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SCATTERING OF SURFACE ELASTIC WAVES BY SURFACE IRREGULARITIES*

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ABSTRACT. The paper deals with the scattering of elastic surface waves (ESW) on rough solid surfaces. The surface profile is modelled by a random field of two-dimensional isotropy. The treatment is based on the Green function method. The results involve closed form expressions describing the frequency power spectrum of displacement distribution along the surface plane.

The continuing development of surface elastic wave devices technology, with growing role of the high frequency region, stimulates the need for a more comprehensive study of the effect of various surface and near-surface inhomogeneities on the propagation characteristics of such waves. Although this topic has been treated for a long time (see, e.g., [1–6]), the existing solutions essentially concern very simplified model schemes, basing predominantly on perturbation techniques. Their validity region is additionally limited by the currently made assumption of both mechanical and electrical isotropy of the substrate. In this paper we use a method based on the Green function technique combined with a well known approach [7] which resides in replacing a rough surface by an ideal one with a randomly distributed stress tensor. This enables us to find consistent closed form expressions for both the scattered field and the frequency power spectrum.

Essentially the method can be regarded as a quite general one. For brevity, however, we restrict our considerations to the case of substrate exhibiting isotropy of its mechanical properties along the surface plane. This is exemplified by hexagonal symmetry crystals with (001) surfaces [8], which are known to be typical anisotropic materials.

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1. FORMULATION OF THE PROBLEM

Let us consider the elastic semispace

$$\{(x_1, x_2, x_3;\ -\infty < x_1,\ x_2 < +\infty,\ x_3 \geqslant \xi(x_1, x_2)\} \tag{1.1}$$

limited by the surface $x_3 = \xi(x_1, x_2)$.

We assume that:

1) the elastic semispace is isotropic in the transverse direction, is homogeneous with density g, and has elastic tensor constants in Voigt notation equal to $c_{11}, c_{12}, c_{13}, c_{33}, c_{44}$;
2) the surface of the medium is free of stresses, meaning that $T_{nn} = T_{ns} = 0$ for $x_3 = \xi(x_1, x_2)$;
3) the actual surface profile is described by a random field $\xi(x_1, x_2)$ with mean value of $m_\xi(x_1, x_2) = 0$ and the known correlation function $K_\xi(\underline{r}, \underline{r}')$; $\underline{r} = (x_1, x_2)$, $\underline{r}' = (x_1', x_2')$.

In our final calculations we shall assume the particular form of the correlation function given by eq. (5.5).

For the random field $\xi(x_1, x_2; \gamma)$ we assume that:

a) $\xi(x_1, x_2; \gamma)$ is the real random field, continuous and differentiable in the mean-square sense;
b) the curvature of almost all the realizations of the random field $\xi(x_1, x_2; \gamma)$ is sufficiently small, and the "amplitude" A of unevenness (for almost all the realizations) is small compared to the Rayleigh wave length λ_R considered. More precisely, we assume the events

$$\left|\frac{A}{\lambda}\right| \ll 1, \qquad \left|\frac{\partial \xi}{\partial x_1}\right| \ll 1, \qquad \left|\frac{\partial \xi}{\partial x_2}\right| \ll 1 \tag{1.2}$$

to occur with probability equal to unity.

For the plane boundary of the elastic semispace $x_3 = 0$ isotropic in the transverse direction, the generalized Rayleigh wave propagating in the direction x_1 has the following displacements components:

$$\begin{aligned} u_1^0 &= (U_1 \exp(-\alpha_1 x_3) + U_2 \exp(-\alpha_2 x_3)) \exp(ik_R x_1 - i\omega t) \\ u_3^0 &= (i\delta_1 U_1 \exp(-\alpha_1 x_3) + i\delta_2 U_2 \exp(-\alpha_2 x_3)) \exp(ik_R x_1 - i\omega t) \end{aligned} \tag{1.3}$$

The meaning of the individual quantities is explained in the Appendix. The boundary conditions mean the vanishing of the stresses at the surface

$$T_{3i} = 0 \quad \text{for} \quad x_3 = 0; \quad i = 1, 3. \tag{1.4}$$

For the rough surface we use reduction to the inhomogeneous conditions at the mean surface $x_3 = 0$. The form of the stresses will be found in the next section (eqs. (2.1) and (2.3)). These small random stresses are responsible for the scattering of the generalized Rayleigh wave.

In our case, the equation for the scattered waves has the following form (see, e.g., [8] eq. (3.6)):

$$u_m(\underline{r}, x_3) = \int_{-\infty}^{+\infty} T_{i3}(\underline{r}', x_3' = 0)\, G_{im}(\underline{r} - \underline{r}', x_3, x_3' = 0)\, d\underline{r}' \qquad (1.5)$$

where G_{im} is the Green function for the elastic semispace.

Introducing the random vector field of displacements $\boldsymbol{u}_m(x_1, x_2)$ and the random vector field of stresses $\boldsymbol{t}_i(x_1, x_2)$ at the surface $x_3 = 0$, we can $(i, m = 1, 2, 3)$ rewrite eq. (1.5) as:

$$\boldsymbol{u}_m(x_1, x_2) = \boldsymbol{t}_i(x_1, x_2) * G_{im}(x_1, x_2) \qquad (1.6)$$

where the asterisk $(*)$ means the operation of two-dimensional convolution.

