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Monotone iterative procedure and systems of a finite number of nonlinear fractional differential equations

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Abstract

The aim of the paper is to present a nontrivial and natural extension of the comparison result and the monotone iterative procedure based on upper and lower solutions, which were recently established in (Wang *et al.* in Appl. Math. Lett. 25:1019-1024, 2012), to the case of any finite number of nonlinear fractional differential equations.

MSC: 26A33; 34A08; 34B15

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1 Introduction

Fractional derivatives and integrals are used for a better description of material properties. In the literature we can find many interesting papers concerning this theory; see *e.g.*, [1–13]. The study of systems involving fractional differential/integral equations is also important as such systems occur in various problems of applied nature; for example, see [14–22]. Some basic theory of fractional differential equations involving the Riemann-Liouville differential operator can be found in [23–25].

In the paper we consider the following system of nonlinear fractional differential equations:

$$\begin{cases} D^\alpha u_1(t) = f_1(t, u_1(t), u_2(t), \dots, u_n(t)), & t \in (0, T], \\ D^\alpha u_2(t) = f_2(t, u_1(t), u_2(t), \dots, u_n(t)), & t \in (0, T], \\ \dots, \\ D^\alpha u_n(t) = f_n(t, u_1(t), u_2(t), \dots, u_n(t)), & t \in (0, T], \\ t^{1-\alpha} u_1(t)|_{t=0} = x_0^1, \quad t^{1-\alpha} u_2(t)|_{t=0} = x_0^2, \quad \dots, \quad t^{1-\alpha} u_n(t)|_{t=0} = x_0^n, \end{cases} \tag{1.1}$$

where D^α is the standard Riemann-Liouville fractional derivative of order α, $0 \leq \alpha \leq 1$, $T > 0$, $f^i \in C([0, T] \times \mathbb{R}^n, \mathbb{R})$, $1 \leq i \leq n$, and $x_0^1, \dots, x_0^n \in \mathbb{R}$ satisfy

$$\sum_{i=2}^{n} x_0^i - x_0^1 \geq 0. \tag{1.2}$$

We investigate system (1.1) with respect to the existence of a solution via the method of upper and lower solutions. There is also presented the concept of an iterative procedure, where the appropriately constructed sequences are convergent to the extreme solution. The paper is a continuation of the investigations in [10] of Wang *et al.*, where the authors examined system (1.1) in the case $n = 2$. After proving the main results we state, for convenience of the reader, the introduced techniques in the case of three nonlinear fractional differential equations and also present a concrete example.

2 Preliminaries

First, let us recall the needed notations and crucial results which will be needed in the next sections of the article.

Denote by $C_{1-\alpha}([0,T])$ the family of all functions $u \in C((0,T])$ such that $t^{1-\alpha}u \in C([0,T])$. A basic theorem concerning the existence of the result and its uniqueness for the linear fractional equation is as follows.

Lemma 2.1 ([23]) *Let* $0 < \alpha \leq 1$, $M \in \mathbb{R}$, *and* $\sigma \in C_{1-\alpha}([0,T])$ *be fixed. Then the linear initial value problem*

$$\begin{cases} D^{\alpha}u(t) = \sigma(t) - Mu(t), & t \in (0,T], \\ t^{1-\alpha}u(t)|_{t=0} = u_0, \end{cases} \quad i,j \in \mathbb{N}, \tag{2.1}$$

has a unique solution, given by the following formula:

$$u(t) = \Gamma(\alpha)u_0 t^{\alpha-1}E_{\alpha,\alpha}\bigl(-Mt^{\alpha}\bigr) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-M(t-s)^{\alpha}\bigr)\sigma(s)\,ds,$$

where $E_{\alpha,\beta}$ *is the Mittag-Leffler function, i.e. the function of the form*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, z \in \mathbb{R}.$$

The comparison result for the initial value problem (2.1) due to Wang *et al.* is as follows.

Lemma 2.2 ([10]) *Let* $0 < \alpha \leq 1$ *and* $M \in \mathbb{R}$ *be given. Then, if* $w \in C_{1-\alpha}([0,T])$ *satisfies*

$$\begin{cases} D^{\alpha}w(t) + Mw(t) \geq 0, & t \in (0,T], \\ t^{1-\alpha}w(t)|_{t=0} \geq 0, \end{cases}$$

then $w(t) \geq 0$ *for all* $t \in (0,T]$.

The same authors also proved the following result, which will be needed in the sequel.

Lemma 2.3 ([10]) *Let* $0 < \alpha \leq 1$, $M \in \mathbb{R}$, *and* $N \geq 0$ *be given. Assume that* $u, v \in C_{1-\alpha}([0,T])$ *satisfy*

$$\begin{cases} D^{\alpha}u(t) \geq -Mu(t) + Nv(t), & t \in (0,T], \\ D^{\alpha}v(t) \geq -Mv(t) + Nu(t), & t \in (0,T], \\ t^{1-\alpha}u(t)|_{t=0} \geq 0, & t^{1-\alpha}v(t)|_{t=0} \geq 0. \end{cases}$$

Then $u(t) \geq 0$, $v(t) \geq 0$ *for all* $t \in (0,T]$.

3 The results

In the sequel we will use the following notation:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i \neq j, \end{cases} \quad i,j \in \mathbb{N}.$$

$C_{1-\alpha}([0,T])^n$ denotes $C_{1-\alpha}([0,T]) \times C_{1-\alpha}([0,T]) \times \cdots \times C_{1-\alpha}([0,T])$ (n times).

Lemma 3.1 *Let $0 < \alpha \leq 1$ be fixed, $M_i \in \mathbb{R}$, $\sigma_i \in C_{1-\alpha}([0,T])$, $i = 1,2,\ldots,n$. Then the linear problem of n equations*

$$\begin{cases} D^\alpha u_1(t) = \sigma_1(t) - M_1 u_1(t) - \sum_{i,j=2}^n M_j \delta_{ji} u_i(t), \quad t \in (0,T], \\ D^\alpha u_j(t) = \sigma_j(t) + (M_j - \sum_{i=1}^n M_i) u_j(t) \\ \qquad - M_j(\sum_{i=1}^n u_i(t) - u_j(t)), \quad t \in (0,T], 2 \leq j \leq n, \\ t^{1-\alpha} u_i(t)|_{t=0} = x_0^i, \quad 1 \leq i \leq n, \end{cases} \tag{3.1}$$

has a unique solution in $C_{1-\alpha}([0,T])^n$.

