ABSTRACT

IRREDUNDANT AND MIXED RAMSEY NUMBERS

by

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November, 2012

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The irredundant Ramsey number $s(m, n)$ is the smallest p such that in every two-coloring of the edges of K_p using colors red (R) and blue (B), either the blue subgraph contains an m-element irredundant set or the red subgraph contains an n-element irredundant set. The mixed irredundant Ramsey number $t(m, n)$ is the smallest number p such that in every two-coloring of the edges of K_p using colors red (R) and blue (B), either the blue subgraph contains an m-element irredundant set or the red subgraph contains an n-element independent set. This thesis provides all known results for irredundant and mixed Ramsey numbers.

IRREDUNDANT AND MIXED RAMSEY NUMBERS

A Thesis

Presented to

The Faculty of the Department of Mathematics

East Carolina University

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts in Mathematics

by

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November, 2012

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ACKNOWLEDGEMENTS

I would like to acknowledge Dr. Johannes Hattingh for showing me the ways of mathematical research while being encouraging, laid back, and fun to work with. To the ECU Mathematics faculty and staff, I wish to express my appreciation for all their encouragement and enthusiasm.

This thesis is dedicated to Dr. Robert Bernhardt, without whose wisdom and guidance I would never have discovered the beauty of mathematics.

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CHAPTER 1: Introduction

A *graph* G is a finite nonempty set of objects, called *vertices* (singular *vertex*), together with a (possibly empty) set of unordered pairs of distinct vertices, called *edges*. The set of vertices of the graph G is called the *vertex set* of G, denoted by $V(G)$, and the set of edges is called the *edge set* of G, denoted by $E(G)$. The edge $e = \{u, v\}$ is said to *join* the vertices u and v. If $e = \{u, v\}$ is an edge of G, then u and v are *adjacent vertices*, while u and e are *incident*, as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are *adjacent edges*. It is convenient to henceforth denote an edge by uv or vu rather than by $\{u, v\}$. The cardinality of the vertex set of a graph G is called the *order* of G and is denoted by $n(G)$, or more simply by n when the graph under consideration is clear, while the cardinality of its edge set is the *size* of G, denoted by $m(G)$ or m. A (n, m)-graph has order n and size m. The graph of order $n = 1$ is called the *trivial graph*. A *nontrivial graph* has at least two vertices.

A *subgraph* of a graph G is a graph all of whose vertices belong to $V(G)$ and all of whose edges belong to $E(G)$. If H is a subgraph of G, then we write $H \subseteq G$. If a subgraph H of G contains all the vertices of G, then H is called a *spanning subgraph* of G.

If G is a graph, we form its *complement*, $\overline{G}$, by taking the vertex set of G and joining two vertices by an edge whenever they are not joined in G. If H is a subgraph of G, then the graph $G \backslash E(H)$ is the *complement of G relative to H*.

An important type of subgraph that we will encounter is an induced subgraph.

If W is a nonempty subset of vertices of a graph G, then the subgraph $\langle W \rangle$ of G *induced* by W is the graph having vertex set W and whose edge set consists of all those edges of G incident with two vertices in W. A subgraph H of G is called a *vertex-induced subgraph*, or simply *induced subgraph*, of G if $H = \langle W \rangle$ for some subset W of $V(G)$. Hence, if H is an induced subgraph of G, then every edge of G incident with two vertices in $V(H)$ belongs to $E(H)$ (so two vertices are adjacent in H if and only if they are adjacent in G). When the context may be unclear, we denote the induced subgraph of G by $G\langle W \rangle$ and the induced subgraph of $\overline{G}$ by $\overline{G}\langle W \rangle$. Similarly, if F is a nonempty subset of edges of G, then the subgraph $\langle F \rangle$ *induced* by F is the graph whose vertex set consists of all those vertices of G incident with an edge in F and whose edge set is F. A subgraph J of a graph G is called an *edge-induced subgraph* of G if $J = \langle F \rangle$ for some subset F of $E(G)$.

A *complete graph* or *clique* is a graph in which every two distinct vertices are adjacent. The complete graph of order n is denoted by K_n and is called an *n-clique*. The *empty graph* is a graph containing no edges.

Let u and v be two (not necessarily distinct) vertices of a graph G. A *u-v walk* in G is a finite, alternating sequence of vertices and edges that begin with the vertex u and ends with the vertex v and in which each edge of the sequence joins the vertex that precedes it to the vertex that follows it in the sequence. The number of edges in the walk is called the *length* of the walk. If all the edges of a walk are different, then the walk is called a *trail*. If, in addition, all the vertices are different, then the trail is called a *path*. A u-v walk is *closed* if $u = v$ and *open* otherwise. A closed walk in which all the edges are different is a *closed trail*. A closed trail which contains at least three vertices is called a *circuit*. A circuit which does not repeat any vertices (except the first and last) is called a *cycle*. The *length* of a cycle (or circuit) is the number of edges in the cycle (or circuit). A cycle of length n is an *n-cycle*. A cycle is *even* if its length is even; otherwise it is *odd*.

A *circulant* graph $C_n\{k_0, k_1\}$ is a graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$, and edge set $\{\{v_i, v_{i+j}\} : i \in \{0, 1, \ldots, n-1\}$ and $j \in \{k_0, k_1\}\}$. All arithmetic on the indices is assumed to be modulo n.

Of particular importance for us will be bipartite graphs. A *bipartite graph* is a graph whose vertex set can be partitioned into two sets V_1 and V_2 (called partite sets) in such a way that each edge of the graph joins a vertex of V_1 to a vertex of V_2. A *complete bipartite graph* is a bipartite graph with partite sets V_1 and V_2 having the added property that every vertex of V_1 is adjacent to every vertex of V_2. If $|V_1| = r$ and $|V_2| = s$, then this graph is denoted by $K(r, s)$ or, more commonly, $K_{r,s}$. A complete bipartite graph of the form $K_{1,s}$ is called a *star graph*. A complete bipartite graph $K_{n,n}$ is called an *n-biclique*. A useful and well-known characterization of bipartite graphs is the following: A nontrivial graph G is bipartite if and only if it contains no odd cycles.

Let v be a vertex of a graph G. The *degree* of v is the number of edges of G incident with v. The degree of v is denoted by $deg_G v$, or simply $d(v)$ if G is clear from the context. The *minimum degree* of G is the minimum degree among the vertices of G and is denoted $\delta(G)$, while the *maximum degree* of G is the maximum degree among the vertices of G and is denoted $\Delta(G)$.

A vertex is called *odd* or *even* depending on whether its degree is odd or even. A vertex of degree 0 in a graph G is called an *isolated vertex* and a vertex of degree 1 is an *end-vertex* of G. We say that a graph is *regular* if all its vertices have the same degree. In particular, if the degree of each vertex is r, then the graph is *regular of degree r* or is *r-regular*.

A well-known and useful theorem in graph theory, called the Handshaking Lemma, states that in any graph, the sum of all the vertex degrees is equal to twice the number of edges. A consequence of the Handshaking Lemma is that in any graph G there is an even number of odd vertices.

We say two graphs, G and H, are *isomorphic* if there is a one-to-one mapping ϕ

from $V(G)$ onto $V(H)$ such that ϕ preserves adjacency; that is, $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. If G and H are isomorphic, then we write $G \cong H$.

A graph G is *connected* if there exists a path in G between any two of its vertices, and is *disconnected* otherwise. Every disconnected graph can be partitioned into connected subgraphs, called *components*. A component of a graph G is a maximal connected subgraph. Two vertices u and v in a graph G are *connected* if $u = v$, or if $u \neq v$ and there is a $u - v$ path in G. The number of components of G is denoted $k(G)$; of course, $k(G) = 1$ if and only if G is connected.

For a connected graph G, we define the *distance d(u,v)* between two vertices u and v as the minimum of the lengths of the $u - v$ paths of G. If G is a disconnected graph, then the distance between two vertices in the same component of G is defined as above. However, if u and v belong to different components of G, then $d(u, v)$ is undefined.

We now introduce the concept of the neighborhood of a vertex.

Let G be a graph. Then the *open neighborhood* of a vertex $v \in V(G)$ is $N(v) = \{u \in V | uv \in E(G)\}$. In general, we define the *open neighborhood of a subset* $X \subseteq V(G)$ by $N(X) = \cup_{x \in X} N(x)$. The *closed neighborhood* of a vertex v is $N[v] = \{v\} \cup N(v)$ and in general, the *closed neighborhood of a subset* $X \subseteq V(G)$ by $N[X] = X \cup N(X)$.

For $x \in X$, the *private neighborhood* of x relative to X is defined as $PN(x, X) = N[x] \backslash N[X - \{x\}]$. The elements of $PN(x, X)$ are the *private neighbors* of x (relative to X).

A set $D \subseteq V(G)$ is a dominating set of G (in which case D is said to *dominate* G) if each vertex in $V(G) \backslash D$ is adjacent to a vertex in D, and D is a minimal dominating set if no proper subset of D dominates G.

The earliest ideas of dominating sets date back to the origins of chess, where one wishes to cover or dominate various opposing pieces or various squares of the chessboard. In 1862 de Jaenisch [13] posed the problem of finding the minimum number of queens that can be placed on a chessboard so that each square of the chessboard is attacked or dominated by at least one of the queens. A graph may be formed from an $n \times n$ chessboard by taking the squares as the vertices and two vertices are adjacent if a queen situated on one square covers the other. Computing the domination number of the latter graph is equivalent to finding the number of queens that can be placed on a chessboard so that each square of the chessboard is is attacked or dominated by at least one of the queens.

The classical problems of covering chessboards with the minimum number of chess pieces rekindled interest in dominating concepts. Ultimately, the theory of domination was formalized by Berge [2] in 1958 and Ore [17] in 1962. Ore coined the term 'domination number', although Berge was the first to define it as the coefficient of external stability.

Some applications for the concept of a dominating set include the following: Berge [1] mentions the problem of keeping a number of strategic locations under surveillance by a set of radar stations. The minimum number of radar stations needed to survey all the locations is the domination number of the associated graph. In a similar vein, Liu [16] discusses the application of domination to communications in a network, where a dominating set represents a set of cities which, acting as transmission stations, can transmit messages to every city in the network.

As a further example, a desirable property for a committee from a collection of people might be that every nonmember know at least one member of the committee, for ease of communication. A committee with this property is a dominating set of the acquaintance graph of the set of people.

The following well-known result characterises dominating sets which are minmal dominating sets:

Proposition 1.1. *[17] A dominating set D is a minimal dominating set if and only if $PN(d, D) \neq \emptyset$ for each $d \in D$.*

This condition motivates the definition of an *irredundant set*:

Let $\langle X \rangle$ be the subgraph of G induced by $X \subseteq V(G)$. A set of vertices X in a graph G is *irredundant* if each vertex $x \in X$ is either isolated in $\langle X \rangle$ or else has a private neighbor $y \in V(G) \backslash X$, which is adjacent to x and to no other vertex of X. In other words, a set $X \subseteq V(G)$ is irredundant if $PN(x, X) \neq \emptyset$ for each $x \in X$.

A set X is *independent* if every two distinct vertices in X are nonadjacent. A set X is *maximal irredundant* if no proper superset of X is irredundant. Thus, a set D is a *minimal dominating* set if and only if it is dominating and irredundant. However, an irredundant set, or even a maximal irredundant set, is not necessarily dominating. It is easy to see that the concept of irredundance extends that of independence, for if X is independent, then $x \in PN(x, X)$ for each $x \in X$, hence X is irredundant. Extremal sets of these types are related by the following two well-known results:

Proposition 1.2. *[2] If X is maximal independent, then X is minimal dominating.*

Proposition 1.3. *[12] If X is minimal dominating, then X is maximal irredundant.*

The *domination number* $\gamma(G)$ and the *upper domination number* $\Gamma(G)$ (*independent domination number* $i(G)$ and *independence number* $\beta(G)$; *irredundance number* $ir(G)$ and *upper irredundance number* $IR(G)$) are defined, respectively, to be the smallest and largest number of vertices in a minimal dominating (maximal independent; maximal irredundant) set of G. The following string of inequalities is obvious from the definitions and the relationships which exist amongst the three concepts (also see [12]):

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

Since irredundance is a generalization of independence and since classical Ramsey numbers can also be defined using independent sets instead of cliques, it seems natural to develop a theory of irredundant Ramsey numbers. Irredundance has received much attention in the literature (see [15] for an extensive bibliography).

1.1 Definitions

Let $G_1, G_2, ..., G_t$ be an arbitrary t-edge coloring of K_p, where for each $i \in \{1, 2, ..., t\}$, G_i is the spanning subgraph of K_p consisting of all edges colored with color i. The *classical Ramsey number* $r(q_1, q_2, ...q_t)$, is usually defined in terms of the existence of cliques of the subgraphs G_i. Since a clique of G_i corresponds to an independent set of the complement $\overline{G_i}$, $r(q_1, q_2, ...q_t)$ may also be defined using independence. In fact, $r(q_1, q_2, ...q_t)$ is the smallest value of p such that for all t-edge colorings of K_p, there is an $i \in \{1, 2, ..., t\}$ for which $\beta(\overline{G_i}) \geq q_i$.

The *irredundant Ramsey number* $s(q_1, q_2, ..., q_t)$ is analogously defined as the smallest p such that for all t-edge colorings of K_p, there is an $i \in \{1, 2, ..., t\}$ for which $IR(\overline{G_i}) \geq q_i$. Since any independent set is irredundant, the irredundant Ramsey numbers exist by Ramsey's theorem and satisfy $s(q_1, q_2, ...q_t) \leq r(q_1, q_2, ..., q_t)$ for all $q_1, q_2, ..., q_t$.

The *mixed Ramsey number* $t(m, n)$, introduced in [10], is the smallest p such that for every graph G of order p, $IR(\overline{G}) \geq m$ or $\beta(G) \geq n$.

We have the following lemma:

Lemma 1.4. *The inequality chain $s(m,n) \leq t(m,n) \leq r(m,n)$ holds for all $m,n \geq 1$.*

Proof. First we will show $t(m,n) \leq r(m,n)$. Let $p = r(m,n)$. In any bicoloring, R and B, of the edges of K_p, either we have an independent set of size m in the blue graph $\langle B \rangle$ or an independent set of size n in the red graph $\langle R \rangle$. Since every independent set is also an irredundant set, then in any two coloring of K_p we have an irredundant set of size m in $\langle B \rangle$ or an independent set of size n in $\langle R \rangle$. By definition, $t = t(m,n)$ is the smallest such number where this is true so $t \leq p = r(m,n)$. Now we will show $s(m,n) \leq t(m,n)$. Let $t = t(m,n)$. Then t is the smallest natural number such that in any red-blue edge coloring of K_t there is an irredundant set of cardinality m in $\langle B \rangle$ or an independent set of cardinality n in $\langle R \rangle$. Since every independent set is also an irredundant set, we have an irredundant set of size n in $\langle R \rangle$. By definition, $s = s(m,n)$ is the smallest number such that this is true so $s(m,n) \leq t = t(m,n)$. Thus, we have $s(m,n) \leq t(m,n) \leq r(m,n)$, as desired. □

The same recurrence inequality which holds for $r(m,n)$ also holds for $s(m,n)$ and $t(m,n)$:

Proposition 1.5. *For all integers $m,n \geq 2$, $x(m,n) \leq x(m-1,n)+x(m,n-1)$ while strict inequality holds if $x(m-1,n)$ and $x(m,n-1)$ are both even, where $x \in \{r,t,s\}$.*

Proof. We illustrate the proof for $x = r$, and remark that the proof is similar when $x \in \{t,s\}$.

Let $N = x(m-1,n) + x(m,n-1)$ and take any bicoloring of K_N in red and blue, (R,B), and let $v \in V(K_N)$. Let M represent the set of vertices adjacent to v with a red edge and let L represent the set of vertices adjacent to v with a blue edge. So, $|M| + |L| + 1 = N = x(m-1,n) + x(m,n-1)$. Now either $|M| \geq x(m-1,n)$ or $|L| \geq x(m,n-1)$ since otherwise $|M| < x(m-1,n)$ and $|L| < x(m,n-1)$

imply $x(m-1,n)+x(m,n-1)-1 = |M|+|L| < x(m-1,n)+x(m,n-1) \leq x(m-1,n)-1+x(m,n-1)-1 = x(m-1,n)+x(m,n-1)-2$, producing a contradiction.

