Criticality-induced universality in ratchets

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(Dated: June 3, 2010)

Conclusive mathematical arguments are presented supporting the ratchet conjecture [R. Chacón, J. Phys. A **40**, F413 (2007)], i.e., the existence of a universal force waveform which optimally enhances directed transport by symmetry breaking. Specifically, such a particular waveform is shown to be *unique* for both temporal and spatial biharmonic forces, and general (*non*-perturbative) laws providing the dependence of the strength of directed transport on the force parameters are deduced for these forces. The theory explains previous results for a great diversity of systems subjected to such biharmonic forces and provides a universal quantitative criterion to optimize *any* application of the ratchet effect induced by symmetry breaking of temporal and spatial biharmonic forces.

PACS numbers: 05.60.-k

Symmetry principles play a fundamental role in the laws of nature by constituting a synthesis of those regularities that are independent of the specific dynamics. Since the beginning of scientific thinking, most of these symmetry principles have been deeply associated with certain "principles of economy" of nature, such as the principle of least action and other variational principles [1]. Curie's principle [2] is just one example of such a symmetry principle. It has been applied for instance to the ubiquitous phenomenon of directed transport by symmetry breaking without any net external force−the so-called ratchet effect [3,4,5]. Directed ratchet transport (DRT) is nowadays understood as a result of the interplay of nonlinearity, symmetry breaking, and non-equilibrium fluctuations, where these fluctuations may include temporal noise [6], spatial disorder [7], and quenched temporal disorder [8]. The space-time symmetries of some generic equations of motion, which have to be broken to allow the appearance of DRT, have recently been proposed [9,10,11] in accordance with Curie's principle. While this symmetry analysis provides useful necessary conditions on the ac (temporal) forces and the static (spatial) potential for DRT to appear, no information at all concerning its strength can be obtained from such a symmetry analysis. This is not altogether surprising since Curie's principle *per se*, in contrast to variational principles, lacks a mathematical formulation connecting quantitatively a measure of the "degree of symmetry" of causes with that of effects, thus impeding the observation of its possible connection (if any) with some natural "principle of economy" [12]. In other words, Curie's principle must be considered as belonging to the philosophical rather than the scientific realm [13]. To overcome this difficulty in the context of DRT, there has recently been proposed a *quantitative* measure of the degree of symmetry breaking (DSB) on which the strength of DRT must depend [14]. It should be stressed that this quantitative relationship between cause (symmetry breaking) and effect (DRT)−the so-called *DSB mechanism* [14]−is absolutely absent in the formulation of Curie's principle. Once one assumes that the breakage of the relevant (for each particular equation of motion) space-time symmetries can be quantified, and that the strength of the resulting DRT (hereafter referred to as $\langle V\rangle$) is proportional to this degree of breakage, the following questions naturally arise: Can one find the parameter values of the periodic zero-mean forces involved that maximally break the relevant symmetries and thus optimally enhance DRT? Are these values universal, implying therefore the existence of a unique optimal force waveform?

In this Letter, a positive response to these questions is provided by completing the mathematical proof of the ratchet conjecture [14], i.e., the existence of a universal force waveform which optimally enhances DRT. The biharmonic temporal force $F(t')/\epsilon = \eta\cos(\omega t' + \varphi_1) + (1-\eta)\cos(2\omega t' + \varphi_2)$ and the biharmonic spatial potential $U(x)/\epsilon = \eta\sin(kx' + \varphi_1) + (1-\eta)\sin(2kx' + \varphi_2)$, $(\epsilon > 0, \eta \in [0,1])$, have been (and still are) overwhelmingly used as standard models in analytical research on ratchets [15]. A simple re-scaling of these biharmonic functions yields