Proof First observe that for any $p_1, p_2, \ldots, p_n \in C_{1-\alpha}([0,T])$ the system

$$\begin{cases} u_1 + u_2 + \cdots + u_n = p_1, \\ u_1 - u_2 + \cdots + u_n = p_2, \\ \ldots, \\ u_1 + u_2 + \cdots - u_n = p_n \end{cases} \tag{3.2}$$

has exactly one solution, which is a consequence of the fact that

$$\det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & \cdots & 1 \\ 1 & 1 & -1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & -1 \end{bmatrix}_{n \times n} = (-2)^{n-1} \neq 0.$$

Next, observe that system (3.1) can be transformed to system (3.2), where $p_1, p_2, \ldots, p_n$ solve the following n problems:

$$\begin{cases} D^\alpha p_1(t) = (\sigma_1(t) + \sigma_2(t) + \cdots + \sigma_n(t)) - (M_1 + M_2 + \cdots + M_n) p_1(t), \\ t^{1-\alpha} p_1(t)|_{t=0} = x_0^1 + x_0^2 + \cdots + x_0^n, \end{cases}$$

$$\begin{cases} D^\alpha p_2(t) = (\sigma_1(t) - \sigma_2(t) + \cdots + \sigma_n(t)) - (M_1 - M_2 + \cdots + M_n) p_2(t), \\ t^{1-\alpha} p_2(t)|_{t=0} = x_0^1 - x_0^2 + \cdots + x_0^n, \end{cases}$$

$$\vdots$$

$$\begin{cases} D^\alpha p_n(t) = (\sigma_1(t) + \sigma_2(t) + \cdots - \sigma_n(t)) - (M_1 + M_2 + \cdots - M_n) p_n(t), \\ t^{1-\alpha} p_n(t)|_{t=0} = x_0^1 + x_0^2 + \cdots - x_0^n. \end{cases}$$

Finally, observe that the solutions of the above equations are unique due to Lemma 2.1, which ends the proof. □

Now we can state and proof the comparison result for system (3.1).

Theorem 3.1 *Let* $0 < \alpha \le 1$, $M_1 \in \mathbb{R}$, $M_2, \dots, M_n \ge 0$, *and let* $u_1, \dots, u_n \in C_{1-\alpha}([0,T])$ *satisfy*

$$\begin{cases} D^\alpha u_1(t) \ge -M_1 u_1(t) + \sum_{i,j=2}^n M_j \delta_{ji} u_i(t), & t \in (0,T], \\ D^\alpha u_s(t) \ge -M_1 u_s(t) + (\sum_{i=2}^n M_i - M_s) u_s(t) & \\ \qquad + M_s(\sum_{i=1}^n u_i(t) - u_s(t)), & 2 \le s \le n, t \in (0,T], \\ t^{1-\alpha} u_s(t)|_{t=0} \ge 0, & 1 \le s \le n. \end{cases} \tag{3.3}$$

Then

$$\sum_{i=1}^n u_i(t) \ge 0, \quad t \in (0,T], \tag{3.4}$$

$$u_s(t) \ge 0, \quad t \in (0,T], 2 \le s \le n, \tag{3.5}$$

$$-u_s(t) + \sum_{i=1}^n u_i(t) \ge 0, \quad t \in (0,T], 2 \le s \le n. \tag{3.6}$$

Proof Put $r(t) = \sum_{s=1}^n u_s(t)$. Using (3.3) we obtain

$$\begin{aligned} D^\alpha r(t) &= \sum_{s=1}^n D^\alpha u_s(t) \\ &\ge -M_1 u_1(t) + \sum_{i,j=2}^n M_j \delta_{ji} u_i(t) - M_1 \sum_{s=2}^n u_s(t) - 2 \sum_{s=2}^n M_s u_s(t) \\ &\quad + \sum_{s=2}^n \sum_{i=2}^n M_i u_s(t) + \sum_{s=2}^n \sum_{i=1}^n M_s u_i(t) \\ &= -M_1 r(t) + \sum_{i,j=2}^n \bigl(M_j \delta_{ji} u_i(t) + M_i u_j(t)\bigr) - 2 \sum_{s=2}^n M_s u_s(t) + \sum_{s=2}^n M_s r(t). \end{aligned}$$

Observe that

$$\sum_{i,j=2, i \ne j}^n \bigl(M_j \delta_{ji} u_i(t) + M_i u_j(t)\bigr) = 0. \tag{3.7}$$

Hence, we obtain

$$\begin{aligned} D^\alpha r(t) &\ge -\left(M_1 - \sum_{s=2}^n M_s\right) r(t) + \sum_{i,j=2, i \ne j}^n \bigl(M_j \delta_{ji} u_i(t) + M_i u_j(t)\bigr) \\ &\quad + \sum_{i,j=2, i=j}^n \bigl(M_j \delta_{ji} u_i(t) + M_i u_j(t)\bigr) - 2 \sum_{s=2}^n M_s u_s(t) \\ &= -\left(M_1 - \sum_{s=2}^n M_s\right) r(t) + 2 \sum_{i=2}^n M_i u_i(t) - 2 \sum_{s=2}^n M_s u_s(t) \\ &= -\left(M_1 - \sum_{s=2}^n M_s\right) r(t). \end{aligned} \tag{3.8}$$

Moreover, observe that

$$t^{1-\alpha}r(t) = \sum_{s=1}^{n} t^{1-\alpha}u_s(t) \geq 0. \tag{3.9}$$

Applying (3.8) and (3.9) to Lemma 2.2 we get (3.4).

Now, consider any $2 \leq s \leq n$ and denote

$$r_s(t) = \sum_{i=1}^{n} u_i(t) - u_s(t), \quad t \in (0, T].$$

By (3.3) we have

$$\begin{aligned}
D^\alpha r_s(t) &= \sum_{i=1}^{n} D^\alpha u_i(t) - D^\alpha u_s(t) = D^\alpha u_1(t) + \sum_{i=2}^{n} D^\alpha u_i(t) - D^\alpha u_s(t) \\
&\geq -M_1u_1(t) + \sum_{i,j=2}^{n} M_j\delta_{ji}u_i(t) - \sum_{i=2}^{n} M_1u_i(t) + M_1u_s(t) \\
&\quad + \sum_{i=2}^{n}\left(\sum_{j=2}^{n} M_j - M_i\right)u_i(t) - \left(\sum_{j=2}^{n} M_j - M_s\right)u_s(t) \\
&\quad + \sum_{i=2}^{n} M_i\left(\sum_{j=1}^{n} u_j(t) - u_i(t)\right) - M_s\left(\sum_{j=1}^{n} u_j(t) - u_s(t)\right) \\
&= -M_1u_1(t) + \sum_{i,j=2}^{n}\bigl(M_j\delta_{ji}u_i(t) + M_ju_i(t)\bigr) - M_1\sum_{i=2}^{n} u_i(t) + M_1u_s(t) + M_su_s(t) \\
&\quad - 2\sum_{i=2}^{n} M_iu_i(t) - u_s(t)\sum_{j=2}^{n} M_j + \sum_{i=2}^{n}\sum_{j=1}^{n} M_iu_j(t) - M_sr_s(t).
\end{aligned}$$

Again, using (3.7), we obtain

$$\begin{aligned}
D^\alpha r_s(t) &\geq -M_1\sum_{i=1}^{n} u_i(t) + M_1u_s(t) + M_su_s(t) - u_s(t)\sum_{j=2}^{n} M_j \\
&\quad + \sum_{i=2}^{n} M_ir_s(t) + u_s(t)\sum_{i=2}^{n} M_i - M_sr_s(t) \\
&= -\left(M_1 - \sum_{i=2}^{n} M_i + M_s\right)r_s(t) + M_su_s(t).
\end{aligned} \tag{3.10}$$

Moreover, observe that (3.3) implies

$$D^\alpha u_s(t) \geq -\left(M_1 - \sum_{i=2}^{n} M_i + M_s\right)u_s(t) + M_sr_s(t). \tag{3.11}$$

Finally, note that (3.10) and (3.11) applied to Lemma 2.3 give (3.5) and (3.6). □

Now, we are in a position to enunciate the main result.

