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This work is dedicated to Professor Leon Mikołajczyk
on the occasion of his 85th birthday.

DYNAMIC PROGRAMMING APPROACH TO STRUCTURAL OPTIMIZATION PROBLEM – NUMERICAL ALGORITHM

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Abstract. In this paper a new shape optimization algorithm is presented. As a model application we consider state problems related to fluid mechanics, namely the Navier-Stokes equations for viscous incompressible fluids. The general approach to the problem is described. Next, transformations to classical optimal control problems are presented. Then, the dynamic programming approach is used and sufficient conditions for the shape optimization problem are given. A new numerical method to find the approximate value function is developed.

Keywords: sufficient optimality condition, elliptic equations, optimal shape control, structural optimization, stationary Navier-Stokes equations, dynamic programming, numerical approximation.

Mathematics Subject Classification: 49K20, 49J20, 93C20, 35L20.

1. INTRODUCTION

Recently, shape optimization problems have received a lot of attention, particularly in relation to a number of applications in physics and engineering that require a focus on shapes instead of on parameters or functions. The goal of these applications is to deform and modify the admissible shapes in order to comply with a given cost function that needs to be optimized. Here the competing objects are shapes, i.e. domains of $\mathbb{R}^N$, instead of functions or controls, as it usually occurs in problems of the calculus of variations or in optimal control theory. This constraint often produces additional difficulties that lead to a lack of existence of a solution and to the introduction of suitable relaxed formulations of the problem. In calculus of variations and optimal control theory we have close at hand many tools to calculate (at least theoretically) the optimal solution for a given cost functional. To mention a few: existence of solutions together with necessary optimality conditions, first order necessary conditions with second order sufficient optimality conditions, field theory (necessary and sufficient optimality conditions), dynamic programming approach (primal and dual). Most of these methods are not developed for shape optimization problems. In the existing literature we can find only approaches that concern the existence of solutions together with necessary optimality conditions see e.g. [2, 5, 14, 15], and second order sufficient optimality conditions see e.g. [4, 6, 7, 9]. To the knowledge of the authors there are no papers using the dynamic programming approach except for the two – [12, 17], where some attempts to use the primal and dual dynamic programming are described.

In the paper, as a model application, we consider state problems related to fluid mechanics, namely the Navier-Stokes equations for viscous incompressible fluids. The main problem is to search for the optimal shape of a given objective. Although the theoretical background of the presented method is expressed in general terms, as an example a structural optimization problem for an elastic body – shape optimization of a dividing tube – is being considered. In that case, the structural optimization problem consists in finding the shape of the boundary of the domain occupied by the body, such that the outflow profile should be close to a given target profile. For an incompressible fluid, conservation laws for momentum and mass are assumed to be in force. The displacement field of the body is governed by the Reynolds-averaged Navier-Stokes (RANS) equations with an algebraic mixing length turbulence. The volume of the body is assumed to be bounded.

In [14, 15] the results concerning the existence, regularity and finite-dimensional approximation of solutions to the mentioned problem are given. Shape optimization of many problems are considered, among others, in [7–9, 14, 19] where necessary optimality conditions, results concerning convergence of the finite-dimensional approximation and numerical results are provided. In [7, 8] a boundary variational approach was proposed in combination with the boundary integral representations of the shape gradient and the shape Hessian. The considered class of model problems allowed the use of boundary integral methods to compute all ingredients of the functional, the gradient, and the Hessian,which arise from the state equation. In combination with a fast wavelet Galerkin method to solve the boundary integral equations, some very efficient first and second order algorithms for shape problems in two and three space dimensions were developed. In the monograph [19], the material derivative method is employed to calculate both the sensitivity of solutions and the derivatives of domain depending functionals with respect to variations of the boundary of the domain occupied by the body. To formulate the necessary optimality condition for shape and topology optimization, the shape and topological derivatives are employed. The notion of the topological derivative and results concerning its application in optimization of elastic structures are reported among others in papers [1, 11, 13, 20–22].

The approach presented in this article is different than the one given in the papers mentioned above. It stays close to the classical optimization problems and gives sufficient optimality conditions of first order, while in [7,8,19] only second order optimality conditions are stated which require more regularity of the data and regularity of the perturbation of domains. We provide a dynamic programming approach to structural optimization problems. This approach allows us to obtain the sufficient conditions for optimality in the problems considered. We believe that the conditions for problems considered in terms of dynamic programming, that we formulate here, have not been provided earlier. There are two main difficulties that must be overcome in structural optimization problems. The first one consists in that we have to perturb the set Ω with boundary condition – compare the one-dimensional case given in e.g. [10]. The second one is that we deal with the stationary Navier-Stokes equation for the state and we have no explicit controls. To overcome these difficulties we introduce to the definition of perturbation of a given set, a control. We do that following the perturbation defined by Zolésio (see e.g. [19, p. 44]) and adding to the part of the original domain a control curve. In this way the deformation and so a corresponding functional depend on control. Thus, the situation is similar to the one in classical optimal control theory. The main difference between other approaches to shape optimization and ours is that we do not need to prove existence of the optimal value, which would necessitate considering a special family of perturbations of a given domain which must be compact in some topology and the functional considered at least lower (upper) semicontinuous with respect to this topology. In our case the constructed deformation needs not be and, in fact, is not compact. Thus we cannot prove the existence result. However, the advantage of our dynamic programming approach is that if we are able to find in any way (simply guess) a function G and the admissible pair $(x^{\varepsilon}(\cdot, w, v^{\varepsilon}), v^{\varepsilon})$ and check that they satisfy (3.23), (3.24) of the verification theorem (Proposition 3.3) then this pair approximates our optimal shape. Moreover, it allows us to describe a new numerical algorithm for that optimization problem.

The paper is organized as follows: in Section 2 a model problem of dividing tube is described. In Section 3 a general shape optimization problem is formulated and next its reduction to a classical control problem is described (Section 3.1). In Section 3.2 a dynamic programming approach is constructed to formulate and proving sufficient conditions for the approximate value function being the approximate solution for the general shape optimization problem. In Section 4 a numerical approximation of the value function is constructed. In Section 5 a numerical algorithm is constructed.

2. MODEL PROBLEM FORMULATION: A DIVIDING TUBE

As a model of the shape optimization problem we consider a problem governed by the Navier-Stokes equations for a viscous incompressible fluid, described and numerically solved in the book [14]. Following [14] we consider two-dimensional fluid flow in a header $\Omega(\alpha)$ (see Figure 1).

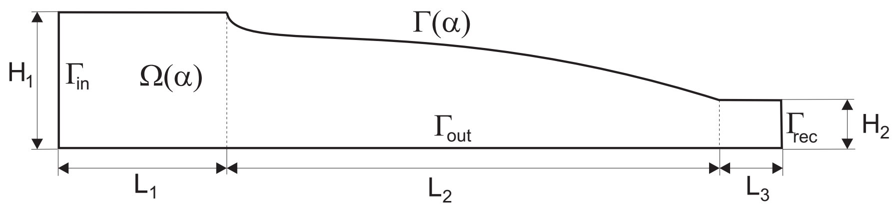

Fig. 1. Dividing tube

The fluid flows in through the part Γ_{in} of the boundary and flows out through the small tubes on the boundary Γ_{out}; there is also a small outflow on Γ_{rec}. The parameters H_1, H_2, L_1, L_2, L_3 defining the geometry are fixed. $\Gamma(\alpha)$ - the only changing part of the boundary $\partial\Omega(\alpha)$ is defined by the formula

$$\Gamma(\alpha) = \left\{x = (x_1, x_2) : L_1 < x_1 < L_1 + L_2,\ x_2 = \alpha(x_1),\ \alpha \in U^{ad}\right\},$$

where

$$U^{ad} = \Big\{\ \alpha \in C^{0,1}([L_1, L_1 + L_2]) : 0 < \alpha_{\min} \leq \alpha \leq \alpha_{\max},\ \alpha(L_1) = H_1, \\ \alpha(L_1 + L_2) = H_2,\ |\alpha'| \leq L_0, \text{ a.e. in } [L_1, L_1 + L_2]\Big\},$$

$\alpha_{\min}$, $\alpha_{\max}$ and L_0 are given parameters. We assume that $\alpha_{\min}, \alpha_{\max} \in \Gamma(\alpha)$. For a compressible fluid (two dimensions) the conservation laws are:

$$\begin{aligned} -\frac{\partial \tau_{ij}(u,p)}{\partial x_j} + \rho u_j \frac{\partial u_i}{\partial x_j} = 0 \quad \text{in} \quad \Omega(\alpha), \quad i = 1, 2, \\ \text{div } u = 0 \quad \text{in} \quad \Omega(\alpha). \end{aligned} \tag{2.1}$$

Here $u = (u_1, u_2)$ is the velocity, $\tau = (\tau_{ij})^2_{i,j=1}$ is the stress tensor, p is the static pressure, ρ is the density of the fluid. The stress tensor, the strain rate tensor $\varepsilon(u)$ and the pressure p satisfy the law

$$\tau_{ij}(u) = 2\mu(u)\varepsilon_{ij}(u) - p\delta_{ij}, \quad i, j = 1, 2,$$

where $\mu(u)$ is the viscosity and $\varepsilon(u) = (\varepsilon_{ij}(u))^2_{i,j=1}$ with $\varepsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. The following boundary conditions are imposed:

$$\begin{aligned} u(x) &= 0 && \text{on} \quad \partial\Omega(\alpha) \backslash (\Gamma_{\text{in}} \cup \Gamma_{\text{out}} \cup \Gamma_{\text{rec}}), \\ u &= u_{\text{in}} && \text{on} \quad \Gamma_{\text{in}}, \\ u &= u_{\text{rec}} && \text{on} \quad \Gamma_{\text{rec}}, \\ u_1 &= 0 && \text{on} \quad \Gamma_{\text{out}}, \\ 2\mu(u)\varepsilon_{2j}(u)\nu_j - p\nu_2 - cu_2^2 &= 0 && \text{on} \quad \Gamma_{\text{out}}. \end{aligned} \tag{2.2}$$

The viscosity $\mu(u) = \mu_0 + \rho l_m^2(\frac{1}{2}\varepsilon_{ij}(u)\varepsilon_{ij}(u))^{1/2}$ with $(\frac{1}{2}\varepsilon_{ij}\varepsilon_{ij})^{1/2}$ being the second invariant of the strain rate tensor and

$$l_m = \frac{1}{2}H(x)\left[0.14 = 0.08\left(1 - \frac{2d(x)}{H(x)}\right)^2 - 0.06\left(1 - \frac{2d(x)}{H(x)}\right)^4\right],$$

where

$$H(x) = \begin{cases} H_1, & 0 \leq x_1 \leq L_1, \\ \alpha(x_1), & L_1 \leq x_1 \leq L_1 + L_2, \\ H_2, & x_1 > L_1 + L_2, \end{cases}$$

and $d(x) = dist(x, \partial\Omega(\alpha) \backslash (\Gamma_{\text{in}} \cup \Gamma_{\text{out}} \cup \Gamma_{\text{rec}}))$.