Our aim is to find the statistical characteristics of the random field $\boldsymbol{u}_m(\underline{r})$, which are connected by eq. (1.6) with the field $\boldsymbol{t}_i(\underline{r})$. The latter field is determined by the random field describing the surface profile in a way which will be considered in the next section.

According to the correlative theory, for solving the problem it is sufficient to find the mean value and the correlaction tensor of the random vector field $\boldsymbol{u}_m(\underline{r})$. The equivalent characteristic in the form of frequency power spectrum of the field to find, is the two-dimensional transform of the correlation tensor.

2. EQUIVALENT STRESS DISTRIBUTION

Sobczyk [7] has proposed the stochastic free surface $\xi(x_1, x_2)$ (i.e., with vanishing stresses) to be replaced by the equivalent surface $x_3 = 0$ with the stresses $T_{3j}(0)$ $(j = 1, 2, 3)$. In a first approximation they have the following form:

$$\begin{aligned} T_{33}(0) &= -\left(\frac{\partial T_{33}^0}{\partial x_3}\right)_{x_3=0} \xi \\ T_{31}(0) &= \left(\frac{\partial T_{31}^0}{\partial x_3}\right)_{x_3=0} \xi - (T_{11}^0)_{x_3=0} \frac{\partial \xi}{\partial x_1} \\ T_{32}(0) &= -(T_{22}^0)_{x_3=0} \frac{\partial \xi}{\partial x_2} \end{aligned} \qquad (2.1)$$

where the quantities in brackets are the stress amplitudes to be induced here by a generalized Rayleigh plane wave. For the transversally isotropic elastic semispace, within Voigt's notation, the relations under consideration can be expressed in the following form:

$$T_{11} = c_{11} u_{1'1} + c_{12} u_{2'2} + c_{13} u_{3'3}$$

$$T_{22} = c_{12} u_{1'1} + c_{11} u_{2'2} + c_{13} u_{3'3}$$

$$\frac{\partial T_{31}}{\partial x_3}=c_{44}(u_{1'33}+u_{3'13}) \tag{2.2}$$

$$\frac{\partial T_{33}}{\partial x_3}=c_{13}u_{1'13}+c_{13}u_{2'23}+c_{33}u_{3'33}$$

Our further discussion will be performed for the T_{22} stress component.

Let us consider the generalized Rayleigh plane wave propagating in the direction x_1. Its displacements components are expressed by eq. (1.3). Omitting the $\exp(-i\omega t)$ term, we then obtain

$$T^0_{22}=c_{12}u^0_{1'1}+c_{13}u^0_{3'3}$$

$$(T^0_{22})_{x_3=0}=i\exp(ik_R x_1)\left[U_1(c_{12}k_R-c_{13}\alpha_1\delta_1)+U_2(c_{12}k_R-c_{13}\alpha_2\delta_2)\right] \tag{2.3a}$$

For isotropy $c_{13}=c_{12}$, and $U_2=-U_1 2\alpha_1\alpha_2/(\alpha_2^2+k_R^2)=-U_1(\alpha_2^2+k_R^2)/(2k_R^2)$, $\delta_1=\alpha_1/k_R$ and $\delta_2=k_R/\alpha_2$, then

$$(T^0_{22})_{x_3=0}=i\exp(ik_R x_1)c_{12}U_1(k_R^2-\alpha_1^2)/k_R \tag{2.4}$$

or, equivalently, introducing the Lame constants $\lambda=c_{12}$, $\mu=c_{44}$ and the amplitude of the dilatation potential A according to the relation $U_1=-Ak_R$ (see Rymarz and Kaliski [9], p. 549), we obtain the relation differing in the imaginary factor only from the one given in [7]: $(T^0_{22})_{x_3=0}=-i\alpha^2 A\exp(ik_R x_1)$, where $\alpha^2=g\omega^2/(\lambda+2\mu)$. This difference results from the manner of representation of the Rayleigh wave in displacements (cf. eq. (1.3) with the results of [9]).

Proceeding in a similar way the remaining quantities can be found. For the transverse isotropy they are as follows:

$$(T^0_{11})_{x_3=0}=i\exp(ik_R x_1)\left[U_1(c_{11}k_R-c_{13}\alpha_1\delta_1)+U_2(c_{11}k_R-c_{13}\alpha_2\delta_2)\right] \tag{2.3b}$$

$$\left(\frac{\partial T_{31}}{\partial x_3}\right)_{x_3=0}=\exp(ik_R x_1)c_{44}\left[U_1(\alpha_1^2+k_R\alpha_1\delta_1)+U_2(\alpha_2^2+k_R\alpha_2\delta_2)\right] \tag{2.3c}$$

$$\left(\frac{\partial T_{33}}{\partial x_3}\right)_{x_3=0}=i\exp(ik_R x_1)\left[U_1(c_{33}\alpha_1^2\delta_1-c_{13}k_R\alpha_1)+U_2(c_{33}\alpha_2^2\delta_2-c_{13}k_R\alpha_2)\right] \tag{2.3d}$$