Now suppose $x(m-1,n)$ and $x(m,n-1)$ are both even and suppose that $x(m,n) = x(m-1,n)+x(m,n-1)$. Let $N' = x(m-1,n)+x(m,n-1)-1$. Then there exists a two coloring (R,B) of K_N such that neither the graph induced by R, $\langle R\rangle$, has an m-clique nor the graph induced by B, $\langle B\rangle$, has an n-clique. Let $v \in V(K_{N'})$, and define M and L as before.

If $|M| \geq x(m-1,n)$, then $\langle M\rangle$ has a red $(m-1)$-clique or a blue n-clique, and so $\langle M\rangle$ has a red m-clique, by considering v, or a blue n-clique, which is a contradiction. So, $|M| \leq x(m-1,n)-1$, and, similarly, $|L| \leq x(m,n-1)-1$.

Suppose $|M| \leq x(m-1,n)-2$ and $|L| \leq x(m,n-1)-2$. Then, $|M|+|L| \leq x(m-1,n)-2+x(m,n-1)-2$. But, $|M|+|L| = N'-1$, so, $x(m-1,n)-2+x(m,n-1)-2 \geq |M|+|L| = N'-1 = ((x(m-1,n)+x(m,n-1))-1)-1$ implying $-4 \geq -2$, a contradiction. Thus, $|M| \geq x(m-1,n)-1$ or $|L| \geq x(m,n-1)-1$. If $|M| \geq x(m-1,n)-1$, then $|M| = x(m-1,n)-1$, and so $x(m,n-1)-1 \leq |L| = N'-1-|M| \leq N'-1-|M| = x(m-1,n)+x(m,n-1)-1-1-(x(m-1,n)-1) = x(m,n-1)-1$, whence $|L| = x(m,n-1)-1$. Similarly, if $|L| = x(m,n-1)-1$, then $|M| = x(m-1,n)-1$. Thus, $d_R(v) = x(m-1,n)-1$ for all $v \in V(K_{N'})$.

So, $\sum_{v\in V(K_{N'})} d_R(v) = 2q(\langle R\rangle)$. Now we have $N \cdot (x(m-1,n)-1) = 2 \cdot n(\langle R\rangle)$. But N' and $x(m-1,n)-1$ are both odd, and the product of two odd numbers is odd, a contradiction. $\square$

Note that unlike the case for $s(m,n)$ and $r(m,n)$, $t(m,n) \neq t(n,m)$ in general.

1.2 Useful Results

In this section we prove results which are used extensively throughout the remainder of the thesis.

For ease of explanation, we sometimes abbreviate $IR(G)$ and $IR(\overline{G})$ to IR and $\overline{IR}$. Also, we frequently refer to the edges of G and the edges of $\overline{G}$ as red edges and blue edges, respectively, and also sometimes denote G by R and $\overline{G}$ by B. By the red neighbors R_v and the blue neighbors B_v of a vertex v, we mean the neighbors of v in R and in B, respectively. So, for each vertex v, $V(G)$ can be partitioned into the sets $V(G) = \{v\} \cup R_v \cup B_v$.

We begin by proving necessary and sufficient conditions for a graph G to satisfy $IR(\overline{G}) \geq m$, but first we introduce some notation. Let $K_{n,n}$, $n \geq 3$, denote the complete bipartite graph with partite sets $U = \{u_1, u_2, ..., u_n\}$ and $W = \{w_1, w_2, ...w_n\}$. Let $C(n)$ be defined by $C(n) = K_{n,n} \backslash \{u_i w_i | i \in \{1, 2, ..., n\}\}$. Denote by $C(n) + K_\ell$ the graph obtained by joining every vertex of K_ℓ to every vertex of $C(n)$; if $\ell = 0$ we take $C(n) + K_\ell$ to mean $C(n)$.

Proposition 1.6. *[10] $\overline{G}$ has an irredundant set of size m if and only if one of the following statements holds:*

(a) $K_m \subseteq G$;

(b) there exist integers k, ℓ with $k \geq 3$, $\ell \geq 0$, and $k + \ell = m$ such that G contains the graph $C(k) + K_\ell$ and G does not contain the edges $u_i w_i$, $i \in \{1, 2, ...k\}$

Proof. Suppose $\overline{G}$ has an irredundant set X of size m. Let k and l be the number of non-isolates and isolates of $\overline{G}\langle X\rangle$, respectively. Note that $k + \ell = m$. $k = 1$ is impossible and if $k = 0$, then X is an independent set of size m in $\overline{G}$, in which case (a) holds. For $k \geq 2$, let $U = \{u_1, u_2, ..., u_\ell\}$ be the set of non-isolates in $\overline{G}\langle X\rangle$ and let w_i be a private neighbor of u_i in $\overline{G}$, $i \in \{1, 2, ...k\}$. The $G\langle X\rangle$ contains the graph

$C(k) + K_\ell$ while it does not contain the edges $u_i w_i$, $i \in \{1, 2, ..., k\}$. Hence, if $k \geq 3$, then (b) holds and if $k = 2$, then $(X - \{u_2\}) \cup \{w_2\}$ is an independent set of size m in $\overline{G}$, in which case (a) holds.

Conversely, if (a) holds then $\overline{G}$ contains an independent, and hence irredundant, set of size m. If (b) holds, then $V(K_\ell) \cup U$ is an irredundant set of size m in $\overline{G}$ as $V(K_\ell)$ is an independent set in $\overline{G}$ and each vertex $u_i \in U$ has a private neighbor w_i in $\overline{G}$, $i \in \{1, 2, ..., k\}$. □

Corollary 1.7. *[3] A graph contains a 3-element irredundant set if and only if its complement contains a K_3 or an induced C_6.*

Proof. By Proposition 1.6, the complement contains a K_3 or a C_6 as a subgraph. If the complement does not contain a K_3, then a C_6 is induced. □

The following result is immediate.

Lemma 1.8. *If (R, B) is a two-coloring of the edges of a complete graph such that $\langle B \rangle$ contains no m-element irredundant set and $\langle R \rangle$ contains no n-element irredundant set, then $\Delta_R < s(m-1, n)$ and $\Delta_B < s(m, n-1)$.*

When applying Corollary 1.7, we refer to a red 6-cycle $v_1 v_2 v_3 v_4 v_5 v_6 v_1$ where the edges $v_1 v_4$, $v_2 v_5$, $v_3 v_6$ are blue as a red 6-cycle with blue diagonals.

A graph G of order p such that $IR(\overline{G}) < m$ and $IR(G) < n$ is called an *(m,n,p)-graph.* Note that if v is any vertex of an *(m,n,p)-graph* G, then the subgraph $\langle R_v \rangle$ of G is an *(m-1,n,deg(v))-graph* while the subgraph $\langle B_v \rangle$ of G is an *(m,n-1,p-1-deg(v))-graph.* This observation is used in the following result.

Proposition 1.9. *[14] If G is an (m,n,p)-graph, $m, n \geq 2$, then*

$$p - s(m, n-1) \leq \delta(G) \leq \Delta(G) \leq s(m-1, n) - 1.$$

Proof. For a vertex v of maximum degree, $\langle R_v \rangle$ is an $(m-1, n, \Delta(G))$-*graph*. Hence, $\Delta(G) < s(m-1, n)$. For a vertex v of minimum degree, $\langle B_v \rangle$ is an $(m, n-1, p-1-\delta(G))$-*graph*. Hence, $p - 1 - \delta(G) < s(m, n-1)$. $\square$

Proposition 1.10. *If G is a graph of order p with $IR(\overline{G}) < m$ and $\beta(G) < n$ for $m, n \geq 2$, then $p - t(m, n-1) \leq \delta(G) \leq \Delta(G) \leq t(m-1, n) - 1$.*

Proof. Similar to that of Proposition 1.9. $\square$

Proposition 1.11. *[14] Suppose that (R, B) is a two-coloring of the edges of a complete graph in which $\langle B \rangle$ contains no 3-element irredundant set. For an arbitrary vertex v, let $Y = R_v$ and let $X = \{x_1, x_2, ..., x_p\} \subseteq B_v$ be such that at most one of the sets $Y_i = \{y | y \in Y, x_i y \in R\}$, $(i = 1, ..., p)$ is empty. Then $\langle R_X \rangle$ is bipartite.*

Proof. First we recall that a graph is bipartite if and only if it contains no odd cycles. Thus, it suffices to show that $\langle R_X \rangle$ contains no odd cycles.

Since, by assumption, $\langle B \rangle$ has no 3-element irredundant set, by Corollary 1.7 its complement, $\langle R \rangle$, does not contain a K_3 or induced C_6. Hence, $\langle B_Y \rangle$ is complete.

We now state three observations that are used during our proof.

(i) If $x_1x_2x_3$ is a path in $\langle X \rangle_R$, then either $Y_1 \subseteq Y_3$ or $Y_3 \subseteq Y_1$. In particular, if Y_1 and Y_3 are nonempty, then so is $Y_1 \cap Y_3$.

Suppose $y_1 \in Y_1 - Y_3$ and $y_3 \in Y_3 - Y_1$. Then $vy_1x_1x_2x_3y_2v$ is an induced C_6 in $\langle R \rangle$, a contradiction.

(ii) Suppose that $x_1x_2x_3x_4x_5$ is a path in $\langle R_X \rangle$ and the edges x_1x_4 and x_2x_5 are in B. Then either $Y_1 \cap Y_3 \cap Y_5 = \emptyset$ or $Y_2 \cap Y_4 = \emptyset$.

If $y_1 \in Y_1 \cap Y_2 \cap Y_3$ and $y_2 \in Y_2 \cap Y_4$, then $y_1x_1x_2y_2x_4x_5y_1$ is an induced C_6 in $\langle R \rangle$.

(iii) Suppose $x_1x_2x_3x_4x_5$ is a path in $\langle R_X \rangle$ and that the edges x_1x_4 and x_2x_5 are in B and each of the sets $Y_1, ..., Y_5$ are nonempty. Then $Y_1 \subset Y_3$.

Otherwise, (i) implies that $Y_3 \subseteq Y_1$, so $Y_1 \cap Y_3 \cap Y_5$ is nonempty since $Y_3 \subseteq Y_1$ implies $Y_3 \cap Y_5$ is nonempty. But $Y_2 \cap Y_4$ is also nonempty which contradicts (ii).

Now suppose $\langle R_X \rangle$ is not bipartite and let $x_1x_2, \ldots, x_{2k+1}x_1$ be its shortest odd cycle. Then all of the chords x_ix_{i+3} are in B. For example, x_1x_4 and x_2x_5 are in B.

We know $k > 1$ since there is no K_3 in $\langle R \rangle$ by assumption.

Now we want to eliminate $k = 2$. Suppose that $x_1x_2x_3x_4x_5x_1$ is a cycle in $\langle R_X \rangle$ and note that we may assume $Y_1, ..., Y_4$ to be nonempty sets. Then by (i) we have vertices $y_1 \in Y_1 \cap Y_3$ and $y_2 \in Y_2 \cap Y_4$. There is no K_3 in $\langle R \rangle$, so edges x_1y_2, x_4y_1, and vx_5 are in $\langle B \rangle$. Now $vy_1x_1x_5x_4y_2v$ is an induced C_6 in $\langle R \rangle$. Hence, $k \neq 2$.

For $k = 3$, we may assume $Y_1, \ldots, Y_6$ nonempty and observe that by (iii) $Y_1 \subseteq Y_3$ and $Y_6 \subseteq Y_4$. (To see the latter, consider the path $x_6x_5x_4x_3x_2$ in $\langle R_X \rangle$ and apply (iii).) Now we have $Y_3 \cap Y_4 = \emptyset$, but (i) gives $Y_1 \cap Y_6 \neq \emptyset$. Thus, $k \neq 3$.

For $k > 3$, we may assume that the path $x_1x_2x_3x_4x_5x_6x_7x_8$ is a segment of an odd cycle in $\langle R_X \rangle$ and that each of the sets $Y_1, \ldots, Y_8$ is nonempty. Then (iii) gives both $Y_4 \subseteq Y_6$ and $Y_6 \subseteq Y_4$ giving us the desired contradiction.

Therefore, $\langle R_X \rangle$ contains no odd cycles and is hence bipartite. $\square$

CHAPTER 2: Irredundant Ramsey Numbers $s(m, n)$

We have $s(1, n) = s(n, 1) = 1$ and $s(2, n) = s(n, 2) = 2$ for all $n \geq 1$.

Theorem 2.1. *[3]* $s(3, 3) = 6$

Proof. Note that $s(3, 3) \leq r(3, 3) = 6$. Now, the graph C_5 contains neither a red K_3 nor a red 6-cycle. Thus, by Corollary 1.7, $IR(\overline{G}) \leq 2$. As C_5 is self-complementary, $IR(G) \leq 2$. Thus, $s(3, 3) \geq 6$. □

Theorem 2.2. *[3]* $s(3, 4) = 8$

Proof. Suppose G is a *(3,4,8)-graph.* As $s(3, 3) = 6$ and $s(2, 4) = 4$, it follows from Proposition 1.9 that $8 - 6 \leq \delta(G) \leq \Delta(G) \leq 4 - 1$ so each vertex of G has degree 2 or 3.

Suppose v has degree 3 in G. All four vertices of B_v send red edges to R_v, for otherwise R_v together with a vertex of B_v would constitute a 4-vertex independent set in G. Thus, at least one of the three vertices of R_v must receive two red edges from B_v. It follows that v is adjacent to a vertex w with $d(w) = 3$.

Since there is no red triangle, $N(v) \cap N(w) = \emptyset$. Let $N(v) = \{v_1, v_2, w\}$ and $N(w) = \{w_1, w_2, v\}$ and let the remaining two vertices of G be x and y.

Case 1: Suppose $xy \in E(G)$ is red. Vertex x sends a red edge to $\{v_1, v_2\}$, for otherwise $\langle\{w, v_1, v_2, x\}\rangle$ is a blue K_4. Assume xv_1 is red. Similarly, to avoid the blue K_4, $\langle\{v, w_1, w_2, x\}\rangle$, we take xw_1 to be red. If yv_1 is red, there would be a red K_3. Thus, yv_1 is red and, similarly, yw_2 is red. Now both v_2w_1 and v_1w_2 must be red as otherwise vv_2yxw_1wv or vv_1xyw_2wv would be a red 6-cycle with blue diagonals. Every vertex in G now has degree 3. Hence, there can be no more red edges. However, $\{v, v_2, w_1, w\}$ is an irredundant set in G, a contradiction of $IR(G) < 4$.

Case 2: Suppose $xy \in E(G)$ is blue. Each vertex of $\{v_1, v_2, w_1, w_2\}$ sends a red edge to either x or y. For example, if v_2x and v_2y are blue, then $\{w, v_2, x, y\}$ is a blue K_4. Now, there is a red edge between $\{v_1, v_2\}$ and $\{w_1, w_2\}$, otherwise these vertices form a blue K_4. We may assume without loss of generality that v_2x and v_2w_1 are red. So w_1x is blue. As w_1 must send a red edge to x or y, the edge w_1y is red and the remaining edges to v_2 and w_1 are blue. The set $X = \{v_1, x, y, w_2\}$ is irredundant in G since $v \in PN(v_1, X)$, $v_2 \in PN(x, X)$, $w_1 \in PN(y, X)$, and $w \in PN(w_2, X)$. This is a contradiction of $IR(G) < 4$.

Thus, G has no vertices of degree 3, so G must be 2-regular. Therefore, G has an independent set of size 4, again contradicting $IR(G) < 4$. Hence, $s(3, 4) \leq 8$.

To show $s(3, 4) \geq 8$ we note that $\overline{IR}(C_7) = 2$ and $IR(C_7) = 3$. □

Suppose the hypothesis of Proposition 1.11 is satisfied. We can claim a p-element set X such that $\langle X \rangle_R$ is bipartite. Then X contains a $\lceil \frac{p}{2} \rceil$-element set that is independent in $\langle R \rangle$. Thus, if $\lceil \frac{p}{2} \rceil \geq n - 1$, this set together with v yields an n-element independent set in $\langle R \rangle$. This observation gives rise to short proofs of the known facts $s(3, 5) \leq 12$ and $s(3, 6) \leq 15$.

Theorem 2.3. *[3]* $s(3, 5) = 12$.

Proof. Consider a bicoloring of K_{12}. By Proposition 1.9:

$$p - s(m, n-1) \leq \delta_R \leq \Delta_R \leq s(m-1, n) - 1$$

$$12 - s(3, 4) \leq \delta_R \leq \Delta_R \leq s(2, 5) - 1$$

$$12 - 8 \leq \delta_R \leq \Delta_R \leq 5 - 1$$

$$4 \leq \delta_R \leq \Delta_R \leq 4$$

.