The shape optimization problem P_m for the dividing tube is as follows (see [14]):

$$\text{find } \alpha^* \in U^{ad} \text{ such that } J(u(\alpha^*)) \leq J(u(\alpha)),\ \alpha \in U^{ad}, \tag{2.3}$$

where

$$J(u(\alpha)) = \int_{\tilde{\Gamma}} (u_2(\alpha) - u_{ad})^2 dx_1, \tag{2.4}$$

$u(\alpha)$ is the solution to (2.1)–(2.2) and $\tilde{\Gamma} \subset \Gamma_{\text{out}}$ given and $u_{ad} = \mathit{constant}$ is a given target profile. $\tilde{\Gamma}$ is used instead of Γ_{out} because in practice the velocity profile must be close to the target profile only.

3. GENERAL PROBLEM AND APPROACH TO IT

In this section the general problem, not limited only to the problem presented in previous section, is described. We consider the following shape optimization problem:

$$\text{minimize} \quad J(\Omega) = \int_{\Omega} L(x, u(x), \nabla u(x)) dx \tag{3.1}$$

subject to

$$\Omega \in \Theta,$$

$$Au(x) = f(x, u(x)), \tag{3.2}$$

$$Bu(x) = \phi(x) \quad \text{on} \quad \partial\Omega, \tag{3.3}$$

where Θ is a certain family of bounded with C^3 boundary domains of $D \subset \mathbb{R}^2$ which will be defined precisely in subsection 3.1 and A is a differential operator e.g. defining the Navier-Stokes equations and B an operator acting on the boundary. We assume that $L : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}$ is Lipschitz continuous with respect to all variables, $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and Lipschitz continuous with respect to the last variable, $\phi(\cdot)$ is continuous on $\partial\Omega$. The assumption that Ω is a domain with C^3 boundary is not essential, it is enough to assume that the set Ω has Lipschitz boundary, however this stronger assumption simplifies some considerations below – we need not to go in to some technical details, which are not essential here. Then, in dividing the tube problem (from the former section) we can simply round the corners.

3.1. REDUCTION OF SHAPE OPTIMIZATION PROBLEM TO A CLASSICAL CONTROL PROBLEM

Let $v_{\min}$ be a C^3 curve lying in Ω with ends of it on the boundary $\partial\Omega$ and entering the boundary in a smooth C^3 way. Next let $v_{\max}$ be a C^3 curve lying out of $\bar{\Omega}$ with ends of it lying on the boundary $\partial\Omega$ between the ends of $v_{\min}$ and entering the boundary in a smooth C^3 way. Denote by Γ_l the portion of $\partial\Omega$ between the end of $v_{\min}$ and the end of $v_{\max}$ and by Γ_r the portion of $\partial\Omega$ between the end of $v_{\max}$ and the end of $v_{\min}$. Moreover, we assume that the domain Φ bounded by: curve $v_{\min}$, Γ_l, curve $v_{\max}$, Γ_r is simply connected. Denote by $\Omega(v_{\min}) = \Omega\backslash\bar{\Phi}$ and by $\Omega(v_{\max}) = \Omega \cup \Phi$ (see Figures 2 and 3).

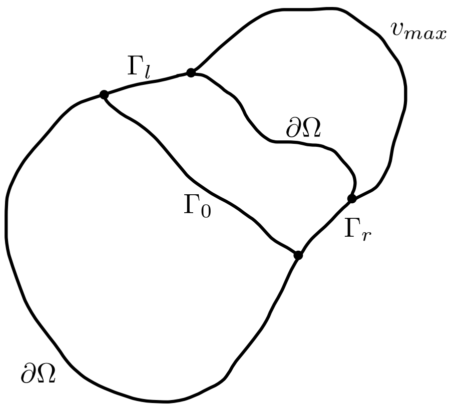

Fig. 2. Notation used in Section 3.1

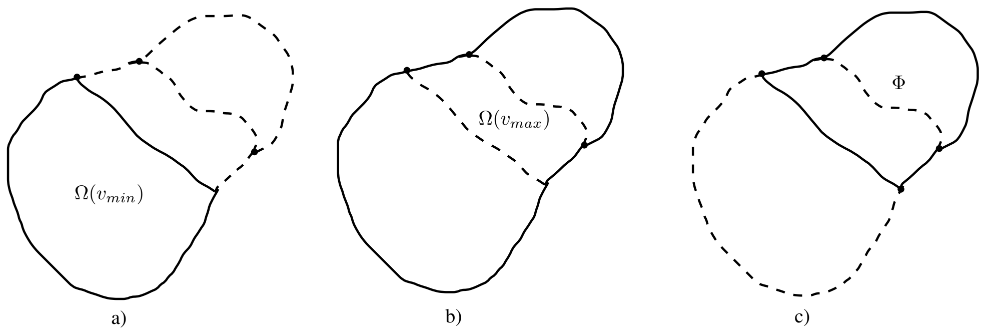

Fig. 3. Notation used in Section 3.1

Of course $\Omega(v_{\min}) \subset \Omega$ and $\Omega \subset \Omega(v_{\max})$. Denote by U the family of all nonintersecting C^3 curves whose graphs are in Φ and having one end on Γ_l and the second on Γ_r and entering both curves in a smooth C^3 way. Elements of U we will denote by v. For $v \in U$ let $\Omega(v) \subset \Omega(v_{\max})$ be a domain bounded by: $\partial\Omega(v_{\min}) \cap \partial\Omega$, portion of Γ_l between end of $v_{\min}$ and end of v, curve v, and portion of Γ_r between end of v and end of $v_{\min}$. Put $\Phi(v) = \Omega(v)\backslash\bar{\Omega}(v_{\min})$, $v \in U$. Let $v_1(s)$, $v_2(s)$, $s \in [0,1]$ be two curves from U parametrized by the same parameter s. The distance between them we define as $dist(v_1, v_2) = \sup_{s\in[0,1]} |v_1(s) - v_2(s)|$. We say that $U \ni v_n \to v \in U$ if $dist(v_n, v) \to 0$ and the same that $\Omega(v_n) \to \Omega(v)$ or $\Phi(v_n) \to \Phi(v)$.

We intend to construct the deformation of $\Omega(v)$, $v \in U$. To this effect let us denote the part of the boundary $\partial\Omega(v_{\min})$ corresponding to the curve $v_{\min}$ as Γ_0 while the part of the boundary $\partial\Omega(v)$ corresponding to the curve v as $\Gamma(v)$. Next the boundary value problem is constructed:

Find $z_{\max} \in C^2(\Omega(v_{\max}))$ such that

$$\begin{cases} \Delta z(x) = 0 & \text{in } \Omega(v_{\max})\backslash\bar{\Omega}(v_{\min}), \\ z(x) = 0 & \text{on } \Gamma_0, \\ z(x) = 1 & \text{on } \Gamma(v_{\max}). \end{cases} \tag{3.4}$$

For $0 \leq t \leq 1$ put

$$z_{\max}^{-1}(t) = \{x \in \Omega(v_{\max})\backslash\bar{\Omega}(v_{\min}) : z_{\max}(x) = t\}.$$

From the equations above, we have

$$\Gamma_0 = z_{\max}^{-1}(0) \quad \text{and} \quad \Gamma(v_{\max}) = z_{\max}^{-1}(1).$$

Next for $v \in U$, $v \neq v_{\min}$, find $z \in C^2(\Omega(v)\backslash\bar{\Omega}(v_{\min}))$ such that

$$\begin{cases} \Delta z(x) = 0 & \text{in } \Omega(v)\backslash\bar{\Omega}(v_{\min}), \\ z(x) = 0 & \text{on } \Gamma_0, \\ z(x) = z_{\max}(x) & \text{on } \Gamma(v). \end{cases} \tag{3.5}$$

Solution to (3.5) belongs to $C^2(\Phi(v))$ and in fact they are restrictions of $z_{\max}$ to $\Phi(v)$. By construction $z(x)$ depends (in a continuous way) on v, we will use the notation $z(x, v)$ for the solution of (3.5), which is obtained for a given v. In that case, v is a parameter and not a variable. The family Θ of sets over which the problem (3.1)–(3.3) is considered is defined as:

$$\Theta = \{\Omega(v) : v \in U\}. \tag{3.6}$$

The sets from Θ are called *admissible sets* for problem (3.1)–(3.3). Following Zolésio [19], for a given $\Phi(v)$, we introduce the field (again v is a parameter, not a variable)

$$V(x, v) = \|\nabla z(x, v)\|^{-2}\nabla z(x, v). \tag{3.7}$$

Then the deformation is defined as

$$T_s(w, v) = x(s, w, v), \quad s \in [0, 1],$$

where $x(\cdot, w, v)$ is a solution to

$$\frac{d}{ds}x(s, w, v) = V(x(s, w, v), v), \quad s \in [0, 1], \ w \in \Gamma_0 \tag{3.8}$$

with the initial condition

$$x(0, w, v) = w, \quad w \in \Gamma_0.$$

Notice, that for a given fixed $w \in \Gamma_0$, the point $x(1, w, v)$ belongs to $\Gamma(v)$. Defining a new functional $I(v)$ as

$$I(v) = J(\Omega(v)) = \int_{\Omega(v)} L(x, u(x), \nabla u(x)) dx \tag{3.9}$$

we can reformulate the problem (3.1)–(3.3) in terms of the family Θ:

$$\text{minimize} \quad \{I(v) : v \in U\} \tag{3.10}$$

subject to

$$\Omega(v) \in \Theta, \tag{3.11}$$

$$Au(x) = f(x, u(x)), \quad x \in \Omega(v), \tag{3.12}$$

$$Bu(x) = \phi(x) \quad \text{on} \quad \partial\Omega(v). \tag{3.13}$$

However, it is still difficult to study (3.10) with the tools of optimal control theory: variables are still domains $\Omega(v)$, in spite of that they are dependent on control v. In the classical control theory variables of functionals are functions. In the next step we will reformulate the functional (3.10) in terms of functions so that we can apply known methods of optimal control theory to our problem (3.10).