$$f(t) = \eta\cos(t) + (1-\eta)\cos(2t + \varphi_{eff}), \tag{1}$$

$$g(x) = \eta\cos(x) + 2(1-\eta)\cos(2x + \varphi_{eff}), \tag{2}$$

where $t = \omega t' + \varphi_1, x = kx' + \varphi_1, f(t) = F(t)/\epsilon, g(x) = -(\epsilon k)^{-1} dU(x)/dx$, and $\varphi_{eff} = \varphi_2 - 2\varphi_1$ is the effective phase. Thus, the breakage of the three relevant symmetries associated with the forces (1) and (2) (i.e., the shift symmetry and the two reversal symmetries, see Refs. [6,9,10,11]) is controlled by only two parameters: η (relative amplitude of the two harmonics) and φ_{eff}. But, unfortunately, changing these parameters implies *also* changing the amplitude and symmetry of the positive and negative parts of the biharmonic forces with respect to the symmetric cases $\eta = \{0,1\}$ (see Figs. 1(a), 2(a), and 3(a)). Since the strength of any transport (induced by symmetry breaking or not, i.e., by non-zero-mean forces) depends upon the amplitude of the driving forces, one concludes that these two effects of changing η or φ_{eff} overlap, so that one will find it difficult to distinguish the contribution to transport that is *purely* due to symmetry breaking. A first step towards clarifying this problematic situation and answering the above questions has been to study the breakage of the relevant symmetries of the elliptic force

$$f_{ellip}(t) = \mathrm{sn}\left(Kt/\pi; m\right) \mathrm{cn}\left(Kt/\pi; m\right), \tag{3}$$

where $K \equiv K(m)$ is the complete elliptic integral of the first kind while $\mathrm{sn}\left(\cdot; m\right)$ and $\mathrm{cn}\left(\cdot; m\right)$ are Jacobian elliptic functions of parameter $m \in [0, 1]$ [16]. Now, m is the single parameter controlling the breakage of the relevant symmetries. Unlike functions (1) and (2), function (3) exhibits the advantageous property that its waveform changes while its amplitude and image remain constant, $f_{ellip}(t) \in [-1/2, 1/2], \forall t$, as the shape parameter m varies from 0 to 1 (see Fig. 1 in Ref. [14]). It has been demonstrated [14] that its maximal (with respect to the integration range) transmitted impulse over a half-period, $I[f] \equiv \left|\int_{T/2} f(t)\, dt\right|$, and its DSB are monotonously decreasing and increasing functions of m, respectively. This result has led to the conclusion that optimal enhancement of DRT is achieved when maximal effective (i.e., *critical*) symmetry breaking occurs, which is in turn a consequence of two reshaping-induced competing effects: the increase of the DSB and the decrease of the maximal transmitted impulse over a half-period, thus implying the existence of a *particular* force waveform which optimally enhances DRT. However, reaching the same conclusion for the biharmonic force (1) is a harder task due to the aforementioned problematic situation.

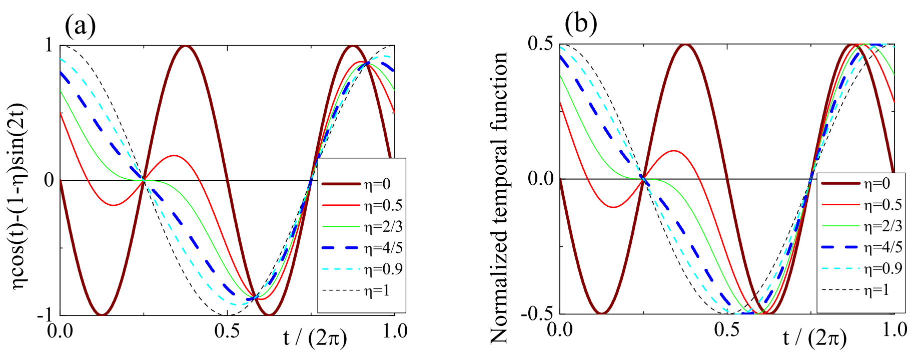

FIG. 1: (Color online). (a) Temporal biharmonic function [Eq. (1)] for the optimal value $\varphi_{eff} = \pi/2$, and (b) the corresponding normalized function [Eq. (5)], versus time for different values of η.

