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New generalized fuzzy metrics and fixed point theorem in fuzzy metric space

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Abstract

In this paper, in fuzzy metric spaces (in the sense of Kramosil and Michalek (Kibernetika 11:336-344, 1957)) we introduce the concept of a generalized fuzzy metric which is the extension of a fuzzy metric. First, inspired by the ideas of Grabiec (Fuzzy Sets Syst. 125:385-389, 1989), we define a new G-contraction of Banach type with respect to this generalized fuzzy metric, which is a generalization of the contraction of Banach type (introduced by M Grabiec). Next, inspired by the ideas of Gregori and Sapena (Fuzzy Sets Syst. 125:245-252, 2002), we define a new GV-contraction of Banach type with respect to this generalized fuzzy metric, which is a generalization of the contraction of Banach type (introduced by V Gregori and A Sapena). Moreover, we provide the condition guaranteeing the existence of a fixed point for these single-valued contractions. Next, we show that the generalized pseudodistance $J : X \times X \to [0, \infty)$ (introduced by Włodarczyk and Plebaniak (Appl. Math. Lett. 24:325-328, 2011)) may generate some generalized fuzzy metric N_J on X. The paper includes also the comparison of our results with those existing in the literature.

Keywords: fuzzy sets; fuzzy metric space; contraction of Banach type; fixed point; generalized fuzzy metrics; fuzzy metrics

1 Introduction

A number of authors generalize Banach's [1] and Caccioppoli's [2] result and introduce the new concepts of contractions of Banach and study the problem concerning the existence of fixed points for such a type of contractions; see *e.g.* Burton [3], Rakotch [4], Geraghty [5, 6], Matkowski [7–9], Walter [10], Dugundji [11], Tasković [12], Dugundji and Granas [13], Browder [14], Krasnosel'skiĭ *et al.* [15], Boyd and Wong [16], Mukherjea [17], Meir and Keeler [18], Leader [19], Jachymski [20, 21], Jachymski and Jóźwik [22], and many others not mentioned in this paper.

In 1975, Kramosil and Michalek [23] introduced the concept of fuzzy metric spaces. It is worth noticing that there exist at least five different concepts of a fuzzy metric space (see Artico and Moresco [24], Deng [25], George and Veeramani [26], Erceg [27], Kaleva and Seikkala [28], Kramosil and Michalek [23]).

In 1989, Grabiec [29] proved an analog of the Banach contraction theorem in fuzzy metric spaces (in the sense of Kramosil and Michalek [23]). In his proof, he used a fuzzy version of Cauchy sequence. It is worth noticing that in the literature in order to prove fixed point theorems in fuzzy metric space, authors used two different types of Cauchy sequences. For details see [30]. The existence of fixed points for maps in fuzzy metric spaces was studied by many authors; see *e.g.* Gregori and Sapena [31], Miheț [32]. Fixed point theory for contractive mappings in fuzzy metric spaces is closely related to the fixed point theory for the same type of mappings in probabilistic metric spaces of Menger type; see Hadžić [33], Sehgal and Bharucha-Reid [34], Schweizer *et al.* [35], Tardiff [36], Schweizer and Sklar [37], Qiu and Hong [38], Hong and Peng [39], Mohiuddine and Alotaibi [40], Wang *et al.* [41], Hong [42], Saadati *et al.* [43], and many others not mentioned in this paper.

In this paper, in fuzzy metric spaces (in the sense of Kramosil and Michalek [23]), we introduce the concept of a generalized fuzzy metric which is the extension of a fuzzy metric. First, inspired by the ideas of Grabiec [29], we define a new G-contraction of Banach type with respect to this generalized fuzzy metric, which is a generalization of a contraction of Banach type (introduced by M Grabiec). Next, inspired by the ideas of Gregori and Sapena [31], we define a new GV-contraction of Banach type with respect to this generalized fuzzy metric, which is a generalization of a contraction of Banach type (introduced by V Gregori and A Sapena). Moreover, we provide the condition guaranteeing the existence of a fixed point for these single-valued contractions. Next, we show that the generalized pseudodistance $J : X \times X \to [0,\infty)$ (introduced by Włodarczyk and Plebaniak [44]) may generate some generalized fuzzy metric N_J on X. Moreover, if we put $J = d$, where $d : X \times X \to [0,\infty)$ is the usual metric, then N_J is a fuzzy metric generated by d.

2 On fixed point theory in Kramosil and Michalek's fuzzy metric spaces and George and Veeramani's fuzzy metric spaces

To begin with, we recall the concept of a fuzzy metric space, which was introduced by Kramosil and Michalek [23] in 1975.

Definition 2.1 [23] The 3-tuple $(X,M,*)$ is a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set in $X^2 \times [0,\infty)$ satisfying the following conditions:

(M1) $\forall_{x,y\in X}\{M(x,y,0) = 0\}$;
(M2) $\forall_{x,y\in X}\{\forall_{t>0}\{M(x,y,t) = 1\} \Leftrightarrow x = y\}$;
(M3) $\forall_{x,y\in X}\forall_{t>0}\{M(x,y,t) = M(y,x,t)\}$;
(M4) $\forall_{x,y,z\in X}\forall_{t,s>0}\{M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)\}$;
(M5) $M(x,y,\cdot) : [0,\infty) \to [0,1]$ is left-continuous, for all $x,y \in X$.

Then M is called a fuzzy metric on X.

Definition 2.2 (I) [29] A sequence $(x_m : m \in \mathbb{N})$ in X is Cauchy in Grabiec's sense (we say G-Cauchy) if

$$\forall_{t>0}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty} M(x_m, x_{m+p}, t) = 1\right\}.$$

(II) [29] A sequence $(x_m : m \in \mathbb{N})$ in X is convergent to $x \in X$ if

$$\forall_{t>0}\left\{\lim_{m\to\infty} M(x_m, x, t) = 1\right\},$$

i.e.,

$$\forall_{t>0}\forall_{\varepsilon>0}\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\left\{M(x_m, x, t) > 1 - \varepsilon\right\}.$$

Of course, since $*: [0,1] \times [0,1] \to [0,1]$ is continuous, by (M4) it follows that the limit is uniquely determined.