For a given control $v \in U$, define $z(\cdot)$ by (3.5). Hence we can write

$$\frac{d}{ds} x(s) = V(x(s), v), \quad s \in [0, 1],\ x(0) = w. \tag{3.14}$$

Formula (3.14) defines a family of trajectories $x(s)$, $s \in [0, 1]$, depending on control function v and initial parameter $w \in \Gamma_0$. Note that as $\partial\Omega(v)$ is of C^3 thus V is of C^1 with respect to x and continuous with respect to v. Hence, by the well known theorem on ODEs, we know that the solution x of (3.14) depends on (w, v) in a continuous way and we shall denote our functions x as $x(\cdot, w, v)$ and call them *states* of the problem (3.10) and v will be called "controls" of this problem. Let us take v_1, $v_2 \in U$ such that $\Omega(v_1) \subset \Omega(v_2)$ and let $x \in \Omega(v_2) \backslash \Omega(v_1)$, then there exists $w_1 \in \Gamma_0$ and trajectory $x(\cdot, w_1, v_1)$ in $[0, 1]$ such that it can be extended with $x(\cdot, w_2, v_2)$ in such a way that $x(1, w_1, v_1) = x(t_1, w_2, v_2)$, for some $t_1 \in (0, 1)$ and there exists $t_2 \in (0, 1)$, $t_1 < t_2$ such that $x(t_2, w_2, v_2) = x$. This new trajectory (it is still absolutely continuous) we will call extension of $x(\cdot, w_1, v_1)$ to x and will denote it by $x(\cdot, w_1, v_1, v_2)$.

The boundary $\Gamma(v)$ is the image of Γ_0 by the map $x(1, \cdot, v)$. Thus, for a given $v \neq v_{\min}$, we have an alternative definition of $\Phi(v) = \Omega(v) \backslash \bar{\Omega}(v_{\min})$:

$$\Phi(v) = \{x : x = x(s, w, v),\ 0 < s < 1,\ w \in \Gamma_0\}.$$

This means that we can construct and study some objects over the set $\Omega(v)$ with the help of the family $F(v)$:

$$F(v) = \{x(s, w, v) : 0 < s < 1,\ w \in \Gamma_0\}$$

of solutions defined by (3.8). The functional (3.9) in terms of the family $F(v)$ can be rewritten as

$$
\begin{aligned}
I(v) &= \int\limits_{\Omega(v_{\min})} L(y, u(y), \nabla u(y))dy + \int\limits_{\Omega(v)\setminus\bar{\Omega}(v_{\min})} L(x, u(x), \nabla u(x))dx = J(F(v)) \\
&= \int\limits_{\Omega(v_{\min})} L(y, u(y), \nabla u(y))dy \\
&\quad + \int\limits_0^1 \int\limits_{\Gamma_0} L(x(s,w,v), u(x(s,w,v)), \nabla u(x(s,w,v))) \left| \frac{\partial}{\partial s} x \ \frac{\partial}{\partial w} x \right| dwds,
\end{aligned}
$$

where u satisfies (3.12)–(3.13) and $\left|\frac{\partial}{\partial s}x \quad \frac{\partial}{\partial w}x\right|$ denotes the determinant of the Jacobian matrix $(\frac{\partial}{\partial s}x\frac{\partial}{\partial w}x)$. We shall denote

$$
\begin{aligned}
&\hat{L}(x(s,w,v), u(x(s,w,v)), \nabla u(x(s,w,v))) \\
&= L(x(s,w,v), u(x(s,w,v)), \nabla u(x(s,w,v_t))) \left| \frac{\partial}{\partial s} x \ \frac{\partial}{\partial w} x \right|
\end{aligned}
$$

and then

$$
\begin{aligned}
\mathbf{I}(v) &= \int\limits_{\Omega(v_{\min})} L(y, u(y), \nabla u(y))dy \\
&\quad + \int\limits_0^1 \int\limits_{\Gamma_0} \hat{L}(x(s,w,v), u(x(s,w,v)), \nabla u(x(s,w,v)))dwds.
\end{aligned}
$$

Therefore we are able to reduce the shape optimal control problem (3.10) to the classical optimal control problem (P):

$$
\text{minimize } \mathbf{I}(v) \tag{3.15}
$$

subject to

$$
\frac{d}{ds}x(s,w,v) = V(x(s,w,v), v), \quad s \in [0,1], \quad x(0,w,v) = w, \quad w \in \Gamma_0, \quad v \in U,
$$

where u satisfies (3.12)–(3.13). In order to formulate any sufficient optimality conditions for (3.15) we apply a classical dynamic programming scheme.

3.2. A DYNAMIC PROGRAMMING APPROACH AS A METHOD OF SOLUTION TO (3.15)

Let us take any $x \in \Omega(v_{\max})\setminus\bar{\Omega}(v_{\min})$ and denote by U_x a subfamily of U such that $x \in v$ for each $v \in U_x$. By (3.14) for each $v \in U_x$ there is a trajectory $x(\cdot, w, v)$ such that $x = x(1, w_v, v)$, for some $w_v \in \Gamma_0$. The problem (P) falls into the category of Lagrange control problems treated in many books (e.g. [10]). Following Chapter IV of this book we define a value function for (3.15), for $x \in \Omega(v_{\max})$:

$$
\begin{aligned}
S(x) = \inf \Bigg\{ & \int\limits_{\Omega(v_{\min})} L(y, u(y), \nabla u(y)) dy \\
& + \int\limits_0^1 \int\limits_{\Gamma_0} \hat{L}(x(s,w,v), u(x(s,w,v)), \nabla u(x(s,w,v))) dw ds \Bigg\},
\end{aligned} \tag{3.16}
$$

where the infimum in (3.16) is taken over all pairs

$$
(x(\cdot, w, v), v) \tag{3.17}
$$

satisfying

$$
\begin{aligned}
&\frac{d}{ds} x(s,w,v) = V(x(s,w,v), v), \quad s \in [0,1], \quad v \in U_x,\ w \in \Gamma_0 \\
&\qquad \text{and for } v \in U_x,\ x(1, w_v, v) = x, \text{ for some } w_v \in \Gamma_0
\end{aligned} \tag{3.18}
$$

and where u satisfies (3.12)–(3.13) in $\Omega(v)$. Each pair $(x(\cdot, w, v), v)$ satisfying (3.18) will be called admissible for the point $x \in \Omega(v_{\max}) \backslash \bar{\Omega}(v_{\min})$. If it happens that $S(\cdot)$ is of C^1 in $\Omega(v_{\max}) \backslash \bar{\Omega}(v_{\min})$ then it satisfies the Hamilton-Jacobi-Bellman equation

$$
\begin{aligned}
\max \Bigg\{ & S_x(x) V(x,v) \\
& - \int\limits_{\Gamma_0} \hat{L}(x(1,w,v), u(x(1,w,v)), \nabla u(x(1,w,v))) dw : v \in U_x \Bigg\} = 0.
\end{aligned} \tag{3.19}
$$

The terminal value for equation (3.19) we assume

$$
S(w) = \int\limits_{\Omega(v_{\min})} L(y, u_{\min}(y), \nabla u_{\min}(y)) dy, \quad w \in \Gamma_0,
$$

where $u_{\min}$ is a solution to (3.12) for $\Omega(v_{\min})$. Moreover, if there exists a pair $(\bar{x}(\cdot, w, \bar{v}), \bar{v})$ satisfying (3.18) and

$$
\begin{aligned}
& S_x(\overline{x}(\tau, \bar{w}, \bar{v})) V(\overline{x}(\tau, \bar{w}, \bar{v}), \bar{v}) \\
& - \int\limits_{\Gamma_0} \hat{L}(\overline{x}(\tau, w, \bar{v}), \bar{u}(\overline{x}(\tau, w, \bar{v})), \nabla \bar{u}(\overline{x}(\tau, w, \bar{v}))) dw = 0, \\
& \tau \in (0,1], \text{ for some } \bar{w} \in \Gamma_0,\ \overline{x}(1, w, \bar{v}) \in \Gamma(\bar{v}),\ w \in \Gamma_0,
\end{aligned}
$$

where $\bar{u}$ is a solution to (3.12) for $\Omega(\bar{v})$, then

$$S(\overline{x}(1,\bar{w},\bar{v})) = \int\limits_{\Omega(v_{\min})} L(y,\bar{u}(y),\nabla\bar{u}(y))dy + \int\limits_0^1\int\limits_{\Gamma_0} \hat{L}(\overline{x}(s,w,\bar{v}),\bar{u}(\overline{x}(s,w,\bar{v})),\nabla\bar{u}(\overline{x}(s,w,\bar{v})))dwds$$

is the optimal value for problem (3.15), and so

$$\mathbf{I}(\bar{v}) = \int\limits_{\Omega(v_{\min})} L(y,\bar{u}(y),\nabla\bar{u}(y))dy + \int\limits_0^1\int\limits_{\Gamma_0} \hat{L}(\overline{x}(s,w,\bar{v}),\bar{u}(\overline{x}(s,w,\bar{v})),\nabla\bar{u}(\overline{x}(s,w,\bar{v})))dwds$$

is an optimal value for problem (3.15) and thus for (3.10). However, in practice, we cannot expect that $S(\cdot)$ is of C^1 in $\Omega(v_{\max})\backslash\bar{\Omega}(v_{\min})$, this is why we are interested in the numerical approximation of $S(\cdot)$. Therefore, we shall look for ε-value function $S_\varepsilon(\cdot)$. For a given $\varepsilon > 0$ we call any $S_\varepsilon : \Omega(v_{\max})\backslash\bar{\Omega}(v_{\min}) \to \mathbb{R}$, ε-value function if

$$S(x) \le S_\varepsilon(x) \le S(x) + \varepsilon, \quad x \in \Omega(v_{\max})\backslash\bar{\Omega}(v_{\min}). \tag{3.20}$$

It is clear that there exists infinitely many ε-value functions $S_\varepsilon(\cdot)$.

