maps $F' \equiv F$ and $G' \equiv G$ then

$$\lceil i_{F'} G' = \lceil i_F G \quad \text{y} \quad \lceil i_{G'} F' = \lceil i_G F.$$

So, denoting by **Set** the category of sets, we have just defined a correspondence

$$
\begin{array}{ccc}
\mathfrak{Q}_Y \times \mathfrak{Q}_Y & \xrightarrow{\triangle} & \textbf{Set} \\
(F,G) & \longrightarrow & \lceil i_F G,
\end{array}
$$

which actually induces a functor.

We can easily characterize the morphisms of $\mathfrak{Q}_Y$ in terms of Δ.

LEMMA 2.1. $\triangle(F,G) = \lceil 0$ *if and only if* $F \longleftarrow G$.

PROOF. If there exists a morphism $F \longleftarrow G$ then it is clear that $i_F G$ is trivial. Conversely, choosing representatives F and G, the hypothesis $\triangle(F,G) = 0$ means that $i_F G$ is trivial. Then, there is a retraction $m : PO \to Y \oplus_F Z$ for i_b, which composed with i_a gives an operator $mi_a : Y \oplus_G Z' \to Y \oplus_F Z$ such that

$$mi_a i_G = mi_b i_F = i_F.$$

Thus, mi_a induces an operator $v : Z' \to Z$ making commutative the diagram

$$
\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "i_F"] & Y \oplus_F Z \arrow[r, "q_F"] & Z \arrow[r] & 0 \equiv F \\
0 \arrow[r] & Y \arrow[r, "i_G"'] \arrow[u, equal] & Y \oplus_G Z' \arrow[r, "q_G"'] \arrow[u, "mi_a"'] & Z' \arrow[r] \arrow[u, "v"'] & 0 \equiv G.
\end{tikzcd}
$$

□

This lemma is just a case of the exactness of the long homology sequence obtained applying the functor $\mathcal{L}(Z', \cdot\,)$ to F.

One has a relatively powerful, although elementary, application of the previous lemma: if F and G are quasi-linear maps such that G is not pull-back of F then $i_F G \not\equiv 0$, what necessarily yields $\operatorname{Ext}_{\mathbf{Q}}(Z', Y \oplus_F Z) \neq 0$. Let us see with an example how to use this idea. Consider a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{j} & R & \longrightarrow & Z & \longrightarrow & 0 \equiv F \\
 & & \| & & & & & & \\
0 & \longrightarrow & Y & \xrightarrow[i]{} & X & \longrightarrow & X/Y & \longrightarrow & 0 \equiv G,
\end{array}$$

in which X is a space with the Dunford-Pettis property (see [**23**] and also Section 5), and R is reflexive. In this case it is not possible an extension J of j to X since that would imply that the operator J would be weakly compact, hence completely continuous, and therefore it cannot be an isomorphism on a reflexive space such as Y. So, G is not pull-back of F and thus $\mathrm{Ext}(X/Y, R) \neq 0$. There are other possible variations; their interest lies in that the twisted sum space R can be a very complicated space; for instance in the situation

$$\begin{array}{ccccccccc}
0 & \longrightarrow & l_2 & \xrightarrow{j} & Z_2 & \longrightarrow & l_2 & \longrightarrow & 0 \equiv F \\
 & & \| & & & & & & \\
0 & \longrightarrow & l_2 & \xrightarrow[i]{} & \mathcal{L}_1 & \longrightarrow & \mathcal{L}_1/l_2 & \longrightarrow & 0 \equiv G
\end{array}$$

it is not possible an extension $J : \mathcal{L}_1 \to Z_2$ of j, and thus $\mathrm{Ext}(\mathcal{L}_1/l_2, Z_2) \neq 0$.

Returning to the diagram 2.1, let us carefully observe the diagonal extension associated to the push-out, which we shall from now on denote $\mathrm{PO}(F, G)$, and described by the maps $Fa \equiv Gb$. The next observation is at the same time crucial and obvious:

▲ **Second observation.** *The push-out diagonal* $\mathrm{PO}(F, G)$ *associated to the objects* F *and* G *of* $\mathfrak{Q}_Y$ *is well-defined:* if we replace $i_F G$ in the diagram 2.1 by an isomorphically equivalent extension D,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y \oplus_F Z & \longrightarrow & PO & \xrightarrow{b} & Z' & \longrightarrow & 0 \equiv i_F G \\
 & & \tau\downarrow & & \downarrow\phi & & \| & & \\
0 & \longrightarrow & Y' & \longrightarrow & X' & \xrightarrow[b']{} & Z' & \longrightarrow & 0 \equiv D,
\end{array}$$

(so, τ and ϕ are isomorphisms making commutative the diagram) then $Gb'\phi = Gb$, in such a way that $Gb'\rfloor = Gb\rfloor$. By symmetry, the same occurs if we replace $i_G F$ by maps E such that $\lceil i_G F = \lceil E$.

Proposition 2.1 (**Diagonal push-out principle**). *The correspondence*

$$\begin{array}{ccc}
\mathfrak{Q}_Y \times \mathfrak{Q}_Y & \xrightarrow{\mathcal{PO}} & \mathbf{Set} \\
(F, G) & \longrightarrow & \mathrm{PO}(FG)\rfloor,
\end{array}$$

induces a functor.

In order to obtain the first of the main results of this chapter , Theorem 2.1, what will really matter is not the fact that $\mathcal{PO}$ is precisely a functor but the fact that the correspondence is well defined. What means that it does not depend upon the representatives: if we replace F and G by equivalent extensions then we get sequences in the cross which are isomorphically equivalent. Thus, the new diagonal push-out extensions shall also be isomorphically equivalent to the original ones.

The diagonal push-out principle yields a solution to the problem set at the beginning of the chapter: if $j : Y \to X$ and $i : Y \to X'$ are embeddings in such a way that i extends through j and viceversa, under which conditions there exists an isomorphism $\rho : X \to X'$ such that $\rho j = i$? We are going to show that, although the extensions of i and j need not be isomorphisms themselves (recall the example 1), they *become*, in a certain sense, in a “good” automorphism of the space $X \oplus X'$. Precisely, there will be an automorphism τ making commutative the diagram

$$\begin{array}{ccc} Y & \xrightarrow{j\oplus 0} & X \oplus X' \\ \| & & \wr\tau \\ Y & \xrightarrow{0\oplus i} & X \oplus X'. \end{array}$$

THEOREM 2.1 (First diagonal principle). *Let Z be a quasi-Banach space and let F, G be two objects of $\mathfrak{Q}_Y$. If $F \longleftarrow G \longleftarrow F$ then $F \oplus 0^G \sim G \oplus 0^F$.*

PROOF. The hypothesis $F \longleftarrow G \longleftarrow F$ means that $\triangle(F,G) = 0 = \triangle(G,F)$; so, choosing representatives F and G, the extensions $i_F G$ and $i_G F$ forming the cross of the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & Y & \xrightarrow{i_G} & Y \oplus_G Z' & \xrightarrow{q_G} & Z' & \longrightarrow & 0 \\ & & {\scriptstyle i_F}\downarrow & & \downarrow{\scriptstyle i_a} & & \| & & \\ 0 & \longrightarrow & Y \oplus_F Z & \xrightarrow[i_b]{} & PO & \xrightarrow[b]{} & Z' & \longrightarrow & 0 \\ & & {\scriptstyle q_F}\downarrow & & \downarrow{\scriptstyle a} & & & & \\ & & Z & = & Z & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

are trivial. Replacing $i_F G$ by the equivalent trivial extension

$$0 \longrightarrow Y \oplus_F Z \longrightarrow (Y \oplus_F Z) \oplus Z' \xrightarrow{\pi'} Z' \longrightarrow 0$$

we obtain, by the second observation, that $Gb\rfloor = G\pi'\rfloor$. Analogously, replacing $i_G F$ by

$$0 \longrightarrow Y \oplus_G Z' \longrightarrow (Y \oplus_G Z') \oplus Z \xrightarrow{\pi} Z \longrightarrow 0$$

one gets $i_G F\rfloor = F\pi\rfloor$. Taking into account now that $G\pi' = G \oplus 0^F$ and $F\pi = F \oplus 0^G$ we conclude that the objects $G \oplus 0^F$ y $F \oplus 0^G$ are strictly isomorphic. In terms of exact sequences one has a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & (Y \oplus_F Z) \oplus (Y \oplus_G Z') & \longrightarrow & Z \oplus (Y \oplus_G Z') & \longrightarrow & 0 \equiv F \oplus 0^G \\ & & \| & & \wr & & \wr & & \\ 0 & \longrightarrow & Y & \longrightarrow & (Y \oplus_G Z') \oplus (Y \oplus_F Z) & \longrightarrow & Z' \oplus (Y \oplus_F Z) & \longrightarrow & 0 \equiv G \oplus 0^F \end{array}$$

of isomorphically equivalent extensions. □

If one prefers a less abstract approach to the problem, the first diagonal principle admits a more concrete version; the cost is, in our opinion, to lose the deeper understanding of the result.

THEOREM 2.2. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j} & X & \longrightarrow & Z & \longrightarrow & 0 \equiv F \\ & & \| & & & & & & \\ 0 & \longrightarrow & Y & \xrightarrow{i} & X' & \longrightarrow & Z' & \longrightarrow & 0 \equiv G \end{array}$$

be two exact sequences so that each of them is pull-back of the other. The exact sequences obtained multiplying in cross by the right

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, equal] & X \oplus X' \arrow[r] \arrow[d, phantom, "\wr"] & Z \oplus X' \arrow[r] \arrow[d, phantom, "\wr"] & 0 \\
0 \arrow[r] & Y \arrow[r, "i"] & X' \oplus X \arrow[r] & Z' \oplus X \arrow[r] & 0
\end{tikzcd}$$

are isomorphically equivalent.

Proof. Since G is pull-back of F, there is an operator γ such that $F\gamma \equiv G$; so we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "j"] & X \arrow[r, "q"] & Z \arrow[r] & 0 \equiv F \\
0 \arrow[r] & Y \arrow[r, "i"] \arrow[u, equal] & X' \arrow[r, "p"'] \arrow[u] & Z' \arrow[r] \arrow[u, "\gamma"'] & 0 \equiv G.
\end{tikzcd}$$

Since F is also a pull-back of G, one has $iF \equiv 0$; what means that the associated pull-back diagonal splits. Thus, one has a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r] & X \oplus Z' \arrow[r, "Q"] & Z \arrow[r] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[u, equal] & X' \oplus Z \arrow[r] \arrow[u, "\phi"] & Z \arrow[r] & 0. \arrow[u, equal]
\end{tikzcd}$$

Now, $Q\phi(x, z) = z$, what implies $FQ\phi \equiv 0_Y^{X'} \oplus F$. On the other hand, since the quotient map Q in a diagonal pull-back sequence is $Q(x, z') = \gamma z' - qx$ one has $FQ(x, z') = F(\gamma z' - qx)$. The quasi-linearity of F makes that, denoting by $\pi_X : X \oplus Z' \to X$ and $\pi_{Z'} : X \oplus Z' \to Z'$ the canonical projections, one has $FQ \equiv F\gamma\pi_{Z'} - Fq\pi_X$; finally, $Fq \equiv 0$ and $F\gamma \equiv G$, in such a way that $FQ \equiv G\pi_{Z'} \equiv G \oplus 0_Y^X$. Thus, $(G \oplus 0_Y^X)\phi \equiv 0_Y^{X'} \oplus F$, which is what we wanted because one has a diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & X \oplus X' \arrow[r] \arrow[d] & Z \oplus X'' \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & Y \arrow[r] & X' \oplus X \arrow[r] & Z' \oplus X \arrow[r] & 0
\end{tikzcd}$$

in which the operator in the middle is an isomorphism by virtue of the 3-lemma. $\square$

Maybe this version of the diagonal principle gives us a clearer approach to the existence of the automorphism of $X \oplus X'$ we were looking for, see 2.2.

2.2. Second diagonal principle. The dual situation to that tackled in the previous section begins with the observation that each couple $F : Z \curvearrowright Y$ and $G : Z \curvearrowright Y'$ of quasi-linear maps generates a commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow{\scriptstyle q_a} & & \uparrow & & \\
0 & \longrightarrow & Y' & \xrightarrow{i_G} & Y \oplus_G Z & \xrightarrow{q_G} & Z & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle q_a} & & \uparrow{\scriptstyle q_F} & & \\
0 & \longrightarrow & Y' & \xrightarrow{b} & PB & \xrightarrow{q_b} & Y \oplus_F Z & \longrightarrow & 0 \\
 & & & & \uparrow{\scriptstyle a} & & \uparrow{\scriptstyle i_F} & & \\
 & & & & Y & = & Y & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0. & &
\end{array}
\tag{6}
$$

▼ **First observation.** The extensions forming the cross Fq_G and Gq_F are again well defined in the sense that they do not depend upon the representatives up to isomorphy classes: if F and G are replaced by equivalent quasi-linear maps $F' \equiv F$ and $G' \equiv G$ one has

$$F'q_{G'}\rfloor = Fq_G\rfloor \quad \text{y} \quad G'q_{F'}\rfloor = Gq_F\rfloor.$$

Thus, we have just defined a correspondence

$$
\begin{array}{ccc}
\mathfrak{Q}^Z \times \mathfrak{Q}^Z & \xrightarrow{\nabla} & \mathbf{Set} \\
(F, G) & \longrightarrow & Gq_F\rfloor,
\end{array}
$$

so natural which it induces a functor.

Through ∇ we can easily characterize the morphisms of $\mathfrak{Q}^Z$:

LEMMA 2.2. $\triangledown(F, G) = 0\rfloor$ *if and only if* $F \longrightarrow G$.

PROOF. If there exists a morphism $F \longrightarrow G$ then it is clear that Gq_F is trivial. Conversely the hypothesis $\triangledown(F, G) = 0$ means, choosing representatives F and G, that the extension Gq_F is trivial. Thus, there is a linear continuous selection s for q_b. Let us now consider selections B, bounded and homogeneous, and L, linear, for q_F. One only has to check that $q_a sB$ and $q_a sL$ are selections (the first one homogeneous and bounded and the second linear) for q_G: $q_G q_a sB = q_F q_b sb = id$ y $q_G q_a sL = q_F q_b sL = id$. Por tanto, $i_G G \equiv q_a s(B - L) \equiv q_a s i_F F$. $\square$

The previous lemma is just one instance of the exactness of the long homology sequence obtained applying the functor $\mathcal{L}(\,\cdot\,, Y')$ to F. It is interesting to observe as an application of the Lemma 2.2 that in terms of ∇ we can decide if there is a morphism $F \longrightarrow G$ between two objects or, alternatively, there exists a set $\nabla(F, G)$ of strictly isomorphic nontrivial objects of $\mathfrak{Q}_Z$.

We focus now our attention on the associated pull-back diagonal sequence in the previous diagram, which we shall denote from now on by $\mathrm{PB}(FG)$; it comes described by the maps $aF \equiv bG$.

▼ **Second observation.** *The diagonal pull-back sequence* $\mathrm{PB}(F, G)$ *associated to the objects* F *and* G *of* $\mathfrak{Q}^Z$ *is well defined*: if we replace Gq_F in the pull-back diagram 6 by an isomophically equivalent extension D,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y' & \xrightarrow{b} & PB & \longrightarrow & Y \oplus_F Z & \longrightarrow & 0 \equiv Gq_F \\
 & & \| & & {\scriptstyle \tau}\uparrow & & \uparrow & & \\
0 & \longrightarrow & Y' & \xrightarrow[d]{} & X' & \longrightarrow & Z' & \longrightarrow & 0 \equiv D,
\end{array}
$$

then $\tau dG = bG$, in such a way that $\lceil dG = \lceil bG$. By symmetry, the same occurs when replacing Fq_G by any other map E such that $Fq_G\rfloor = E\rfloor$.

Proposition 2.2 (**Diagonal pull-back principle**). *The correspondence*

$$\begin{array}{ccc} \mathfrak{Q}^Z \times \mathfrak{Q}^Z & \xrightarrow{\mathcal{PB}} & \mathbf{Set} \\ (F,G) & \longrightarrow & \lceil \mathrm{PB}(F,G) \end{array}$$

induces a functor.

This is orientative of the naturalness of all our constructions; however, what really matters here is not that $\mathcal{PB}$ is a functor, but the fact that it provides a well-defined correspondence, in the sense that it does not depend on the representatives. If F and G are replaced by equivalent sequences, the extensions forming the cross are isomorphically equivalent and the same occurs with the diagonal pull-back sequences obtained .

The diagonal pull-back principle provides a solution to the problem we set at the beginning of the chapter: given two quotient maps $q : X \to Z$ and $p : X' \to Z$ such that each can be lifted through the other, under which conditions there would be an isomorphism $\varrho : X \to X'$ so that $p\varrho = q$? As we have already observed, the lifted operators need not be isomorphisms; nevertheless, we shall show next the existence of an automorphism τ of $X \oplus X'$ making commutative the digram

$$\begin{tikzcd} X \oplus X' \arrow[r, "q\oplus 0"] \arrow[d, "\tau\wr"'] & Z \arrow[d, equal] \\ X \oplus X' \arrow[r, "0\oplus p"] & Z. \end{tikzcd}$$

Theorem 2.3 (Second diagonal principle). *Let Z be a quasi-Banach space and let F, G two objects of $\mathfrak{Q}^Z$. If $F \longrightarrow G \longrightarrow F$ then $0_G \oplus F \sim 0_F \oplus G$.*

Proof. The hypothesis $F \longrightarrow G \longrightarrow F$ means that $\nabla(F,G) = 0 = \nabla(G,F)$; that is, fixing representatives F y G, that the extensions Gq_F and Fq_G in the cross of the diagram

$$\begin{tikzcd} & & 0 & 0 & \\ 0 \arrow[r] & Y' \arrow[r, "i_G"] \arrow[d, equal] & Y \oplus_G Z \arrow[r, "q_G"] \arrow[u] & Z \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & Y' \arrow[r] & PB \arrow[r] \arrow[u] & Y \oplus_F Z \arrow[r] \arrow[u, "q_F"'] & 0 \\ & & Y \arrow[u] \arrow[r, equal] & Y \arrow[u, "i_F"'] & \\ & & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

are trivial. Replacing Gq_F by $0^F_{Y'} : Y \oplus_F Z \curvearrowright Y'$ and Fq_G by $0^G_Y : Y' \oplus_G Z \curvearrowright Y$ one obtains isomorphically equivalent extensions

$$\begin{tikzcd} 0 \arrow[r] & (Y \oplus_F Z) \oplus Y' \arrow[r] \arrow[d, "\tau"] & (Y \oplus_F Z) \oplus (Y' \oplus_G Z) \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \equiv 0_F \oplus G \\ 0 \arrow[r] & (Y' \oplus_G Z) \oplus Y \arrow[r] & (Y' \oplus_G Z) \oplus (Y \oplus_F Z) \arrow[r] & Z \arrow[r] & 0 \equiv 0_G \oplus F. \end{tikzcd}$$

□

We also present here a more concrete proof of the diagonal principle.

THEOREM 2.4. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X & \xrightarrow{q} & Z & \longrightarrow & 0 \equiv F \\ & & & & & & \| & & \\ 0 & \longrightarrow & Y' & \longrightarrow & X' & \xrightarrow[q']{} & Z & \longrightarrow & 0 \equiv G. \end{array}$$

be two exact sequence such that each of them is push-out of the other. The exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' \oplus Y & \longrightarrow & X' \oplus X & \longrightarrow & Z & \longrightarrow & 0 \\ & & \wr & & \wr & & \| & & \\ 0 & \longrightarrow & X \oplus Y' & \longrightarrow & X \oplus X' & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

obtained multiplying in cross by the left are isomorphically equivalent.

PROOF. Since there exists an operator α such that $F \equiv \alpha G$ one has $Fq' \equiv 0$, and therefore there would be an isomorphism τ making commutative the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{d} & Y' \oplus X & \longrightarrow & X' & \longrightarrow & 0 \equiv Fq' \\ & & \| & & \big\downarrow \tau & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & Y \oplus X' & \longrightarrow & X' & \longrightarrow & 0 \equiv 0_Y^{X'}; \end{array}$$

here d is defined by $d(y) = (\alpha y, -iy)$. Since $\tau d(y) = (y, 0)$, it follows that $\tau dF \equiv F \oplus 0_{X'}^{Z}$. On the other hand, $dF = (\alpha F, -iF) = (G, 0)$ or, equivalently, $dF = G \oplus 0_X^Z$. All together we have

$$\tau(G \oplus 0_X^Z) \equiv F \oplus 0_{X'}^{Z},$$

what means the existence of a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X \oplus Y' & \longrightarrow & X \oplus X' & \longrightarrow & Z & \longrightarrow & 0 \\ & & \big\downarrow \tau & & \big\downarrow & & \| & & \\ 0 & \longrightarrow & X \oplus Y & \longrightarrow & X \oplus X' & \longrightarrow & Z & \longrightarrow & 0, \end{array}$$

in which the operator in the middle must be an isomorphism by the 3-lemma. □

3. Automorphic spaces

The notion of automorphic spaces stems from the paper [**57**], in which Lindenstrauss and Rosenthal show, among other things, that c_0 has the following property: every isomorphism between two subspaces of c_0 extends to an automorphism of c_0. See also [**58**, 2.f.].

DEFINITION 3.1 (Automorphic space). *A quasi-Banach space X shall be called automorphic if every isomorphism between two infinite codimensional subspaces extends to an automorphism of the space; i.e., given two embeddings $i, j : Y \hookrightarrow X$ such that X/iY y X/jY have infinite dimension, there exists an automorphism τ of X such that $\tau i = j$.*

The first diagonal principle yields a clearer proof for the Lindenstrauss-Rosenthal theorem [**57**]; moreover, it uncovers its homological nature.

THEOREM 3.1. *The space c_0 is automorphic.*

PROOF. Let $i : Y \to c_0$, $j : Y \to c_0$ be two embeddings and let F_i, F_j the quasi-linear maps they induce. Since c_0 is separably injective, we have $jF_i \equiv 0 \equiv iF_j$, and thus by the Lemma 2.1

one gets $F_i \longleftarrow F_j \longleftarrow F_i$. Applying the first diagonal principle one gets that the extensions obtained multiplying in cross

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & c_0 \oplus c_0 & \longrightarrow & c_0/i(Y) \oplus c_0 & \longrightarrow & 0 \equiv F_i \oplus 0^{c_0} \\ & & \| & & \wr & & \wr & & \\ 0 & \longrightarrow & Y & \longrightarrow & c_0 \oplus c_0 & \longrightarrow & c_0/j(Y) \oplus c_0 & \longrightarrow & 0 \equiv F_j \oplus 0^{c_0} \end{array}$$

are isomorphically equivalent. On the other hand, since the quotient operator $q_i : c_0 \to c_0/i(c_0)$ cannot be weakly compact, using a classical result of Pelczynski [**64**] one gets a subspace M_i of c_0 isomorphic to c_0, and therefore complemented, on which q_i is an isomorphism; that means that F_i can be written as

$$0 \longrightarrow Y \xrightarrow{\ i\ } c_0 \oplus M_i \xrightarrow{\ q_i\ } c_0/i(Y) \oplus M_i \longrightarrow 0 \equiv F_i$$

The same applies to F_j.It is easy to check now thatF_i and $F_i \oplus 0^{c_0}$ are isomorphically equivalent, as well as F_j and $F_j \oplus 0^{c_0}$. □

The other example of automorphic space is, for obvious reasons por razones obvias, the Hilbert space l_2. It is worth to observe that, again for obvious reasons related to size,the nonseparable versions $c_0(I)$ and $l_2(I)$ cannot be automorphic spaces. In [**57**, section 3] one can find several non-automorphic spaces. An open problem that still stands open is the conjecture formulated in [**57, 58**]:

Problem. Are c_0 and l_2 the only automorphic spaces?

This problem will not be solved in this thesis, although it has been one of the propellers of the work we have developed. Recently, Tokarev [**76**] affirmed to have constructed new automorphic spaces. The idea of his construction seems good: take as starting point the Gurarii space of almost universal disposition for finite dimensional spaces and, using an amalgamation process (basically passing to the inductive limit in a construction resembling that of Bourgain and Pisier [**7**]) to finally obtain a space of universal disposition for all its subspaces. Nevertheless, the push-out process is not easy to handle and in the end Tokarev arguments are extremely difficult to follow.

3.1. Partially automorphic spaces and the first diagonal principle. There exist spaces with a "partially automorphic behaviour" such as the Lindenstrauss-Rosenthal result in [**57**] for l_∞, or that of Lindenstrauss in [**53**] for l_1 show. We shall see those results now. As it happened with c_0, the first diagonal principle underlies to all those results; that seems to indicate that the automorphic character of a space is at least close to be a homological property.