(III) [29] A fuzzy metric space in which every G-Cauchy sequence is convergent is called complete in Grabiec's sense (G-complete for short).

Some interesting observations on these definitions can be found in [45].

In 1989, Grabiec [29] established the following extension of Banach's result in Kramosil and Michalek's fuzzy metric space.

Theorem 2.1 (Fuzzy Banach contraction theorem, Grabiec [29]) *Let* $(X, M, *)$ *be a G-complete fuzzy metric space such that*

$$\forall_{x,y \in X} \Big\{ \lim_{t \to \infty} M(x,y,t) = 1 \Big\}.$$

Let $T: X \to X$ *be a mapping satisfying*

(G1) $\exists_{k \in (0,1)} \forall_{x,y \in X} \forall_{t>0} \{ M(T(x), T(y), kt) \geq M(x,y,t) \}$.

Then T *has a unique fixed point.*

Next, we recall the concept of a fuzzy metric space, which was introduced by George and Veeramani [26] in 1994.

Definition 2.3 [26] The 3-tuple $(X, M, *)$ is a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

(M1) $\forall_{x,y \in X} \forall_{t>0} \{ M(x,y,t) > 0 \}$;

(M2) $\forall_{x,y \in X} \{ \forall_{t>0} \{ M(x,y,t) = 1 \} \Leftrightarrow x = y \}$;

(M3) $\forall_{x,y \in X} \forall_{t>0} \{ M(x,y,t) = M(y,x,t) \}$;

(M4) $\forall_{x,y,z \in X} \forall_{t,s>0} \{ M(x,z,t+s) \geq M(x,y,t) * M(y,z,s) \}$;

(M5) $M(x,y,\cdot): (0,\infty) \to [0,1]$ is continuous, for all $x, y \in X$.

Then M is called a fuzzy metric on X.

Definition 2.4 (I) [26] Let $(X, M, *)$ be a fuzzy metric space. The open ball $B(x,r,t)$ for $t > 0$ with center $x \in X$ and radius r, $0 < r < 1$, is defined as

$$B(x,r,t) = \big\{ y \in X : M(x,y,t) > 1 - r \big\}.$$

The family $\{ B(x,r,t) : x \in X, 0 < r < 1, t > 0 \}$ is a neighborhood's system for a Hausdorff topology on X, which we call induced by the fuzzy metric M.

(II) [31] A sequence $(x_m : m \in \mathbb{N})$ in X is Cauchy in George and Veeramani's sense (we say GV-Cauchy) if

$$\forall_{\varepsilon>0} \forall_{t>0} \exists_{m_0 \in \mathbb{N}} \forall_{n,m \geq m_0} \big\{ M(x_n, x_m, t) > 1 - \varepsilon \big\}.$$

(III) [31] A fuzzy metric space in which every GV-Cauchy sequence is convergent is called complete in George and Veeramani's sense (GV-complete for short).

In 2002, Gregori and Sapena [31] established the following extension of Banach's result in George and Veeramani's fuzzy metric spaces.

Theorem 2.2 (Fuzzy Banach contraction theorem, Gregori and Sapena [31]) *Let* $(X, M, *)$ *be a GV-complete fuzzy metric space in which fuzzy contractive sequences, i.e.,*

$$\exists_{k\in[0,1]}\forall_{t>0}\forall_{m\in\mathbb{N}}\left\{\frac{1}{M(x_{m+1},x_{m+2},t)}-1\leq k\left(\frac{1}{M(x_m,x_{m+1},t)}-1\right)\right\},$$

are GV-Cauchy. Let $T : X \to X$ *be a mapping satisfying*

(G2) $\exists_{k\in(0,1)}\forall_{x,y\in X}\forall_{t>0}\{\frac{1}{M(T(x),T(y),t)}-1\leq k(\frac{1}{M(x,y,t)}-1)\}$.

Then T *has a unique fixed point.*

3 On generalized fuzzy metric and fixed point theory in Kramosil and Michalek's fuzzy metric spaces and George and Veeramani's fuzzy metric spaces

Now in Kramosil and Michalek's fuzzy metric space we introduce the concept of a generalized fuzzy metric on X. Next, we define a new kind of completeness of the space.

Definition 3.1 Let $(X, M, *)$ be a fuzzy metric space. The map N is said to be a *G-generalized fuzzy metric on* X if the following three conditions hold:

(N1) $\forall_{x,y,z\in X}\forall_{t,s>0}\{N(x,z,t+s)\geq N(x,y,t)*N(y,z,s)\}$;

(N2) $N(x,y,\cdot):[0,\infty)\to[0,1]$ is left-continuous, for all $x, y \in X$;

(N3) for any sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X such that

$$\forall_{t>0}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty}N(x_m,x_{m+p},t)=1\right\} \tag{3.1}$$

and

$$\forall_{t>0}\left\{\lim_{m\to\infty}N(x_m,y_m,t)=1\right\}, \tag{3.2}$$

we have

$$\forall_{t>0}\left\{\lim_{m\to\infty}M(x_m,y_m,t)=1\right\}. \tag{3.3}$$

Remark 3.1 If $(X, M, *)$ is a fuzzy metric space, then the fuzzy metric M is a G-generalized fuzzy metric on X. However, there exists a G-generalized fuzzy metric on X which is not a fuzzy metric on X (for details see Example 4.3).

Definition 3.2 (I) A sequence $(x_m : m \in \mathbb{N})$ in X is N-Cauchy in Grabiec's sense (we say N-G-Cauchy) if

$$\forall_{t>0}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty}N(x_m,x_{m+p},t)=1\right\}.$$

(II) A sequence $(x_m : m \in \mathbb{N})$ in X is N-convergent to $x \in X$ if

$$\forall_{t>0}\left\{\lim_{m\to\infty}N(x_m,x,t)=1\right\}.$$

(III) A fuzzy metric space is called N-G-complete if each N-G-Cauchy sequence $(x_m : m \in \mathbb{N})$ in X is N-convergent to some $x \in X$ and

(NC) $\forall_{t>0}\{\lim_{m\to\infty}N(x_m,x,t)=\lim_{m\to\infty}N(x,x_m,t)=1\}$.

Now we prove the auxiliary lemma.

