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Fixed points of dynamic processes of set-valued F-contractions and application to functional equations

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Abstract
The article is a continuation of the investigations concerning F-contractions which have been recently introduced in [Wardowski in Fixed Point Theory Appl. 2012:94,2012]. The authors extend the concept of F-contractive mappings to the case of nonlinear F-contractions and prove a fixed point theorem via the dynamic processes. The paper includes a non-trivial example which shows the motivation for such investigations. The work is summarized by the application of the introduced nonlinear F-contractions to functional equations.

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Keywords: set-valued F-contraction; dynamic process; fixed point; functional equation

1 Introduction

Metric fixed point theory is a branch of mathematics, which is widely used not only in other mathematical theories, but also in many practical problems of natural sciences and engineering. One of the most common applications of fixed points of contractive mappings defined for different types of spaces is the verification of the existence and uniqueness of solutions of differential, integral or functional equations. The multiplicity of these nonlinear problems entails the search for more and better tools, which is nowadays very noticeable in the literature. One of such tools was recently delivered by Wardowski [1], where the author introduced a new type of contractive mapping called F-contraction, *i.e.*, the mapping $T: X \to X$ with a domain in a metric space (X,d) satisfying for all $x,y \in X$ such that $d(Tx,Ty) > 0$ the contractive condition of the form

$$\tau + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big) \tag{1}$$

for some $\tau > 0$ and $F: \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing, *i.e.*, for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;

(F2) For each sequence (α_n) of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Substituting in (1) the appropriate mapping F satisfying (F1)-(F3), we obtain different types of non-equivalent contractions. See the following examples of F-contractions for concrete mappings F:

$$F(t) = \ln t, \quad t > 0, \forall_{x,y\in X}\, d(Tx, Ty) \le e^{-\tau} d(x,y), \tag{2}$$

$$F(t) = \ln t + t, \quad t > 0, \forall_{x,y\in X}\, \frac{d(Tx, Ty)}{d(x,y)} e^{d(Tx,Ty)-d(x,y)} \le e^{-\tau}, \tag{3}$$

$$F(t) = -\frac{1}{\sqrt{t}}, \quad t > 0, \forall_{x,y\in X}\, d(Tx, Ty) \le \frac{1}{(1+\tau\sqrt{d(x,y)})^2} d(x,y),$$

$$F(t) = \ln\bigl(t^2 + t\bigr), \quad t > 0, \forall_{x,y\in X}\, \frac{d(Tx, Ty)(d(Tx,Ty)+1)}{d(x,y)(d(x,y)+1)} \le e^{-\tau}. \tag{4}$$

From (F1) and (1) it is easy to conclude that every F-contraction T is a contractive mapping, *i.e.*, $d(Tx, Ty) < d(x,y)$ for all $x,y \in X$, $Tx \neq Ty$. Consequently, every F-contraction is a continuous mapping. If F_1, F_2 are the mappings satisfying (F1)-(F3), $F_1(\alpha) \le F_2(\alpha)$ for all $\alpha > 0$ and a mapping $G = F_2 - F_1$ is nondecreasing, then every F_1-contraction T is F_2-contraction; for details, see [1]. For the aforementioned mapping T, Wardowski proved the following theorem.

Theorem 1.1 (Wardowski [1]) *Let (X,d) be a complete metric space, and let $T\colon X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ a sequence $(T^n x_0)$ is convergent to x^*.*

Theorem 1.1 is a generalization of the Banach contraction principle, since for the mapping F of the form $F(t) = \ln t$, an F-contraction mapping becomes the Banach contraction condition (2). In [1] the author provided also the example showing the essentiality of this generalization.

In the literature one can find some interesting papers concerning F-contractions; *e.g.*, in [2] the authors investigated F-contractions of Hardy-Rogers type. Some of the articles present the application of F-contractions to integral and functional equations [3], iterated function systems [4], Volterra-type integral equations [5] and integral inequalities [6].

