ACTA UNIVERSITATIS LODZIENSIS
FOLIA OECONOMICA 152, 2000

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LOGICAL MANY-VALUEDNESS VERSUS PROBABILITY

Abstract. The aim of the papers is to present and discuss the most direct issues on relation between logical many-valuedness and logical probability i.e. probability related to propositions. Having introduced the reader into the realm of many-valued logics, we outline two faces of the problem. One is that logical values must not be identified with the probability values, the other concerns the so-called subjective probability which, as shown by Giles, may be interpreted within the infinite-valued logic of Łukasiewicz.

The mathematical probability calculus in its simplest form resembles many-valued logic. Therefore, the question of a connection between probability and many-valuedness emerges quite naturally. The aim of the paper is to present and discuss the most direct issues on relation between logical many-valuedness and logical probability i.e. the probability related to propositions. Section 1 is a short introduction into the realm of logical many-valuedness. Section 2 is devoted to the three-valued Łukasiewicz logic, which serves as a preparatory example for the sequel. In Section 3 we present the arguments showing that logical values of any many-valued logic must not be identified with the probability values. Section 4 provides an overview of an ingenious construction by Giles showing the way in which the so-called subjective probability may be interpreted within the infinite-valued logic of Łukasiewicz.

1. PRINCIPAL MOTIVATIONS FOR MANY-VALUED LOGIC

The assumption stating that to every proposition it may be ascribed exactly one of the two logical values, *truth* or *falsity*, called the *principle of bivalence*, constitutes the basis of classical logic. It determines both the subject matter and the scope of applicability of the classical logic and it found its expression through the two honoured logic laws: *law of the excluded middle*,

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$$(EM) \quad p \wedge \neg p$$

and the *principle of contradiction*,

$$(CP) \quad \neg (p \wedge \neg p).$$

Given the classical understanding of the logical connectives, *EM* and *CP* may be read as stating that of the two propositions p and $\neg p$: at least one is true and at least one is false, respectively.

The most natural and straightforward step beyond the two-valued logic is the introduction of more logical values, rejecting simultaneously the principle of bivalence. The roots of many-valued logics can be traced back to Aristotle (4^{th} century BC) who considered *future contingents* sentences like

"There will be a sea-battle tomorrow".

The Philosopher from Stagira emphasizes the fact that such sentences describing accidental events are neither actually true nor actually false. Consequently, he suggests that there is a third logical status of propositions.

The prehistory of many-valuedness falls on the Middle Ages and first serious attempts to create three-valued logical constructions appeared at the turn of the XIXth century. The final thoroughly successful formulation of the three-valued logic was made by Łukasiewicz in 1913, see Łukasiewicz (1920). Independently Post (1920) introduced a family of finite-valued logics. Finally, two years later Łukasiewicz constructed logics having infinitely many logical values, see Section 4. Nowadays, the area of many-valued logic is an autonomous field of investigation, see e.g. Malinowski (1993).

2. THREE-VALUED LOGIC OF ŁUKASIEWICZ

The actual introduction of a third logical value next to truth and falsity, was preceded by thorough philosophical studies of the problems of induction and the theory of probability. Łukasiewicz, a fierce follower of indeterminism, finally introduced the third logical value to be assigned to non-determined propositions; specifically, to propositions describing casual future events, i.e. *future contingents*. Łukasiewicz (1920) refers to Aristotle *future contingents* and analyses the sentence: "I shall be in Warsaw at noon on 21 December of the next year". He argues that at the moment of the utterance this sentence is neither true nor false, since otherwise would get fatalist conclusions about necessity or impossibility of the contingent future events.

At the early stage Łukasiewicz interpreted the third logical value as "possibility" or "indeterminacy". Following intuitions of these concepts, he extended the classical interpretation of negation and implication in the following tables[1]:

x	$-x$
0	1
1/2	1/2
1	0

$\rightarrow$	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

The other connectives of disjunction, conjunction and equivalence were introduced through the sequence of the following definitions:

$$\alpha \vee \beta = (\alpha \rightarrow \beta) \rightarrow \beta,$$
$$\alpha \wedge \beta = \neg(\neg \alpha \vee \neg \beta),$$
$$\alpha \equiv \beta = (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha).$$

Their tables are as follows:

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

$\wedge$	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

$\equiv$	0	1/2	1
0	1	1/2	0
1/2	1/2	1	1/2
1	0	1/2	1

A valuation of formulas in the three-valued logic is any function v: $For \rightarrow \{0, 1/2, 1\}$ compatible with the above tables. A *tautology* is a formula which under any valuation v takes on the *designated* value 1.