Thus, $\langle R \rangle$ is 4-regular. Let v be any vertex of G. Every vertex of B_v sends red edges to R_v, for otherwise R_v together with a vertex of B_v would constitute a 5-vertex independent set in G, a contradiction of $IR(G) < 5$. Therefore, Proposition 1.9 can be applied using $X = B_v$. We thus find an independent set of order $\lceil \frac{7}{2} \rceil + 1 = 5$, contradicting $IR(G) < 5$.

To show $s(3,5) \geq 12$, we display an 11-vertex graph with $\overline{IR} = 2$ and $IR = 4$ in Figure 2.1. $\square$

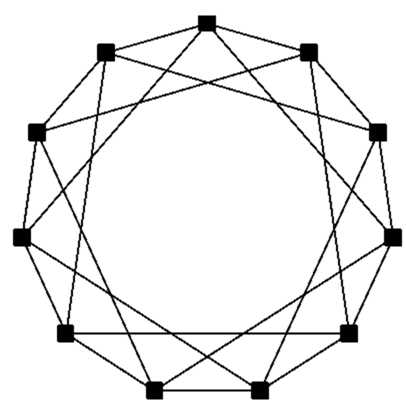


Figure 2.1: An 11-vertex graph with $\overline{IR} = 2$ and $IR = 4$.

Theorem 2.4. *[4]* $s(3,6) = 15$.

Proof. We look at a bicoloring of K_{15}. Applying Proposition 1.9 we have,

$$p - s(m, n-1) \leq \delta_R \leq \Delta_R \leq s(m-1, n) - 1$$

$$15 - s(3,5) \leq \delta_R \leq \Delta_R \leq s(2,6) - 1$$

$$15 - 12 \leq \delta_R \leq \Delta_R \leq 6 - 1$$

$$3 \leq \delta_R \leq \Delta_R \leq 5.$$

So $\langle R \rangle$ has 15 vertices all of which have degree 3, 4, or 5. As no graph has an odd number of odd vertices, there exists at least one vertex, v, of degree 4. We want to avoid a 6-element independent set in $\langle R \rangle$. If there were three vertices of B_v, each adjacent to all of the vertices of R_v in $\langle B \rangle$, then there would be a 6-element independent set in $\langle R \rangle$ if any two of these three were adjacent in $\langle B \rangle$ and a 3-element independent set in $\langle B \rangle$ otherwise. Thus, at least eight vertices are sending red edges. That is, at most two of the vertices of B_v are completely joined to R_v in $\langle B \rangle$. We may apply Proposition 1.11 to $X \subset B_v$ with $|X|$=9. Thus we obtain an independent set in $\langle R \rangle$ with $\lceil 9/2 \rceil + 1 = 6$ vertices. Thus, $s(3,6) \leq 15$.

We present a 14-vertex graph that has $\overline{IR} = 2$ and $IR = 5$ in Figure 2.2 to show $s(3,6) \geq 15$. (It is tedious, but possible, to prove this by hand, but it can also be verified by computer using the program of [8].) $\square$

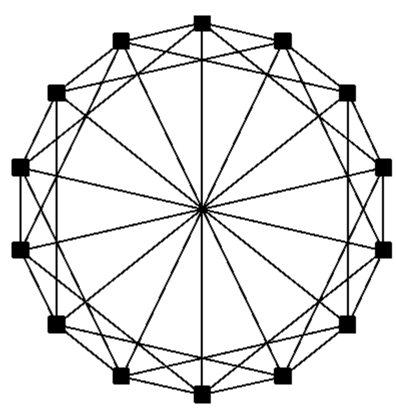


Figure 2.2: A 14-vertex graph with $\overline{IR} = 2$ and $IR = 5$.

Theorem 2.5. *[11]* $s(3,7) = 18$.

First we give some results that are used throughout the proof.

Lemma 2.6. *If G is a (3,7)-graph of order 18, then $3 \leq \delta(G) \leq \Delta(G) \leq 6$.*

Proof. By Proposition 1.9 we have

$$18 - s(3,6) \leq \delta(G) \leq \Delta(G) \leq s(2,7) - 1$$

giving the desired result. $\square$

Lemma 2.7. *Suppose G satisfies $IR(\overline{G}) < 3$ and v is a vertex of degree at least 2. If $v_1, v_2,$ and $v_3 \in V(G)\backslash N[v]$ and $v_1v_2, v_2v_3 \in E(G)$, then either $N(v_1) \cap N(v) \subseteq N(v_3) \cap N(v)$ or $N(v_3) \cap N(v) \subseteq N(v_1) \cap N(v)$.*

Proof. Suppose there are vertices u_1 and u_2 satisfying $u_1 \in (N(v_1)\backslash N(v_3)) \cap N(v)$ and $u_2 \in (N(v_3)\backslash N(v_1)) \cap N(v)$. Then the 6-cycle $vu_1v_1v_2v_3u_2v$, where $vv_2, u_1v_3, v_1u_2 \notin E(G)$ implies that $IR(\overline{G}) \geq 3$, a contradiction of our assumption that $IR(\overline{G}) < 3$. $\square$

It suffices to show that $s(3,7) \leq 18$ as the circulant graph $C_{17}\{1,4\}$ is a $(3,7)$-graph implying $s(3,7) \geq 18$.

We now present the proof that $s(3,7) \leq 18$ due to Chen and Rousseau in [7].

Proof. We assume to the contrary that G is a (3,7)-graph with 18 vertices. Then $3 \leq \delta(G) \leq \Delta(G) \leq 6$ by Lemma 2.6. Now, let $v \in G$ with $d(v) = \Delta(G)$. Let $d(u,v)$ denote the distance in G from u to v and for each positive integer i set $V_i = \{u|d(u,v) = i\}$ and $V_{>i} = \cup_{j>i}V_j$.
Since $d(v) \leq 6$ we have that $|V_{>1}| \geq 11$. Then, $|V_2| \leq 9$ by Proposition 1.11. As G is a (3,7)-graph and $N(v)$ is an independent set, it follows that $G\langle V_{>2}\rangle$ is a $(3, 7-\Delta(G))$-graph.

Claim 2.8. *The degree of v is $d(v) = \Delta(G) = 4$.*

Proof. Since $G\langle V_{>2}\rangle$ is a $(3, 7-\Delta(G))$-graph, we have $d(v) \leq 4$ as $s(3,1) = 1$ and $s(3,2) = 3$. Suppose $d(v) = \Delta(G) = 3$. Then, $|V_2| \leq 2|N(v)| = 6$. Since $G\langle V_{>2}\rangle$ is a

(3,4)-graph and $s(3,4) = 8$, we have $|V_{>2}| \leq 7$. Hence,

$$18 = |V(G)| \leq 1 + 3 + 6 + 7 = 17,$$

a contradiction.

Claim 2.9. *Either* $|V_2| = 9$ *and* $|V_{>2}| = 4$ *or* $|V_2| = 8$ *and* $|V_{>2}| = 5$. *In addition,* $V_{>3} = \emptyset$.

Proof. As $d(v) = 4$, it follows that $G\langle V_{>2}\rangle$ is a (3,3)-graph and $|V_{>2}| \leq 5$. By Proposition 1.11, we have that $|V_2| \leq 9$. Since $|V_2| + |V_{>2}| = 13$, there are two cases: (a) $|V_2| = 9$ and $|V_{>2}| = 4$ or (b) $|V_2| = 8$ and $|V_{>2}| = 5$. If (a) holds and $w \in V_{>3}$, then Proposition 1.11 yields a 7-element independent set consisting of v, w, and five vertices from V_2. Thus, $V_{>3} = \emptyset$. In case (b), we must have $G\langle V_{>2}\rangle \cong C_5$ and it follows that $V_{>3} = \emptyset$ as $\delta(G) \geq 3$.

Let (X, Y) be a bipartition of $G\langle V_2\rangle$. Let c denote the number of components of $G\langle V_2\rangle$ and for $i = 1, 2, 3, ..., c$, let (X_i, Y_i) be bipartitions of these components. We may assume $V_2 = X \cup Y$, with $X = \cup X_i$, $Y = \cup Y_i$, and $|X_i| \geq |Y_i|$ for $i = 1, 2, 3, ..., c$. (We note that Y_i may be empty if $|X_i| = 1$). If S and T are disjoint sets of vertices in G, we say that there is an ST edge if $N(S) \cap T \neq \emptyset$.

Claim 2.10. *The bipartition* (X, Y) *must satisfy* $4 \leq |X| \leq 5$ *and* $3 \leq |Y| \leq 4$. *If* $|X| = 5$, *then* $V_3 \subset N(X)$. *If* $|Y| = 4$, *then for every nonadjacent pair* $W = \{w_i, w_j\} \subset V_3$ *there is a* WX *edge and a* WY *edge.*

Proof. Claim 2.10 follows as $8 \leq |V_2| \leq 9$ and the independence number of $G\langle V_2 \cup V_3\rangle \leq 5$.

Claim 2.11. *For any vertex* $w \in V_3$ *and any connected component* (X_i, Y_i) *of* (X, Y), *either* $N(w) \cap X_i = \emptyset$ *or* $N(w) \cap Y_i = \emptyset$.

Proof. Claim 2.11 follows as $G\langle V_2 \cup \{w\}\rangle$ is a bipartite graph.

Claim 2.12. *If $|Y| = 4$ and $|X_i| \geq 2$, then $|N(x) \cap V_3| \leq 1$ for any $x \in X_i$.*

Proof. Suppose there exists a vertex $x_1 \in X_i$ such that $N(x_1) \cap V_3 \supset W = \{w_1, w_2\}$. By Claim 2.11, there is no WY_i edge. Thus, by Claim 2.10, as $|Y| = 4$, there is a $W(Y \backslash Y_i)$ edge, say wz_3 where $w \in W$. Also, there is an edge joining W to $X \backslash X_i$, say $w'z_2$ where $w' \in W$, since otherwise $W \cup Y_i \cup (X \backslash X_i)$ is an independent set of at least 6 vertices. Indeed, if $|X| = 4$, then $|W| = 2$ and $|X_i| = a = |Y_i|$. We have

$$|W| + |Y_i| + |X \backslash X_i| = 2 + a + (|X| - a) = 2 + |X| \geq 2 + 4 = 6.$$

Now, if $|X| = 5$, then as $|W| = 2$ and $a = |X_i| = |Y_i| + 1$ we have

$$|W| + |Y_i| + |X \backslash X_i| = 2 + a - 1 + |X| - a = |X| + 1 \geq 5 + 1 = 6.$$

Since x_1 is adjacent to w_1 and w_2, and to at least one vertex in Y_i, x_1 is adjacent to precisely one vertex in V_1, say u_1. Now $d(x_1) = \Delta(G) = 4$. Since $|X_i| \geq 2$, there exists $x_2 \neq x_1$ such that $x_2 \in X_i$. Then, as $H = \langle X_i \cup Y_i\rangle$ is connected, there exists a path P from x_1 to x_2. Let x_1, z_1, x'_2, x_2 be the vertices of P. We have that x_1 and x'_2 (which may equal x_2) must have the common neighbor in $u_1 \in V_1$. Now x_1wz_3 is a path, so x_1 and z_3 must have the common neighbor $u_1 \in V_1$. Now, $d(u_1) = \Delta(G) = 4$. But, $x_1w'z_2$ is a path so x_1 and z_2 must have a common neighbor V_1 which must also be u_1, implying $d(u_1) \geq 5$, a contradiction.

We proceed with the proof divided into cases according to the structure of connected components and values of $|V_2|$ and $|V_3|$.

Case 2.13. $|V_2| = 9$ *and* $|V_3| = 4$.

As the independence number of $G\langle V_2\rangle$ is at most 5, it follows that $|X| = 5$ and $|Y| = 4$. Without loss of generality, we assume that $|X_1| = |Y_1| + 1$ and $|X_i| = |Y_i|$ for $i \neq 1$. By Claim 2.10, $N(X) \supset V_3$. Furthermore, in this case, we have $N(X_1) \supset V_3$, since if $w \in V_3$ and $w \notin N(X_1)$, then $G\langle V_2 \cup \{w\}\rangle$ would contain a 6-element independent set by Claim 2.11.

Suppose that (X, Y) is connected, that is, $|X_1| = 5$ and $|Y_1| = 4$. Then, by Claim 2.11, there is no V_3Y_1 edge. Hence there is a pair of independent vertices in $G\langle V_3\rangle$ and $G\langle Y \cup V_3\rangle$ contains a 6-element independent set, a contradiction.

Thus $1 \leq |X_1| \leq 4$. Since $N(X_1) \supset V_3$ and the neighborhood of any vertex in G is an independent set, $|X_1| \geq 2$. If $2 \leq |X_1| \leq 3$, then since $N(X_1) \supset V_3$, there exists a vertex $x_1 \in X_1$ such that $|N(x_1) \cap V_3| \geq 2$. This contradicts Claim 2.12. Therefore, $|X_1| = 4$, $|Y_1| = 3$, and $|X_2| = |Y_2| = 1$. Now let $X_1 = \{x_1, x_2, x_3, x_4\}$, $X_2 = \{x_5\}$, $Y_1 = \{y_1, y_2, y_3\}$ and $Y_2 = \{y_4\}$. There is no Y_1V_3 edge by Claim 2.11. We consider the graph induced by the set $\{w_1, w_2, w_3, w_4, x_5, y_4\}$. This graph contains no triangle, so it must contain a 3-element independent set. Three such vertices together with Y_1 constitute a 6-element independent set, a contradiction.

Case 2.14. $|V_2| = 8$ *and* $|V_3| = 5$ *with* $|X| = |Y| = 4$.

Since $G\langle V_3\rangle$ is a (3,3)-graph, it is a 5-cycle. Let $G\langle V_3\rangle = w_1w_2w_3w_4w_5w_1$. In this case, $|X_i| = |Y_i|$ for $i = 1, 2, ..., c$. We say that a pair of nonadjacent vertices $W = \{w_i, w_j\} \subset V_3$ has the property $P(k)$ if there is a WX_k edge and a WY_k edge. For each nonadjacent pair $\{w_i, w_j\}$ there is at least one k for which $\{w_i, w_j\}$ has property $P(k)$. Since otherwise, $G\langle V_2 \cup \{w_i, w_j\}\rangle$ has a 6-element independent set by Claim 2.11.

Notice that C_5 has five pairs of nonadjacent vertices and $c \leq 4$. Then for some k there are two pairs of nonadjacent vertices, say $\{w_1, w_3\}$ and $\{w_i, w_j\}$, having property

$P(k)$, by the pigeonhole principle.

Now we show that $|X_k| = |Y_k| \geq 2$. Suppose, instead, that $X_k = \{x_1\}$ and $Y_k = \{y_1\}$. We may assume $w_1x_1 \in E(G)$ and $w_3y_1 \in E(G)$. Since G contains no K_3, $w_2 \notin \{w_i, w_j\}$. Hence, without loss of generality, we may assume $w_i = w_1$ and $w_j = w_4$. Then we must have $w_4y_1 \in E(G)$ so $\{y_1, w_3, w_4\}$ is a triangle, a contradiction.

As $|X_k| = |Y_k| \geq 2$, we have $c \leq 3$. In this case, three pairs of nonadjacent vertices in V_3, say $\{w_1, w_3\}$, $\{w_1, w_4\}$, and $\{w_i, w_j\}$, have property $P(k)$ for some k by the pigeonhole principle. Without loss of generality, we assume that $w_1 \in N(Y_k)$ and $\{w_3, w_4\} \subseteq N(X_k)$. By Claim 2.12, $|N(x) \cap V_3| \leq 1$ for $x \in X_k$. Note that in this case, the same argument shows that $|N(y) \cap V_3| \leq 1$ for $y \in Y_k$. Assume that $\{x_1w_3, x_2w_4, y_1w_1\} \subset E(G)$.

Now suppose $|X_k| = |Y_k| = 2$. Since $G\langle X_k \cup Y_k\rangle$ is connected, we may assume $x_1y_1 \in E(G)$. The 6-cycle $w_1y_1x_1w_3w_4w_5w_1$, where $w_1w_3, y_1w_4, x_1w_5 \notin E(G)$, implies $IR(\overline{G}) \geq 3$, a contradiction.