2 Preliminaries

Throughout the article denoted by $\mathbb{R}$ is the set of all real numbers, by $\mathbb{R}_+$ is the set of all positive real numbers and by $\mathbb{N}$ is the set of all natural numbers. (X,d) (X for short) is a metric space with a metric d.

Denote by $N(X)$ a collection of all nonempty subsets of X, by $C(X)$ a collection of all nonempty closed subsets of X.

Let $T\colon X \to N(X)$ be a set-valued mapping, and let $x_0 \in X$ be arbitrary and fixed. Define

$$D(T,x_0) := \bigl\{(x_n)_{n\in\mathbb{N}\cup\{0\}} \subset X \colon \forall_{n\in\mathbb{N}}\, x_n \in Tx_{n-1}\bigr\}.$$

Each element of $D(T,x_0)$ is called a *dynamic process* of T starting at x_0. The dynamic process $(x_n)_{n\in\mathbb{N}\cup\{0\}}$ will be written simply (x_n). If T is a single-valued mapping, then the dynamic process is uniquely defined and is of the form $(T^n x_0)$ (see [7] and [8]).

Example 2.1 Consider a Banach space $X = C([0,1])$ with a norm $\|x\| = \sup_{t\in[0,1]} |x(t)|$, $x \in X$. Let $T\colon X \to 2^X$ be such that for any $x \in X$, Tx is a family of the functions $t \mapsto$

$c\int_0^t x(s)\,ds$, where $c \in [0,1]$, *i.e.*,

$$(Tx)(t) = \left\{c\int_0^t x(s)\,ds\colon c \in [0,1]\right\}, \quad x \in X$$

and let $x_0(t) = t$, $t \in [0,1]$. Then the sequence $(\frac{1}{n!(n+1)!}t^{n+1})$ is a dynamic process of the operator T starting at x_0.

An element $x^* \in X$ is called a *fixed point of* T if $x^* \in Tx^*$. A mapping $f\colon X \to \mathbb{R}$ is called $D(T,x_0)$*-dynamic lower semicontinuous at* $u \in X$ if, for any dynamic process $(x_n) \in D(T,x_0)$ and for any subsequence (x_{n_i}) of (x_n) convergent to u, we have $f(u) \leq \liminf_{i\to\infty} f(x_{n_i})$. We say that f is $D(T,x_0)$*-dynamic lower semicontinuous* if f is $D(T,x_0)$-dynamic lower semicontinuous at each $u \in X$. The mapping f is called *lower semicontinuous* if for any sequence $(x_n) \subset X$ and $u \in X$ such that $x_n \to u$, we have $f(u) \leq \liminf_{n\to\infty} f(x_n)$.

3 The results

In the following we will consider only the dynamic processes $(x_n) \in D(T,x_0)$ satisfying the following condition:

$$\text{(D)} \quad \forall_{n\in\mathbb{N}} \left[d(x_n,x_{n+1}) > 0 \Rightarrow d(x_{n-1},x_n) > 0\right].$$

Remark 3.1 Observe that if the investigated process does not satisfy property (D), then there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0},x_{n_0+1}) > 0$ and $d(x_{n_0-1},x_{n_0}) = 0$. Then we get

$$x_{n_0-1} = x_{n_0} \in Tx_{n_0-1},$$

which implies the existence of a fixed point. Therefore considering the dynamic processes satisfying condition (D) does not reduce a generality of our research.

We will use the notion of set-valued F-contraction in the following sense.

Definition 3.1 Let $x_0 \in X$ and let $F\colon \mathbb{R}_+ \to \mathbb{R}$ satisfy (F1)-(F3). The mapping $T\colon X \to C(X)$ is called a set-valued F-contraction with respect to a dynamic process $(x_n) \in D(T,x_0)$ if there exists a function $\tau: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\forall_{n\in\mathbb{N}} \left[d(x_n,x_{n+1}) > 0 \Rightarrow \tau\big(d(x_{n-1},x_n)\big) + F\big(d(x_n,x_{n+1})\big) \leq F\big(d(x_{n-1},x_n)\big)\right].$$

Observe that, due to Remark 3.1, the above contractive condition is well defined.