Lemma 3.1 *Let $(X, M, *)$ be a fuzzy metric space and let the map N be a G-generalized fuzzy metric on X. Then for each $x, y \in X$ the following property holds*:

$$\left\{\forall_{t>0}\left\{N(x,y,t) = 1 \wedge N(y,x,t) = 1\right\} \Rightarrow \{x = y\}\right\}.$$

Proof Let $x, y \in X$ such that

$$\forall_{t>0}\left\{N(x,y,t) = 1 \wedge N(y,x,t) = 1\right\} \tag{3.4}$$

be arbitrary and fixed. By (N1) and (3.4), we get

$$\forall_{t>0}\left\{N(x,x,t) \geq N\left(x,y,\frac{t}{2}\right) * N\left(y,x,\frac{t}{2}\right) = 1 * 1 = 1\right\}. \tag{3.5}$$

Defining the sequences $(x_m = x : m \in \mathbb{N})$ and $(y_m = y : m \in \mathbb{N})$, from (3.5) and (3.4) we have

$$\forall_{t>0}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty} N(x_m, x_{m+p}, t) = 1\right\}$$

and

$$\forall_{t>0}\left\{\lim_{m\to\infty} N(x_m, y_m, t) = 1\right\}.$$

Hence, the properties (3.1) and (3.2) hold. Therefore, by (N3), we see that

$$\forall_{t>0}\left\{\lim_{m\to\infty} M(x_m, y_m, t) = 1\right\}$$

which, by the definition of the sequences $(x_m = x : m \in \mathbb{N})$ and $(y_m = y : m \in \mathbb{N})$, gives

$$\forall_{t>0}\left\{M(x,y,t) = 1\right\}.$$

Hence, by (M2), we conclude that $x = y$. □

The main result of the paper is the following.

Theorem 3.1 *Let $(X, M, *)$ be a fuzzy metric space, and let N be a G-generalized fuzzy metric on X such that*

$$\forall_{x,y\in X}\left\{\lim_{t\to\infty} N(x,y,t) = 1\right\}. \tag{3.6}$$

Let $T : X \to X$ be an N-G-contraction of Banach type, i.e., T is a mapping satisfying

(B1) $\exists_{k\in(0,1)}\forall_{x,y\in X}\forall_{t>0}\{N(T(x), T(y), kt) \geq N(x,y,t)\}$.

We assume that a fuzzy metric space $(X, M, *)$ is N-G-complete. Then T has a unique fixed point $w \in X$, and for each $x \in X$, the sequence $(x_m = T^m(x_0) : x_0 = x, m \in \mathbb{N})$ is convergent to w. Moreover, $N(w, w, t) = 1$, for all $t > 0$.

Proof The proof will be divided into four steps.

Step I. *We see that for each $x \in X$ the sequence $(x_m = T^m(x_0) : x_0 = x, m \in \mathbb{N})$ satisfies*

$$\exists_{k\in(0,1)}\forall_{t>0}\forall_{m\in\mathbb{N}}\left\{N(x_m, x_{m+1}, kt) \geq N\left(x_0, x_1, \frac{t}{k^{m-1}}\right)\right\}. \tag{3.7}$$

Indeed, let $x_0 = x \in X$ be arbitrary and fixed and let $(x_m = T^m(x_0) : m \in \mathbb{N})$. Let $k \in (0,1)$ be as in (B1), and let $m \in \mathbb{N}$ and $t > 0$ be arbitrary and fixed. From (B1) we obtain

$$\begin{aligned} N(x_m, x_{m+1}, kt) &= N\bigl(T(x_{m-1}), T(x_m), kt\bigr) \geq N(x_{m-1}, x_m, t) \\ &= N\left(T(x_{m-2}), T(x_{m-1}), k\frac{t}{k}\right) \geq N\left(x_{m-2}, x_{m-1}, \frac{t}{k}\right) \\ &= N\left(T(x_{m-3}), T(x_{m-2}), k\frac{t}{k^2}\right) \\ &\geq N\left(x_{m-3}, x_{m-2}, \frac{t}{k^2}\right) \\ &\geq \cdots \geq N\left(x_0, x_1, \frac{t}{k^{m-1}}\right). \end{aligned}$$

Consequently, the property (3.7) holds.

Step II. *We see that for each $x \in X$ the sequence $(x_m = T^m(x_0) : x_0 = x, m \in \mathbb{N})$ is N-G-Cauchy, i.e., it satisfies*

$$\forall_{t>0}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty} N(x_m, x_{m+p}, t) = 1\right\}. \tag{3.8}$$

Indeed, let $x_0 = x \in X$ be arbitrary and fixed and let $(x_m = T^m(x_0) : m \in \mathbb{N})$. Let $m, p \in \mathbb{N}$ and $t > 0$ be arbitrary and fixed. Then by (N1) and (3.7) we calculate

$$\begin{aligned} N(x_m, x_{m+p}, t) &\geq N\left(x_m, x_{m+1}, \frac{t}{p}\right) * \overset{(p)}{\cdots} * N\left(x_{m+p-1}, x_{m+p}, \frac{t}{p}\right) \\ &\geq N\left(x_0, x_1, \frac{t}{pk^m}\right) * \overset{(p)}{\cdots} * N\left(x_0, x_1, \frac{t}{pk^{m+p-1}}\right). \end{aligned}$$

Now, using (3.6) we obtain

$$\lim_{m\to\infty} N(x_m, x_{m+p}, t) \geq 1 * \overset{(p)}{\cdots} * 1 = 1.$$

Thus (3.8) holds.