THEOREM 3.2 (Lindenstrauss-Rosenthal, [**57**]). *Let i and j be two embeddings of a Banach space Y into l_∞ in such a way that neither l_∞/iY nor l_∞/jY are reflexive. There is an automorphism τ of l_∞ such that $\tau i = j$. If, however, both quotients l_∞/iY y l_∞/jY are reflexive, the automorphism τ exists if and only if the Fredholm index of any extension of j to all l_∞ through i is* 0. *If one of the quotients is reflexive but not the other, no automorphism τ can exist.*

PROOF. Let us consider $i, j : Y \to l_\infty$ two embeddings and F_i, F_j the corresponding induced quasi-linear maps. Since l_∞ is injective, $jF_i \equiv 0 \equiv iF_j$; equivalently, there is a commutative diagram of semiequivalent extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\ i\ } & l_\infty & \xrightarrow{\ q_i\ } & l_\infty/i(Y) & \longrightarrow & 0 \equiv F_i \\ & & \| & & \uparrow\downarrow & & \uparrow\downarrow & & \\ 0 & \longrightarrow & Y & \xrightarrow{\ j\ } & l_\infty & \xrightarrow{\ q_j\ } & l_\infty/j(Y) & \longrightarrow & 0 \equiv F_j. \end{array}$$

Applying the second diagonal principle we get that the sequences obtained multiplying in cross $F_i \oplus 0^{l_\infty}$ and $F_j \oplus 0^{l_\infty}$ are isomorphically equivalent

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & l_\infty \oplus l_\infty \arrow[r] \arrow[d, phantom, "\wr"] & l_\infty/i(Y) \oplus l_\infty \arrow[r] \arrow[d, phantom, "\wr"] & 0 \equiv F_i \oplus 0^{l_\infty} \\ 0 \arrow[r] & Y \arrow[r] & l_\infty \oplus l_\infty \arrow[r] & l_\infty/j(Y) \oplus l_\infty \arrow[r] & 0 \equiv F_j \oplus 0^{l_\infty} \end{tikzcd}$$

• *None of the quotients* $\mathbf{l}_\infty/\mathbf{i}(\mathbf{Y})$, $\mathbf{l}_\infty/\mathbf{j}(\mathbf{Y})$ *is reflexive:* in this case the quotient operators q_i and q_j are not weakly compact and, using a combination of a classical result of Pelczynski [**?**] plus one of Rosenthal (see [**?**]), they must be isomorphisms on copies M_i and M_j of l_∞, necessarily complemented. Thus, the extensions F_i and F_j can be described as

$$0 \longrightarrow Y \xrightarrow{\ i\ } l_\infty \oplus M_i \xrightarrow{\ q_i\ } l_\infty/i(Y) \oplus M_i \longrightarrow 0 \equiv F_i.$$

It is therefore clear that F_i and $F_i \oplus 0^{l_\infty}$ are isomorphically equivalent, as well as F_j and $F_j \oplus 0^{l_\infty}$.

• *The quotient* $\mathbf{l}_\infty/\mathbf{i}(\mathbf{Y})$, $\mathbf{l}_\infty/\mathbf{j}(\mathbf{Y})$ *are reflexive:* For this part any injective space I can play the role of l_∞. Let therefore $i : Y \to I_i$ be embeddings whose cokernels $R_i = I_i/i(Y)$ and $R_j = I_j/j(Y)$ are reflexive. From the diagonal principle it follows that the extensions in the diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "i\oplus 0"] \arrow[d, equal] & I_i \oplus I_j \arrow[r] \arrow[d, phantom, "\wr"] & R_i \oplus I_j \arrow[r] \arrow[d, "\mu"] & 0 \equiv F_i \oplus 0^{I_j} \\ 0 \arrow[r] & Y \arrow[r, "j\oplus 0"] & I_j \oplus I_i \arrow[r] & R_j \oplus I_i \arrow[r] & 0 \equiv F_j \oplus 0^{I_i}. \end{tikzcd} \tag{7}$$

are isomorphically equivalent. From the fact that every operator $I \to R$ is strictly singular (the injective space has Dunford-Pettis property, the operator is weakly compact and thus completely continuous) and the decomposition principle of Edelstein-Wojtaszczyk [**58**], it follows that there exist spaces $E_i < I_i$ and $Z_i < R_j$ such that $\mu(R_i) = Z_i \oplus E_i$; and also spaces $E_j < I_j$ and $Z_j < R_i$ such that $\mu^{-1}(R_j) = Z_j \oplus E_j$. Necessarily, E_i and E_j shall be finite dimensional spaces and thus the isomorphism μ acts as follows: if $R_i = Z_j \oplus G_j$ and $R_j = Z_i \oplus G_i$ then $\mu(Z_j) = Z_i$, $\mu(G_j) = E_i$ and $\mu(E_j) = G_i$. That leaves the diagram obtained from the first diagonal principle simplified to a diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "i\oplus 0"] \arrow[d, equal] & I_i \oplus E_j \arrow[r] \arrow[d, phantom, "\wr"] & Z_j \oplus G_j \oplus E_j \arrow[r] \arrow[d, "\mu"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "j\oplus 0"] & I_j \oplus E_i \arrow[r] & Z_i \oplus G_i \oplus E_i \arrow[r] & 0. \end{tikzcd}$$

formed by isomorphically equivalent sequences. Graphically, μhas the form

$$\begin{tikzcd}[column sep=tiny] Z_j \arrow[d, "\beta"'] & \oplus & G_j \arrow[d, "\gamma"'] & \oplus & E_j \arrow[d, "\alpha"] \\ Z_i & \oplus & E_i & \oplus & G_i \end{tikzcd}$$

Now, if the dimensions $dimG_j = dimE_i$ and $dimE_j = dimG_i$ coincide, we can ‘straighten" the isomorphism μ as follows: if $\lambda : E_i \to G_i$ is an isomorphism, then the map $\nu : Z_j \oplus G_j \to Z_i \oplus G_i$ defined by

$$\nu(z, g) = (\beta z, \lambda\gamma g)$$

is an isomorphism $R_i \to R_j$. To simplify notation, in what follows we shall write $F_j \oplus 0$ and $F_i \oplus 0$ to refer to the extensions appearing in the diagram 7; also, B, B_k, $k = 0, 1, ..., 4$, represent homogeneous and bounded maps $Z_j \oplus G_j \oplus E_j \to Y$. One has

$$\begin{aligned}
(F_j\nu \oplus 0)(z,g,h) &= F_j(\beta z, 0) + F_j(0, \lambda\gamma g) + B_0(z,g,h)\\
&= (F_j \oplus 0)(\beta z, 0, 0) + (F_j \oplus 0)(0, \lambda\gamma g, 0) + B_0(z,g,h)\\
&= (F_j \oplus 0)(\beta z, 0, 0) + (F_j \oplus 0)(0, \lambda\gamma g, 0) + (F_j \oplus 0)(0,0,\gamma g) + B_0(z,g,h)\\
&= (F_j \oplus 0)(\beta z, 0, 0) + (F_j \oplus 0)(0,0,\gamma g) + (F_j \oplus 0)(0, \alpha h, 0) -\\
&\quad\ (F_j \oplus 0)(0, \alpha h, 0) + B_1(0,g,0) + B_0(z,g,h)\\
&= (F_j \oplus 0)(\beta z, \alpha h, \gamma g) + B_3(z,g,h) + B_2(0,0,h) + B_1(0,g,0) + B_0(z,g,h)\\
&= (F_j \oplus 0)\mu(z,g,h) + B_4(z,g,h)\\
&\equiv (F_i \oplus 0)(z,g,h) + B(z,g,h),
\end{aligned}$$

and therefore $F_j\nu \equiv F_i$. $\square$

The second result we mentioned appears in the paper [**53**] of Lindenstrauss with the purpose of solving a problem posed by Lindenstrauss and Rosenthal in [**56**] about the classification of $\mathcal{L}_1$-spaces. Translated to our language, the problem is: if we consider the projective presentation of $L_1(0,1)$:

$$0 \longrightarrow D_1 \longrightarrow l_1 \longrightarrow L_1(0,1) \longrightarrow 0$$

and then we consider the projective presentations of the successive kernels

$$0 \longrightarrow D_{n+1} \longrightarrow l_1 \longrightarrow D_n \longrightarrow 0,$$

Can it be $D_n \simeq D_{n+1}$? The negative answer will follows from the fact that l_1 is "automorphic for subspaces having $\mathcal{L}_1$ cokernels", as we show next:

THEOREM 3.3 (Lindenstrauss, [**53**]). *Let $i, j : D \to l_1$ be two embeddings such that the quotients $l_1/i(D)$ and $l_1/j(D)$ are of type $\mathcal{L}_1$. There exists an automophism τ of l_1 such that $\tau \circ j = i$.*

PROOF. The hypothesis give us a diagrama which, with some abuse of notation, is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \xrightarrow{\ i\ } & l_1 & \xrightarrow{\ q_i\ } & \mathcal{L}_1 & \longrightarrow & 0 \equiv F_i\\
 & & \| & & & & & & \\
0 & \longrightarrow & D & \xrightarrow[\ j\ ]{} & l_1 & \xrightarrow[\ q_j\ ]{} & \mathcal{L}_1' & \longrightarrow & 0 \equiv F_j.
\end{array}$$

One has that $jF_i \equiv 0 \equiv iF_j$ since $\mathcal{Z}(\mathcal{L}_1, l_1) = 0$, and thus both sequences are semiequivalent. The first diagonal principle makes isomorphically equivalent the two sequences in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \xrightarrow{\ i\ } & l_1 \oplus l_1 & \longrightarrow & \mathcal{L}_1 \oplus l_1 & \longrightarrow & 0 \equiv F_i \oplus 0^{l_1}\\
 & & \| & & \wr & & \wr & & \\
0 & \longrightarrow & D & \xrightarrow[\ j\ ]{} & l_1 \oplus l_1 & \longrightarrow & \mathcal{L}_1' \oplus l_1 & \longrightarrow & 0 \equiv F_j \oplus 0^{l_1}.
\end{array}$$

Finally, since every $\mathcal{L}_1$-space contains a complemented copy of l_1, on which the quotient operators can be inverted, one gets that F_i and $F_i \oplus 0^{l_1}$ are isomorphically equivalent, as well as F_j and $F_j \oplus 0^{l_1}$. $\square$

It is clear that an isomorphism $D_n \simeq D_m$ eventually would lead to $D_1 \simeq L_1$, what is impossible since L_1 is not a subspace of l_1.

3.2. Partially automorphic spaces and automorphy classes. We intend now to fix the notion of partially automorphic space. The results of the previous section suggest to define weaker forms of the property *to be automorphic.* Precisely, we shall focus on the property of being automorphic with respect to a given class of subspaces or operators. We shall see that the new definitions uncover some new ideas about the problem.

Given two quasi-Banach spaces Y, X we call sheaf $\lhd$ to a collection of embeddings $Y \to X$. Sometimes shall be simpler to choose a class $\mathcal{C}$ of operators and consider the sheaf $\lhd_{\mathcal{C}}$ of embeddings $Y \to X$ in $\mathcal{C}$. If we need to remark all together we shall write

$$Y \lhd_{\mathcal{C}} \quad X.$$

We shall say that two embeddings $i, j : Y \to X$ have the same **isomorphy class** if the induced quasi-linear maps F_i and F_j are isomorphically equivalent. A quasi-Banach space X shall be said Y-automorphic if all embeddings of the sheaf $Y \lhd_{\mathcal{L}} X$ have the same isomorphy class ($\mathcal{L}$ denotes the class of all operators).

Let us give one step further and to try to discard the subspace Y. We shall say that a class $\mathcal{A}$ of operators is an **automorphy class** for X if for every quasi-Banach space Y, all embeddings of the sheaf $Y \lhd_{\mathcal{A}} X$ have the same isomorphy class. Sometimes we shall also say that X is $\mathcal{A}$-automorphic.

A space X can have several automorphy classes. Naturally, X shall be automorphic if it has just one automorphy class $\mathcal{A} = \mathcal{L}$.

The concept of partially automorphic space suggest to use the number of automorphy classes as an indicator of the automorphic character of the space. It is clear that the automorphy classes of a space X form a partition of the set of all embeddings into X. Thus, a spaces shall be *more* automorphic as smaller is the number of automorphy classes. It seems natural to ask:

Question: How are the spaces with a finite or countable quantity of automorphy classes?

The first examples that comes to one's mind is that of Hilbert spaces. The Hilbert space $l_2(\aleph_n)$ has $n+1$ automorphy classes, corresponding with the embeddings with cokernel having density character $\aleph_0, \aleph_1, \ldots, \aleph_n$. Analogously, $l_2(\aleph_\omega)$ has a countable quantity of automorphy classes. Less trivial is the following result:

PROPOSITION 3.1. *The space l_∞ has a countable quantity of automorphy classes.*

PROOF. Let $\mathcal{R}$ be the class of all embeddings into l_∞ with reflexive cokernel. And let us consider for each Banach space Y_α (when possible) an embedding $i_\alpha : Y_\alpha \to l_\infty$ of $\mathcal{R}$. A simple observation is that given $j \in \mathcal{R}$, every extension $\widehat{j}$ of j through i_α is a Fredholm operator (it has finite dimensional kernel and cokernel). It can be checked (see e.g. [**58**]) that the Fredholm index $I(\widehat{j})$ is independent of the extension of j considered; it has therefore meaning to define $\mathrm{ind}_\alpha j$ as the Fredholm index of any extensión of j through i_α. For each $k \in \mathbb{Z}$ we form the sheaf

$$\lhd_{k,\alpha} = \{j \in \mathcal{R} : \quad \mathrm{ind}_\alpha j = k\}.$$

Let us see that the set$\mathcal{F}_k$[1] defined by

$$\mathcal{F}_k = \bigcup_\alpha \lhd_{k,\alpha}$$

is an automorphy class. With that purpose we recall the second part of Theorem 3.2 which ensures that for each α the elements in the sheaf $\lhd_{0,\alpha}$ are in the same isomorphy class. Fix an integer k and consider $i, j \in \lhd_{k,\alpha}$. Let $\widehat{i}$ be an extension of i through i_α and let $\widetilde{j}$ b e an extension of j through i. It is clear that $\widetilde{j} \circ \widehat{i}$ is an extension of j through i_α. Using the general properties of the Fredholm index, one gets that

$$k = I(\widetilde{j} \circ \widehat{i}) = I(\widetilde{j}) + I(\widehat{i}).$$

Thus, $I(\widetilde{j}) = 0$, and then i and j belong to the same isomorphy class.

[1]The classes $\lhd_{k,i}$ are not equivalence classes; if $j \in \lhd_{k,i}$ then $i \in \lhd_{-k,j}$

If $\mathcal{F}$ is the class of all embeddings $i : Y \to l_\infty$ having non-reflexive cokernel, we already know from Theorem 3.2 that $\mathcal{F}$ is an automorphy class. Therefore, the automorphy classes of l_∞ are

$$\mathcal{F}, \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \dots \mathcal{F}_{-1}, \mathcal{F}_{-2}, \dots$$

□

It is tantalizing to conjecture that $c_0(\aleph_n)$ has $n+1$ automorphy classes. The idea shall be to use the decomposition lemma of Johnson and Zippin [**39**] plus some combinatorics.

One space which at (very) first sight seems automorphic, although it is clearly not, is $c_0 \oplus l_2$. This observations suggests some questions:

Problems:

(1) ¿Which are the automorphy classes of $c_0 \oplus l_2$?
(2) What does it happen if we replace "product" by "l_2-amalgam" $l_2(c_0)$?
(3) If (A_n) is a collection of finite dimensional spaces, Has the space $l_\infty(A_n)$ the property of being $c_0(A_n)$-automorphic?
(4) Still simpler and apparently easier: Has the space $C[0,1]$ the property of being l_2 automorphic? See section 5.1 in Chapter 4 for a thorough discussion of the problem.

3.2.1. **Automorphic spaces using a notion of spectrum.**

Definition 3.2. *We define the spectre of a quasi-Banach space X as the set of objects F of the category for which X is isomorphic to $Y \oplus_F Z$, that is*

$$Spec(X) = \{F : Z \curvearrowright Y : Y \oplus_F Z \backsimeq X\}.$$

The spectrum gives us another form of regarding the notion of automorphic space.

Lemma 3.1. *A quasi-Banach space X is automorphic if each couple of elements of its spectrum $F : Z \curvearrowright Y$ and $F' : Z' \curvearrowright Y'$ for which Y and Y' are isomorphic, are strictly isomorphic.*

Curiously, the notion of spectrum suggests a possible notion of "quasi-automorphic" space.

Definition 3.3. *A quasi-Banach space X shall be called quasi-automorphic if each two elements*
$F : Z \curvearrowright Y$, $F' : Z' \curvearrowright Y'$ in its spectrum for which Y is isomorphic to Y' and Z is isomorphic to Z' are strictly isomorphic.

4. Co-automorphic spaces

In addition to the result for c_0, which suggested the definition of automorphic space, and the result for l_∞, which suggested the concept of partially automorphic space, it appears in [**57**] another result for l_1: every isomorphism between two proper quotients of l_1 with infinite dimensional kernel lifts to an automorphism of l_1. Such result suggests the definition of co-automorphic space.

Definition 3.4 (Co-automorphic space). *A quasi-Banach space X is said to be co-automorphic if every isomorphism between two of its proper quotients can be lifted to an automorphism of the space; namely, given two quotient maps $q, p : X \to Z$, with infinite dimensional kernel, there exist an automorphism τ of X such that $p\tau = q$.*

By duality, it is clear that Theorem 2 in [**57**] also has homological nature; that we show obtaining the result from the second diagonal principle.

Theorem 4.1. *The space l_1 is co-automorphic.*

Proof. Let $q, q' : l_1 \to Z$ be two quotient maps with infinite dimensional kernel, and let F_q, $F_{q'}$ the quasi-linear induced maps. Since l_1 is projective, $F_q q' \equiv 0 \equiv F_{q'} q$, that is, the extensions in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & l_1 & \xrightarrow{q} & Z & \longrightarrow & 0 \equiv F_q \\ & & \uparrow\downarrow & & \uparrow\downarrow & & \| & & \\ 0 & \longrightarrow & Y' & \longrightarrow & l_1 & \xrightarrow{q'} & Z & \longrightarrow & 0 \equiv F_{q'} \end{array}$$

are semiequivalent. Applying the second diagonal principle we get that $0_{l_1} \oplus F_q$ and $0_{l_1} \oplus F_{q'}$ are strictly isomorphic, and thus we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & l_1 \oplus Y & \xrightarrow{i} & l_1 \oplus l_1 & \xrightarrow{q} & Z & \longrightarrow & 0 \equiv 0_{l_1} \oplus F_q \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & l_1 \oplus Y' & \xrightarrow{j} & l_1 \oplus l_1 & \xrightarrow{q'} & Z & \longrightarrow & 0 \equiv 0_{l_1} \oplus F_{q'} \end{array}$$

formed by isomorphically equivalent extensions. On the other hand, since every closed infinite dimensional subspace of l_1 contains a subspace isomorphic to l_1 and complemented in l_1, there will be a subspace M of Y isomorphic to l_1 and complemented in l_1, so that F_q can be written as

$$0 \longrightarrow M \oplus Y \longrightarrow M \oplus l_1 \xrightarrow{q} Z \longrightarrow 0 \equiv F_q.$$

And analogously for $F_{q'}$. It is immediate to verify that F_q and $0_{l_1} \oplus F_q$ are isomorphically equivalent, as well as $F_{q'}$ y $0_{l_1} \oplus F_{q'}$. $\square$

In addition to the conjecture about automorphic spaces, it remains unsolved the dual conjecture also formulated by Lindenstrauss and Rosenthal in [**57**].

Problem: Are l_1 and l_2 the only co-automorphic Banach ?

4.1. Partially co-automorphic spaces and the second diagonal principle. Let us see next how the second diagonal principle unifies most of the results in the literature about"partially co-automorphic spaces". Again, this fact reveals that all of them have a common homologcal nature; in spite of the fact that the original proof, quite complicated some times, did not apparently have any common point.

Firstly, Kalton observes in [**41**] that the l_p-spaces have, for $0 < p < 1$, a behaviour close to that of l_1. After the proof we did for l_1 there is no surprise in the extra hypothesis.

THEOREM 4.2. *Kalton* [**41**] *Let $0 < p < 1$ and let q and Q be two quotient maps from l_p onto a space X not isomorphic to l_p. If $\ker q$ and $\ker Q$ contain copies of l_p complemented in l_p then there exist an automorphism τ of l_p such that $q\tau = Q$.*

PROOF. Let F_q and F_Q be the quasi-linear maps induced by q and Q, respectively. Since l_p, $0 < p < 1$, is projective in the category of p-Banach spaces, one has $F_q \longrightarrow F_Q \longrightarrow F_q$ and, therefore, the second diagonal principle yields the existence of a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & l_p \oplus \ker q & \longrightarrow & l_p \oplus l_p & \xrightarrow{q} & Z & \longrightarrow & 0 \equiv 0_{l_p} \oplus F_q \\ & & \wr & & \wr & & \| & & \\ 0 & \longrightarrow & l_p \oplus \ker Q & \longrightarrow & l_p \oplus l_p & \xrightarrow{Q} & Z & \longrightarrow & 0 \equiv 0_{l_p} \oplus F_Q \end{array}$$

formed by isomorphically equivalent extensions. The extra hypothesis about $\ker q$ and $\ker Q$ allow us to use the same argument as in Theorem 4.1; thus, F_q can be written as

$$0 \longrightarrow M \oplus \ker q \longrightarrow M \oplus l_p \xrightarrow{q} Z \longrightarrow 0 \equiv F_q$$

where M is a subspace of $\ker q$ isomorphic to l_p and complemented in l_p. The same happens with F_Q. Therefore, F_q and $0_{l_p} \oplus F_q$ are isomorphically equivalent, as well as F_Q and $0_{l_p} \oplus F_Q$. $\square$

Another result much more delicate for the spaces $L_p = L_p(0,1)$ when $0 \le p < 1$ appears in [**45**].

THEOREM 4.3 (Kalton-Peck,[**46**]). *Let $0 \leq p < 1$ and let q and Q be two quotient maps from L_p onto a quasi-Banach space X such that* $\ker q$ *and* $\ker Q$ *are either q-Banach spaces for some $q > p$ or ultrasumand spaces; then there exists an automorphism τ of L_p such that $q\tau = Q$.*

PROOF. The proof is based on the result of Kalton and Peck in [**46**] asserting that $\mathrm{Ext}(L_p, A) = 0 = \mathcal{L}(L_p, A)$ when A is either an ultrasumand or a q-Banach space for some $p < q$. This makes $F_q \longrightarrow F_Q \longrightarrow F_q$. Applying the second diagonal principle one has that the extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_p \oplus \ker q & \longrightarrow & L_p \oplus L_p & \xrightarrow{q} & Z & \longrightarrow & 0 \\ & & \wr & & \wr & & \| & & \\ 0 & \longrightarrow & L_p \oplus \ker Q & \longrightarrow & L_p \oplus L_p & \xrightarrow{Q} & Z & \longrightarrow & 0 \end{array}$$

are isomorphically equivalent. Moreover, the hypotheses about the kernels imply $\mathcal{L}(L_p, \ker q) = 0 = \mathcal{L}(L_p, \ker Q)$. It thus follows that an isomorphism $L_p \oplus \ker q \to L_p \oplus \ker Q$ becomes an isomorphism between the kernels. So, F_q and F_Q are isomorphically equivalent. □

4.1.1. **A new technique: Natural transformations of the functors $\mathcal{Q}(\mathbf{X}, \cdot\,)$.** In [**19**] it is obtained a new proof of Theorem 4.3 by using a unusual technique: characterizing the natural transformations of different functors $\mathcal{Q}(X, \cdot\,)$. The interest of that technique goes beyond the concrete result it provides.

Let us first observe that an operator $T : Y \to X$ induces a natural transformation $\eta : \mathcal{Q}(X,\cdot) \to \mathcal{Q}(Y,\cdot)$ in the form $\eta_A(W) = WT$. The first result is modelled upon theorem 10.1 and proposition 10.3 in [**32**].

LEMMA 4.1. *Let X and Y be p-Banach spaces. If one considers the functors $\mathcal{Q}(X,\cdot)$ and $\mathcal{Q}(Y,\cdot)$ acting between the categories $\mathbf{Q_p} \to \mathbf{Vect}$, every natural transformation $\eta : \mathcal{Q}(X,\cdot) \to \mathcal{Q}(Y,\cdot)$ comes induced by an operator $T : Y \to X$ in the form $\eta_Z(W) = WT$*

PROOF. Let $\eta : \mathcal{Q}(X,\cdot) \to \mathcal{Q}(Y,\cdot)$ be a natural transformation and

$$0 \longrightarrow K(X) \xrightarrow{i} l_p(\Gamma) \xrightarrow{q} X \longrightarrow 0 \equiv F$$

a projective presentation of X. Taking homology with the functor $\mathcal{L}(Y, \cdot\,)$ one gets the exact sequence

$$0 \to \mathcal{L}(Y, K(X)) \to \mathcal{L}(Y, l_p(\Gamma)) \to \mathcal{L}(Y, X) \xrightarrow{F^*} \mathcal{Q}(Y, K(X)) \xrightarrow{i^*} \mathcal{Q}(Y, l_p(\Gamma)) \to \cdots.$$

Observe that $\eta_{K(X)}(F) \in \ker i^*$ by the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{Q}(X, K(X)) & \xrightarrow{\eta_{K(X)}} & \mathcal{Q}(Y, K(X)) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ \mathcal{Q}(X, l_p(\Gamma)) & \xrightarrow[\eta_{l_p(\Gamma)}]{} & \mathcal{Q}(Y, l_p(\Gamma)), \end{array}$$

since $i^*\eta_{K(X)}(F) = \eta_{l_p} i^*(F) = 0$. The exactness of the long homology sequence at $\mathcal{Q}(Y, K(X))$ implies the existence of an operator $\alpha : Y \to X$ such that $\eta_{K(X)}(F) = F\alpha$. Let us see that for every quasi-linear map $W : X \curvearrowright Z$ one has $\eta_Z(W) = W\alpha$.