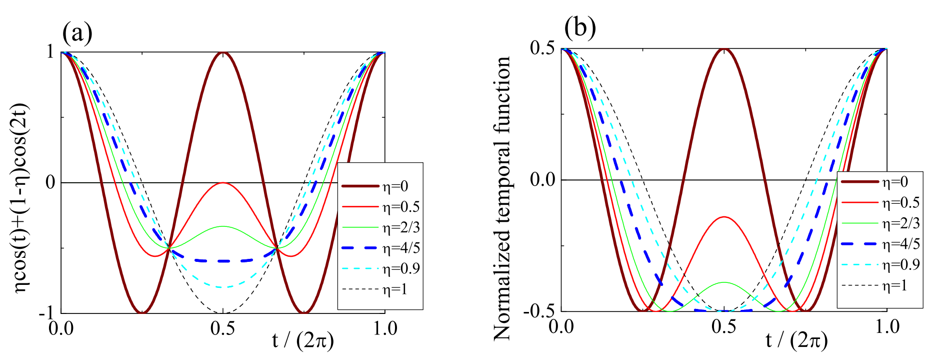

FIG. 2: (Color online). (a) Temporal biharmonic function [Eq. (1)] for the least favourable value $\varphi_{eff} = 0$, and (b) the corresponding normalized function [Eqs. (7) and (8)], versus time t for different values of η.

While it has been shown for Hamiltonian systems subjected to symmetric spatial potentials and force (1) [14] that when $(1-\eta)/\eta \geqslant 1/2$ and $\varphi_{eff} = \{\pi/2, 3\pi/2\}$ the DSB is maximal, it is shown in this Letter that for such optimal values of φ_{eff} the maximal transmitted impulse over a half-period exhibits a single maximum at $(1-\eta)/\eta = 1/2$ (i.e. $\eta = 2/3$) while the amplitude of the force (1) is held constant, concluding again that optimal enhancement of DRT is achieved when critical symmetry breaking occurs. Consider, for example, the case $\varphi_{eff} = \pi/2$. Clearly, one needs an affine transformation to renormalize the function $\eta\cos(t) - (1-\eta)\sin(2t)$ in order to change its image from

$[-M(\eta), M(\eta)]$, where

$$M(\eta) \equiv \frac{3\eta + N(\eta)}{32(1-\eta)}\sqrt{32 + 30\eta^2 + 2\eta[N(\eta) - 32]},$$
$$N(\eta) \equiv \sqrt{32 - 64\eta + 33\eta^2}, \tag{4}$$

to $[-1/2, 1/2]$ for $0 \leqslant \eta \leqslant 1$ (in accordance with both the amplitude and the image of the elliptic force (3)). One therefore obtains the normalized function

$$f^*_{\varphi_{eff}=\pi/2}(t) \equiv \frac{f_{\varphi_{eff}=\pi/2}(t)}{2M(\eta)}. \tag{5}$$

Figure 1(b) shows a plot of $f^*_{\varphi_{eff}=\pi/2}(t)$. One readily finds that its corresponding maximal transmitted impulse over a half-period,

$$I[f^*_{\varphi_{eff}=\pi/2}](\eta) = \frac{1}{M(\eta)}, \tag{6}$$

exhibits a single maximum at $\eta = 2/3$, confirming thus the above scenario of optimal enhancement of DRT by critical symmetry breaking (see Fig. 4 below for a plot of $I[f^*_{\varphi_{eff}=\pi/2}] - 1$ versus η).

Since the breakage of the shift symmetry of any biharmonic function [Eqs. (1) and (2)] occurs for *all* values of the effective phase, one could expect on the basis of the DSB mechanism that the above scenario should hold for any other value of the effective phase, including the least favourable ones, i.e., those values not breaking the corresponding relevant reversal symmetry [14]. It is shown here that this is indeed the case by considering the illustrative value $\varphi_{eff} = 0$. Using now a different affine transformation, one renormalizes the function $\eta\cos(t) + (1-\eta)\cos(2t)$ to change its image from $\left[-\frac{9\eta^2-16\eta+8}{8-8\eta}, 1\right]$ for $0 \leqslant \eta \leqslant 4/5$ and $[1-2\eta, 1]$ for $4/5 \leqslant \eta \leqslant 1$ to $[-1/2, 1/2]$ for $0 \leqslant \eta \leqslant 1$. Noting again that this renormalization process is unique, one straightforwardly obtains the normalized function

$$f^*_{\varphi_{eff}=0}(t) \equiv \frac{f_{\varphi_{eff}=0}(t) - 1/2 - R(\eta)/2}{1 - R(\eta)}, \tag{7}$$

$$R(\eta) \equiv \left\{ \begin{array}{c} \frac{9\eta^2-16\eta+8}{8\eta-8}, \ 0 \leqslant \eta \leqslant 4/5 \\ 1-2\eta, \ 4/5 \leqslant \eta \leqslant 1 \end{array} \right\}. \tag{8}$$