Now let us introduce as follows the random field corresponding to the stresses at the surface:

$$t_1(\underline{r})=\left[T_1\xi(\underline{r})-iT_1'\frac{\partial\xi(\underline{r})}{\partial x_1}\right]\exp(ik_R x_1) \tag{2.5a}$$

$$t_2(\underline{r})=\left[-iT_2\frac{\partial\xi(\underline{r})}{\partial x_2}\right]\exp(ik_R x_1) \tag{2.5b}$$

$$t_3(\underline{r}) = -iT_3\,\xi(\underline{r})\exp(ik_R x_1) \tag{2.5c}$$

where $\xi(\underline{r})$ is the random field for the known parameters describing the properties of the rough surface, and the values T_i, T_1' $(i=1, 2, 3)$ are equal to:

$$T_1 = c_{44}[U_1(\alpha_1^2 + k_R\,\alpha_1\,\delta_1) + U_2(\alpha_2^2 + k_R\,\alpha_2\,\delta_2)] \tag{2.6a}$$

$$T_1' = U_1(c_{11}\,k_R - c_{13}\,\alpha_1\,\delta_1) + U_2(c_{11}\,k_R - c_{13}\,\alpha_2\,\delta_2) \tag{2.6b}$$

$$T_2 = U_1(c_{12}k_R - c_{13}\,\alpha_1\,\delta_1) + U_2(c_{12}\,k_R - c_{13}\,\alpha_2\,\delta_2) \tag{2.6c}$$

$$T_3 = U_1(c_{33}\,\alpha_1^2\,\delta_1 - c_{13}\,k_R\,\alpha_1) + U_2(c_{33}\,\alpha_2^2\,\delta_2 - c_{13}\,k_R\,\alpha_2) \tag{2.6d}$$

3. RELATION BETWEEN THE STATISTICAL CHARACTERISTICS OF THE DISPLACEMENTS AND STRESSES

The vector of the expectation value $m_{u_m}(\underline{r})$ will be found by taking the expectation value of the expressions at both sides of the equation (see eq. (1.6)):

$$u_m(\underline{r}) = \int_{-\infty}^{+\infty} t_i(\underline{a})\,G_{im}(\underline{r}-\underline{a})\,d\underline{a} \tag{3.1}$$

then

$$m_{u_m} = \int_{-\infty}^{+\infty} m_{t_i}(\underline{a})\,G_{im}(\underline{r}-\underline{a})\,d\underline{a} = m_{t_i}(\underline{r}) * G_{im}(\underline{r}) \tag{3.2}$$

Hence $m_{u_m}(\underline{r})$ is the response of the system to $m_{t_i}(\underline{r})$.

To calculate the correlation tensor $R_{u_m u_n}(\underline{r}, \underline{r}')$, the two sides of eq. (3.1) are multiplied by $u_n^*(\underline{r}')$

$$u_m(\underline{r}) = \int_{-\infty}^{+\infty} t_i(\underline{a})\,G_{im}(\underline{r}-\underline{a})\,u_n^*(\underline{r}')\,d\underline{a} \tag{3.3}$$

with subsequent averaging to obtain

$$\begin{aligned} R_{u_m u_n}(\underline{r}, \underline{r}') &= \int_{-\infty}^{+\infty} R_{t_i u_n}(\underline{a}, \underline{r}')\,G_{im}(\underline{r}-\underline{a})\,d\underline{a} \\ &= R_{t_i u_n}(\underline{r}, \underline{r}') * G_{im}(\underline{r}) \end{aligned} \tag{3.4}$$

An identical procedure is applied to the product

$$t_i(\underline{r})\,u_n^*(\underline{r}') = \int_{-\infty}^{+\infty} t_i(\underline{r})\,t_j^*(\underline{a})\,G_{jn}^*(\underline{r}', \underline{a})\,d\underline{a} \tag{3.5}$$

giving

$$\begin{aligned} R_{t_i u_n}(\underline{r}, \underline{r}') &= \int_{-\infty}^{+\infty} R_{t_i t_j}(\underline{r}, \underline{a})\,G_{jn}^*(\underline{r}', \underline{a})\,d\underline{a} \\ &= R_{t_i t_j}(\underline{r}, \underline{r}') * G_{jn}^*(\underline{r}') \end{aligned} \tag{3.6}$$

On insertion of (3.6) into (3.4) one obtains

$$R_{u_m u_n}(\underline{r}, \underline{r}') = R_{t_i t_j}(\underline{r}, \underline{r}') * G^*_{jn}(\underline{r}') * G_{im}(\underline{r}) \tag{3.7}$$

It is easy to show that the following equation holds

$$R_{u_m u_n}(\underline{r}, \underline{r}') = R_{t_i t_j}(\underline{r}, \underline{r}') * G_{im}(\underline{r}) * G^*_{jn}(\underline{r}') \tag{3.8}$$

It is seen from the above considerations that to find the fundamental characteristics of the random field $u_m(\underline{r})$, at first both the vector of the mean value and the correlation tensor of the random field $t_m(\underline{r})$ are needed.

