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Finite-time singularity versus global regularity for hyper-viscous Hamilton–Jacobi-like equations

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Abstract
The global regularity for the two- and three-dimensional Kuramoto–Sivashinsky equations is one of the major open questions in nonlinear analysis. Inspired by this question, we introduce in this paper a family of hyper-viscous Hamilton–Jacobi-like equations parametrized by the exponent in the nonlinear term, p, where in the case of the usual Hamilton–Jacobi nonlinearity, $p = 2$. Under certain conditions on the exponent p we prove the short-time existence of weak and strong solutions to this family of equations. We also show the uniqueness of strong solutions. Moreover, we prove the blow-up in finite time of certain solutions to this family of equations when the exponent $p > 2$. Furthermore, we discuss the difference in the formation and structure of the singularity between the viscous and hyper-viscous versions of this type of equation.

Mathematics Subject Classification: 35Q53, 35K55

1. Introduction

The Kuramoto–Sivashinsky equation (KSE)

$$\phi_t + \Delta^2\phi + \Delta\phi + \tfrac{1}{2}|\nabla\phi|^2 = 0, \tag{1.1}$$

subject to the appropriate initial and boundary conditions, is an amplitude equation that arises when studying the propagation of instabilities in hydrodynamics and combustion theory.

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Specifically, it appears in hydrodynamics as a model for the flow of thin soap films flowing down an inclined surface, and in combustion theory as a model for the propagation of flame fronts [20,29]. To avoid dealing with the average of the solution to this equation, most authors consider, instead, the system of equations for the evolution of $u = \nabla\phi$

$$u_t + \Delta^2 u + \Delta u + \tfrac{1}{2}\nabla|u|^2 = 0, \tag{1.2}$$

which is also called the KSE. In the one-dimensional case, equation (1.2) was studied by several authors both analytically and computationally (see, e.g. [5–7, 9, 10, 14–16, 18, 19, 25, 26, 31, 32] and references therein). In this case, it has been shown that the long-term dynamics of this equation are finite-dimensional. In particular, it possesses a globally invariant, finite-dimensional exponentially attracting inertial manifold. Thus, the long-term dynamics of this equation are equivalent to those of a finite-dimensional ordinary differential system.

The question of global regularity of (1.1) or (1.2) in the two-dimensional, or higher, case is one of the major challenging problems in nonlinear analysis of partial differential equations. Since $u = \nabla\phi$, equation (1.2) can be written as:

$$\boldsymbol{u}_t + \Delta^2\boldsymbol{u} + \Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = 0, \tag{1.3}$$

in which the nonlinearity takes a more familiar advection form. Let us assume that it is not difficult to prove the short-time well-posedness for all regular initial data, or global well-posedness for small initial data, for any of equations, (1.1), (1.2) or (1.3), at any spatial dimension, subject to appropriate boundary conditions, such as periodic boundary conditions. (See also the work of [28] for global well-posedness for 'small' but not 'too-small' initial data in two-dimensional thin domains, subject to periodic boundary conditions.) However, the major challenge is to show the global well-posedness for (1.2) or (1.3) in the two- and higher-dimensional cases. It is clear that the main obstacle in this challenging problem is not due to the destabilizing linear term Δu. In fact, one can equally consider the system:

$$\boldsymbol{u}_t + \Delta^2\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = 0 \tag{1.4}$$

or the equation

$$\phi_t + \Delta^2\phi + \tfrac{1}{2}|\nabla\phi|^2 = 0. \tag{1.5}$$

Now, equations (1.4) and (1.5) are more familiar. These are hyper-viscous versions of the Burgers–Hopf system of equations:

$$\boldsymbol{u}_t - \Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = 0 \tag{1.6}$$

or its scalar version

$$\phi_t - \Delta\phi + \tfrac{1}{2}|\nabla\phi|^2 = 0. \tag{1.7}$$

Using the maximum principle for $|u(x,t)|^2$ one can easily show the global regularity for (1.6) in one, two and three dimensions, subject to periodic or homogeneous Dirichlet boundary conditions [21]. Similarly, using the Cole–Hopf transformation $v = \mathrm{e}^{-\phi/2} - 1$, one can convert equation (1.7) into the heat equation in the variable v and hence conclude the global regularity in the cases of the Cauchy problem, periodic boundary conditions or homogeneous Dirichlet boundary conditions (see, e.g. [21] and references therein). However, it is clear that the maximum principle does not apply to equation (1.4) and the Cole–Hopf transformation does not apply to (1.5); hence, the global regularity for (1.4) or (1.5) in two and three dimensions is still an open question. Inspired by this question, and by virtue of (1.5), we consider in this paper the hyper-viscous generalization of the Hamilton–Jacobi equation to the initial boundary value problem with L^2 initial data (2.2)–(2.4).

In section 2, we introduce the problem under consideration and our functional setting. In section 3, under certain constraints on the exponent p, we employ in section 3 the Galerkin approximation procedure to establish the short-time existence of weak and strong solutions to the initial boundary value problem (2.2)–(2.4). We observe that all the weak solutions instantaneously become strong solutions. Moreover, we show in section 3 the uniqueness of strong solutions. The uniqueness of weak solutions remains an open question. In section 4, we show that certain solutions to the problem (2.2)–(2.4) blow-up in finite time, provided $p > 2$. It is worth mentioning that the same results are proved by Souplet [30] to the following generalization of the viscous Hamilton–Jacobi equation:

$$u_t - \Delta u = |\nabla u|^p \qquad \text{in } Q_\infty, \tag{1.8}$$

$$u = 0 \qquad \text{on } \Gamma_\infty, \tag{1.9}$$

$$u(x, 0) = u_0(x) \qquad \text{in } \Omega, \tag{1.10}$$

where Ω, Γ_∞ and Q_∞ are defined below in section 2. However, there is an essential difference in the structure of the formation of singularities in problems (1.8)–(1.10) and (2.2)–(2.4). First, we observe that regardless of the value of p, $p \geqslant 0$, problem (1.8)–(1.10) satisfies a maximum principle, and hence the $L^\infty(\Omega)$ norm of the solutions to problem (1.8)–(1.10) remain bounded for as long as the solutions exist. Thus, the solutions to (1.8)–(1.10) that blow-up in finite time must develop their singularities in one of their spatial derivatives, while the $L^\infty(\Omega)$ norm remains finite. On the contrary, for problem (2.2)–(2.4), we show that at the blow-up time, the $L^2(\Omega)$ norm of the solution and therefore the $L^\infty(\Omega)$ norm of the solution must tend to infinity. This is a consequence of the fact that we obtain a lower bound on the existence time which depends only on the L^2 norm of the initial data u_0. Notice that in this case, given the boundary condition (2.3), some derivatives should also blow-up at the same time. This remarkable observation is in a sense consistent with the common general belief that the hyper-viscous operator Δ^2 smooths the formation of singularities in the finer/smaller spatial scales faster than does the viscous operator $(-\Delta)$. This is, of course, valid provided the solution remains bounded in the $L^\infty(\Omega)$ norm, which is not the case for problem (2.2)–(2.4) since we lost the maximum principle.

As we stressed above, our main case of concern is equation (1.4) or (1.5), i.e. the equation (2.2) when $p = 2$. The question of global existence for problem (2.2)–(2.4), in the case $p = 2$, is still open, while we have global regularity in this case, as we mentioned earlier, for equations (1.6), (1.7) and (1.8) (when $p = 2$). In section 5, we consider this case subject to radial symmetry. In particular, we show global existence for radial initial data in a radially symmetric domain that excludes a neighbourhood of the origin. Thus, even in this restricted case, the question of global well-posedness for equation (1.5) is still open. In particular, one is tempted to look for a radially symmetric self-similar solution, which might lead to a singularity in finite time, a subject of future research.

Finally, it is worth noting that by replacing the term $|\nabla u|^p$ in (1.8) or in (2.2) by the nonlocal term $|(-\Delta)^{1/2}u|^p$ one gets equations that are, roughly speaking, of the same type and structure as (1.8) and (2.2). However, it is shown in [27] that in the situation of nonlocal equations, i.e. where the nonlinear term is $|(-\Delta)^{1/2}u|^p$, certain solutions blow-up in finite time, for $p > 1$ and at any spatial dimension including the one-dimensional case.

2. Notations

Let Ω be a smooth, bounded, open domain in $\mathbb{R}^n$, p a given positive number and

$$Q_t = \Omega \times (0, t), \qquad \Gamma_t = \partial\Omega \times (0, t), \qquad \Omega_t = \Omega \times \{t\}. \tag{2.1}$$

We consider the hyper-viscous Hamilton–Jacobi-type initial boundary value problem

$$u_t + \Delta^2 u = |\nabla u|^p \qquad \text{in } Q_\infty, \tag{2.2}$$

$$u = \Delta u = 0 \qquad \text{on } \Gamma_\infty, \tag{2.3}$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega, \tag{2.4}$$

where $|\nabla u| = (\nabla u, \nabla u)^{1/2}$ and $(\cdot,\cdot)$ is the usual Euclidean dot product in $\mathbb{R}^n$.

We will assume that

$$1 \leqslant p < \frac{n+8}{n+2}. \tag{2.5}$$

Here, we will use the usual notation $\|u(\cdot,t)\|_{s,q}$ for the norm of u in the Sobolev space $W^{s,q}(\Omega_t)$.

We introduce the space

$$E = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega). \tag{2.6}$$

By classical results of elliptic regularity the dot product $\langle u, v\rangle = \int_\Omega \Delta u \Delta v \,\mathrm{d}x$ makes E a Hilbert space. We will denote the dual space of E by E'.

Next, we introduce the concepts of weak and strong solution. In both cases we will require only enough regularity to be able to make sense of the quantities involved in equation (2.2).

Definition 1. *A weak solution to problem (2.2)–(2.4) in the interval* $[0,T)$ *with* $u_0 \in L^2(\Omega)$ *is a function* $u \in L^2((0,T); E) \cap L^\infty((0,T); L^2(\Omega))$ *for which* $\partial u/\partial t \in L^2((0,T); W^{-n-2,2}(\Omega))$, $|\nabla u|^p \in L^1(Q_T)$ *and* $u \in L^2_{\mathrm{loc}}((0,T); W^{4,2}(\Omega))$. *The boundary conditions are satisfied in the sense of traces and initial condition (2.4) in the weak sense. The partial differential equation is satisfied in the sense that for any* $\phi(x,t) \in C^\infty(Q_T)$ *with compact support in* Q_T *the following integral equality holds:*

$$\int_0^T \int_\Omega \frac{\partial u}{\partial t}\phi \,\mathrm{d}x\,\mathrm{d}s + \int_0^T \int_\Omega \Delta u \Delta \phi \,\mathrm{d}x\,\mathrm{d}s = \int_0^T \int_\Omega |\nabla u|^p \phi \,\mathrm{d}x\,\mathrm{d}s. \tag{2.7}$$

Next, we will define strong solutions. Here, the emphasis is on being able to make sense of the term $u_t u$. This will require higher integrability requirements on u_t.