On one hand $W \equiv \phi F$ for some operator $\phi : K(X) \to Z$. The commutativity of the diagram

$$\begin{array}{ccc} \mathcal{Q}(X, K(X)) & \xrightarrow{\eta_{K(X)}} & \mathcal{Q}(Y, K(X)) \\ {\scriptstyle \phi^*}\downarrow & & \downarrow{\scriptstyle \phi^*} \\ \mathcal{Q}(X, Z) & \xrightarrow[\eta_Z]{} & \mathcal{Q}(Y, Z) \end{array}$$

implies $\eta_Z(W) = \eta_Z(\phi F) = \eta_Z \phi^*(F) = \phi^* \eta_{K(X)}(F) = \phi F \alpha = W\alpha$. □

The next result translates a classical: [**32**, thm.10.4])

PROPOSITION 4.1. *Let X and Y be p-Banach spaces. Two functors $\mathcal{Q}(X, \cdot)$ and $\mathcal{Q}(Y, \cdot)$ acting between the categories $\mathbf{Q_p}$ and* **Vect** *are naturally equivalent if and only if there exist sets I, J such that $X \oplus l_p(I) \simeq Y \oplus l_p(J)$.*

PROOF. A natural equivalence $\eta : \mathcal{Q}(X,\cdot) \to \mathcal{Q}(Y,\cdot)$ induces at the same time natural equivalences $\eta^n : \mathcal{Q}^n(X,\cdot) \to \mathcal{Q}^n(Y,\cdot)$ between the iterated derived spaces (see [**32, 11**] and also Chapter 5). After the previous lemma, η comes induced by an operator $\alpha : Y \to X$. It may happen that α is a quotient map or not. If it is, the kernel K_α is a projective space: for every Z, the exactness of the homology sequence

$$\mathcal{L}(Y,Z) \to \mathcal{L}(\mathrm{K}_\alpha, Z) \to \mathcal{Q}(X,Z) \overset{\eta_Z}{\to} \mathcal{Q}(Y,Z) \to \mathcal{Q}(\mathrm{K}_\alpha, Z) \to \mathcal{Q}^2(X,Z) \overset{\eta_Z^2}{\to} \mathcal{Q}^2(Y,Z)$$

together with the fact that η_Z y η_Z^2 are isomorphisms imply $\mathcal{Q}(\mathrm{K}_\alpha, Z) = 0$.
Now, since for every Z the map η_Z is an isomorphism, $\mathcal{L}(Y,Z) \to \mathcal{L}(\mathrm{K}_\alpha, Z)$ is surjective and $0 \to \mathrm{K}_\alpha \to Y \overset{\alpha}{\to} X \to 0$ splits. Then $Y = \mathrm{K}_\alpha \oplus X$; from where $Y = l_p(I) \oplus X$ for some index set I. If, on the contrary, α is not surjective, we shall make pull-back with a projective presentation $0 \to K \to l_p(J) \overset{Q}{\to} X \to 0$ of X to obtain the diagonal pull-back extension

$$0 \longrightarrow PB \longrightarrow Y \oplus l_p(J) \xrightarrow{\alpha \oplus Q} X \longrightarrow 0.$$

Using the previous result one gets $Y \oplus l_p(J) = X \oplus l_p(I)$.
The converse is clear. □

Up to here an expected result, in some sense. The novelty begins when we realize that for $L_p = L_p(0,1),\ 0 < p < 1$ one can obtain a dual-like result. To simplify notation, given a subspace A of L_p we shall write

$$0 \longrightarrow A \xrightarrow{i_A} L_p \xrightarrow{q_A} L_p/A \longrightarrow 0 \equiv F_A.$$

LEMMA 4.2. *Let A and B q-Banach subspaces of L_p, $0 < p < q \le 1$. Let us consider the functors $\mathcal{Q}(L_p/A,\cdot)$ and $\mathcal{Q}(L_p/B,\cdot)$ acting between the categories $\mathbf{Q_q} \to$* **Vect**. *Every natural transformation $\mathcal{Q}(L_p/A,\cdot) \to \mathcal{Q}(L_p/B,\cdot)$ comes induced by an operator $B \to A$.*

PROOF. The key of the result lies in that if A is a q-Banach subspace of L_p for $0 < p < q \le 1$ then there is a natural equivalence $\nu_A^{-1} : \mathcal{L}(A,\cdot) \to \mathcal{Q}(L_p/A,\cdot)$, where the functors $\mathcal{L}(A,\cdot)$ and $\mathcal{Q}(L_p/A,\cdot)$ are considered acting between the categories $\mathbf{Q_q} \to \mathbf{Vect}$. This follows from the long homology sequence obtained from the functor $\mathcal{L}(\cdot\,, X)$ applied to $0 \to A \to L_p \to L_p/A \to 0$:

$$0 \to \mathcal{L}(L_p/A, X) \to \mathcal{L}(L_p, X) \to \mathcal{L}(A,X) \to \mathcal{Q}(L_p/A, X) \to \mathcal{Q}(L_p, X) \to \cdots$$

Since $\mathcal{L}(L_p, X) = 0 = \mathcal{Q}(L_p, X)$ it follows from [**11**] that $\mathcal{L}(A,X) \simeq \mathcal{Q}(L_p/A, X)$. The isomorphism comes defined as $\nu_A^{-1}(T) = TF_A$. It is not difficult to check that ν_A^{-1} defines a natural equivalence. Its inverse $\nu_A : \mathcal{Q}(L_p/A,\cdot) \to \mathcal{L}(A,\cdot)$ is slightly awkward to describe, but in the end everything is reduced to find, given a quasi-linear map $W : L_p/A \curvearrowright Z$, an operator $\nu^A(W) : A \to Z$ such that $W \equiv \nu^A(W) F_A$.
Let $\eta : \mathcal{Q}(L_p/A,\cdot) \to \mathcal{Q}(L_p/B,\cdot)$ be a natural transformation. The operator we are looking for to represent η is going to be

$$\nu_B\left(\eta_A(F_A)\right) : B \to A$$

where $\eta_A(F_A) \equiv \nu_B\left(\eta_A(F_A)\right) F_B$. The form in which η acts represented by that operator is: from the commutative diagram

$$\begin{array}{ccc} \mathcal{Q}(L_p/A, A) & \xrightarrow{\eta_A} & \mathcal{Q}(L_p/B, A) \\ \Big\downarrow{\scriptstyle \nu(W)^*} & & \Big\downarrow{\scriptstyle \nu(W)^*} \\ \mathcal{Q}(L_p/A, Z) & \xrightarrow{\eta_Z} & \mathcal{Q}(L_p/B, Z), \end{array}$$

we see that for$W : L_p/A \curvearrowright Z$ one has

$$\eta_Z(W) = \eta_Z(\nu(W)F_A) = \nu(W)\eta_A(F_A) = \nu(W)\nu(\eta_A(F_A))\,F_B.$$

□

And, as before:

PROPOSITION 4.2. *Let A and B q-Banach subspaces of L_p, for $0 < p < q \leq 1$. Two functors $\mathcal{Q}(L_p/A, \cdot)$ and $\mathcal{Q}(L_p/B, \cdot)$ acting between the categories $\mathbf{Q_q} \to \mathbf{Vect}$ are naturally equivalent if and only if A and B are isomorphic.*

PROOF. Let η be a natural equivalence with inverse η^{-1} (having the same form as η). Given a quasi-linear map W one gets $W \equiv \eta^{-1}(\eta(W))$, and thus:

$$\nu(W)F_A \equiv \eta^{-1}(\nu(W)\nu(\eta(F_A))\,F_B) \equiv \nu(W)\nu(\eta(F_A))\,\nu\left(\eta^{-1}(F_B)\right)F_A.$$

Next, if T is an operator with range in a q-Banach space and $TF_A \equiv 0$ then $T = 0$ since $\mathcal{L}(L_p, \,\cdot\,) = 0$. Thus

$$\nu(W) = \nu(W)\nu(\eta(F_A))\,\nu\left(\eta^{-1}(F_B)\right).$$

Since for each space Z the map $\nu_{A,Z} : \mathcal{Q}(L_p/A, Z) \to \mathcal{L}(A, Z)$ is an isomorphism, choosing W in such a way that $\nu(W)$ is injective one gets

$$\nu(\eta(F_A))\,\nu\left(\eta^{-1}(F_B)\right) = 1.$$

Reasoning analogously with $\eta\eta^{-1}$ one would get $\nu\left(\eta^{-1}(F_B)\right)\nu(\eta(F_A)) = 1$, which shows that que $\nu(\eta(F_A))$ actually was an isomorphism. □

Finally, we can give a proof of the character partially co-automorphic of L_p

THEOREM 4.4. *Given q-Banach subspaces A and B of L_p for $0 < p < q \leq 1$, the extensions $0 \to A \to L_p \to L_p/A \to 0$ and $0 \to B \to L_p \to L_p/B \to 0$ are isomorphically equivalent if and only if A and B are isomorphic; and if and only if the functors $\mathcal{Q}(L_p/A, \cdot)$ and $\mathcal{Q}(L_p/B, \cdot)$ acting between the categories $\mathbf{Q_p} \to \mathbf{Vect}$ are naturally equivalent.*

PROOF. The proof combines two argumentations: since the functors $\mathbf{Q}_p \to \mathbf{Vect}$ are naturally equivalent, we get $(L_p/A) \oplus l_p \simeq (L_p/B) \oplus l_p$; from that, L_p/A and L_p/B are also isomorphic. Let us call $\psi : L_p/B \to L_p/A$ the isomorphism that induces the natural transformation $\eta_Z(W) = W\psi$. On the other hand, since the functors $\mathbf{Q}_q \to \mathbf{Vect}$ for $p < q$ also are naturally equivalent, there will be another isomorphism $\phi : B \to A$ that induces the same natural transformation as $\eta_Z(W) = \nu_Z(W)\phi F_B$. So,

$$\eta_A(F_A) = F_A\psi = \phi F_B.$$

. □

The following argument of Kalton, produced in a private conversation, shows that the results we have shown are optimal: the closed subspace of L_p generated by the Rademacher functions R is l_2, as well as the *span* G of the Gaussian variables. An operator $L_p \to L_p$ must send order-bounded sequences into order-bounded sequences. However while the Rademacher functions are order bounded the Gaussian are not. Thus, there is no isomorphisms in L_p extending an isomorphism $R \to G$.

The attempt to clarify the partially automorphic character of L_1 presents some difficulties. For instance, if q and Q are two quotient maps of L_1 onto Z such that the kernels are ultrasumands, it is clear that the diagonal principle applies to give isomorphicallly equivalent extensions:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L_1 \oplus \ker q & \longrightarrow & L_1 \oplus L_1 & \xrightarrow{q} & Z & \longrightarrow & 0 \\
 & & \wr & & \wr & & \| & & \\
0 & \longrightarrow & L_1 \oplus \ker Q & \longrightarrow & L_1 \oplus L_1 & \xrightarrow{Q} & Z & \longrightarrow & 0
\end{array}$$

What is now difficult is to get that also the sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker q & \longrightarrow & L_1 & \xrightarrow{q} & Z & \longrightarrow & 0\\ & & \wr & & \wr & & \| & & \\ 0 & \longrightarrow & L_1\oplus\ker Q & \longrightarrow & L_1\oplus L_1 & \xrightarrow{Q} & Z & \longrightarrow & 0\end{array}$$

be isomorphically equivalent. In principle, no reasonable hypothesis on Z, such as not containing L_1 (which is not a 3-space property, see [**16**]), seems to work.

There is however a result of Kislyakov [**51**] that can be understood as a show of the partially co-automorphic character of L_1. One just has to replace "ultrasumand" by "reflexive".

THEOREM 4.5. *Let A and B be two reflexive subspaces of $L_1(\mu)$ such that $L_1(\mu)/A = L_1(\mu)/B$. There exists finite dimensional subspaces E and E' such that the extensions*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A\oplus E & \longrightarrow & L_1(\mu)\oplus E & \longrightarrow & L_1(\mu)/A & \longrightarrow & 0\\ & & \wr & & \wr & & \| & & \\ 0 & \longrightarrow & B\oplus E' & \longrightarrow & L_1(\mu)_1\oplus E' & \longrightarrow & L_1(\mu)/B & \longrightarrow & 0\end{array}$$

are isomorphically equivalent.

PROOF. By Lindenstrauss's lifting principle [**53**] one gets a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & L_1(\mu) & \xrightarrow{q} & L_1(\mu)/A & \longrightarrow & 0\equiv F_q\\ & & \uparrow\downarrow & & \uparrow\downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & L_1(\mu) & \longrightarrow & L_1(\mu)/B & \longrightarrow & 0\equiv F_p.\end{array}$$

Applying the diagonal principle we obtain that the extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A\oplus L_1(\mu) & \longrightarrow & L_1(\mu)\oplus L_1(\mu) & \longrightarrow & L_1(\mu)/A & \longrightarrow & 0\equiv 0_{L_1(\mu)}\oplus F_q\\ & & \wr & & \wr & & \| & & \\ 0 & \longrightarrow & B\oplus L_1(\mu) & \longrightarrow & L_1(\mu)\oplus L_1(\mu) & \longrightarrow & L_1(\mu)/B & \longrightarrow & 0\equiv 0_{L_1(\mu)}\oplus F_p\end{array}\tag{8}$$

are isomorphically equivalent. A dualization of the diagram 8 and reasoning as in the third case of Theorem 3.2 we obtain finite dimensional spaces E_1^* and E_2^* such that $F_q^*\oplus E_1^*$ and $F_p^*\oplus E_2^*$ (with some abuse of notation) are isomorphically equivalent, from where it follows that $F_p\oplus E_1$ and $F_q\oplus E_2$ are also isomorphically equivalent. □

Now, it is clear that when $\mathrm{ind}_{q^*}(p*) = 0$ then the isomorphisms can be "straightened" to get that the original extensions are isomorphically equivalent (the w*-continuous character does not change by a finite dimensional perturbation).

4.2. Co-automorphic spaces using isomorphy classes. Given X and Z objects of $\mathbf{Q}$ we say that an *inverted sheaf* $\rhd$ is a collection of quotient maps $X\to Z$. Given a class $\mathcal{C}$ of operators, we shall sometimes denote by $\rhd_{\mathcal{C}}$ the sheaf (if there is no confusion we omit "inverted") of quotient maps $X\to Z$ of $\mathcal{C}$. If it is necessary to refer to all the variables, we shall write

$$X\rhd_{\mathcal{C}} Z.$$

We say that two quotient maps $q: X\to Z$ and $q': X'\to Z'$ belong to the same **isomorphy class** if the induced objects F_q and $F_{q'}$ are strictly isomoprhic.

A quasi-Banach space X shall be called Z-co-automorphic if all quotient maps of the sheaf $X\rhd_{\mathcal{L}} Z$ have the same isomorphy class.

We shall say that a class of operators $\mathcal{C}$ is a **co-automorphy class** for X if for every quasi-Banach Z the quotient operators of the sheaf $X\rhd_{\mathcal{C}} Z$ belong to the same isomorphy class. We shall also say that X is $\mathcal{C}$-co-automorphic.

A space X may have several co-automorphy classes. Of course, X shall be co-automorphic if it only has as co-automorphy class the set of all quotinet maps defined on X.

We shall interpret the number of co-automorphy classes of X as a measure of the co-automorphic character of the space. The co-automorphy classes of X form a partition of the set of all quotient maps on X. So, we understand that a space is *more* co-automorphic as smaller is the number of co-automorphy classes it has. A question emerges automatically:

Question: Besides the Hilbert case, does there exist a non-co-automorphic space with a finite or countable number of automorphy classes?

Unlike the automorphic case, and besides Hilbert spaces, we do not know co-automorphic spaces with a countable quantity of co-automorphy classes. Certainly, L_1 has at least a countable quantity; as many as $\mathcal{F}_k$ in l_∞. In this case, for each $k \in \mathbb{Z}$ we have the class formed by the quotients q with reflexive kernel such that $q^* \in \mathcal{F}_k$. Unfortunately what we do not know is the analogue of the class $\mathcal{F}$ of l_∞.

Problems:

(1) How many co-automorphy classes do there exist in $l_1 \oplus l_2$.
(2) What does it happen if we replace "product" by vector amalgam $l_2(l_1)$?

4.2.1. **Co-automorphuic spaces using the spectrum.** The notion of spectrum offers a new perspective of the concept of co-automorphic space:

LEMMA 4.3. *A quasi-Banach space X is co-automorphic if given two elements $[F : Z \curvearrowright Y]$, $[F' : Z' \curvearrowright Y']$ in the spectrum of X with Z and Z' isomorphic then F and F' are strictly isomorphic.*

What would be the corresponding notion of *quasi-co-automorphic* space coincides with that quasi-automorphic space we already defined in Section 3.2.1.

5. Applications of the diagonal principles

5.1. Categorical applications.

5.1.1. *Isomorphisms of the category.* The first application of the diagonal principles is to obtain a characterization of the isomorphisms of the categories $\mathfrak{Z}$ y $\mathfrak{Q}$:

PROPOSITION 5.1. *$F \longleftrightarrow F'$ if and only if there exist G and G' such that $F \overset{e}{\sim} G \sim G' \overset{e}{\sim} F'$. That is, two objects are isomorphic if and only if they are elementarily isomorphic to two isomorphically equivalent objects.*

PROOF. Let $(\alpha, \gamma) : F \rightrightarrows F'$ be the isomorphism between F and F'. One has:

$$F \overset{e}{\sim} 0_{\alpha F} \oplus F \oplus 0_{F'} \sim 0_F \oplus \alpha F \oplus 0_{F'} \equiv 0_F \oplus F'\gamma \oplus 0_{F'} \sim 0_F \oplus F' \oplus 0_{F'\gamma} \overset{e}{\sim} F'$$

□

5.1.2. *On the naturalness of the definition of the functor* Ext. Since the early stages of the development of homological algebra the algebraist know about the possibility of defining the functors Ext via projective objects (in categories with *enough* projective objects) and via injective objects (again, in categories with *enough* injective objects). They also know that the choice of the projective presentation has no effect since the resulting functors Ext are naturally equivalent (see [**32**] or else [**61**]). And the same occurs for injective presentations.

The diagonal principles establish that the reason for that naturalness lies deeper: If

$$0 \longrightarrow A \longrightarrow P \longrightarrow X \longrightarrow 0$$

and

$$0 \longrightarrow B \longrightarrow P' \longrightarrow X \longrightarrow 0$$

are two projective presentations of X the second diagonal principle establishes that the sequences in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A\oplus P' & \longrightarrow & P\oplus P' & \longrightarrow & X & \longrightarrow & 0 \\ & & {\scriptstyle\phi}\Big\uparrow & & {\scriptstyle\psi}\Big\uparrow & & \| & & \\ 0 & \longrightarrow & B\oplus P & \longrightarrow & P'\oplus P & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

are isomorphically equivalent. That implies the existence of a natural transformation

$$\mathcal{L}(A,\cdot) \longrightarrow \mathcal{L}(B,\cdot)$$

which will carry an operator $T : A \to \heartsuit$ to $\phi^*(T \oplus 0)|_B$. This correspondence respects the equivalence relation "$T \approx 0$ *if and only if* T *extends to* P" (resp. P') that defines the functors Ext; it is clear that if T extends to an operator $\widehat{T} : P \to \heartsuit$ then $\phi^*(T\oplus 0)|_B$ extends to $\psi^*(\widehat{T}\oplus 0)|_{P'}$. We have in fact a natural equivalence $\mathcal{L}(A,\cdot) \longrightarrow \mathcal{L}(B,\cdot)$; therefore, the corresponding functors Ext are also naturally equivalent.

Still clearer is the case of separable spaces: it is clear that isomorphically equivalent sequences define "isomorphically equivalent" homology sequences. In particular, if ϕ is the isomorphisms between the kernels el of the projective presentations then $\phi^* : \mathcal{L}(A,\cdot) \longrightarrow \mathcal{L}(B,\cdot)$ is the isomrphism between the spaces of operators that respects the equivalence relation and, therefore, induces an isomomorphism between the spaces Ext. For a deeper discussion of the topic the reader can go to [**61**].

And the same applies to injective presentations.

5.2. Applications to the study of the Dunford-Pettis property. The diagonal principles fit quite well for the study of Banach space properties stable by complemented subspaces. Among them, one of the more interesting is the Dunford-Pettis property. Most of what one needs to know about this property can be found in [**23**] or else in [**16**, Chapter 6], and can be summed up in:

(1) A Banach space X has the Dunford-Pettis property (in short, DPP) if every weakly compact operator $X \to Y$ is completely continuous.
(2) The DPP is inherited by complemented subspaces. And also by locally complemented subspaces.
(3) If X^* has DPP then X has DPP, but not conversely. There exist essentially two examples of this situation: one is Stegall's example $l_1(l_2^n)$, which is a Schur space whose dual $l_\infty(l_2^n)$ contains complemented copies of l_2 as can be seen in [**23, 17**]); the other is the dual of $c_0\widehat{\otimes}_\pi c_0$, since it was proved in [**29**] that $(c_0\widehat{\otimes}_\pi c_0)^{**}$ fails to have the DPP.

The homological techniques seem to be quite well adapted to the study of some of the several open problems about the Dunford-Pettis property.

Problem 1. Does the quotient l_∞/l_1 have the DPP?

We speak of "the" quotient since the isomorphism $l_1 \to l_\infty$ has nonreflexive cokernel and therefore l_∞ is l_1-automorphic. Thus, if $K(c_0)$ is the kernel of a quotient map $l_1 \to c_0$ then $l_\infty/l_1 \simeq K(c_0)^*$, and thus the previous question coincides with a question going back to [**17**]:

Problem 2. Does $K(c_0)^*$ have DPP?

Even, we observe that we know that $K(c_0)$ has DPP just because it is a subspace of l_1. Kalton and Pelczynski drop that lucky factor and ask in [**50**]:

Problem 3. If $0 \to K_0 \to L_1 \to c_0 \to 0$ is an exact sequence, does K_0 have DPP?

Although it is easy to put examples in which the answer is affirmative, it is not to put nontrivial examples (here "nontrivial" means "not elementarily isomorphic to the projective presentation of c_0"). Kalton and Pelczynksi give in [**50**] a quotient map $L_1(G) \to c_0(S)$, the Fourier transform which, when G is locally compact and abelian and S a Sidon set of the dual group of G, has kernel with the DPP.

If we try to study the DPP of the quotient space l_∞/l_1 we observe that the classical techniques do not work; indeed, there is a result of Diestel [**23**] establishing that if an extension $0 \to Y \to X \to Z \to 0$ has the space X with DPP while Y does not contain l_1 then X/Y has DPP. One of the consequences is Kislyakov's result [**51**] asserting that quotients of an $\mathcal{L}_\infty$-space by a reflexive subspace have DPP, as well as all its iterated duals. Therefore, the same shall happen with the kernels of quotients of an $\mathcal{L}_1$ over a reflexive space. But not, in principle, over c_0.

The kernel $K(X)$ of a quotient operator $L_1 \to X$ keeps some of the properties of X. In [**50**, Lema 2.1 y Prop. 2.3] Kalton and Pelczynski prove that if $0 \to K(X) \to l_1(\Gamma) \to X \to 0$ is a projective presentation of an ultrasummandwith the Radon-Nikodym property then $K(X)$ must also be an ultrasummand. Let us see something more:

Lemma 5.1. *Let Z be an ultrasummand with the Radon-Nikodym property (RNP). The kernel of any quotient operator $L_1(\mu) \to Z$ is an ultrasummand.*

Proof. Observe the diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\delta_K"'] & L_1(\mu) \arrow[r, "q"] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \equiv F \\
0 \arrow[r] & K^{**} \arrow[r] \arrow[d, equal] & PO \arrow[r] & Z \arrow[r] \arrow[d, equal] & 0 \equiv \delta_K F \\
0 \arrow[r] & K^{**} \arrow[r] \arrow[d, equal] & PB \arrow[r, "v"'] \arrow[d, "w"] & Z \arrow[r] \arrow[d, "\delta_Z"] & 0 \equiv F^{**}\delta_Z \\
0 \arrow[r] & K^{**} \arrow[r] & L_1(\mu)^{**} \arrow[r, "q^{**}"'] & Z^{**} \arrow[r] & 0 \equiv F^{**}.
\end{tikzcd}$$

The extensions $\delta_K F$ and $F^{**}\delta_Z$ are equivalent. Since Z is an ultrasumand, there is a projection $p : Z^{**} \to Z$. Since $q^{**}w = \delta_Z v$ one will have $pq^{**}w = p\delta_Z v = v$; and since Z has RNP and $L_1(\mu)^{**}$ is an abstract L-space, v factorizes through some $l_1(\Gamma)$-space. So, $Fv \equiv 0$. It is moreover clear that $F^{**}\delta_Z q \equiv 0$. The diagonal principle directly yields that $K \oplus PB$ is isomorphic to $K^{**} \oplus L_1(\mu)$, from where it follows that K is complemented in a dual, and thus it must be an ultrasummand. □

We hasten to remark that the space L_1 in the Lemma cannot be replaced by an arbitrary $\mathcal{L}_1$-space. The property of being an ultrasumand is crucial: take, for instance, a sequence $0 \to D_1 \to l_1 \to L_1 \to 0$; the space D_1 is an $\mathcal{L}_1$-space that is not an ultrasummand. Therefore, the kernel K in an extension $0 \to K \to D_1 \to l_2 \to 0$ cannot be neither an ultrasummand: otherwise, see [**9**], D_1 shall be an ultrasummand.