4. RELATION BETWEEN THE POWER SPECTRA OF DISPLACEMENTS AND STRESSES

The power spectrum of the random field $u_m(\underline{r})$, which we denote by $\tilde{R}_{u_m u_n}$, is the Fourier transform of the correlation tensor $R_{u_m u_n}$ given by eq. (3.8). To find it, we will use the relation connecting the convolution with the Fourier transformation (see, for example, [10], p. 58 eq. (1.109)):

$$\iint_{-\infty}^{+\infty} F(\xi, \eta)\, G(\xi, \eta) \exp[i(\xi x + \eta y)]\, d\xi\, d\eta = \iint_{-\infty}^{+\infty} f(u, v)\, g(x-u, y-v)\, du\, dv \tag{4.1}$$

In the latter expression the Fourier 2D transforms (denoted by capital letters) are connected with the originals (denoted by lower case letters) as follows:

$$F(\xi, \eta) = \iint_{-\infty}^{+\infty} f(x, y) \exp(-i(\xi x + \eta y))\, dx\, dy \tag{4.2a}$$

$$f(x, y) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} F(\xi, \eta) \exp[i(\xi x + \eta y)]\, d\xi\, d\eta \tag{4.2b}$$

On the basis of (3.8) and (4.1), we obtain:

$$R_{u_m u_n}(\underline{r}, \underline{r}') = \iiiint_{-\infty}^{+\infty} R_{t_i t_j}(x_1 - u, x_2 - v; x_1' - u', x_2' - v')\, G_{im}(u, v) \times G^*_{jn}(u', v')\, du\, dv\, du' dv' \tag{4.3}$$

If the random field t_i is homogeneous spatially, then

$$R_{t_i t_j}(\underline{r}, \underline{r}') = R_{t_i t_j}(\underline{r} - \underline{r}) \tag{4.4}$$

and eq. (4.3) goes over into the following form:

$$I = R_{u_m u_n}(\underline{r}, \underline{r}') = \iiiint_{-\infty}^{+\infty} R_{t_i t_j}[(x_1 - x_1') - (u - u'), (x_2 - x_2') - (v - v')] \times G_{im}(u, v)\, G^*_{jn}(u', v')\, du\, dv\, du' dv' \tag{4.5}$$

Let us introduce the new variables: $\underline{r} - \underline{r}' = \underline{\rho} = (\rho_1, \rho_2)$; $u_r = u - u'$; $v_r = v - v'$, then

$$I=\iiiint_{-\infty}^{+\infty} R_{t_it_j}(\rho_1-u_r,\ \rho_2-v_r)\, G_{im}(u,v)\, G^*_{jn}(u-u_r,\ v-v_r)\, du\, dv\, du_r\, dv_r \tag{4.6}$$

Using (4.1) and the symmetry of the Green function with respect to its arguments, we can write

$$I=\iiiint_{-\infty}^{+\infty} R_{t_it_j}(\rho_1-u_r,\ \rho_2-v_r)\, \tilde{G}_{im}(k_1,\ k_2)\, \tilde{G}^*_{jn}(k_1,\ k_2) \times \exp\left[i(k_1 u_r+k_2 v_r)\right] du_r\, dv_r\, dk_1\, dk_2 \tag{4.7}$$

where the tilde stands for the 2D Fourier transform. Using (4.1) once again for the elimination of the integral over u_r and v_r and the Fourier representation of the Dirac delta $\delta(k-k')$ distribution, we obtain

$$I=\iiiint_{-\infty}^{+\infty} \tilde{R}_{t_it_j}(k_1',\ k_2')\, \tilde{G}_{im}(k_1,\ k_2)\, \tilde{G}^*_{jn}(k_1,\ k_2)\, \delta(k_1-k_1')\, \delta(k_2-k_2') \times (2\pi)^2 \exp\left[i(k_1'\rho_1+k_2'\rho_2)\right] dk_1'\, dk_2'\, dk_1\, dk_2 \tag{4.8}$$

By making use of the filtration properties of the Dirac delta, and performing the Fourier 2D transformation for both sides of eq. (4.8), we obtain the power spectrum as given by the relation:

$$\tilde{R}_{u_mu_n}(k_1,\ k_2)=\tilde{R}_{t_it_j}(k_1,\ k_2)\, \tilde{G}_{im}(k_1,\ k_2)\, \tilde{G}^*_{jn}(k_1,\ k_2) \tag{4.9}$$

5. THE FUNDAMENTAL STATISTICAL CHARACTERISTICS OF THE RANDOM STRESS FIELD

From eq. (2.5) the following relations for the components of the vector of the mean value for the field t_i $(i=1,\ 2,\ 3)$ result:

$$m_{t_1}=\left[T_1\, m_\xi(x_1,\ x_2)-iT_1'\frac{\partial m_\xi(x_1,\ x_2)}{\partial x_1}\right]\exp(ik_R x_1) \tag{5.1a}$$

$$m_{t_2}=\left[-iT_2\frac{\partial m_\xi(x_1,\ x_2)}{\partial x_2}\right]\exp(ik_R x_1) \tag{5.1b}$$

$$m_{t_3}=\left[-iT_3\, m_\xi(x_1,\ x_2)\right]\exp(ik_R x_1) \tag{5.1c}$$

To illustrate the calculation of the correlation tensor of the random vector field t_i, let us consider the element $R_{t_1t_2}(\underline{r},\underline{r}')$. We than have