Theorem 3.2 *Suppose that there exist $u_0^1, u_0^2, \ldots, u_0^n \in C_{1-\alpha}([0,T])$, $u_0^1 \le \sum_{i=2}^n u_0^i$, satisfying*

$$\begin{cases} D^\alpha u_0^1(t) \le f_1(t, u_0^1(t), u_0^2(t), \ldots, u_0^n(t)), & t \in (0,T], \\ D^\alpha u_0^s(t) \ge f_s(t, u_0^1(t), u_0^2(t), \ldots, u_0^n(t)), & t \in (0,T], 2 \le s \le n, \\ t^{1-\alpha} u_0^1(t)|_{t=0} \le x_0^1, \\ t^{1-\alpha} u_0^s(t)|_{t=0} \ge x_0^s, \quad 2 \le s \le n, \end{cases} \tag{3.12}$$

and there exist $M_1 \in \mathbb{R}$, $M_2, \ldots, M_n > 0$ such that

(i)

$$f_1(t, \alpha_1, \ldots, \alpha_n) - f_1(t, \beta_1, \ldots, \beta_n) \ge -M_1(\alpha_1 - \beta_1) - \sum_{i,j=2}^n M_j \delta_{ji}(\alpha_i - \beta_i), \tag{3.13}$$

(ii)

$$\begin{aligned} &f_s(t, \alpha_1, \ldots, \alpha_n) - f_s(t, \beta_1, \ldots, \beta_n) \\ &\quad \ge \left(-M_1 + \sum_{i=2}^n M_i - M_s\right)(\alpha_s - \beta_s) - M_s\left(\alpha_1 - \beta_1 + \alpha_s - \beta_s - \sum_{i=2}^n (\alpha_i - \beta_i)\right), \end{aligned}$$

where $\alpha_i, \beta_i \in \mathbb{R}$, $1 \le i \le n$ satisfy for all $t \in [0,T]$ and $2 \le s \le n$,

$$\begin{aligned} &u_0^1(t) - \left(\sum_{i=2}^n u_0^i(t) - u_0^s(t)\right) \le \beta_1 - \left(\sum_{i=2}^n \beta_i - \beta_s\right) \le \alpha_1 - \left(\sum_{i=2}^n \alpha_i - \alpha_s\right) \le u_0^s(t), \\ &u_0^1(t) - \left(\sum_{i=2}^n u_0^i(t) - u_0^s(t)\right) \le \alpha_s \le \beta_s \le u_0^s(t), \end{aligned}$$

(iii)

$$\begin{aligned} &\sum_{s=2}^n f_s\big(t, u^1(t), u^2(t), \ldots, u^n(t)\big) - f_1\big(t, u^1(t), u^2(t), \ldots, u^n(t)\big) \\ &\quad \ge \left(-M_1 + \sum_{s=2}^n M_s\right)\left(\sum_{s=2}^n u^s(t) - u^1(t)\right), \end{aligned} \tag{3.14}$$

where

$$u_0^1 - \left(\sum_{i=2}^n u_0^i - u_0^s\right) \le u^1 - \left(\sum_{i=2}^n u^i - u^s\right) \le u^s \le u_0^s, \quad 2 \le s \le n.$$

Then there exists a solution $(\bar{u}^1, \bar{u}^2, \ldots, \bar{u}^n)$ of system (1.1) *such that*

$$(n-1)u_0^1 - (n-2)\sum_{i=2}^n u_0^i \le \bar{u}^1 \le \sum_{i=2}^n u_0^i, \qquad u_0^1 - \sum_{i=2}^n u_0^i + u_0^s \le \bar{u}^s \le u_0^s, \quad 2 \le s \le n.$$

Moreover, there exist iterative sequences $(u_k^1), (u_k^2), \ldots, (u_k^n)$ such that $u_k^i \to \bar{u}^i$, $k \to \infty$, $i = 1, 2, \ldots, n$, uniformly on compact subsets of $(0,T]$.

Proof Let us first consider the linear system of the form

$$\begin{cases} D^\alpha u^1(t) = f_1(t, u_0^1(t), u_0^2(t), \dots, u_0^n(t)) + M_1 u_0^1(t) + \sum_{i,j=2}^n M_j \delta_{ji} u_0^i(t) \\ \qquad - M_1 u^1(t) - \sum_{i,j=2}^n M_j \delta_{ji} u^i(t), \quad t \in (0, T], \\ D^\alpha u^s(t) = f_s(t, u_0^1(t), u_0^2(t), \dots, u_0^n(t)) + M_1 u_0^s(t) + (\sum_{i=2}^n M_i - M_s) u^s(t) \\ \qquad + M_s(u_0^1(t) + \sum_{i=2}^n u^i(t) - u^s(t)) - M_1 u^s(t) - (\sum_{i=2}^n M_i - M_s) u_0^s(t) \\ \qquad - M_s(u^1(t) + \sum_{i=2}^n u_0^i(t) - u_0^s(t)), \quad t \in (0, T], 2 \le s \le n, \\ t^{1-\alpha} u^s(t)|_{t=0} = x_0^s, \quad 1 \le s \le n, \end{cases} \tag{3.15}$$

where $u^1, u^2, \dots, u^n \in C_{1-\alpha}([0, T])$. Due to Lemma 3.1 there exists a system of solutions $(u_1^1, u_1^2, \dots, u_1^n) \in C([0, T])^n$ for system (3.15). Using induction we obtain the sequence $(u_k^1, u_k^2, \dots, u_k^n) \in C([0, T])^n$, $k \in \mathbb{N}$, satisfying