Thus $|X_k| = |Y_k| \geq 3$. In this case, we have $c \leq 2$. Now there are four pairs of nonadjacent vertices satisfying $P(k)$ for some k. In particular, we have $N(X_k \cup Y_k) \supseteq V_3$. If $|X_k| = |Y_k| = 3$, let $X_k = \{x_1, x_2, x_3\}$ and $Y_k = \{y_1, y_2, y_3\}$. Without loss of generality, we may assume that $|N(X_k) \cap V_3| \geq 3$, $x_1w_1 \in E(G)$, and $x_3w_3 \in E(G)$. Now we must have $|N(X_k) \cap V_3| = 3$ and $|N(x_1) \cap V_3| = |N(x_2) \cap V_3| = |N(x_3) \cap V_3| = 1$ by Claim 2.12. Suppose $u \in V_1 \cap N(x_1) \cap N(x_3)$. Then the 6-cycle $ux_1w_1w_2w_3x_3u$ where $uw_2, x_1w_3, w_1x_3 \notin E(G)$ implies that $IR(\overline{G}) \geq 3$, a contradiction.

Thus, there exist vertices $u_1, u_2 \in V_1$ such that $u_1 \in N(x_1) \backslash N(x_3)$ and $u_2 \in N(x_3) \backslash N(x_1)$. If $y \in Y_k$ is adjacent to both x_1 and x_3, then the 6-cycle $x_1yx_3u_2vu_1x_1$ with $x_1u_2, yv, x_3u_1 \notin E(G)$ implies $IR(\overline{G}) \geq 3$, a contradiction.

Since $G(X_k \cup Y_k)$ is connected, we may assume y_1 is adjacent to both x_1 and x_2

and y_3 is adjacent to both x_2 and x_3. Thus, $N(x_2) \cap V_1 \supseteq (N(x_1) \cup N(x_3)) \cap V_1$ (since otherwise, there is a 6-cycle with no adjacent pair of opposite vertices). Now $|N(x_2) \cap V_1| \geq 2$, $|N(x_2) \cap Y_1| \geq 2$, and $|N(x_2) \cap V_3| = 1$ implying $d(x_2) > 4$ which contradicts $\Delta(G) = 4$.

Thus $X_k = X$ and $Y_k = Y$ so (X, Y) is connected. We have either $N(w_i) \cap X = \emptyset$ or $N(w_i) \cap Y = \emptyset$ for any vertex $w_i \in V_3$, by Claim 2.11. Since $G\langle V_3 \rangle$ is a 5-cycle, there are two nonadjacent vertices $w_i, w_j \in V_3$ so that either $(N(w_i) \cup N(w_j)) \cap X = \emptyset$ or $(N(w_i) \cup N(w_j)) \cap Y = \emptyset$, which contradicts Claim 2.10.

Case 2.15. $|X| = 5$ *and* $|Y| = 3$.

By Claim 2.10, $N(X) \supset V_3$. We assume $|X_1| > |Y_1|$, and $|X_2| = |Y_2| + 1$ if $|X_1| = |Y_1| + 1$.

Subcase 2.16. $|X_1| = |Y_1| + 2$

In this case, $N(X_1) \supseteq V_3$. As (X_1, Y_1) is connected, there is a vertex, say y_1, such that $|N(y_1) \cap X_1| \geq 3$. Assume that $N(y_1) \supseteq \{x_1, x_2, x_3\}$. Then by Lemma 2.7 there is a vertex $u_1 \in V_1$ such that $N(u_1) \supseteq \{x_1, x_2, x_3\}$. Thus, $N(u_1) = \{v, x_1, x_2, x_3\}$.

If $|X_1| = 3$, there are two vertices, x_1 and x_2, such that $|N(\{x_1, x_2\}) \cap V_3| = 4$. Without loss of generality, we may assume that $N(x_1) \cap V_3 = \{w_1, w_3\}$ and $N(x_2) \cap V_3 = \{w_2, w_4\}$. Now the 6-cycle $u_1 x_1 w_1 w_5 w_4 x_2 u_1$ where $u_1 w_5, x_1 w_4, w_1 x_2 \notin E(G)$ implies that $IR(\overline{G}) \geq 3$, a contradiction.

If $|X_1| = 4$, we denote $X_1 = \{x_1, x_2, x_3, x_4\}$, $Y_1 = \{y_1, y_2\}$, $X_2 = \{x_5\}$, and $Y_2 = \{y_3\}$. As $\Delta(G) = 4$, we have $x_4 y_1 \notin E(G)$ so $x_4 y_2 \in E(G)$. Without loss of generality, assume that $x_3 y_2 \in E(G)$. Then $N(x_4) \cap V_1 \subseteq N(x_3) \cap V_1$, by Lemma 2.7. Let $N(x_4) \cap V_1 = \{u_2\}$. Then $N(u_2) \supseteq \{v, x_3, x_4\}$. Thus, $|N(u_2) \cap \{x_5, y_3\}| \leq 1$ since $d(u_2) \leq 4$. Assume that $y_3 \notin N(u_2)$. Then, since $|N(y_3) \cap V_3| \leq 2$, there are two nonadjacent vertices, w_1 and w_3, which are not adjacent to y_3. Since $N(X_1) \supseteq V_3$,

$N(\{w_1, w_3\}) \cap Y_1 = \emptyset$. Hence $Y \cup \{u_1, u_2, w_1, w_3\}$ forms an independent set of 7 vertices, a contradiction.

Thus $X_1 = X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y_1 = Y = \{y_1, y_2, y_3\}$. As $N(X) \supseteq V_3$, $N(Y) \cap V_3 = \emptyset$. Without loss of generality, we assume that $\{x_4y_2, x_3y_2\} \subseteq E(G)$. Since $x_4u_1 \notin E(G)$, $N(x_4) \cap V_1 \subseteq N(x_3) \cap V_1$. Assume $u_2 \in N(x_4) \cap V_1$. If there is a vertex x_i, $i = 1, 2$, such that $N(x_i) \cap V_1 \supseteq N(x_3) \cap V_1$, then $N(u_2) = \{v, x_i, x_3, x_4\}$. Hence $Y \cup \{u_1, u_2, w_1, w_3\}$ forms an independent set of 7 vertices, a contradiction.

Thus we have $N(x_3) \cap V_1 \supseteq N(x_i) \cap V_1$ for each $i = 1, 2, 4$. Since $d(x_3) \leq 4$, we have $N(x_3) \cap V_1 = \{u_1, u_2\}$. Notice that (X, Y) is connected. By Lemma 2.7, $N(x_3) \cap V_1 \supseteq N(x_5) \cap V_1$. In particular, we have $u_2x_5 \in E(G)$. Again, $Y \cup \{u_1, u_2, w_1, w_3\}$ forms an independent set, a contradiction.

Subcase 2.17. $|X_1| = |Y_1| + 1 = 3$, *and* $|X_2| = |Y_2| + 1 = 2$

Let $X_1 = \{x_1, x_2, x_3\}$, $Y_1 = \{y_1, y_2\}$, $X_2 = \{x_4, x_5\}$, and $Y_2 = \{y_3\}$. Since there is no independent set of six vertices in $G\langle V_{>1}\rangle$, we have $N(X) \supset V_3$ and for any nonadjacent pair of vertices $W = \{w_i, w_j\}$ in V_3 there is a WX_1 edge and a WX_2 edge. Thus $|N(X_i) \cap V_3| \geq 3$ for $i = 1, 2$.

By Lemma 2.7, there is a vertex $u_1 \in V_1$ such that $N(u_1) \supseteq \{x_4, x_5\}$. Since $|N(X_2) \cap V_3| \geq 3$, we assume $N(x_5) \cap V_3 = \{w_1, w_3\}$. As $N(\{w_1, w_3\}) \cap X_1 \neq \emptyset$, we assume $w_1x_3 \in E(G)$. By Lemma 2.7, $u_1x_3 \in E(G)$. Now $N(u_1) = \{v, x_3, x_4, x_5\}$. Without loss of generality, assume $N(x_3) \cap N(x_2) \cap Y_1 \neq \emptyset$. Since $x_2u_1 \notin E(G)$, there is a vertex $u_2 \in V_1$ such that $N(u_1) \supseteq \{x_2, x_3\}$. Hence, $N(x_3) = \{u_1, u_2, w_1\} \cup (N(x_3) \cap Y_1)$ and $N(x_2) \cap V_1 = \{u_2\}$. By Lemma 2.7, we have $u_2 \in N(x_1) \cap V_1$ since (X_1, Y_1) is connected. Thus $N(u_2) = \{v, x_1, x_2, x_3\}$.

If $|N(x_1) \cap V_3| = 2$, assume that $N(x_1) \cap V_3 = \{w_i, w_j\}$ for nonadjacent w_i and w_j. Then $N(x_1) \cap V_1 = \{u_2\}$. By Lemma 2.7, $(N(w_1) \cup N(w_3)) \cap \{x_4, x_5\} = \emptyset$, a contradiction. Thus $|N(x_1) \cap V_3| = 1$. Similarly, $|N(x_2) \cap V_3| = |N(x_3) \cap V_3| = 1$. As $|N(X_1) \cap V_3| \geq 3$, there are two nonadjacent vertices in $N(X_1) \cap V_3$. Without loss of generality, assume $x_1w_2, x_2w_4 \in E(G)$. The 6-cycle $u_2x_1w_2w_3w_4x_2u_2$ where opposite vertices are nonadjacent, implies $IR(\overline{G}) \geq 3$, a contradiction.

Subcase 2.18. $|X_1| = |X_2| = 2$ *and* $|X_3| = |Y_1| = |Y_2| = |Y_3| = 1$.

Let $X_1 = \{x_1, x_2\}$, $X_2 = \{x_3, x_4\}$, $X_3 = \{x_5\}$, $Y_1 = \{y_1\}$, $Y_2 = \{y_2\}$, and $Y_3 = \{y_3\}$. Then $\{x_1y_1, x_2y_1, x_3y_2, x_4y_2, x_5y_3\} \subset E(G)$. Clearly, $N(X \backslash \{x_5\}) \supset V_3$. Hence we may assume $|N(x_1) \cap V_3| = 2$. Without loss of generality, assume $N(x_1) \cap V_3 = \{w_1, w_3\}$. So $(N(w_1) \cup N(w_2)) \cap Y_1 = \emptyset$. Since $d(x_1) \leq 4$, $|N(x_1) \cap V_1| = 1$. Let $N(x_1) \cap V_1 = \{u_1\}$. By Lemma 2.7, $u_1x_2 \in E(G)$.

Suppose $N(\{w_1, w_3\}) \cap X_2 = \emptyset$. As there is no independent set of six vertices in $G\langle V_{>1}\rangle$, $N(w_1) \cup N(w_3) \supset \{x_5, y_3\}$. By Lemma 2.7, $N(u_1) \supset \{x_5, y_3\}$, implying $d(u_1) \geq 5$, a contradiction. Hence $N(\{w_1, w_3\}) \cap X_2 \neq \emptyset$. Without loss of generality, assume $x_3w_1 \in E(G)$. Then $u_1x_3 \in E(G)$ by Lemma 2.7, and $N(u_1) = \{v, x_1, x_2, x_3\}$. So $N(x_4) \cap V_1 \subset N(x_3) \cap V_1$, by Lemma 2.7. Since $|N(x_3) \cap V_1| \leq 2$, we may assume $N(x_3) \cap V_1 = \{u_1, u_2\}$ and $N(x_4) \cap V_1 = \{u_2\}$.

Then $N(x_3) \cap V_3 = \{w_1\}$. Since $N(X \backslash \{x_5\}) \supset V_3$, we have $N(x_2) \cup N(x_4) \supset \{w_2, w_4, w_5\}$. In particular, there is $i = 2$ or 4 such that $|N(x_i) \cap \{w_2, w_4, w_5\}| = 2$.

If $i = 2$, then $N(x_2) \cap V_1 = \{u_1\}$. By Lemma 2.7 and as $N(x_3) = \{w_1, y_2, u_1, u_2\}$, $(N(x_2) \cap V_3) \cup \{x_3, x_4, x_5, y_1, v\}$ is an independent set, a contradiction. Thus, $|N(x_4) \cap \{w_2, w_4, w_5\}| = 2$. As $N(x_1) \cap \{w_2, w_4, w_5\} = \emptyset$, $N(x_2) \cap N(x_4) \cap V_3 \neq \emptyset$. Similarly, we can show that $N(x_2) \cap V_3 \subseteq N(x_4) \cap V_3$. Thus, $V_3 = N(X_1 \cup X_2) \cap V_3 = N(\{x_1, x_4\}) \cap V_3 \neq V_3$, a contradiction.

Subcase 2.19. $|X_1| = 1, |Y_1| = 0, |X_2| = |Y_2| + 1$, *and* $|X_i| = |Y_i|$ *for* $i \neq 1, 2$.

Set $X_1 = \{x_1\}$. Since $N(x_1)$ is an independent set, $|N(x_1) \cap V_3| \leq 2$. We assume

$N(x_1) \cap V_3 \subseteq \{w_2, w_5\}$. By Proposition 1.11, $N(X_2) \supset \{w_1, w_3, w_4\}$. In particular, $|X_2| \geq 2$. Since $|(V_2 \backslash \{x_1\}) \cup \{w_1, w_3\}| = 9$, $G\langle (V_2 \backslash \{x_1\}) \cup \{w_1, w_3\}\rangle$ is not bipartite. Hence there is a $k \neq 1, 2$ such that either $N(w_1) \cap X_k \neq \emptyset$ and $N(w_3) \cap Y_k \neq \emptyset$ or $N(w_1) \cap Y_k \neq \emptyset$ and $N(w_3) \cap X_k \neq \emptyset$.

Without loss of generality, we assume $N(w_1) \cap X_3 \neq \emptyset$ and $N(w_3) \cap Y_3 \neq \emptyset$. Since $|N(w_1) \cap V_2| \leq 2$, $N(w_1) \cap V_2 \subset X_2 \cup X_3$. Thus $N(w_4) \cap Y_3 \neq \emptyset$ as $G\langle V_2 \backslash \{x_1\}\rangle \cup \{w_1, w_4\}$ is not bipartite. Since there is no triangle in G and $w_3 w_4 \in E(G)$, we have $|Y_3| \geq 2$. Hence, $|X_2| = |X_3| = |Y_3| = 2$ and $|Y_2| = 1$. Set $X_2 = \{x_2, x_3\}$, $X_3 = \{x_4, x_5\}$, $Y_2 = \{y_1\}$, and $Y_3 = \{y_2, y_3\}$.

Since $N(x_2) \cup N(x_3) \supset \{w_1, w_3, w_4\}$, without loss of generality we assume $x_2 w_1 \in E(G)$. Since $N(w_3) \cap N(w_4) = \emptyset$, without loss of generality we assume that $\{x_2 w_1, x_2 w_3, x_3 w_4, x_4 w_1, y_2 w_3, y_3 w_4\} \subseteq E(G)$. Since $d(x_2) \leq 4$ we have $|N(x_2) \cap V_1| = 1$. Let $N(x_2) \cap V_1 = \{u_1\}$. By Lemma 2.7, $N(u_1) \supset \{x_3, x_4, y_2\}$. Thus $N(u_1) \supset \{v, x_2, x_3, x_4, y_2\}$, which contradicts $d(u_1) \leq 4$.

□

For the proof of $s(4, 4) \leq 13$ we need an algorithm described in [9]. This algorithm constructs all the $(4, 4, 13)$-graphs G in which a vertex v has degree five with the vertices in $N_R(v) = \{1, 2, 3, 4, 5\}$ having degrees d_1, d_2, d_3, d_4, d_5 respectively where $d_i \in \{5, 6, 7\}$ for $i = 1, 2, 3$. It takes as input all $(3, 4, 5)$-graphs and all $(3, 4, 7)$-graphs and gives as output all $(4, 4, 13)$-graphs as specified, if any exist. Note that if G is a $(4, 4, p)$-graph and v is a vertex with degree d in G, then $\overline{G}$ is also a $(4, 4, p)$-graph and v has degree $p - 1 - d$ in $\overline{G}$. Thus, if we show that no $(4, 4, p)$-graph has a vertex of degree d, we will also have shown that no $(4, 4, p)$-graph has a vertex of degree $p - 1 - d$.

