Izvestiya: *Mathematics* **69**:3 439–486

Izvestiya RAN: *Ser. Mat.* **69**:3 3–54
DOI 10.1070/IM2005v069n03ABEH000535

Harmonic analysis of causal operators and their spectral properties

A. G. Baskakov and I. A. Krishtal

Abstract. The definition and study of causal operators are based on the representation theory of group algebras. We study the structure of the spectra of causal operators, obtain conditions for causal invertibility and state criteria for a causal operator to belong to the radical.

§ 1. Introduction

Causal (or Volterra) linear operators are used in system theory [1] and in the study of various classes of functional differential equations [2]–[5]. They are usually defined in terms of chains of invariant subspaces indexed by points of the set $\mathbb{R}$ of real numbers or the set $\mathbb{Z}$ of integers. It should be mentioned that there are papers dealing with Volterra operators on Hilbert spaces whose authors construct the chains of invariant subspaces rather than axiomatize their existence [6].

In this paper we define and study causal operators using the representation theory of Abelian groups (Banach modules over group algebras) and, in particular, their spectral theory. The class of operators defined here contains not only many classes of causal operators studied earlier (see [1]–[5] and the references there), but also some new classes.

In this paper we make systematic use of the concept of the Beurling spectrum of vectors and operators in representation spaces (Banach modules) and thereby develop a technique for the investigation of linear operators In particular, the definition of causal operators is made in terms of the Beurling spectrum: causal operators are defined to be operators whose Beurling spectrum (with respect to some representation in the space of operators) is contained in a certain semigroup (of operators with a "lower-triangular matrix").

The main results of this paper deal with the problem of causal invertibility of operators (in particular, with the study of the structure of inverse operators), the structure of their spectra, and conditions under which they belong to the radical of the algebra of causal operators.

This research was carried out with the financial support of the Russian Foundation for Basic Research (grant no. 04-01-00141).

AMS 2000 *Mathematics Subject Classification.* 47E05, 46H25, 47A35, 47D03, 47A06, 34G10, 47A10.

§ 2. Banach modules over group algebras

Let X be a complex Banach space and let $\operatorname{End} X$ be the Banach algebra of bounded linear operators acting in X. Let $\mathbb{G}$ be a locally compact Abelian group and $\widehat{\mathbb{G}}$ its dual, the group of continuous unitary characters of $\mathbb{G}$ (see [7]–[9]). Let us note that the binary operations on $\mathbb{G}$ and $\widehat{\mathbb{G}}$ are usually written in additive form. We denote by $L_1(\mathbb{G})$ the Banach algebra of (equivalence classes of) complex functions defined on $\mathbb{G}$ and integrable with respect to the Haar measure on $\mathbb{G}$. The role of multiplication is played by the convolution of functions. We denote by $\hat{f}\colon \widehat{\mathbb{G}} \to \mathbb{C}$ the Fourier transform of $f \in L_1(\mathbb{G})$.

We assume that the space X is a *non-degenerate* Banach $L_1(\mathbb{G})$-module [8], [10] whose structure is *associated* with some isometric representation $T\colon \mathbb{G} \to \operatorname{End} X$. This means that conditions (i) and (ii) stated in the following assumption hold for X (in this paper we consider only $L_1(\mathbb{G})$-modules for which this assumption holds).

Assumption 2.1. *The following three conditions hold for the Banach $L_1(\mathbb{G})$-module X:*

(i) *the equation $fx = 0$, $f \in L_1(\mathbb{G})$, implies that $x \in X$ is equal to zero* (*that is*, *X is non-degenerate*),

(ii)

$$T(g)(fx) = \bigl(S(g)f\bigr)x = f\bigl(T(g)x\bigr) \tag{2.1}$$

for all $f \in L_1(\mathbb{G})$, $x \in X$ and $g \in \mathbb{G}$, where $S(g)$ is the shift operator by $g \in \mathbb{G}$ on $L_1(\mathbb{G})$, that is, $S(g)f(s) = f(s+g)$, $s, g \in \mathbb{G}$, $f \in L_1(\mathbb{G})$ (*that is, the module structure on X is associated with the representation $T\colon \mathbb{G} \to \operatorname{End} X$*),