$$t_1(\underline{r})\, t_2^*(\underline{r}')=\left[iT_1\, T_2\, \xi(\underline{r}_1)\frac{\partial \xi^*(\underline{r}')}{\partial x_2'}+T_1'\, T_2\frac{\partial \xi(\underline{r})}{\partial x_1}\frac{\partial \xi^*(\underline{r}')}{\partial x_2'}\right] \times \exp\left[ik_R(x_1-x_1')\right] \tag{5.2}$$

and on averaging

$$R_{t_1t_2}(\underline{r},\underline{r}')=[iT_1\,T_2\,R_{\xi,\,\partial\xi/\partial x_2'}(\underline{r},\underline{r}') + T_1'\,T_2\,R_{\partial\xi/\partial x_1,\,\partial\xi/\partial x_2'}(\underline{r},\underline{r}')\exp[ik_R(x_1-x_1')] \tag{5.3}$$

Using the properties of the correlation function (Bass and Fuks [1], p. 46), we can write:

$$R_{t_1t_2}(\underline{r},\underline{r}')=\left[iT_1\,T_2\frac{\partial}{\partial x_2'}K_\xi(\underline{r},\underline{r}')+T_1'\,T_2\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2'}K_\xi(\underline{r},\underline{r}')\right] \times\exp[ik_R(x_1-x_1')] \tag{5.4}$$

where $K_\xi(\underline{r},\underline{r}')$ is the correlation function of the random field $\xi(\underline{r})$. In particular, we will assume that $\xi(\underline{r})$ is the isotropic Gaussian field, of the mean value $m_\xi(\underline{r})=0$ and the correlation function $K_\xi(\rho)$ given by the following

$$K_\xi(\rho)\equiv K_\xi^N(\rho)=\frac{A}{2\pi\sigma^2}\exp[-\rho^2/(2\sigma^2)] \tag{5.5}$$

where $\sqrt{2}\sigma$ has the meaning of the correlation radius of the rough surface, and A is the value characterizing the roughnesses. For such a surface, after some manipulations we obtain:

$$R_{t_1t_2}(\underline{r},\underline{r}')=\left[-T_1'\,T_2\frac{\rho_1\rho_2}{\sigma^4}+iT_1\,T_2\frac{\rho_2}{\sigma^2}\right]K_\xi^N(\rho)\exp(ik_R\,\rho_1) \tag{5.6a}$$

Proceeding in the same way for remaining elements of the correlation tensor, we read

$$R_{t_1t_1}(\underline{r},\underline{r}')=\left[T_1^2-\frac{T_1'^2}{\sigma^2}\left(\frac{\rho_1^2}{\sigma^2}-1\right)\right]K_\xi^N(\rho)\exp(ik_R\,\rho_1) \tag{5.6b}$$

$$R_{t_2t_2}(\underline{r},\underline{r}')=\left[-\frac{T_2^2}{\sigma^2}\left(\frac{\rho_2^2}{\sigma^2}-1\right)\right]K_\xi^N(\rho)\exp(ik_R\,\rho_1) \tag{5.6c}$$

$$R_{t_3t_3}(\underline{r},\underline{r}')=T_3^2\,K_\xi^N(\rho)\exp(ik_R\,\rho_1) \tag{5.6d}$$

$$R_{t_2t_1}(\underline{r},\underline{r}')=R_{t_1t_2}(\underline{r},\underline{r}') \tag{5.6e}$$

$$R_{t_1t_3}(\underline{r},\underline{r}')=\left[-T_1'\,T_3\frac{\rho_1}{\sigma^2}+iT_1\,T_3\right]K_\xi^N(\rho)\exp(ik_R\,\rho_1) \tag{5.6f}$$

$$R_{t_3t_1}(\underline{r},\underline{r}')=-R_{t_1t_3}(\underline{r},\underline{r}') \tag{5.6g}$$

$$R_{t_2t_3}(\underline{r},\underline{r}')=\left[-T_2\,T_3\frac{\rho_2}{\sigma^2}\right]K_\xi^N(\rho)\exp(ik_R\,\rho_1) \tag{5.6h}$$

$$R_{t_3t_2}(\underline{r},\underline{r}')=-R_{t_2t_3}(\underline{r},\underline{r}) \tag{5.6i}$$

where $\underline{\rho}=(\rho_1,\rho_2)$.

Denoting by R_{ij} the matrix $R_{ij}=R_{t_i t_j}(\underline{r},\underline{r})$, $i, j=1, 2, 3$, from eq. (5.6) we find:

$$R_{ij}=\frac{A}{2\pi\sigma^2}\begin{bmatrix} T_1^2+T_1'^2/\sigma^2 & , & 0 & , & iT_1T_3 \\ 0 & , & T_2^2/\sigma^2 & , & 0 \\ -iT_1T_3 & , & 0 & , & T_2^3 \end{bmatrix} \tag{5.7}$$

It is easy to show that the R_{ij} matrix is Hermitian.