Step III. *Next we see that for each $x \in X$ the sequence $(x_m = T^m(x_0) : x_0 = x, m \in \mathbb{N})$ is convergent to a fixed point of T.*

Indeed, let $x_0 = x \in X$ be arbitrary and fixed and let $(x_m = T^m(x_0) : m \in \mathbb{N})$. By Step II the sequence $(x_m : m \in \mathbb{N})$ is N-G-Cauchy in X. By the N-G-completeness of X (Definition 3.2(III)), there exists $w \in X$ such that $(x_m : m \in \mathbb{N})$ is N-convergent to w (*i.e.*, $\forall_{t>0}\{\lim_{m\to\infty} N(x_m, w, t) = 1\}$). Moreover, by (NC), we get

$$\forall_{t>0}\left\{\lim_{m\to\infty} N(x_m, w, t) = \lim_{m\to\infty} N(w, x_m, t) = 1\right\}. \tag{3.9}$$

Next, using (N1) and (B1) we calculate

$$\begin{aligned}\forall_{t>0}\forall_{m\in\mathbb{N}}\Big\{&N\big(T(w),w,t\big)\geq N\left(T(w),T(x_m),\frac{t}{2}\right)*N\left(T(x_m),w,\frac{t}{2}\right)\\&=N\left(T(w),T(x_m),\frac{t}{2}\right)*N\left(x_{m+1},w,\frac{t}{2}\right)\\&\geq N\left(w,x_m,\frac{t}{2k}\right)*N\left(x_{m+1},w,\frac{t}{2}\right)\Big\},\end{aligned}$$

which, by (3.9), gives

$$\forall_{t>0}\left\{N\big(T(w),w,t\big)\geq \lim_{m\to\infty}N\left(w,x_m,\frac{t}{2k}\right)*\lim_{m\to\infty}N\left(x_{m+1},w,\frac{t}{2}\right)=1*1=1\right\}. \quad (3.10)$$

Similarly, using (N1) and (B1) we calculate

$$\begin{aligned}\forall_{t>0}\forall_{m\in\mathbb{N}}\Big\{&N\big(w,T(w),t\big)\geq N\left(w,T(x_m),\frac{t}{2}\right)*N\left(T(x_m),T(w),\frac{t}{2}\right)\\&=N\left(w,x_{m+1},\frac{t}{2}\right)*N\left(T(x_m),T(w),\frac{t}{2}\right)\\&\geq N\left(w,x_{m+1},\frac{t}{2}\right)*N\left(x_m,w,\frac{t}{2k}\right)\Big\},\end{aligned}$$

which, by (3.9), gives

$$\forall_{t>0}\left\{N\big(w,T(w),t\big)\geq \lim_{m\to\infty}N\left(w,x_{m+1},\frac{t}{2}\right)*\lim_{m\to\infty}N\left(x_m,w,\frac{t}{2k}\right)=1*1=1\right\}. \quad (3.11)$$

Now, from (3.10), (3.11), and Lemma 3.1 we obtain $w=T(w)$, *i.e.*, w is a fixed point of T in X. Moreover, by (N1), (3.10), and (3.11), we obtain

$$\forall_{t>0}\left\{N(w,w,t)\geq N\left(w,T(w),\frac{t}{2}\right)*N\left(T(w),w,\frac{t}{2}\right)=1*1=1\right\}. \quad (3.12)$$

Now, if we define the sequence $(y_m=w:m\in\mathbb{N})$, then by (3.8) and (3.9) we have

$$\forall_{t>0}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty}N(x_m,x_{m+p},t)=1\right\}$$

and

$$\forall_{t>0}\left\{\lim_{m\to\infty}N(x_m,y_m,t)=1\right\}.$$

Therefore (3.1) and (3.2) hold, so by (N3) we have $\forall_{t>0}\{\lim_{m\to\infty}M(x_m,y_m,t)=1\}$, which gives

$$\forall_{t>0}\left\{\lim_{m\to\infty}M(x_m,w,t)=1\right\}.$$

Step IV. *Finally we see that w is a unique fixed point of T in X and $N(w,w,t)=1$, for all $t>0$.*

Indeed, assume that $T(v) = v$ for some $v \in X$. Then using (B1) we obtain

$$\forall_{t>0}\forall_{m\in\mathbb{N}}\left\{1 \geq N(v,w,t) = N\big(T(v),T(w),t\big) \geq N\left(v,w,\frac{t}{k}\right) = N\left(T(v),T(w),\frac{t}{k}\right) \geq N\left(v,w,\frac{t}{k^2}\right) \geq \cdots \geq N\left(v,w,\frac{t}{k^m}\right)\right\},$$

which, by (N2) and (3.6), gives

$$\forall_{t>0}\left\{1 \geq N(v,w,t) \geq \lim_{m\to\infty} N\left(v,w,\frac{t}{k^m}\right) = 1\right\}.$$

Similarly, using (B1), (N3), and (3.6) we calculate $\forall_{t>0}\{1 \geq N(w,v,t) \geq \lim_{m\to\infty} N(w,v,\frac{t}{k^m}) = 1\}$. Hence,

$$\forall_{t>0}\big\{N(v,w,t) = 1 \wedge N(w,v,t) = 1\big\}.$$

Next, applying Lemma 3.1, we get $v = w$, thus the fixed point of T is unique. Moreover, by (3.12) we get $\forall_{t>0}\{N(w,w,t) = 1\}$. □

Remark 3.2 It is worth noticing that in George and Veeramani's fuzzy metric space we may introduce the concept of a generalized fuzzy metric (in the sense of George-Veeramani) on X (for short, GV-generalized fuzzy metric). Let $(X,M,*)$ be a fuzzy metric space. The map N is said to be a *GV-generalized fuzzy metric on X* if the following three conditions hold:

($N_{GV}1$) $\forall_{x,y,z\in X}\forall_{t,s>0}\{N(x,z,t+s) \geq N(x,y,t) * N(y,z,s)\}$;
($N_{GV}2$) $N(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous, for all $x, y \in X$;
($N_{GV}3$) for any sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X such that

$$\forall_{\varepsilon>0}\forall_{t>0}\exists_{m_0\in\mathbb{N}}\forall_{n>m\geq m_0}\big\{N(x_n,x_m,t) > 1-\varepsilon\big\} \tag{3.13}$$

and

$$\forall_{t>0}\forall_{\varepsilon>0}\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\big\{N(x_m,y_m,t) > 1-\varepsilon\big\}, \tag{3.14}$$

we have

$$\forall_{t>0}\forall_{\varepsilon>0}\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\big\{M(x_m,y_m,t) > 1-\varepsilon\big\}. \tag{3.15}$$

Remark 3.3 Using similar considerations, we may introduce the concepts of N-Cauchy sequences in George and Veeramani's sense and N-GV-completeness. Precisely: (I) A sequence $(x_m : m \in \mathbb{N})$ in X is N-Cauchy in George and Veeramani's sense (we say N-GV-Cauchy) if

$$\forall_{\varepsilon>0}\forall_{t>0}\exists_{m_0\in\mathbb{N}}\forall_{n>m\geq m_0}\big\{N(x_m,x_n,t) > 1-\varepsilon\big\}.$$

(II) A fuzzy metric space is called N-GV-complete, if each N-GV-Cauchy sequence $(x_m : m \in \mathbb{N})$ in X is N-convergent to some $x \in X$ and $\forall_{t>0}\{\lim_{m\to\infty} N(x_m,x,t) = \lim_{m\to\infty} N(x,x_m,t) = 1\}$.