Theorem 2.20. *[9] $s(4, 4) = 13$.*

Proof. Suppose G is a $(4,4,13)$-graph. Since $s(3,4) = 8$, it follows from Proposition 1.9 that $5 \leq \delta(G) \leq \Delta(G) \leq 7$. The algorithm described above can be used to show that G has no vertex of degree 5 and hence also no vertex of degree 7. A slight adjustment of the algorithm shows that G cannot be 6-regular. It follows that $s(4,4) \leq 13$. An example of a $(4,4,12)$-graph which shows that $s(4,4) > 12$ appears in the following figure. □

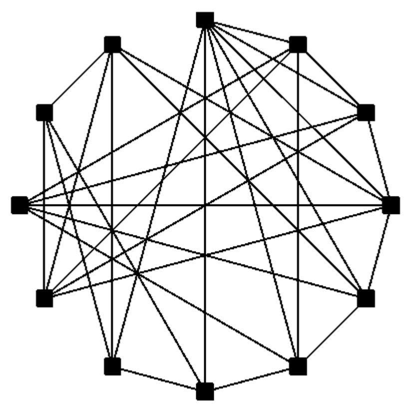


Figure 2.3: An 11-vertex graph with $\overline{IR} = 2$ and $IR = 4$.

CHAPTER 3: Mixed Ramsey Numbers $t(m, n)$

In this chapter, we provide all known results of the mixed Ramsey number. We recall that the *mixed Ramsey number* is the smallest p such that for every graph G of order p, $IR(\overline{G}) \geq m$ or $\beta(G) \geq n$.

We note that $t(1, n) = t(n, 1) = 1$ and $t(2, n) = t(n, 2) = n$ for all $n > 1$.

Theorem 3.1. *[10]*

(a) t(3,3)=6

(b) t(3,4)=9

(c) t(4,3)=8

(d) t(5,3)=13.

Proof. (a) Follows from the observation that $s(3, 3) \leq t(3, 3) \leq r(3, 3)$ and $s(3, 3) = r(3, 3) = 6$.

(b) First we have that $t(3, 4) \leq r(3, 4) = 9$ and we may easily verify that the graphs G_1 and G_3 depicted in Figure 3.1 satisfy $\beta(G_1) = \beta(G_3) = 3$ and (by Corollary 1.7) $IR(\overline{G_1}) = IR(\overline{G_3}) = 2$. Thus, $t(3, 4) > 8$.

(c) As $s(4, 3) \leq t(4, 3) \leq r(4, 3)$ we have $8 \leq t(4, 3) \leq 9$. We also note that the graphs G_1, G_2, and G_3 in Figure 3.1 are the only 8-vertex graphs G with $\beta(\overline{G}) = 2$ and $\beta(G) = 3$. Therefore, $\overline{G_1}$, $\overline{G_2}$, and $\overline{G_3}$ are the only 8-vertex graphs G with $\beta(G) = 2$ and $\beta(\overline{G}) = 3$. It is easy to see that each of G_1, G_2, and G_3 have an irredundant set of cardinality four (an irredundant set is denoted by the circular vertices). As $IR(G) \geq \beta(G)$, every 8-vertex graph G therefore satisfies $\beta(G) \geq 3$ or $IR(\overline{G}) \geq 4$. Hence, $t(4, 3) = 8$.

(d) By Proposition 1.5, $t(5, 3) \leq t(4, 3) + t(5, 2) = 8 + 5 = 13$. The graph G depicted in Figure 3.2 is a 12-vertex graph with $\beta(\overline{G}) = 2$ and $IR(G) = 4$. (We can easily verify this by computer (see [8]) or directly, a routine but tedious exercise.) □

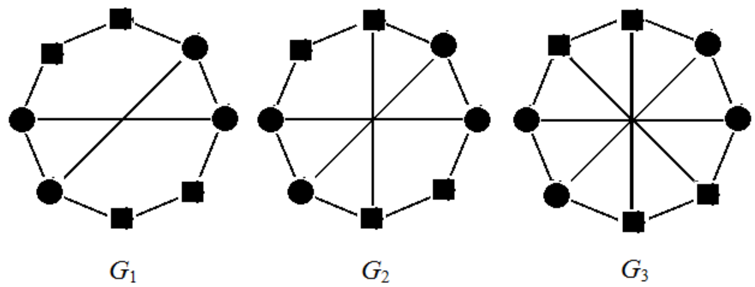


Figure 3.1: Graphs G_1, G_2 and G_3 used in the proof of Theorem 3.1.

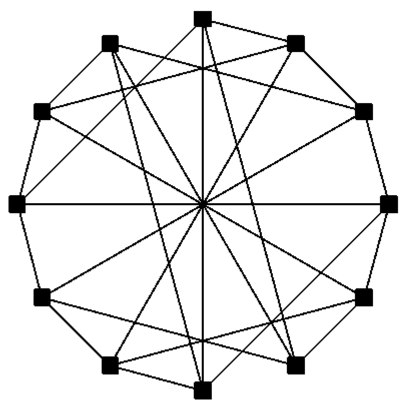


Figure 3.2: A graph G with $\beta(\overline{G}) = 2$ and $IR(G) = 4$.

Theorem 3.2. *[10]* $t(3,5) = 12$.

Proof. Suppose G is a 12-vertex graph with $IR(\overline{G}) < 3$ and $\beta(G) < 5$. Since $t(3,4) = 9$ and $t(2,5) = 5$, it follows from Proposition 1.10 that $12-9 \leq \delta(G) \leq \Delta(G) \leq 5-1$ so $3 \leq \delta(G) \leq \Delta(G) \leq 4$.

Suppose v has degree 4. Then each vertex of B_v must send a red edge to R_v, for otherwise R_v together with a vertex of B_v would constitute an independent set of cardinality five, contradicting $\beta(G) < 5$. Thus, Proposition 1.11 can be applied using $X = B_v$. We therefore find an independent set of $\lceil \frac{7}{2} \rceil + 1 = 5$ vertices, a contradiction of $IR(G) < 5$.

Hence, G is 3-regular. Let v be any vertex of G. At most two vertices of B_v send no red edges to R_v, for otherwise R_v together with two vertices of B_v would constitute an independent set of cardinality five. Therefore B_v contains a 7-vertex set X that complies with the hypothesis of Proposition 1.11. It follows that there exists an independent set with $\lceil \frac{7}{2} \rceil + 1 = 5$ vertices, a contradiction. □

Theorem 3.3. $t(3,6) = 15$

Proof. The proof of $t(3,6) \leq 15$ is similar to the proof of $s(3,6) \leq 15$ in Theorem 2.4. That $t(3,6) \geq 15$ follows from $s(3,6) \leq t(3,6)$ and $s(3,6) = 15$. □

3.1 Mixed Ramsey Numbers $t(3,7)$ and $t(3,8)$

The following is based on [5].

In this chapter, we show that $t(3,7) = 18$ and $t(3,8) = 22$.

Using the fact that $s(m,n) \leq t(m,n) \leq r(m,n)$ and Propostion 1.5 we know that

$$18 = s(3,7) \leq t(3,7) \leq \min\{t(2,7) + t(3,6), r(3,7)\} = \min\{7 + 15, 23\} = 22,$$

$$18 = s(3,7) \leq s(3,8) \leq t(3,8) \leq \min\{t(2,8)+t(3,7), r(3,8)\} = \min\{8+22, 28\} = 28.$$

Before presenting the proofs, we first prove a corollary of Proposition 1.11:

Corollary 3.4. *If there is a star in $\langle V_{>1}(v)\rangle_{red}$ with three end-vertices x_1, x_2, and $x_3 \in V_2(v)$, then x_1, x_2, and x_3 are joined by means of red edges to a common vertex in $V_1(v)$.*

Proof. If x_1, x_2, and x_3 are not joined by means of red edges to a common vertex in $V_1(v)$, then each pair of vertices from the set $\{x_1, x_2, x_3\}$ must have a distinct common neighbor in $V_1(v)$ by Proposition 1.11. But then these three common neighbors in $V_1(v)$ and $\{x_1, x_2, x_3\}$ form a red 6-cycle with blue diagonals in $\langle V_1(v) \cup V_2(v)\rangle$, a contradiction by Corollary 1.7. $\square$

As $\langle V_2(v) \cup \{u\}\rangle_{\text{red}}$ is a bipartite graph for any $u \in V_{>2}(v)$ by Proposition 1.11, it follows that $\langle V_2(v)\rangle_{\text{red}}$ must itself be bipartite. In the proofs of $t(3,7) = 18$ and $t(3,8) = 22$, we use the symbol c to denote the number of components of $\langle V_2(v)\rangle_{\text{red}}$ and we denote the bipartitions of these components by (X_ℓ, Y_ℓ), for all $\ell = 1, ..., c$. We may assume, without loss of generality, that $|X_\ell| \geq |Y_\ell|$ for all $\ell = 1, ..., c$. Define $X = \cup_{\ell=1}^{c} X_\ell$ and $Y = \cup_{\ell=1}^{c} Y_\ell$. Then $|X| \geq |Y|$. We have the following six useful results.

Lemma 3.5. *Let v be any vertex of a $(3, n, p)$-graph and suppose $x \in V_{>2}(v)$.*

(a) If x sends a red edge to X_ℓ, then x sends no red edge to Y_ℓ and vice versa for any $\ell = 1, ..., c$.

(b) If $|X| \geq n - 2$ and there is exactly one $\ell \in \{1, ..., c\}$ such that $|X_\ell| > |Y_\ell|$, then each vertex in $V_3(v)$ sends a red edge to X_ℓ.

(c) If $|Y| \geq n - 3$, then there exists, for each edge uw in $\langle V_3(v)\rangle_{blue}$, an $\ell \in \{1, ..., c\}$ such that u sends a red edge to X_ℓ and w sends a red edge to Y_ℓ.

(d) If $|Y| \geq n-3$ and there is an odd cycle in $\langle V_{\geq 3}(v)\rangle_{blue}$, then the pairs of red edges sent to $V_2(v)$ by the edges of this cycle according to part (c) above go to at least two components of the bipartite graph $\langle V_2\rangle_{red}$.

(e) If $|Y| \geq n-3$, $\Delta(R) = 4$ and Z is a partite set of a component of $\langle V_2(v)\rangle_{red}$ such that $|Z| \geq 2$, then any vertex $z \in Z$ sends at most one red edge to $V_3(v)$.

(f) If $|X| \geq n - 2$, then $V_{>3}(v) = \emptyset$.

Proof. (a) If the statement of the lemma is false, then an odd cycle results in the bipartite graph $\langle V_2(v) \cup \{x\}\rangle_{\text{red}}$ guaranteed by Proposition 1.11.

(b) Suppose $|X| \geq n-2$ and that there is exactly one $\ell \in \{1, ..., c\}$ such that $|X_\ell| > |Y_\ell|$ and that there is a vertex $u \in V_3(v)$ sending no red edge to X_ℓ. It follows by part (a) that u does not send a red edge to both X_i and Y_i, for all $i = 1, ..., c$. Therefore, we may select from X_i or Y_i the part, say Z_i, from each component $i \in \{1, ..., c\}$ sending no red edge to u. But then $|\bigcup_{i=1}^{c} Z_i| = \sum_{i=1}^{c} |X_i| \geq n-2$ and hence $\{u, v\} \cup (\bigcup_{i=1}^{c} Z_i)$ is an independent set of cardinality at least n in the red subgraph of the $(3, n, p)$-graph, a contradiction.

(c) Suppose $|Y| \geq n-3$ and that there is a blue edge uw in $V_3(v)$, but no $i \in \{1, ..., c\}$ for which u sends a red edge to X_i and w sends a red edge to Y_i. Then we may select from X_i or Y_i the part, say Z_i, from each component $i \in \{1, ..., c\}$ sending no red edge to either u or to w by part (a). But then $|\bigcup_{i=1}^{c} Z_i| \geq \sum_{i=1}^{c} |Y_i| \geq n-3$ and hence $\{u, v, w\} \cup (\bigcup_{i=1}^{c} Z_i)$ is an independent set of cardinality at least n in the red subgraph of the $(3, n, p)$-graph, a contradiction.

(d) Suppose all the pairs of red edges sent to $V_2(v)$ by the edges of the odd blue cycle in $\langle V_{\geq 3} \rangle$ according to part (c) above go to the same component, say (X', Y'), of $V_2(v)$. Then it follows by part (a) above that each vertex of the blue cycle in $\langle V_{\geq 3} \rangle$ sends red edges to either X' or Y', but not to both. Therefore, the vertices of the blue cycle in $\langle V_{\geq 3} \rangle$ send red edges to X' or Y' in alternating fashion as one traverses the blue cycle, but this is impossible since the blue cycle is odd.

(e) Suppose Z and Z' are the partite sets of a component of the bipartite graph $\langle V_2(v) \rangle_{\text{red}}$ such that $|Z| \geq 2$. Since $\langle Z \rangle_{\text{red}}$ is connected, there is a red path $z_1 z' z_2$ in $\langle V_2(v) \rangle_{\text{red}}$ with $z_1, z_2 \in Z$ and $z' \in Z'$. If z_1 is joined by means of red edges to two vertices $w, w' \in V_3(v)$, then ww' is a blue edge (in order to avoid the formation of a red K_3). But then we may assume by part (c) that the blue edge ww' sends a red edge wx to a vertex $x \in X_\ell$, and another red edge $w'y$ to a vertex $y \in Y_\ell$. Moreover, $x, y \notin Z$ by part (a). It also follows by Proposition 1.11 that each pair of endpoints of the red paths xwz_1, $z_1w'y$ and $z_2z'z_1$ must each have a (not necessarily distinct) common neighbor in $V_1(v)$, but this is a contradiction, because then z_1 or one of these common neighbors will have a red degree larger than $\Delta(R) = 4$.

(f) Suppose $|X| \geq n-2$ and that $u \in V_{>3}(v)$. Then $\{u, v\} \cup X$ is an independent set of cardinality at least n in the red subgraph of the $(3, n, p)$-graph, a contradiction. $\square$

By combining the results of Lemma 3.5, we have the following useful result.

Corollary 3.6. *If $|X| \geq n-2$, $|Y| \geq n-3$ and there is exactly one $\ell \in \{1, ..., c\}$ such that $|X_\ell| > |Y_\ell|$, then*

(a) the pair of red edges sent by any edge in $\langle V_3(v)\rangle_{blue}$ to $V_2(v)$ necessarily goes to a balanced component of $\langle V_2(v)\rangle_{red}$, i.e., not to the component (X_ℓ, Y_ℓ).

(b) $|X_\ell| \geq |V_3(v)|$ if $\Delta(R) = 4$ and $|X_\ell| \geq 2$.

3.1.1 The Ramsey number $t(3, 7)$

Suppose there exists a $(3, 7, 18)$-graph. Let G and $\overline{G}$ be the red and blue subgraphs, respectively, and denote the minimum and maximum red degrees of G respectively by $\delta(G)$ and $\Delta(G)$. Suppose v is a vertex of red degree $\Delta(G)$. As $t(2, 7) = 7$ and $t(3, 6) = 15$, it follows by Proposition 1.10 that

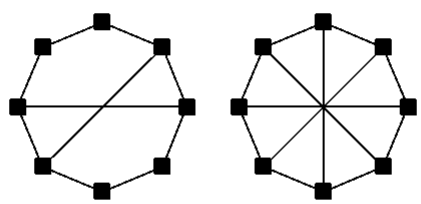
$$3 \leq \delta(G) \leq \Delta(G) \leq 6.$$

It is, however, possible to improve the bounds on $\Delta(G)$.

Lemma 3.7. $3 \leq \delta(G) \leq \Delta(G) \leq 4$

Proof. Suppose first that $\Delta(G) = 6$. Then $V_{>2}(v) = \emptyset$, for the existence of an element $v \in V_{>2}(v)$ would induce an independent set $\{v\} \cup V_{>2}(v)$ of cardinality 7 in G. It follows by Proposition 1.11 that $\langle V_2(v)\rangle_{\text{red}}$ is bipartite and since $|V_2(v)| = 11$, this bipartite graph has a partite set, say Z, of cardinality at least 6. But then $\{v\} \cup Z$ is an independent set of cardinality at least 7 in G, a contradiction.