$$\begin{cases} D^\alpha u_k^1(t) = f_1(t, u_{k-1}^1(t), u_{k-1}^2(t), \dots, u_{k-1}^n(t)) + M_1 u_{k-1}^1(t) + \sum_{i,j=2}^n M_j \delta_{ji} u_{k-1}^i(t) \\ \qquad - M_1 u_k^1(t) - \sum_{i,j=2}^n M_j \delta_{ji} u_k^i(t), \quad t \in (0, T], \\ D^\alpha u_k^s(t) = f_s(t, u_{k-1}^1(t), u_{k-1}^2(t), \dots, u_{k-1}^n(t)) + M_1 u_{k-1}^s(t) \\ \qquad + (\sum_{i=2}^n M_i - M_s) u_k^s(t) + M_s(u_{k-1}^1(t) + \sum_{i=2}^n u_k^i(t) - u_k^s(t)) \\ \qquad - M_1 u_k^s(t) - (\sum_{i=2}^n M_i - M_s) u_{k-1}^s(t) \\ \qquad - M_s(u_k^1(t) + \sum_{i=2}^n u_{k-1}^i(t) - u_{k-1}^s(t)), \quad t \in (0, T], 2 \le s \le n, \\ t^{1-\alpha} u_k^s(t)|_{t=0} = x_0^s, \quad 1 \le s \le n. \end{cases} \tag{3.16}$$

Now, put $p_1^1 = u_1^1 - u_0^1$, $p_1^s = u_0^s - u_1^s$, $2 \le s \le n$. From (3.12) and (3.15), for all $t \in (0, T]$, we obtain

$$\begin{aligned} D^\alpha p_1^1(t) &= D^\alpha u_1^1(t) - D^\alpha u_0^1(t) \\ &= f_1\bigl(t, u_0^1(t), u_0^2(t), \dots, u_0^n(t)\bigr) + M_1 u_0^1(t) + \sum_{i,j=2}^n M_j \delta_{ji} u_0^i(t) \\ &\quad - M_1 u_1^1(t) - \sum_{i,j=2}^n M_j \delta_{ji} u_1^i(t) - D^\alpha u_0^1(t) \\ &\ge -M_1 p_1^1(t) + \sum_{i,j=2}^n M_j \delta_{ji} p_1^i(t), \\ D^\alpha p_1^s(t) &= D^\alpha u_0^s(t) - D^\alpha u_1^s(t) \\ &= D^\alpha u_0^s(t) - f_s\bigl(t, u_0^1(t), u_0^2(t), \dots, u_0^n(t)\bigr) - M_1 u_0^s(t) \\ &\quad - \left(\sum_{i=2}^n M_i - M_s\right) u_1^s(t) - M_s \left(u_0^1(t) + \sum_{i=2}^n u_1^i(t) - u_1^s(t)\right) + M_1 u_1^s(t) \\ &\quad + \left(\sum_{i=2}^n M_i - M_s\right) u_0^s(t) + M_s \left(u_1^1(t) + \sum_{i=2}^n u_0^i(t) - u_0^s(t)\right) \\ &\ge -M_1 p_1^s(t) + \left(\sum_{i=2}^n M_i - M_s\right) p_1^s(t) + M_s \left(\sum_{i=1}^n p_1^i(t) - p_1^s(t)\right) \quad \text{for all } 2 \le s \le n, \\ t^{1-\alpha} p_1^1(t)|_{t=0} &= t^{1-\alpha} u_1^1(t)|_{t=0} - t^{1-\alpha} u_0^1(t)|_{t=0} \ge x_0^1 - x_0^1 = 0, \\ t^{1-\alpha} p_1^s(t)|_{t=0} &= t^{1-\alpha} u_0^s(t)|_{t=0} - t^{1-\alpha} u_1^s(t)|_{t=0} \ge x_0^s - x_0^s = 0, \quad 2 \le s \le n. \end{aligned}$$

Hence, using Theorem 3.1, we have

$$u_1^s \le u_0^s, \quad 2 \le s \le n \tag{3.17}$$

and

$$u_1^1 - u_0^1 + \sum_{i=2}^{n} \left(u_0^i - u_1^i\right) \ge u_0^s - u_1^s, \quad 2 \le s \le n. \tag{3.18}$$

Consider now $q_1 = \sum_{i=2}^{n} u_1^i - u_1^1$. Using (3.14) and (3.15) we have

$$\begin{aligned}
D^\alpha q_1(t) &= \sum_{s=2}^{n} u_1^s(t) - u_1^1(t) = \sum_{s=2}^{n} D^\alpha u_1^s(t) - D^\alpha u_1^1(t) \\
&= \sum_{s=2}^{n} f_s\left(t, u_0^1(t), u_0^2(t), \dots, u_0^n(t)\right) + \sum_{s=2}^{n} M_1 u_0^s(t) \\
&\quad + \sum_{s=2}^{n} \left(\sum_{i=2}^{n} M_i - M_s\right) u_1^s(t) + \sum_{s=2}^{n} M_s \left(u_0^1(t) + \sum_{i=2}^{n} u_1^i(t) - u_1^s(t)\right) \\
&\quad - \sum_{s=2}^{n} M_1 u_1^s(t) - \sum_{s=2}^{n} \left(\sum_{i=2}^{n} M_i - M_s\right) u_0^s(t) \\
&\quad - \sum_{s=2}^{n} M_s \left(u_1^1(t) + \sum_{i=2}^{n} u_0^i(t) - u_0^s(t)\right) - f_1\left(t, u_0^1(t), u_0^2(t), \dots, u_0^n(t)\right) \\
&\quad - M_1 u_0^1(t) - \sum_{i,j=2}^{n} M_j \delta_{ji} u_0^i(t) + M_1 u_1^1(t) + \sum_{i,j=2}^{n} M_j \delta_{ji} u_1^i(t) \\
&= \sum_{s=2}^{n} f_s\left(t, u_0^1(t), u_0^2(t), \dots, u_0^n(t)\right) - f_1\left(t, u_0^1(t), u_0^2(t), \dots, u_0^n(t)\right) \\
&\quad - \left(M_1 - \sum_{s=2}^{n} M_s\right) q_1(t) + \left(M_1 - \sum_{s=2}^{n} M_s\right) \left(\sum_{s=2}^{n} u_0^s(t) - u_0^1(t)\right) \\
&\ge - \left(M_1 - \sum_{s=2}^{n} M_s\right) q_1(t).
\end{aligned}$$

Moreover, (1.2) implies

$$t^{1-\alpha} q_1(t)|_{t=0} = \sum_{i=2}^{n} t^{1-\alpha} u_1^i(t)|_{t=0} - t^{1-\alpha} u_1^1(t)|_{t=0} = \sum_{i=2}^{n} x_0^i - x_0^1 \ge 0.$$

Now, from Lemma 2.2 we conclude

$$u_1^1(t) \le \sum_{i=2}^{n} u_1^i(t) \quad \text{for all } t \in [0, T]. \tag{3.19}$$