We tell that for a given $x \in \Omega(v_{\max})\backslash\bar{\Omega}(v_{\min})$ the pair $(x^\varepsilon(\cdot,w,v^\varepsilon),v^\varepsilon)$, such that $x^\varepsilon(1,w_\varepsilon,v^\varepsilon) = x$, is ε-optimal if

$$S(x^\varepsilon(1,w,v^\varepsilon)) + \varepsilon \ge \int\limits_{\Omega(v_{\min})} L(y,\overset{\varepsilon}{u}(y),\nabla\overset{\varepsilon}{u}(y))dy + \int\limits_0^1\int\limits_{\Gamma_0} \hat{L}(x^\varepsilon(l,w,v^\varepsilon),\overset{\varepsilon}{u}(x^\varepsilon(l,w,v^\varepsilon)),\nabla\overset{\varepsilon}{u}(x^\varepsilon(l,w,v^\varepsilon)))dwdl$$

and $v^\varepsilon \in U_x$, where $\overset{\varepsilon}{u}$ is a solution to (3.12) for $\Omega(v^\varepsilon)$. In the next proposition we show that our ε-value function $S_\varepsilon(\cdot)$ has analogous properties to the classical value function.

Proposition 3.1. *Let $(x(\cdot,w,v),v)$ be any admissible pair for x defined in $[0,1]$ i.e. $v \in U_x$ and $w_1 \in \Gamma_0$ such that $x(1,w_1,v) = x$. Then along $x(s,w_1,v)$, $s \in [0,1]$, for any $s_1 \le s_2$ $s_1, s_2 \in [0,1]$ we have*

$$S_\varepsilon(x(s_2,w_1,v)) - \int\limits_0^{s_2}\int\limits_{\Gamma_0} \hat{L}(x(l,w,v),u(x(l,w,v)),\nabla u(x(l,w,v)))dwdl \le S_\varepsilon(x(s_1,w_1,v)) - \int\limits_0^{s_1}\int\limits_{\Gamma_0} \hat{L}(x(l,w,v),u(x(l,w,v)),\nabla u(x(l,w,v)))dwdl + \varepsilon,$$

where u is a solution to (3.12) *for* $\Omega(v)$. *Moreover, along an* ε*-optimal pair* $(x^\varepsilon(\cdot,w,v^\varepsilon),v^\varepsilon)$ *such that* $v^\varepsilon \in U_x$ *we have the inequality:*

$$S_\varepsilon(x^\varepsilon(s,w_\varepsilon,v^\varepsilon)) \le \int\limits_{\Omega(v_{\min})} L(y,u^\varepsilon(y),\nabla u^\varepsilon(y))dy$$

$$+\int\limits_0^s\int\limits_{\Gamma_0} \hat{L}(x^\varepsilon(l,w,v^\varepsilon),u^\varepsilon(x^\varepsilon(l,w,v^\varepsilon)),\nabla u^\varepsilon(x^\varepsilon(l,w,v^\varepsilon)))dwdl+\varepsilon, \quad s\in[0,1],$$

where $w_\varepsilon \in \Gamma_0$ *is such that* $x^\varepsilon(1,w_\varepsilon,v^\varepsilon) = x$ *and* u^ε *is a solution to* (3.12)–(3.13) *in* $\Omega(v^\varepsilon)$.

Proof. We prove only the first part of the proposition as the second one is simply a consequence of the first and the definition of an ε-optimal pair. Let $(x(\cdot,w,v),v)$ be any admissible pair for x, i.e. defined in $[0,1]$ such that $x(1,w_1,v)=x$, for some $w_1\in\Gamma_0$. According to the definition of an admissible pair, the pair $(x(s,w,v),v)$, $s\in[0,s_1]$ is also admissible for the point $x(s_2,w_1,v)$. Next, take any admissible pair $(x^2(s,w,v^2),v^2)$, $s\in[0,s_1]$ for the point $(s_1,x(s_1,w_1,v))$ such that $x^2(s_1,w_2,v^2) = x(s_1,w_1,v)$, for some $w_2\in\Gamma_0$. Hence

$$S_\varepsilon(x(s_2,w_1,v)) \le \int\limits_{s_1}^{s_2}\int\limits_{\Gamma_0} \hat{L}(x(l,w,v),u(x(l,w,v)),\nabla u(x(l,w,v)))dwdl$$

$$+\int\limits_{\Omega(v_{\min})} L(y,u^2(y),\nabla u^2(y))dy$$

$$+\int\limits_0^{s_1}\int\limits_{\Gamma_0} \hat{L}(x^2(l,w,v^2),u^2(x^2(l,w,v^2)),\nabla u^2(x^2(l,w,v^2)))dwds+\varepsilon,$$

where u^2 is a solution to (3.12) for $\Omega(v^2)$. As the pair $(x^2(s,w,v^2),v^2)$, $s\in[0,s_1]$, was chosen in an arbitrary way and $x^2(s_1,w_2,v^2)=x(s_1,w_1,v)$, we have

$$S_\varepsilon(x(s_2,w_1,v)) - \int\limits_0^{s_2}\int\limits_{\Gamma_0} \hat{L}(x(l,w,v),u(x(l,w,v)),\nabla u(x(l,w,v)))dwdl$$

$$\le \inf\left\{ \int\limits_{\Omega(v_{\min})} L(y,\hat{u}(y),\nabla\hat{u}(y))dy \right.$$

$$\left. +\int\limits_0^{s_1}\int\limits_{\Gamma_0} \hat{L}(\widehat{x}(l,w,\hat{v}),\hat{u}(\widehat{x}(l,w,\hat{v})),\nabla\hat{u}(\widehat{x}(l,w,\hat{v})))dwds \right\}$$

$$-\int\limits_0^{s_1}\int\limits_{\Gamma_0} \hat{L}(x(l,w,v),u(x(l,w,v)),\nabla u(x(l,w,v)))dwdl+\varepsilon,$$

where the infimum is taken over all admissible pairs $(\widehat{x}(s,w,\hat{v}),\hat{v})$, $s \in [0,s_1]$ for the point $x(s_1,w_1,v)$ and $\hat{u}$ is a solution to (3.12) for $\Omega(\hat{v})$. The last inequality implies the first assertion of the proposition. $\square$

It can be noticed easily that the conditions stated in the above proposition are in fact necessary conditions of ε-optimality. It turns out that they are also sufficient conditions of ε-optimality.

Proposition 3.2. *Let $G(\cdot)$ be any function defined in $\Omega(v_{\max})\backslash\bar{\Omega}(v_{\min})$. Assume that for any $s_1 \le s_2$, $s_1, s_2 \in [0,1]$ and any admissible pair $(x(\cdot,w,v),v)$ for x, i.e. defined in $[0,1]$ and $x(1,w_1,v) = x$, for some $w_1 \in \Gamma_0$, we have*

$$\begin{aligned} &G(x(s_2,w_1,v)) - \int_0^{s_2}\int_{\Gamma_0} \hat{L}(x(l,w,v),u(x(l,w,v)),\nabla u(x(l,w,v)))dwdl \\ &\le G(x(s_1,w_1,v)) - \int_0^{s_1}\int_{\Gamma_0} \hat{L}(x(l,w,v),u(x(l,w,v)),\nabla u(x(l,w,v)))dwdl + \varepsilon, \end{aligned} \tag{3.21}$$

where u is a solution to (3.12) for $\Omega(v)$ and $G(w) = \int_{\Omega(v_{\min})} L(y,u_{\min}(y),\nabla u_{\min}(y))dy$, $w \in \Gamma_0$. If there exists an admissible pair $(x^\varepsilon(\cdot,w,v^\varepsilon),v^\varepsilon)$ for x, i.e. defined in $[0,1]$ and $x^\varepsilon(1,w_1,v^\varepsilon) = x$ such that

$$\begin{aligned} G(x^\varepsilon(1,w_1,v^\varepsilon)) \ge &\int_{\Omega(v_{\min})} L(y,u^\varepsilon(y),\nabla u^\varepsilon(y))dy \\ &+ \int_0^1\int_{\Gamma_0} \hat{L}(x^\varepsilon(l,w,v^\varepsilon),u^\varepsilon(x^\varepsilon(l,w,v^\varepsilon)),\nabla u^\varepsilon(x^\varepsilon(l,w,v^\varepsilon)))dwdl, \end{aligned} \tag{3.22}$$

where u^ε is a solution to (3.12) for $\Omega(v^\varepsilon)$, then $(x^\varepsilon(\cdot,w,v^\varepsilon),v^\varepsilon)$ is an ε-optimal pair for $S_\varepsilon(x) = G(x)$.