Definition 2. *A strong solution in* Q_T *to problem (2.2)–(2.4) with* $u_0 \in L^2(\Omega)$ *is a weak solution that also satisfies* $\partial u/\partial t \in L^2((0,T); E')$.

It is clear from our definitions that the boundary conditions would be satisfied in the sense of traces. Also, a weak solution to the problem is $C^{1,4}_{t,x}$ in the interior of Q_T; therefore, the partial differential equation would be satisfied in the usual sense.

3. Local existence and uniqueness

Theorem 1. *Assume (2.5). For any* $u_0 \in L^2(\Omega)$, *there exists at least a maximal weak solution* u *of (2.2)–(2.4).*

Theorem 2.

1. *Let* $u_0 \in L^2(\Omega)$. *If*

$$1 \leqslant p \leqslant 2 \quad \text{for } n \leqslant 3 \qquad \text{and} \qquad 1 \leqslant p < \frac{n}{n-2} \quad \text{for } n \geqslant 4, \tag{3.1}$$

   then every weak solution to problem (2.2)–(2.4) is a strong solution.
2. *Assume (3.1), then strong solutions are unique.*
3. *Under assumption (2.5), for any* $u_0 \in W^{2,2}(\Omega)$, *every weak solution of (2.2)–(2.4) is a strong solution. Furthermore, in this case* $u \in L^\infty((0,T); W^{2,2}(\Omega))$.

4. *Under assumption (2.5), for any $u_0 \in L^2(\Omega)$, every weak solution of (2.2)–(2.4) instantaneously becomes a strong solution. That is, for any $\tau > 0$ we have $\partial u/\partial t \in L^2((\tau, T); E')$.*

Theorem 3. *Assume (3.1) and $u_0 \in L^2(\Omega)$. Let u be any solution of (2.2)–(2.4) and denote by $T^* = T^*(u)$ its maximal existence time. Then,*

$$T^* \geqslant C\|u_0\|_{0,2}^{-\gamma}, \tag{3.2}$$

where $\gamma = \gamma(p, n) > 0$ and $C = C(p, \Omega)$. Moreover, if $T^* < \infty$, then

$$\|u(\cdot, t)\|_{0,2} \geqslant C'(T^* - t)^{-1/\gamma} \qquad \text{on } [0, T^*). \tag{3.3}$$

Before proving these theorems we wish to start with some auxiliary results that consist of some *a priori* estimates.

Lemma 1. *Assume that (2.5) holds and let $u \in L^\infty((0, T); L^2(\Omega))$ be a smooth solution to problem (2.2)–(2.4). Then, there exists a constant C independent of u such that*

$$\int_{\Omega_t} u^2(x, t)\,\mathrm{d}x \leqslant \int_\Omega u_0^2(x)\,\mathrm{d}x + C\int_0^t \left(\int_\Omega u^2(x, s)\,\mathrm{d}x\right)^\sigma \mathrm{d}s, \tag{3.4}$$

where

$$\sigma = 1 + \frac{4(p-1)}{n+8-(n+2)p}.$$

Proof of the lemma. Since u is assumed to be smooth we can multiply (2.2) by u and integrate by parts so that we get

$$\frac{1}{2}\int_{\Omega_t} u^2(x, t)\,\mathrm{d}x + \int_0^t\int_\Omega (\Delta u)^2\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int_\Omega u_0^2\,\mathrm{d}x + \int_0^t\int_\Omega u|\nabla u|^p\,\mathrm{d}x\,\mathrm{d}s. \tag{3.5}$$

We intend to estimate the term

$$\int_0^T\int_\Omega u|\nabla u|^p\,\mathrm{d}x\,\mathrm{d}s.$$

Applying Hölder's inequality we obtain

$$\left|\int_\Omega u|\nabla u|^p\,\mathrm{d}x\right| \leqslant \left(\int_\Omega |u|^s\,\mathrm{d}x\right)^{1/s}\cdot\left(\int_\Omega |\nabla u|^{ps'}\,\mathrm{d}x\right)^{1/s'},$$

where $(1/s) + (1/s') = 1$. We choose s_1 such that $W^{s_1,2}(\Omega)$ is included in $W^{1,ps'}(\Omega)$. That is, we assume $ps' \geqslant 2$ and we set

$$s_1 = -\frac{n}{ps'} + \frac{n}{2} + 1. \tag{3.6}$$

We will use the following interpolation inequality (see, e.g. [33] p 186):

$$\|u\|_{s_1,2} \leqslant c\|u\|_{0,2}^\theta \cdot \|u\|_{2,2}^{1-\theta}, \tag{3.7}$$

where $s_1 = 2(1-\theta)$ for some $\theta \in (0, 1)$.

Assuming $s_1 < 2$, we then have that for $s_1 = 2(1-\theta)$, and s' given by (3.6)

$$\left(\int_\Omega |\nabla u|^{ps'}\,\mathrm{d}x\right)^{1/s'} \leqslant c\|u\|_{0,2}^{\theta p}\cdot\|u\|_{2,2}^{(1-\theta)p}.$$

On the other hand, using interpolation inequalities and embedding results for Sobolev spaces (see, e.g. [33] pp 186, 328) and assuming $0 \leqslant \frac{1}{2} - 1/s < 2/n$ we find that

$$\left(\int_\Omega |u|^s\,\mathrm{d}x\right)^{1/s} \leqslant c\|u\|_{0,2}^{\theta_1}\cdot\|u\|_{2,2}^{1-\theta_1},$$

where

$$\frac{1}{s} = \frac{1}{2} - \frac{2(1-\theta_1)}{n}$$

for some $\theta_1 \in (0, 1)$.

Therefore,

$$\left|\int_\Omega u \cdot |\nabla u|^p \, \mathrm{d}x\right| \leqslant c\|u\|_{0,2}^{\theta_1+\theta p} \cdot \|u\|_{2,2}^{(1-\theta_1)+p(1-\theta)}$$
$$\leqslant c\|u\|_{0,2}^{\theta_1+p\theta} \cdot \|u\|_{2,2}^{1+p-(p\theta+\theta_1)}. \tag{3.8}$$

We need to have that $1 + p - (p\theta + \theta_1) < 2$. A direct calculation shows that this holds whenever (2.5) is satisfied.

We will assume that

$$\theta_1 = 1 - \frac{n}{4} + \frac{n(s'-1)}{2s'} \qquad \text{and} \qquad \theta = \frac{1}{2} - \frac{n}{4} + \frac{n}{2s'p}. \tag{3.9}$$

Using Young's inequality ($|ab| \leqslant c_1|a|^q + c_2|b|^{q'}$) we then have

$$\left|\int_\Omega |u||\nabla u|^p \, \mathrm{d}x\right| \leqslant c\|u\|_{0,2}^{2\sigma} + \frac{1}{2}\|\Delta u\|_{0,2}^2, \tag{3.10}$$

where we made use of the fact that $\|u\|_{2,2} \simeq \|\Delta u\|_{0,2}$, which is a classical elliptic regularity result for functions that vanish on the boundary.

From (3.8), (3.9) and (3.10), it follows that

$$\sigma = \frac{\theta_1 + p\theta}{2} \times \left(1 - \frac{(1 + p - (p\theta + \theta_1))}{2}\right)^{-1}.$$

Using (3.9), we find, after doing some elementary calculations, that

$$\sigma = \frac{-4 - n + np - 2p}{-8 + 2p - n + np} = 1 + \frac{4(p-1)}{n + 8 - (n+2)p},$$

we will observe that $\sigma > 1$ for $p > 1$.

We then deduce from (3.5), (3.8) and (3.10) that

$$\int_{\Omega_t} u^2(x,t)\,\mathrm{d}x + \int_0^t \int_\Omega (\Delta u)^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant \int_\Omega u_0^2 \,\mathrm{d}x + C\int_0^t \left(\int_\Omega u^2 \,\mathrm{d}x\right)^\sigma \mathrm{d}s, \tag{3.11}$$

which proves the lemma.

Remark 1. We will explore the particular case $n = 2$ before continuing.

If we set $p = (\alpha(n+8)/(2+n))$ for some $(2+n)/(n+8) \leqslant \alpha < 1$. Then, we will have that

$$\sigma = \frac{3}{5(1-\alpha)}.$$

Lemma 2. *Assume that $p > 1$, that (2.5) holds and let $u \in L^\infty((0,T); L^2(\Omega))$ be a smooth solution to problem (2.2)–(2.4). Then, there exists a constant C, independent of u, and a time $T^* = 1/(\sigma-1)\|u_0\|_{0,2}^{2(\sigma-1)}C$ such that for all $t < T^*$,*

$$\int_{\Omega_t} u^2(x,t)\,\mathrm{d}x \leqslant \left(\frac{\|u_0\|_{0,2}^{2(\sigma-1)}}{1-(\sigma-1)\|u_0\|_{0,2}^{2(\sigma-1)}Ct}\right)^{1/(\sigma-1)} < \infty \tag{3.12}$$

and

$$\int_0^t \int_\Omega (\Delta u)^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant \|u_0\|_{0,2}^2 + Ct\left(\frac{\|u_0\|_{0,2}^{2(\sigma-1)}}{1-(\sigma-1)\|u_0\|_{0,2}^{2(\sigma-1)}Ct}\right)^{\sigma/(\sigma-1)} < \infty. \tag{3.13}$$

Proof of the lemma. For $\sigma \neq 1$, the solution of the initial value problem

$$v'(t) = C(v(t))^{\sigma}, \qquad v(0) = \int_{\Omega} u_0^2(x)\,\mathrm{d}x \tag{3.14}$$

is given by

$$v(t) = \left(\frac{\|u_0\|_{0,2}^{2(\sigma-1)}}{1-(\sigma-1)\|u_0\|_{0,2}^{2(\sigma-1)}Ct} \right)^{1/(\sigma-1)}.$$

We then deduce from estimate (3.4) of lemma 1 and Gronwall's integral inequality (see, e.g. [24] p 86]) that

$$\int_{\Omega_t} u^2(x,t)\,\mathrm{d}x \leqslant v(t) = \left(\frac{\|u_0\|_{0,2}^{2(\sigma-1)}}{1-(\sigma-1)\|u_0\|_{0,2}^{2(\sigma-1)}Ct} \right)^{1/(\sigma-1)}. \tag{3.15}$$

Estimate (3.13) follows from (3.15) and estimate (3.11).