Now using similar arguments to the corresponding ones appearing in Section 3 and in the paper of Gregori and Sapena [31] we may conclude the following fixed point theorem in George and Veeramani's fuzzy metric space.

Theorem 3.2 *Let $(X,M,*)$ be a fuzzy metric space, and let N be a GV-generalized fuzzy metric on $X^2 \times [0,\infty)$ such that N-fuzzy contractive sequences, i.e.,*

$$\exists_{k\in(0,1)}\forall_{t>0}\forall_{m\in\mathbb{N}}\left\{\frac{1}{N(x_{m+1},x_{m+2},t)}-1\leq k\left(\frac{1}{N(x_m,x_{m+1},t)}-1\right)\right\},$$

are N-GV-Cauchy. Let $T:X\to X$ be an N-GS-contraction of Banach type (in the sense of Gregori and Sapena), i.e., a mapping satisfying

(B2) $\exists_{k\in(0,1)}\forall_{x,y\in X}\forall_{t>0}\{\frac{1}{N(T(x),T(y),t)}-1\leq k(\frac{1}{N(x,y,t)}-1)\}$.

We assume that a fuzzy metric space $(X,M,*)$ is N-GV-complete. Then T has a unique fixed point $w\in X$, and for each $x\in X$, the sequence $(x_m=T^m(x_0):x_0=x,m\in\mathbb{N})$, is convergent to w. Moreover, $N(w,w,t)=1$, for all $t>0$.

4 Examples illustrating Theorem 3.2 and some comparisons

Now, we will present some examples illustrating the concepts that have been introduced so far. We will show a fundamental difference between Theorem 2.2 and Theorem 3.2. Examples will show that Theorem 3.2 is the essential generalization of Theorem 2.2. First, we recall an example of the standard fuzzy metric induced by the metric d.

Example 4.1 [31, Definition 2.5] Let X be a metric space. Let $*$ be the usual product on $[0,1]$. Then the 3-tuple $(X,M_d,*)$ where

(MD) $M_d(x,y,t)=\frac{t}{t+d(x,y)}$, $x,y\in X$,

is a George and Veeramani fuzzy metric space (standard fuzzy metric space), and M_d is fuzzy metric on X.

Recently, in 2011, Włodarczyk and Plebaniak introduced the concept of generalized pseudodistances which, in a natural way, are extensions of metrics. For details see [44]. We recall the concept of a generalized pseudodistance.

Definition 4.1 Let X be a metric space with a metric $d:X\times X\to[0,\infty)$. The map $J:X\times X\to[0,\infty)$ is said to be a *generalized pseudodistance on X* if the following two conditions hold:

(J1) $\forall_{x,y,z\in X}\{J(x,z)\leq J(x,y)+J(y,z)\}$;

(J2) for any sequences $(x_m:m\in\mathbb{N})$ and $(y_m:m\in\mathbb{N})$ in X such that

$$\lim_{n\to\infty}\sup_{m>n}J(x_n,x_m)=0 \tag{4.1}$$

and

$$\lim_{m\to\infty}J(x_m,y_m)=0, \tag{4.2}$$

we have

$$\lim_{m\to\infty}d(x_m,y_m)=0. \tag{4.3}$$

We recall also the following remark.

Remark 4.1 (A) If (X,d) is a metric space, then the metric $d : X \times X \to [0,\infty)$ is a generalized pseudodistance on X. However, there exists a generalized pseudodistance on X which is not a metric (see Example 4.2).

(B) From (J1) and (J2) it follows that if $x \neq y$, $x, y \in X$, then

$$J(x,y) > 0 \vee J(y,x) > 0.$$

Indeed, if $J(x,y) = 0$ and $J(y,x) = 0$, then $J(x,x) = 0$, since, by (J1), we get $J(x,x) \leq J(x,y) + J(y,x) = 0 + 0 = 0$. Now, defining $x_m = x$ and $y_m = y$ for $m \in \mathbb{N}$, we conclude that (4.1) and (4.2) hold. Consequently, by (J2), we get (4.3), which implies $d(x,y) = 0$. However, since $x \neq y$, we have $d(x,y) \neq 0$. Contradiction.

(C) From (B) it follows that if $x \neq y$, then

$$\forall_{x,y\in X}\bigl\{\{J(x,y) = 0 \wedge J(y,x) = 0\bigr\} \Rightarrow \{x = y\}\bigr\}.$$

Now we introduce and use some particular kind of generalized pseudodistance to construct the generalized fuzzy metrics.

Example 4.2 Let X be a metric space with metric $d : X \times X \to [0,\infty)$. Let $E \subset X$ be a bounded and closed set, containing at least two different points, be arbitrary and fixed. Let $c, k > 0$ be such that $k > c > \delta(E)$, where $\delta(E) = \sup\{d(x,y) : x, y \in E\}$ are arbitrary and fixed. Define the map $J : X \times X \to [0,\infty)$ as follows:

$$J(x,y) = \begin{cases} d(x,y) & \text{if } \{x,y\} \cap E = \{x,y\}; \\ c & \text{if } x \notin E \wedge y \in E; \\ k & \text{if } x \in E \wedge y \notin E; \\ c+k & \text{if } \{x,y\} \cap E = \emptyset. \end{cases} \tag{4.4}$$

We can show that the map J is a generalized pseudodistance on X. Indeed, let $x, y \in X$ be arbitrary and fixed. We consider the following four cases:

Case 1. If $J(x,y) = d(x,y)$, then by (4.4) we obtain $\{x,y\} \in E$, so by the triangle inequality for d, we get $d(x,y) \leq d(x,z) + d(z,y)$ (if $z \in E$), and $d(x,y) < c < k = J(x,z)$ (if $z \notin E$, since $c > \delta(E)$). In consequence, in both situations

$$J(x,y) = d(x,y) \leq J(x,z) + J(z,y).$$

Case 2. If $J(x,y) = c$, then by (4.4) we obtain $x \notin E$ and $y \in E$, so by (4.4) $J(x,z) = c$ (if $z \in E$) and $J(x,z) = c + k$ (if $z \notin E$). In consequence, in both situations

$$J(x,y) = c \leq J(x,z) + J(z,y).$$

Case 3. If $J(x,y) = k$, then by (4.4) we obtain $x \in E$ and $y \notin E$, so by (4.4), $J(z,y) = k$ (if $z \in E$) and $J(z,y) = c + k$ (if $z \notin E$). In consequence, in both situations

$$J(x,y) = k \leq J(x,z) + J(z,y).$$

Case 4. If $J(x,y) = c + k$, then by (4.4) we obtain $x \notin E$ and $y \notin E$, so by (4.4), $J(x,z) = c$, $J(z,y) = k$ (if $z \in E$) and $J(x,z) = J(z,y) = c + k$ (if $z \notin E$). In consequence, in both situations

$$J(x,y) = c + k \leq J(x,z) + J(z,y).$$

Therefore, $\forall_{x,y,z\in X}\{J(x,y) \leq J(x,z) + J(z,y)\}$, *i.e.*, the condition (J1) holds.

For proving that (J2) holds we assume that the sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ in X satisfy (4.1) and (4.2). Then, in particular, (4.2) yields

$$\forall_{0<\varepsilon<c}\exists_{m_0=m_0(\varepsilon)\in\mathbb{N}}\forall_{m\geq m_0}\big\{J(v_m,u_m) < \varepsilon\big\}. \tag{4.5}$$

By (4.5) and (4.4), since $\varepsilon < c$, we conclude that

$$\forall_{m\geq m_0}\big\{E \cap \{v_m,u_m\} = \{v_m,u_m\}\big\}. \tag{4.6}$$

From (4.6), (4.4), and (4.5), we get $\forall_{0<\varepsilon<c}\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\{d(v_m,u_m) < \varepsilon\}$. Therefore, the sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ satisfy (4.3). Consequently, the property (J2) holds.

In the remaining part of the work, the generalized pseudodistance defined by (4.4) will be called a generalized pseudodistance generated by d.

Example 4.3 Let (X,d) be a standard metric space. Let $J : X \times X \to [0,\infty)$ be a generalized pseudodistance on X generated by d (*i.e.*, defined in Example 4.2). Let $*$ be a continuous t-norm given by $a * b = ab$. Then the N_J where

$$N_J(x,y,t) = \frac{t}{t + J(x,y)}, \tag{4.7}$$

$x,y \in X$, is a GV-generalized fuzzy metric on X.

Part I. We prove ($N_{GV}1$).

Let $x,y,z \in X$ be arbitrary and fixed. By (J1) we get

$$J(x,z) \leq J(x,y) + J(y,z). \tag{4.8}$$

Assume that there exist $t_0 > 0$ and $s_0 > 0$ such that $N(x,y,t_0) * N(y,z,s_0) > N(x,z,t_0 + s_0)$. This, by (4.7), gives

$$\frac{t_0}{t_0 + J(x,y)} \cdot \frac{s_0}{s_0 + J(y,z)} > \frac{t_0 + s_0}{t_0 + s_0 + J(x,z)}.$$

Hence by a simple calculation we obtain a contradiction. In consequence (N1) and ($N_{GV}1$) hold.

Part II. We prove ($N_{GV}2$).

Let $x,y \in X$ be arbitrary and fixed. Then for $l = J(x,y) \in [0,\infty)$ we have

$$N_J(x,y,t) = \frac{t}{t + l}, \quad t \in [0,\infty).$$

Thus, $N_J(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous, for each $x,y \in X$. In consequence (N2) and ($N_{GV}2$) hold.

Part III. Next we prove ($N_{GV}3$).

We assume that the sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X satisfy (3.13) and (3.14). Then, in particular, (3.14) yields

$$\forall_{t>0}\forall_{0<\varepsilon<1-\frac{t}{t+c}}\exists_{m_0(\varepsilon)\in\mathbb{N}}\forall_{m\geq m_0}\{N_J(x_m,y_m,t)>1-\varepsilon\}. \tag{4.9}$$

Since $\varepsilon < 1 - \frac{t}{t+c}$, by a simple calculation we have

$$1-\varepsilon > \frac{t}{t+c}. \tag{4.10}$$

Next, from (4.9) and (4.10) we obtain

$$\forall_{t>0}\forall_{0<\varepsilon<1-\frac{t}{t+c}}\exists_{m_0(\varepsilon)\in\mathbb{N}}\forall_{m\geq m_0}\left\{N_J(x_m,y_m,t)>\frac{t}{t+c}\right\}. \tag{4.11}$$

Now, let $m \geq m_0$. We obtain $N_J(x_m, y_m, t) > \frac{t}{t+c}$, next, by (4.7) we have $\frac{t}{t+J(x_m,y_m)} > \frac{t}{t+c}$, so $t + J(x_m, y_m) < t + c$ and finally $J(x_m, y_m) < c$, which, by (4.4), gives $J(x_m, y_m) = d(x_m, y_m)$. Therefore $N_J(x_m, y_m, t) = M(x_m, y_m, t)$.

Hence, using (4.9) we obtain

$$\forall_{t>0}\forall_{0<\varepsilon<1-\frac{t}{t+c}}\exists_{m_0(\varepsilon)\in\mathbb{N}}\forall_{m\geq m_0}\{M_d(x_m,y_m,t)>1-\varepsilon\}.$$

Consequently (3.15) holds. Hence ($N_{GV}3$) holds.