Proof. Let $(x(\cdot,w,v),v)$ be any admissible pair for x, i.e. defined in $[0,1]$ and $x(1,w_1,v,v_2) = x$ $(w_1 \in \Gamma_0)$. Then by (3.21)

$$\begin{aligned} &G(x(1,w_1,v)) - \int_{\Omega(v_{\min})} L(y,u(y),\nabla u(y))dy \\ &- \int_0^1\int_{\Gamma_0} \hat{L}(x(l,w,v),u(x(l,w,v)),\nabla u(x(l,w,v)))dwdl \le \varepsilon, \end{aligned}$$

where u is a solution to (3.12) for $\Omega(v)$. Thus

$$G(x(1,w_1,v)) \leq \inf \left\{ \int\limits_{\Omega(v_{\min})} L(y,\hat{u}(y),\nabla\hat{u}(y))dy \right.$$
$$\left. + \int\limits_0^1 \int\limits_{\Gamma_0} \hat{L}(\widehat{x}(l,w,\hat{v}),\hat{u}(\widehat{x}(l,w,\hat{v})),\nabla\hat{u}(\widehat{x}(l,w,\hat{v})))dwdl \right\} + \varepsilon,$$

where the infimum is taken over all admissible pairs $(\widehat{x}(s,w,\hat{v}),\hat{v})$, $s\in[0,1]$ for the point x and where $\hat{u}$ is a solution to (3.12) for $\Omega(\hat{v})$. For the pair $(x^{\varepsilon}(\cdot,w,v^{\varepsilon}),v^{\varepsilon})$ we have (3.22), therefore this pair is an ε-optimal pair for $S_{\varepsilon}(x)=G(x)$. □

Now, we formulate and prove the so-called *verification theorem* for problem (P).

Proposition 3.3. *Let $G(\cdot)$, defined in $\Omega(v_{\max})\backslash\bar{\Omega}(v_{\min})$, be a C^1 solution of the inequality*

$$0 \leq \max \left\{ G_x(x)V(x,v) \right.$$
$$\left. - \int\limits_{\Gamma_0} \hat{L}(x(1,w,v),u(x(1,w,v)),\nabla u(x(1,w,v)))dw : v\in U_x \right\} \leq \varepsilon, \tag{3.23}$$

where u is a solution to (3.12) *for $\Omega(v)$ and boundary condition*

$$G(w) = \int\limits_{\Omega(v_{\min})} L(y,u_{\min}(y),\nabla u_{\min}(y))dy,\ \ w\in\Gamma_0,$$

where $u_{\min}$ is a solution to (3.12) *for $\Omega(v_{\min})$. If there exists a pair $(x^{\varepsilon}(s,w,v^{\varepsilon}),v^{\varepsilon})$, $s\in[0,1]$, satisfying* (3.18) *and for some $w_{\varepsilon}\in\Gamma_0$,*

$$\begin{aligned} 0 \leq\; & G_x(x^{\varepsilon}(s,w_{\varepsilon},v^{\varepsilon}))V(x^{\varepsilon}(s,w_{\varepsilon},v^{\varepsilon}),v^{\varepsilon}) \\ & - \int\limits_{\Gamma_0} \hat{L}(x^{\varepsilon}(s,w,v^{\varepsilon}),u^{\varepsilon}(x^{\varepsilon}(s,w,v^{\varepsilon})),\nabla u^{\varepsilon}(x^{\varepsilon}(s,w,v^{\varepsilon})))dw \leq \varepsilon, \end{aligned} \tag{3.24}$$

where u^{ε} is a solution to (3.12) *for $\Omega(v^{\varepsilon})$, then*

$$\int\limits_{\Omega(v_{\min})} L(y,u^{\varepsilon}(y),\nabla u^{\varepsilon}(y))dy$$
$$+ \int\limits_0^1 \int\limits_{\Gamma_0} \hat{L}(x^{\varepsilon}(s,w,v^{\varepsilon}),u^{\varepsilon}(x^{\varepsilon}(s,w,v^{\varepsilon})),\nabla u^{\varepsilon}(x^{\varepsilon}(s,w,v^{\varepsilon})))dwds$$

is the ε-optimal value for problem (3.15) *and thus for* (3.10). *Moreover, the pair $(x^{\varepsilon}(s,w,v^{\varepsilon}),v^{\varepsilon})$ is ε-optimal for the ε-value function $S_{\varepsilon}(x^{\varepsilon}(1,w_{\varepsilon},v^{\varepsilon})) = G(x^{\varepsilon}(1,w_{\varepsilon},v^{\varepsilon}))$.*

Proof. Take any admissible pair $(x(\cdot,w,v),v)$ and w_ε as in (3.24). Then by (3.23), for $\tau \in [0,1]$,

$$\begin{aligned}\frac{d}{d\tau}G(x(\tau,w_\varepsilon,v)) &= G_x(x(\tau,w_\varepsilon,v))V(x(\tau,w_\varepsilon,v),v)\\ &\le \int\limits_{\Gamma_0} \hat{L}(x(\tau,w,v),u(x(\tau,w,v)),\nabla u(x(\tau,w,v)))dw+\varepsilon.\end{aligned}$$

Hence we infer that $G(\cdot)$ satisfies (3.21). Similarly by (3.24) we get (3.22). Thus, by Proposition 3.2, we get the assertion of the proposition. ☐

Remark 3.4. It is clear that, in practice, finding any solution to (3.21), directly, is almost impossible, similarly to find an admissible pair $(x^\varepsilon(\cdot,w,v^\varepsilon),v^\varepsilon)$ satisfying (3.24). This is why in the next section we develop a numerical approximation of the value function. However the above verification theorem can also be of use if we are able to find in any way (simply guess) a function G and the admissible pair $(x^\varepsilon(\cdot,w,v^\varepsilon),v^\varepsilon)$ and check that they satisfy (3.24), (3.23). This theorem is also the essential part of the numerical approximation.

4. NUMERICAL APPROXIMATION OF THE VALUE FUNCTION

This section is an adaptation of the method developed by Pustelnik in his Ph.D. thesis [18] for the numerical approximation of the value function for the Bolza problem from optimal control theory.

Let us define the following set

$$T = \left\{x\colon x \in \Omega(v_{\max})\backslash\bar{\Omega}(v_{\min})\right\}.$$

Since $\Omega(v_{\max})\backslash\bar{\Omega}(v_{\min})$ is bounded, the set $\bar{T}$ is compact. Let $T \ni x \to g(x)$ be an arbitrary function of class C^1 in $\bar{T}$ such that $g(x)=c$, $x\in\Gamma_0$, where c is some constant which we will determine later. For a given function g, we define $(x,v)\to G_g(x,v)$ as

$$\begin{aligned}G_g(x,v) &= g_x(x)V(x,v)\\ &\quad-\int\limits_{\Gamma_0}\hat{L}(x(1,w,v),u(x(1,w,v)),\nabla u(x(1,w,v)))dw,\end{aligned}\tag{4.1}$$

$v \in U_x$, where $x(\cdot,w,v)$, u are defined as in (3.18). Next, we define the function $x \to F_g(x)$ as

$$F_g(x) = \max\left\{G_g(x,v)\colon v\in U_x\right\}.\tag{4.2}$$

Note that by the assumptions on L and V, the function F_g is continuous in T. By continuity of F_g and compactness of $\bar{T}$, there exist k_d and k_g such that

$$k_d \le F_g(x) \le k_g \text{ for } x\in\Omega(v_{\max})\backslash\bar{\Omega}(v_{\min}).$$

4.1. DEFINITION OF COVERING OF T

Let $\eta > 0$ be fixed and $\{q_j^\eta\}_{j\in\mathbb{Z}}$ be a sequence of real numbers such that $q_j^\eta = j\eta$, $j \in \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers). Denote

$$J = \{j \in \mathbb{Z} : \text{there is } x \in T \text{ such that } j\eta < F_g(x) \le (j+1)\eta\},$$

i.e.

$$J = \{j \in \mathbb{Z} : \text{there is } x \in T \text{ such that } q_j^\eta < F_g(x) \le q_{j+1}^\eta\}.$$

Next, let us divide the set T into the sets $P_j^{\eta,g}$, $j \in J$, as follows

$$P_j^{\eta,g} := \left\{x \in T : q_j^\eta < F_g(x) \le q_{j+1}^\eta\right\}, \quad j \in J.$$

As a consequence, we have for all $i,j \in J$, $i \ne j$, $P_i^{\eta,g} \cap P_j^{\eta,g} = \varnothing$, $\bigcup_{j\in J} P_j^{\eta,g} = T$ an obvious proposition:

Proposition 4.1. *For each $x \in T$ there exists an $\varepsilon > 0$, such that a ball with center x and radius ε is either contained in one set $P_j^{\eta,g}$, $j \in J$, only or contained in a union of two sets $P_{j_1}^{\eta,g}$, $P_{j_2}^{\eta,g}$, $j_1, j_2 \in J$. In the latter case $|j_1 - j_2| = 1$.*

4.2. DISCRETIZATION OF F_g

Define in T a function

$$h^{\eta,g}(x) = -q_{j+1}^\eta,\ x \in P_j^{\eta,g}, \quad j \in J. \tag{4.3}$$

Then, by the construction of the covering of T, we have

$$0 \le F_g(x) + h^{\eta,g}(x) \le \eta, \quad x \in T. \tag{4.4}$$

Let $(x(\cdot,w,v),v)$ be any admissible pair with the trajectory defined in $[0,1]$, starting at the point $x(0,w,v))$, $w \in \Gamma_0$ fixed. We show that there exists an increasing sequence of m points $\{\tau_i\}_{i=1,\dots,m}$, $\tau_1 = 0$, $\tau_m = 1$, such that for $\tau \in [\tau_i, \tau_{i+1}]$

$$\begin{aligned} |F_g(x(\tau_i,w,v)) - F_g(x(\tau,w,v))| &\le \frac{\eta}{2}, \quad i = 2,\dots,m-2,\\ |F_g(x(\tau_2,w,v)) - F_g(x(\tau,w,v))| &\le \frac{\eta}{2}, \quad \tau \in (\tau_1,\tau_2],\\ |F_g(x(\tau_{m-1},w,v)) - F_g(x(\tau,w,v))| &\le \frac{\eta}{2}, \quad \tau \in [\tau_{m-1},\tau_m). \end{aligned} \tag{4.5}$$