Remark 2. Notice that $\sigma = 1$ corresponds to $p = 1$. In this case, we have that $v(t) = v(0)\mathrm{e}^{Ct}$.

The next lemma will be needed to prove theorem 2.

Lemma 3. *Assume that*

$$1 \leqslant p \leqslant 2 \quad \text{for } n \leqslant 3 \qquad \text{and} \qquad 1 \leqslant p < \frac{n+8}{n+2} \quad \text{for } n \geqslant 4 \tag{3.16}$$

and let $u \in L^{\infty}(0,T;L^2(\Omega))$ be a weak solution to problem (2.2)–(2.4). Then, $|\nabla u|^p \in L^2((0,T);L^r(\Omega))$, where $r = 1$ for $n \leqslant 3$, and $r \geqslant 2n/(n+4)$ for $n \geqslant 4$. Furthermore, there exist constants C and q independent of u such that

$$\int_0^T \left(\int_{\Omega} (|\nabla u|^p)^r\,\mathrm{d}x \right)^{2/r} \mathrm{d}s \leqslant C \sup_{0\leqslant t\leqslant T} \|u(.,t)\|_{0,2}^q \int_0^T \|u(.,s)\|_{2,2}^2\,\mathrm{d}s. \tag{3.17}$$

Proof of the lemma. From interpolation inequalities and embedding results for Sobolev spaces it follows that

$$\left(\int_{\Omega} |\nabla u|^{p\sigma}\,\mathrm{d}x \right)^{1/\sigma} \leqslant c\|u\|_{s,2}^p \leqslant c\|u\|_{0,2}^{\theta p} \cdot \|u\|_{2,2}^{(1-\theta)p}, \tag{3.18}$$

where

$$s = -\frac{n}{p\sigma} + \frac{n}{2} + 1 \qquad \text{and} \qquad \mathrm{s} = 2(1-\theta) \text{ for some } \theta \in (0, 1/2]. \tag{3.19}$$

It then follows that

$$\theta = \frac{1}{2} - \frac{n}{4} + \frac{n}{2\sigma p}. \tag{3.20}$$

We will now separate the case $n \leqslant 3$ from the case $n \geqslant 4$.

Case 1: $n \leqslant 3$. Setting $\delta = (2-p)/2n$, a direct calculation shows that for $\sigma = (2/p) + \delta$ and $0 \leqslant \delta \leqslant (1/2n)$, the constraints $\theta \in (0, 1/2]$ and $\sigma \geqslant 1$ are satisfied. Tedious but easy calculations show that $p(1-\theta) \leqslant 1$.

Using Hölder's inequality we then find that

$$\left(\int_{\Omega} |\nabla u|^p\,\mathrm{d}x \right) \leqslant c|\Omega|^{1-1/\sigma}\|u\|_{0,2}^{\theta p} \cdot \|u\|_{2,2}^{(1-\theta)p}, \tag{3.21}$$

from which (3.17) can be easily deduced.

Case 2: $n \geqslant 4$. Setting $\delta = p \cdot (n+2)/(n+8)$, a direct calculation shows that for

$$\sigma = \frac{2n}{\delta(n+8)-4} \qquad \text{and} \qquad \frac{n+2}{n+8} \leqslant \delta < 1$$

the constraints $\theta = 1 - (1/p) \in (0, 1/2)$ and $\sigma \geqslant 2n/(n+4)$ are satisfied. Notice that $p(1-\theta) = 1$.

Once again, (3.17) can easily be deduced from (3.18).

Next, let us prove theorem 1.

Proof of theorem 1. We will use the Galerkin method for establishing the existence of a solution. For this purpose, we let w_i, $i = 1, 2, 3, \ldots$, be the eigenfunctions of the Laplace operator in $W_0^{1,2}(\Omega)$ orthonormalized with respect to the $L^2(\Omega)$ norm. It is well known that this set of eigenfunctions constitutes a basis of $L^2(\Omega)$. We set $E_m := span\{w_1, \ldots, w_m\}$. For fixed k we look for a function $u_k = \sum_{i=1}^k a_{i,k}(t) w_i(x)$, which solves the Galerkin truncated system

$$\int_\Omega \frac{\partial u_k}{\partial t}\phi \,\mathrm{d}x + \int_\Omega \Delta u_k \Delta\phi \,\mathrm{d}x = \int_\Omega |\nabla u_k|^p \phi \,\mathrm{d}x, \tag{3.22}$$

$$a_{i,k}(0) = \int_\Omega u_0 w_i \,\mathrm{d}x \qquad i = 1, \ldots, k. \tag{3.23}$$

for every test function $\phi \in E_k$.

This is a system of nonlinear ordinary differential equations for the k unknown coefficients $a_{i,k}(t)$. For $p \geqslant 1$, this system of ordinary differential equations satisfies the conditions of the Picard theorem. Therefore, it has a unique local solution $a_i^{(k)}(t)$, $i = 1, \ldots, k$ in some interval about $t = 0$.

Since $u_k \in E_k$ it can be used as a test function in (3.22). Following the same steps as in the proof of lemma 2 we conclude that, for every k fixed, u_k is in $L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); W^{2,2}(\Omega))$ for all

$$T \leqslant T_k^* = \frac{1}{(\sigma-1)\|u_k(x,0)\|_{0,2}^{2(\sigma-1)} C}.$$

Since $\|u_k(x,0)\|_{0,2} \leqslant \|u_0(x)\|_{0,2}$ for all k, it follows that the T_k^* are uniformly bounded from below by

$$T^* = \frac{1}{(\sigma-1)\|u_0(x)\|_{0,2}^{2(\sigma-1)} C}. \tag{3.24}$$

Proceeding as we did in the proofs of (3.12) and (3.13), we find that, for $\tau < T^*$ fixed, u_k is bounded in $L^\infty((0,\tau); L^2(\Omega)) \cap L^2((0,\tau); W^{2,2}(\Omega))$ independently of k.

From the weak compactness of the sequence u_k it follows that there exists a sub-sequence, denoted again by u_k, and a function $u(x,t)$ such that for any $t < T^*$

$$u_k \longrightarrow u \quad \text{as } k \to \infty \qquad \text{in } L^\infty_{\text{weak-star}}((0,t); L^2_{\text{weak}}(\Omega)), \tag{3.25}$$

$$u_k \longrightarrow u \quad \text{as } k \to \infty \qquad \text{in } L^2_{\text{weak}}((0,t); W^{2,2}_{\text{weak}}(\Omega)). \tag{3.26}$$

We intend to show that the sequence u_k converges to a solution of equation (2.2). Because of the presence of the nonlinear term $|\nabla u|^p$ in equation (2.2), the estimates we already have will not be enough to establish the desired result.

Next, we will derive an estimate of $\partial u_k/\partial t$.

For this purpose we let ϕ be a function in $W_0^{n+2,2}(\Omega)$ and we decompose $\phi = \phi_k + (\phi - \phi_k)$, where ϕ_k is the L^2 projection of ϕ onto the space E_k. It is well known (see [22]) that thanks to the special choice of the sequence w_i we have

$$\|\phi_k\|_{n+2,2} \leqslant c\|\phi\|_{n+2,2}, \tag{3.27}$$

where c is a constant independent of ϕ.

Because of the orthogonality property of the functions w_i we have that

$$\int_{\Omega_t} \frac{\partial u_k}{\partial t}\phi \,\mathrm{d}x = \int_{\Omega_t} \frac{\partial u_k}{\partial t}\phi_k \,\mathrm{d}x$$

and since $\phi_k \in E_k$ it follows from (3.22) that

$$\int_{\Omega_t} \frac{\partial u_k}{\partial t}\phi \,\mathrm{d}x = -\int_{\Omega_t} \Delta u_k \Delta\phi_k \,\mathrm{d}x + \int_{\Omega_t} |\nabla u_k|^p \phi_k \,\mathrm{d}x. \tag{3.28}$$

We need to estimate the last term in the equality above.

From the Hölder inequality and the embedding of $W^{n+2,2}$ into L^∞ for any n, we find that for any $q > 1$

$$\left|\int_\Omega |\nabla u_k|^p \phi_k \,\mathrm{d}x\right| \leqslant \|\nabla u_k\|_{0,pq}^p \|\phi_k\|_{0,q'} \leqslant c\|\nabla u_k\|_{0,pq}^p \|\phi_k\|_{n+2,2}|\Omega|^{1/q'}, \tag{3.29}$$

where c is independent of ϕ and k.

Proceeding as we did in the proof of lemma 1, we find that

$$\|\nabla u_k\|_{0,pq} \leqslant c\|u_k\|_{s,2} \leqslant c\|u_k\|_{0,2}^{(1-\theta)} \cdot \|u_k\|_{2,2}^{\theta}, \tag{3.30}$$

where

$$\frac{1}{pq} = \frac{1}{2} - \frac{s-1}{n}.$$

We then find that $s = (npq - 2n + 2pq)/2pq$. We also have that $s = 2\theta$, and therefore $\theta = (npq - 2n + 2pq)/4pq$. Hence, we have that

$$\left|\int_\Omega |\nabla u_k|^p \phi_k \,\mathrm{d}x\right| \leqslant c\|u_k\|_{0,2}^{p(1-\theta)} \|u_k\|_{2,2}^{p\theta} \|\phi_k\|_{n+2,2} \tag{3.31}$$

$$\leqslant c\|u_k\|_{0,2}^{(2pq-npq+2n)/(4q)} \|u_k\|_{2,2}^{(npq-2n+2pq)/(4q)} \|\phi_k\|_{n+2,2}. \tag{3.32}$$

We recall that we are assuming that $p < (n+8)/(n+2)$.

The constraint that $s \in (1, 2)$ is satisfied whenever we choose q such that

$$\frac{2}{p} \leqslant q \quad \text{for } n \leqslant 2 \qquad \text{and} \qquad \frac{2}{p} \leqslant q \leqslant \frac{2n}{(n-2)p} \quad \text{for } n \geqslant 3. \tag{3.33}$$

We would also like to have that $p\theta < 2$. This is always the case whenever either $(n+2)p - 8 \leqslant 0$ or

$$q < \frac{2n}{(n+2)p - 8}. \tag{3.34}$$

For $p < (n+8)/(n+2)$ and q satisfying (3.33) and (3.34), then, in (3.32) we have that the exponent of $\|u_k\|_{2,2}$ satisfies $0 < (npq - 2n + 2pq)/(4q) < 2$. Assuming that u_k is uniformly bounded in $L^\infty((0, t); L^2(\Omega))$ and in $L^2((0, t); W^{2,2}(\Omega))$, we deduce from the above that

$$\left|\int_\Omega |\nabla u_k|^p \phi_k \,\mathrm{d}x\right| \leqslant cF(t)\|\phi_k\|_{n+2,2}, \tag{3.35}$$

where for $\gamma = (2/p\theta) > 1$, $F(t)$ is bounded in $L^\gamma(0, t)$ independently of k.