6. POWER SPECTRUM OF THE RANDOM VECTOR FIELD t_i

As can be seen from eq. (4.9), the power spectrum of the random field t_i is given by the 2D Fourier transform of the correlation tensor $R_{t_i t_j}$ $(i, j=1, 2, 3)$. Performing the transformation, we obtain the following elements of the tensor of the power spectrum $\tilde{R}_{t_i t_j}$:

$$\tilde{R}_{t_1t_1}(\underline{k})=E\{T_1^2+T_1'^2(k_1-k_R)^2\} \tag{6.1a}$$

$$\tilde{R}_{t_2t_2}(\underline{k})=ET_2^2k_2^2 \tag{6.1b}$$

$$\tilde{R}_{t_3t_3}(\underline{k})=ET_3^2 \tag{6.1c}$$

$$\tilde{R}_{t_1t_2}(\underline{k})=E\{T_1'T_2(k_1-k_R)k_2+T_1T_2k_2\} \tag{6.1d}$$

$$\tilde{R}_{t_2t_1}(\underline{k})=\tilde{R}_{t_1t_2}(\underline{k}) \tag{6.1e}$$

$$\tilde{R}_{t_1t_3}(\underline{k})=iE\{T_1'T_3(k_1-k_R)+T_1T_3\} \tag{6.1f}$$

$$\tilde{R}_{t_3t_1}(\underline{k})=-\tilde{R}_{t_1t_3}(\underline{k}) \tag{6.1g}$$

$$\tilde{R}_{t_2t_3}(\underline{k})=iET_2T_3k_2 \tag{6.1h}$$

$$\tilde{R}_{t_3t_2}(\underline{k})=-\tilde{R}_{t_2t_3}(\underline{k}) \tag{6.1i}$$

where

$$E=A\exp\left[-\frac{(k_1-k_R)^2+k_2^2}{2(1/\sigma)^2}\right] \tag{6.2}$$

and $\underline{k}$ stands for (k_1, k_2).

7. POWER SPECTRUM OF THE RANDOM FIELD OF DISPLACEMENTS FOR A TRANSVERSELY ISOTROPIC MEDIUM

The elements of the correlation tensor $\tilde{R}_{u_\alpha u_\beta}(k_1, k_2)$ can be found in a simpler way then directly from eq. (4.9) by making use of the isotropy of the medium in the plane parallel to the surface.

To see this let us subject the set of equations (4.9) to the operation of rotation by means of the matrix

$$S_{ab}(\underline{k})=\begin{bmatrix}\hat{k}_1 & , & \hat{k}_2 & , & 1\\ -\hat{k}_2 & , & \hat{k}_1 & , & 0\\ 0 & , & 0 & , & 1\end{bmatrix} \tag{7.1a}$$

$$S_{ab}^{-1}(\underline{k})=\begin{bmatrix}\hat{k}_1 & , & -\hat{k}_2 & , & 0\\ \hat{k}_2 & , & \hat{k}_1 & , & 0\\ 0 & , & 0 & , & 1\end{bmatrix} \tag{7.1b}$$

where $\hat{k}_1=k_1/k$, $\hat{k}_2=k_2/k$. For brevity we use the following notations

$$\tilde{U}_{mn}=R_{u_m u_n}(k_1, k_2) \tag{7.2}$$

$$\tilde{t}_{ij}=S_{ia}(\underline{k})\,R_{t_a t_b}(\underline{k})\,S_{bj}^{-1}(\underline{k}) \tag{7.3}$$

As a result of the above transformation we obtain

$$\tilde{U}_{mn}=(2\pi)^2\tilde{t}_{ij}(\underline{k})\,g_{im}(k)\,g_{jn}^{*}(k) \tag{7.4}$$

where

$$g_{im}(k)=S_{ia}(\underline{k})\,\tilde{G}_{ab}(\underline{k})\,S_{bm}^{-1}(\underline{k}) \tag{7.5a}$$

$$g_{jn}^{*}(k)=S_{ja}(\underline{k})\,\tilde{G}_{ab}^{*}(\underline{k})\,S_{bn}^{-1}(\underline{k}) \tag{7.5b}$$

This subterfuge eliminates some of the elements of the Green function matrix, and the remaining ones are made dependent on the value of the vector k only. The non-zero elements of the reduced Green function matrix are as follows [11, 8]: $g_{11}, g_{13}, g_{22}, g_{31}, g_{33}$.

Using eq. (7.3) for calculating the elements of the matrix $\tilde{t}_{ij}$ we get

$$\tilde{t}_{11}=\frac{1}{k^2}E\{k_1^2T_1^2+T_1'^2(k_1-k_R)^2k_1^2+k_2^4T_2^2 + 2k_1k_2[T_1'T_2(k_1-k_R)k_2+T_1T_2k_2]\} \tag{7.6a}$$

$$\tilde{t}_{22}=\frac{1}{k^2}E\{k_2^2T_1^2+T_1'^2(k_1-k_R)^2k_2^2+k_1^2k_2^2T_2^2 + 2k_1k_2[T_1'T_2(k_1-k_R)k_2+T_1T_2k_2]\} \tag{7.6b}$$

$$\tilde{t}_{33}=ET_3^2 \tag{7.6c}$$

$$\tilde{t}_{12}=\frac{1}{k^2}E\{(k_1^2-k_2^2)[T_1'T_2(k_1-k_R)k_2+T_1T_2k_2] + k_1k_2[T_2^2k_2^2-T_1^2-T_1'^2(k_1-k_R)^2]\} \tag{7.6d}$$