Example 4.4 Let $(X, M_d, *)$ be a standard fuzzy metric space, where $X = [0,1]$, $*$ be a continuous t-norm given by $a * b = ab$. Let the closed set $E = [0, \frac{3}{8}] \subset X$ and let $J : X \times X \to [0,\infty)$ be given by

$$J(x,y) = \begin{cases} d(x,y) & \text{if } \{x,y\}\cap E = \{x,y\}; \\ 2 & \text{if } x \notin E \wedge y \in E; \\ 3 & \text{if } x \in E \wedge y \notin E; \\ 5 & \text{if } \{x,y\}\cap E = \emptyset. \end{cases} \tag{4.12}$$

Let N_J be defined by

$$N_J(x,y,t) = \frac{t}{t+J(x,y)}. \tag{4.13}$$

Let $T : X \to X$ be a single-valued map given by

$$T(x) = \begin{cases} 0 & \text{for } x \in [0,\frac{1}{2}]; \\ \frac{3}{2}x - \frac{3}{4} & \text{for } x \in (\frac{1}{2},\frac{3}{4}]; \\ \frac{3}{8} & \text{for } x \in (\frac{3}{4},1], \end{cases} \quad x \in X. \tag{4.14}$$

(A) By Example 4.2, J is a generalized pseudodistance on X. Next, by Example 4.3, N_J is a GV-generalized fuzzy metric on X.

(B) We observe that T is N_J-GS-contraction of Banach type, *i.e.*, T satisfies the condition (B2). The proof will be divided into two steps.

Step I. First, we show that T satisfies the following conditions:

$$\exists_{\lambda\in[0,1)}\forall_{x,y\in X}\{J(T(x),T(y))\leq\lambda J(x,y)\}. \tag{4.15}$$

Indeed, let $\lambda=\frac{1}{4}$ and let $x,y\in X$ be arbitrary and fixed. We consider the following two cases:

Case 1. If $\{x,y\}\cap E=\{x,y\}$ then by (4.12), $J(x,y)=d(x,y)$. Moreover, since $\max\{x,y\}<\frac{1}{2}$ thus by (4.14), $T(x)=T(y)=0\in E$. Hence, by (4.12), we obtain

$$J(T(x),T(y))=J(0,0)=d(0,0)=0\leq\frac{1}{4}d(x,y)=\lambda J(x,y). \tag{4.16}$$

Case 2. If $\{x,y\}\cap E\neq\{x,y\}$ then by (4.12), $J(x,y)\in\{2,3,5\}$. Moreover, since

$$\forall_{x,y\in X}\left\{\max\{T(x),T(y)\}\leq\frac{3}{8}\right\},\qquad\{T(x),T(y)\}\cap E=\{T(x),T(y)\},$$

and by (4.12), $J(T(x),T(y))=d(T(x),T(y))<\delta(E)=3/8$. Hence we obtain

$$J(T(x),T(y))=d(T(x),T(y))\leq\frac{3}{8}<\frac{1}{2}=\frac{1}{4}\cdot 2=\frac{1}{4}\cdot\min\{2,3,5\}\leq\lambda J(x,y). \tag{4.17}$$

Concluding, from (4.16) and (4.17), we obtain (4.15).

Step II. We show that T satisfies the following conditions:

$$\exists_{k\in(0,1)}\forall_{x,y\in X}\forall_{t>0}\left\{\frac{1}{N(T(x),T(y),t)}-1\leq k\left(\frac{1}{N(x,y,t)}-1\right)\right\}. \tag{4.18}$$

Let $k=\lambda=\frac{1}{4}$. Let $x,y\in X$, $t>0$ be arbitrary and fixed. By (4.15) we know that $J(T(x),T(y))\leq\lambda J(x,y)$. Hence, we obtain the following chain of equivalences:

$$\begin{aligned}&\{J(T(x),T(y))\leq\lambda J(x,y)\}\\ &\quad\Leftrightarrow\quad\left\{\frac{J(T(x),T(y))}{t}\leq k\frac{J(x,y)}{t}\right\}\\ &\quad\Leftrightarrow\quad\left\{\frac{t+J(T(x),T(y))}{t}-1\leq k\left[\frac{t+J(x,y)}{t}-1\right]\right\}\\ &\quad\Leftrightarrow\quad\left\{\frac{1}{N_J(T(x),T(y),t)}-1\leq k\left[\frac{1}{N_J(x,y,t)}-1\right]\right\}.\end{aligned}$$

Hence, the condition (4.18) is true, and the map T is N_J-GS-contraction of Banach type.

(C) Observe that T is not contraction of Banach type (in the sense of Gregori and Sapena), *i.e.*, T does not satisfy the condition (G2). Indeed, suppose that T is contraction of Banach type (in the sense of Gregori and Sapena). Then there exists $k_0\in[0,1)$ such that

$$\forall_{t>0}\forall_{x,y\in X}\left\{\frac{1}{M_d(T(x),T(y),t)}-1\leq k_0\left[\frac{1}{M_d(x,y,t)}-1\right]\right\}. \tag{4.19}$$

In particular, for $x_0=1/2$ and $y_0=3/4$, by (4.14), we have $T(x_0)=0$, $T(y_0)=3/8$. Hence $d(T(x_0),T(y_0))=3/8$. Moreover, $d(x_0,y_0)=1/4$ and consequently, for each $t>0$, by (MD) and (4.19), we have

$$\begin{aligned}\frac{3/8}{t} &= \frac{t+3/8}{t} - 1 = \frac{1}{M_d(T(x),T(y),t)} - 1 \\ &\leq k_0\left[\frac{1}{M_d(x,y,t)} - 1\right] \\ &= k_0\frac{t+1/4}{t} - 1 = k_0\frac{1/4}{t}.\end{aligned}$$

Hence $3/2 \leq k_0$, which is impossible (recall $k_0 \in [0,1)$).

(D) Now we see that $(X, M_d, *)$ is GV-complete standard fuzzy metric space.

Indeed, we see that (X,d) is complete metric space, thus by [31, Result 4.3] we conclude that the standard fuzzy metric space $(X, M_d, *)$ is GV-complete.

(E) Next, we observe that the fuzzy metric space $(X, M, *)$ is N-GV-complete.