On the other hand,

$$\left|\int_\Omega \Delta u_k \Delta\phi_k \,\mathrm{d}x\right| \leqslant c\|u_k\|_{2,2}\cdot\|\phi_k\|_{n+2,2}. \tag{3.36}$$

Assuming that the dual of $W_0^{2+n,2}(\Omega)$ is $W^{-2-n,2}(\Omega)$, we deduce from (3.27), (3.28), (3.35) and (3.36) that $\partial u_k/\partial t$ is uniformly bounded in $L^{\gamma_1}((0,t); W^{-2-n,2}(\Omega))$, where $\gamma_1 = \min(\gamma, 2)$.

It then follows from Aubin's lemma (see, e.g. [22] p 57) that u_k is compact in the strong topology of $L^2(0, T, W^{2-\epsilon,2}(\Omega))$ for any $\epsilon > 0$. Therefore, for any r such that $r < \infty$ when $n \leqslant 2$ and $r < 2n/(n-2)$ when $n > 2$, there is a subsequence of u_k, denoted again by u_k, such that ∇u_k converges to ∇u strongly in $L^2(0, t; L^r(\Omega))$.

In order to show that the limit function u is a solution to the partial differential equation (2.2) we will show next that $\forall\psi \in C^\infty(Q_T)$ we have that $\int_0^t\int_\Omega |\nabla u_k|^p\psi \,\mathrm{d}x\,\mathrm{d}s$ converges to $\int_0^t\int_\Omega |\nabla u|^p\psi\,\mathrm{d}x\,\mathrm{d}s$ as k goes to infinity. Notice that from lemma 3, we get that $|\nabla u|^p \in L^1(Q_T)$.

Let $\psi \in C^\infty(Q_T)$. Then,

$$\begin{aligned}\left|\int_\Omega (|\nabla u_k|^p - |\nabla u|^p)\psi\,\mathrm{d}x\right| &\leqslant c\|\psi\|_{0,\infty}\|(|\nabla u_k| - |\nabla u|)\|_{0,r}\|(|\nabla u_k| + |\nabla u|)\|_{0,(p-1)r'}^{(p-1)}\\ &\leqslant c\|\psi\|_{0,\infty}\|\nabla u_k - \nabla u\|_{0,r}\|(|\nabla u_k| + |\nabla u|)\|_{0,(p-1)r'}^{(p-1)},\end{aligned}$$

where r' is the conjugate of r.

Now, using interpolation inequalities, again we find that

$$\|\nabla u_k\|_{0,(p-1)r'} \leqslant c\|u_k\|_{s,2} \tag{3.37}$$
$$\leqslant c\|u_k\|_{0,2}^{1-\theta}\|u_k\|_{2,2}^{\theta} \tag{3.38}$$

with

$$\frac{1}{(p-1)r'} = \frac{1}{2} - \frac{s-1}{n} \qquad \text{and} \qquad s = 2\theta.$$

We then find that

$$\theta = \frac{1}{4}\left(n + 2 - \frac{2n}{(p-1)r'}\right).$$

We need to choose $r' \geqslant 1$ such that $s \in [1, 2]$ and $(p-1)\theta \leqslant 1$.

We recall that p is subject to assumption (2.5).

In the case where $n \leqslant 2$, we only impose that $r' > 1$, and we find that the conditions above are met whenever

$$\frac{3}{2}(p-1) - 2 \leqslant \frac{1}{r'} \leqslant \frac{p-1}{2} \qquad \text{for } n = 1, \tag{3.39}$$
$$(p-2) \leqslant \frac{1}{r'} \leqslant \frac{(p-1)}{2} \qquad \text{for } n = 2. \tag{3.40}$$

For the case $n \geqslant 3$ we will also require that $r \leqslant 2n/(n-2)$ so we will denote $r = \tau(2n/(n-2))$ and require that $((n-2)/2n) \leqslant \tau < 1$. We recall that $(1/r') = 1 - (1/r)$ and we find that the constraints above are satisfied whenever τ is chosen such that the following conditions are satisfied:

$$\frac{2n}{n-2} - \frac{n}{n-2}(p-1) \leqslant \frac{1}{\tau} \leqslant \frac{2n}{n-2} - (p-1) \qquad \text{for } n \geqslant 3 \quad \text{and} \quad p \leqslant 2, \tag{3.41}$$
$$\frac{2n}{n-2} - \frac{n}{n-2}(p-1) \leqslant \frac{1}{\tau} \leqslant \frac{2n}{n-2} - (p-1) + \frac{4}{n-2}(2-p) \qquad \text{for } n \geqslant 3 \quad \text{and} \quad p \geqslant 2. \tag{3.42}$$

Using (2.5) and (3.39)–(3.42), it is always possible to choose θ such that $\theta(p-1) < 1$. Then, from (3.38) it follows that

$$\left|\int_\Omega (|\nabla u_k|^p - |\nabla u|^p)\psi \,\mathrm{d}x\right| \leqslant c\|\psi\|_{0,\infty}\|\nabla u_k - \nabla u\|_{0,r}(1 + \|u_k\|_{2,2} + \|u\|_{2,2}), \tag{3.43}$$

where we made use of the fact that u_k is uniformly bounded in $L^\infty((0,t); L^2(\Omega))$.

Therefore, for any $\psi \in C^\infty(Q_T)$

$$\int_0^t \int_\Omega (|\nabla u_k|^p - |\nabla u|^p)\psi \,\mathrm{d}x\,\mathrm{d}s \longrightarrow 0 \qquad \text{as } k \to \infty \tag{3.44}$$

We then deduce from (3.28) that the limit u of the sequence u_k satisfies

$$\int_0^t \int_{\Omega_t} \frac{\partial u}{\partial t}\psi \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Omega_t} \Delta u \Delta\psi \,\mathrm{d}x\,\mathrm{d}s = \int_0^t \int_{\Omega_t} |\nabla u|^p \psi \,\mathrm{d}x\,\mathrm{d}s \tag{3.45}$$

for all $\psi \in C^\infty(Q_T) \cap L^2(0,t; W_0^{1,2}(\Omega))$.

We also have that u belongs to $W^{1,2}(0,t; W^{-2,2}(\Omega))$, whence its trace at $t=0$ is well-defined, and that $u(x,0) = \lim_{k\to\infty} u_k(x,0) = u_0(x)$.

To finish proving that the function u is a solution to the boundary value problem (2.2)–(2.4) we still need to show that u satisfies $\Delta u = 0$ on Γ_t. For this purpose, we will need to have a stronger estimate on the sequence u_k. This is the purpose of the next lemma.

Lemma 4. *Let τ and T be numbers such that $0 < \tau < T < T^*$, and let u_k be the sequence of solutions to the Galerkin system (3.22)–(3.23). Then, there exists a positive number M_τ such that*

$$\int_\tau^T \int_\Omega (\Delta^2 u_k)^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant M_\tau. \tag{3.46}$$

Proof of the lemma. Let τ and T be fixed. For each k we know from (3.13) that $\|u_k(\cdot,t)\|_{2,2}$ is bounded in $L^2(0,\tau)$ uniformly with respect to k. Therefore, there exists a time $t_k \in (0,\tau)$ such that

$$\|u_k(\cdot,t_k)\|_{2,2} \leqslant \frac{1}{\tau}\int_0^\tau \|u_k(\cdot,t)\|_{2,2}\,\mathrm{d}s \leqslant c, \tag{3.47}$$

where c is independent of k.

Since $\Delta u_k \in E_k$ we can set $\phi = \Delta^2 u_k$ in (3.22) and integrate over (t_k, T). We then get that

$$\begin{aligned}\int_\Omega (\Delta u_k(x,T))^2 \,\mathrm{d}x + 2\int_{t_k}^T \int_\Omega (\Delta^2 u_k)^2 \,\mathrm{d}x\,\mathrm{d}s \\ = \int_\Omega (\Delta u_k(x,t_k))^2 \,\mathrm{d}x + 2\int_{t_k}^T \int_\Omega (\Delta^2 u_k)|\nabla u_k|^p \,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{3.48}$$

We need to estimate the last term in the inequality above. First, we get from using Young's inequality and (3.13) that

$$\int_\Omega (\Delta u_k(x,T))^2 \,\mathrm{d}x + \int_{t_k}^T \int_\Omega (\Delta^2 u_k)^2 \,\mathrm{d}x\,\mathrm{d}s \leqslant \int_\Omega (\Delta u_k(x,t_k))^2 \,\mathrm{d}x + \int_{t_k}^T \int_\Omega |\nabla u_k|^{2p} \,\mathrm{d}x\,\mathrm{d}s. \tag{3.49}$$

Using Sobolev embedding and interpolation inequalities, we then find that

$$\|\nabla u_k\|_{0,2p} \leqslant c\|u_k\|_{0,2}^{1-\theta}\|u_k\|_{4,2}^\theta \leqslant c\|u_k\|_{0,2}^{1-\theta}\|\Delta^2 u_k\|_{0,2}^\theta \tag{3.50}$$

for

$$\theta = \frac{1}{4} + \frac{n}{8}\left(1 - \frac{1}{p}\right).$$

Raising both sides to the power $2p$ and using Young's inequality we then get that

$$\|\nabla u_k\|_{0,2p}^{2p} \leqslant c\|u_k\|_{0,2}^{2p(1-\theta)q} + \tfrac{1}{2}\|\Delta^2 u_k\|_{0,2}^2, \tag{3.51}$$

where q is the conjugate of $q' = 2/2p\theta$. An elementary calculation shows that $q = 8/(8-2p-np+n)$. It is easy to verify that for $1 \leqslant p < (n+8)/(n+2)$ we have that $4/3 \leqslant q < \infty$.