A plot of $f^*_{\varphi_{eff}=0}(t)$ is shown in Fig. 2(b). One finds that the corresponding maximal transmitted impulse over a half-period,

$$I[f^*_{\varphi_{eff}=0}](\eta) = \left(\frac{\pi}{2}\right)\frac{1+R(\eta)}{1-R(\eta)}, \tag{9}$$

also exhibits a single maximum at $\eta = 2/3$ with $I[f^*_{\varphi_{eff}=0}](\eta = 2/3) = \pi/6$, confirming again the above scenario of optimal enhancement of DRT by critical symmetry breaking (see Fig. 4 below for a plot of $I[f^*_{\varphi_{eff}=0}]$ versus η).

If the above scenario is universal (in a sense to be specified below), it must also be found for the spatial force (2) since the forces (1) and (2) exhibit the same symmetry properties, and hence the DSB is the same for both types of forces. For the sake of comparison with the latter temporal case, consider again the value $\varphi_{eff} = 0$. In this case, one has to renormalize the function $\eta\cos(x) + 2(1-\eta)\cos(2x)$ to change its image from $\left[-\frac{33\eta^2-64\eta+32}{16-6\eta}, 2-\eta\right]$ for $0 \leqslant \eta \leqslant 8/9$ and $[2-3\eta, 2-\eta]$ for $8/9 \leqslant \eta \leqslant 1$ to $[-1/2, 1/2]$ for $0 \leqslant \eta \leqslant 1$. The resulting normalized function is now

$$g^*_{\varphi_{eff}=0}(x) \equiv \frac{g_{\varphi_{eff}=0}(x) - (2-\eta)/2 - P(\eta)/2}{2 - \eta - P(\eta)}, \tag{10}$$

$$P(\eta) \equiv \left\{ \begin{array}{c} \frac{33\eta^2-64\eta+32}{16\eta-16}, \ 0 \leqslant \eta \leqslant 8/9 \\ 2-3\eta, \ 8/9 \leqslant \eta \leqslant 1 \end{array} \right\}. \tag{11}$$

Figure 3(b) shows a plot of $g^*_{\varphi_{eff}=0}(x)$. Accordingly, the corresponding maximal transmitted impulse over a half-period is now

$$I[g^*_{\varphi_{eff}=0}](\eta) = \left(\frac{\pi}{2}\right)\frac{2-\eta+P(\eta)}{2-\eta-P(\eta)}, \tag{12}$$

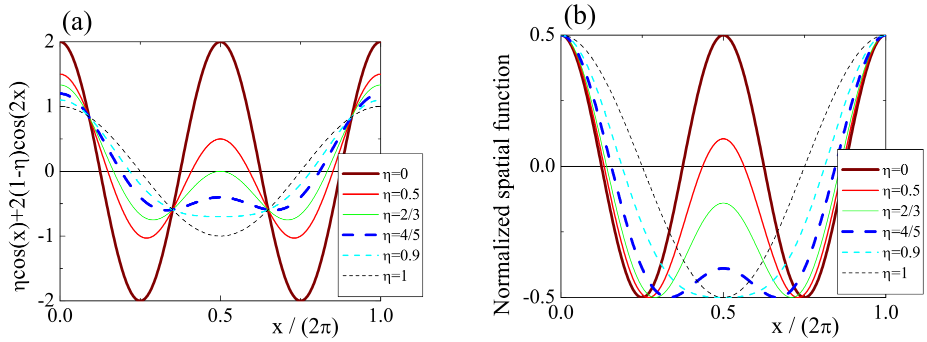

FIG. 3: (Color online). (a) Spatial biharmonic function [Eq. (2)] for the least favourable value $\varphi_{eff} = 0$, and (b) the corresponding normalized function [Eqs. (10) and (11)], versus space x for different values of η.