The main result of the paper is the following.

Theorem 3.1 *Let* (X,d) *be a complete metric space,* $T\colon X \to C(X)$, $x_0 \in X$ *and let* $F\colon \mathbb{R}_+ \to \mathbb{R}$ *satisfy* (F1)-(F3). *Assume that*:

1. *There exist a function* $\tau: \mathbb{R}_+ \to \mathbb{R}_+$ *and a dynamic process* $(x_n) \in D(T,x_0)$ *such that*
   (H1) T *is a set-valued* F*-contraction with respect to* (x_n);
   (H2) $\forall_{t\geq 0}\ \liminf_{s\to t^+} \tau(s) > 0$.
2. *A mapping* $X \ni x \mapsto d(x, Tx)$ *is* $D(T,x_0)$*-dynamic lower semicontinuous.*

Then there exists a fixed point of T.

Proof Consider a dynamic process (x_n) of the mapping T starting at x_0 satisfying condition (H1). Observe that if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then the existence of a fixed point is clear. Hence we can assume that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$.

From (H1) we get

$$F\big(d(x_n, x_{n+1})\big) \le F\big(d(x_{n-1}, x_n)\big) - \tau\big(d(x_{n-1}, x_n)\big) < F\big(d(x_{n-1}, x_n)\big).$$

Thus the sequence $(d(x_n, x_{n+1}))$ is decreasing and hence convergent.

We show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. From (H2) there exists $b > 0$ and $n_0 \in \mathbb{N}$ such that $\tau(d(x_n, x_{n+1})) > b$ for all $n > n_0$. Thus, we obtain for all $n > n_0$ the following inequalities:

$$\begin{aligned} F\big(d(x_n, x_{n+1})\big) &\le F\big(d(x_{n-1}, x_n)\big) - \tau\big(d(x_{n-1}, x_n)\big) \\ &\le F\big(d(x_{n-2}, x_{n-1})\big) - \tau\big(d(x_{n-2}, x_{n-1})\big) - \tau\big(d(x_{n-1}, x_n)\big) \\ &\le \cdots \le F\big(d(x_0, x_1)\big) - \tau\big(d(x_0, x_1)\big) - \cdots - \tau\big(d(x_{n-1}, x_n)\big) \\ &= F\big(d(x_0, x_1)\big) - \big(\tau\big(d(x_0, x_1)\big) + \cdots + \tau\big(d(x_{n_0-1}, x_{n_0})\big)\big) \\ &\quad - \big(\tau\big(d(x_{n_0}, x_{n_0+1})\big) + \cdots + \tau\big(d(x_{n-1}, x_n)\big)\big) \\ &\le F\big(d(x_0, x_1)\big) - (n - n_0)b. \end{aligned} \tag{5}$$

From the above we have $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$ and, by (F2), $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

From (F3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1})^k F\big(d(x_n, x_{n+1})\big) = 0. \tag{6}$$

By (5) the following holds for all $n > n_0$:

$$\begin{aligned} &d(x_n, x_{n+1})^k F\big(d(x_n, x_{n+1})\big) - d(x_n, x_{n+1})^k F\big(d(x_0, x_1)\big) \\ &\quad \le d(x_n, x_{n+1})^k \big(F\big(d(x_0, x_1)\big) - (n - n_0)b\big) - d(x_n, x_{n+1})^k F\big(d(x_0, x_1)\big) \\ &\quad = -d(x_n, x_{n+1})^k (n - n_0)b \le 0. \end{aligned}$$