Combining (3.51) and (3.49) we get that

$$\int_\Omega (\Delta u_k(x,T))^2\,\mathrm{d}x + \frac{1}{2}\int_{t_k}^T \int_\Omega (\Delta^2 u_k)^2\,\mathrm{d}x\,\mathrm{d}s \leqslant \int_\Omega (\Delta u_k(x,t_k))^2\,\mathrm{d}x + c\int_{t_k}^T \|u_k\|_{0,2}^{2p(1-\theta)q}\,\mathrm{d}s. \tag{3.52}$$

Assuming that $\|u_k\|_{0,2}$ is in $L^\infty(0,T)$ and that its norm in this space is bounded uniformly with respect to k we then have that

$$\int_\Omega (\Delta u_k(x,T))^2 + \frac{1}{2}\int_{t_k}^T \int_\Omega (\Delta^2 u_k)^2\,\mathrm{d}x\,\mathrm{d}s \leqslant \int_\Omega (\Delta u_k(x,t_k))^2\,\mathrm{d}x + cT. \tag{3.53}$$

We now use the fact that by our choice of t_k we have that $\|u_k(.,t_k)\|_{2,2}^2 \leqslant (1/\tau)\int_0^\tau \|u_k(.,t)\|_{2,2}^2\,\mathrm{d}s \leqslant c$, where c is independent of k.

Therefore, we have that

$$\int_\tau^T \int_\Omega (\Delta^2 u_k)^2\,\mathrm{d}x\,\mathrm{d}s \leqslant M_\tau, \tag{3.54}$$

where M_τ is independent of k. This ends the proof of lemma 4.

We will now finish the proof of the existence of a weak solution.

From the basis we used in our Galerkin approximation it is immediate that $u_k = \Delta u_k = 0$ on Γ_{T^*}. Now, for $\tau > 0$ we have, from lemma 4, that there exists a subsequence u_{k_n} which converges to u in $L^2_{\text{weak}}((\tau,T); W^{4,2}_{\text{weak}}(\Omega))$. Therefore by taking a sequence of τ_n which converges to zero as n goes to infinity and using the usual diagonal procedure we can find a subsequence u_{k_m} such that for any $\tau > 0$, u_{k_m} will converge weakly in $L^2((\tau,T); W^{4,2}(\Omega))$ to u. Furthermore, by virtue of (3.54), we also have

$$\int_\tau^T \int_\Omega (\Delta^2 u)^2\,\mathrm{d}x\,\mathrm{d}s \leqslant M_\tau,$$

as required by weak solutions. From all the above we deduce that $\forall\tau > 0$, $\Delta u = 0$ on Γ as an element of $L^2(\tau,T; W^{3/2,2}(\Gamma))$. This finishes the proof of theorem 1.

Proof of theorem 2.

Remark 3. In the case $p = 1$, the proofs are very simple we will concentrate on the case $p > 1$.

Since u is a weak solution, by definition, $u \in L^2((0,T); E)$. Also, for any $t > 0$, and any $v \in E$, we have that

$$\int_{\Omega_t} v\Delta^2 u\,\mathrm{d}x = \int_{\Omega_t} \Delta v \Delta u\,\mathrm{d}x. \tag{3.55}$$

Hence, $\Delta^2 u \in L^2((0,T); E')$.

Now let $v \in L^2((0,T),E)$. Then by embedding theorems for Sobolev spaces we have $E \subset C^0(\bar{\Omega})$ for $n \leqslant 3$, $E \subset L^\delta(\Omega))$ $\forall \delta \in [1,\infty)$ for $n=4$, and $E \subset L^\delta(\Omega))$ with $1 \leqslant \delta \leqslant 2n/(n-4)$ for $n>4$.

It then follows from lemma 3 that $|\nabla u|^p \in L^2((0,T);\ E')$.

We then deduce from the partial differential equation that $u_t \in L^2((0,T);\ E')$.

Next, we will prove uniqueness.

Let u_1, u_2 be two strong solutions corresponding to the same initial u_0. We denote $w = u_1 - u_2$. Taking the difference of the equations satisfied by each function, we get that

$$\frac{\partial w}{\partial t} + \Delta^2 w = |\nabla u_1|^p - |\nabla u_2|^p. \tag{3.56}$$

Notice that all of the terms appearing in equation (3.56) are in $L^2((0,T);E')$. Since the function w is in $L^2((0,T);E)$, a well-known lemma from Lions–Magenes [23] (see also [32]) implies that the function $\|w(t)\|_{L^2}$ is absolutely continuous and that $\mathrm{d}/\mathrm{d}t\|w(t)\|_{L^2}^2 = 2\langle \partial w/\partial t, w\rangle_{E'}$. Therefore, by taking the action of equation (3.56) on w and integrating by parts we find that

$$\begin{aligned}\frac{1}{2}\int_{\Omega_t} w^2(x,t)\,\mathrm{d}x + \int_0^t\int_{\Omega_t}(\Delta w)^2\,\mathrm{d}x\,\mathrm{d}s &= \int_0^t\int_{\Omega_t}(|\nabla u_1|^p - |\nabla u_2|^p w\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant c\int_0^t\int_{\Omega_t}(|\nabla u_1|^{p-1} + |\nabla u_2|^{p-1})|\nabla w||w|\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{3.57}$$

The uniqueness will be derived using Gronwall's inequality. For this purpose, we need to derive some estimates of the last term in the inequality above.

$$\int_{\Omega_t}|\nabla u_1|^{p-1}|\nabla w||w|\,\mathrm{d}x \leqslant c\|\nabla u_1\|_{0,2(p-1)\alpha'}^{p-1}\|\nabla w\|_{0,2\alpha}\|w\|_{0,2}, \tag{3.58}$$

where α' is the conjugate of α.

Using the Sobolev embedding theorem together with interpolation inequalities we have that

$$\|\nabla w\|_{0,2\alpha} \leqslant c\|w\|_{0,2}^{1-\theta}\|w\|_{2,2}^{\theta} \leqslant c\|w\|_{0,2}^{1-\theta}\|\Delta w\|_{0,2}^{\theta} \tag{3.59}$$

for $\theta = \frac{1}{2} + (n(\alpha-1)/4\alpha)$. In order that $\theta \in (0,1)$ we will require that for $n>2$ there holds

$$\alpha \leqslant \frac{n}{n-2} \Longleftrightarrow 1 - \frac{1}{\alpha} = \frac{1}{\alpha'} \leqslant \frac{2}{n}. \tag{3.60}$$

Combining (3.57) with (3.59) and (3.58) and using Young's inequality we find that

$$\begin{aligned}\frac{1}{2}\int_{\Omega_t} w^2(x,t)\,\mathrm{d}x + \int_0^t\int_{\Omega_t}(\Delta w)^2\,\mathrm{d}x\,\mathrm{d}s &\leqslant \frac{1}{2}\int_0^t \|\Delta w\|_{0,2}^{\theta q'}\,\mathrm{d}x\,\mathrm{d}s\\ &+c\int_0^t\left(\|\nabla u_1\|_{0,2(p-1)\alpha'}^{q(p-1)} + \|\nabla u_2\|_{0,2(p-1)\alpha'}^{q(p-1)}\right)\|w\|_{0,2}^{(2-\theta)q}\,\mathrm{d}s,\end{aligned} \tag{3.61}$$

where we use the notation $'$ to refer to conjugates, and c refers to a generic constant.

Choosing $q' = 2/\theta$, $q = 2/(2-\theta)$ (where θ is as above) and absorbing the term $\int_0^t \|w\|_{2,2}^{\theta q'}\,\mathrm{d}x\,\mathrm{d}s$ on the right-hand side, we find from (3.61) that

$$\int_{\Omega_t} w^2(x,t)\,\mathrm{d}x \leqslant c\int_0^t\left(\|\nabla u_1\|_{0,2(p-1)\alpha'}^{q(p-1)} + \|\nabla u_2\|_{0,2(p-1)\alpha'}^{q(p-1)}\right)\|w\|_{0,2}^2\,\mathrm{d}s. \tag{3.62}$$

We now estimate $\|\nabla u_1\|_{0,2(p-1)\alpha'}$ in terms of $\|u_1\|_{0,2}$, and $\|u_1\|_{2,2}$. Proceeding as we did in estimating $\|\nabla w\|_{0,2\alpha}$ we get

$$\|\nabla u_1\|_{0,2(p-1)\alpha'} \leqslant c\|u_1\|_{0,2}^{(1-\gamma)}\|u_1\|_{2,2}^{\gamma},$$

with

$$\gamma = \frac{1}{2} + \frac{n((p-1)\alpha - \alpha + 1)}{4(p-1)\alpha}.$$

In order to have $\gamma \in (0, 1)$ we will require that (2.5) be satisfied and that

$$(p-1)\left(1 - \frac{2}{n}\right) \leqslant \frac{1}{\alpha'} \leqslant (p-1)\left(1 + \frac{2}{n}\right). \tag{3.63}$$

We then get from (3.62) that

$$\begin{aligned}\int_{\Omega_t} w^2(x,t)\,\mathrm{d}x &\leqslant c\int_0^t \left(\|u_1\|_{0,2}^{q(p-1)(1-\gamma)}\|u_1\|_{2,2}^{q(p-1)\gamma} + \|u_2\|_{0,2}^{q(p-1)(1-\gamma)}\|u_2\|_{2,2}^{q(p-1)\gamma}\right)\|w\|_{0,2}^2\,\mathrm{d}s \\ &\leqslant c\int_0^t \left(\|u_1\|_{2,2}^{q(p-1)\gamma} + \|u_2\|_{2,2}^{q(p-1)\gamma}\right)\|w\|_{0,2}^2\,\mathrm{d}s. \end{aligned}\tag{3.64}$$

To obtain (3.64) we used that $\|u_1(t,\cdot)\|_{0,2}$, and $\|u_2(t,\cdot)\|_{0,2}$, are bounded in $L^\infty(0,T)$, and that $q(p-1)(1-\gamma) > 0$. By the choice of q we have $(2-\theta)q = 2$. Now elementary calculations show that

$$q(p-1)\gamma = \frac{-2(2\alpha p - 2\alpha + n\alpha p - 2n\alpha + n)}{-6\alpha + n\alpha - n}. \tag{3.65}$$

Owing to (2.5) notice that $q(p-1)\gamma \leqslant 2$ whenever $1/\alpha' \leqslant 6/n$. Therefore, assuming that (3.60) and (3.63) are satisfied, we then have that

$$\int_{\Omega_t} w^2(x,t)\,\mathrm{d}x \leqslant c\int_0^t \left(\|u_1\|_{2,2}^{q(p-1)\gamma} + \|u_2\|_{2,2}^{q(p-1)\gamma}\right)\|w\|_{0,2}^2\,\mathrm{d}s \tag{3.66}$$

with $\|u_1(s,\cdot)\|_{2,2}^{q(p-1)\gamma} \in L^1(0,t)$, $\|u_2(s,\cdot)\|_{2,2}^{q(p-1)\gamma} \in L^1(0,t)$ and $\int_{\Omega_t} w^2(x,0)\,\mathrm{d}x = 0$, from which we deduce by Gronwall's inequality that $\int_{\Omega_t} w^2(x,t)\,\mathrm{d}x = 0$, for all $t > 0$.