The set $Ł_3$ of tautologies of three-valued logic of Łukasiewicz differs from *TAUT*. So, for instance, neither the law of the excluded middle, nor the principle of contradiction is in $Ł_3$. To see this, it suffices to assign 1/2 for p: any such valuation also associates 1/2 with *EM* and *CP*. The thorough-going refutation of these two laws was intended, in Łukasiewicz's opinion, to codify the principles of indeterminism.

To close up it would be in order to add that most of the three-valued propositional logics are compatible with the Łukasiewicz logic having the same characterization (the table) of disjunction and, most often, also the table of negation.

[1] The truth-tables of binary connectives $*$ are viewed as follows: the value of α is placed in the first vertical line, the value of β in the first horizontal line and the value of $\alpha * \beta$ at the intersection of the two lines.

3. NON-CLASSICAL LOGICAL VALUES AND PROBABILITY OF PRPOSITIONS

It is interesting to note that still before the construction of his three-valued logic, Łukasiewicz classified as *undefinite* the propositions with free nominal variables and assigned to them fractional "logical" values indicating the proportions between the number of actual variable values verifying a proposition and the number of all possible values of that variable. Clearly, only finite domains were admitted and values were relative: Łukasiewicz values depend on the set of individuals actually evaluated. So, for example, the value of the proposition '$x^2 - 1 = 0$' amounts to 1/2 in the set $\{-1, 0\}$ and to 2/3 in the set $\{-1, 0, 1\}$.

The mathematical probability calculus in its simplest form resembles many-valued logic. Therefore, the question of a connection between probability and many-valuedness emerges quite naturally. Łukasiewicz (1913) invented a theory of logical probability. The differentiating feature of thus comprehended probability is that it refers to propositions and not to events. The continuators of Łukasiewicz's conception, Reichenbach and Zawirski among them, exerted much effort to create a many-valued logic within which logical probability could find a satisfactory interpretation, see e.g. Zawirski (1934a), (1934b), Reichenbach (1935). The Reichenbach-Zawirski conception is based on the assumption that there is a function *Pr* ranging over the set of propositions of a given *standard* propositional language, with values from the real interval [0, 1]. The postulates for *Pr* are:

P1. $0 \leqslant Pr(p) \leqslant 1$,
P2. $Pr(p \vee \neg p) = 1$,
P3. $Pr(p \vee q) = Pr(p) + Pr(q)$ if p and q are mutually exclusive ($Pr(p \wedge q) = 0$,
P4. $Pr(p)Pr(q)$ when p and q are logically equivalent.

From P1–P4 it is possible to infer other expected properties of *Pr*. If then we identify the logical value $v(p)$ with the measure of probability $Pr(p)$ then for $Pr(p) = 1/2$ from the properties mentioned we would get that

$$1/2 \vee 1/2 = Pr(p \vee \neg \mathrm{p}) = 1 \quad \text{and} \quad 1/2 \vee 1/2 = Pr(p \vee p) = Pr(p) = 1/2.$$

Consequently, logical probability must not be identified with logical values of any ordinary extensional many-valued logic.

4. INFINITE-VALUED ŁUKASIEWICZ LOGIC AND SUBJECTIVE PROBABILITY

In 1922 Łukasiewicz generalizes his logical construction and defines the family of finite n-valued logics having as their values the sets $\{0, {}^{1}/_{n-1}, ..., {}^{n-2}/_{n-1}, 1\}$ and two infinite-valued logic: $\aleph_0$- and $\aleph_1$-valued. The first is based on the set of all rational numbers of the interval [0, 1] and the second the whole real interval [0, 1].

The functions corresponding to the connectives are defined in all these systems, including infinite-valued, by the following formulas:

(i) $\neg x = 1 - x,$

$x \rightarrow y = \min(1,\ 1 - x + y),$

(ii) $x \vee y = (x \rightarrow y) \rightarrow y = \max(x,\ y),$

$x \wedge y = \neg(\neg x \vee \neg y) = \min(x,\ y),$

$x \equiv y = (x \rightarrow y) \wedge (y \rightarrow x) = 1 - |x - y|.$

The introduction of new many-valued logics was not supported by any separate argumentation – Łukasiewicz did not give new reasons for the choice of more logical values. It would be, however, easy to see, that these generalizations were correct: for $n = 3$ one gets exactly the matrix of his 1920' three-valued logic.