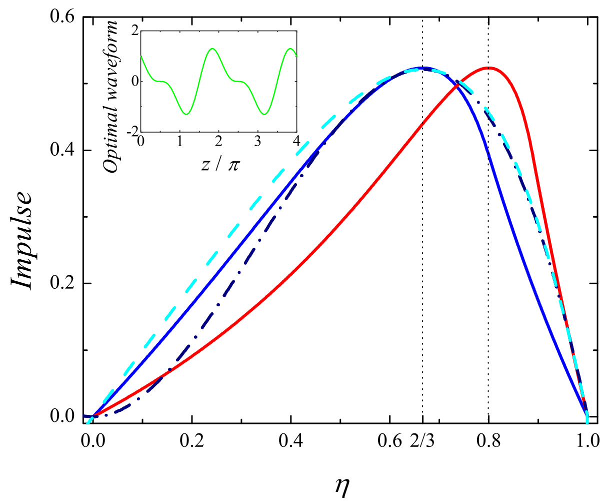

FIG. 4: (Color online). Impulse functions $I[f^{*}_{\varphi_{eff}=0}](\eta)$ [blue (black) line, Eq. (9)], $I[g^{*}_{\varphi_{eff}=0}](\eta)$ [red (gray) line, Eq. (12)], $3.37(I[f^{*}_{\varphi_{eff}=\pi/2}]-1)$ [dashed line, cf. Eq. (6)], and $3.52\eta^2(1-\eta)$ (dashed-dotted line) versus η. The inset show the optimal waveform $\cos(z) - 0.5\sin(2z)\,, z = \{t, x\}$, for both temporal and spatial biharmonic forces [Eqs. (1) and (2)].

which exhibits a single maximum at $\eta = 4/5$ with $I[g^{*}_{\varphi_{eff}=0}](\eta = 4/5) = \pi/6$, i.e., the *same* value as $I[f^{*}_{\varphi_{eff}=0}](\eta = 2/3)$ (see Fig. 4 for a plot of $I[g^{*}_{\varphi_{eff}=0}]$ versus η). Observe that the generality of the present DRT scenario is additionally supported by the remarkable conclusion that the two biharmonic forces (1) and (2) have the *same optimal waveform* at their respective optimal values of η and φ_{eff} (see Fig. 4, inset). In this sense, such an optimal waveform is universal, i.e., once one has identified the relevant spatio-temporal symmetries to be broken in a given equation of motion containing the force (1) or the force (2), this optimal waveform maximally enhances the ratchet effect in that any other waveform (i.e., any choice of the parameters η and φ_{eff} other than the optimal) yields a lower ratchet effect while the remaining equation parameters are held constant.

The present theory explains on the basis of a simple criticality scenario all previously published results for a great diversity of systems (see, in particular, the references cited in Ref. [14]). Additionally, this theory is coherent with, and explains in a general setting, the general validity of the scaling law $\langle V\rangle \sim \eta^2(1-\eta)$ deduced in Refs. [14] and [17] for a temporal biharmonic force with *small* amplitudes. Indeed, one can now expect $\langle V\rangle \sim S(\eta)$ for *arbitrary* amplitudes, where $S(\eta)$ exhibits features similar to those of the functions $I[f^{*}_{\varphi_{eff}=0}](\eta)$ and $I[g^{*}_{\varphi_{eff}=0}](\eta)$ for temporal and spatial biharmonic forces, respectively, i.e., $S(\eta = 0, 1) = 0$ while $S(\eta)$ exhibits a single maximum at $\eta = 2/3$ and $\eta = 4/5$ for temporal and spatial biharmonic forces, respectively. It also explains: (i) the effectiveness of the traditionally used ratchet potential $V(x) = V_0 [\sin(2\pi x/L) + 0.25 \sin(4\pi x/L)]$ [4-6]; (ii) the experimentally obtained optimal parameters of the driving potential, biharmonic in both space and time, of a quantum ratchet [18]; and (iii) the directed ratchet transport strength of matter-wave solitons formed in a Bose-Einstein condensate [19]. Experimental confirmation of the present findings can be readily obtained, for example in the context of cold atoms in optical lattices [20]. A theory providing a universal quantitative criterion to optimize the ratchet effect induced by symmetry breaking of temporal and spatial biharmonic forces was demonstrated, paving the way for any future optimal application of the ratchet effect.

Acknowledgments

The author acknowledges useful conversations with M. Rietmann and R. Carretero-González.

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