Indeed, it is a direct consequence of two facts: Lipschitz continuity of $x(\cdot,w,v)$ with a common Lipschitz constant and continuity of F_g. From (4.5) we infer that for each $i \in \{1,\dots,m-1\}$ if $x(\tau_i,w,v) \in P_j^{\eta,g}$ for a certain $j \in J$, then we have for $\tau \in [\tau_i, \tau_{i+1})$

$$x(\tau,w,v) \in P_{j-1}^{\eta,g} \cup P_j^{\eta,g} \cup P_{j+1}^{\eta,g}.$$

Define

$$h^{\eta,g}(x(\tau_1,w,v)) = h^{\eta,g}(x(\tau,w,v)) \text{ for some } \tau \text{ near } \tau_1$$

and

$$h^{\eta,g}(x(\tau_m,w,v)) = h^{\eta,g}(x(\tau,w,v)) \text{ for some } \tau \text{ near } \tau_m.$$

Thus for $\tau \in [\tau_i, \tau_{i+1}]$

$$h^{\eta,g}(x(\tau_i,w,v)) - \eta \le h^{\eta,g}(x(\tau,w,v)) \le h^{\eta,g}(x(\tau_i,w,v)) + \eta, \tag{4.6}$$

and so, for $i \in \{2,\dots,m-1\}$

$$h^{\eta,g}(x(\tau_i,w,v)) - h^{\eta,g}(x(\tau_{i-1},w,v)) = \eta^i_{x(\cdot,w,v)}, \tag{4.7}$$

where $\eta^i_{x(\cdot,w,v)}$ is equal to $-\eta$ or 0 or η. Integrating (4.6) we get for each $i \in \{1,\dots,m-1\}$

$$\begin{aligned}[h^{\eta,g}(x(\tau_i,w,v)) - \eta](\tau_{i+1}-\tau_i) &\le \int_{\tau_i}^{\tau_{i+1}} h^{\eta,g}(x(\tau,w,v))d\tau \\ &\le [h^{\eta,g}(x(\tau_i,w,v)) + \eta](\tau_{i+1}-\tau_i)\end{aligned}$$

and, as a consequence,

$$\begin{aligned}\sum_{i\in\{1,\dots,m-1\}} [h^{\eta,g}(x(\tau_i,w,v))(\tau_{i+1}-\tau_i)] - \eta &\le \int_0^1 h^{\eta,g}(x(\tau,w,v))d\tau \\ &\le \sum_{i\in\{1,\dots,m-1\}} [h^{\eta,g}(x(\tau_i,w,v))(\tau_{i+1}-\tau_i)] + \eta.\end{aligned}$$

Now, we will present the expression $\sum_{i\in\{1,\dots,m-1\}}[h^{\eta,g}(x(\tau_i,w,v))(\tau_{i+1}-\tau_i)]$ in a different, more useful form. By performing simple calculations, we get the two following equalities:

$$\begin{aligned}&\sum_{i\in\{2,\dots,m-1\}} [h^{\eta,g}(x(\tau_i,w,v)) - h^{\eta,g}(x(\tau_{i-1},w,v))]\tau_m \\ &= -h^{\eta,g}(x(\tau_1,w,v))\tau_m + h^{\eta,g}(x(\tau_{m-1},w,v))\tau_m, \\ &\sum_{i\in\{2,\dots,m-1\}} [h^{\eta,g}(x(\tau_i,w,v)) - h^{\eta,g}(x(\tau_{i-1},w,v))](-\tau_i) \\ &= \sum_{i\in\{1,\dots,m-1\}} [h^{\eta,g}(x(\tau_i,w,v))(\tau_{i+1}-\tau_i)] \\ &\quad + h^{\eta,g}(x(\tau_1,w,v))\tau_1 - h^{\eta,g}(x(\tau_{m-1},w,v))\tau_m.\end{aligned} \tag{4.8}$$

From (4.8) we get

$$\begin{aligned}&\sum_{i\in\{2,\dots,m-1\}} [h^{\eta,g}(x(\tau_i,w,v)) - h^{\eta,g}(x(\tau_{i-1},w,v))](\tau_m-\tau_i) \\ &= \sum_{i\in\{1,\dots,m-1\}} [h^{\eta,g}(x(\tau_i,w,v))(\tau_{i+1}-\tau_i)] - h^{\eta,g}(x(\tau_1,w,v))(\tau_1-\tau_m),\end{aligned}$$

and next, we obtain

$$\begin{aligned}
&\sum_{i\in\{2,\dots,m-1\}} [h^{\eta,g}(x(\tau_i,w,v)) - h^{\eta,g}(x(\tau_{i-1},w,v))](\tau_m-\tau_i) \\
&\quad + h^{\eta,g}(x(\tau_1,w,v))(\tau_m-\tau_1) - \eta(\tau_m-\tau_1) \\
&\le \int_{\tau_1}^{\tau_m} h^{\eta,g}(x(\tau,w,v))d\tau \\
&\le \sum_{i\in\{2,\dots,m-1\}} [h^{\eta,g}(x(\tau_i,w,v)) - h^{\eta,g}(x(\tau_{i-1},w,v))](\tau_m-\tau_i) \\
&\quad + h^{\eta,g}(x(\tau_1,w,v))(\tau_m-\tau_1) + \eta(\tau_m-\tau_1)
\end{aligned}$$

and, taking into account (4.7), we infer that

$$\begin{aligned}
&\sum_{i\in\{2,\dots,m-1\}} \eta^i_{x(\cdot,w,v)}(\tau_m-\tau_i) + h^{\eta,g}(x(\tau_1,w,v))(\tau_m-\tau_1) - \eta(\tau_m-\tau_1) \\
&\le \int_{\tau_1}^{\tau_m} h^{\eta,g}(x(\tau,w,v))d\tau \\
&\le \sum_{i\in\{2,\dots,m-1\}} \eta^i_{x(\cdot,w,v)}(\tau_m-\tau_i) + h^{\eta,g}(x(\tau_1,w,v))(\tau_m-\tau_1) + \eta(\tau_m-\tau_1).
\end{aligned} \tag{4.9}$$

We would like to stress that (4.9) is very useful from a numerical point of view: we can estimate the integral $h^{\eta,g}(\cdot,\cdot)$ along any trajectory $x(\cdot,w,v)$ as a sum of a finite number of values, where each value consists of a number from the set $\{-\eta,0,\eta\}$ multiplied by $\tau_m-\tau_i$. Moreover, for two different trajectories: $x(\cdot,w^1,v^1)$, $x(\cdot,w^2,v^2)$, the expressions

$$\sum_{i\in\{2,\dots,m-1\}} \eta^i_{x(\cdot,w^1,v^1)}(\tau_m-\tau_i) + h^{\eta,g}(x(\tau_1,w^1,v^1))(\tau_m-\tau_1)$$

and

$$\sum_{i\in\{2,\dots,m-1\}} \eta^i_{x(\cdot,w^2,v^2)}(\tau_m-\tau_i) + h^{\eta,g}(x(\tau_1,w^2,v^2))(\tau_m-\tau_1)$$

are identical if

$$h^{\eta,g}(x(\tau_1,w^1,v^1)) = h^{\eta,g}(x(\tau_1,w^2,v^2)) \tag{4.10}$$

and

$$\eta^i_{x(\cdot,w^1,v^1)} = \eta^i_{x(\cdot,w^2,v^2)} \text{ for all } i\in\{2,\dots,m-1\}. \tag{4.11}$$

The last one means that in the set B of all trajectories $x(\cdot,w,v)$, $w\in\Gamma_0$, $v\in U_x$, we can introduce an equivalence relation r: we say that two trajectories $x(\cdot,w^1,v^1)$ and $x(\cdot,w^2,v^2)$, $w^1,w^2\in\Gamma_0$, $v^1,v^2\in U_x$ are equivalent if they satisfy (4.10) and (4.11). We denote the set of all disjoint equivalence classes by B_r. The cardinality of B_r, denoted by $\|B_r\|$, is finite and bounded from above by 3^{m+1}.

Define

$$X = \Big\{ x = (x_1, \ldots, x_{m-1}) : x_1 = 0,\ x_i = \eta^i_{x^j},$$
$$i = 2, \ldots, m-1, \quad x^j \in B_r,\ j = 1, \ldots, \|B_r\| \Big\}.$$

It is easy to see that the cardinality of X is finite.

The considerations above allow us to estimate the approximation of the value function.