Letting $n \to \infty$ in the above and using (6), we obtain

$$\lim_{n\to\infty} n d(x_n, x_{n+1})^k = 0. \tag{7}$$

Now, let us observe that from (7) there exists $n_1 \in \mathbb{N}$ such that $n d(x_n, x_{n+1})^k \le 1$ for all $n \ge n_1$. Consequently, we have

$$d(x_n, x_{n+1}) \le \frac{1}{n^{1/k}} \quad \text{for all } n \ge n_1. \tag{8}$$

In order to show that (x_n) is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \ge n_1$. From the definition of the metric and from (8), we get

$$\begin{aligned} d(x_m, x_n) &\le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &< \sum_{i=n}^{\infty} d(x_{i+1}, x_i) \le \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

From the above and from the convergence of the series $\sum_{i=1}^{\infty} 1/i^{\frac{1}{k}}$, we receive that (x_n) is a Cauchy sequence.

From the completeness of X there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Denote $s_n = d(x_n, x_{n+1})$, $n \in \mathbb{N}$. Obviously, $s_n \to 0$, $n \to \infty$. Thus, by $D(T, x_0)$-dynamic lower semicontinuity of the mapping $X \ni x \mapsto d(x, Tx)$, we have

$$d(x^*, Tx^*) \le \liminf_{n\to\infty} d(x_n, Tx_n) \le \liminf_{n\to\infty} s_n = 0.$$

The closedness of Tx^* implies $x^* \in Tx^*$. □

The direct consequence of Theorem 3.1 for single-valued maps is the following.

Corollary 3.1 *Let* (X, d) *be a complete metric space,* $T\colon X \to X$, $x_0 \in X$, *and let* $F\colon \mathbb{R}_+ \to \mathbb{R}$ *satisfy* (F1)-(F3). *Assume that*:

1. *There exists a function* $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*
   (P1) $\forall_{n\in\mathbb{N}}\ [d(T^n x_0, T^{n+1} x_0) > 0 \Rightarrow \tau(d(T^{n-1} x_0, T^n x_0)) + F(d(T^n x_0, T^{n+1} x_0)) \le F(d(T^{n-1} x_0, T^n x_0))]$;
   (P2) $\forall_{t\ge 0}\ \liminf_{s\to t^+} \tau(s) > 0$;
2. $X \ni x \mapsto d(x, Tx)$ *is* $D(T, x_0)$*-dynamic lower semicontinuous.*

Then there exists a fixed point of T.

Corollary 3.2 *Let* (X, d) *be a complete metric space,* $T\colon X \to X$, *and let* $F\colon \mathbb{R}_+ \to \mathbb{R}$ *satisfy* (F1)-(F3). *Assume that*:

1. *There exists a function* $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*
   (C1) $\forall_{x\in X}\ [d(Tx, T^2x) > 0 \Rightarrow \tau(d(x, Tx)) + F(d(Tx, T^2x)) \le F(d(x, Tx))]$;
   (C2) $\forall_{t\ge 0}\ \liminf_{s\to t^+} \tau(s) > 0$;
2. $X \ni x \mapsto d(x, Tx)$ *is lower semicontinuous.*

Then there exists a fixed point of T.

4 Example

In this section we present an example which shows the motivation for investigating nonlinear F-contractions. It is worthy to note that the presented example does not apply to the mapping F, which gives the contractive condition of the known type in the literature. The example also shows that considering τ as a non-constant function significantly extends the applicability of Theorem 1.1 and Corollaries 3.1 and 3.2.

Example 4.1 First, we recursively define a sequence (x_n) as follows:

$$x_1 = 1,$$
$$x_2 = \frac{1}{2},$$
$$x_n = x_{n-1} - \frac{1 - x_{n-1}}{1 + x_{n-1}}(x_{n-2} - x_{n-1})^2, \quad n = 3, 4, \ldots.$$

One can verify that

$$\forall_{n\in\mathbb{N}}\ 0 < x_{n+1} < x_n \le 1. \tag{9}$$

Hence, we can put

$$x_\infty := \lim_{n\to\infty} x_n.$$

Let $X = \{x_n : n \in \mathbb{N}\} \cup \{x_\infty\}$. Together with the standard metric, X constitutes a complete metric space. Consider a mapping $T: X \to X$ by the formulae

$$Tx_n = x_{n+1},$$
$$Tx_\infty = x_\infty.$$

Take $x_0 = x_1$. Observe that (x_n) is a dynamic process of the mapping T starting at x_1, which is convergent to x_∞. Moreover, note that $X \ni x \mapsto d(x, Tx)$ is $D(T, x_0)$-dynamic lower semicontinuous.