(iii) $\|fx\| \leqslant \|f\|_1 \|x\|$, $f \in L_1(\mathbb{G})$, $x \in X$, *where $\|f\|_1$ is the norm of f in $L_1(\mathbb{G})$.*

Let $T\colon \mathbb{G} \to \operatorname{End} X$ be a strongly continuous isometric representation. Then the formula

$$T(f)x = fx = \int_{\mathbb{G}} f(g)T(-g)x\,dg, \qquad f \in L_1(\mathbb{G}), \quad x \in X, \tag{2.2}$$

defines on X the structure of a Banach $L_1(\mathbb{G})$-module for which Assumption 2.1 holds, and this structure is associated with T.

Lemma 2.2. *Every* (*non-degenerate*) *Banach $L_1(\mathbb{G})$-module has precisely one representation associated with it.*

Proof. Let X be a (Banach) $L_1(\mathbb{G})$-module and let $T_1, T_2\colon \mathbb{G} \to \operatorname{End} X$ be representations associated with it. Consider an arbitrary $x \in X$ and $g \in \mathbb{G}$. Let $x_k = T_k(g)x$, $k = 1, 2$. It follows from (2.1) that

$$fx_1 = T_1(g)(fx) = S(g)(fx) = f\bigl(T_2(g)x\bigr) = fx_2, \qquad f \in L_1(\mathbb{G}),$$

that is, $f(x_1 - x_2) = 0$ for all $f \in L_1(\mathbb{G})$. Since X is non-degenerate, we have $x_1 = T_1(g)x = x_2 = T_2(g)x$, which completes the proof of the lemma.

Remark 2.3. We denote by (X, T) the Banach $L_1(\mathbb{G})$-module X with associated representation T.

Remark 2.4. Instead of requiring that the associated representation $T\colon \mathbb{G} \to \operatorname{End} X$ for $L_1(\mathbb{G})$-modules be isometric, we can require that it be bounded. In this case T is isometric with respect to the norm $\|x\|_* = \sup_{g\in\mathbb{G}} \big\|T(g)x\big\|$, $x \in X$, which is equivalent to the original norm on X.

Consider the Banach algebra $M(\mathbb{G})$ of bounded Borel measures on $\mathbb{G}$, where the role of multiplication is played by the convolution of measures, with the canonical embedding of $L_1(\mathbb{G})$ in $M(\mathbb{G})$. Assume that the (isometric) representation T: $\mathbb{G} \to \operatorname{End} X$ is strongly continuous (that is, the map $g \mapsto T(g)x$ is continuous for all $x \in X$). The formula

$$T(\mu)x = \mu x = \int_{\mathbb{G}} T(-g)x\,\mu(dg), \qquad \mu \in M(\mathbb{G}), \quad x \in X \tag{2.3}$$

(see also formula (2.2)) defines a homomorphism T from $M(\mathbb{G})$ to $\operatorname{End} X$. The use of the same symbol T is justified by the following fact: if μ_g is the Dirac measure concentrated at the point $(-g)$ of $\mathbb{G}$, then (2.3) implies that $T(\mu_g) = T(g)$.

Formula (2.3) defines on X the structure of a Banach $M(\mathbb{G})$-module and $\|\mu x\| \leqslant \|\mu\|\,\|x\|$ for all $\mu \in M(\mathbb{G})$ and $x \in X$. A similar, but more general, approach is described below in Example 2.12.

Remark 2.5. If the representation $T\colon \mathbb{G} \to \operatorname{End} X$ is not assumed to be strongly continuous and $M_d(\mathbb{G})$ is the subalgebra of $M(\mathbb{G})$ formed by the discrete measures, then the same formula (2.3) defines on X the structure of a Banach $M_d(\mathbb{G})$-module. We can assume that $M_d(\mathbb{G})$ coincides with (is isomorphic to) the algebra $L_1(\mathbb{G}_d)$, where $\mathbb{G}_d$ is the group $\mathbb{G}$ equipped with the discrete topology. The group $\widehat{\mathbb{G}_d}$ dual to it is the Bohr compactification (Bohr compactum) of $\widehat{\mathbb{G}}$. Hence, X is an $L_1(\mathbb{G}_d)$-module. This module will be denoted by (X, T_d), where $T_d\colon \mathbb{G}_d \to \operatorname{End} X$, $T_d(g) = T(g)$, $g \in \mathbb{G}_d$. It follows from Assumption 2.1 that X is an $\big(L_1(\mathbb{G}) \oplus M_d(\mathbb{G})\big)$-module.