The researches of Giles in the early 1970's directed towards finding a logic appropriate for the formalization of physical theories, quantum mechanics including, resulted in a very convincing interpretation of the $\aleph_0$-valued Łukasiewicz logic, see Giles (1974). The main point of Giles' approach consists in the so-called dispersive physical interpretation of standard logical language: each prime proposition in a physical theory is associated through the rules of interpretation with a certain experimental procedure which ends in one of the two possible outcomes, "yes" and "no". The tangible meaning of a proposition is related to the observers and expressed in terms of subjective probability. In the case of prime propositions it is determined from the values of probability of success ascribed by observers in respective experiment, whereas in the case of compound propositions it is determined from the rules of obligation formulated in the *dialogue logic* (see Lorenz (1961). The formalization starts with an assumption that

(*) *all prime propositions are definite for all speakers (observers) taken into consideration*

and that speakers are committed to pay certain sum of money for every single assertion of a prime proposition, when the experiment associated with it results in "no". The meaning of compound propositions is then appointed by the rules of debate of two participants: a given person and their partner who can be a fate as well. The rules of obligation generate a game, which starts with an utterance of a compound proposition. For the standard connectives they are the following:

Assertion	Obligation (Commitment)
$p \vee q$	undertaking to assert either p or q at one's own choice
$p \wedge q$	undertaking to assert either p or q at the opponent's choice
$p \to q$	agreement to assert q if the opponent will assert p
$\neg p$	agreement to pay $1 to opponent if they will assert p.

Giles translates subjective probability into "risk values": assigning to prime propositions risk values is a valuation. Subsequently, he employs results of game theory and shows that each valuation of prime propositions has an unique extension onto the whole language guaranteeing both participants no increase in the risk value of the initial position (a formula whose utterance starts the game). Thus, Giles establishes that for every formula and each participant an *optimal strategy* exists.
The risk value function $\langle\ \ \rangle$ is defined for any formulas α, β through the schemes:

$$\begin{aligned}\langle\alpha\to\beta\rangle &= \max\{0,\ \langle\beta\rangle-\langle\alpha\rangle,\\ \langle\alpha\vee\beta\rangle &= \min\{\langle\alpha\rangle,\ \langle\beta\rangle\},\\ \langle\alpha\wedge\beta &= \max\{\langle\alpha\rangle,\ \langle\beta\rangle\},\\ \langle\neg\alpha\rangle &= 1-\langle\alpha\rangle.\end{aligned}$$

The formulae corresponding to propositions, the utterance of which may lead only to not losing final positions, are referred to as *tautologies*. Now, using of the equality

$$pr(\alpha) = \min\{1,\ 1-\langle\alpha\rangle\}$$

one may change risk value associations with the subjective probability valuations:

$$\begin{aligned}pr(\alpha\to\beta) &= \min\{1,\ 1-pr(\alpha)+pr(\beta)\},\\ pr(\alpha\vee\beta) &= \max\{pr(\alpha),\ pr(\beta)\},\\ pr(\alpha\wedge\beta) &= \min\{pr(\alpha),\ pr(\beta)\},\\ pr(\neg\alpha) &= 1-pr(\alpha).\end{aligned}$$

A moment's reflection shows that *pr* if a valuation of $\aleph_0$-valued Łukasiewicz logic and, therefore, the set of tautologies of Giles' dialogue logic coincides with the set of tautologies of Łukasiewicz logic. In probabilistic terms the property of being the tautology is the property of those formulas whose probability ammounts to 1 independently of the values assigned to prime propositions as its components.

5. CONCLUSION

The considerations in Section 3 show that logical probability i.e. the probability associated to propositions must not be identified with logical values of any ordinary extensional many-valued logic. On the other hand, the results by Giles open new possibilities. They show that the so-called subjective probability (of a speaker) associated with events and verified by elementary experiments found a satisfactory interpretation as logical value of the infinite-valued Łukasiewicz logic.

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