$$\tilde{t}_{21}=\tilde{t}_{12} \tag{7.6e}$$

$$\tilde{t}_{13}=i\frac{1}{k}E\{k_1[T_1'T_3(k_1-k_R)+T_1T_3]+k_2(T_2T_3k_2)\} \tag{7.6f}$$

$$\tilde{t}_{31} = -\tilde{t}_{13} \tag{7.6g}$$

$$\tilde{t}_{23} = i\frac{1}{k}E\{-k_2[T_1' T_3(k_1-k_R)+T_1 T_3]+k_1(T_2 T_3 k_2)\} \tag{7.6h}$$

$$\tilde{t}_{32} = -\tilde{t}_{23} \tag{7.6i}$$

From (7.5) we obtain

$$U_{11} = (2\pi)^2[\tilde{t}_{11} g_{11} g_{11}^* + \tilde{t}_{13} g_{11} g_{31}^* + \tilde{t}_{31} g_{31} g_{11}^* + \tilde{t}_{33} g_{31} g_{31}^*] \tag{7.7a}$$

$$U_{22} = (2\pi)^2 \tilde{t}_{22} g_{22} g_{22}^* \tag{7.7b}$$

$$U_{33} = (2\pi)^2[\tilde{t}_{11} g_{13} g_{13}^* + \tilde{t}_{13} g_{13} g_{33}^* + \tilde{t}_{31} g_{33} g_{13}^* + \tilde{t}_{33} g_{33} g_{33}^*] \tag{7.7c}$$

$$U_{12} = (2\pi)^2[\tilde{t}_{12} g_{11} g_{22}^* + \tilde{t}_{32} g_{31} g_{22}^*] \tag{7.7d}$$

$$U_{21} = (2\pi)^2[\tilde{t}_{21} g_{22} g_{11}^* + \tilde{t}_{23} g_{22} g_{31}^*) \tag{7.7e}$$

$$U_{13} = (2\pi)^2[\tilde{t}_{11} g_{11} g_{13}^* + \tilde{t}_{13} g_{11} g_{33}^* + \tilde{t}_{31} g_{31} g_{13}^* + \tilde{t}_{33} g_{31} g_{33}^*] \tag{7.7f}$$

$$U_{31} = (2\pi)^2[\tilde{t}_{11} g_{13} g_{11}^* + \tilde{t}_{13} g_{13} g_{31}^* + \tilde{t}_{31} g_{33} g_{11}^* + \tilde{t}_{33} g_{33} g_{31}^*] \tag{7.7g}$$

$$U_{23} = (2\pi)^2[\tilde{t}_{21} g_{22} g_{13}^* + \tilde{t}_{23} g_{22} g_{33}^*] \tag{7.7h}$$

$$U_{32} = (2\pi)^2[\tilde{t}_{12} g_{13} g_{22}^* + \tilde{t}_{32} g_{33} g_{22}^*]. \tag{7.7i}$$

To make more transparent the form of the above equations, we introduce the following notations: $g_{11}=g_1$, $g_{22}=g_2$, $g_{33}=g_3$, $g_{13}=ig_5$, $g_{31}=ig_5'$, where the quantities g_1, g_2, g_3, g_5 and g_5' are real numbers (cf. [8] expressions D.11). Similarly

$$\begin{array}{lll} \tilde{t}_{11}=t_1, & \tilde{t}_{12}=t_6, & \tilde{t}_{13}=it_5 \\ \tilde{t}_{21}=t_6, & \tilde{t}_{22}=t_2, & \tilde{t}_{23}=it_4 \\ \tilde{t}_{31}=-it_5, & \tilde{t}_{32}=-it_4, & \tilde{t}_{33}=t_3 \end{array}$$

where t_1, t_2, t_3, t_4, t_5 and t_6 are real numbers too.

In this notation eq. (7.7) reduces to:

$$U_{11} = (2\pi)^2[t_1 g_1^2 + 2t_5 g_1 g_5' + t_3 g_5'^2] \tag{7.8a}$$

$$U_{22} = (2\pi)^2 t_2 g_2^2 \tag{7.8b}$$

$$U_{33} = (2\pi)^2[t_1 g_5^2 + 2t_5 g_3 g_5 + t_3 g_3^2] \tag{7.8c}$$

$$U_{12} = (2\pi)^2[t_6 g_1 g_2 + t_4 g_5' g_2] \tag{7.8d}$$

$$U_{21} = U_{12} \tag{7.8e}$$

$$U_{13} = -i(2\pi)^2[t_1 g_1 g_5 + t_5 g_1 g_3 + t_5 g_5 g_5' + t_3 g_3 g_5'] \tag{7.8f}$$

$$U_{31} = -U_{13} \tag{7.8g}$$

$$U_{23} = -i(2\pi)^2[t_6 g_2 g_5 + t_4 g_2 g_3] \tag{7.8h}$$

$$U_{32} = -U_{23} \tag{7.8i}$$

It follows from this that the matrix of the power spectrum of the random displacement field is Hermitian. Physically, it has the meaning of the average density of the acoustic energy scattered into a unit interval of the wave vector $\underline{k}$.