Suppose next that $\Delta(G) = 5$. If $\langle V_{>2}(v)\rangle_{\text{red}}$ has two independent vertices u and w. Then $\{u, w\} \cup V_1(v)$ is an independent of cardinality 7 in G, a contradiction. Hence, if $|V_{>2}(v)| \geq 3$, then $V_{>2}(v)$ induces a red K_3 in G. Therefore, $|V_{>2}(v)| \leq 2$, and so $|V_2(v)| = 18 - 1 - 5 - |V_{>2}(v)| \geq 10$. As $\langle V_2(v) \cup \{z\}\rangle_{\text{red}}$ is bipartite for any vertex $z \in V_{>2}(v)$ by Proposition 1.11, it must have a partite set, say Z', of cardinality at least 6. But then $\{v\} \cup Z'$ is an independent set of cardinality at least 7 in G, again a contradiction. □



(a) E_{10} (b) E_{11}

Figure 3.3: If $\Delta(G) = 3$, then $\langle V_{>2}(v)\rangle_{\text{red}}$ is isomorphic to E_{10} or E_{11} [[6], Table 5]
.

Lemma 3.8. $\Delta(G) = 4$

Proof. Suppose $\Delta(G) = 3$. Then it follows by Lemma 3.7 that G is 3-regular and so $|V_2(v)| \leq 6$. However, if $|V_2(v)| < 6$, then $|V_{>2}(v)| \geq 9$, and as $t(3, 4) = 9$, it follows that there is an irredundant set Z of cardinality 3 in $\langle V_{>2}(v)\rangle_{\text{blue}}$ or an independent set Z' of cardinality 4 in $\langle V_{>2}(v)\rangle_{\text{red}}$. In the former case, Z is also an irredundant set of cardinality 3 in $\overline{G}$, a contradiction. In the latter case, $Z' \cup V_1(v)$ is an independent set of cardinality 7 in G, a contradiction. Thus, $|V_2(v)| = 6$ and $|V_{>2}(v)| = 8$.

According to [[6], Table 5] $\langle V_{>2}(v)\rangle_{\text{red}}$ must therefore be isomorphic to the red subgraph of one of only two possible $(3, 4, 8)$-graphs; these red subgraphs E_{10} and E_{11}

are shown in Figure 3.3. Clearly, $\langle V_{>2}(v)\rangle_{\text{red}} \ncong E_{11}$ as at least one vertex of $V_{>2}(v)$ must be adjacent to a vertex of $V_2(v)$, but E_{11} is already cubic. If $\langle V_{>2}(v)\rangle_{\text{red}} \cong E_{10}$, then, since each vertex of $V_2(v)$ is adjacent to a vertex in $V_1(v)$ and G is cubic, there is only one way to draw the edges between $V_1(v)$ and $V_2(v)$, as shown in Figure 3.4. Also, since G is cubic, all vertices of degree 2 in E_{10} must be in $V_3(v)$, and all vertices of degree 3 in E_{10} must be in $V_4(v)$, as shown in Figure 3.4. Since y_1 is adjacent to exactly one vertex in $V_2(v)$, we may assume without loss of generality that y_1 is not adjacent to either x_3 or x_4. But then $\{v_1, v_3, x_3, x_4, y_1, z_1, z_3\}$ is an independent set of cardinality 7 in G, a contradiction. $\square$

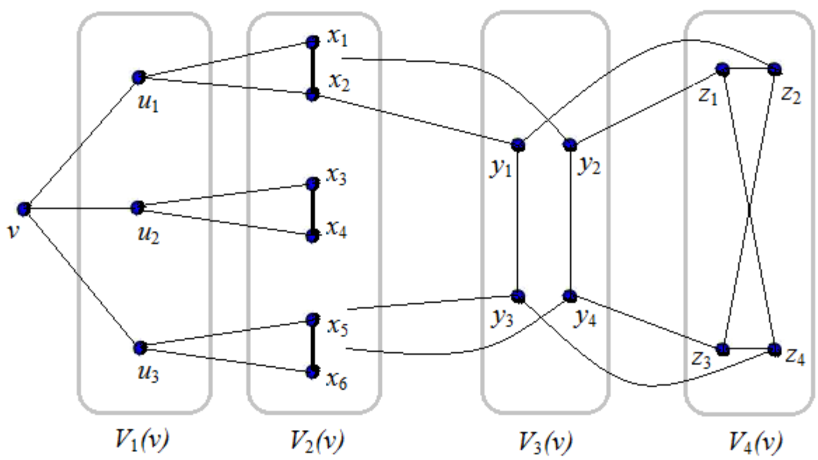


Figure 3.4: Part of the (3,7,18)-graph $(G, \overline{G})$ if $\Delta(G) = 3$.

The following properties of G may be deduced from Lemma 3.8.

Lemma 3.9. *$V_1(v)$ is an independent set of cardinality 4 in G. Furthermore, $8 \leq |V_2(v)| \leq 9$, $4 \leq |V_3(v)| \leq 5$, and $V_{>3}(v) = \emptyset$.*

Proof. It follows by Lemma 3.8 that $|V_1(v)| = 4$. In order to avoid triangles in $\langle \{v\} \cup V_1(v)\rangle_{\text{red}}$, it follows that $\langle V_{>2}(v)\rangle_{\text{red}}$ must be edgeless.

Now suppose $|V_{>2}(v)| \geq 6$. As $t(3,3) = 6$, it follows that, in order to avoid a red K_3

in G, the subgraph $\langle V_{>2}(v)\rangle_{\text{red}}$ must have an independent set, say Z, of cardinality 3. But then the set $V_1(v) \cup Z$ of cardinality 7 is independent in G. This contradiction shows that $|V_{>2}(v)| \leq 5$, and hence $|V_2(v)| \geq 8$.

Suppose next that $|V_2(v)| \geq 10$. Then $\langle V_2(v) \cup \{w\}\rangle_{\text{red}}$ is bipartite for any vertex $w \in V_{>2}(v)$ by Proposition 1.11, and hence has a partite set, say Z', of cardinality at least 6. But then $\{v\} \cup Z'$ is an independent set of cardinality at least 7 in G. This contradiciton shows that $|V_2(v)| \leq 9$ and hence that $|V_{>2}(v)| \geq 4$.

If $|V_2(v)| = 9$, then $\langle V_2(v)\rangle_{\text{red}}$ has a partite set of cardinality at least 5 and hence it follows by Lemma 3.5 (f) that $V_{>3}(v) = \emptyset$. Suppose, therefore, that $|V_2(v)| = 8$ and hence $|V_{>2}(v)| = 5$. Then, in order to avoid red triangles, $\langle V_{>2}(v)\rangle_{\text{red}}$ must be a 5-cycle. However, if any vertex of this 5-cycle is in $\langle V_{>3}(v)\rangle_{\text{red}}$, then that vertex will have degree 2 in G, contradicting the result of Lemma 3.7. This contradiction shows that $V_{>3}(v)$ must be empty. □

We may now prove our first main result of this section.

Theorem 3.10. *$t(3, 7) = 18$.*

Proof. It follows by Lemma 3.9 that there are two cases to consider.

Case i: $|V_1(v)| = 4$, $|V_2(v)| = 9$, and $|V_3(v)| = 4$. This case may be proven to be impossible by following the exact same arguments as in Case 1 of the proof that $s(3, 7) \leq 18$ in [4], because in these arguments no irredundant set of cardinality 7 is ever avoided which is not also an independent set of cardinality 7.

Case ii: $|V_1(v) = 4$, $|V_2(v)| = 8$, and $|V_3(v)| = 5$. This case may be proven to be impossible by following the exact same arguments as in Case 2 and 3 of the proof that $s(3, 7) \leq 18$ in [4] for the same reason as cited above. □

3.1.2 The Ramsey number $t(3,8)$

In this section, we show that $t(3,8) = 22$. We begin by producing a $(3,8,21)$-graph in the first subsection, showing that $t(3,8) > 21$. We show in the following subsection that if a $(3,8,22)$-graph exists, each vertex of such a coloring must have red degree 4 or 5. This is followed by a proof in the third subsection that no vertex of a $(3,8,22)$-graph can, in fact, have red degree 5, and hence that the red subgraph of such a coloring must be 4-regular. It is finally shown in the last subsection, by considering a number of exhaustive cases, that the assumption of the existence of a 4-regular subgraph of a $(3,8,22)$-graph leads to a contradiction in each case, implying that $t(3,8) \leq 22$.

The lower bound $t(3,8) > 21$

Consider the graph H of order 21 in Figure 3.5. It is easily verifiable that H is triangle-free and has no 6-cycle in which all three diagonals are absent. It therefore follows by Corollary 1.7 that H has no irredundant set of cardinality 3. Furthermore, H has no independent set of order 8, so that the red-blue edge coloring $(H, \overline{H})$ is a $(3,8,21)$-graph.

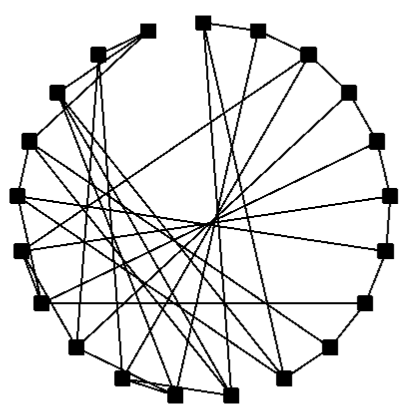


Figure 3.5: The red subgraph H of a $(3,8,21)$-graph $(H, \overline{H})$.

Properties of any $(3, 8, 22)$-graph

Suppose there exists a $(3, 8, 22)$-graph G, and denote the minimum and maximum degrees of G by $\delta(G)$ and $\Delta(G)$, respectively. Suppose v is a vertex of red degree $\Delta(G)$ in this coloring. Then it follows by Proposition 1.10 that

$$4 \leq \delta(G) \leq \Delta(G) \leq 7.$$

The coloring G has the following properties.

Lemma 3.11. *$V_1(v)$ is an independent set of G, $|V_2(v)| \leq 11$ and $|V_{>2}(v)| < t(3, 8 - \Delta(G))$.*

Proof. $V_1(v)$ is an independent set in G, because it induces a clique in $\overline{G}$ in order to avoid triangles in $\langle \{v\} \cup V_1(v) \rangle_{\text{red}}$, which are prohibited by Corollary 1.7. Furthermore, $|V_1(v)| = \Delta(G)$.

Suppose $|V_2(v)| \geq 12$ and let $w \in V_{>2}(v)$. Then it follows, by Proposition 1.11, that $X = V_2(v) \cup \{w\}$ induces a bipartite subgraph of order at least 13 in G. One of the partite sets, say A, of this bipartite subgraph has cardinality at least 7. But then the set $A \cup \{v\}$ is an independent set of cardinality at least 8 in G, a contradiction. Hence, $|V_2(v)| \leq 11$.

Now suppose $|V_{>2}(v)| \geq t(3, 8 - \Delta(G))$. Then, $\langle V_{>2}(v) \rangle_{\text{red}}$ possesses an independent set I of cardinality $8 - \Delta(G)$. But then $V_1(v) \cup I$ is an independent set of cardinality 8 in G, a contradiction. Hence $|V_{>2}(v)| < t(3, 8 - \Delta(G))$. □

It is possible to improve the bounds on $\Delta(G)$.

Lemma 3.12. *$4 \leq \delta(G) \leq \Delta(G) \leq 5$.*

Proof. Suppose $\Delta(G) = 7$. Then $|V_1(v)| = 7$, $|V_2(v)| \leq 11$ and $|V_{>2}(v)| < t(3, 1) = 1$ by Lemma 3.11, and so $|V_1(v)| + |V_2(v)| + |V_{>2}(v)| < 7 + 11 + 1 = 19$, a contradiction.

Next suppose $\Delta(G) = 6$. Then $|V_1(v)| = 6$, $|V_2(v)| \leq 11$ and $|V_{>2}(v)| < t(3,2) = 3$ by Lemma 3.11, and so $|V_1(v)| + |V_2(v)| + |V_{>2}(v)| < 6 + 11 + 3 = 20$, again a contradiction. $\square$

The maximum degree of G is not 5

Suppose $\Delta(G) = 5$. Then it follows by Lemma 3.11 that $|V_1(v)| = 5$, $|V_2(v)| \leq 11$ and $|V_{>2}(v)| \leq 5$. But since $|V_1(v)| + |V_2(v)| + |V_{>2}(v)| = 21$, it must hold that $|V_2(v)| = 11$ and $|V_{>2}(v)| = 5$.

The subgraph $\langle V_2(v)\rangle_{\text{red}}$ of G is bipartite by Proposition 1.11. Suppose $\langle V_2(v)\rangle_{\text{red}}$ comprises c components and denote the partite sets of $\langle V_2(v)\rangle_{\text{red}}$ by $X = \bigcup_{\ell=1}^{c} X_\ell$ and $Y = \bigcup_{\ell=1}^{c} Y_\ell$. Then we may assume that $|X| = 6$ and $|Y| = 5$, and that $\langle V_2(v)\rangle_{\text{red}}$ has exactly one component, say (X_c, Y_c), for which $|X_c| = |Y_c| + 1$, while all other components are balanced (that is, $|X_\ell| = |Y_\ell|$ for all $\ell = 1, ..., c-1$). Note that $|V_{>3}(v)| = \emptyset$ by Lemma 3.5 (f). Hence, $\langle V_3(v)\rangle_{\text{red}}$ must be a 5-cycle, in order to avoid triangles in G and $\overline{G}$. Furthermore, $|X_c| \geq 3$, for if $|X_c| \leq 2$, then it would follow by Lemma 3.5 (b) that at least three vertices in $V_3(v)$ send red edges to some vertex in X_c, thus forming a triangle in G. Since $\langle V_3(v)\rangle_{\text{blue}}$ contains an odd cycle by Lemma 3.5 (d), we conclude that $\langle V_2(v)\rangle_{\text{red}}$ must have at least two balanced components (that is, at least three components in total). Note that, since $\langle V_3(v)\rangle_{\text{red}}$ is a 5-cycle, only one pair of red edges can go to a (1,1)-component of $\langle V_2\rangle_{\text{red}}$ (for otherwise a K_3 will be forced in G). In view of these restrictions and $|X_c| \geq 3$, it necessarily follows that

$$|X_1| = |Y_1| = 1,\ |X_2| = |Y_2| = |Y_3| = 2,\ \text{and}\ |X_3| = 3. \tag{3.1}$$

Note that the component (X_2, Y_2) must receive four pairs of red edges from $V_3(v)$, since the component (X_1, Y_1) can only receive on such pair of edges.

We show that the cardinalities in (3.1) lead to a contradiction, and hence that our supposition that $\Delta(G) = 5$ was wrong. Denote the 5-cycle of $\langle V_3(v)\rangle_{\text{red}}$ by $w_1w_2w_3w_4w_5w_1$. In order to avoid triangles in G, it follows by Lemma 3.5 (a) that the pairs of red edges sent by the five edges of $\langle V_3(v)\rangle_{\text{blue}}$ to $V_2(v)$ must occur in alternating fashion between the partite sets X_2 and Y_2, as shown in Figure 3.6. But then the edge x_2y_2 must be blue in order to avoid a red 6-cycle $x_2y_2w_4w_2w_5w_3x_2$ with blue diagonals; notice that the edges x_2w_2 and y_2w_5 are blue by Lemma 3.5 (a). Similarly, the edge x_2y_1 must be blue in order to avoid a red 6-cycle $x_2y_1w_2w_4w_1w_3x_2$ with blue diagonals. But then x_2 is isolated in the component (X_2, Y_2) of $\langle V_2(v)\rangle_{\text{red}}$, a contradiction.

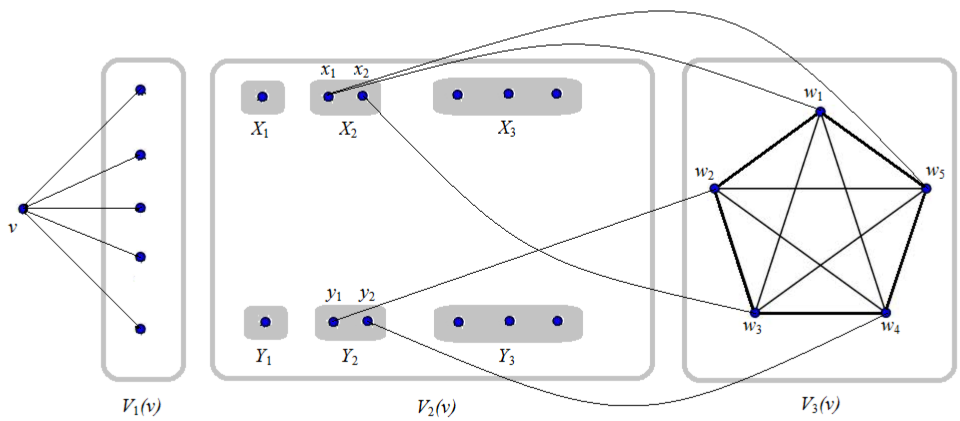


Figure 3.6: A part of $(G, \overline{G})$ if $\Delta(G) = 5$.