Combining (3.17) and (3.18) with (3.19) we obtain for all $2 \le s \le n$ the inequalities

$$u_0^1 - \left(\sum_{i=2}^{n} u_0^i - u_0^s\right) \le u_1^1 - \left(\sum_{i=2}^{n} u_1^i - u_1^s\right) \le u_1^s \le u_0^s.$$

Let $2 \leq s \leq n$ be fixed and suppose now that for some $k \in \mathbb{N}$ the following inequalities hold:

$$u_{k-1}^1 - \left(\sum_{i=2}^n u_{k-1}^i - u_{k-1}^s \right) \leq u_k^1 - \left(\sum_{i=2}^n u_k^i - u_k^s \right) \leq u_k^s \leq u_{k-1}^s. \tag{3.20}$$

Denote $p_{k+1}^1 = u_{k+1}^1 - u_k^1$, $p_{k+1}^s = u_k^s - u_{k+1}^s$, $2 \leq s \leq n$. From (3.13), (3.16), and (3.20) we obtain

$$\begin{aligned}
D^\alpha p_{k+1}^1(t) &= D^\alpha u_{k+1}^1(t) - D^\alpha u_k^1(t) \\
&= f_1\big(t, u_k^1(t), u_k^2(t), \ldots, u_k^n(t)\big) + M_1 u_k^1(t) + \sum_{i,j=2}^n M_j \delta_{ji} u_k^i(t) - M_1 u_{k+1}^1(t) \\
&\quad - \sum_{i,j=2}^n M_j \delta_{ji} u_{k+1}^i(t) - f_1\big(t, u_{k-1}^1(t), u_{k-1}^2(t), \ldots, u_{k-1}^n(t)\big) - M_1 u_{k-1}^1(t) \\
&\quad - \sum_{i,j=2}^n M_j \delta_{ji} u_{k-1}^i(t) + M_1 u_k^1(t) + \sum_{i,j=2}^n M_j \delta_{ji} u_k^i(t) \\
&\geq -M_1\big(u_{k1}^1(t) - u_{k-1}^1(t)\big) - \sum_{i,j=2}^n M_j \delta_{ji}\big(u_k^i(t) - u_{k-1}^i(t)\big) + M_1 u_k^1(t) \\
&\quad + \sum_{i,j=2}^n M_j \delta_{ji} u_k^i(t) - M_1 u_{k+1}^1(t) - \sum_{i,j=2}^n M_j \delta_{ji} u_{k+1}^i(t) - M_1 u_{k-1}^1(t) \\
&\quad - \sum_{i,j=2}^n M_j \delta_{ji} u_{k-1}^i(t) + M_1 u_k^1(t) + \sum_{i,j=2}^n M_j \delta_{ji} u_k^i(t) \\
&= -M_1 p_{k+1}^1(t) + \sum_{i,j=2}^n M_j \delta_{ji} p_{k+1}^i(t), \\
D^\alpha p_{k+1}^s(t) &= D^\alpha u_k^s(t) - D^\alpha u_{k+1}^s(t) \\
&\geq \left(-M_1 + \sum_{i=2}^n M_i - M_s \right)\big(u_{k-1}^s(t) - u_k^s(t)\big) - M_s \Bigg(u_{k-1}^1(t) - u_k^1(t) \\
&\quad + u_{k-1}^s(t) - u_k^s(t) - \sum_{i=2}^n \big(u_{k-1}^i - u_k^i\big) \Bigg) + M_1 u_{k-1}^s(t) + \left(\sum_{i=2}^n M_i - M_s \right) u_k^s(t) \\
&\quad + M_s \left(u_{k-1}^1(t) + \sum_{i=2}^n u_k^i(t) - u_k^s(t) \right) - M_1 u_k^s(t) - \left(\sum_{i=2}^n M_i - M_s \right) u_{k-1}^s(t) \\
&\quad - M_s \left(u_k^1(t) + \sum_{i=2}^n u_{k-1}^i(t) - u_{k-1}^s(t) \right) - M_1 u_k^s(t) - \left(\sum_{i=2}^n M_i - M_s \right) u_{k+1}^s(t) \\
&\quad - M_s \left(u_k^1(t) + \sum_{i=2}^n u_{k+1}^i(t) - u_{k+1}^s(t) \right) + M_1 u_{k+1}^s(t) + \left(\sum_{i=2}^n M_i - M_s \right) u_k^s(t) \\
&\quad + M_s \left(u_{k+1}^1(t) + \sum_{i=2}^n u_k^i(t) - u_k^s(t) \right) \\
&= -M_1 p_{k+1}^s(t) + \left(\sum_{i=2}^n M_i - M_s \right) p_{k+1}^s(t) + M_s \left(\sum_{i=1}^n p_{k+1}^i(t) - p_{k+1}^s(t) \right).
\end{aligned}$$

Also observe that $t^{1-\alpha}p^1_{k+1}(t)|_{t=0} = t^{1-\alpha}p^s_{k+1}(t)|_{t=0} = 0$, which, together with the above, due to Theorem 3.1, gives

$$u^s_{k+1} \le u^s_k, \quad 2 \le s \le n, \tag{3.21}$$

$$u^s_k - u^s_{k+1} \le \sum_{i=2}^n \left(u^i_k - u^i_{k+1}\right) + u^1_{k+1} - u^1_k. \tag{3.22}$$

Consider now $q_k = \sum_{i=2}^n u^i_k - u^1_k$. Using the same arguments as with q_1 we obtain

$$D^\alpha q_k(t) \ge -\left(M_1 - \sum_{s=2}^n M_s\right) q_k(t)$$

and

$$t^{1-\alpha} q_k(t)|_{t=0} \ge 0,$$

which, due to Lemma 2.2, gives

$$u^1_k \le \sum_{s=2}^n u^i_k. \tag{3.23}$$

Summarizing, by (3.21)-(3.23) and induction, we obtain the following inequalities describing the sequences $(u^s_k)_{k \in \mathbb{N}\cup\{0\}}$:

$$\begin{aligned} u^1_0 - \left(\sum_{i=2}^n u^i_0 - u^s_0\right) &\le u^1_1 - \left(\sum_{i=2}^n u^i_1 - u^s_1\right) \\ &\le \cdots \le u^1_k - \left(\sum_{i=2}^n u^i_k - u^s_k\right) \\ &\le u^s_k \le \cdots \le u^s_1 \le u^s_0, \end{aligned} \tag{3.24}$$

where $2 \le s \le n$. The inequalities (3.24) imply

$$\lim_{k\to\infty} u^s_k(t) = \bar{u}^s(t), \quad s = 2, \ldots, n.$$

Observe that

$$u^1_0 - \left(\sum_{i=2}^n u^i_0 - u^s_0\right) \le \bar{u}^s \le u^s_0, \quad s = 2, \ldots, n.$$

In order to show that the sequence (u^1_k) is convergent observe first that from (3.24) there exists a function x^* such that

$$\lim_{k\to\infty} \left(u^1_k(t) - \sum_{i=2}^{n-1} u^i_k(t)\right) = x^*(t).$$