Now, from the definition of the sequence (x_n) and (9), we obtain for all $n \in \mathbb{N}$

$$\begin{aligned}
&\frac{d(T^n x_0, T^{n+1}x_0)(d(T^n x_0, T^{n+1}x_0)+1)}{d(T^{n-1}x_0, T^n x_0)(d(T^{n-1}x_0, T^n x_0)+1)} \\
&\quad = \frac{x_{n+1}-x_{n+2}}{x_n - x_{n+1}} \times \frac{x_{n+1}-x_{n+2}+1}{x_n - x_{n+1}+1} \\
&\quad = \frac{1-x_{n+1}}{1+x_{n+1}} \times (x_n - x_{n+1})^2 \times \frac{1}{x_n - x_{n+1}} \times \frac{x_{n+1}-x_{n+2}+1}{x_n - x_{n+1}+1} \\
&\quad = (x_n - x_{n+1}) \times \frac{1-x_{n+1}}{1+x_{n+1}} \times \frac{x_{n+1}-x_{n+2}+1}{x_n - x_{n+1}+1} \\
&\quad < (x_n - x_{n+1}) \times \frac{1-x_{n+1}+x_n}{1+x_{n+1}-x_{n+2}} \times \frac{x_{n+1}-x_{n+2}+1}{x_n - x_{n+1}+1} \\
&\quad = x_n - x_{n+1} = \mathrm{e}^{-\tau(d(T^{n-1}x_0, T^n x_0))},
\end{aligned}$$

where $\tau: \mathbb{R}_+ \to \mathbb{R}_+$ is of the form

$$\tau(t) = \begin{cases} -\ln t & \text{for } t \in (0, \frac{1}{2}), \\ \ln 2 & \text{for } t \in [\frac{1}{2}, \infty). \end{cases}$$

Finally, observe that taking $F(t) = \ln(t^2 + t)$, $t > 0$ (see (4)) we get that T satisfies the following contractive condition:

$$\tau\big(d\big(T^{n-1}x_0, T^n x_0\big)\big) + F\big(d\big(T^n x_0, T^{n+1}x_0\big)\big) \le F\big(d\big(T^{n-1}x_0, T^n x_0\big)\big).$$

Therefore, the assumptions of Corollary 3.1 are fulfilled and x_∞ is a fixed point of T.

5 Application

Now, we apply our results in order to prove the existence of a solution of the following functional equation:

$$q(x) = \sup_{y\in D}\big\{f(x,y) + G\big(x, y, q\big(\eta(x,y)\big)\big)\big\}, \quad x \in W, \tag{10}$$

where $f: W \times D \to \mathbb{R}$ and $G: W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded, $\eta: W \times D \to W$, W and D are Banach spaces. Equations of the type (10) find their application in mathematical optimization, computer programming and in dynamic programming, which gives tools for solving boundary value problems arising in engineering and physical sciences (see, *e.g.*, [9]).

Let $B(W)$ denote the set of all bounded real-valued functions on W. The pair $(B(W), \|\cdot\|)$, where

$$\|h\| = \sup_{x\in W} |h(t)|, \quad h \in B(W),$$

is a Banach space.

In order to show the existence of a solution of equation (10), we consider the operator $T: B(W) \to B(W)$ of the form

$$(Th)(x) = \sup_{y\in D} \{ f(x,y) + G(x,y,h(\eta(x,y))) \} \tag{11}$$

for all $h \in B(W)$ and $x \in W$. Obviously, T is well defined, since f and G are bounded.