Our next result extends Kislyakov's theorem previously mentioned and completes those of Kalton and Pelczynski [**50**]:

Proposition 5.2. *Let $0 \to K \to \mathcal{L}_1 \to X \to 0$ be an exact sequence in which X is an ultrasummand with RNP. Then K has the DPP.*

PROOF. Observe the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(X) & \longrightarrow & l_1(\Gamma) & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & \mathcal{L}_1 & \longrightarrow & X & \longrightarrow & 0. \end{array}$$

We have just seen that $K(X)$ is an ultrasumand. From that,using Lindenstrauss lifting principle [**53**], one has $\mathrm{Ext}(\mathcal{L}_1, K(X)) = 0$. Applying the diagonal principle one gets that $K(X) \oplus \mathcal{L}_1$ is isomorphic to $K \oplus l_1(\Gamma)$. Since $K(X)$ has the Schur property, $K(X) \oplus \mathcal{L}_1$ has DPP, hence $K \oplus l_1(\Gamma)$ also has it, as well as K. □

However, the Diestel-Kislyakov theorem is slightly better regarding the fact that when X is reflexive then it yields the DPP for$K(X)$ and for all its iterated duals. This ceases to be true when X is just a separable dual (optimal conditions to be al ultrasummand with RNP). The example $0 \to l_1(l_2^n) \to l_1 \to X \to 0$ shows it: since l_2 is a quotient of l_∞ there will be a "uniform" sequence of extensions $0 \to D_n \to l_\infty^n \to l_2^n \to 0$. We form its c_0-product

$$0 \to c_0(D_n) \to c_0 \to c_0(l_2^n) \to 0.$$

The dual sequence is the one we are looking for. The space $X = c_0(D_n)^*$ ia separable dual while $l_1(l_2^n)^* = l_\infty(l_2^n)$ has not DPP.

We pose now the question: Which quotients of $\mathcal{L}_\infty$-spaces have DPP? The Diestel-Kislyakov theorem ensures that quotients of $\mathcal{L}_\infty$-spaces by reflexive subspaces have DPP, and all their iterated duals too. If the subspace does not contain l_1, still the result yields the DPP for the quotient, but the information about its dual is lost. An Asplund space is one whose separable subspaces have separable duals; thus, an Asplund space cannot contain l_1. Let us prove that the dual of a quotient $\mathcal{L}_\infty/A$ of an $\mathcal{L}_\infty$-space by an Asplund subspace has DPP.

PROPOSITION 5.3. *Let A be an Asplund space. The dual of every quotient $\mathcal{L}_\infty/A$ has DPP.*

PROOF. The dual of an Asplund space has RNP. We obtain the result applying thus Proposition 5.2 to the dual sequence

$$0 \longrightarrow A^\perp \longrightarrow \mathcal{L}_1 \longrightarrow A^* \longrightarrow 0$$

□

Unfortunately, the bidual of $\mathcal{L}_\infty/A$ needs not to have the DPP as the example $0 \to c_0(K_n) \to c_0 \to c_0(l_2^n) \to 0$ shows.

The question of whether the bidual $K(c_0)^{**}$ of the kernel of a projective presentation of c_0 has DPP means to ask if $(l_\infty/l_1)^*$ has DPP. Let us see that, for what this question matters, the space l_∞ plays no special role.

PROPOSITION 5.4. *Let E be a subspace of two spaces $\mathcal{L}_\infty$ and $\mathcal{L}'_\infty$ of $\mathcal{L}_\infty$-type.*

1. *If $\mathcal{L}_\infty$ and $\mathcal{L}'_\infty$ are injective, the quotients $\mathcal{L}_\infty/E$ and $\mathcal{L}'_\infty/E$ have DPP simultaneously.*
2. *If $\mathcal{L}'_\infty$ is a subspace of $\mathcal{L}_\infty$ and $\mathcal{L}_\infty/E$ have DPP then $\mathcal{L}'_\infty/E$ has DPP.*
3. *The dual spaces $(\mathcal{L}_\infty/E)^*$ and $(\mathcal{L}'_\infty/E)^*$ have DPP simultaneosuly.*

PROOF. The first affirmation directly follows from the diagonal principle. To check the others, let us consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & E \arrow[r, equal] \arrow[d] & E \arrow[d] & & \\
0 \arrow[r] & \mathcal{L}'_\infty \arrow[r] \arrow[d] & \mathcal{L}_\infty \arrow[r] \arrow[d] & \mathcal{L}_\infty/\mathcal{L}'_\infty \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{L}'_\infty/E \arrow[r] \arrow[d] & \mathcal{L}_\infty/E \arrow[r] \arrow[d] & \mathcal{L}_\infty/\mathcal{L}'_\infty \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

Since the middle row locally splits, the lower row locally splits. So, $(\mathcal{L}_\infty/E)^* = (\mathcal{L}'_\infty/E)^* \oplus (\mathcal{L}_\infty/\mathcal{L}'_\infty)^*$ and recall that $\mathcal{L}_\infty/\mathcal{L}'_\infty$ is an $\mathcal{L}_\infty$-space as a quotient of two $\mathcal{L}_\infty$-spaces. □

We are pointing at the following problem. In general, a quotient $\mathcal{L}_\infty/E$ is not uniquely defined: if we think for a moment about $C[0,1]$, there exist, in principle, many quotients $C[0,1]/l_2$. We shall pose in the next Chapter, section 5.1, the question if they must be isomorphic.

Question: If X and Y are two isomorphic subspaces of $C[0,1]$, Does $C[0,1]/X$ have DPP if and only if $C[0,1]/Y$ has DPP?.

The reason to focus our attention on $C[0,1]$ is that in order to have a simultaneity result (in the sense of Proposition 5.4) it seems necessary to have a "big" $\mathcal{L}_\infty$ space. One should not forget that the Bourgain-Pisier [**7**] construction produces for each separable space E a separable $\mathcal{L}_\infty(E)$-space such that the quotient $\mathcal{L}_\infty(E)/E$ has the Schur property. At the same time, since there exist extensions $0 \to D_2 \to C[0,1] \to l_2 \to 0$, it is clear that the implication (2) in Proposition 5.4 does not admit a converse.

We finally observe that the statement (1) of Proposition 5.4 implies that "There is a quotient l_∞/X with DPP" and "All quotients l_∞/X have DPP" are equivalent statements.

Proposition 5.5. *Let X be a separable Banach space. They are equivalent:*

(1) *Given a projective presentation $0 \to K(X) \to l_1 \to X \to 0$ of X, the space $K(X)^*$ has DPP.*
(2) *Given an extension $0 \to W \to \mathcal{L}_1 \to X \to 0$ the space W^* has DPP.*
(3) *For every exact sequence $0 \to X \to C[0,1] \to C[0,1]/X \to 0$ the space $C[0,1]/X$ has DPP.*

Proof. We already know that (1) and (2) are equivalent and that (1) implies (3) (all thanks to Proposition 5.4). To show that (3) implies (1) it is enough to show that every separable subspace l_∞/X^* is contained in a subspace with DPP. Let thus $S \to l_\infty/X^*$ be a separable subspace, and let us form the pull-back diagram

$$\begin{tikzcd}
0 \arrow[r] & X^* \arrow[r] & l_\infty \arrow[r] & l_\infty/X^* \arrow[r] & 0 \\
0 \arrow[r] & X^* \arrow[r] \arrow[u, equal] & PB \arrow[r] \arrow[u] & S \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

We have that $PB \subset l_\infty$ is a separable space such that $S = PB/X^*$. If A denotes the smallest commutative C^*-algebra that PB generates in l_∞, it turns out that $A = C(K)$ for some metrizable compact K. Milutin's theorem yields that $C(K)$ is isomorphic to a complemented subspace of $C[0,1]$. From that, A/X is, in turn, isomorphic to a complemented subspace of $C[0,1]/X$, and it will necessarily have DPP. □

CHAPTER 4

Extensión of C(K)-valued operators

1. Introduction

The classical extension problem for operators is the following: given an operator $T : Y \to E$ and an embedding $j : Y \to X$ we ask if T can be extended to X through j; that is, if there exists an operator $\widehat{T} : X \to E$ such that $\widehat{T}j = T$. The problem also admits a formulation in terms of morphisms of $\mathfrak{Q}$: let us recall that extending an operator $T : Y \to E$ is equivalent to the existence of a push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j} & X & \longrightarrow & Z & \longrightarrow & 0 \equiv F \\ & & {\scriptstyle T}\big\downarrow & & \big\downarrow & & \| & & \\ 0 & \longrightarrow & E & \longrightarrow & PO & \longrightarrow & Z & \longrightarrow & 0 \equiv \mathfrak{T} \end{array}$$

in which $\mathfrak{T}$ is trivial. Thus, in te context of $\mathfrak{Q}$ the extension problem is to find out if an operator $T : Y \to E$ induces a null morphism $F \longrightarrow \mathfrak{T}$.

As a rule, operators between quasi-Banach spaces do not extend. However, there are interesting situations in which they do. For instance:

- If Y is complemented in X, every operator on Y extend to X.
- If E is injective then every operator $T : Y \to E$ extends through any embeddding $Y \to X$.
- If $T : Y \to E$ is 2-summing then T extends through any embedding $j : Y \to X$.

As these examples show, there are three variables in an extension problem: the class $\mathfrak{I}$ of emeddings; the class $\mathfrak{E}$ of range spaces; the class $\mathfrak{U}$ of operators we want to extend.

In particular, for the choice $\mathfrak{I} = \{j\}$, $\mathfrak{E} = \{E\}$ and $\mathfrak{U} = \mathcal{L}$ (all operators) the problem admits a formulation in terms of the exactness of functor $\mathcal{L}(\,\cdot\,, E)$:

$$\text{Is the functor } \mathcal{L}(\,\cdot\,, E) \text{ exact on } 0 \to Y \xrightarrow{j} X \to Z \to 0?$$

Our objective in this chapter is to study the extension problem for operators with range $\mathrm{C}(K)$; that is, we set $\mathfrak{E}$ as the class of $\mathrm{C}(K)$-spaces, with K a Hausdorff compact. For the variable $\mathfrak{I}$ we shall study two instances: embedding $H \to c_0$ and embeddings $q^* : W \to l_1$, with q^* the adjoint of a quotient $q : c_0 \to W_*$; that is, W is a $w(l_1, c_0)$-closed subspace of l_1.

Lindenstrauss and Pelczynski showed in [**55**] that every operator defined on a subspace of c_0 with range $C(K)$ extends to c_0. The corresponding result for $w(l_1, c_0)$-closed subspaces of l_1 was proved by Johnson and Zippin in [**38**]. The proofs of both results are complex and quite different, although intuition suggests that the second result is *dual* of the first one. We shall present an homological proof that unifies *by duality* both results.

The idea behind our proof is the following: each result shall be established under the additional hypothesis that the exact sequences involved admit a certain finite dimensional decomposition; then we shall use results reducing the general case to the previous one.

Then, we shall study how much the range of the variables $\mathfrak{I}$, $\mathfrak{E}$ and $\mathfrak{U}$ can be enlarged in such a way that the Lindenstrauss-Pelczynski and Johnson-Zippin theorems remain valid. For instance, the fact that injective spaces must be of type $\mathcal{L}_\infty$ brings this class to our attention;

since Johnson-Zippin had already shown their theorem for $\mathcal{L}_\infty$-valued operators, we ask about if the Lindenstrauss-Peckzynski theorem remain valid in this case.

1.1. Zippin's lemma.

DEFINITION 4.1 (**Almost trivial extension**). *We shall say that an extension* $0 \to Y \to X \to Z \to 0$ *λ-almost splits if there is a constant λ such that every operator $T : Y \to C(K)$ admits an extension $\widehat{T}$ to X such that $\|\widehat{T}\| \leq \lambda\|T\|$. We shall also say that Y is λ-almost complemented in X. A quasi-lineal map $F : Z \curvearrowright Y$ sshall be called λ-almost trivial if* $\mathrm{dist}(TF, \mathbf{L}) \leq \lambda\|T\|$. *An extension $0 \to Y \to X \to Z \to 0 \equiv F$ shall be termed almost-trivial if it is λ-almost trivial or if λ-almost splits for some $\lambda > 0$.*

The correspondence between both notions for $0 \to Y \to Y \oplus_F \to Z \to Z \equiv F$ is that if the extension λ-almost splits then the map is λ-almost trivial; and if the map is λ-almost trivial then the canonical extension is $\lambda + 1$-almost trivial.

Our approach to the Lindenstrauss-Pelczynski and Johnson-Zippin theorems passes through the characterization of almost-trivial extensions that Zippin formulates and proves in [**79**]. This result is more or less implicit in Lindenstrauss [**54**, Chapter VII]; it can be traced back to Pelczynski [**66**, 2.8] and even as far as Dunford-Schwartz [**25**, VI.7.1].

LEMMA 1.1. *An extension $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0$ is λ-almost trivial if and only if there is a w^*-continuous map $\omega : B_{Y^*} \to \lambda B_{X^*}$ such that $j^*\omega = id$; this map shall be called a w^*-selector for j^*.*

PROOF. Let $\omega : B_{Y^*} \to \lambda B_{X^*}$ be a w^*-selector for j^*. Then, $\tau(x)(k) = \omega(T^*k)(x)$ defines an operator $\tau : X \to C(K)$ with $\|\tau\| \leq \lambda\|T\|$ and such that $\tau(jy)(k) = \omega(T^*k)(jy) = j^*\omega(T^*k)(y) = T^*k(y) = Ty(k)$. Conversely, assume that every operator $T : Y \to C(K)$ admits an extension to X. If we consider the canonical map $\delta : Y \to C(B_{Y^*})$, $\delta(y)(y^*) = y^*(y)$, and take an extension D to X with $\|D\| \leq \lambda$, the map $\omega(y^*)(x) = D(x)(y^*)$ shall be a w^*-selector for j^* since $\omega(y^*)(jy) = D(jy)(y^*) = y^*(y)$. □

Zippin uses this criterion to obtain in [**80, 81**] different proofs of the Lindenstrauss- Pełczyński theorem. The criterion is inspired by the most classical possible situation: the extension

$$0 \longrightarrow E \xrightarrow{\ \delta\ } C(B_{E^*}) \longrightarrow Q \longrightarrow 0 \equiv \mathsf{C}_E,$$

is 1-almost trivial. Indeed, the map $\omega : B_{E^*} \to B_{C(B_{E^*})}$ defined as $\omega(x^*)(f) = f(x^*)$ is a w^*-selector for δ^*. Moreover,

COROLLARY 1.1. *An extension $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0 \equiv F$ is almost trivial if and only if there is a morphism $\mathsf{C}_Y \longleftarrow F$.*

PROOF. It is clear that if F is pull-back of C_Y then F is almost trivial. Conversely, if F is almost trivial the canonical embedding $\delta : Y \to C(B_{Y^*})$ extends to X through j and F is a pull-back of C_Y. □

It is worth mentioning two results about the relationships between almost complementation and morphisms of $\mathfrak{Q}$. The first one we have just mentioned.

LEMMA 1.2. *Assume that $F \longleftarrow G$, and let F be almost trivial then G is almost trivial*

The second one is essentially [**38**, Prop.1.1]; it shall be proved in part *ii)* of our next lemma 2.1.

LEMMA 1.3. *Assume that $F \longrightarrow G$ through a morphsim induced by a surjective operator; if F is almost trivial then G is almost trivial.*

2. The Lindenstrauss-Pelczynski theorem

This section turns around the result that Lindenstrauss and Pelczynski prove in [**55**]:

THEOREM 2.1. *Let H be a subspace of c_0 and let $T : H \to C(K)$ be an operator. For each $\varepsilon > 0$ the operator T admits an extension $\widehat{T}$ to c_0 such that $\|\widehat{T}\| \leq (1+\varepsilon)\|T\|$.*

The original proof is an adaptation of the Hahn-Banach's theorem to the case of operators $T : H \to C(K)$ with H a subspace of c_0. It consists in extending T to one more dimension: given $\varepsilon > 0$ and an element $p \in c_0$ not in H, there is an extension $T_p : H + [p] \to C(K)$ con norma $\|T_p\| \leq \|T\| + \varepsilon$. Now, using the separability of c_0 one concludes the existence of an extension $\widehat{T} : c_0 \to C(K)$ with $\|\widehat{T}\| \leq (1+\varepsilon)\|T\|$. An example of Johnson and Zippin in [**39**] shows that one cannot reach an equal norm extension ($\varepsilon = 0$). In that same paper Johnson and Zippin extend the Lindenstrauss- Pelczynski theorem to subspaces of $c_0(\Gamma)$ using an argument that avoids the one dimensional jumps.

2.1. An homological approach to the Lindenstrauss-Pelczynski theorem. We need some results to perform our homological proof of the theorem:

LEMMA 2.1. *Let us have a push-out diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A \arrow[d, "a"] \arrow[r, phantom, "="] & A \arrow[d, "i"] & & \\
0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "b"] & X \arrow[r] \arrow[d, "c"] & Z \arrow[r] \arrow[d, equal] & 0 \equiv F \\
0 \arrow[r] & B \arrow[r, "d"'] \arrow[d] & C \arrow[r] \arrow[d] & Z \arrow[r] & 0 \equiv G \\
 & 0 \arrow[d, phantom, "|||"] & 0 \arrow[d, phantom, "|||"] & & \\
 & H & V, & &
\end{tikzcd}$$

i) *If the functor $\mathcal{L}(\,\cdot\,, \heartsuit)$ is exact on the extensions V and G then $\mathcal{L}(\cdot, \heartsuit)$ is exact on F. In particular, if the extensions V and G are almost trivial then the same occurs to F.*

ii) *If the functor $\mathcal{L}(\,\cdot\,, \heartsuit)$ is exact on F then it is exact on G. In particular, if F is almost trivial then the same occurs to G.*

PROOF. **i)** Let $T : Y \to \heartsuit$ be an operator. By hypothesis $TaV \equiv 0$, thus there exists $R : X \to \heartsuit$ an extension of Ta through i. Since $(Rj - T)a = 0$ there is an operator $S : B \to \heartsuit$ such that $T - Rj = Sb$. We obtain that T extends to X since, by hypothesis, $0 \equiv SG \equiv SbF = (T - Rj)F = TF$. **ii)** Given an operator $T : B \to \heartsuit$ the composition Tb extends to $\hat{T} : X \to \heartsuit$ through j. Since $\hat{T}i = \hat{T}ja = Tbj = 0$ there is an operator $\tau : C \to \heartsuit$ such that $\tau c = \hat{T}$. Thus $\tau db = \tau cj = \hat{T}j = Tb$, hence $\tau d = T$. □

It shall be useful to observe that some properties of the w*-topology in l_p, $1 \leq p < \infty$, pass to l_p-sums. Given an l_p-sum $\ell_p(X_n)$ we denote $\pi_j : \ell_p(X_n) \to X_j$ the natural projections.

LEMMA 2.2. *Let $(E_n^*)_n$ be a sequence of duals and let $(x_k)_k$ be a bounded sequence in $l_p(E_n^*)$, $1 \leq p \leq \infty$. The sequence $(x_k)_k$ is w^*-null if and only if the sequences $(\pi_j(x_k))_k$ are w^*-null.*

Proof. We make firs the case $p > 1$. It is clear that if (x_k) is w^*-null, the sequences $(\pi_j(x_k))_k$ are also w^*-null. Conversely, let x be an element of $l_{p^*}(E_n^*)$ that we write as $x = \lim s_n$ in such a way that (s_n) are finitely supported (that is, $\pi_j(s_n) = 0$ except for a finite quantity of indices j). If for all j the sequence $(\pi_j(x_k))_k$ is w^*-null in E_j, fixed $\varepsilon/2 > 0$, there exist $N_1 > 0$ and $N_2 > 0$ such that for all $n > N_1$ one has $|x_k(x - s_n)| \leq \varepsilon/2$, and for all $k > N_2$ one has $|x_k(s_n)| \leq \varepsilon/2$. Thus, $|x_k(p)| \leq \varepsilon$. The case $p = 1$ is entirely analogous choosing x in $c_0(E_n^*)$. □

Proposition 2.1. *Let λ be a positive constant. The c_0-product and l_p-product, $1 \leq p \leq \infty$), of a $\mathcal{Q}$-bounded family (F_n) of λ-almost trivial maps is λ-almost trivial.*

Proof. Let us consider the extentensions $0 \to A_n \stackrel{j_n}{\to} C_n \to B_n \to 0 \equiv F_n$. For each n let $\omega_n : B_{A_n^*} \to \lambda B_{C_n^*}$ be a $w*$-selector for j_n^*. We construct the l_p-product

$$0 \longrightarrow l_p(A_n) \xrightarrow{\chi} l_p(C_n) \longrightarrow l_p(B_n) \longrightarrow 0 \equiv l_p(F_n)$$

of the family (F_n). It follows from lemma 2.2 that the map $\Omega : B_{l_1(A_n^*)} \to \lambda B_{l_1(C_n^*)}$ defined by $\Omega[(a_n^*)] = [\omega_n(a_n^*)]$ is a w^*-selector for χ^*. □

Taking into account lemmata 1.2 and 1.3 we obtain:

Corollary 2.1. *The l_∞-product in $\mathcal{Q}^Z$ of λ-almost trivial maps is λ-almost trivial. The l_1-coproduct in $\mathcal{Q}_Y$ of λ-almost trivial maps is λ-almost trivial.*

We are ready to prove the Lindenstrauss-Pelczynski theorem.

Theorem 2.2. *Every extension $0 \to H \to c_0 \to c_0/H \to 0$ almost splits.*

Proof. We organize the proof in two steps:

♦ *Every extension $0 \to c_0(A_n) \to c_0 \to Q \to 0 \equiv F$, with A_n finite-dimensional spaces almost splits*: Fix $\varepsilon > 0$ and fix for each A_n an embedding $j_n : A_n \to l_\infty^{m(n)}$ such that $\|j_n\| \leq 1+\varepsilon$. The Lindenstrauss-Rosenthal theorem asserts that F is isomorphically equivalent to the c_0-product

$$0 \longrightarrow c_0(A_n) \xrightarrow{\chi} c_0(l_\infty^{m(n)}) \longrightarrow c_0(l_\infty^{m(n)}/A_n) \longrightarrow 0 \equiv c_0(F_n),$$

of the extensions

$$0 \longrightarrow A_n \xrightarrow{j_n} l_\infty^{m(n)} \longrightarrow l_\infty^{m(n)}/A_n \longrightarrow 0 \equiv F_n$$

Quite clearly χ is the embedding that takes the value $(j_n a_n)_n$ on $(a_n)_n \in c_0(A_n)$. There is no loss of generality identifying F and $c_0(F_n)$. Thus, Proposition 2.1 asserts that F is almost trivial because the Bartle-Graves selection principle (see [**3**, Prop. 1.19 (ii)]) allows us to obtain a continuous selection $\omega_n : B_{A_n^*} \to (1+\varepsilon)B_{l_\infty^{m(n)*}}$ for each j_n^*; being A_n^* finite-dimensional spaces this shall also be w^*-continuous. In other words, each extension F_n is $(1+\varepsilon)$-almost trivial.

♦ *Every extension $0 \to H \to c_0 \to c_0/H \to 0$ almost splits*: A combination of results of Johnson, Rosenthal and Zippin (see 1.g.2 y 2.d.1 in [**58**]) yields that every subspace H of c_0 admits a representation $0 \to c_0(A_n) \to H \to c_0(B_n) \to 0$ in which the spaces A_n and B_n are finite dimensional for all $n \in \mathbb{N}$. We construct the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & c_0(A_n) \arrow[r, equal] \arrow[d, "u"'] & c_0(A_n) \arrow[d] & & \\
0 \arrow[r] & H \arrow[r, "j"] \arrow[d, "p"'] & c_0 \arrow[r] \arrow[d, "q"] & c_0/H \arrow[r] \arrow[d, equal] & 0 \equiv \mathcal{H} \\
0 \arrow[r] & c_0(B_n) \arrow[r, "w"'] \arrow[d] & c_0/c_0(A_n) \arrow[r] \arrow[d] & c_0 \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

Using Lemma 2.1 and the result [**35**] asserting that every quotient of c_0 is a subspace of c_0 we can conclude that the functor $\mathcal{L}(\cdot, C(K))$ is exact on $\mathcal{H}$. □

The advantage of this homological approach in comparison with the original one of Lindenstrauss and Pelckzynski is that it is clean. Moreover, it shall provide us buy duality a proof for the Johnson-Zippin theorem. The disadvantage of this proof is the difficulty of getting a good estimate (as in [**55**]) for the norm of the extension: given $\varepsilon > 0$, we obtained through Proposición 2.1 an estimate $(1+\varepsilon)$ for subspaces $c_0(A_n) \hookrightarrow c_0$ "well placed". However, the price we pay for using the diagonal principle is to spoil the estimate. The main advantage of the delicate work of Zippin in [**81**] i aimed to obtain a *final* estimate of $1+\varepsilon$.

2.2. Beyond the Lindenstrauss-Pelczynski theorem. The extension problem that Theorem 2.2 solves is defined by the data

- $\mathfrak{I}$ is the class of embeddings $H \hookrightarrow c_0$;
- $\mathfrak{E} = C(K)$;
- $\mathfrak{U} = \mathcal{L}$.

We pose now some questions with the aim of drawing the boundaries of the theorem.

2.2.1. *Can the class $\mathfrak{E}$ be enlarged?* Let us start with a definition.