Hence, putting $\bar{u}^1 = x^* + \sum_{s=2}^{n-1} \bar{u}^s$, we have

$$\begin{aligned}\lim_{k\to\infty}\left(u_k^1(t)-\bar{u}^1(t)\right) &= \lim_{k\to\infty}\left(u_k^1(t)-x^*(t)-\sum_{s=2}^{n-1}\bar{u}^s(t)+\sum_{s=2}^{n-1}u_k^s(t)-\sum_{s=2}^{n-1}u_k^s(t)\right)\\ &= \lim_{k\to\infty}\left(u_k^1(t)-\sum_{s=2}^{n-1}u_k^s(t)-x^*(t)+\sum_{s=2}^{n-1}\left(u_k^s(t)-\bar{u}^s(t)\right)\right)\\ &= \lim_{k\to\infty}\left(u_k^1(t)-\sum_{s=2}^{n-1}u_k^s(t)-x^*(t)\right)+\sum_{s=2}^{n-1}\lim_{k\to\infty}\left(u_k^s(t)-\bar{u}^s(t)\right)=0.\end{aligned}$$

In order to show the uniform convergence of sequences $(u_k^2), (u_k^3), \dots, (u_k^n)$, observe that from (3.24) and from the fact that $u_k^s \to \bar{u}^s$, $s = 2, 3, \dots, n$, we have

$$\bar{u}^s \le u_k^s \le \cdots \le u_1^s \le u_0^s \quad \text{for all } k \in \mathbb{N}.$$

Then, the uniform convergence of sequences (u_k^s), $s = 2, 3, \dots, n$, on a compact subset of $(0, T]$ is a straightforward consequence of Dini's theorem, which states that if a monotone sequence of continuous functions is convergent on a compact set, then it converges uniformly.

Showing a uniform convergence of (u_k^1) requires some observations. Take any $2 \le s \le n$ and denote

$$h_k = u_k^1 - \left(\sum_{i=2}^{n} u_k^i - u_k^s\right), \quad k \in \mathbb{N} \cup \{0\}.$$

From (3.24) and the convergence of $(u_k^1), \dots, (u_k^n)$ we have

$$h_0 \le h_1 \le \cdots \le h_k \le \bar{u}^1 - \left(\sum_{i=2}^{n} \bar{u}^i - \bar{u}^s\right).$$

Applying again Dini's result we get the uniform convergence of (h_k) on every compact subset of $(0, T]$. Finally note that

$$u_k^1 = h_k + \left(\sum_{i=2}^{n} u_k^i - u_k^s\right), \quad k \in \mathbb{N},$$

and thus (u_k^1) is uniformly convergent on a compact subset of $(0, T]$ to $\bar{u}^1$ as a linear combination of sequences uniformly convergent.

Moreover, observe that the limit functions satisfy the properties

$$\begin{aligned}&(n-1)u_0^1 - (n-2)\sum_{i=2}^{n} u_0^i \le \bar{u}^1 \le \sum_{i=2}^{n} u_0^i,\\ &u_0^1 - \sum_{i=2}^{n} u_0^i + u_0^s \le \bar{u}^s \le u_0^s, \quad 2 \le s \le n.\end{aligned}$$

Taking k to ∞ in (3.16) we see that $(\bar{u}^1, \bar{u}^2, \dots, \bar{u}^n)$ is a system of solutions of system (1.1). Also observe that from (3.24) we have the following relations between the limit functions:

$$\bar{u}^1 - \left(\sum_{i=2}^{n} \bar{u}^i - \bar{u}^s \right) \le \bar{u}^s, \quad 2 \le s \le n,$$

which ends the proof. □

Remark 3.1 Observe that using the same methods as in the proof of Theorem 3.2 we can see that $(\bar{u}^1, \bar{u}^2, \dots, \bar{u}^n)$ is an extremal solution of system (1.1) in the sense that if $(u^1, \dots, u^n)$ were any other solution such that

$$u_0^1 - \left(\sum_{i=2}^{n} u_0^i - u_0^s \right) \le u^1 - \left(\sum_{i=2}^{n} u^i - u^s \right) \le u_0^s, \qquad u_0^1 - \left(\sum_{i=2}^{n} u_0^i - u_0^s \right) \le u^s \le u_0^s$$

for any $2 \le s \le n$, then we would have

$$\bar{u}^1 - \left(\sum_{i=2}^{n} \bar{u}^i - \bar{u}^s \right) \le u^1 - \left(\sum_{i=2}^{n} u^i - u^s \right), \quad u^s \le \bar{u}^s, 2 \le s \le n.$$

4 The system of three fractional differential equations

In order to see the nature of the iterative procedure introduced in the proof of Theorem 3.2, we consider the case $n = 3$.

Corollary 4.1 *If there exist $u_0, v_0, w_0 \in C_{1-\alpha}([0,T])$, $u_0 \le v_0 + w_0$ such that*

$$\begin{cases} D^\alpha u_0(t) \le f(t, u_0(t), v_0(t), w_0(t)), & t \in (0,T], \\ D^\alpha v_0(t) \ge g(t, u_0(t), v_0(t), w_0(t)), & t \in (0,T], \\ D^\alpha w_0(t) \ge h(t, u_0(t), v_0(t), w_0(t)), & t \in (0,T], \\ t^{1-\alpha} u_0(t)|_{t=0} \le x_0, \\ t^{1-\alpha} v_0(t)|_{t=0} \ge y_0, \\ t^{1-\alpha} w_0(t)|_{t=0} \ge z_0, \end{cases} \tag{4.1}$$

and there exist $M \in \mathbb{R}$, $N, S \ge 0$ satisfying

$$f(t, \alpha_1, \alpha_2, \alpha_3) - f(t, \beta_1, \beta_2, \beta_n) \ge -M(\alpha_1 - \beta_1) + (-N + S)(\alpha_2 - \beta_2) + (N - S)(\alpha_3 - \beta_3),$$

$$g(t, \alpha_1, \alpha_2, \alpha_3) - g(t, \beta_1, \beta_2, \beta_3) \ge -N(\alpha_1 - \beta_1) + (-M + S)(\alpha_2 - \beta_2) + N(\alpha_3 - \beta_3),$$

$$h(t, \alpha_1, \alpha_2, \alpha_3) - h(t, \beta_1, \beta_2, \beta_3) \ge -S(\alpha_1 - \beta_1) + S(\alpha_2 - \beta_2) + (-M + N)(\alpha_3 - \beta_3),$$

where $\alpha_i, \beta_i \in \mathbb{R}$, $1 \le i \le 3$ satisfy, for all $t \in [0,T]$,

$$u_0(t) - w_0(t) \le \beta_1 - \beta_3 \le \alpha_1 - \alpha_3 \le v_0(t), \qquad u_0(t) - w_0(t) \le \alpha_2 \le \beta_2 \le v_0(t),$$

$$u_0(t) - v_0(t) \le \beta_1 - \beta_2 \le \alpha_1 - \alpha_2 \le w_0(t), \qquad u_0(t) - v_0(t) \le \alpha_3 \le \beta_3 \le w_0(t)$$

and

$$(g + h - f)(t, u, v, w) \ge (-M + N + S)(v + w - u), \tag{4.2}$$

where

$$u_0(t) - w_0(t) \le u - w \le v \le v_0(t),$$
$$u_0(t) - v_0(t) \le u - v \le w \le w_0(t).$$

