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MINIMAL SEQUENT CALCULI FOR ŁUKASIEWICZ'S FINITELY-VALUED LOGICS*

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Abstract

The primary objective of this paper, which is an addendum to the author's [8], is to apply the general study of the latter to Łukasiewicz's n-valued logics [4]. The paper provides an analytical expression of a $2(n-1)$-place sequent calculus (in the sense of [10, 9]) with the cut-elimination property and a strong completeness with respect to the logic involved which is most compact among similar calculi in the sense of a complexity of systems of premises of introduction rules. This together with a quite effective procedure of construction of an *equality determinant* (in the sense of [5]) for the logics involved to be extracted from the constructive proof of Proposition 6.10 of [6] yields an equally effective procedure of construction of both Gentzen-style [2] (i.e., 2-place) and Tait-style [11] (i.e., 1-place) minimal sequent calculi following the method of translations described in Subsection 4.2 of [7].

1. Introduction

Here we entirely follow the general study [8] extending it to Łukasiewicz's finitely-valued logics [4] in addition to Dunn's finitely-valued normal extensions of *RM* [1] as well as Gödel's finitely-valued logics [3] completely

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studied in [8]. Łukasiewicz's logics do deserve a particular emphasis because, as opposed to Dunn's and Gödel's logics, they do *all* have both equality determinant (in the sense of [5]) and singularity determinant (in the sense of [7])(cf. Proposition 6.10 of [6] and Corollary 6.2 of [7] for positive results as well as Propositions 6.5 and 6.8 therein for negative ones), in which case many-place sequent calculi (in the sense of [10, 9]) to be constructed following [8] for the former logics are naturally translated into both Gentzen-style [2](i.e., 2-place) and Tait-style [11] (i.e., 1-place) sequent calculi according to Subsections 4.2.1 and 4.2.2 of [7].

2. Main results

$L = \{\neg, \wedge, \vee, \supset\}$. Take any $n \geqslant 2$. Here we deal with the matrix underlying algebra $\mathfrak{A}_n$ specified as follows. The carrier A_n of $\mathfrak{A}_n$ is set to be n. Finally, operations of $\mathfrak{A}_n$ are defined as follows:

$$\begin{aligned}
\neg^{\mathfrak{A}_n} a &\triangleq n-1-a,\\
a \wedge^{\mathfrak{A}_n} b &\triangleq \min(a,b),\\
a \vee^{\mathfrak{A}_n} b &\triangleq \max(a,b),\\
a \supset^{\mathfrak{A}_n} b &\triangleq \min(n-1, n-1-a+b),
\end{aligned}$$

for all $a, b \in A_n$.

LEMMA 2.1. *For any $i \in n \setminus \{0\}$ and any $j \in n \setminus \{n-1\}$, we have the following introduction rules for $\mathcal{M}^{\mathfrak{A}_n}$:*

$$\frac{\{\{I_{n-1-i}{:}p_0\}\}}{\{F_i{:}\neg p_0\}} \qquad \frac{\{\{F_{n-1-j}{:}p_0\}\}}{\{I_j{:}\neg p_0\}}$$

$$\frac{\{\{F_i{:}p_0\},\{F_i{:}p_1\}\}}{\{F_i{:}(p_0 \wedge p_1)\}} \qquad \frac{\{\{I_j{:}p_0, I_j{:}p_1\}\}}{\{I_j{:}(p_0 \wedge p_1)\}}$$

$$\frac{\{\{F_i{:}p_0, F_i{:}p_1\}\}}{\{F_i{:}(p_0 \vee p_1)\}} \qquad \frac{\{\{I_j{:}p_0\},\{I_j{:}p_1\}\}}{\{I_j{:}(p_0 \vee p_1)\}}$$

$$\frac{\{\{I_{n-2-k}{:}p_0, F_{i-k}{:}p_1\} \mid 0 \leqslant k < i\}}{\{F_i{:}(p_0 \supset p_1)\}}$$

$$\frac{\{\{F_{n-l}{:}p_0, I_{j-l}{:}p_1\} \mid 0 < l \leqslant j\} \cup \{\{F_{n-1-j}{:}p_0\},\{I_j{:}p_1\}\}}{\{I_j{:}(p_0 \supset p_1)\}}$$

PROOF: Let $i \in n \setminus \{0\}$ and $j \in n \setminus \{n-1\}$. Checking (1) of [8] for the introduction rules of types $s{:}\gamma$, where $s \in \{F_i, I_j\}$ and $\gamma \in \{\neg, \wedge, \vee\}$, is trivial. As for those of types $s{:}\supset$, where $s \in \{F_i, I_j\}$, take any $a, b \in n$. Remark that $(a \supset^{\mathfrak{A}_n} b) \in F_i \Leftrightarrow n-1-a+b \geqslant i$. Likewise, $(a \supset^{\mathfrak{A}_n} b) \in I_j \Leftrightarrow n-1-a+b \leqslant j$.