DEFINITION 4.2 (**Spaces of type $\mathcal{LP}$**). *We shall say that a space X is an $\mathcal{LP}$ space if all operators from subspaces of c_0 into X can be extended to c_0.*

LEMMA 2.3. *Every $\mathcal{LP}$-space is an $\mathcal{L}_\infty$-space*

PROOF. Let $T : Y \to \mathcal{LP}$ be a compact operator from a subspace $Y \to X$ of a separable space; then T factorizes as BA through some subspace $i : H \to c_0$. Being an $\mathcal{LP}$ space allows one to extends B to an operator $B_1 : c_0 \to \mathcal{LP}$, while Sobczyk's theorem gives us an extension $A_1 : X \to c_0$ of iA. The composition $B_1A_1 : X \to \mathcal{LP}$ extends T. Using Lindenstrauss's characterization [**54**], $\mathcal{LP}$ must be an $\mathcal{L}_\infty$-space. □

► We ask now if every $\mathcal{L}_\infty$ space is an $\mathcal{LP}$ space. Let us give an example showing that the answer is no. The construction is based on the Bourgain- Pisier construction [**7**] showing that every separable Banach space X admits an extension

$$0 \longrightarrow X \longrightarrow \mathcal{L}_\infty(X) \longrightarrow \mathcal{L}_\infty(X)/X \longrightarrow 0,$$

in which $\mathcal{L}_\infty(X)$ is a separable $\mathcal{L}_\infty$-space and $\mathcal{L}_\infty(X)/X$ has the Schur property.

PROPOSITION 2.2. *Let $0 \to H \to c_0 \to c_0/H \to 0 \equiv \mathcal{H}$ be an extension in which c_0/H is not isomorphic to c_0. There exists an operator from H into an $\mathcal{L}_\infty$-space that cannot be extended to c_0.*

Proof. Let $0 \to H \xrightarrow{i} \mathcal{L}_\infty(H) \to S \to 0 \equiv \mathcal{H}_\infty$ be the Bourgain- Pisier extension for H. Assume that i extends to c_0, so that there exists a morphism $\mathcal{H}_\infty \longleftarrow \mathcal{H}$. Since c_0 is separably injective, $\mathcal{H} \longleftarrow \mathcal{H}_\infty$. Applying the first diagonal principle one gets that the spaces $\mathcal{L}_\infty(H) \oplus c_0/H$ y $c_0 \oplus S$ are isomorphic. In particular, c_0/H is a complemented subspace of $c_0 \oplus S$. Since S and c_0 are totally incomparable by the Schur property of S the decomposition theorem of Edelstein-Wojtasczyk (see [**58**, Theorem 2.c.13]) ensures that c_0/H is isomorphic to some $A \oplus B$ with A complemented in c_0 and B complemented in S. Since c_0/H is a subespace of c_0, B can only be finite dimensional, hence $c_0/H \simeq c_0$, against the hypothesis. □

In addition to $C(K)$-spaces, it is clear that complemented subspace of $C(K)$-spaces and separably injective spaces are also $\mathcal{LP}$-spaces. It is unknown if there exist complemented subspaces of $C(K)$-spaces that are not $C(K)$-spaces. Moreover, the reader might be surprised by the distinction between those two classes (complemented subspace of $C(K)$-spaces and separably injective spaces); especially regarding the fact that every injective space is complemented in some $C(K)$-space. Let us show that the distinction is necessary.

Proposition 2.3. *There exists a separably injective space that is not complemented in any $C(K)$-space.*

Proof. Let us considert the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & l_\infty & \longrightarrow & l_\infty/c_0 & \longrightarrow & 0 \equiv \mathcal{I} \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \lambda\cdot 1} & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & P(\lambda) & \longrightarrow & l_\infty/c_0 & \longrightarrow & 0 \equiv F_\lambda. \end{array}$$

Benyamini shows in [**2**] that $P(\lambda$ is no less than λ^{-1}-complemented in any $C(K)$-space. Thus, the c_0-product of the family $(F_{n^{-1}})$ is an extension

$$0 \longrightarrow c_0(c_0) \longrightarrow c_0(P(n^{-1})) \longrightarrow c_0(l_\infty/c_0) \longrightarrow 0 \equiv c_0(F_{n^{-1}})$$

in which $c_0(c_0)$ as well as $c_0(l_\infty/c_0)$ are $C(K)$-spaces. However, $c_0(P(n^{-1}))$ cannot be complemented in any $C(K)$-space. Finally, the three spaces are separably injective as it follows from 5.1, where it was shown that the c_0-sum of separably injective spaces is separably injective, plus a 3-space argument. □

Still, there exist other $\mathcal{LP}$-spaces. The next result appears announced in [**55**, remark 2, p.234] with only an indication for the proof: Isometric preduals of L_1 are $\mathcal{LP}$-spaces.

► It is quite natural to ask whether the previous clases (complemented subspaces of a $C(K)$, separably injective spaces and isometric preduals of L_1) exhaust $\mathcal{LP}$-spaces. Once again the answer is no:

Example. Let $0 \to l_2 \to \mathcal{L}_\infty(l_2) \to S \to 0$ be the Bourgain-Pisier extension for l_2. Every subspace H of c_0 verifies $\mathcal{L}(H, l_2) = \mathcal{K}(H, l_2)$, and also $\mathcal{L}(H, S) = \mathcal{K}(H, S)$ because S is a Schur space. On the other hand it is not difficult to check that $\mathcal{L}(H, \cdot\,) = \mathcal{K}(H, \cdot\,)$ is a 3-space property (see [**16**, §6.1 y §6.7]). In particular $\mathcal{L}(H, \mathcal{L}_\infty(l_2)) = \mathcal{K}(H, \mathcal{L}_\infty(l_2))$ and by Lindenstrauss's classical result [**54**], compact operators $H \to \mathcal{L}_\infty(l_2)$ extend to c_0. Moreover, since $\mathcal{L}_\infty(l_2)$ does not contain c_0 (another 3-space property, see [**16**]), it cannot be a complemented subspace of a $C(K)$-space, let alone separably injective. It cannot be an isometric L_1-predual since it has a Schur quotient.

Regarding those examples it makes sense to pose:

Problem: Characterize $\mathcal{LP}$-spaces.

And, more precisely, we arrive to one of the key problems that remain open in this memoir: since Johnson and Zippin show in [**36**] that every separable isometric predual of L_1 is isometric to a quotient of $C(\Delta)$, where Δ is the Cantor set, we ask:

Problema (LP2). Are $\mathcal{L}_\infty$ quotients of $C[0,1]$ spaces of type $\mathcal{L}P$?

2.2.2. *Can the class $\mathfrak{I}$ be enlarged?* Let us observe once more that if F is almost trivial and there is a morphism $F \longleftarrow G$ of $\mathfrak{Q}$ then G is almost trivial. Since Sobczyk's theorem implies that every extension $G : Z \to H$ with $H \hookrightarrow c_0$ and Z separable is pull-back of the natural extension $F : c_0/H \curvearrowright H$ one sees that the result remains valid when $\mathfrak{I}$ is the class of all embeddings $H \hookrightarrow S$ in which S is any separable space. Put in another way, *every subspace of c_0 is almost complemented in any separable superspace.* Although we cannot ignore the role of separability (for instance, the subspaces of c_0 are not almost complemented in l_∞), not everything has been lost: $\mathfrak{I}$ can actually cover all embeddings $H \to c_0(\Gamma)$, with Γ any index set. This was shown by Johnson and Zippin in [**39**]:

PROPOSITION 2.4. *Let Γ be any index set. Every extension $0 \to H \to c_0(\Gamma) \to Z \to 0$ almost splits.*

We do not know if the homological argument used in the proof of Theorem 2.2 can be modified to cover this case.

Problem: Is it possible a homogical approach to Proposition 2.4?

Moreover, there is a remarkable question about the $\mathcal{LP}$ spaces:

Question. Given a subspace H of $c_0(\Gamma)$, does every operator $H \to \mathcal{LP}$ an extension to $c_0(\Gamma)$?

3. The Johnson-Zippin theorem

One of the questions Zippin posed at te end of the paper [**80**] is intimately connected with the behaviour of the functor Ext. The question is: when is an extension $0 \to E \to l_1 \to Z \to 0$ almost trivial? Equivalently, taking into account the description of the functor $\mathrm{Ext}_{\mathbf{B}}(Z, \cdot\,)$ in terms of projective presentations: which Banach spaces Z verify the equation $\mathrm{Ext}_{\mathbf{B}}(Z, C(K)) = 0$ for each Hausdorff compact K? Johnson and Zippin gave a partial solution to the problem in [**38**]. We shall give an answer, in unexpected terms, in Section 4.

THEOREM 3.1 (Johnson-Zippin). *Every operator defined on a $w(\ell_1, c_0)$-closed subspace of ℓ_1 with range on a $\mathcal{L}_\infty$-space can be extended to ℓ_1.*

Johnson and Zippin's proof is basically technical, quite long and, in principle, completely different from that Lindenstrauss and Pełczyński gave for their result in [**55**]. Nevertheless, we shall show that both theorems hare a common homological nature.

3.1. An homological approach to the Johnson-Zippin theorem.

Let us begin reminding a result which can be seen in [**11**].

LEMMA 3.1. *The property $\mathcal{Z}(\,\cdot\,, \heartsuit) = 0$ is a 3-space property.*

PROOF. Let $F : Z \curvearrowright Y$ and $G : Y \oplus_F Z \curvearrowright \heartsuit$ be objects of $\mathfrak{Q}$. By hypothesis, $Gj_F \equiv 0$ and thus the exactness of the long homology sequence at the term $\mathcal{Z}(Y \oplus_F Z, \heartsuit)$ ensures the existence of $H : Z \curvearrowright \heartsuit$ such that $Hq_F \equiv G$; but, by hypothesis again, $H \equiv 0$, what is enough to conclude. □

We shall use once more the notation $\mathcal{L}_\infty$ to refer to a unspecified $\mathcal{L}_\infty$-space. The homological formulation that corresponds to Theorem 3.1 is:

THEOREM 3.2. *Let H be a subspace of c_0. Then $\mathcal{Z}(H^*, \mathcal{L}_\infty) = 0$.*

PROOF. Since H admits a representation $0 \to c_0(A_n) \to H \to c_0(B_n) \to 0$ with A_n and B_n finite dimensional spaces, the dual H^* decomposes in the form $0 \to \ell_1(B_n^*) \to H^* \to \ell_1(A_n^*) \to 0$. Using Lemma 3.1 it is enough to prove $\mathcal{Z}(l_1(E_n), \mathcal{L}_\infty) = 0$ for any family (E_n) of finite dimensional spaces. The identification

$$\mathcal{Z}(l_1(E_n), \mathcal{L}_\infty) \xleftrightarrow{\bigoplus} l_\infty^{\bowtie}(Z(E_n, \mathcal{L}_\infty)),$$

induced by the coproduct ensures that every element F of $\mathcal{Z}(\ell_1(E_n), \mathcal{L}_\infty)$ is the coproduct of the family (Fj_n), where $j_n : E_n \to l_1(E_n)$ are the natural inclusions. Since the elements of

$Z(\heartsuit, \mathcal{L}_\infty)$ locally split it follows (see [**10**]) that $l_\infty^{\bowtie}(Z(E_n, \mathcal{L}_\infty)) = 0$. Since the coproduct $\bigoplus$ establishes an isomorphism of vector spaces, we conclude that $\mathcal{Z}(l_1(E_n), \mathcal{L}_\infty) = 0$. $\square$

In this case it is possible to obtain a reasonably good estimate of the norm of the extension operator using nonlinear arguments. See [**13**]. Those techniques yield an estimate $3 + \varepsilon$ for the norm of the extension of an operator $T : H^\perp \to C(K)$ with respect to an exact sequence $0 \to H^\perp \to \ell_1 \to H^* \to 0$ where H is a subspace of c_0. THat is the same estimate that Johnson and Zippin obtain in [**38**], although the proof in [**13**] is perhaps simpler. Nevertheless, they are able to obtain under the additional hypothesis that H^* has the approximation property, a better estimate: there is an extension operator $\tilde{T}$ with $\|\tilde{T}\| \leq (1+\varepsilon)\|T\|$. They also ask whether such estimate is also possible when H^* does not have the A.P. It would be interesting to see an homological approach to these problems.

3.2. Beyond the Johnson-Zippin theorem. Let us recall that the extension problem solved with the Johnson- Zippin theorem comes described by the data:

- $\mathfrak{I} = w(l_1, c_0)$-continuous embeddings $H^\perp \to l_1$;
- $\mathfrak{E} = \mathcal{L}_\infty$;
- $\mathfrak{U} = \mathcal{L}$.

3.2.1. **About the range of $\mathfrak{I}$.** In [**38**], Johnson and Zippin had already observed that their result remains valid for extensions $0 \to D \to S \to H^* \to 0$ with S separable and H a subspace of c_0. Let us see that the separability assumption can be spared. To do that, we first observe that every projective presentation of H^* i almost trivial, since any two projective presentations are semiequivalent. Next, we use Lemma 1.3 to obtain

PROPOSITION 3.1. *If H is a subspace of c_0, every extension $0 \to A \to B \to H^* \to 0$ is almost trivial.*

We know that $\mathfrak{I}$ cannot be enlarged to cover all embeddings $W \hookrightarrow l_1$ because Kalton proves in [**44**] that $\mathcal{Z}(X, C[0,1]) \neq 0$ for each separable Banach space X without the Schur property. It seems to be so far unknown the existence of an uncomplemented subspace of l_1 not isomorfo to a $w(l_1, c_0)$-closed subspace of l_1 (Johnson and Zippin even ask in [**38**] if every subspace of l_1 isomorphic to a $w(l_1, c_0)$-closed subspace is almost complemented). To change the predual of l_1 has serious consequences since a $w(l_1, C(\omega^\omega))$-closed subspace of l_1 is not necessarily almost complemented: let us consider the extension $0 \to S \xrightarrow{j} C(\omega^\omega) \to Q \to 0$, where S is Schreier's space; the dual extension $0 \to Q^* \to l_1 \to S^* \to 0$ cannot be almost trivial since S^* fails Schur property because S fails the DPP.

Thus, the problem of the characterization of almost complemented subspaces of l_1, that is, those subspaces $K(X) \subset l_1$ such that $\mathcal{Z}(X, C(K)) = 0$ for every $C(K)$-space, remains open. A result of Kalton appeared in [**44**] is "partially converse" to the Johnson-Zippin theorem, and shows that the hypothesis on $K(X)$ in the Johnson-Zippin theorem cannot be weakened "too much": precisely, he shows that every almost complemented subspace $K(X)$ of l_1 induces a quotient X with the strong Schur property (SSP); and if, moreover, X admits a "uniform finite dimensional decomposition" (UFDD) then X is the dual of a subspace of c_0.

Since the the only way in which $\mathrm{Ext}_{\mathbf{B}}(X, C(K)) = 0$ is possible is that X has SSP, it makes sense to ask:

Question: Does there exist a subspace X of l_1 such that $\mathcal{Z}(X, C(K)) \neq 0$?

It is possible to be surprised when during the next chapter we display a subspace X of $l_1(\Gamma)$ for which $\mathcal{Z}(X, c_0) \neq 0$.

3.2.2. **About the equation $\mathrm{Ext}_\square(\mathbf{X}, \mathbf{C}(\mathbf{K})) = \mathbf{0}$.** Let us observe that when $\square = \mathbf{Q}$, unlike what occurs for $\square = \mathbf{B}$, we know of the existence of nontrivial elements in $\mathrm{Ext}_{\mathbf{Q}}(X, C(K))$ for any subspace X of l_1, including l_1 itself. For instance, the process of construction of singular objects we reviewed in Chapter 2 allows us to construct a nontrivial element of $\mathrm{Ext}_{\mathbf{Q}}(l_1, C[0,1])$ starting with a nontrivial map $l_1 \curvearrowright \mathbb{R}$.

4. Characterization of the spaces X such that $\mathcal{Z}(\mathbf{X}, \mathbf{C}(\mathbf{K})) = \mathbf{0}$

Our aim now is to characterize the spaces X whose extensions by any $C(K)$-space split. As a previous step we shall present an alternative and, in our opinion, more natural, construction of the initial object for the class $\mathcal{Z}(X, C(K))$ obtained in [**12**].

We shall consider on the space $Z(X, \mathbb{R})$ of scalar z-linear map, as well as in its subspace $ZL(X, \mathbb{R})$ of z-linear maps in canonical form with respect to a given Hamel basis (e_γ) of X, the topology w^* of the pointwise convergence; so that $F = w^* - \lim F_\alpha$ if and only if for each point $x \in X$ one has $F(x) = \lim F_\alpha(x)$. We shall say that an application $Z(X, \mathbb{R}) \to Z(Y, \mathbb{R})$ is w^*-continuous when it transforms w^*-convergent nets into w^*-convergent nets.

Lemma 4.1. *The unit ball of $[ZL(X, \mathbb{R}), Z(\cdot)]$ is w^*-compact.*

Proof. We consider the embedding

$$\begin{array}{ccc} ZL(X, \mathbb{R}) & \xrightarrow{j} & \mathbb{R}^X \\ F & \to & (Fx)_{x \in X}. \end{array}$$

If $x = \sum_\gamma x_\gamma e_\gamma$ then $|Fx| \leq Z(F) \sum_\gamma |x_\gamma|$, and thus the image $j(B_{ZL(X,\mathbb{R})})$ is contained in $\prod_{x \in X} \left[- \sum_\gamma |x_\gamma|, \sum_\gamma |x_\gamma| \right]$. That $j(B_{ZL(X,\mathbb{R}))})$ is closed follows from the fact that the two properties 'to be z-linear" and "to be in canonical form" come defined by conditions respecting pointwise convergence. □

In Chapter 5 we shall see some good reasons to denote the unit ball $B_{ZL(X,\mathbb{R})}$ by B_{X^z}. We shall use from now on this notation and, unless stated otherwise, we shall understand that B_{X^z} carries the w^*-topology. We define the map $\Delta : X \curvearrowright C(B_{X^z})$ as

$$\Delta(x)(F) = Fx.$$

That $\Delta(x)$ is w^*-continuous is clear. That Δ is z-linear follows from the inequality $|\Delta(\sum_{i=1}^n x_i)(F) - \sum_{i=1}^n \Delta(x_i)(F)| = |F(\sum_{i=1}^n x_i) - \sum_{i=1}^n F(x_i)| \leq Z(F) \sum_{i=1}^n \|x_i\|$. We shall say that Δ is a **$\mathbf{C}(\cdot)$-presentation** of the space X. Occasionally we shall write Δ_X to distinguish $C(\cdot)$- presentations of different spaces.

The initial character of $\Delta : X \curvearrowright C(B_{X^z})$ appears in the next proposition:

Proposition 4.1. *Given a z-linear map in canonical form $F : X \curvearrowright C(K)$ there exists an operator $\varphi_F : C(B_{X^z}) \to C(K)$ so that $\varphi_F \Delta = F$.*

Proof. . Given $F : X \curvearrowright C(K)$ in canonical form we define the operator $\varphi_F(f)(k) = f(\delta_k F)$. It is clear that $\varphi_F(f)$ is a continuous function on K. Moreover $\varphi_F \Delta(x)(k) = \Delta(x)(\delta_k F) = F(x)(k)$. □

The reader will see when next characterization Theorem 4.1 comes that we needed a more precise characterization of the initial object of $\mathcal{Z}(X, C(K))$ more explicit than that in [**12**]. An initial object for $\mathcal{Q}(X, C(K))$ can be constructed in an analogous way with only minor modifications in the estimates. A variation interesting to us Una appears when considering z-linear maps on finite dimensional spaces E din canonical convex form with respect to a given set (a_i) including a basis for E. We momentarily denote $ZLC(E, \mathbb{R})$ this subspace of of $ZL(E, \mathbb{R})$; it is easy to check that its unit ball $B_{E^{zc}}$ is w^*-closed. We have now a new z-linear map $\Delta^c : E \curvearrowright C(B_{E^{zc}})$ defined as $\Delta^c(e)(F^c) = F^c(e)$, which factorizes as $R\Delta$, where $R : C(B_{E^z}) \to C(B_{E^{zc}})$ is just the natural restriction map. This map has the same initial character with respect to $ZLC(E, C(K))$ as Δ had with respect to $ZL(E, C(K))$: given a z-linear map in its canonical convex form $F^c : E \curvearrowright C(K)$ there exists an operator φ_{F^c} such that $\varphi_{F^c} \Delta^c = F^c$; in particular, given $F^c : E \curvearrowright \mathbb{R}$ there exists a functional δ_{F^c} such that $\delta_{F^c} \Delta^c = F^c$.

We present now a couple of applications of the existence of this initial object $\Delta : X \curvearrowright C(B_{X^z})$. The first one is the characterization result announced in the title of this section. One more definition will help.

Let us recall that given a subspace Y of a Banach space X, a λ-*metric projection* on X is a (homogeneous) map $m : X \to Y$ such that $\forall x \in X$, $\|x - m(x)\| \leq \lambda\|x\|$. A metric projection is a λ-metric projection for some λ.

Definition 4.3. *A z-metric projection on a Banach space X is a metric projection on $Z(X, \mathbb{R})$ with respect to the subspace X'*

A w^*-continuous z-metric projection is a z-metric projection which is $w^* - w(X', X)$-continuous.

Theorem 4.1. *A Banach space X admits a w^*-continuous z-metric projection if and only if for every compact Hausdorff space K one has $\mathcal{Z}(X, C(K)) = 0$.*

Proof. Let $F : X \curvearrowright C(K)$ be a z-linear map and let $m : Z(X, \mathbb{R}) \to X'$ be a w^*-continuous z-metric projection. We define a linear map $L : X \to C(K)$ by means of

$$L(x)(k) = m(\delta_k F)(x).$$

The application L is well defined since $L(x)$ is a continuous function: if $k = \lim k_\alpha$ in K then $L(x)(k) = m(\delta_k F)(x) = m(w^* - \lim \delta_{k_\alpha} F)(x) = \lim m\left(\delta_{k_\alpha} F\right)(x) = \lim L(x)(k_\alpha)$. Moreover, $|F(x)(k) - L(x)(k)| = |\delta_k F(x) - m(\delta_k F)(x)| \leq \lambda Z(\delta_k F)\|x\|$, hence $\|F - L\| \leq \lambda Z(F)$.
Conversely, if $\mathcal{Z}(X, C(K)) = 0$ for every $C(K)$-space then $\mathcal{Z}(X, C(B_{X^z})) = 0$. Let $L : X \to C(B_{X^z})$ be a linear map so that $\|\Delta - L\| < +\infty$. The map $m : Z(X, \mathbb{R}) \to X'$ given by

$$m(F) = Z(F)\delta_{Z(F)^{-1}(F - L_F)} L + L_F$$

defines a w^*-continuous z-metric projection , as we shall prove now. It is a metric projection since

$$\begin{aligned} |Fx - m(F)x| &= |Z(F)\frac{Fx - L_F x}{Z(F)} + L_F x - Z(F)\delta_{Z(F)^{-1}(F - L_F)} Lx - L_F x| \\ &\leq Z(F)|\delta_{Z(F)^{-1}(F - L_F)}\Delta x - \delta_{Z(F)^{-1}(F - L_F)} Lx| \\ &\leq Z(F)\|\Delta - L\|\|x\|. \end{aligned}$$

It is w^*-continuous because the linearization process is w^*-continuous; thus, if $F = w^* - \lim F_\alpha$ then $\lim m(F_\alpha)(x) = \lim Lx(F_\alpha - L_{F_\alpha}) + L_{F_\alpha}(x) = Lx(F - L_F) + L_F(x) = m(F)(x)$. □

The second application is a key result that allowed us to make a non-linear approach to Sobczyk's theorem and the vectorial versions we obtained in Chapter 1. The next result can be understood as the reciprocal, in some sense, of a classical result essentially due to Kalton [**40**]; see also [**10**].

Lemma 4.2 (Change of convergence Lemma). *Let $F_n : E \curvearrowright Y$ a sequence of z-linear maps in canonical convex form on a finite dimensional space. If $F = w^* - \lim F_n$ then $F = \|\cdot\| - \lim F_n$.*

Proof. The proof goes in four steps. The two first have the purpose of establishing the result for $\mathbb{R}$-valued maps. The third step establishes the change between $Z(\cdot)$-convergence to norm-convergence. The fourth step yields a simple extension to finite dimensional valued maps.
First step. *Let $F : E \curvearrowright C(K)$ be a z-linear map on a finite dimensional space in so that the range of F is finite dimensional. We can assume without loss of generality that $K = [0, 1]$. Let us prove that if $k = \lim k_n$ then $\delta_k F = Z(\cdot) - \lim \delta_{k_n} F$.* To do that, let us observe that $F(B_E)$ is a bounded set on a finite dimensional space, and thus its closed convex hull is a compact, hence equicontinuous, set of $C(K)$. We therefore have

$$\forall \varepsilon\ \exists \delta > 0 : \forall x : \|x\| \leq 1,\ \forall p, q : |p - q| < \delta \Rightarrow |\delta_p Fx - \delta_q Fx| < \varepsilon.$$

Thus, if $\lim p_n = p$ then

$$\forall \varepsilon\ \exists N \in \mathbb{N} : \forall x : \|x\| \leq 1,\ \forall n > N \Rightarrow |\delta_{p_n} Fx - \delta_p Fx| < \varepsilon$$

In this way $Z\left(\delta_{p_n}F - \delta_p F\right) = Z\left((\delta_{p_n} - \delta_p)F\right) \le \varepsilon$ since

$$\left|(\delta_{p_n} - \delta_p)F(\sum x_i)\right| = \left|(\delta_{p_n} - \delta_p)F\left(\frac{\sum x_i}{\sum \|x_i\|}\right)\right| \sum \|x_i\| \le \epsilon \sum \|x_i\|.$$

Second step. We use now the initial object $\Delta^c : E \to C(B_{E^{zc}}, w^*)$. If F_n is a collection of z-linear $\mathbb{R}$-valued maps in convex canonical form such that $F = w^* - \lim F_n$ then we are in the hypotheses of the previous step and we get $\delta_F \Delta^c = Z(\cdot) - \lim \delta_{F_n}\Delta^c$; or, what is the same, $F = Z(\cdot) - \lim F_n$.