Then there exists a solution

$$\left(u^*, v^*, w^*\right) \in [2u_0 - v_0 - w_0, v_0 + w_0] \times [u_0 - w_0, v_0] \times [u_0 - v_0, w_0]$$

of (4.1) *and the sequences* $(u_n) \subseteq [2u_0 - v_0 - w_0, v_0 + w_0]$, $(v_n) \subseteq [u_0 - w_0, v_0]$, $(w_n) \subseteq [u_0 - v_0, w_0]$ *such that* $u_n \to u^*$, $v_n \to v^*$, $w_n \to w^*$ *uniformly on compact subsets of* $(0, T]$. *Moreover, the following inequalities hold*:

$$u_0 - v_0 \le u_1 - v_1 \le \cdots \le u_n - v_n \le \cdots \le u^* - v^* \le w^* \le \cdots \le w_n \le \cdots \le w_1 \le w_0,$$
$$u_0 - w_0 \le u_1 - w_1 \le \cdots \le u_n - w_n \le \cdots \le u^* - w^* \le v^* \le \cdots \le v_n \le \cdots \le v_1 \le v_0.$$

4.1 Example

Consider the nonlinear problem of the form

$$\begin{cases} D^{0.5}u(t) = \Gamma(1.5)^{-1}v(t) - \Gamma(1.5)^{-1}w(t) + (v(t) - t)^2 + 2(t - w(t) + u(t))^2, \\ D^{0.5}v(t) = \Gamma(1.5)^{-1}v(t) + (v(t) - t)^2 + (t - w(t) + u(t))^2, \\ D^{0.5}w(t) = -\Gamma(1.5)^{-1}u(t) + \Gamma(1.5)^{-1}v(t) + (t - w(t) + u(t))^2, \\ t^{0.5}u(t)|_{t=0} = t^{0.5}v(t)|_{t=0} = t^{0.5}w(t)|_{t=0} = 0, \end{cases} \tag{4.3}$$

where $t \in [0,1]$. Taking

$$f(t,u,v,w) = \Gamma(1.5)^{-1}v - \Gamma(1.5)^{-1}w + (v - t)^2 + 2(t - w + u)^2,$$
$$g(t,u,v,w) = \Gamma(1.5)^{-1}v + (v - t)^2 + (t - w + u)^2,$$
$$h(t,u,v,w) = -\Gamma(1.5)^{-1}u + \Gamma(1.5)^{-1}v + (t - w + u)^2$$

and

$$u_0(t) = 0, \qquad v_0(t) = w_0(t) = t, \quad t \in [0,1],$$

we obtain, for all $t \in [0,1]$,

$$D^{0.5}u_0(t) = 0 = f\left(t, u_0(t), v_0(t), w_0(t)\right),$$
$$D^{0.5}v_0(t) = \frac{\sqrt{t}}{\Gamma(1.5)} \ge \frac{t}{\Gamma(1.5)} = g\left(t, u_0(t), v_0(t), w_0(t)\right),$$
$$D^{0.5}w_0(t) = \frac{\sqrt{t}}{\Gamma(1.5)} \ge 0 = h\left(t, u_0(t), v_0(t), w_0(t)\right).$$

Next, for all $\alpha_i, \beta_i \in \mathbb{R}$, $1 \le i \le 3$ such that

$$-t \le \beta_1 - \beta_3 \le \alpha_1 - \alpha_3 \le t, \qquad -t \le \alpha_2 \le \beta_2 \le t,$$
$$-t \le \beta_1 - \beta_2 \le \alpha_1 - \alpha_2 \le t, \qquad -t \le \alpha_3 \le \beta_3 \le t,$$

one can calculate that

$$\begin{aligned}
&f(t,\alpha_1,\alpha_2,\alpha_3) - f(t,\beta_1,\beta_2,\beta_3) \geq \Gamma(1.5)^{-1}(\alpha_2-\beta_2) - \Gamma(1.5)^{-1}(\alpha_3-\beta_3),\\
&g(t,\alpha_1,\alpha_2,\alpha_3) - g(t,\beta_1,\beta_2,\beta_3) \geq \Gamma(1.5)^{-1}(\alpha_2-\beta_2),\\
&h(t,\alpha_1,\alpha_2,\alpha_3) - h(t,\beta_1,\beta_2,\beta_3) \geq -\Gamma(1.5)^{-1}(\alpha_1-\beta_1) + \Gamma(1.5)^{-1}(\alpha_2-\beta_2).
\end{aligned}$$

Therefore it is sufficient to take in Corollary 4.1 $M = 0, N = 0, S = \Gamma(1.5)^{-1}$. Finally observe that condition (4.2) also holds. Thus, the system of fractional differential equations (4.3) has a solution $(u^*, v^*, w^*) \in [-2t, 2t] \times [-t, t] \times [-t, t]$.

Now, using the proof of Theorem 3.2 and Lemma 3.1, we can derive the iterative procedure (u_k, v_k, w_k) convergent to the solution (u^*, v^*, w^*). First observe that the sequences (u_k), (v_k), (w_k) satisfy the following system of linear equations:

$$\begin{aligned}
&D^{0.5}u_k = f(t,u_{k-1},v_{k-1},w_{k-1}) - \Gamma(1.5)^{-1}v_{k-1} + \Gamma(1.5)^{-1}w_{k-1} + \Gamma(1.5)^{-1}v_k - \Gamma(1.5)^{-1}w_k,\\
&D^{0.5}v_k = g(t,u_{k-1},v_{k-1},w_{k-1}) - \Gamma(1.5)^{-1}v_{k-1} + \Gamma(1.5)^{-1}v_k,\\
&D^{0.5}w_k = h(t,u_{k-1},v_{k-1},w_{k-1}) + \Gamma(1.5)^{-1}u_{k-1} - \Gamma(1.5)^{-1}v_{k-1} - \Gamma(1.5)^{-1}u_k + \Gamma(1.5)^{-1}v_k,\\
&t^{0.5}u_k(t)|_{t=0} = t^{0.5}v_k(t)|_{t=0} = t^{0.5}w_k(t)|_{t=0} = 0,
\end{aligned}$$