G is not 4-regular

If G is 4-regular, then it follows by Lemma 3.11 that $|V_2(v)| \leq 11$ and $|V_{>2}(v)| < t(3,4) = 9$. Therefore, if G is 4-regular, then there are five cases to consider, as outlined in Table 3.1. Note that $|Y| \leq |X| \leq 6$ in order to avoid an independent set $\{v\} \cup X$ of cardinality 8 in G; hence the five cases in the table.

Lemma 3.13. *Case I in Table 3.1 is impossible.*

Case	$\lvert V_1(v)\rvert$	$\lvert V_2(v)\rvert$	$\lvert V_{\geq 3}(v)\rvert$	$\lvert X\rvert$	$\lvert Y\rvert$	Considered
I	4	11	6	6	5	in Lemma 3.13
IIa	4	10	7	5	5	in Lemma 3.14
IIb	4	10	7	6	4	in Lemma 3.15
IIIa	4	9	8	5	4	in Lemma 3.18
IIIb	4	9	8	6	3	in Lemma 3.18

Table 3.1: Five cases to consider if G is 4-regular.

Proof. In Case I in Table 3.1 it follows by Lemma 3.5 (f) that $V_{>3}(v) = \emptyset$ and hence that $|V_3(v)| = 6$. Furthermore, since $6 = |X| > |Y| = 5$, there is exactly one component of the bipartite graph $\langle V_2(v)\rangle_{\text{red}}$, say (X_k, Y_k), for which $|X_k| > |Y_k|$, while all other components have partite sets of equal cardinalities. It follows by Lemma 3.5 (b) that $|X_k| \neq 1$, since $\Delta(G) = 4$. Hence it follows by Corollary 3.6 (b) that $|X_k| \geq |V_3(v)| = 6$. But since $|X_k| \leq |X| = 6$, we must have that $|X_k| = 6$, so that $\langle V_2(v)\rangle_{\text{red}}$ has only one component. Furthermore, since $r(3,3) = 6$ and since $\langle V_3(v)\rangle_{\text{red}}$ contains no K_3, it follows that $\langle V_3(v)\rangle_{\text{blue}}$ contains a K_3, contradicting Lemma 3.5 (d). $\square$

Lemma 3.14. *Case IIa in Table 3.1 is impossible.*

Proof. As $\langle V_{\geq 3}(v)\rangle_{\text{red}}$ contains no independent set of cardinality 4, it must be the red subgraph of a $t(3,4)$-avoidance graph, *i.e.*, one of the graphs $E_1 - E_8$ in [[6], Figure 1(a)-(h)]. Note that there are only eight avoidance graphs of order 7 for $s(3,4)$ [[6], Table 3], and since $\beta(G) \leq IR(G)$ for any graph G, it follows that this set of avoidance $s(3,4)$-graphs is also a complete set of avoidance $t(3,4)$-graphs of order 7. However, since $\Delta(E_i) \leq 3$ for all $i = 1, ..., 8$ (as G is 4-regular) and since only vertices of E_i with degree 4 can be in $\langle V_{>3}(v)\rangle_{\text{red}}$, we have that $V_{>3}(v) = \emptyset$.

Furthermore, it follows by Lemma 3.5 (d) that the pairs of red edges sent by the edges of a triangle in $\langle V_3(v)\rangle_{\text{blue}}$ to $V_2(v)$ must go to at least two different components of $\langle V_2(v)\rangle_{\text{red}}$. This implies that each triangle in $\langle V_3(v)\rangle_{\text{blue}}$ sends at least five red edges to $V_2(v)$. Note that if such a triangle sends exactly five red edges to $V_2(v)$, then its vertices send respectively 1, 2, and 2 red edges to $V_2(v)$ as they are traversed around the triangle. The complement of each of the avoidance graphs E_3, E_4, E_7 , and E_8 has a triangle violating the above condition. Therefore, $\langle V_3(v)\rangle_{\text{red}}$ must be isomorphic to E_1, E_2, E_5, or E_6, shown in Figure 3.7.

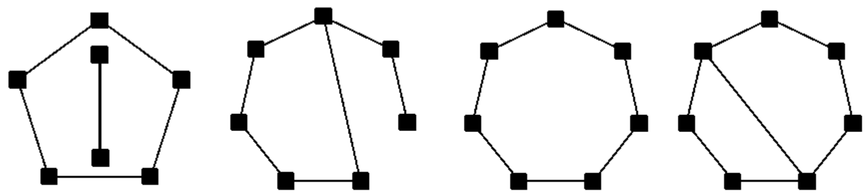


Figure 3.7: In case IIa of Table 3.1 $\langle V_3(v)\rangle_{\text{red}}$ must be isomorphic to E_1, E_2, E_5, or E_6.

Let A and B be two disjoint subsets of the vertex set of G. Then we denote by $E^r(A,B)$ the number of edges of G joining vertices in A with vertices in B, while $E^r(A)$ denotes the number of edges of G joining two vertices of A. Since the sum of the vertex degrees in G over $V_3(v)$ is

$$|V_3(v)| \times \delta(G) = 7 \times 4 = E^r(V_2(v), V_3(v)) + 2E^r(V_3(v)), \tag{3.2}$$

and

$$\begin{aligned} E^r(G) &= E^r(\{v\}, V_1(v)) + E^r(V_1(v), V_2(v)) + E^r(V_2(v)) + E^r(V_2(v), V_3(v)) + E^r(V_3(v)) \\ &= 4 + 12 + E^r(V_2(v)) + (28 - 2E^r(V_3(v))) + E^r(V_3(v)) \\ &= 44 + E^r(V_2(v)) - E^r(V_3(v)) \\ &= 44, \end{aligned}$$

as $E^r(G) = \frac{22\times 4}{2} = 44$. Thus,

$$E^r(V_2(v)) = E^r(V_3(v)). \tag{3.3}$$

Furthermore, we observe from Figure 3.7 that $E^r(V_3(v)) = 6,\ 7,\ \text{or } 8$, and hence we have three subcases to consider.

Subcase i: $E^r(V_3(v)) = 6$. In this subcase, it follows by (3.2) that $E^r(V_2(v), V_3(v)) = 28 - 12 = 16$. No vertex $x \in V_2(v)$ can receive three red edges from $V_3(v)$, for otherwise x would be isolated in $\langle V_2(v)\rangle_{\text{red}}$, which is impossible, since all components of $\langle V_2(v)\rangle_{\text{red}}$ are balanced in Case IIa. It follows by Lemma 3.5 (e) that each vertex in $V_2(v)$ receiving two red edges from $V_3(v)$ must be in a $(1,1)$-component of $\langle V_2(v)\rangle_{\text{red}}$. Furthermore, if $\langle V_2(v)\rangle_{\text{red}}$ has more than three $(1,1)$-components, then the maximum number of edges in $\langle V_2(v)\rangle_{\text{red}}$ is 5, contradicting the fact that $E^r(V_2(v)) = 6$. Therefore, to accommodate all 16 red edges from $V_3(v)$, it follows that $\langle V_2(v)\rangle_{\text{red}}$ necessarily comprises three $(1,1)$-components and one $(2,2)$-component, in which case each of the six vertices in the $(1,1)$-components of $\langle V_2(v)\rangle_{\text{red}}$ receives exactly two red edges from $V_3(v)$.

Let $\langle\{x_1, y_1\}\rangle_{\text{red}}$ be a $(1,1)$-component of $\langle V_2(v)\rangle_{\text{red}}$, and let $N(x_1) \cap V_3(v) = \{w_1, w_2\}$ and $N(y_1) \cap V_3(v) = \{w_3, w_4\}$. Note that $w_1,\ w_2,\ w_3,$ and w_4 are all distinct in order to avoid triangles in G. It follows by Lemma 3.5 (c) that the blue edge w_1w_2 must send a pair of red edges, w_1x_2 and w_2y_2, to some component of $\langle V_2(v)\rangle_{\text{red}}$. Furthermore, it follows by Proposition 1.11 and the fact that x_1 has degree 4 that x_1, x_2, and y_2 must all have one common neighbor, u, in $V_1(v)$. Note that since x_2 and y_2 are both adjacent to u, both x_2 and y_2 must be in the $(2,2)$-component of $\langle V_2(v)\rangle_{\text{red}}$ in which case x_2y_2 must be blue in order to avoid a red K_3 in G. Similarly, as the $(2,2)$-component of $\langle V_2(v)\rangle_{\text{red}}$ contains exactly three red edges, the blue edge w_3w_4

must also send red edges to x_2 and y_2, but this contradicts Lemma 3.5 (e).

Subcase ii: $E^r(V_3(v)) = 7$. In this subcase it follows by (3.2) that $E^r(V_2(v), V_3(v)) = 28 - 14 = 14$. Again, since no vertex in $V_2(v)$ can receive three red edges from $V_3(v)$, there must be at least four vertices, x_1, y_1, x_3, and y_3, in $V_2(v)$ which each receives two red edges from $V_3(v)$. Hence there are at least two $(1,1)$-components in $\langle V_2(v)\rangle_{\text{red}}$. Since $E^r(V_2(v)) = 7$, there can be at most three $(1,1)$-components in $\langle V_2(v)\rangle_{\text{red}}$. Without loss of generality, let x_1 and y_1 be in the same $(1,1)$-component of $\langle V_2(v)\rangle_{\text{red}}$, and let w_1, w_2, w_3, and w_4 be the neighbors of x_1 and y_1 in $V_3(v)$. As in subcase i, the blue edge w_1w_2 must send a pair of red edges, say w_1x_2 and w_2y_2, to some component of $\langle V_2(v)\rangle_{\text{red}}$. Also, x_1, x_2, and y_2 must have a common neighbor, say u, in $V_1(v)$. Since $V_3(v)$ contains exactly seven vertices, it follows by the pigeonhole principle that at least one of x_3 or y_3 must be joined by means of a red edge to one of w_1, w_2, w_3, or w_4. We may therefore assume that x_3w_1 is red. But then x_1, x_2, and x_3 must all have a common neighbor in $V_1(v)$, implying $d(x_1) > 4$ and $d(u) > 4$, a contradiction of $\Delta(G) = 4$.

Subcase iii: $E^r(V_3(v)) = 8$ (implying $E^r(V_2(v)) = 8$ and $\langle V_3(v)\rangle_{\text{red}} \cong E_6$). In this subcase, it follows by (3.2) that $E^r(V_2(v), V_3(v)) = 28 - 16 = 12$. There are at least two vertices, x_1 and x_2 say, in $V_2(v)$ that each receive two red edges from $V_3(v)$. Label the vertices of $V_3(v)$ as in Figure 3.8.

We consider to which pairs of vertices in $V_3(v)$ the vertices x_1 and x_2 can send red edges. Note that x_1 cannot send red edges to pairs of vertices of the form $\{w_4, w_i\}$ or $\{w_7, w_j\}$, where w_4w_i and w_7w_j are blue edges, since w_4 or w_7 will be saturated in terms of its red degree, therefore either contradicting Lemma 3.5 (c) or forming a red K_3 in G. Hence the vertex x_1 must send red edges to two nonadjacent vertices in $\{w_1, w_2, w_3, w_5, w_6\}$. However, x_1 may not send red edges to the following pairs of vertices:

- $\{w_2, w_6\}$, for otherwise $x_1w_2w_3w_4w_5w_6x_1$ would form a red 6-cycle with blue diagonals,
- $\{w_2, w_5\}$, by symmetry, for the same reason as above,
- $\{w_1, w_5\}$, for otherwise $x_1w_1w_2w_3w_4w_5x_1$ would form a red 6-cycle with blue diagonals,
- $\{w_1, w_6\}$, by symmetry, for the same reason as above.

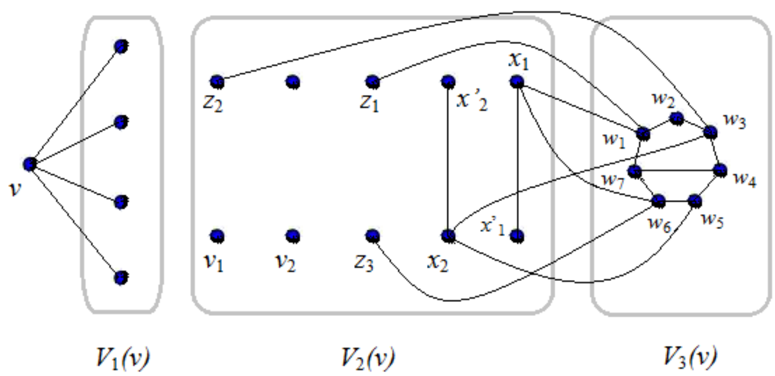


Figure 3.8: A subgraph of G in Subcase iii.

Therefore, the only pairs of vertices in $V_3(v)$ to which the vertex x_1 may send red edges are $\{w_1, w_6\}$, $\{w_1, w_3\}$, or $\{w_3, w_5\}$. First consider the case where x_1 sends red edges to $\{w_1, w_3\}$ and x_2 sends red edges to $\{w_1, w_6\}$ or $\{w_3, w_5\}$. Suppose, without loss of generality, that x_2 sends red edges to $\{w_1, w_6\}$. Then, w_1 is saturated in terms of its red degree, which means that the blue edge w_1w_3 cannot send a pair of red edges to $V_2(v)$ as dictated by Lemma 3.5. We conclude, without loss of generality, that x_1 must send red edges to $\{w_1, w_6\}$, while x_2 sends red edges to $\{w_3, w_5\}$ (see Figure 3.8).

Note that x_1x_2 must be blue, for otherwise $x_1x_2w_5w_4w_7w_1x_1$ would form a red 6-cycle with blue diagonals. Therefore x_1 and x_2 are in different $(1, 1)$-components of $\langle V_2(v)\rangle_{\text{red}}$, and the blue edges w_1w_6 and w_3w_5 must send pairs of red edges to the remaining components of $\langle V_2(v)\rangle_{\text{red}}$. Suppose, without loss of generality, that w_1, w_3, and w_6 send red edges to z_1, $z_2 \in X$ and $z_3 \in Y$, as shown in Figure 3.8. Then we can select two vertices, v_1 and v_2, from Y which are not joined by means of red edges to the saturated vertices $\{w_1, w_3, w_6\}$, which means that $\langle\{v, v_1, v_2, x'_1, x'_2, w_1, w_5, w_6\}\rangle_{\text{blue}}$ is a clique of order 8 in $\overline{G}$, a contradiction. □

Lemma 3.15. *Case IIb in Table 3.1 is impossible.*

Proof. In this case, there are two possibilities to consider, namely where $\langle V_2(v)\rangle_{\text{red}}$ has two unbalanced components, or where $\langle V_2(v)\rangle_{\text{red}}$ has one unbalanced component. We consider the case where $\langle V_2(v)\rangle_{\text{red}}$ has two unbalanced components first.

Let (X_i, Y_i) be the components in question with $|X_i| = |Y_i| + 1$ for $i = 1, 2$. It follows by Lemma 3.5 (f) that $V_{>3}(v) = \emptyset$. Every vertex $w \in V_3(v)$ must send a red edge to $X_1 \cup X_2$ in order to avoid the clique of order 8 in $\overline{G}$ induced by $\{v, w\} \cup X_1 \cup X_2$ together with the partite sets of the balanced components of $\langle V_2(v)\rangle_{\text{red}}$ which do not receive red edges from w. Let t denote the number of red edges incident with vertices in $X_1 \cup X_2$. Then

$$\begin{aligned} t &= E^r(X_1 \cup X_2, V_3(v)) + E^r(X_1 \cup X_2, Y_1 \cup Y_2) + E^r(X_1 \cup X_2, V_1(v)) \\ &\geq 7 + (2|X_1| - 2) + (2|X_2| - 2) + |X_1| + |X_2| \\ &= 3 + 3|X_1| + 3|X_2|. \end{aligned}$$

Let $\epsilon = t - (3 + 3|X_1| + 3|X_2|)$. So, $3 + 3|X_1| + 3|X_2| + \epsilon \leq 4|X_1| + 4|X_2|$ implies

$$3 + \epsilon \leq |X_1| + |X_2|. \tag{3.4}$$

We show that there is a pair of vertices in $V_3(v)$ which have a common neighbor $x_1 \in X_i$ such that $|X_i| \geq 2$, for some $i \in \{1, 2\}$. Since $|X_1| + |X_2| \geq 3$, we have that $|X_1| \geq 2$ or $|X_2| \geq 2$. Assume, without loss of generality, that $|X_1| \geq 2$. Suppose every vertex in X_1 sends at most one red edge to $V_3(v)$. Then the remaining vertices in $V_3(v)$ send red edges to X_2. If $|X_2| = 1$ (implying $|X_1| = 5$, $|Y_1| = |Y| = 4$, $|Y_2| = 0$), then $|\{v\} \cup N^r(X_2) \cup X_1| \geq 1 + 2 + 5 = 8$ in G, a contradiction. So, $|X_2| \geq 2$. Since $|V_3(v)| = 7$ and $|X| = 6$ ($|X_1| \leq 4$), there exist two vertices $w_1,\ w_2 \in V_3(v)$ that send red edges to a vertex $x_1 \in X_2$. Assume therefore, without loss of generality, that $x_1 \in X_2$, x_1w_1, x_1w_2 are red, and $|X_2| \geq 2$.