Third step. *If $F_n : E \curvearrowright X$ are z-linear maps in canonical form on a finite dimensional space E (there is no need to ask that X be finite dimensional too) and $F = Z(\cdot) - \lim F_n$ then* $F = \|\cdot\| - \lim F_n$. The result cannot be simpler: if we set $F = 0$ to simplify then

$$\|F_n(p)\| = \|F_n(\sum p_\gamma e_\gamma)\| \le Z(F_n) \sum |p_\gamma| \le Z(F_n)\,\mathrm{dist}(E, l_1^{\dim E})\|p\|$$

Fourth step. We consider now z-linear maps $F_n : E \curvearrowright Y_n$ in canonical convex for with respect to a given set (a_n) containing a basis of E. One has that $\sup_n \dim[\mathrm{Im} F_n] < +\infty$. There is a finite dimensional space l_∞^M in which all the spaces Y_n can be placed in an almost isometric form. We can consider then $F_n : E \curvearrowright l_\infty^M$. Let $(\delta_j)_{j=1,\ldots,M}$ be the collection of evaluation functionals on the coordinates of l_∞^M. Since $\delta_j F_n$ is in canonical convex form one has $\delta_j F_n = \delta_{\delta_j F_n}\Delta^c$. Moreover, if $F = w^* - \lim F_n$ then one also has $\delta_j F = w^* - \lim \delta_j F_n$ for all $1 \le j \le M$; thus, from the previous results one gets $\delta_j F = \|\cdot\| - \lim \delta_j F_n$ for all $1 \le j \le M$. Finally

$$\begin{aligned}
\|F_n - F\| &= \sup_{\|x\|\le 1} \|F_n x - Fx\| \\
&= \sup_{\|x\|\le 1} \sup_{1\le j\le M} |\delta_j F_n x - \delta_j F x| \\
&= \sup_{1\le j\le M} \sup_{\|x\|\le 1} |\delta_j F_n x - \delta_j F x| \\
&= \sup_{1\le j\le M} \|\delta_j F_n - \delta_j F\|
\end{aligned}$$

which is everything what is needed . □

5. Further applications

5.1. Almost complementation and automorphy clases of C[0, 1]. The Lindenstrauss-Pelczynski theorem can be read as follows: *the class of operators $\mathcal{L}(H, C[0,1])$ from subspaces H of c_0 form an automophy class in* $C[0,1]$. In [**55**] they conjecture what we can interpret as: "the previous one is the only automorphy class in $C[0,1]$". More precisely:

Conjecture [**55**, Sect. 3, remark]**:** Let X be a separable space. The space C$[0,1]$ is X-automorphic if and only if X is a subspace of c_0.

Proposition 5.1. *Let X be a separable Banach space that does not contain l_1. The space $C[0,1]$ is X-automorphic if and only if every extension* $0 \to X \to C[0,1] \to C[0,1]/X \to 0 \equiv F$ *is almost trivial.*

Proof. If $C[0,1]$ is X-automorphic, using Milutin's theorem we get that F is isomorphically equivalent to $0 \to X \xrightarrow{\delta} C(B_{X^*}) \to Q \to 0 \equiv \complement_X$; the conclusion follows since $\complement_X$ is almost trivial. Conversely, let $0 \to X \to C[0,1] \xrightarrow{q_1} Q_1 \to 0 \equiv F_1$ and $0 \to X \to C[0,1] \xrightarrow{q_2} Q_2 \to 0 \equiv F_2$ be two extensions. The hypothesis means $F_1 \longleftarrow F_2 \longleftarrow F_1$, and thus the diagonal principle yields the existence of a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & C[0,1]\oplus C[0,1] & \longrightarrow & Q_1 \oplus C[0,1] & \longrightarrow & 0 \equiv F_1 \oplus 0^{C[0,1]} \\
 & & \| & & \wr & & \wr & & \\
0 & \longrightarrow & X & \longrightarrow & C[0,1]\oplus C[0,1] & \longrightarrow & Q_2 \oplus C[0,1] & \longrightarrow & 0 \equiv F_2 \oplus 0^{C[0,1]},
\end{array}$$

formed by isomorphically equivalent extensions. We observe now that since Q_1 y Q_2 contain l_1 (by a 3-space argument), the spaces Q_1^* and Q_2^* are not separable; therefore (see [**71**]) there exist subspaces C_1 and C_2 of $C[0,1]$ isomorphic to $C[0,1]$ such that q_1 is an isomorphism on

C_1 and q_2 is an isomorphism on C_2. Using a result of Pelczynski [**65**] the space C_1 contains another copy of $C[0,1]$ which is complemented in the original $C[0,1]$; and the same occurs with Q_2. It is easy to check, using the same argument we used to prove that c_0 i automorphic, that F_1 is isomorphically equivalent to $F_1 \oplus 0^{C[0,1]}$; as well as F_2 and $F_2 \oplus 0^{C[0,1]}$, and this concludes the proof. $\square$

Since l_2 is the only automorphic space we know, besides c_0, it makes sense to ask whether $C[0,1]$ is l_2-automorphic; or, equivalently:

Problem: Is every extension $0 \to \ell_2 \to C[0,1] \to C[0,1]/\ell_2 \to 0$ almost trivial?

This problem appears posed by Johnson and Zippin in [**38**, Prob. 4.2]. It is curious and so we notice it that, after the last results, either the Lindenstrauss-Pelczynski conjecture 5.1 has negative answer or the Johnson-Zippin question has a negative answer.

All in all, the same question for operators $T : l_2 \to \mathcal{L}_\infty$ has negative answer: if we consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & l_2 & \xrightarrow{i} & C(K) & \longrightarrow & C(K)/l_2 & \longrightarrow & 0 \\
 & & \| & & & & & & \\
0 & \longrightarrow & l_2 & \xrightarrow[j]{} & \mathcal{L}_\infty(l_2) & \longrightarrow & S & \longrightarrow & 0,
\end{array}$$

in which $\mathcal{L}_\infty(\ell_2)$ is the Bourgain-Pisier space associated to l_2. Since S has the Schur property, $\mathcal{L}_\infty(\ell_2)$ cannot contain c_0 and therefore every extension $J : C(K) \to \mathcal{L}_\infty(\ell_2)$ of j would be a weakly compact operator, hence completely continuous by the DPP of $\mathcal{L}_\infty$-spaces, and Ji could not be the identity of ℓ_2.

5.2. Other almost trivial extensions. Let us briefly describe the very few known examples of *almost trivial* extensions (and some *not almost trivial*) in addition to those already studied.

▲ An interesting result was stated by Zippin in [**79, 80**], although not explicitly proved in the literature until [**13**]: every extension $0 \to W \to \ell_p \to \ell_p/W \to 0$, for $1 \leq p < \infty$, is almost trivial.

▲ In [**79**], Zippin proves that for every separable space E there exists an almost trivial extension $0 \to E \to X(E) \to Z(E) \to 0$ in which both $X(E)$ and $Z(E)$ admit a finite dimensional decomposition.

▲ The Proposición 2.1 generates more almost trivial extensions. A couple of examples: given a subspace H of c_0 the extensions $0 \to l_p(H) \to l_p(c_0) \to l_p(c_0/H) \to 0$, for $1 \leq p \leq \infty$, and $0 \to c_0(H^\perp) \to c_0(l_1) \to c_0(H^*) \to 0$ are almost trivial.

▲ A simple argument as before, using the fact that every operator $\ell_\infty \to C[0,1]$ is weakly compact, shows that if X is a separable space without the Schur property then X is not almost complemented in ℓ_∞.

CHAPTER 5

On the extension of z-linear maps

1. Introducction

The extension problem for quasi-linear maps can be understood as the *non linear* analogue (we'll say of order 2) of the extension problem for operators: given a quasi-linear map $F : Z \curvearrowright Y$ and an embedding $j : Z \to W$, we ask if there is a quasi-linear map $\widehat{F} : W \curvearrowright Y$ such that $\widehat{F}j \equiv F$. We'll say that $\widehat{F}$ is an extension of F through $j : Z \to W$. The following lemma shows that if there is extension then there is *exact extension*:

LEMMA 1.1. *A quasi-linear map $F : Z \curvearrowright Y$ admits an extension to W through j if and only if there exists a quasi-linear map $\check{F}$ such that $\check{F}j = F$ (we shall say that $\check{F}$ is an exact extension of F).*

PROOF. Let $\widehat{F}$ be a quasi-linear map such that $\widehat{F}j = F + B + L$ with $B : Z \curvearrowright Y$ bounded homogeneous and $L : Z \curvearrowright Y$ linear. Since j admits projections l, π linear, and bounded and homogeneous (respectively), it is evident that $\check{F} = \widehat{F} - B\pi - Ll$ extends F exactly. $\square$

It is clear that if F admits an extension W through j, the same happens with any equivalent version of F. Moreover, any equivalent version $G \equiv \widehat{F}$ is still an extension of F. So, the extension problem for quasi-linear maps admits a formulation in the category $\mathfrak{Q}$: we'll say that an object G is an extension of the object F through $j : Z \to W$ if j induces a morphism $G \longleftarrow F$.

The extension problem for quasi-linear maps comes outlined by two variables: **a)** The class $\mathfrak{E}$ of *range* spaces where the maps take values, and **b)** the class $\mathfrak{I}$ of embeddings through which one extends the maps. Unlike the case of extension of operators, we have just two variables to outline the problem: the reason is that there is not a consistent theory to identify classes of quasi-linear maps. We just know the two types defined in this work: $\mathcal{A}$-trivial, strictly singular/cosingular ...

As a rule, a quasi-linear map cannot be extended: the simplest example is perhaps a non-trivial quasi-linear map $F : l_1 \curvearrowright \mathbb{R}$ that cannot be extended to any K-superspace, like $C[0, 1]$, since a nontrivial object cannot have a trivial one as extension.

From now on we'll work with z-linear maps. It is important to recall that even in this case the maps do not necessarily admit an extension: as an example we can consider any nontrivial z-linear map $l_2 \curvearrowright l_2$, which cannot be extended to, say, $L_1(0, 1)$ through any embedding since $\mathcal{Z}(\mathcal{L}_1, l_2) = 0$.

There are however some interesting situations in which it is evident the existence of extension. For instance, we can observe that every z-linear map can be extended to *some* part (although maybe not through any embedding):

- ► Every object $F : Z \curvearrowright Y$ of $\mathfrak{Z}$ admits an extension to a certain quotient $I(Y)$ of an injective space: just pick an embedding $u : Y \oplus_F Z \to I$ and take a look at the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & I & \longrightarrow & I(Y) & \longrightarrow & 0 \\ & & \| & & \uparrow u & & \uparrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & Y \oplus_F Z & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

- ▶ We shall prove in Proposition 4.2 that if Y is a Banach space such that some quotient $I(Y)$ is injective, then every z-linear map $\heartsuit \curvearrowright Y$ extends to any superspace. When $I(Y)$ is just separably injective, F extends to any superspace in which $\heartsuit$ is coseparable (we'll say that A is coseparable in B (or that B is a coseparable superspace of A) if B/A is separable.
- ▶ Every object $F : Z \curvearrowright Y$ admits extension to any superspace W where Z is complemented: if $\pi : W \to Z$ is a projection then $F\pi$ extends F. In particular, the coproduct $\bigoplus F_i$ in $\mathfrak{Z}_Y$ of a family $(F_i)_i$ containing F is an extension of F. In a sense, this extension is not interesting since both are ismorphic objects in $\mathfrak{Z}$.
- ▶ Every trivial object $T : Z \curvearrowright Y$ extends to any superspace W of Z: just take $0 : W \curvearrowright Y$. Following lemma 1.1, T admits exact extension.

A natural question in this context is:

Question: When a trivial object admits a nontrivial extension? The answer is obviously "never" if we want to extend $F : Z \curvearrowright Y$ to a superspace W for which $\mathcal{Z}(W, Y) = 0$. We know no general answer further of some few particular situations. Let us remark that no trivial object admits a singular extension.

1.1. The extension problem in terms of exactness of the functor $\mathcal{Z}(\cdot, \clubsuit)$. Let us consider $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0 \equiv F$. Let us recall that the problem of the extension of all operators $Y \to \diamondsuit$ to X through a given embedding j admits a formulation in terms of the exactness of the functor $\mathcal{L}(\cdot, \diamondsuit)$ on F: is the sequence

$$0 \longrightarrow \mathcal{L}(Z, \diamondsuit) \xrightarrow{q^*} \mathcal{L}(X, \diamondsuit) \xrightarrow{j^*} \mathcal{L}(Y, \diamondsuit) \longrightarrow 0$$

exact at $\mathcal{L}(Y, \diamondsuit)$; namely, is the morphism j^* induced by j is surjective. The analogous problem of the extension of z-linear maps $Y \curvearrowright \diamondsuit$ to X through an embedding j can be described in terms of the exactness of the functor $\mathcal{Z}(\cdot, \diamondsuit)$ on F:

Problem 1. Is the sequence

$$\mathcal{Z}(Z, \diamondsuit) \xrightarrow{q^*} \mathcal{Z}(X, \diamondsuit) \xrightarrow{j^*} \mathcal{Z}(Y, \diamondsuit) \longrightarrow 0$$

exact in $\mathcal{Z}(Y, \diamondsuit)$? That is, is j^* surjective? In particular, it is quite natural the question:

Problem 2. Does the exactness of $\mathcal{L}(\cdot, \diamondsuit)$ on F imply the exactness of $\mathcal{Z}(\cdot, \diamondsuit)$ on F?

In this chapter we shall focus on the problem of extension of z-linear maps with range $C(K)$, K a compact Hausdorff. So, after having investigated the Lindenstrauss-Pelczynski theorem in Chapter 3 (let us recall, *the functor $\mathcal{L}(\cdot, C(K))$ is exact on every exact sequence* $0 \to H \xrightarrow{j} c_0 \xrightarrow{q} c_0/H \to 0 \equiv F$), we ask now if $\mathcal{Z}(\cdot, C(K))$ is exact on F; equivalently:

Problem LP2: Is exact the sequence

$$0 \longrightarrow \mathcal{Z}(c_0/H, C(K)) \xrightarrow{q^*} \mathcal{Z}(c_0, C(K)) \xrightarrow{j^*} \mathcal{Z}(H, C(K)) \longrightarrow 0?$$

From now on we shall refer to this as the *Lindenstrauss-Pelczynski problem of orden 2* (in short, LP2).

The same questions can be posed with respect to $w(l_1, c_0)$-continuous embeddings in l_1, which corresponds in *level 1* to the Johnson-Zippin theorem (see Chapter 3). This extension problem of z-linear maps could be called the *Johnson-Zippin problem of order 2.* However, in this case the answer comes immediately since a subspace M of l_1 which is the dual of a quotient of c_0 is also the dual of a subspace of c_0, hence $\mathcal{Z}(M, C[0,1]) = 0$.

Still, there is a unifying, quite natural, formulation for the problems of extension of operators and z-linear maps. Such formulation comes in terms of the *homology sequence.*

2. Elementary homology at level 2

The existence of the homology sequence in the categories **B** of Banach spaces and **Q** of quasi-Banach spaces can be seen explicitly constructed in [**11**]. We recall here that given an exact sequence $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0 \equiv F$ and a Banach space $\diamondsuit$ there is an exact infinite sequence

$$\begin{array}{lllllll} 0 & \longrightarrow & \mathcal{L}(Z,\diamondsuit) & \xrightarrow{q^*} & \mathcal{L}(X,\diamondsuit) & \xrightarrow{j^*} & \mathcal{L}(Y,\diamondsuit) \\ & \xrightarrow{F_1^*} & \mathcal{Z}(Z,\diamondsuit) & \xrightarrow{q^*} & \mathcal{Z}(X,\diamondsuit) & \xrightarrow{j^*} & \mathcal{Z}(Y,\diamondsuit) \\ & \xrightarrow{F_2^*} & \mathcal{Z}^2(Z,\diamondsuit) & \xrightarrow{q^*} & \mathcal{Z}^2(X,\diamondsuit) & \xrightarrow{j^*} & \mathcal{Z}^2(Y,\diamondsuit) \longrightarrow \cdots \end{array}$$

in the category **Vect** of vector spaces. The morphism F_1^* is called *first connecting morphism* and comes defined by $F_1^*(T) = TF$. The morphism F_2^* is called *second connecting morphism* and we shall wait a little to give its definition. We shall use the notation $\mathcal{Z}^2(A,B)$ to refer to the space, well known to algebraists (see [**32, 59, 74**]), $\mathrm{Ext}^2(A,B)$. The reason of this choice is not only aesthetic, it is due to the fact that the elements of $\mathrm{Ext}^2(A,B)$ can be described by the composition of two z-linear maps, modulus a certain equivalence equivalence relation (corresponding to Yoneda's for four term exact sequences). Accepted this, the second connceting morphism shall be $F_2^*(G) = GF$.

The fact that we only present the homology sequence up to level two (technically, up to the second derived functor of Hom) is because that is everything we need to treat the extension problems in which we are interested. Indeed, Banach space theory occurs inside those three first sections of the homology sequence: the **classical theory** occurs inside *level* 0, the section that corresponds to the functor $\mathcal{L}(\cdot,\diamondsuit)$ is where problems related to spaces and operators occur; the theory of exact sequences occurs in *level* 1, the section corresponding to the functor $\mathcal{Z}$; about *level* 2, corresponding to the functor $\mathcal{Z}^2$, it is much more what is unknown than what is known (a theory explicitly developed for the functor $\mathcal{Z}^{\mathbf{2}} : \mathbf{B}\times\mathbf{B} \to \mathbf{Set}$ does not exist). Since everything remains to be done, let us introduce in this section the basic elements of *homology at level 2.*

The questions **1** and **2** stated in the previous section can be formulated now in terms of the homology sequence as follows:
Problem 1: Which pairs $(F, \diamondsuit)$ verify $F_1^* = 0$?
Problem 2: Which pairs $(F, \diamondsuit)$ verify that $F_1^* = 0$ implies $F_2^* = 0$?

To study the extension problem for z-linear maps in these terms it turns out to be essential to know the meaning of 0 in $\mathcal{Z}^2(Q,Y)$. That is our next objective.

2.1. The space $\mathcal{Z}^{\mathbf{2}}(\mathbf{A},\mathbf{B})$. Given two objects $F : Z \curvearrowright B$ and $G : A \curvearrowright Z$ of $\mathfrak{Z}$,

$$\begin{array}{lllllllll} 0 & \longrightarrow & B & \longrightarrow & X & \xrightarrow{q} & Z & \longrightarrow & 0 \equiv F \\ 0 & \longrightarrow & Z & \xrightarrow{j} & W & \longrightarrow & A & \longrightarrow & 0 \equiv G, \end{array} \tag{9}$$

that we combine in one object of $\mathcal{Z}^2(Q,Y)$:

$$0 \longrightarrow B \longrightarrow X \xrightarrow{q} Z \xrightarrow{j} W \longrightarrow A \longrightarrow 0 \equiv FG$$

where we write $\equiv FG$ to remark that the new object has been, in some sense, constructed by composition of a representative of F and one of G. For this reason we shall sometimes write $FG : A \curvearrowright\curvearrowright B$ to refer to it.

We can represent FG as an exact sequence through the diagram induced by the composition jq:

$$0 \longrightarrow B \longrightarrow X \xrightarrow{jq} W \longrightarrow A \longrightarrow 0 \equiv FG.$$

One has:

Lemma 2.1. *The following are equivalent*

(1) *F extends to W through j*
(2) *G lifts to X through q*
(3) *There is a commutative diagram*

(10)
$$\begin{tikzcd}
 & & \widetilde{G} \arrow[d, phantom, "\equiv"] & G \arrow[d, phantom, "\equiv"] & \\
 & & 0 & 0 & \\
 & & A \arrow[u] \arrow[r, phantom, "="] & A \arrow[u] & \\
0 \arrow[r] & B \arrow[r] & \square \arrow[r] \arrow[u] & W \arrow[r] \arrow[u] & 0 \equiv \widehat{F} \\
0 \arrow[r] & B \arrow[r] \arrow[u, equal] & X \arrow[r, "q"] \arrow[u] & Z \arrow[r] \arrow[u, "j"] & 0 \equiv F \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

PROOF. The implications $1, 2 \Longrightarrow 3$ are clear since the morphisms $\widehat{F} \longleftarrow F$ and $\widetilde{G} \to G$ in $\mathfrak{Z}$ induced by j and q, respectively, yield the same diagram 2.1. Reciprocally, from the existence of a diagram like 2.1 it follows that $\widehat{F}$ is an extension of F through j and $\widetilde{G}$ is a lifting of G through q. $\square$

It is worth to remark that the *concatenation* FG of objects F and G of $\mathfrak{Z}$ verifies the conditions of the theorem if either F or G (or both) are trivial.

Given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \equiv FG \\
0 \arrow[r] & B \arrow[r] & X' \arrow[r] & W' \arrow[r] & A \arrow[r] & 0 \equiv F'G'
\end{tikzcd}$$

we shall write $FG \looparrowright F'G'$. If the arrows go in the opposite direction we write $FG \looparrowleft F'G'$. The relation $\looparrowright$ is not an equivalence relation since it is not symmetric. The reader can go to any textbook [**32, 59**] to check that the associated equivalence relation $\bumpeq$ of four-term exact sequences comes defined by:

$$FG \bumpeq F'G' \Longleftrightarrow \exists\, F_1G_1 \;:\; \text{or } FG \looparrowright F_1G_1 \looparrowleft F'G' \quad \text{ó} \quad FG \looparrowleft F_1G_1 \looparrowright F'G'$$

We shall denote by $\mathcal{Z}^2(A, B)$ the set of $\bumpeq$-equivalence classes of sequences $FG : A \to B$. Again, one can find in [**32, 59**] a natural way to endow $\mathcal{Z}^2(Q, Y)$ of a vector space structure. From this fact we are only interested in the identification of the element 0: it is precisely the class having a representative FG satisfying a diagram

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & A \arrow[u] \arrow[r, equal] & A \arrow[u] & \\
0 \arrow[r] & B \arrow[r] & \square \arrow[r] \arrow[u] & W \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[u, equal] & X \arrow[r] \arrow[u] & Z \arrow[r] \arrow[u, "j"] & 0 \equiv F \\
 & & 0 \arrow[u] & 0 \arrow[u] \arrow[d, phantom, sloped, "\equiv"] & \\
 & & & G. &
\end{tikzcd}$$

That is, the element FG is 0 if the maps F and G verify the conditions of Lemma 2.1. It is clear that if either F or G are trivial, then $FG \backsim 0$. The converse is not true; it is enough to consider a singular application $F : Z \curvearrowright Y$ and making pull-back to an uncomplemented subspace $j : H \hookrightarrow Z$. Taking now the map $G : Z/jH \curvearrowright H$ induced by j, we obtain an element $(Fj)G \backsim 0$ de $\mathcal{Z}^2(Z/jH, Y)$ such that $Fj \not\equiv 0 \not\equiv G_j$. There exist, however, examples of elements FG for which the implication $[FG \backsim 0] \Longrightarrow$ [o bien $G = 0$ ó $F = 0$] holds:

Observation1. Let I be an injective space. Let us consider the following element of $\mathcal{Z}^2(A, B)$:

$$0 \longrightarrow B \longrightarrow I \xrightarrow{q} I(B) \longrightarrow X \longrightarrow A \longrightarrow 0 \equiv \mathcal{I}_B G.$$

Then:

$$\mathcal{I}_B G \backsim 0 \Longleftrightarrow G \text{ lifts through } q \Longleftrightarrow G \equiv 0.$$

Observation2. Let P be a projective Banach space. The following element of $\mathcal{Z}^2(A, B)$:

$$0 \longrightarrow B \longrightarrow X \longrightarrow K(A) \xrightarrow{j} P \longrightarrow A \longrightarrow 0 \equiv G\mathcal{P}_A.$$

verifies:

$$G\mathcal{P}_A \backsim 0 \Longleftrightarrow G \text{ extends through } j \Longleftrightarrow G \equiv 0.$$

Keeping in mind all this elements of level 2 homology, it makes full meaning the sentence "the second connecting morphism $F_2^* = 0$" which appears in the homological formulations of problems **1**. y **2**. at the beginning of the section.

2.2. Characterization theorem for $\mathcal{Z}^2(\mathbf{A}, \mathbf{B})$. The result we prove now, well-known from homological algebra, provides a tool to give useful definitions of the second derived functor in terms of the first derived functor. We shall use it to reduce the problem of extension of z-linear maps to a problem of extension of operators.

THEOREM 2.1. *Let A and B be two Banach spaces. Given a projective presentation $\mathcal{P}_A : A \curvearrowright K(A)$ and an injective presentation $\mathcal{I}_Y : I(B) \curvearrowright B$ then one has*

$$\mathcal{Z}^2(A, B) \simeq \mathcal{Z}(K(A), B) \simeq \mathcal{Z}(A, I(B)).$$

PROOF. During the proof $\simeq$ means "isomorphism" of vector spaces.