A compatibility condition between conditions (3.60) and (3.63) requires that

$$(p-1)\left(1 - \frac{2}{n}\right) \leqslant \frac{1}{\alpha'} \leqslant \frac{2}{n}. \tag{3.67}$$

This is easily seen to be satisfied whenever condition (3.1) holds.

To prove part 3 of theorem 2, we follow similar steps to those in the proof of theorem 1. Since $u_0 \in W^{2,2}(\Omega)$, one can easily establish a similar estimate to (3.52) to reach

$$\int_\Omega (\Delta u(x,t))^2\,\mathrm{d}x + \frac{1}{2}\int_0^t\int_\Omega (\Delta^2 u)^2\,\mathrm{d}x\,\mathrm{d}s \leqslant \int_\Omega (\Delta u(x,0))^2\,\mathrm{d}x + c\int_0^t \|u\|_{0,2}^{2p(1-\theta)q}\,\mathrm{d}s, \tag{3.68}$$

for every $t \in [0, T)$. Here,

$$\theta = \frac{1}{4} + \frac{n}{8}\left(1 - \frac{1}{p}\right) \qquad \text{and} \qquad q = \frac{8}{8 - 2p - np + n}.$$

Again, it is easy to verify that for $1 \leqslant p < (n+8)/(n+2)$ we have that $4/3 \leqslant q < \infty$. Thanks to definition 1, and to (3.68), we conclude that $u \in L^\infty((0,T); W^{2,2}(\Omega))$, from which we can easily complete the proof of part 3 of theorem 2.

The proof of part 4 of theorem 2 can easily be deduced from the previous parts.

Proof of theorem 3. Let u be a fixed maximal weak solution of (2.2)–(2.4) and $T := T^*(u)$ its maximal existence time.

For each fixed $t_0 \in (0, T)$, let w_{t_0} be the maximal solution constructed in theorem 1 (see (3.24)), with initial condition $w_{t_0}(t_0) = u(t_0)$. On the one hand, as a consequence of the proof of theorem 1, we have:

$$T^*(w_{t_0}) - t_0 \geqslant C\|u(t_0)\|_{0,2}^{-\gamma}.$$

On the other hand, clearly we have $u(t_0) \in W^{2,2}$ (for a.e. t_0), so that by theorem 2 (point 3) w_{t_0} is a strong solution. Also, by theorem 2 (point 4), we know that u is a strong solution for $t \geqslant t_0$. It then follows from theorem 2 (point 2) that u and w_{t_0} coincide on their common existence interval. Since u is a maximal solution, it follows that:

$$T \geqslant T^*(w_{t_0}) \geqslant t_0 + C\|u(t_0)\|_{0,2}^{-\gamma}, \qquad \text{for a.e. } t_0 \in (0, T).$$

This yields

$$\|u(t_0)\|_{0,2} \geqslant C^{'}(T - t_0)^{-1/\gamma},$$

that is (3.3).

Since, by definition 1, $u(t)$ converges to u_0 weakly in L^2 as $t \to 0$, we have:

$$\|u_0\|_{0,2} \geqslant \limsup_{t\to 0} \|u(t)\|_{0,2}$$

and (3.2) follows from (3.3).

4. Finite time blow-up

We will show here that under certain assumptions the solution u to problem (2.2)–(2.4) blows up in finite time. For this purpose, we start by introducing some notation and recalling some well-known results.

It is well known (see, e.g. [17] and the references therein) that under the assumptions we made on Ω, the eigenvalue problem

$$-\Delta\psi = \lambda\psi \qquad \psi \in W_0^{1,2}(\Omega) \tag{4.1}$$

has a smallest positive eigenvalue $\lambda = \lambda_1$ and that the associated eigenfunction ϕ does not vanish in Ω. Notice that $\phi \in W^{2,2}(\Omega) \cap W^{1,\infty}(\Omega)$. We, therefore, can choose a ϕ such that $\phi > 0$ in Ω and $\int_\Omega \phi\, \mathrm{d}x = 1$. Furthermore, it can be proved (see [2, 3, 30] and the references therein) that

$$\int_\Omega \phi^{-\alpha}\, \mathrm{d}x = C(\alpha, \Omega) < \infty \qquad \forall\alpha \in (0, 1). \tag{4.2}$$

Proposition 1. *Assume $p > 2$ and let $u_0 \in L^2(\Omega)$ satisfy $\int_\Omega u_0(x)\phi(x)\, \mathrm{d}x > M = M(\Omega, p) > 0$ sufficiently large. Then, problem (2.2)–(2.4) cannot admit a globally defined weak solution. Indeed, there exists $T^\# = T^\#(M) > 0$ such that either u ceases to exist before $T^\#$, or the quantity $z(t) = \int_\Omega u(x, t)\phi(x)\, \mathrm{d}x$ satisfies $\lim_{t\to T^\#-} z(t) = +\infty$.*

Proof. The proof follows the well-known technique of Kaplan introduced in [17]. Multiplying equation (2.2) by ϕ and integrating over Ω we find

$$\int_{\Omega_t} u_t\phi\, \mathrm{d}x + \int_{\Omega_t} (\Delta^2 u)\phi\, \mathrm{d}x = \int_{\Omega_t} |\nabla u|^p\phi\, \mathrm{d}x.$$

Integrating the term $\int_{\Omega_t}(\Delta^2 u)\phi\,\mathrm{d}x$ by parts enough times, and using that ϕ is an eigenfunction we find that

$$\int_\Omega (\Delta^2 u)\phi\,\mathrm{d}x = \int_\Omega u(\Delta^2\phi)\,\mathrm{d}x = (\lambda_1)^2\int_\Omega u\phi\,\mathrm{d}x.$$

Therefore, setting $z(t) = \int_\Omega u(x,t)\phi(x)\,\mathrm{d}x$, we have that

$$z'(t) + \lambda_1^2 z(t) = \int_\Omega |\nabla u|^p\phi(x)\,\mathrm{d}x. \tag{4.3}$$

Using Poincaré's inequality, one can then show that

$$\int_\Omega |\nabla u|^p\phi(x)\,\mathrm{d}x \geqslant c\left|\int_\Omega u(x,t)\phi\,\mathrm{d}x\right|^p, \tag{4.4}$$

where c is a positive constant. See, for example, [2, 3, 30] and the references therein for a complete proof of this estimate. For convenience, we will provide here a quick sketch of this proof: note that

$$\int_\Omega |\nabla u|\,\mathrm{d}x \leqslant \int_\Omega |\nabla u|\phi^{1/p}\phi^{-1/p}\,\mathrm{d}x \leqslant \left(\int_\Omega |\nabla u|^p\phi\,\mathrm{d}x\right)^{1/p}\left(\int_\Omega \phi^{-p'/p}\,\mathrm{d}x\right)^{1/p'}, \tag{4.5}$$

where p' is the conjugate of p.

Now,

$$|z(t)|^p \leqslant \|\phi\|^p_{L^\infty(\Omega)}\left(\int_\Omega |u|\,\mathrm{d}x\right)^p \leqslant \|\phi\|^p_{L^\infty} c\left(\int_\Omega |\nabla u|\,\mathrm{d}x\right)^p, \tag{4.6}$$

where Poincaré's inequality was used. Observe that (4.4) follows from (4.5) and (4.6).

Combining (4.4) with (4.3) we get that

$$z'(t) + \lambda_1^2 z(t) \geqslant c|z(t)|^p. \tag{4.7}$$

Now if $z(0) \geqslant (\lambda_1^2/c)^{1/(p-1)} \equiv M$ then it follows from the inequality above that $z(t) \geqslant (\lambda_1^2/c)^{1/(p-1)}$, $\forall t > 0$.

Therefore, we have that there exist constants a_0 and a_1 such that

$$z'(t) \geqslant a_0(z - a_1)^p$$

and $z(0) \geqslant a_1$. Hence,

$$z(t) \geqslant a_1 + \left(\frac{1}{A - Bt}\right)^{1/(p-1)},$$

where $A = (z(0) - a_1)^{(1-p)}$ and $B = (p-1)a_0$.

Consequently, either $z(t)$ ceases to exit before the time $T_{\text{blow-up}} = A/B$ or it becomes infinite at the time $T_{\text{blow-up}} = A/B$.

Remark 4. We observe that a similar proof is used in [30] to prove the blow-up of certain solutions to problem (1.8)–(1.10), for $p > 2$. However, since each solution to problem (1.8)–(1.10), for $p > 2$, satisfies a maximum principle, the $L^\infty(\Omega)$ norm of the solution remains finite for as long as the solution exists. Since the solution, nonetheless, blows up, it follows that some of the derivatives of the solution must become singular in a finite time.

Next we will show that under certain conditions genuine blow-up, in the sense of [12], of the solution does occur.

Theorem 4. *Assume $n \leqslant 3$ and $2 < p < (n+8)/(n+2)$. Let $u_0 \in L^2(\Omega)$ satisfy $\int_\Omega u_0(x)\phi(x)\,\mathrm{d}x > M = M(\Omega, p) > 0$ sufficiently large. Then, for any maximal weak solution u of the problem (2.2)–(2.4), it holds $T* < \infty$. Furthermore, u satisfies*

$$\lim_{t\to T^*} \|u(\cdot,t)\|_{0,2} = \infty \qquad \textit{and therefore } \lim_{t\to T^*} \|u(\cdot,t)\|_\infty = \infty. \tag{4.8}$$

Proof. This is an immediate consequence of proposition 1. Indeed, if $T^* = \infty$ it would imply that $u \in L^\infty(0, T; L^2(\Omega))$ for all $T < \infty$, which clearly contradicts proposition 1. As for (4.8) it directly follows from theorem 1.

Remark 5. It is clear from part 3 of theorem 2 (see also (3.68)) and theorem 4 that for as long as the $L^\infty(\Omega)$ norm of the solution u remains finite, the $W^{2,2}(\Omega)$ norm of the solution remains finite as well. That is, the derivatives do not become singular before the $L^\infty(\Omega)$ norm blows up. This is different from the behaviour of the singular solutions to problem (1.8)–(1.10) as is observed in remark 4, and as we have already mentioned in section 1.