which can be equivalently transformed to the system

$$\begin{cases}
u_k + v_k + w_k = p_k,\\
u_k - v_k + w_k = q_k,\\
u_k + v_k - w_k = r_k,
\end{cases}$$

where p_k, q_k, r_k are the solutions of the following systems:

$$\begin{cases}
D^{0.5}p_k = (f+g+h)(t,u_{k-1},v_{k-1},w_{k-1})\\
\qquad\qquad + \Gamma(1.5)^{-1}u_{k-1} - 3\Gamma(1.5)^{-1}v_{k-1} + \Gamma(1.5)^{-1}w_{k-1} - \Gamma(1.5)^{-1}p_k,\\
t^{0.5}p_k(t)|_{t=0} = 0,
\end{cases}$$

$$\begin{cases}
D^{0.5}q_k = (f-g+h)(t,u_{k-1},v_{k-1},w_{k-1})\\
\qquad\qquad + \Gamma(1.5)^{-1}u_{k-1} - \Gamma(1.5)^{-1}v_{k-1} + \Gamma(1.5)^{-1}w_{k-1} - \Gamma(1.5)^{-1}q_k,\\
t^{0.5}q_k(t)|_{t=0} = 0,
\end{cases}$$

$$\begin{cases}
D^{0.5}r_k = (f+g-h)(t,u_{k-1},v_{k-1},w_{k-1})\\
\qquad\qquad - \Gamma(1.5)^{-1}u_{k-1} - \Gamma(1.5)^{-1}v_{k-1} + \Gamma(1.5)^{-1}w_{k-1} + \Gamma(1.5)^{-1}r_k,\\
t^{0.5}r_k(t)|_{t=0} = 0.
\end{cases}$$

The solutions of the above systems, due to Lemma 2.1, are given by the formulas

$$\begin{aligned}
p_k(t) = {}& \int_0^t (t-s)^{-0.5}E_{0.5,0.5}\bigl(-\Gamma(1.5)^{-1}(t-s)^{0.5}\bigr)\bigl((f+g+h)\bigl(s,u_{k-1}(s),v_{k-1}(s),w_{k-1}(s)\bigr)\\
&+ \Gamma(1.5)^{-1}u_{k-1}(s) - 3\Gamma(1.5)^{-1}v_{k-1}(s) + \Gamma(1.5)^{-1}w_{k-1}(s)\bigr)\,ds,\\
q_k(t) = {}& \int_0^t (t-s)^{-0.5}E_{0.5,0.5}\bigl(-\Gamma(1.5)^{-1}(t-s)^{0.5}\bigr)\bigl((f-g+h)\bigl(s,u_{k-1}(s),v_{k-1}(s),w_{k-1}(s)\bigr)\\
&+ \Gamma(1.5)^{-1}u_{k-1}(s) - \Gamma(1.5)^{-1}v_{k-1}(s) + \Gamma(1.5)^{-1}w_{k-1}(s)\bigr)\,ds,
\end{aligned}$$

$$r_k(t) = \int_0^t (t-s)^{-0.5} E_{0.5,0.5}\big(\Gamma(1.5)^{-1}(t-s)^{0.5}\big)\big((f+g-h)\big(s, u_{k-1}(s), v_{k-1}(s), w_{k-1}(s)\big) \\ - \Gamma(1.5)^{-1}u_{k-1}(s) - \Gamma(1.5)^{-1}v_{k-1}(s) + \Gamma(1.5)^{-1}w_{k-1}(s)\big)\,ds.$$

In consequence, the iterative sequences are of the form

$$\begin{aligned} u_k(t) &= \frac{1}{2}(q_k + r_k) \\ &= \frac{1}{2}\int_0^t (t-s)^{-0.5}\big[E_{0.5,0.5}\big(-\Gamma(1.5)^{-1}(t-s)^{0.5}\big)\big((f-g+h)\big(s, u_{k-1}(s), v_{k-1}(s), w_{k-1}(s)\big) \\ &\quad + \Gamma(1.5)^{-1}u_{k-1}(s) - \Gamma(1.5)^{-1}v_{k-1}(s) + \Gamma(1.5)^{-1}w_{k-1}(s)\big) \\ &\quad + E_{\alpha,\alpha}\big(\Gamma 1.5^{-1}(t-s)^{\alpha}\big)\big((f+g-h)\big(s, u_{k-1}(s), v_{k-1}(s), w_{k-1}(s)\big) \\ &\quad - \Gamma(1.5)^{-1}u_{k-1}(s) - \Gamma(1.5)^{-1}v_{k-1}(s) + \Gamma(1.5)^{-1}w_{k-1}(s)\big)\big]\,ds, \\ v_k(t) &= \frac{1}{2}(p_k - q_k) \\ &= \frac{1}{2}\int_0^t (t-s)^{-0.5}\big[E_{0.5,0.5}\big(-\Gamma(1.5)^{-1}(t-s)^{0.5}\big)\big((f+g+h)\big(s, u_{k-1}(s), v_{k-1}(s), w_{k-1}(s)\big) \\ &\quad + \Gamma(1.5)^{-1}u_{k-1}(s) - 3\Gamma(1.5)^{-1}v_{k-1}(s) + \Gamma(1.5)^{-1}w_{k-1}(s)\big) \\ &\quad - E_{0.5,0.5}\big(-\Gamma(1.5)^{-1}(t-s)^{0.5}\big)\big((f-g+h)\big(s, u_{k-1}(s), v_{k-1}(s), w_{k-1}(s)\big) \\ &\quad + \Gamma(1.5)^{-1}u_{k-1}(s) - \Gamma(1.5)^{-1}v_{k-1}(s) + \Gamma(1.5)^{-1}w_{k-1}(s)\big)\big]\,ds, \\ w_k(t) &= \frac{1}{2}(p_k - r_k) \\ &= \frac{1}{2}\int_0^t (t-s)^{-0.5}\big[E_{0.5,0.5}\big(-\Gamma(1.5)^{-1}(t-s)^{0.5}\big)\big((f+g+h)\big(s, u_{k-1}(s), v_{k-1}(s), w_{k-1}(s)\big) \\ &\quad + \Gamma(1.5)^{-1}u_{k-1}(s) - 3\Gamma(1.5)^{-1}v_{k-1}(s) + \Gamma(1.5)^{-1}w_{k-1}(s)\big) \\ &\quad - E_{0.5,0.5}\big(\Gamma(1.5)^{-1}(t-s)^{0.5}\big)\big((f+g-h)\big(s, u_{k-1}(s), v_{k-1}(s), w_{k-1}(s)\big) \\ &\quad - \Gamma(1.5)^{-1}u_{k-1}(s) - \Gamma(1.5)^{-1}v_{k-1}(s) + \Gamma(1.5)^{-1}w_{k-1}(s)\big)\big]\,ds. \end{aligned}$$

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author formulated and proved all the results in the article, produced the illustrative example, wrote the manuscript, and read and approved it.

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