- $\mathcal{Z}^2(\mathbf{A}, \mathbf{B}) \simeq \mathcal{Z}(\mathbf{K}(\mathbf{A}), \mathbf{B})$: Let $\mathcal{P}_A : A \curvearrowright K(A)$ be a projective presentation of A. We define an application $\theta : \mathcal{Z}^2(A, B) \to \mathcal{Z}(K(A), B)$ as

$$\theta(FG) = F\psi_G,$$

where ψ_G is any operator representing the morphism $\mathcal{P}_A \to G$ en $\mathfrak{Z}$ as it is shown in the diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & B & \longrightarrow & PB & \longrightarrow & K(A) & \longrightarrow & P & \longrightarrow & A & \longrightarrow & 0 \equiv \mathcal{P}_A\theta(FG) \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \psi_G} & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & D & \xrightarrow{q} & E & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \equiv FG, \end{array}$$

obtained making the pull-back $F\psi_G$. It is not difficult to check that θ is well-defined: $\theta(FG)$ does not depend neither on the choice of the representative ψ_G of the morphism $\mathcal{P}_B \to G$; nor on the choice of the representatives F, G of the element of $\mathcal{Z}^2(A, B)$. The inverse application shall be $\chi : \mathcal{Z}(K(A), B) \to \mathcal{Z}^2(A, B)$ given by

$$\chi(H) = H\mathcal{P}_A.$$

Again, χ is also well defined. Let us consider on $\mathcal{Z}^2(A, B)$ the vector space structure *imported* from $\mathcal{Z}(K(A), B)$ through the bijection θ. This precisely coincides with that is naturally defined in $\mathrm{Ext}^2(A, B)$. It is clear now that θ is linear.

- $\mathcal{Z}^{\mathbf{2}}(\mathbf{A}, \mathbf{B}) \simeq \mathcal{Z}(\mathbf{A}, \mathcal{I}(\mathbf{B}))$: Let $\mathcal{I}_B : I(B) \curvearrowright B$ be an injective presentation of B. We define the map $\omega : \mathcal{Z}^2(A, B) \to \mathcal{Z}(A, I(B))$ by means of

$$\omega(FG) = \xi_F G,$$

where ξ_F is any operator representing the morphism $\mathcal{I}_B \leftarrow F$ of $\mathfrak{Z}$ as it shows the diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & B & \longrightarrow & I & \longrightarrow & I(B) & \longrightarrow & PO & \longrightarrow & A & \longrightarrow & 0 \equiv \mathcal{I}_B\omega(FG) \\ & & \| & & \uparrow & & \uparrow{\scriptstyle \xi_F} & & \uparrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & D & \longrightarrow & E & \xrightarrow{j} & C & \longrightarrow & A & \longrightarrow & 0 \equiv FG, \end{array}$$

constructed making the push-out $\xi_F G$. It is easy to check that ω is well defined and does not depend neither on the representative ξ_F of the morphism $\mathcal{I}_B \leftarrow F$, nor on the representative FG chosen. The inverse map shall be $\zeta : \mathcal{Z}(A, I(B)) \to \mathcal{Z}^2(A, B)$ defined by

$$\zeta(H) = \mathcal{I}_B H.$$

Since the natural vector space structure of $\mathcal{Z}^2(A, B)$ coincides with that *imported* from $\mathcal{Z}(A, \mathcal{I}(B))$ through ω, this map is linear. $\square$

COROLLARY 2.1. *Given a Banach space X the following conditions are equivalent:*

1. *Every z-linear map $F : \heartsuit \curvearrowright X$ extends to any superspace $\heartsuit$.*
2. *For every Banach space $\diamondsuit$ one has $\mathcal{Z}^2(\diamondsuit, X) = 0$.*
3. *For every subspace $K \subset l_1(\Gamma)$ one has $\mathcal{Z}(K, X) = 0$*
4. *Every quotient $I(X)$ is injective.*

It is worth to recall also the"separable version":

COROLLARY 2.2. *Given a Banach space X the following are equivalent*

1. *Every z-linear map $F : \heartsuit \curvearrowright X$ extends to any coseparable superspace of $\heartsuit$.*
2. *For every separable $\diamondsuit$ one has $\mathcal{Z}^2(\diamondsuit, X) = 0$.*
3. *For every subspace K of l_1 one has $\mathcal{Z}(M, X) = 0$.*
4. *Every quotient $I(X)$ is separably injective.*

An interesting example that shows that those two cases are different is given by the extension $0 \to c_0 \to l_\infty \to l_\infty/c_0 \to 0 \equiv \mathcal{I}_0$. It is well-known that l_∞/c_0 is separably injective (see Corollary 2.3 below) but not injective; it is thus clear that $\mathcal{I}_0$ can be extended to any coseparable superspace of l_∞/c_0, although there exist superspaces where it does not extend. In particular, $\mathcal{I}_0$ cannot be extended to any injective superspace $l_\infty(\Gamma)$; otherwise, we'll have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & c_0 & \longrightarrow & \Box & \longrightarrow & l_\infty(\Gamma) & \longrightarrow & 0 \equiv G \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & c_0 & \xrightarrow{i} & l_\infty & \longrightarrow & l_\infty/c_0 & \longrightarrow & 0 \equiv \mathcal{I}_0,
\end{array}$$

from which we deduce that the diagonal pull-back sequence iG is trivial (since l_∞ is injective). Thus, l_∞/c_0 would be complemented in some injective space $l_\infty \oplus l_\infty(\Gamma)$, which is impossible.

That l_∞/c_0 is separably injective is, in addition to a well-known result, a particular case of the following corollary of Theorem 2.1:

COROLLARY 2.3. *The quotient SI/SI' of two separably injective spaces SI and SI' is separably injective.*

PROOF. Let us consider the sequence $0 \to SI' \to SI \to SI/SI' \to 0 \equiv E$ and let $F : S \curvearrowright SI/SI'$ be a z-linear map defined on a separable Banach space S. The image of F by the isomorphism $\theta : Ext^2(S, SI') \to \mathcal{Z}(K(S), SI')$ is described by the diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & SI' & \longrightarrow & PB & \longrightarrow & K(S) & \longrightarrow & l_1 & \longrightarrow & S & \longrightarrow & 0 \equiv \theta(EF)\mathcal{P}_S \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle\phi} & & \downarrow & & \| & & \\
0 & \longrightarrow & SI' & \longrightarrow & SI & \xrightarrow{q} & SI/SI' & \longrightarrow & X & \longrightarrow & S & \longrightarrow & 0 \equiv EF.
\end{array}$$

Since SI' is separably injective , the extension $\theta(EF)$ is trivial and thus $\theta(EF)\mathcal{P}_S \backsimeq 0 \backsimeq EF$, and therefore F is trivial. $\square$

We can fix for later use some names: we'll say that a space X is **2-injective** (resp. **2-separably injective**) if $\mathcal{Z}^2(\cdot, X) = 0$ (resp. $\mathcal{Z}^2(S, X) = 0$ for each separable space S). General properties of the homology sequence immediately yield that every injective space is 2-injective and every separably injective space is 2-separably injective. The last theorem implies that X is 2-injective if and only if some/every quotient $I(X)$ is injective; and X is 2-separably injective if and only if some/every quotient $I(X)$ is separably injective.

PROPOSITION 2.1. *There exist 2-injective spaces that are not even $\mathcal{L}_\infty$-spaces.*

PROOF. Quite clearly, the task is to find non $\mathcal{L}_\infty$-spaces Y having injective quotients $I(Y)$. Let us recall the existence of a sequence

$$0 \longrightarrow D \longrightarrow c_0 \longrightarrow c_0 \longrightarrow 0 \equiv B_0$$

predual of the nontrivial sequence $B_0^* \equiv B$ constructed by Bourgain in [**4**] that provides an uncomplemented copy of l_1 inside l_1. The space D^{**} verifies $l_\infty/D^{**} = l_\infty$. Thus, every z-linear map $F : \heartsuit \curvearrowright D^{**}$ extends to any Banach superspace. On the other hand, D is not a $\mathcal{L}_\infty$-space because $D^* = Q$ is not a $\mathcal{L}_1$-space since the dual sequence $0 \to l_1 \to l_1 \to Q \to 0 \equiv B_0^*$ does not split. $\square$

Further interesting examples of 2-injective but not injective spaces, although this time of type $\mathcal{L}_\infty$ are provided by Rosenthal sequences constructed in [**73**]:

$$0 \longrightarrow \mathcal{J}_G \longrightarrow l_\infty \longrightarrow C(G) \longrightarrow 0,$$

where $C(G)$ an injective space of continuous functions which is not a dual. Therefore, every object $F : \heartsuit \curvearrowright \mathcal{J}_G$ extends to any Banach superspace.

3. Equivalent formulations of the problem of extension for z-linear maps

We are going to introduce now several notions which shall help us to give different formulations of the extension problem for z-linear maps.

3.1. Projective presentation in $\mathfrak{Z}$. Up to now we have used the notation $K(Z)$ to denote the kernel of a quotient $l_1(\Gamma) \to Z$ without entering into the problem of the nature of the correspondence $Z \to K(Z)$. It could be understood, not without some difficulties, as a functor $\mathbf{B} \to \mathbf{Set}$; however, the right interpretation of $K(\cdot)$ follows after a classical result, see [**32**]; we shall present such result explicitly in the category $\mathbf{B}$, although in an alternate form.

LEMMA 3.1. *Let $F : Z \curvearrowright Y$ be a z-linear map and let $\mathcal{P}_Z : Z \curvearrowright K(Z)$, $\mathcal{P}_Y : Y \curvearrowright K(Y)$ be projective presentations of Z and Y, respectively. There is a commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K(Y) \arrow[r] \arrow[d] & P' \arrow[r, "q'"] \arrow[d] & Y \arrow[r] \arrow[d, "i"'] & 0 \equiv \mathcal{P}_Y \\
0 \arrow[r] & K(Y \oplus_F Z) \arrow[r] \arrow[d] & P \arrow[r, "Q"] \arrow[d] & Y \oplus_F Z \arrow[r] \arrow[d, "q"'] & 0 \equiv \mathcal{P}_{Y \oplus_F Z} \\
0 \arrow[r] & K(Z) \arrow[r] \arrow[d] & P'' \arrow[r, "q''"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \equiv \mathcal{P}_Z \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

PROOF. We choose a lifting ψ of q'' to X through q. We define a linear continuous map $Q : P' \oplus P'' \to Y \oplus_F Z$ by means of $Q(p', p'') = iq'p' - \psi p''$. It is easy to check that Q is surjective. In particular, the induced sequence

$$0 \longrightarrow PB \longrightarrow P' \oplus P'' \xrightarrow{Q} Y \oplus_F Z \longrightarrow 0 \equiv \mathcal{P}_{Y \oplus_F Z}$$

is just the diagonal sequence associated to the pull-back diagram

$$\begin{tikzcd}
P'' \arrow[r, "\psi"] & Y \oplus_F Z \\
PB \arrow[r] \arrow[u] & P'. \arrow[u, "iq'"']
\end{tikzcd}$$

It can be verified without difficulty that $\mathcal{P}_{Y \oplus_F Z}$ is precisely the projective presentation of $Y \oplus_F Z$ appearing in the statement of the result. $\square$

Let us observe that this lemma is actually defining a correspondence: to each application $F : Z \curvearrowright Y$ assigns a z-linear map $\mathcal{K}(F) : K(Z) \curvearrowright K(Y)$ associated to the exact sequence of the kernels

$$0 \longrightarrow K(Y) \xrightarrow{k(j_F)} K(Y \oplus_F Z) \xrightarrow{k(q_F)} K(Z) \longrightarrow 0 \equiv \mathcal{K}(F)$$

as in the diagram. Still, this correspondence can be described in a cleaner form: the object $\mathcal{K}(F)$ is just the image by $\theta : \mathcal{Z}^2(Z, K(Y)) \to \mathcal{Z}(K(Z), K(Y))$ of $\mathcal{P}_Y F$; that is, the $\equiv$-equivalence class of the extension S appearing in the diagram:

(11)

$$\begin{tikzcd}
0 \arrow[r] & K(Y) \arrow[r] \arrow[d, equal] & PB \arrow[r] \arrow[d] & K(Z) \arrow[r] \arrow[d, "\phi"] & P'' \arrow[r, "q''"] \arrow[d, "\psi"] & Z \arrow[r] \arrow[d, equal] & 0 \equiv S\mathcal{P}_Z \\
0 \arrow[r] & K(Y) \arrow[r] & P' \arrow[r, "q"] & Y \arrow[r] & Y \oplus_F Z \arrow[r] & Z \arrow[r] & 0 \equiv \mathcal{P}_Y F.
\end{tikzcd}$$

It is not hard to check that S is indeed equivalent to the application $\mathcal{K}(F)$ obtained as the Lemma 3.1 indicates. Thus, fixing projective presentations for each Banach space we have obtained a new functor:

PROPOSITION 3.1. *The correspondence*

$$\mathfrak{Z} \xrightarrow{\mathcal{K}(\cdot)} \mathfrak{Z},$$

that assigns to each object F the object $\mathcal{K}(F)$ induces a functor.

PROOF. That $\mathcal{K}(\cdot)$ is well defined is clear since the morphism θ was also well-defined. Moreover, it verifies $\mathcal{K}(0) = 0$. It can be checked without special difficulty that $\mathcal{K}(\cdot)$ is indeed a functor; that is, given any morphism $(\alpha, \gamma) : F \rightrightarrows G$ in $\mathfrak{Z}$ there is a morphism $\mathcal{K}(\alpha, \gamma) : \mathcal{K}(F) \rightrightarrows \mathcal{K}(G)$. □

Looking again the Diagram 11 we get more: different projective presentations $\mathcal{P}'_Y : Y \curvearrowright K'(Y)$ y $\mathcal{P}'_Z : Z \curvearrowright K'(Z)$ yield maps $K'(Z) \curvearrowright K'(Y)$ isomorphically equivalent to $\mathcal{K}(F)$. Thus, $\mathcal{K}(\cdot)$ can also be understood as a functor $\mathfrak{Z} \to \mathbf{Set}$ assigning to each object F the isomorphy class $\lfloor K(F) \rceil$.

Let us also observe that when $F \equiv F'$ one has $\mathcal{P}_{Y \oplus_F Z} \sim \mathcal{P}_{Y \oplus_{F'} Z}$. This remark, and the fact that $\mathcal{K}(\cdot)$ is well defined, means that we shall not distinguish from now on the diagrams constructed after lemma 3.1 for equivalent maps $F \equiv F'$. It has therefore meaning to speak of **projective presentations of an object** F that we shall denote $\mathcal{P}(F)$.

It is interesting the interpretation in the category $\mathfrak{Z}$ that the projective presentation of an object F has; actually we can represent $\mathcal{P}_F$ in two different forms as a composition of morphisms of $\mathfrak{Z}$:

$$\triangleright \qquad \mathcal{P}(F) :\ \mathcal{P}_Y \rightrightarrows \mathcal{P}_{Y \oplus_F Z} \rightrightarrows \mathcal{P}_Z$$

$$\triangleright \qquad \mathcal{P}(F) :\ \mathcal{K}(F) \rightrightarrows 0 \rightrightarrows F,$$

where 0 is the trivial object $0 : P'' \to P'$.

By dualization of the statement and proof of Lemma 3.1 it is easy to obtain a functor

$$\mathfrak{Z} \xrightarrow{\mathcal{I}(\cdot)} \mathfrak{Z}$$

that assigns, after fixing injective presentations for each Banach space, to each object $F : Z \curvearrowright Y$ $\mathfrak{Z}$ the object

$$0 \longrightarrow I(Y) \longrightarrow I(Y \oplus_F Z) \longrightarrow I(Z) \longrightarrow 0 \equiv \mathcal{I}(F)$$

where $I(Y)$ and $I(Z)$ are the quotient spaces I/Y and I/Z, respectively. If one prefers $\mathcal{I} : \mathfrak{Z} \to \mathfrak{Z}$ can be described in terms of the isomorphism $\zeta : \mathcal{Z}^2(A, B) \to \mathcal{Z}(A, I(B))$; in these terms $\mathcal{I}$ assigns to each F the object $\zeta(F\mathcal{I}_Z)$.
The dual diagrams constructed in this case shall be called *injective presentations of the object* F and shall be denoted $\mathcal{I}(F)$. These injective presentations can be represented through the composition of morphisms in two different ways:

$$\triangleright \qquad \mathcal{I}(F) :\ \mathcal{I}_Y \rightrightarrows \mathcal{I}_{Y \oplus_F Z} \rightrightarrows \mathcal{I}_Z$$

$$\triangleright \qquad \mathcal{I}(F) :\ F \rightrightarrows 0 \rightrightarrows \mathcal{I}(F),$$

Returning to the problem of extension of z-linear maps, it is quite curious that the form in which injective and projective presentation are involved are rather different.

3.2. The z-Dual of a Banach space. Following, in part, the theory of lipschitz maps [**3, 28**], of metric projections [**14**] and the construction of the Banach envelope of a quasi-Banach space, we define the **z-dual** of a Banach space X as the Banach space

$$X^z = [ZL(X, \mathbb{R}), Z(\cdot)]$$

of z-linear maps in canonical form with respect to a given Hamel basis (e_γ) formed by norm 1 elements. we have already seen in the previous chapter that the unit ball of X^z endowed with the w^*-topology is a compact space, and thus $X^z = [ZL(X,\mathbb{R}), Z(\cdot)]$ is a dual space. Its canonical predual can be easily described. To this end, let us consider the map

$$\Omega : X \curvearrowright (X^z)^*$$

defined as $\Omega(x)(F) = F(x)$. It is clear that $\Omega(x)$ is linear; its continuity follows from the estimate

$$\|\Omega(x)\| = \sup_{Z(F)\leq 1} |F(x)| \leq Z(F) \sum_\gamma |x_\gamma|.$$

It is also clear that Ω is z-linear (in fact $Z(\Omega) \leq 1$) and is in its canonical form.
The closed subspace spanned by the functionals $\Omega(x)$ in $(X^z)^*$ shall be denoted $co_z(X)$ and it is the canonical predual of X^z, as we show next: if $u : co_z(X) \to (X^z)^*$ is the canonical inclusion then $u^*_{|X^z}$ is an isomorphism:

- *It is injective*: Since $u^*(F) = 0 \Longrightarrow \forall x \in X\ F(x) = 0 \Longrightarrow F = 0$.
- *It s continuous*: since
$$\|u^*(F)\| = \sup_{\|\sum_i \lambda_i \Omega(x_i)\| \leq 1} |\sum_i \lambda_i \Omega(x_i)(F)| \leq Z(F)$$
- *It is surjective*: If $\mu \in co_z(X)^*$ then $F(x) = \mu(\Omega(x))$ defines a z-linear map such that $u^*(F) = \mu$.

We shall call co_z-**presentation** of X to the z-linear map $\Omega : X \curvearrowright co_z(X)$. We shall sometimes write Ω_X instead of Ω if we want to remark the dependence upon the space X. The *envelope* $co_z(\cdot)$ has several universal properties. The first one follows from being a predual of X^z:

Proposition 3.2. *Each z-linear map $F : X \curvearrowright \mathbb{R}$ in canonical form defines a unique linear continuous functional $\phi_F : co_z(X) \to \mathbb{R}$ in such a way that $F = \phi_F \Omega$.*

Something more is true: $\Omega : X \curvearrowright co_z(X)$ is an initial object in $\mathfrak{Z}$.

Proposition 3.3. *For each z-linear map $F : X \curvearrowright X'$ there is a morphism $\Omega \to F$ en $\mathfrak{Z}$.*

Proof. Just observe that if $u : co_z(X) \to (X^z)^*$ is the canonical inclusion then $u\Omega : X \curvearrowright (X^z)^*$ has the property that for each z-linear map $F : X \curvearrowright Y$ there is an operator $\phi_F : (X^z)^* \to (Y)^{**}$ given by $\phi_F(f)(y^*) = f(y^*F)$ in such a way that $\phi_F u\Omega \equiv \delta_Y F$, where $\delta_Y : Y \hookrightarrow Y^{**}$ is the canonical inclusion. It only remains to see that $\phi_{|co_z(X)}$ takes values in Y, which is easy:

$$\phi_F(\Omega(x))(y^*) = \Omega(x)(y^*F) = y^*(F(x)) = \delta_{F(x)}(y^*).$$

□

Finally, regarding functorial properties of $co_z(\cdot)$ we have:

Proposition 3.4. *The correspondence $\mathbf{co_z}(\cdot) : \mathbf{B} \to \mathbf{B}$ that assigns to a Banach space X the space $co_z(X)$ is a functor.*

Proof. Given an operator $T : X \to X'$ there is an operator $co_z(T) : co_z(X) \to co_z(X')$ that, moreover, induces a commutative diagram

$$\begin{array}{ccc} co_z(X) & \curvearrowleft & X \\ {\scriptstyle co_z(T)}\downarrow & & \downarrow{\scriptstyle T} \\ co_z(X') & \curvearrowleft & X'. \end{array}$$

□

It is easy to see that if $j : Y \hookrightarrow X$ is an embedding then also $co_z(j) : co_z(Y) \hookrightarrow co_z(X)$ is an embedding:

$\|co_z(j)(\sum \lambda_i \Omega_Y(y_i))\| = \sup_{Z_X(F)\leq 1} |\sum \lambda_i F(jy_i)| = \sup_{Z_Y(F)\leq 1} |\sum \lambda_i F(y_i)| = \|\sum \lambda_i \Omega_Y(y_i)\|$,

just recalling the existence of exact extension for scalar z-linear maps. However, it seems unlikely that $co_z(\ \cdot\ )$ is an exact functor $\mathbf{B} \to \mathbf{B}$: were $0 \to co_z(Y) \to co_z(X) \to co_z(Z) \to 0$ exact then it would be exact the sequence $0 \to Z^z \to X^z \to Y^z \to 0$ of their z-duals, something that does not necessarily happen. It is therefore interesting to observe that $co_z(\cdot)$ behaves as an "almost exact" functor $\mathbf{B} \to \mathbf{B}$:

PROPOSITION 3.5. *For each z-linear map in canonical form $F : X \curvearrowright X'$ there exists a z-linear map in canonical form $co_z(F) : co_z(X) \curvearrowright co_z(X')$ defined by $co_z(F) = \Omega\phi_F$.*

PROOF. Let $F : X \curvearrowright X'$ be a z-linear map in canonical form. We define $co_z(F)$ as the pull-back z-linear map $\Omega_Y \phi_F$ shown in the diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & co_z(Y) & \longrightarrow & PB & \longrightarrow & co_z(Z) & \longrightarrow & \boxtimes & \longrightarrow & Z & \longrightarrow & 0 \equiv co_z(F)\Omega_Z \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \psi_F} & & \downarrow{\scriptstyle \psi} & & \| & & \\
0 & \longrightarrow & co_z(Y) & \longrightarrow & \boxminus & \xrightarrow{q} & Y & \longrightarrow & Y \oplus_F Z & \longrightarrow & Z & \longrightarrow & 0 \equiv \Omega_Y F.
\end{array}$$

□

We shall write $\zeta(j) : co_z(Y) \to co_z(Y) \oplus_{co_z(F)} co_z(Z)$ to denote the embedding appearing in the extension $co_z(F)$. this last proposition allows us to interpret $\mathbf{co_z}(\cdot)$ as a functor $\mathfrak{Z} \to \mathfrak{Z}$ just as it happened with $\mathcal{K}(\cdot)$; for each morphism $F \rightrightarrows G$ en $\mathfrak{Z}$ there is a morphism $co_z(\alpha, \gamma) : co_z(F) \rightrightarrows co_z(G)$.

3.2.1. **Linearization of z-linear maps.** The functors $\mathcal{K}(\cdot)$ and $co_z(\cdot)$ can be understood as linearization processes for z-linear maps. Moreover, they are in some sense equivalent: since $\Omega_X : X \to co_z(X)$ is initial it must be isomorphic to the projective presentation $\mathcal{P}_X$ of X. Thus, the second diagonal principle yields an isomorphism $0_P \oplus \Omega \sim 0_T \oplus \mathcal{P}_X$, where $T = co_z(X) \oplus_\Omega X$. That is to say, the sequences in the following diagram are isomorphically equivalent

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P \oplus co_z(X) & \longrightarrow & P \oplus T & \longrightarrow & X & \longrightarrow & 0 \\
 & & \wr & & \wr & & \| & & \\
0 & \longrightarrow & T \oplus K(X) & \xrightarrow{i} & T \oplus P & \longrightarrow & X & \longrightarrow & 0.
\end{array}$$

3.3. The equivalence theorem. We begin with the formulation of the extension problem of a z-linear map F through an embedding j in terms of projective presentations, injective presentations and the envelope co_z. To this end, let $F : Y \curvearrowright E$ be an object of $\mathfrak{Z}$, $j : Y \to X$ an embedding, $G : Z \curvearrowright Y$ the object induced by j, and let us consider projective presentations $\mathcal{P}_Y$ of Y and $\mathcal{P}_X$ of X. Finally, let $\mathcal{I}_E$ be an injective presentation of E. One has:

THEOREM 3.1. *The following assertions are equivalent:*

1. *F extends to X through j*
2. *$G_2^*(F) = 0$, where $G_2^* : \mathcal{Z}(Y, E) \to \mathcal{Z}^2(Z, E)$ is the second connecting morphism.*
3. *Every representative $\psi : K(Y) \to E$ of the morphism $\mathcal{P}_Y \to F$ extends to through $k(j) : K(Y) \hookrightarrow K(X)$.*
4. *Every representative $\xi : Y \to I(E)$ of the morphism $\mathcal{I}_E \leftarrow F$ extends through $j : Y \hookrightarrow X$.*
5. *Every representative $\phi_F : co_z(Y) \to E$ of the morphism $\Omega_Y \to F$ extends through $co_z(j) : co_z(Y) \hookrightarrow co_z(X)$.*

PROOF. The equivalence $1 \iff 2$ has already been proved in lemma 2.1.