As $|X_2| \geq 2$ there is a red path x_1yx_2 with $x_1,\ x_2 \in X_2$ and $y \in Y_2$. By Proposition 1.11, $N^r(x_1) \cap N^r(x_2) \cap V_1(v) = \{u\} \in V_1(v)$ thus saturating the vertex x_1. Now, the vertices w_1 and w_2 can send at most one red edge to $V_2(v) \backslash X_2$, for otherwise u or x_1 will be oversaturated in order to accomodate the common neighbors in $V_1(v)$ of the endpoints of the red paths of order 3 formed in $V_2(v) \cup V_3(v)$, as necessitated by Proposition 1.11. However, if this additional red edge does not go to X_1, or if w_1 and w_2 do not send additional red edges to $V_3(v)$, then a clique of order 8 is induced in $\overline{G}$ by $\{v, w_1, w_2\} \cup X_1 \cup X_2$ together with the partite sets of the balanced components of $\langle V_2(v) \rangle_{\text{red}}$ that do not receive a red edge from either w_1 or w_2.

Assume therefore, without loss of generality, that w_2x_3 is red, for some $x_3 \in X_1$. Then, ux_3 is red by Proposition 1.11. Note that $\epsilon \geq 1$ since w_2 now sends two edges to $X_1 \cup X_2$. We now consider, as subcases, the possible cardinalities of X_1 and X_2. Note that if $|X_1| \geq 2$, then $\epsilon \geq 2$ since x_3 is part of a red path of order three whose endpoints have a common neighbor in $V_1(v)$ other than u, by Proposition 1.11.

Before continuing, we note the following two useful observations.

Observation 3.16. *Let $S = \{p_1, p_2, p_3, p_4, p_5\}$. If $\langle S \rangle_{red} \subseteq \langle V_2(v) \rangle_{red}$ is a path $p_1p_2p_3p_4p_5$ of order 5, then the pairs of vertices $\{p_1, p_3\}$, $\{p_2, p_4\}$, and $\{p_3, p_5\}$ all have* distinct *common neighbors in $V_1(v)$.*

Proof. It follows by Proposition 1.11 that all three pairs of vertices must have common neighbors in $V_1(v)$. To avoid triangles in G, only the pairs of vertices $\{p_1, p_3\}$ and $\{p_3, p_5\}$ can possibly share a common neighbor in $V_1(v)$. Suppose p_1, p_3, and p_5 all have the same common neighbor, u_1, and that p_2 and p_4 have the same common neighbor, u_2, in $V_1(v)$. Then $u_1p_1p_2u_2p_4p_5u_1$ is a red 6-cycle with blue diagonals, a contradiction. $\square$

Observation 3.17. *If $y \in Y$ and $|X_i| = |Y_i| + 1$ for $i = 1, 2$, then $E^r(\{y\}, X) \leq 2$.*

Proof. Suppose, to the contrary, that there is a vertex $y_2 \in Y$ which sends red edges to $x_1,\ x_2,\ x_3 \in X$. Then x_1, x_2, and x_3 must have a common neighbor, $u_1 \neq u$, in $V_1(v)$ by Corollary 3.4. But then $\langle \{u, u_1\} \cup Y \cup \{w_1, w_2\} \rangle_{\text{blue}}$ is a clique of order 8 in $\overline{G}$, a contradiction. $\square$

From the above observations it is easy to see that $\langle V_2(v) \rangle_{\text{red}}$ cannot contain a $(4, 3)$-component (or larger): It follows by Observation 3.17 that $\langle X_i \cup Y_i \rangle_{\text{red}}$ is either a path of order 7 or else X_i contains a vertex x which sends three red edges to Y_i. In both cases all the common neighbors cannot be accommodated (either because x is oversaturated in terms of its red degree or there are not enough vertices in $V_1(v)$, as may be seen by applying Observation 3.16 on the three subpaths of order 5 of $p_1, ..., p_7$ starting with p_1, p_2, and p_3, respectively). We therefore complete the proof of the lemma by considering two subcases.

Subcase i: $|X_1| = 2$. Since $\epsilon \geq 1$, it follows by (3.4) that $|X_1| + |X_2| \geq 4$ and hence $|X_1| \geq 2$. But then $\epsilon \geq 2$ implying that $|X_1| \geq 3$ by (3.4). Therefore, $|X_1| = 3$, and it follows by Observation 3.17 that $\langle X_1 \cup Y_1 \rangle_{\text{red}}$ is a path, $p_1p_2p_3p_4p_5$, of order 5. But then it follows by Observation 3.16 that $\epsilon \geq 3$, since p_3 sends two red edges to $V_1(v)$, contradicting the cardinality of $|X_1|$ in view of (3.4).

Subcase ii: $|X_2| = 3$. In this subcase $\langle X_2 \cup Y_2 \rangle_{\text{red}}$ is a path of order 5, so $\epsilon \geq 2$. Thus, $|X_1| + |X_2| \geq 5$, and so $|X_1| \geq 2$. If $|X_1| = 2$, then $\epsilon \geq 3$, again a contradiction, as above. We conclude that $|X_1| = 3$. But if $\langle X_1 \cup Y_1 \rangle_{\text{red}}$ and $\langle X_2 \cup Y_2 \rangle_{\text{red}}$ each contains a path of order 5, then the required number of unique common neighbors forces $d(v) > 4 = \Delta(G)$, a contradiction.

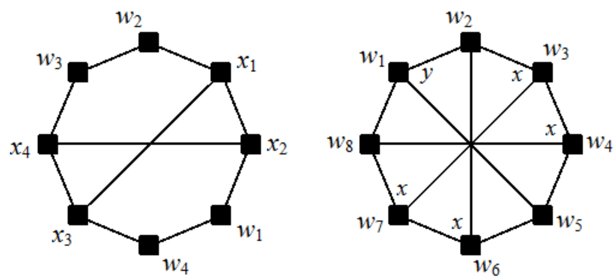
The final possibility to consider is when $\langle V_2(v) \rangle_{\text{red}}$ has only one unbalanced component, (X_1, Y_1), with $|X_1| = |Y_1| + 2$. Using a similar argument to that used to obtain (3.4), it may be shown that $|X_1| \geq 4 + \epsilon$. As $|X_1| \leq 6$, we have that $\epsilon \leq 2$. Notice that if $\epsilon = 2$, then $\langle V_2(v) \rangle_{\text{red}}$ comprises only one component, and $V_3(v)$ sends no red edges to Y. But then $V_3(v) \cup Y \cup \{v\}$ induces a clique of order 8 in $\overline{G}$ since $\langle V_3(v) \rangle_{\text{blue}}$ must contain a triangle. We therefore conclude that $\epsilon \leq 1$.

We complete the proof by showing that the above inequality cannot be satisfied. The subgraph $\langle X_1 \cup Y_1 \rangle_{\text{red}}$ must either be a connected graph containing a cycle or must contain an induced path of order 5, $p_1p_2p_3p_4p_5$. In the former case, $\epsilon \geq 1$. In the latter case, it follows by Observation 3.16 that p_3 sends two red edges to $V_1(v)$ and so again, $\epsilon \geq 1$. Note that it now follows that $|X_1| = 5$. Also, as before, there must be two vertices in $V_3(v)$, w_1 and w_2, which send red edges to $x_1 \in X_1$. Using a similar argument as in the subcase with two unbalanced components, it follows that w_1 or w_2 must send a red edge to $X_1 - \{x_1\}$ in order to avoid a clique of order 8 in $\overline{G}$, implying $\epsilon \geq 2$. $\square$

Lemma 3.18. *Cases IIIa and IIIb in Table 3.1 are impossible.*

Proof. In both cases, $|V_{\geq 3}(v)| = 8$, so $\langle V_{\geq 3}(v) \rangle_{\text{red}}$ has to be the red subgraph of a $(3, 4, 8)$-coloring, *i.e.*, one of the graphs E_{10} or E_{11} in Figure 3.9, for otherwise a triangle would result in G or else a clique of order 8 would be induced in $\overline{G}$ by the vertices in $V_1(v)$ together with four vertices in $V_{\geq 3}(v)$. Since neither E_{10} nor E_{11} has a vertex of degree 4, it follows in both cases that, in fact, $V_{\geq 4}(v) = \emptyset$.

We first consider the possibility that $\langle V_3(v) \rangle_{\text{red}} \cong E_{10}$ with vertices labeled as



(a) E_{10} (b) E_{11}

Figure 3.9: The only two possibilities for the red subgraph of a (3,4,8)-graph [[6], Table 4].

in Figure 3.9. Each of the vertices w_1, w_2, w_3, and w_4 has two neighbors in $V_2(v)$. We show that these neighbors are, in fact, distinct, *i.e.*, that there are eight such neighbors in total. Without loss of generality, we show only that the neighbors of w_1 are distinct from those of w_2, w_3, and w_4. First, w_1 and w_4 cannot have a common neighbor in $V_2(v)$, for otherwise a triangle would result in G. Furthermore, if w_1 and w_3 have a common neighbor, y, in $V_2(v)$, then the red 6-cycle with blue diagonals $w_3yw_1w_4x_3x_4w_3$ results in $(G, \overline{G})$, unless the edge x_3y is red. But, $w_3yw_1x_2x_1w_2w_3$ is similarly a red 6-cycle with blue diagonals in $(G, \overline{G})$, unless the edge x_1y is red. But x_1y and x_3y cannot both be red, for this would oversaturate the vertex y. A similar argument shows that w_1 and w_2 cannot have a common neighbor (in this case the two red 6-cycles with blue diagonals are $w_2yw_1x_2x_4w_3w_2$ and $w_2yw_1w_4x_3x_1w_2$).

Define $Z_1 = (N(w_1) \cap V_2(v)) \cup \{x\}$ and $Z_i = N(w_i) \cap V_2(v)$ for all $i \in \{2, 3, 4\}$, and let xv_1 be red without loss of generality, as shown in Figure 3.10. Then each pair of vertices in Z_i must have a common neighbor, v_i, in $V_1(v)$ by Proposition 1.11, for all $i \in \{1, 2, 3, 4\}$. Note that $N(w_2) \cup \{w_1, v_1\}$, $N(w_3) \cup \{w_4, v_1\}$, and $N(w_4) \cup \{w_3, v_1\}$ each forms a clique of order 6 in $\overline{G}$. Therefore, in order to avoid a clique of order 8 in $\overline{G}$, there must be a red edge between Z_2 and $\{v_3, v_4\}$, between Z_3 and $\{v_2, v_4\}$, and

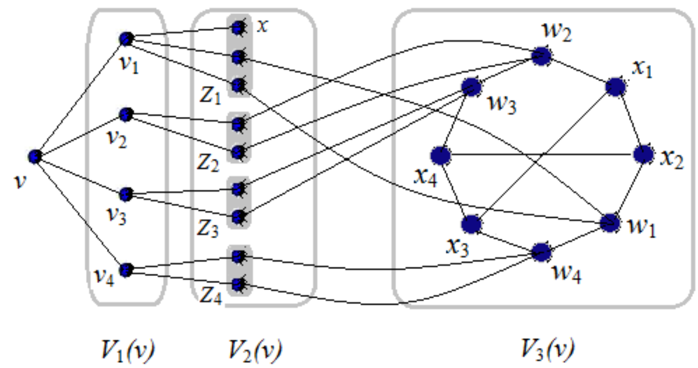


Figure 3.10: A part of $(G, \overline{G})$ in the cases IIIa and IIIb, if $\langle V_3(v)\rangle_{red} \cong E_{11}$.

Z_4 and $\{v_2, v_3\}$. Hence there are three red edges between $Z_2 \cup Z_3 \cup Z_4$ and $\{v_2, v_3, v_4\}$. Since G is 4-regular, there cannot be any red edges between Z_1 and $\{v_2, v_3, v_4\}$. But then a clique of order 8 is induced in $\overline{G}$ by the vertices in $Z_1 \cup \{v_2, v_3, v_4, w_3, w_4\}$, a contradiction.

Consider next the possibility that $\langle V_3(v)\rangle_{\text{red}} \cong E_{11}$ with vertices labelled $w_1, ..., w_8$ as in Figure 3.9(b). In Case IIIa of Table 3.1 the vertices in $\{v, w_i, w_j\} \cup X$ will induce a clique of order 8 in $\overline{G}$ if a blue edge $w_i w_j$ in $V_3(v)$ sends both its red edges to Y. Similarly, the vertices in $\{v, w_i, w_j, w_k\} \cup Y$ will induce a clique of order 8 in $\overline{G}$ if a blue triangle $w_i w_j w_k$ in $V_3(v)$ sends all its red edges to X in Case IIIa of Table 3.1. Label the vertices in $V_3(v)$ by means of the symbols x and y to indicate whether the vertices send red edges to X or Y, respectively. Thus, in order to avoid a clique of order 8 in $\overline{G}$, the vertices in $V_3(v)$ should be labeled x and y in such a way that the endpoints of every blue edge in $V_3(v)$ are not both labeled y, and such that the vertices of a blue triangle in $V_3(v)$ are not all labeled x. We show that this is not possible. Since not all vertices in $V_3(v)$ can be labeled x, some vertex, w_1 say, must be labeled y. To avoid labeling both endpoints of blue edges in $V_3(v)$ with the symbol y, the vertices w_3, w_4, w_6, and w_7 must all be labeled x. Furthermore, in order to avoid labeling all the vertices of the triangles $\langle\{w_3, w_6, w_8\}\rangle_{\text{blue}}$ and $\langle\{w_2, w_4, w_7\}\rangle_{\text{blue}}$ with the symbol x, the vertices w_2 and w_8 must both be labeled y. But then both endpoints of the blue edge w_2w_8 in $V_3(v)$ are labeled y, a contradiction. $\square$

The following result therefore holds in view of Lemmas 3.13, 3.14, 3.15, 3.18.

Theorem 3.19. *$t(3, 8) = 22$.*

CHAPTER 4: Conclusion

Using the results of this thesis and various sources, we have the following table:

Irredundant Ramsey Number	Mixed irredundant Ramsey Number	Classical Ramsey Number
$s(3,3) = 6$	$t(3,3) = 6$	$r(3,3) = 6$
$s(3,4) = 6$	$t(4,3) = 8$	$r(3,4) = 9$
$s(3,5) = 12$	$t(3,4) = 9$	$r(3,5) = 14$
$s(3,6) = 7$	$t(3,5) = 12$	$r(3,6) = 18$
$s(3,7) = 18$	$t(5,3) = 13$	$r(3,7) = 23$
$s(4,4) = 13$	$t(3,6) = 15$	$r(3,8) = 28$
	$t(6,3) = 17$	$r(3,9) = 36$
	$t(4,4) = 14$	$r(4,4) = 18$
	$t(3,7) = 18$	$r(4,5) = 25$
	$t(3,8) = 22$	

Table 4.1: Ramsey numbers known exactly.

Using the results of the previous chapter and the fact that $s(m,n) \leq t(m,n) \leq r(m,n)$, it also follows that

$$\begin{array}{ll} 14 \leq t(4,5) \leq 25, & 14 \leq t(5,4) \leq 25, \\ 18 \leq t(7,3) \leq 23, & 18 \leq t(8,3) \leq 28, \\ 18 \leq s(3,8) \leq 22, & 13 \leq s(4,5) \leq 25. \end{array}$$

These are the six smallest unknown Ramsey numbers involving the graph theoretic notion of irredundance, and are certainly worthy of further investigation.

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