We will prove the following theorem.

Theorem 5.1 *Let* $T: B(W) \to B(W)$ *be an operator defined by* (11) *and assume that the following conditions are satisfied*:

1. $X \ni x \mapsto \|x - Tx\|$ *is lower semicontinuous*;
2. *There exists a function* $C: \mathbb{R}_+ \to \mathbb{R}_+$ *such that*
   (a) $\forall_{t\geq 0}\ \liminf_{s\to t^+} C(s) > 0$;
   (b) $\forall_{t>0}\ C(t) < t$;
   (c) $\forall_{h,k\in B(W)}\ \forall_{x\in W}\ \forall_{y\in D}\ |G(x,y,h(x)) - G(x,y,k(x))| \leq \|h-k\| - C(\|h-k\|)$.

Then the functional equation (10) *has a bounded solution.*

Proof Let $\lambda > 0$ be arbitrary, $x \in W$ and $h \in B(W)$. Without loss of generality, we can assume that $Th \neq h$. There exist $y_1, y_2 \in D$ such that

$$(Th)(x) < f(x,y_1) + G(x,y_1,h(\eta(x,y_1))) + \lambda,$$
$$(T^2h)(x) < f(x,y_2) + G(x,y_2,(Th)(\eta(x,y_2))) + \lambda,$$
$$(Th)(x) \geq f(x,y_2) + G(x,y_2,h(\eta(x,y_2))),$$
$$(T^2h)(x) \geq f(x,y_1) + G(x,y_1,(Th)(\eta(x,y_1))).$$

Then we get

$$\begin{aligned}(Th)(x) - (T^2h)(x) &< G(x,y_1,h(\eta(x,y_1))) - G(x,y_1,(Th)(\eta(x,y_1))) + \lambda \\ &\leq |G(x,y_1,h(\eta(x,y_1))) - G(x,y_1,(Th)(\eta(x,y_1)))| + \lambda \\ &\leq \|h - Th\| - C(\|h - Th\|) + \lambda\end{aligned}$$

and

$$\begin{aligned}(T^2h)(x) - (Th)(x) &< G(x,y_2,(Th)(\eta(x,y_2))) - G(x,y_2,h(\eta(x,y_2))) + \lambda \\ &\leq |G(x,y_2,(Th)(\eta(x,y_2))) - G(x,y_2,h(\eta(x,y_2)))| + \lambda \\ &\leq \|h - Th\| - C(\|h - Th\|) + \lambda\end{aligned}$$

for all $\lambda > 0$. Hence

$$\left|(Th)(x) - \left(T^2h\right)(x)\right| \le \|h - Th\| - C\left(\|h - Th\|\right)$$

and

$$\left\|Th - T^2h\right\| \le \|h - Th\| - C\left(\|h - Th\|\right).$$

Therefore

$$\left\|Th - T^2h\right\| \le \|h - Th\|$$

and

$$\left\|Th - T^2h\right\| - \|h - Th\| \le -C\left(\|h - Th\|\right).$$

In consequence, we get

$$\left\|Th - T^2h\right\| e^{\|Th - T^2h\| - \|h - Th\|} \le \|h - Th\| e^{-C(\|h - Th\|)}.$$

Using simple calculations, we obtain

$$C\left(\|h - Th\|\right) + \ln\left\|Th - T^2h\right\| + \left\|Th - T^2h\right\| \le \ln\|h - Th\| + \|h - Th\|,$$

which is equivalent to

$$\tau\left(\|h - Th\|\right) + F\left(\left\|Th - T^2h\right\|\right) \le F\left(\|h - Th\|\right),$$

for $F(t) = \ln t + t$, $t > 0$ (see (3)) and $\tau = C$. Corollary 3.2 implies the existence of a bounded solution of equation (11). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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