1 $\Longleftrightarrow$ **3**. Let us observe the diagram

$$\begin{tikzcd}
0 \arrow[r] & K(Y) \arrow[r] \arrow[d, "k(j)"'] & P' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "j"'] & 0 \equiv \mathcal{P}_Y \\
0 \arrow[r] & K(X) \arrow[r] & P \arrow[r] & X \arrow[r] \arrow[d, equal] & 0 \equiv \mathcal{P}_X \\
0 \arrow[r] & E \arrow[r] \arrow[d, equal] & \square \arrow[r] & X \arrow[r] & 0 \equiv \widehat{F} \\
0 \arrow[r] & E \arrow[r] & W \arrow[r] \arrow[u] & Y \arrow[r] \arrow[u, "j"] & 0 \equiv F.
\end{tikzcd}$$

Let $\psi : K(Y) \to E$ an operator such that $F \equiv \psi \mathcal{P}_Y$. If F admits an extension $\widehat{F}$ through j then there exists an operator $T : K(X) \to E$ such that $T\mathcal{P}_X \equiv \widehat{F}$. That means that

$$Tk(j)\mathcal{P}_Y \equiv T\mathcal{P}_X j \equiv \widehat{F}j \equiv F.$$

Thus, there is an extension S of $Tk(j) - \psi$ a P'. Since P' is complemented in P, take $\pi : P \to P'$ a projection. If $i_Y : K(Y) \to P'$ and $i_X : K(X) \to P$ are the canonical inclusions then $T - S\pi i_X : K(X) \to E$ is an extension of ψ through $k(j)$:

$$[T - S\pi i_X]k(j) = Tk(j) - S\pi i_X k(j) = Tk(j) - Si_Y = \psi.$$

Conversely, if there exists an operator $T : K(X) \to E$ that extends some representative ψ of $\mathcal{P}_Y \longrightarrow F$ then it is clear that $T\mathcal{P}_X$ extends F through j.

1 $\Longleftrightarrow$ **4**. The injective version of the previous diagram is

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, equal] & I \arrow[r, "Q"] & I(E) \arrow[r] & 0 \equiv \mathcal{I}_E \\
0 \arrow[r] & E \arrow[r] \arrow[d, equal] & \square \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "\upsilon"] & 0 \equiv \widehat{F} \\
0 \arrow[r] & E \arrow[r] & W \arrow[r] \arrow[u] & Y \arrow[r] \arrow[u, "j"] & 0 \equiv F.
\end{tikzcd}$$

It is clear that when a representative ξ of the morphism $\mathcal{I}_E \longleftarrow F$ admits an extension υ to X through j, the application $\mathcal{I}_E\upsilon$ is an extension to X of F. And conversely, given an extension $\widehat{F}$ of F there exists an operator $\upsilon : X \to I(E)$making commutative the diagram above. Let ξ be any representative of $\mathcal{I}_E \longleftarrow F$. There exists a lifting $W : Y \to I$ of $\upsilon j - \xi$ through $Q : I \to I(E)$. Since I is injective, there is an extension $R : X \to I$ of W trough j. Finally, one has that $\upsilon - QR$ is an extension of ξ through $j : Y \to X$;

$$\upsilon j - QRj = \upsilon j - QW = \xi.$$

1 $\Longleftrightarrow$ **5**. Considering the embedding $co_z(j)$ induced by j, we have a commutative diagram

$$\begin{tikzcd}
co_z(X) \curvearrowleft X : \Omega_X \\
co_z(Y) \curvearrowleft Y : \Omega_Y \arrow[u, "co_z(j)", "j"'] \arrow[d, "\phi_F"', equal] \\
E \curvearrowleft Y : F.
\end{tikzcd}$$

It is obviously true that if $\widehat{\phi_F}$ is an extension of a representative of the morphism ϕ_F to $co_z(X)$ through $co_z(j)$, the map $\widehat{\phi_F}\Omega_X$ is an extension of F through j:

$$\widehat{\phi_F}\Omega_X j \equiv \widehat{\phi_F} co_z(j)\Omega_Y \equiv F.$$

Let us see the converse: let $\chi : co_z(Y) \to E$ be a representative of the morphism $\Omega_Y \longrightarrow F$, i.e., $\chi\Omega_Y \equiv F$ or, equivalently, $\chi\Omega_Y = F+B+L$ for some maps $B, L : Y \to E$ bounded-homogeneous and linear, respectively. Since F extends through j, there exists an exact extension $\widehat{F}$ of $F+B+L$ (that is, $\widehat{F}j = F + B + L$). Let $\phi_{\widehat{F}} : co_z(X) \to E$ the operator such that $\phi_{\widehat{F}}\Omega_X = \widehat{F}$. One has

$$\phi_{\widehat{F}} co_z(j)\Omega_Y = \phi_{\widehat{F}}\Omega_X j = \widehat{F}j = \chi\Omega_Y.$$

The conclusion follows from the definition of Ω_Y. □

The equivalences of Theorem 3.1 reveal an important aspect of the relationships between the classical problem of extension of operators (level 1) and that of extension of z-linear maps (level 2)

4. The extension problem: dependence on the quality of the embedding

As we already mentioned in the introduction, the extension problem for z-linear maps admits two variables: **a)** The class $\mathfrak{Y}$ of *range* spaces where the maps take values, and **b)** the class $\mathfrak{I}$ of embeddings *through* which the maps are extended. We shall focus now on different aspects related with variable (b); in the next section we shall consider problems related with variable (a).

There are two types of embeddings $Y \to X$ which are especially interesting: those inducing almost-split extensions $0 \to Y \to X \to Z \to 0 \equiv G$ and those inducing locally split extensions. We saw in Corollary 2 of Lemma 1.1 that an extension $G : Z \curvearrowright Y$ almost-splits if and only if it is a pull-back of $\mathtt{C}_Y$; i.e., there is a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & C(B_{Y^*}) & \longrightarrow & Q(Y) & \longrightarrow & 0 \equiv \mathtt{C}_Y \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \equiv G
\end{array}$$

Analogously, G locally splits if and only if there is a pull-back diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & Y^{**} & \longrightarrow & H(Y) & \longrightarrow & 0 \equiv H_Y \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \equiv G
\end{array}$$

It immediately follows:

PROPOSITION 4.1. *Let $F : Y \curvearrowright \diamondsuit$ be a z-linear map.*

(1) *F extends to any superspace W in which Y is almost complemented if and only if F extends to $C(B_{Y^*})$ through the canonical embedding.*
(2) *F extends to any superspace W in which Y is locally complemented if and only if F extends to Y^{**} through the canonical embedding.*

Let us see now to which extent to extend a z-linear map through an embedding and to extend an operator through that same embedding are equivalent things. Consider the diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \diamondsuit & \longrightarrow & I & \longrightarrow & I(\diamondsuit) & \longrightarrow & PO & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle u} & & \uparrow & & \| & & \\
0 & \longrightarrow & \diamondsuit & \longrightarrow & \blacksquare & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \equiv FG
\end{array} \tag{12}$$

An application of theorem 3.1 yields that F extends to X if and only if the operator u extends to X. If we replace the injective presentation of $\Diamond$ by any other extension in such a way that we get a diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \Diamond & \longrightarrow & \square & \longrightarrow & E & \longrightarrow & PO & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow u & & \uparrow & & \| & & \\ 0 & \longrightarrow & \Diamond & \longrightarrow & \blacksquare & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & 0 \equiv FG. \end{array}$$

then to have an extension of u to X is a sufficient condition, but not necessary, to have an extension of F to X. The cases in which we have at our disposal extension of operators give us:

Proposition 4.2. *With the same notation as before there is extension of F to X in the following cases:*

a) *E is injective.*
b) *E is separably injective and Z is separable.*
c) *G splits.*
d) *G locally splits and u is weakly compact.*
e) *G almost-splits and E is a complemented subspace of some $C(K)$.*
f) *Y is a subspace of c_0, Z is separable and E is a space of type $\mathfrak{LP}$.*
g) *$X = l_1$, Y is $w(l_1, c_0)$ closed and E is an $\mathcal{L}_\infty$-space.*
h) *E is an $\mathcal{L}_\infty$-space and u is compact.*
i) *u is 2-summing.*

Let us see some examples:

b) Let us consider a sequence $0 \to N \to l_\infty \to l_2 \to 0$ and we choose an element $F \not\equiv 0$ of $\mathcal{Z}(N, c_0)$. From Proposition 4.2 it follows that F extends to l_∞ because l_∞/c_0 is separably injective; therefore, it also extends to any Banach superspace.

i) Every z-linear map $\mathcal{L}_\infty \curvearrowright N$ extends to any Banach superspace because the operators $\mathcal{L}(\mathcal{L}_\infty, l_2)$ are 2-summing.

b) Not every map $F : Y \curvearrowright \Diamond$ extends to superspaces X in which Y is almost-complemented: to see this it is enough to take an extension $0 \to N \to l_\infty \to Y \to 0 \equiv F$ in which Y is not an $\mathcal{L}_\infty$-space and then the extension $0 \to Y \xrightarrow{\delta} C(B_{Y^*}) \to Q(Y) \to 0$ to obtain that F cannot be extended through δ; otherwise, Y would be a complemented subspace of the product $l_\infty \oplus C(B_{Y^*})$ and would also be an $\mathcal{L}_\infty$-space.

The previous example is nothing but a particular case of some Observations we have already made: the injective presentation $\mathcal{I}_Y : I(Y) \curvearrowright Y$ extends to X if and only if $I(Y)$ is complemented in X. Dually, the projective presentation $\mathcal{P}_Z : Z \curvearrowright K(Z)$ lifts to X' if and only if $K(Z)$ is complemented in X'.

4.1. Extending to $\mathbf{c_0}$. A typical almost trivial sequence is $0 \to H \to c_0 \to c_0/H \to 0$. One of the problems we mentioned at the beginning of the chapter was about the possibility of extending to c_0 the objects F defined on its subspaces. From Proposition 4.2, f), and from the observation of Diagram 12 one has:

Corollary 4.1. *The quotient space $I(Y)$ has property $\mathfrak{LP}$ if and only if every z-linear map $F : H \curvearrowright Y$ defined on a subspace H of c_0 extends to c_0.*

Proof. It is clear from f) that if $I(Y)$ is of type $\mathfrak{LP}$ then F extends to c_0. Conversely, a look at the Diagram 12 when $\Diamond = Y$ and $G = F_0$ yields that given an operator $T : H \to I(Y)$ one has $\mathcal{I}_Y T F_0 \approx 0$, and thus T extends to c_0. $\square$

In fact, having extension to c_0 is equivalent to having extension to any coseparable superspace. After Sobzyck's theorem one has:

Proposition 4.3. *Let $F : Z \curvearrowright Y$ be an object of $\mathfrak{Z}$. If there is a morphism $G \longleftarrow F$ in $\mathfrak{Z}$ with $G : c_0 \curvearrowright Y$, the map F extends to any coseparable superspace of Z.*

5. Extending $C(K)$-valued z-linear maps

Recalling the results of extension of operators with range $C(K)$, we ask now about the extension of z-linear maps with range $C(K)$ through almost trivial maps. It would be convenient to keep in mind through the section that maps $F : Z \curvearrowright C(K)$ defined on separable spaces can be assumed without loss of generality to have range in $C[0,1]$, since it is possible to find a version of F having separable range.

Let us consider the vector space $Z(X,Y)$ endowed with the w*-topology we defined as: a sequence $(F_n)_n$ of elements of $\mathcal{Z}(X,Y)$ is w*-convergent to F if for every point $x \in X$ the sequence $(F_n(x))$ converges to $F(x)$ in Y. Let us recall that every element $F \in Z(X,\mathbb{R})$ is trivial, that is, the space $Z(X,\mathbb{R})$ admits a decomposition

$$Z(X,\mathbb{R}) = \mathbf{B}(X,\mathbb{R}) + X'$$

where $\mathbf{B}(X,\mathbb{R})$ is the space of homogeneous bounded maps $X \to \mathbb{R}$.

Let $j : Y \to X$ be an embedding. We observe that although the functor $Z(\cdot,\mathbb{R})$ is not exact, the induced map $j^* : Z(X,\mathbb{R}) \to Z(Y,\mathbb{R})$ is surjective since a trivial object extends (exactly) to any superspace. We shall say that a map $\omega : B_{Z(Y,\mathbb{R})} \to \lambda B_{Z(X,\mathbb{R})}$, $\lambda > 0$ is a $w^* - selector$ for j^* if ω is w^*-continuous and verifies $i^*\omega = id$. If it is necessary to remark the constant λ we shall say that it is a λ- w*-selector.

We consider now an extension $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0 \equiv F$. We are going to relate the existence of a w^*-continuous selector for j^* and the extension problem for $C(K)$-valued z-linear maps through j. We begin with an easy observation:

Lemma 5.1. *There is a w^*-continuous selector $\omega : B_{Z(Y,\mathbb{R})} \to \lambda B_{Z(X,\mathbb{R})}$ for j^* if and only if there is a w^*-continuous selection $\omega' : B_{ZL(Y,\mathbb{R})} \to \lambda B_{ZL(X,\mathbb{R})}$ for $j^* : ZL(X,\mathbb{R}) \to ZL(Y,\mathbb{R})$.*

Proof. It is clear that if we have a w^*-selector ω for $j^* : Z(X,\mathbb{R}) \to Z(Y,\mathbb{R})$ then $\omega'(F) = \omega(F) - L_{\omega(F)}$ is a w^*-continuous selection for $j^* : ZL(X,\mathbb{R}) \to ZL(Y,\mathbb{R})$. And, conversely, if there exists a w^*-continuous selection ω' para $j^* : ZL(X,\mathbb{R}) \to ZL(Y,\mathbb{R})$ (which we shall also call w^*-selector), then $\omega(F) = \omega'(F - L_F) + L_F l$, where $l : X \to Y$ is a linear retraction for j, is the new selector we were looking for. □

Theorem 5.1 (Zippin's lemma at level 2). *Let $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0 \equiv G$ be. Every z-linear map $F : Y \curvearrowright C(K)$ extends to a z-linear map $\widehat{F} : X \curvearrowright C(K)$ with $Z(\widehat{F}) \leq \lambda Z(F)$ if and only if there is a w^*-selector $B_{Y^z} \to \lambda B_{X^z}$ for j^*.*

Proof. Let $\omega : B_{Y^z} \to \lambda B_{X^z}$ be a w^*-selector for j^*. Given a z-linear map $F : Y \curvearrowright C(K)$ with $Z(F) \leq 1$, we can define the extension $G : X \curvearrowright C(K)$ of F through j:

$$G(x)(k) = \omega(\delta_k F)(x).$$

The map G is well-defined since, for each $x \in X$, $G(x)$ is a continuous function on K. Moreover, G is a z-linear map in its canonical form: the z-linearity G is a consequence of the estimate

$$Z(G) \leq \sup\{Z(\omega(\delta_k F) : k \in K\} \leq \lambda Z(F).$$

Finally, it is clear that G extends F.
To obtain the reciprocal we observe that it is only necessary to know how to extend the initial map $\Delta_Y : Y \curvearrowright C(B_{Y^z})$. Assume then that $\Theta : X \curvearrowright C(B_{Y^z})$ is a z-linear extension of Δ_Y a través de j. We can define the map $\omega : B_{Y^z} \to Z(\Theta)B_{X^z}$ as

$$\omega(F)(x) = \Theta(x)(F).$$

One clearly has $j^*\omega = id$; moreover, ω is w^*-continuous: if $w^* - \lim F_n = F$ then

$$\lim \omega(F_n)(x) = \lim \Theta(x)(F_n) = \Theta(x)(F) = \omega(F)(x).$$

because $\lim \Theta(x)(F_n) = \Theta(x)(f)$ since $\Theta(x)$ is w^*-continuous. □

The reader might have been surprised by the fact that a w^*-continuous extension process is not automatically guaranteed: after all, all z-linear maps $Y \to \mathbb{R}$ are trivial $F = B + L$, and therefore can be (exactly) extended to X (just taking $m : X \to Y$ a homogeneous bounded projection, $l : X \to Y$ a linear projection and putting then $\widehat{F} = Bm + Ll$). The problem is that this extension process is not, in general, w^*-continuous; that is, if $F = w^* - \lim F_n$ and $F_n = B_n + L_n$ then it does not automatically follow that $L = w^* - \lim L_n$.

On the other hand, choosing as L an optimal approximation to F, i.e., $\|B\| = \|F - L\| \leq Z(F)$ (see [**9**]) and since m can be chosen verifying $\|m\| \leq 1 + \varepsilon$ we get an estimate of $Z(\widehat{F})$

$$Z(\widehat{F}) = Z(Bm) \leq \|B\|\|m\| \leq Z(F)(1 + \varepsilon).$$

All this means that in order to have a w^*-continuous extension process it would be enough to have a w^*-continuous metric projection $Z(Y, \mathbb{R}) \to Y'$. But, as we have already seen in Chapter 4, that is exactly equivalent to the fact $\mathcal{Z}(Y, C(K)) = 0$ for all $C(K)$-spaces. If so, that clearly explains why there is extension.

Throughout this chapter we have seen that for every Banach space X there exist two initial objects in $\mathfrak{Z}^Z$: projective presentations and co_z-presentations of Z. Restricting our attention to the class $\mathfrak{Z}^Z_{C(K)}$ we have one more initial object which, moreover, belongs to the class: the $\mathcal{C}(\cdot)$-presentation of Z. Let us see that the extension problem for z-linear maps with range in a $C(K)$-space can be formulated in terms of those three presentations.

Theorem 5.2. *Let us consider* $0 \to Y \xrightarrow{j} X \xrightarrow{q} Z \to 0 \equiv G$ *and the associated objects* $0 \to K(Y) \xrightarrow{k(j)} K(X) \to K(Z) \to 0 \equiv \mathcal{K}(G)$ *and* $0 \to co_z(Y) \xrightarrow{co_z(j)} co_z(X) \to Q \to 0 \equiv \mathcal{S}$. *They are equivalent:*

(1) *There exists a λ-w^*-selector for $j^* : Z(X, \mathbb{R}) \to Z(Y, \mathbb{R})$.*
(1') *Every z-linear map $F : Y \curvearrowright C(K)$ extends through j (i.e., $G_2^* = 0$) in such a way that $Z(\widehat{F}) \leq \lambda Z(F)$.*
(2) *There exists a λ-w^*-selector for the quotient $k(j)^* : K(X)^* \to K(Y)^*$.*
(2') *The object $\mathcal{K}(G)$ is λ-almost trivial.*
(3) *There exists a λ-w^*-selector for the quotient operator $co_z(j)^* : X^z \to Y^z$.*
(3') *The object $\mathcal{S}$ is λ-almost trivial.*

Proof. It is clear that every statement and the statement *prime* are equivalent. The equivalence between (1) and (1') is Zippin's lemma of order 2 (Theorem 5.1), while the equivalence between (2) and (2′) and between (3) and (3′) is precisely Zippin's lemma. Moreover, the two statements in (1') follows from (1) and (2) in Theorem 3.1 (case $E = C(K)$); statement (2') follows from (3) in Theorem 3.1, and (3') follows from (5) in Theorem 3.1. All this gives the equivalences (**1**) $\Longleftrightarrow$ (**2**) $\Longleftrightarrow$ (**3**). $\square$

Let us remark that if we set $\lambda = 1$ there is one more equivalence: $j^* : Z(X, \mathbb{R}) \to Z(Y, \mathbb{R})$ *admits a 1-w^*-selector if and only if the natural embeddding $j^{**} : C(B_{Y^z}) \to C(B_{X^z})$ admits a projection.*

Proof. Look at the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C(B_{X^z}) \arrow[r] & \blacksquare \arrow[r] & X \arrow[r] & 0 \equiv \Delta_X \\
0 \arrow[r] & C(B_{Y^z}) \arrow[r] \arrow[u, "(j^*)^*"'] & \square \arrow[r] \arrow[u] & Y \arrow[r] \arrow[u, "j"] & 0 \equiv \Delta_Y.
\end{tikzcd}$$

If π is a retraction for j^{**} then, given a z-linear map $G \equiv \varphi\Delta_Y : Y \to C(K)$, the map $\widehat{G} = \varphi\pi\Delta_X$ is an extension of G to X:

$$\widehat{G}j \equiv \phi\pi\Delta_X j \equiv \phi\pi j^{**}\Delta_Y \equiv \phi\Delta_Y \equiv G.$$

Conversely, if ω is a 1-w^*-selector for j^*, the induced operator between the spaces of continuous functions $\omega^* : C(B_{X^z}) \to C(B_{Y^z})$ is a projection for j^{**}. $\square$

After Theorem 5.2 it is natural to wonder: ¿Do the functors $\mathcal{K}(\cdot)$ and $co_z(\cdot)$ respect almost-complementation? More precisely:

Question: If F is almost trivial, Is $\mathcal{K}(F)$ almost trivial? And, given an almost complemented subspace $j : Y \hookrightarrow X$, Is $co_z(j) : co_z(Y) \hookrightarrow co_z(X)$ almost complemented?

Observe that this question applied to $0 \to H \to c_0 \to c_0/H \to 0 \equiv F$ is just the statement of the problem **LP2**.

On the other hand, we have already seen in Proposition 4.1 that it would be enough to obtain a positive answer to the extension problem for z-linear maps $F : Z \curvearrowright C(K)$ through the natural embedding $Z \hookrightarrow C(B_{Z^*})$ to guarantee the extension of F to c_0.

Question: Does any z-linear map $X \curvearrowright C(K)$ extend to $C(B_{X^*})$?

6. Resumé of questions related to the problem to which this thesis converges

Throughout this thesis we have formulated several problems in different languages. It should be clear by now that many of them address to a single problem which we have called the *Lindenstrauss-Pelczynski problem of order* 2 or, in short, **LP2**: ¿Is it possible to extend a z-linear map $F : H \curvearrowright C[0,1]$ from a subspace H of c_0 to the whole space?

The LP2 problem would have an affirmative answer if any of the following problems stated through the thesis would have an affirmative answer:

(1) Every z-linear map $F : X \curvearrowright C(K)$ extends through the canonical inclusion $X \to C(B_{X^*})$.
(2) The space $l_\infty/C[0,1]$ is an $\mathcal{LP}$ space.
(3) $C[0,1]$ is 2-separably injective.
(4) $\mathrm{Ext}(K, C[0,1]) = 0$ holds for each subspace K of l_1.
(5) For each object $F : X \curvearrowright C[0,1]$ there is an object $G : C(K) \curvearrowright C[0,1]$ such that $G \longleftarrow F$.
(6) The functor $\mathcal{K}(\,\cdot\,)$ respects almost complementation.
(7) The functor $co_z(\,\cdot\,)$ respects almost trivial embeddings.

An affirmative answer to problem 1 would imply, in particular, that $l_\infty(\Gamma)/C(K)$ is a complemented subspace of some space of continuous functions on a compact space, being thus an $\mathcal{LP}$-space, answering 2 affirmatively. If 3 is true then $l_\infty/C[0,1]$ is separably injective; also, 4 follows from 3. The argument against 4 is the existence of subspaces K of $l_1(\Gamma)$ for which $\mathrm{Ext}(K, c_0) \neq 0$. The Lindenstrauss-Pelczynski theorem ensures that an affirmative answer to 5 yields the solution to LP2. It is conceivable that the functors $\mathcal{K}(\,\cdot\,)$ and $co_z(\,\cdot\,)$ respect almost-complementation. Against, it stands the diagram

$$\mathcal{P}(F) : \mathcal{K}(F) \rightrightarrows 0 \rightrightarrows F,$$

which , in some sense shows that between an object F and its projective presentation is 0, what makes difficult to pass significative information from one to the other.

Other far reaching problems would also give LP2 as a by-product:

- Obtain the z-dual of $c_0(A_n)$. In particular, we want a decomposition formula for $c_0(A_n)^z$ in terms of A_n^z. Against we have the plausible non-existence of product in $\mathfrak{Z}_{\mathbb{R}}$.
- Show the existence of a "functorial" Bartle-Graves continuous selection process; here functorial means that if we have a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{q_1} & Z_1 \\ {\scriptstyle p}\big\uparrow & & \big\uparrow{\scriptstyle q} \\ X_2 & \xrightarrow[q_2]{} & Z_2 \end{array}$$

composed of quotient operators then given a continuous selection $s_1 : Z_1 \to X_1$ for q_1 with norm $1+\varepsilon$ then there exists $s_2 : Z_2 \to X_2$, a continuous selection for q_2 with norm $1+\varepsilon$ in such a way $ps_2 = s_1 q$. It is not that difficult to get one with norm $2+\varepsilon$.

This functorial Bartle-Graves would provide a proof for LP2 analogous, at level 2, to the homological proof we presented for the Lindenstrauss-Pelczynski theorem. Moreover, it provides a proof for a stronger result that implies all the rest:
- Every z-linear map $F : X \curvearrowright C(K)$ extends to any coseparable superspace of X.

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