Suppose $n-1-a+b \geqslant i$, that is, $n-1-i+b \geqslant a$. Consider any $0 \leqslant k < i$. Suppose $a \in F_{n-1-k} = n \setminus I_{n-2-k}$, that is, $a \geqslant n-1-k$. Combining two inequalities, we get $k \geqslant i-b$, that is, $b \in F_{i-k}$.

Conversely, assume $n-1-a+b < i$, in which case $n-1-a < i$ too. As $0 \leqslant n-1-a$, we can choose $k \triangleq n-1-a$. If a was in I_{n-2-k}, we would have $0 \leqslant -1$. Likewise, by the inequality under assumption, if b was in F_{i-k}, we would have $b > b$. Thus, both $a \notin I_{n-2-k}$ and $b \notin F_{i-k}$.

Remark that (1) of [8] for the introduction rule of type $I_j{:}\supset$ is equivalent to the following condition:

$$n-1-a+b \leqslant j \Leftrightarrow \forall l \in (j+2) : a \leqslant n-l-1 \Rightarrow b \leqslant j-l \tag{2.1}$$

for all $a, b \in A_n$.

First, suppose $n-1-a+b \leqslant j$, that is, $n-1-j+b \leqslant a$. Consider any $l \in (j+2)$. Assume $a \leqslant n-l-1$. Combining two inequalities, we get $b \leqslant j-l$ as required.

Finally, assume $n-1-a+b > j$. Put $l \triangleq \min(n-1-a, j+1)$. Then, $l \in (j+2)$. Moreover, $a \leqslant n-l-1$. If b was not greater than $j-l$, we would have $l+b \leqslant j$, in which case $l \leqslant j$, and so $l = n-1-a$, in which case $n-1-a+b \leqslant j$. The contradiction with the inequality under assumption shows that $b > j-l$. Thus, (2.1) holds. This completes the argument. □

Notice that each of the sets of premises of rules involved in the formulation of Lemma 2.1 consists of functional S_n-signed $\emptyset$-sequents of some type $V \subseteq \mathrm{Var}$ and forms an anti-chain with respect to $\preccurlyeq$. Then, by Theorem 2.15(ii) of [8], Lemma 2.1 yields

Theorem 2.2. *For any $i \in n \setminus \{0\}$ and any $j \in n \setminus \{n-1\}$:*

$$
\begin{aligned}
P^{\mathfrak{A}_n}_{F_i:\neg} &= \{\{I_{n-1-i}{:}p_0\}\}, \\
P^{\mathfrak{A}_n}_{I_j:\neg} &= \{\{F_{n-1-j}{:}p_0\}\}, \\
P^{\mathfrak{A}_n}_{F_i:\wedge} &= \{\{F_i{:}p_0\}, \{F_i{:}p_1\}\}, \\
P^{\mathfrak{A}_n}_{I_j:\wedge} &= \{\{I_j{:}p_0, I_j{:}p_1\}\}, \\
P^{\mathfrak{A}_n}_{F_i:\vee} &= \{\{F_i{:}p_0, F_i{:}p_1\}\}, \\
P^{\mathfrak{A}_n}_{I_j:\vee} &= \{\{I_j{:}p_0\}, \{I_j{:}p_1\}\}, \\
P^{\mathfrak{A}_n}_{F_i:\supset} &= \{\{I_{n-2-k}{:}p_0, F_{i-k}{:}p_1\} \mid 0 \leqslant k < i\}, \\
P^{\mathfrak{A}_n}_{I_j:\supset} &= \{\{F_{n-l}{:}p_0, I_{j-l}{:}p_1\} \mid 0 < l \leqslant j\} \cup \{\{F_{n-1-j}{:}p_0\}, \{I_j{:}p_1\}\}
\end{aligned}
$$

This provides the minimal $2(n-1)$-place sequent calculus for $\mathfrak{A}_n$. Notice that $P^{\mathfrak{A}_n}_{I_{n-2}:\supset}$ has exactly n elements. Remark that, in case $n = 2$, the resulted calculus coincides with Gentzen's classical calculus LK [2].

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