Remark 6. Estimate (4.4) does not hold for the case $p \leqslant 2$. First, we present a direct proof for the case of $p < 2$. To see this, let $n = 1$ and $\Omega = (0, \pi)$, so that $\phi(x) = \sin(x)$. Now consider the sequence of functions $u_k(x) = kx$, for $0 \leqslant x \leqslant 1/k$, $u_k(x) = 1$, for $1/k \leqslant x \leqslant \pi - 1/k$, and $u_k(x) = (\pi - x)k$ for $\pi - 1/k \leqslant x \leqslant \pi$. It is then easy to see that

$$\lim_{k\to\infty} \int_\Omega |\nabla u_k|^p \phi(x)\,\mathrm{d}x = 0 \tag{4.9}$$

while

$$\lim_{k\to\infty} \int_\Omega u_k \phi(x)\,\mathrm{d}x = 2 \neq 0. \tag{4.10}$$

For the case $p = 2$ we give an indirect proof. Suppose (4.4) is true for $p = 2$. Following the work of Souplet [30] and applying this inequality one can show that certain solutions to equation (1.10) (for $p = 2$), i.e. equation (1.7), blow-up in finite time. This is certainly not true, because, as we have mentioned in section 1, the scalar Burgers equation (1.7) has global regularity.

Remark 7. The theorem of this section would still hold for $\Delta^2 u$ replaced by $(-\Delta)^k u$, k integer and appropriate boundary conditions.

5. Global existence of a radial solution in an annulus with Neumann boundary conditions

In this section, we will consider the case where Ω is an annulus. We will assume that

$$\Omega = \{x \in \mathbb{R}^2 \text{ such that } 0 < r_0 < \|x\| < R_1\}, \tag{5.1}$$

where r_0 and R_1 are given positive numbers.

We will then consider problem (2.2)–(2.4), in Ω with $p = 2$, but with Neumann boundary conditions, i.e.

$$u_t + \Delta^2 u = |\nabla u|^2 \qquad \text{in } Q_\infty, \tag{5.2}$$

$$\frac{\partial u}{\partial r} = \frac{\partial \Delta u}{\partial r} = 0 \qquad \text{on } \Gamma_\infty, \tag{5.3}$$

$$u(x, 0) = u_0(x) \qquad \text{in } \Omega. \tag{5.4}$$

Notice that now $\Gamma = \{x \text{ such that } \|x\| = r_0 \text{ or } \|x\| = R_1\}$. In this section, we will assume that the initial condition u_0 is a radial function, i.e.

$$u_0(x) = u_0(r). \tag{5.5}$$

Following a procedure similar to the one introduced in section 3, one can show the short-time existence and uniqueness of solutions to system (5.2)–(5.4) for any smooth initial condition (not necessarily a radial function). Since the above problem is equivariant under rotation and since u_0 is assumed to be a radial function, one can search for radial solutions as an ansatz to this end and obtain the following reduced radial system of PDEs:

$$u_t + \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 u = \left|\frac{\partial u}{\partial r}\right|^2 \qquad \text{in } Q_\infty, \tag{5.6}$$

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) u = 0 \qquad \text{on } \Gamma_\infty, \tag{5.7}$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega, \tag{5.8}$$

$$u_0(x) = u_0(r). \tag{5.9}$$

Once we establish the existence of solution to the reduced radial problem (5.6)–(5.9), by the uniqueness of the solutions to problem (5.2)–(5.4), we may conclude that this radial solution is the only solution to problem (5.2)–(5.4). Later, we show that this radial solution exists globally in time, and by this, we establish the global existence and uniqueness of solutions to problem (5.2)–(5.4) with radial initial data. Based on the above observation, we will deal, from now on, only with the ansatz radial solution and the reduced radial system (5.6)–(5.9).

Next, we will derive some *a priori* estimates.

Lemma 5. *Let $u(r,t)$ be a radially symmetric solution to problem (5.2)–(5.4), so that (5.6)–(5.9) hold. Then,*

$$\int_{r_0}^{R_1} u_r^2(r,t)\,\mathrm{d}r \leqslant \mathrm{e}^{c\cdot t} \int_{r_0}^{R_1} u_r^2(r,0)\,\mathrm{d}r \tag{5.10}$$

and

$$\int_0^T \int_{r_0}^{R_1} u_{rrr}^2(r,t)\,\mathrm{d}r\,\mathrm{d}t \leqslant 2\mathrm{e}^{c\cdot T} \int_{r_0}^{R_1} u_r^2(r,0)\,\mathrm{d}r, \tag{5.11}$$

where the constant c depends only on the domain Ω.

Proof of the lemma. Multiplying equation (5.2) (or (5.6)) by $(1/r)u_{rr}$ and integrating by parts in space we get

$$\frac{1}{2}\int_{r_0}^{R_1} u_r^2(r,t)\,\mathrm{d}r + \int_0^t \int_{r_0}^{R_1} \nabla(\Delta u)\nabla\left(\frac{1}{r}u_{rr}\right) r\,\mathrm{d}r\,\mathrm{d}s = \frac{1}{2}\int_{r_0}^{R_1} u_r^2(r,0)\,\mathrm{d}r. \tag{5.12}$$

Here, we have used the following immediate consequence of our boundary conditions (5.3):

$$\int_{r_0}^{R_1} u_r^2(r,t) u_{rr}(r,t)\,\mathrm{d}r = 0. \tag{5.13}$$

Assuming that the function u is radially symmetric and that the Laplace operator restricted to such functions is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

and that

$$\nabla = \frac{\vec{r}}{|\vec{r}|}\frac{\partial}{\partial r}$$

we find that

$$\int_{r_0}^{R_1} \nabla(\Delta u)\nabla\left(\frac{1}{r}u_{rr}\right) r\,\mathrm{d}r = \int_{r_0}^{R_1}\left(u_{rrr}+\frac{u_{rr}}{r}-\frac{u_r}{r^2}\right)\left(u_{rrr}-\frac{u_{rr}}{r}\right)\mathrm{d}r$$
$$= \int_{r_0}^{R_1}(u_{rrr})^2\,\mathrm{d}r - \int_{r_0}^{R_1}\left(\frac{u_{rr}}{r}\right)^2\mathrm{d}r - \int_{r_0}^{R_1}\left(u_{rrr}\frac{u_r}{r^2}\right)\mathrm{d}r$$
$$+ \int_{r_0}^{R_1}\left(\frac{u_r u_{rr}}{r^3}\right)\mathrm{d}r. \tag{5.14}$$

It then follows from (5.12) that

$$\frac{1}{2}\int_{r_0}^{R_1} u_r^2(r,t)\,\mathrm{d}r + \int_0^t\int_{r_0}^{R_1}(u_{rrr})^2\,\mathrm{d}r\,\mathrm{d}s = \frac{1}{2}\int_{r_0}^{R_1}u_r^2(r,0)\,\mathrm{d}r + \int_0^t\int_{r_0}^{R_1}\left(\frac{u_{rr}}{r}\right)^2\mathrm{d}r\,\mathrm{d}s$$
$$+ \int_0^t\int_{r_0}^{R_1}\left(u_{rrr}\frac{u_r}{r^2}\right)\mathrm{d}r\,\mathrm{d}s - \int_0^t\int_{r_0}^{R_1}\left(\frac{u_r u_{rr}}{r^3}\right)\mathrm{d}r\,\mathrm{d}s. \tag{5.15}$$

Using the Cauchy–Schwarz and Young inequalities to estimate the term $|\int_{r_0}^{R_1}(u_{rrr}(u_r/r^2))\,\mathrm{d}r|$ and integrating the term $\int_{r_0}^{R_1}(u_r u_{rr}/r^3)\,\mathrm{d}r$ by parts in space we get from (5.15) that

$$\frac{1}{2}\int_{r_0}^{R_1} u_r^2(r,t)\,\mathrm{d}r + \frac{1}{2}\int_0^t\int_{r_0}^{R_1}(u_{rrr})^2\,\mathrm{d}r\,\mathrm{d}s \leqslant \frac{1}{2}\int_{r_0}^{R_1}u_r^2(r,0)\,\mathrm{d}r + \int_0^t\int_{r_0}^{R_1}\left(\frac{u_{rr}}{r}\right)^2\mathrm{d}r\,\mathrm{d}s$$
$$+\frac{1}{2}\int_0^t\int_{r_0}^{R_1}\left(\frac{u_r}{r^2}\right)^2\mathrm{d}r\,\mathrm{d}s + \frac{3}{2}\int_0^t\int_{r_0}^{R_1}\left(\frac{u_r^2}{r^4}\right)\mathrm{d}r\,\mathrm{d}s. \tag{5.16}$$

Next, we will estimate the term $\int_{r_0}^{R_1}(u_{rr}/r)^2\,\mathrm{d}r$. Working in $W_0^{2,2}(r_0,R_1)$ we find from interpolation inequalities of the type of (3.7) that

$$\int_{r_0}^{R_1}(u_{rr})^2\,\mathrm{d}r \leqslant c_1\left(\int_{r_0}^{R_1}u_r^2\,\mathrm{d}r\right)^{1/2}\left(\int_{r_0}^{R_1}u_{rrr}^2\,\mathrm{d}r\right)^{1/2}, \tag{5.17}$$

where the constant c_1 depends only on the domain. Using the Young inequality we then get

$$\int_{r_0}^{R_1}\left(\frac{u_{rr}}{r}\right)^2\mathrm{d}r \leqslant \frac{4c_1^2}{r_0^4}\int_{r_0}^{R_1}u_r^2\,\mathrm{d}r + \frac{1}{4}\int_{r_0}^{R_1}u_{rrr}^2\,\mathrm{d}r. \tag{5.18}$$

It then follows from (5.16) that

$$\frac{1}{2}\int_{r_0}^{R_1} u_r^2(r,t)\,\mathrm{d}r + \frac{1}{4}\int_0^t\int_{r_0}^{R_1}(u_{rrr})^2\,\mathrm{d}r\,\mathrm{d}s \leqslant \frac{1}{2}\int_{r_0}^{R_1}u_r^2(r,0)\,\mathrm{d}r$$
$$+\left(\frac{4c_1^2}{r_0^4}+\frac{2}{r_0^4}\right)\int_0^t\int_{r_0}^{R_1}{u_r}^2\,\mathrm{d}r\,\mathrm{d}s. \tag{5.19}$$

Setting

$$c = 2\left(\frac{4c_1^2}{r_0^4}+\frac{2}{r_0^4}\right),$$

we then have that

$$\int_{r_0}^{R_1}u_r^2(r,t)\,\mathrm{d}r \leqslant \int_{r_0}^{R_1}u_r^2(r,0)\,\mathrm{d}r + c\int_0^t\int_{r_0}^{R_1}u_r^2\,\mathrm{d}r\,\mathrm{d}s, \tag{5.20}$$

from which we deduce (5.10) by Gronwall's inequality.

The estimate (5.11) can be deduced from (5.10) and (5.19) by elementary calculations.