APPENDIX

GENERALIZED RAYLEIGH WAVE FOR TRANSVERSELY ISOTROPIC MEDIUM

A transversely isotropic elastic semispace can be exemplified by any hexagonal medium with sixfold rotation axis in the x_3 direction, i.e., a medium belonging to one of the following crystallographic classes: 6, $\bar{6}$, 6/*m*, 6*mm*, $\bar{6}$*m*2, 62, 6/*mm* (the Herman-Maugin's notation).

The generalized Rayleigh plane wave of wave vector k_R, propagating in the x_1 direction has the displacements components given by eq. (1.3). Here, for simplicity of presentation, the wave vector k_R will be replaced by k. The values α_1 and α_2 describe the wave in the x_3 direction and are equal to the following [11, 12]:

$$\alpha_1^2=\tfrac{1}{2}[x+(x^2-4y^2)^{1/2}] \tag{A.1a}$$

$$\alpha_2^2=\tfrac{1}{2}[x-(x^2-4y^2)^{1/2}] \tag{A.1b}$$

where

$$x=\gamma_1^2+\gamma_4^2-k^2\frac{(c_{13}+c_{44})^2}{c_{33}\,c_{44}};\quad y^2=\gamma_1^2\gamma_4^2 \tag{A.2a}$$

and

$$\gamma_1^2=\frac{c_{11}\,k^2-g\omega^2}{c_{44}};\quad \gamma_4^2=\frac{c_{44}\,k^2-g\omega^2}{c_{33}} \tag{A.2b}$$

The values U_1 and U_2 describing the Rayleigh wave amplitude are proportional to each other:

$$\frac{U_2}{U_1}=-\frac{\alpha_1+k\delta_1}{\alpha_2+k\delta_2}=\frac{c_{13}\,k-c_{33}\,\delta_1\,\alpha_1}{c_{13}\,k-c_{33}\,\delta_2\,\alpha_2} \tag{A.3}$$

where

$$\delta_1=-\frac{(c_{13}+c_{44})\,k\alpha_1}{(\gamma_4^2-\alpha_1^2)\,c_{33}}=\frac{c_{44}(\gamma_1^2-\alpha_1^2)}{(c_{13}+c_{44})\,k\alpha_1} \tag{A.4a}$$

$$\delta_2=-\frac{(c_{13}+c_{44})\,k\alpha_2}{(\gamma_4^2-\alpha_2^2)\,c_{33}}=\frac{c_{44}(\gamma_1^2-\alpha_2^2)}{(c_{13}+c_{44})\,k\alpha_2} \tag{A.4b}$$

It is a complicated task to find the vector k being the root of the equation (see [11]):

$$D(w\omega)=0 \tag{A.5}$$

where $w^2=g\omega^2/(k^2c_{44})$. The quantity $D(w\omega)$ is given by:

$$D(w\omega)=-\frac{g\omega^2}{w^2}\frac{\alpha_1'-\alpha_2'}{\alpha_1'\alpha_2'c_{13}(c_{13}+c_{44})}\left(\frac{c_{11}}{c_{44}}-w^2\right)W \tag{A.6}$$

where

$$W=w^2c_{44}+(c_{33}c_{44})^{1/2}\frac{(1-w^2)^{1/2}}{\left(\frac{c_{11}}{c_{44}}-w^2\right)^{1/2}}\left(w^2+\frac{c_{13}^2-c_{11}c_{33}}{c_{33}c_{44}}\right) \tag{A.7}$$

$$\alpha_i=\alpha_i' k \quad \text{for} \quad i=1,2$$

Guillot [13] has found that the equation $W=0$ has the single solution w_R^2 in the interval [0,1), and if the system reduces to the isotropic one, the above equation is that for the Rayleigh surface wave.

The third-degree equation for $u=w_R^2$ has the form [14]:

$$u^3\gamma^2(\gamma'-1)+u^2\gamma[\gamma-\gamma'-2(\delta\delta'-1)]+u(\delta\delta'-1)[2\gamma-\delta\delta'-1)]+(\delta\delta'-1)^2=0 \tag{A.8}$$

where $c_{44}/c_{11}=\gamma$, $c_{44}/c_{33}=\gamma'$, $c_{13}/c_{11}=\delta$, and $c_{13}/c_{33}=\delta'$. For isotropic medium $c_{33}=c_{11}$, $c_{13}=c_{12}$, $c_{44}=(c_{11}-c_{12})/2$ and we obtain the well known equation [9]:

$$\gamma^2(\gamma-1)\left[u^2-8u^2+8u(3-2\gamma)-16(1-\gamma)\right]=0 \tag{A.9}$$

CONCLUSIONS

The method enables the performing of a relatively exact analysis of ESW scattering from rough surfaces as modelled by a normal isotropic random field.

Given in closed form, the relations defining the power spectrum of the random field of displacements enable one, for a transversely isotropic media, to obtain the coefficients for each of the wave mode. Since they are highly complicated, the use of computers is necessary.

The results obtained can be applied with a good approximation for even moderately weak piezoelectric crystals, in spite of disregarding electric components.

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