Theorem 4.2. *We have the following estimation, for any $w_1 \in \Gamma_0$,*

$$\min_{x \in B_r, w_0 \in \Gamma_0} \left(- \int_{\tau_1}^{\tau_m} h^{\eta,g}(x(\tau, w_0, v)) d\tau - g(x(\tau_m, w_0, v)) \right)$$
$$\leq \max_{x \in B_r} \Bigg\{ \int_{\tau_1}^{\tau_m} \left(- \int_{\Gamma_0} \hat{L}(x(s, w, v), u(x(s, w, v)), \nabla u(x(s, w, v))) dw \right) ds$$
$$- g(x(\tau_1, w_1, v)) \Bigg\}$$
$$\leq \max_{x \in B_r, w_0 \in \Gamma_0} \left(- \int_{\tau_1}^{\tau_m} h^{\eta,g}(x(\tau, w_0, v)) d\tau - g(x(\tau_m, w_0, v)) \right) + \eta(\tau_m - \tau_1),$$

where u is a solution to (3.12) *for $\Omega(v)$.*

Proof. By inequality (4.4)

$$0 \leq F_g(x) + h^{\eta,g}(x) \leq \eta,$$

we have

$$-h^{\eta,g}(x) \leq F_g(x) \leq -h^{\eta,g}(x) + \eta.$$

Integrating the last inequality along any $x(\cdot, w_0, \tilde{v})$ in the interval $[\tau_1, \tau_m]$, we get

$$- \int_{\tau_1}^{\tau_m} h^{\eta,g}(x(\tau, w_0, \tilde{v})) d\tau \leq \int_{\tau_1}^{\tau_m} \Bigg(\max_{v \in U_x} \Big\{ g_x(x(\tau, w_0, \tilde{v})) V(x(\tau, w_0, \tilde{v}), v)$$
$$- \int_{\Gamma_0} \hat{L}(x(\tau, w, v), u(x(\tau, w, v)), \nabla u(x(\tau, w, v))) dw \Big\} \Bigg) d\tau$$
$$\leq - \int_{\tau_1}^{\tau_m} h^{\eta,g}(x(\tau, w_0, \tilde{v})) d\tau + \eta(\tau_m - \tau_1),$$

where u is a solution to (3.12) for $\Omega(v)$. Thus,

$$\begin{aligned}
&-\int_{\tau_1}^{\tau_m} h^{\eta,g}(x(\tau,w_0,\tilde v))d\tau\\
&\le \max_{v\in U_x}\int_{\tau_1}^{\tau_m}\Big(g_x(x(\tau,w_0,\tilde v))V(x(\tau,w_0,\tilde v),v)\\
&\qquad\qquad -\int_{\Gamma_0}\hat L(x(\tau,w,v),u(x(\tau,w,v)),\nabla u(x(\tau,w,v)))dw\Big)d\tau\\
&\le -\int_{\tau_1}^{\tau_m} h^{\eta,g}(x(\tau,w_0,\tilde v))d\tau+\eta(\tau_m-\tau_1).
\end{aligned}$$

Hence, we get two inequalities

$$\begin{aligned}
&\min_{x\in B_r,w_0\in\Gamma_0}\left(-\int_{\tau_1}^{\tau_m} h^{\eta,g}(x(\tau,w_0,\tilde v))d\tau-g(x(\tau_m,w_0,\tilde v))\right)\\
&\le \min_{x\in B_r,w_0\in\Gamma_0}\max_{v\in U_x}\int_{\tau_1}^{\tau_m}\Bigg(-g(x(\tau_m,w_0,\tilde v))\\
&\qquad +g_x(x(\tau,w_0,\tilde v))V(x(\tau,w_0,\tilde v),v)\\
&\qquad -\int_{\Gamma_0}\hat L(x(\tau,w,v),u(x(\tau,w,v)),\nabla u(x(\tau,w,v)))dw\Bigg)d\tau
\end{aligned}$$

and

$$\begin{aligned}
&\max_{x\in B_r,w_0\in\Gamma_0}\max_{v\in U_x}\int_{\tau_1}^{\tau_m}\Bigg(-g(x(\tau_m,w_0,\tilde v))\\
&\qquad +g_x(x(\tau,w_0,\tilde v))V(x(\tau,w_0,\tilde v),v)\\
&\qquad -\int_{\Gamma_0}\hat L(x(\tau,w,v),u(x(\tau,w,v)),\nabla u(x(\tau,w,v)))dw\Bigg)d\tau\\
&\le \max_{x\in B_r,w_0\in\Gamma_0}\left(-\int_{\tau_1}^{\tau_m} h^{\eta,g}(x(\tau,w_0,\tilde v))d\tau-g(x(\tau_m,w_0,\tilde v))\right)+\eta(\tau_m-\tau_1).
\end{aligned}$$

Both inequalities imply that, for any $w_1 \in \Gamma_0$,

$$\min_{x\in B_r, w_0\in\Gamma_0} \max_{v\in U_x} \int_{\tau_1}^{\tau_m} \Bigg(- g(x(\tau_m, w_0, \tilde{v})) + g_x(x(\tau, w_0, \tilde{v}))V(x(\tau, w_0, \tilde{v}), v)$$
$$- \int_{\Gamma_0} \hat{L}(x(\tau, w, v), u(x(\tau, w, v)), \nabla u(x(\tau, w, v)))dw\Bigg) d\tau$$
$$\leq \max_{v\in U_x} \Bigg(- g(x(\tau_m, w_1, v)) + \int_{\tau_1}^{\tau_m} \Bigg(g_x(x(\tau, w_1, v))V(x(\tau, w_1, v), v)$$
$$- \int_{\Gamma_0} \hat{L}(x(\tau, w, v), u(x(\tau, w, v)), \nabla u(x(\tau, w, v)))dw\Bigg) d\tau\Bigg)$$
$$\leq \max_{x\in B_r} \max_{, w_0\in\Gamma_0 v\in U_x} \int_{\tau_1}^{\tau_m} \Bigg(- g(x(\tau_m, w_0, \tilde{v})) + g_x(x(\tau, w_0, \tilde{v}))V(x(\tau, w_0, \tilde{v}), v)$$
$$- \int_{\Gamma_0} \hat{L}(x(\tau, w, v), u(x(\tau, w, v)), \nabla u(x(\tau, w, v)))dw\Bigg) d\tau.$$

As a consequence of the above we get, for any $w_1 \in \Gamma_0$,

$$\min_{x\in B_r, w_0\in\Gamma_0} \Bigg(- \int_{\tau_1}^{\tau_m} h^{\eta,g}(x(\tau, w_0, \tilde{v}))d\tau - g(x(\tau_m, w_0, \tilde{v}))\Bigg)$$
$$\leq \max_{x\in B_r} \Bigg\{ \int_{\tau_1}^{\tau_m} \Bigg(- \int_{\Gamma_0} \hat{L}(x(\tau, w, v), u(x(\tau, w, v)), \nabla u(x(\tau, w, v)))dw\Bigg) d\tau$$
$$- g(x(\tau_1, w_1, v))\Bigg\}$$
$$\leq \max_{x\in B_r, w_0\in\Gamma_0} \Bigg(- \int_{\tau_1}^{\tau_m} h^{\eta,g}(x(\tau, w_0, v))d\tau - g(x(\tau_m, w_0, \tilde{v}))\Bigg) + \eta(\tau_m - \tau_1)$$

and thus the assertion of the theorem follows. □

Now, we use the definition of an equivalence class to reformulate the theorem above in a way that is more useful in practice. To this effect let us note that, by the definition of an equivalence relation r, we have

$$\min_{x\in B_r} \Big\{ - \sum_{i=2,\dots,m-1} \eta_x^i(\tau_m - \tau_1)\Big\} = \min_{x\in X} \Big\{ - \sum_{i\in\{1,\dots,m-1\}} x^i(\tau_m - \tau_1)\Big\}$$

and

$$\max_{x\in B_r}\Big\{-\sum_{i=2,\dots,m-1}\eta_x^i(\tau_m-\tau_1)\Big\}=\max_{x\in X}\Big\{-\sum_{i\in\{1,\dots,m-1\}}x^i(\tau_m-\tau_1)\Big\}.$$

Taking into account (4.9) we get

$$\begin{aligned}
&\min_{x\in X}\Big\{-\sum_{i\in\{1,\dots,m-1\}}x^i(\tau_m-\tau_i)\Big\}+\min_{x\in B_r,w_0\in\Gamma_0}\{-h^{\eta,g}(x(\tau_1,w_0,v))(\tau_m-\tau_1)\\
&\quad -g(x(\tau_m,w_0,v))\}-\eta(\tau_m-\tau_1)\\
&\le\min_{x\in B_r}\Big\{-\int_{\tau_1}^{\tau_m}h^{\eta,g}(x(\tau,w_0,v))d\tau-g(x(\tau_m,w_0,v))\Big\}\\
&\le\min_{x\in X}\Big\{-\sum_{i\in\{1,\dots,m-1\}}x^i(\tau_m-\tau_i)\Big\}+\max_{x\in B_r,w_0\in\Gamma_0}\{-h^{\eta,g}(x(\tau_1,w_0,v))(\tau_m-\tau_1)\\
&\quad -g(x(\tau_m,w_0,v))\}+\eta(\tau_m-\tau_1)
\end{aligned}$$

and a similar formula for supremum. Applying that to the result of the theorem above, we obtain the following estimation, for any $w_1\in\Gamma_0$,

$$\begin{aligned}
&\min_{x\in X}\Big\{-\sum_{i\in\{1,\dots,m-1\}}x^i(\tau_m-\tau_i)\Big\}+\min_{x\in B_r,w_0\in\Gamma_0}\{-h^{\eta,g}(x(\tau_1,w_0,v))(\tau_m-\tau_1)\\
&\quad -g(x(\tau_m,w_0,v))\}-2\eta(\tau_m-\tau_1)\\
&\le\max_{x\in B_r}\Big\{\int_{\tau_1}^{\tau_m}\Big(-\int_{\Gamma_0}\hat{L}(x(\tau,w,v),u(x(\tau,w,v)),\nabla u(x(\tau,w,v)))dw\Big)d\tau\\
&\quad -g(x(\tau_1,w_1,v))\Big\}\\
&\le\max_{x\in X}\Big\{-\sum_{i\in\{1,\dots,m-1\}}x^i(\tau_m-\tau_i)\Big\}+\max_{x\in B_r,w_0\in\Gamma_0}\{-h^{\eta,g}(x(\tau_1,w_0,v))(\tau_m-\tau_1)\\
&\quad -g(x(\tau_m,w_0,v_{x_{n+1}}))\}+\eta(\tau_m-\tau_1).
\end{aligned}\tag{4.12}$$

Thus, we come to the main theorem of this section, which allows us to reduce an infinite dimensional problem to a finite dimensional one.