Indeed, let $(x_m : m \in \mathbb{N})$ be a sequence such that $(x_m : m \in \mathbb{N})$ is N_J-GV-Cauchy, *i.e.*,

$$\forall_{\varepsilon>0}\forall_{t>0}\exists_{m_0\in\mathbb{N}}\forall_{n>m\geq m_0}\left\{N_J(x_m,x_n,t) > 1-\varepsilon\right\}. \tag{4.20}$$

Now, by (4.13) and (4.20) we have

$$\forall_{\varepsilon>0}\forall_{t>0}\exists_{m_0\in\mathbb{N}}\forall_{n>m\geq m_0}\left\{\frac{t}{t+J(x_m,x_n,t)} > 1-\varepsilon\right\}. \tag{4.21}$$

Hence, in particular, (4.21) yields

$$\forall_{t>0}\forall_{0<\varepsilon<\min\{\frac{2}{2+t},1\}}\exists_{m_0\in\mathbb{N}}\forall_{n>m\geq m_0}\left\{\frac{t}{t+J(x_m,x_n,t)} > 1-\varepsilon\right\}.$$

Hence by (4.12) we get

$$\exists_{m_0\in\mathbb{N}}\forall_{n>m\geq m_0}\left\{\{x_m,x_n\}\cap E = \{x_m,x_n\}\right\},$$

which gives $\forall_{m\geq m_0}\{x_m \in E\}$. Moreover, by (4.20), after simple calculations we see that the sequence $(x_m : m \in \mathbb{N})$ is GV-Cauchy. Now from (D) we obtain the result that there exists $x \in X$ such that

$$\forall_{t>0}\left\{\lim_{m\to\infty} M_d(x_m,x,t) = 1\right\}. \tag{4.22}$$

Now, from (4.22) and (MD) we know that $\lim_{m\to\infty} x_m = x$. Moreover, since E is a closed set, we obtain $x \in E$. Hence

$$\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\left\{\{x_m,x\}\cap E = \{x_m,x\}\right\},$$

which, by (4.12), gives

$$\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\left\{J(x_m,x) = J(x,x_m) = d(x_m,x)\right\}. \tag{4.23}$$

Finally, by (4.13) and (4.23) we have $N_J(x_m,x,t) = \frac{t}{t+J(x_m,x)} = \frac{t}{t+J(x,x_m)} = N_J(x,x_m,t)$. Hence, by (4.23) we obtain

$$\forall_{t>0}\left\{\lim_{m\to\infty} N_J(x_m,x,t) = \lim_{m\to\infty} N_J(x,x_m,t) = \lim_{m\to\infty}\frac{t}{t+J(x,x_m)} = \lim_{m\to\infty}\frac{t}{t+d(x,x_m)} = \lim_{m\to\infty} M_d(x,x_m,t) = 1\right\}.$$

Hence we find that $(X,M,*)$ *is* N-GV-*complete*.

(F) Now we see that each N-fuzzy contractive sequence $(x_m : m\in\mathbb{N})$ is N-GV-Cauchy. Indeed, let $(x_m : m\in\mathbb{N})$ be an N-fuzzy contractive sequence, *i.e.*,

$$\exists_{k\in(0,1)}\forall_{t>0}\forall_{m\in\mathbb{N}}\left\{\frac{1}{N(x_{m+1},x_{m+2},t)} - 1 \le k\left(\frac{1}{N(x_m,x_{m+1},t)} - 1\right)\right\}.$$

Hence,

$$\exists_{k\in(0,1)}\forall_{t>0}\forall_{m\in\mathbb{N}}\left\{\frac{t+J(x_{m+1},x_{m+2})}{t} - 1 \le k\left(\frac{t+J(x_m,x_m)}{t} - 1\right)\right\},$$

which gives

$$\exists_{k\in(0,1)}\forall_{t>0}\forall_{m\in\mathbb{N}}\left\{J(x_{m+1},x_{m+2}) \le kJ(x_m,x_{m+1})\right\}.$$

Now, by (4.12),

$$\exists_{m_0\in\mathbb{N};m_0\le 4}\forall_{m\ge m_0}\{x_m\in E\} \tag{4.24}$$

and

$$\exists_{k\in(0,1)}\forall_{t>0}\forall_{m\ge m_0}\left\{d(x_{m+1},x_{m+2}) \le kd(x_m,x_{m+1})\right\}.$$

Hence, the sequence $(x_m : m\in\mathbb{N})$ is contractive in (X,d), thus (by the completeness of (X,d)) convergent. Consequently, $(x_m : m\in\mathbb{N})$ is Cauchy in X. Therefore $(x_m : m\in\mathbb{N})$ is GV-Cauchy in $(X,M_d,*)$, *i.e.*,

$$\forall_{\varepsilon>0}\forall_{t>0}\exists_{m_1\in\mathbb{N}}\forall_{n,m\ge m_1}\left\{M_d(x_n,x_m,t) > 1-\varepsilon\right\}. \tag{4.25}$$

Now let $t>0$ and $\varepsilon>0$ be arbitrary and fixed. Then there exists $m_2 = \max\{m_0,m_1\}$ such that, by (4.24) and (4.25), we obtain

$$\forall_{n>m\ge m_2}\left\{N_J(x_n,x_m,t) = M_d(x_n,x_m,t) > 1-\varepsilon\right\}.$$

Hence the sequence $(x_m : m\in\mathbb{N})$ is N-GV-Cauchy.

(G) Finally, we observe that all assumptions of Theorem 3.2 are satisfied. The point $w=0$ is a fixed point of T in X. Moreover, for each $x\in X$, the sequence $(x_m = T^m(x_0) : x_0 = x, m\in\mathbb{N})$ satisfies condition $\forall_{m\ge 3}\{x_m = 0\}$. Hence, by (MD), we obtain $\forall_{t>0}\{\lim_{m\to\infty} M(x_m,x,t) = \lim_{m\to\infty}\frac{t}{t+d(0,x_m)} = 1\}$. In consequence, for each $x\in X$, the sequence $(x_m = T^m(x_0) : x_0 = x, m\in\mathbb{N})$ is convergent (in the standard fuzzy metric space $(X,M_d,*)$) to w.

Remark 4.2 (I) We observe that if we put $N_J = M_d$ in Theorem 3.2, then we find that Theorems 3.2 and 2.2 are identical.

(II) The introduction of the concept of a generalized fuzzy metric is essential. If X and T are such as in Example 4.4, then we can show that T is an N_J-GS-contraction of Banach type, but it is not a contraction of Banach type with respect to M_d (see Example 4.4(B), (C)). Hence, we see that our theorem is a generalization of Theorem 2.2 (Gregori and Sapena [31]).

Competing interests
The author declares that they have no competing interests.

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