Lemma 6. *Let $u(r,t)$ be a radially symmetric solution to problem (5.2)–(5.4). Then,*

$$\int_{r_0}^{R_1} u^2(r,t)\,\mathrm{d}r \leqslant \mathrm{e}^t \frac{R_1}{r_0}\int_{r_0}^{R_1} u^2(r,0)\,\mathrm{d}r + (t\mathrm{e}^t+1)\frac{16c^2R_1^2}{r_0^2}\mathrm{e}^{4ct}\left(\int_{r_0}^{R_1} u_r^2(r,0)\,\mathrm{d}r\right)^3, \quad (5.21)$$

where the constant c depends only on the domain Ω.

Proof of the lemma. Multiplying equation (5.2) by u and integrating by parts in space we get

$$\frac{1}{2}\int_{r_0}^{R_1} u^2(r,t)r\,\mathrm{d}r + \int_0^t\int_{r_0}^{R_1}(\Delta u)^2 r\,\mathrm{d}r\,\mathrm{d}s = \frac{1}{2}\int_{r_0}^{R_1} u^2(r,0)r\,\mathrm{d}r + \int_0^t\int_{r_0}^{R_1} u(u_r)^2 r\,\mathrm{d}r\,\mathrm{d}s. \quad (5.22)$$

We will estimate the last term on the right-hand side of the above equality:

$$\begin{aligned}\left|\int_0^t\int_{r_0}^{R_1} u(u_r)^2 r\,\mathrm{d}r\,\mathrm{d}s\right| &\leqslant R_1\int_0^t\left(\|u(\cdot,s)\|_{0,\infty}\int_{r_0}^{R_1}(u_r)^2\,\mathrm{d}r\right)\mathrm{d}s\\ &\leqslant R_1\mathrm{e}^{ct}\left(\int_{r_0}^{R_1} u_r^2(r,0)\,\mathrm{d}r\right)\int_0^t\|u(\cdot,s)\|_{0,\infty}\,\mathrm{d}s, \quad (5.23)\end{aligned}$$

where (5.10) was used. Using embedding results for Sobolev spaces and interpolation inequalities (see, e.g. [1]) we have that

$$\|u(\cdot,t)\|_{0,\infty} \leqslant c\left(\int_{r_0}^{R_1} u^2(r,t)\,\mathrm{d}r\right)^{1/2}\left(\int_{r_0}^{R_1} u_{rrr}^2(r,t)\,\mathrm{d}r\right)^{1/2}. \quad (5.24)$$

Using (5.11) and the Cauchy–Schwarz inequality we then find that

$$\int_0^t\|u(\cdot,t)\|_{0,\infty}\,\mathrm{d}s \leqslant c\left(\int_0^t\int_{r_0}^{R_1} u^2(r,t)\,\mathrm{d}r\,\mathrm{d}s\right)^{1/2}(2\mathrm{e}^{c\cdot t})^{1/2}\left(\int_{r_0}^{R_1} u_r^2(r,0)\,\mathrm{d}r\right)^{1/2}. \quad (5.25)$$

We then get from (5.22)–(5.25) that

$$\begin{aligned}\int_{r_0}^{R_1} u^2(r,t)\,\mathrm{d}r &\leqslant \frac{R_1}{r_0}\int_{r_0}^{R_1} u^2(r,0)\,\mathrm{d}r\\ &\quad+\frac{\sqrt{2}cR_1}{r_0}\mathrm{e}^{2ct}\left(\int_{r_0}^{R_1} u_r^2(r,0)\,\mathrm{d}r\right)^{3/2}\left(\int_0^t\int_{r_0}^{R_1} u^2(r,t)\,\mathrm{d}r\,\mathrm{d}s\right)^{1/2}\\ &\leqslant \frac{R_1}{r_0}\int_{r_0}^{R_1} u^2(r,0)\,\mathrm{d}r + \frac{8c^2R_1^2}{r_0^2}\mathrm{e}^{4ct}\left(\int_{r_0}^{R_1} u_r^2(r,0)\,\mathrm{d}r\right)^3\\ &\quad+\left(\int_0^t\int_{r_0}^{R_1} u^2(r,s)\,\mathrm{d}r\,\mathrm{d}s\right). \quad (5.26)\end{aligned}$$

By integration it follows that

$$\int_0^t\int_{r_0}^{R_1} u^2(r,s)\,\mathrm{d}r\,\mathrm{d}s \leqslant (\mathrm{e}^t-1)\frac{R_1}{r_0}\int_{r_0}^{R_1} u^2(r,0)\,\mathrm{d}r + t\mathrm{e}^t\frac{8c^2R_1^2}{r_0^2}\mathrm{e}^{4ct}\left(\int_{r_0}^{R_1} u_r^2(r,0)\,\mathrm{d}r\right)^3. \quad (5.27)$$

The lemma immediately follows from substituting (5.27) in (5.26).

We are now ready to give the main theorem of this section.

Theorem 5. *Assume that the initial condition u_0 is a radially symmetric function and that $u_0 \in W^{1,2}(\Omega)$. Then, there exists a function $u(r,t)$ defined for all $t > 0$, such that*

$$u \in L^\infty_{\mathrm{loc}}([0,\infty); W^{1,2}(\Omega)), \tag{5.28}$$

$$u \in L^2_{\mathrm{loc}}([0,\infty); W^{3,2}(\Omega)) \tag{5.29}$$

and u is the unique radially symmetric solution to problem (5.2)–(5.4). Furthermore, u satisfies estimates (5.10), (5.11) and (5.21).

Proof of theorem 5. We will first show the existence of a solution for a short time. Then, we will show that such a solution in fact exists for all time.

We will use the Galerkin method to show the existence of a solution for a short time. For this purpose, we let w_i, $i = 1, 2, \ldots$, be an orthonormalized basis for $L^2(\Omega)$. It is well known that we can choose the special basis made of functions that satisfy

$$\Delta\psi_i = \mu_i\psi_i \quad \text{in } \Omega, \qquad \frac{\partial\psi_i}{\partial r} = 0 \quad \text{on } \partial\Omega. \tag{5.30}$$

It is easy to see that these functions are radially symmetric. We proceed as we did in the proof of theorem 1 and use the same notation as we did there. For k fixed we look for a function $u_k = \sum_{i=1}^k a_{i,k}(t) w_i(r)$ such that

$$\int_\Omega \frac{\partial u_k}{\partial t}\phi\,\mathrm{d}x + \int_\Omega \Delta u \Delta\phi\,\mathrm{d}x = \int_\Omega (\nabla u_k)^2\phi\,\mathrm{d}x, \tag{5.31}$$

$$a_{i,k}(0) = \int u_0 w_i\,\mathrm{d}x \qquad i = 1, \ldots, k \tag{5.32}$$

for all $\phi \in E_k$.

As before, the existence and uniqueness of u_k follows from the Picard theorem. That the function u_k satisfies the boundary conditions follows from the choice of the special basis.

Since $u_k \in E_k$ it can be used as a test function in (5.31). Setting $\phi = u_k$ in (5.31), proceeding as we did in the proof of 3.4, and using the estimate

$$\|u\|_{1,4} \leqslant c\|u\|_{0,2}^{1/4}(\|u\|_{0,2} + \|\Delta u\|_{0,2})^{3/4}, \tag{5.33}$$

we get that

$$\int_{\Omega_T} u_k^2(x,T)\,\mathrm{d}x + \int_0^T\int_\Omega (\Delta u_k)^2\,\mathrm{d}x\,\mathrm{d}s \leqslant \int_\Omega u_k^2(x,0)\,\mathrm{d}x \tag{5.34}$$

$$+ c\int_0^T \left(\int_\Omega u_k^2\,\mathrm{d}x\right)^3 \mathrm{d}s, \tag{5.35}$$

where the constant c is independent of k.

Using differential inequalities as we did in the proof of lemma 1, we then find that there exists a time $T^* > 0$, which depends on the initial condition $u(x,0)$, such that the sequence u_k is uniformly bounded in $L^\infty((0,T^*); L^2(\Omega)) \cap L^2((0,T^*); W^{2,2}(\Omega))$. We then deduce that there exists a subsequence u_k and a function u such that

$$u_k \longrightarrow u \quad \text{as} \quad k \to \infty \qquad \text{in } L^\infty_{\text{weak-star}}(0,t; L^2_{\text{weak}}(\Omega)), \tag{5.36}$$

$$u_k \longrightarrow u \quad \text{as} \quad k \to \infty \qquad \text{in } L^2_{\text{weak}}(0,t; W^{2,2}_{\text{weak}}(\Omega)). \tag{5.37}$$

Then, one needs to show that the limit function does satisfy the partial differential equation. This can be done in the same way as we did in the proof of theorem 1. Since no new difficulty arises we will not repeat that proof here. Similarly, the uniqueness is handled in the same way as in theorem 2 and we will therefore not repeat its proof here.

That the solution exists for all time is a direct consequence of the *a priori* estimates (5.10) and (5.21).

Remark 8. The global existence result of this section is also true in space dimension three in a shell domain between two concentric spheres. The proof is similar to the one we used in the case of space dimension two.

Remark 9. Using standard energy methods and estimates one can easily show the global regularity of solutions to our problem (5.2)–(5.4) in the one-dimensional case in the interval $\Omega = (a, b)$.

Remark 10. Here, we were unable to obtain global existence of radial solutions to problem (5.2)–(5.4) for the case where Ω is a disk/ball. However, in a forthcoming paper we will study the global existence of radial solutions to the modified equation

$$\frac{\partial u}{\partial t} + \Delta^2 u = |r^{n-1}\nabla u|^2 \tag{5.38}$$

in a disk/ball. Whether there is a connection between the global existence of radial solutions to (5.38) and the potential formation of singularity at the origin for certain radial solutions to problem (5.2)–(5.4) for the case where Ω is a disk/ball is a subject of future research.

It is worth observing that the restriction of equation (5.38) to radial solutions might be viewed as a higher dimensional generalization of the following one-dimensional equation:

$$\frac{\partial u}{\partial t} + u_{xxxx} = (u_x)^2.$$

To see this, the term u_x^2 may be viewed, morally speaking, as the the anti-derivative of the one-dimensional Laplacian of u squared, subject to the boundary condition (5.7). That is, $u_x^2(x) = (\int^x u_{yy}\,\mathrm{d}y)^2$. Using the radial symmetry and the boundary condition (5.7), then, the corresponding analogue of the above integral formula in space dimension n would be

$$|r^{n-1}\nabla u|^2 = \left(\int \frac{1}{r^{n-1}}(r^{n-1}u_r)_r r^{n-1}\,\mathrm{d}r\right)^2. \tag{5.39}$$

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