Theorem 4.3. *Let $\eta>0$. Assume that there are $\theta>0$, $\bar{v}$ and $\bar{w}\in\Gamma_0$ such that*

$$\begin{aligned}
&\min_{x\in X}\Big\{\sum_{i\in\{1,\dots,m-1\}}x^i(\tau_m-\tau_i)\Big\}+\min_{x\in B_r,w_0\in\Gamma_0}\{h^{\eta,g}(x(\tau_1,w_0,v))+g(x(\tau_m,w_0,v))\}\\
&\ge\max_{x\in X}\Big\{\sum_{i\in\{1,\dots,m-1\}}x^i(\tau_m-\tau_i)\Big\}\\
&+\max_{x\in B_r,w_0\in\Gamma_0}\{h^{\eta,g}(x(\tau_1,w_0,v))\ +g(x(\tau_m,w_0,v))\}+\theta,
\end{aligned}\tag{4.13}$$

$$\max_{x\in B_r, w_0\in\Gamma_0}\{h^{\eta,g}(x(\tau_1,w_0,v))+g(x(\tau_m,w_0,v))\} = \{h^{\eta,g}(x(\tau_1,\bar{w},\bar{v}))+g(x(\tau_m,\bar{w},\bar{v}))\}.$$

Then

$$-(\eta+\theta)+h^{\eta,g}(x(\tau_1,\bar{w},\bar{v}))+g(x(\tau_m,\bar{w},\bar{v}))+\max_{x\in X}\Big\{\sum_{i\in\{1,\ldots,m-1\}} x^i(\tau_m-\tau_i)\Big\} \quad (4.14)$$

is the ε*-optimal value at* $x(\tau_m,\bar{w},\bar{v})$ *for* $\varepsilon = 3\eta+\theta$ *with*

$$g(w) = \int\limits_{\Omega(v_{\min})} L(y,\bar{u}(y),\nabla\bar{u}(y))dy, \quad w\in\Gamma_0,$$

where $\bar{u}$ *is a solution to* (3.12) *for* $\Omega(\bar{v})$.

Proof. From the formulae (4.12), (4.13) and equality $\tau_m-\tau_1=1$ we infer

$$\begin{aligned}
&\max_{x\in X}\left\{\sum_{i\in\{1,\ldots,m-1\}} x^i(\tau_m-\tau_i)\right\}\\
&+\max_{x\in B_r, w_0\in\Gamma_0}\left\{h^{\eta,g}(x(\tau_1,w_0,v))+g(x(\tau_m,w_0,v))\right\}+2\eta\\
&\geq \min_{x\in B_r}\left\{\int\limits_{\tau_1}^{\tau_m}\left(\int\limits_{\Gamma_0}\hat{L}(x(\tau,w,v),u(x(\tau,w,v)),\nabla u(x(\tau,w,v)))dw\right)d\tau\right.\\
&\qquad\left.+\int\limits_{\Omega(v_{\min})} L(y,\bar{u}(y),\nabla\bar{u}(y))dy\right\}\\
&\geq \max_{x\in X}\left\{\sum_{i\in\{1,\ldots,m-1\}} x^i(\tau_m-\tau_i)\right\}+\max_{x\in B_r, w_0\in\Gamma_0}\{h^{\eta,g}(x(\tau_1,w_0,v))\\
&\quad +g(x(\tau_m,w_0,v))\}-\eta-\theta.
\end{aligned}$$

Next, using the definition of the value function (3.16), we get (4.14). □

5. THE SHAPE OPTIMIZATION PROBLEM P_m

In this paragraph we are going to summarize the results presented in that section in a form of a numerical algorithm. The algorithm itself does not have a form of computer code or pseudocode and has to be tailored to the precise form of the problem being solved. Therefore it is rather a mathematical framework serving as a guidance for developing computer oriented algorithms.

The algorithm follows a sequence of steps detailed below. The general approach is iterative – generate the mesh, repeat all the steps and obtain the approximation together with the number indicating the quality of the approximation, and if the quality is not sufficient – generate a denser mesh and repeat the algorithm.

1. Create a mesh M covering the domain $\Omega(v_{\max}) \setminus \bar{\Omega}(v_{\min})$ – during further computations only nodes from the mesh are taken into consideration. The mesh can be generated using various mesh generating methods, and the quality of the mesh generation will impact the speed of the program (in terms of the number of necessary iterations entailing generating denser and denser meshes but also the time necessary for running a single iteration over the generated mesh).
2. Create a set W of points $x \in M \cap \Gamma_0$.
3. Create a finite set U of curves that cover the generated mesh (i.e. for evey $m \in M$ there exists $v \in U$ such that $m \in v$). The shape of the curves beetween the nodes is not important in theory, but in practice for the fastest convergence of the algorithm the graph of U should approximate the set $\Omega(v_{\max}) \setminus \bar{\Omega}(v_{\min})$ as close as possible. We have found out that it is very good to use a family of Bézier (spline) curves to this effect.
4. For every $v \in U$ and every $w \in W$ generate a trajectory for which $x(0, w, v) = w$ and $x(1, w, v) \in v$ (i.e. the trajectory starts at Γ_0 and ends at v). Because of assumption from the point 1, trajectories consist of a finite number of points coresponding to (some) nodes from mesh. Denote the set of points representing trajectories as T. It approximates the set of points representing all trajectories for the problem under consideration. The quality of this approximation depends on the quality of the mesh.
5. For every point $x \in T$ find the set U_x
6. For every point $x \in T$ calculate
$$F_g(x) = \max \{G_g(x, v) \colon v \in U_x\}.$$
This calculation is also an approximation, and an appropriate numerical algorithm should be used to achieve best convergence in the whole method.
7. Find θ and $\bar{v}$ for which inequality (4.13) from Theorem 4.3 holds and calculate the ε-optimal value.This means using the approximate values obtained in previous steps to calculate the discrete maximum and minimum in formula (4.13). The values $\bar{v}$ and θ can be used to obtain the ε - optimal value from the formula (4.14) together with the ε representing the precision of the approximation. If the precision is not sufficient, the whole algorithm should be repeated with denser mesh and more precise approximation methods in subsequent steps.

REFERENCES

[1] M. Burger, B. Hackl, W. Ring, *Incorporating topological derivatives into level set methods*, Journal of Computational Physics **194** (2004) 1, 344–362.

[2] D. Bucur, G. Buttazzo, *Variational Methods in Shape Optimization Problems*, Birkhäuser, 2005.

[3] C. Castaing, M. Valadier, *Convex analysis and measurable multifunctions*, Springer--Verlag, 1977.

[4] M. Dambrine, *On variations of the shape Hessian and sufficient conditions for the stability of critical shapes*, Rev. R. Acad. Cien. Serie A. Mat. RACSAM **96** (2002) 1, 95–121.

[5] M.C. Delfour, J.P. Zolésio, *Shapes and Geometries: Analysis, Differential Calculus and Optimization*, Adv. Des. Control, SIAM, Philadelphia, 2001.

[6] K. Eppler, *Second derivatives and sufficient optimality conditions for shape functionals*, Control Cybernet. **29** (2000), 485–512.

[7] K. Eppler, H. Harbrecht, *Numerical solution of elliptic shape optimization problems using wavelet-based BEM*, Optim. Methods Softw. **18** (2003), 105–123.

[8] K. Eppler, H. Harbrecht, *2nd order shape optimization using wavelet BEM*, Optim. Methods Softw. **21** (2006), 135–153.

[9] K. Eppler, H. Harbrecht, R. Schneider, *On convergence in elliptic shape optimization*, SIAM J. Control Optim. **46** (2007) 1, 61–83.

[10] W.H. Fleming, R.W. Rishel, *Deterministic and Stochastic Optimal Control*, New York, Springer-Verlag, 1975.

[11] P. Fulmański, A. Laurin, J.F. Scheid, J. Sokolowski, *A level set method in shape and topology optimization for variational inequalities*, Int. J. Appl. Math. Comput. Sci. **17** (2007), 413–430.

[12] P. Fulmański, A. Nowakowski, *Dual dynamic approach to shape optimization*, Control Cybernet. **35** (2006) 2, 205–218.

[13] S. Garreau, P. Guillaume, M. Masmoudi, *The topological asymptotic for PDE systems: the elasticity case*, SIAM J. Control Optim. **39** (2001), 1756–1778.

[14] J. Haslinger, R. Mäkinen, *Introduction to Shape Optimization. Theory, Approximation and Computation*, SIAM Publications, Philadelphia, 2003.

[15] I. Hlavaček, J. Haslinger, J. Nečas, J. Lovišek, *Solving of variational Inequalities in Mechancs*, Mir, Moscow, 1996 [in Russian].

[16] H. Maurer, J. Oberle, *Second order sufficient conditions for optimal control problems with free final time: the Riccati approach*, SIAM J. Control Optim. **41** (2002), 380–403.

[17] A. Nowakowski, *Shape optimization of control problems described by wave equations*, Control Cybernet. **37** (2008) 4, 1045–1055.

[18] J. Pustelnik, *Approximation of optimal value for Bolza problem*, Ph.D. Thesis, 2009 [in Polish].

[19] J. Sokołowski, J.P. Zolésio, *Introduction to Shape Optimiation*, Springer-Verlag, 1992.

[20] J. Sokołowski, A. Żochowski, *Optimality conditions for simultaneous topology and shape optimization*, SIAM J. Control Optim. **42** (2003) 4, 1198–1221.

[21] J. Sokołowski, A. Żochowski, *On Topological Derivative in Shape Optimization*, [in:] T. Lewiśki, O. Sigmund, J. Sokołowski, A. Żochowski, *Optimal Shape Design and Modelling*, Academic Printing House EXIT, Warsaw, Poland, 2004, 55–143.

[22] J. Sokołowski, A. Żochowski, *A Modeling of topological derivatives for contact problems*, Numer. Math. **102** (2005) 1, 145–179.

[23] G. Stadler, *Semismooth Newton and augmented Lagrangian methods for a simplified friction problem*, SIAM J. Optim. **15** (2004) 1, 39–62.

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