Definition 2.6. A vector x of the Banach $L_1(\mathbb{G})$-module (X, T) is said to be *T-continuous* if the function

$$\varphi_x\colon \mathbb{G} \to X, \qquad \varphi_x(g) = T(g)x, \quad g \in \mathbb{G},$$

is continuous at the zero of $\mathbb{G}$ (and so uniformly continuous on $\mathbb{G}$).

We denote the set of T-continuous vectors of X by X_c or $(X, T)_c$. This set is a closed *submodule* of X, that is, X_c is a closed linear subspace of X invariant under the operators $T(f)$, $T(g)$, $f \in L_1(\mathbb{G})$, $g \in \mathbb{G}$.

Lemma 2.7. *Let* (X, T) *be a Banach* $L_1(\mathbb{G})$*-module. Then* (2.2) *holds for every* $x \in X_c$.

Proof. It is clear that formula (2.2) defines the structure of a Banach $L_1(\mathbb{G})$-module on X_c. It follows from (2.2) that

$$T_c\colon \mathbb{G} \to \operatorname{End} X_c, \qquad T_c(g)x = T(g)x, \qquad x \in X_c,$$

is a representation associated with the $L_1(\mathbb{G})$-module X_c. By Lemma 2.2, it is unique. The lemma is proved.

In the following examples we consider the Banach modules frequently used in this paper and introduce some basic function spaces.

Example 2.8. Let Σ be the σ-algebra of Borel subsets of $\widehat{\mathbb{G}}$ and $E\colon \Sigma \to \operatorname{End} X$ a bounded countably additive projector-valued measure (see [11]). Then the formula

$$T(g)x = \int_{\widehat{\mathbb{G}}} \gamma(g)\, dE(\gamma)x, \qquad x \in X, \tag{2.4}$$

defines a bounded (isometric by Remark 2.4) strongly continuous representation $T\colon \mathbb{G} \to \operatorname{End} X$. Hence, X is an $L_1(\mathbb{G})$-module. Formulae (2.2) and (2.4) imply that

$$fx = \int_{\mathbb{G}} f(g)T(-g)x\, dg = \int_{\widehat{\mathbb{G}}} \hat{f}(\gamma)\, dE(\gamma)x \tag{2.5}$$

for all $f \in L_1(\mathbb{G})$ and $x \in X$.

The next example and Example 2.10 with $\mathbb{G} = \mathbb{Z}$ are special cases of this.

Example 2.9. Let $\mathcal{E} = \{E_n, n \in \Omega\}$ be a family of projectors on the Banach space X, where $\Omega \subseteq \mathbb{Z}$ is a non-empty (possibly finite) subset that is a resolution of the identity, which means that $E_iE_j = 0$ for $i \neq j$ and for every $x \in X$ the series $\sum_{k\in\Omega} E_kx$ converges unconditionally to x. Hence, the quantity

$$C(\mathcal{E}) = \sup\left\| \sum_{k\in\Omega} \gamma_k E_k \right\| < \infty$$

is finite (the equivalent renormalization mentioned above enables us to assume that this quantity is equal to unity). Here the supremum is taken over the finite sets of complex numbers (γ_k) belonging to $\mathbb{T} = \{\gamma \in \mathbb{C}\colon |\gamma| = 1\}$ (the algebraic operation in this group is written in multiplicative form). The formula

$$U(\gamma)x = \sum_{n\in\Omega} \gamma^n E_n x, \qquad x \in X, \quad \gamma \in \mathbb{T}, \tag{2.6}$$

defines a bounded (isometric after renormalization) strongly continuous representation $U\colon \mathbb{T} \to \operatorname{End} X$ of the compact group $\mathbb{T}$ (its dual group is identified with $\mathbb{Z}$). The structure of an $L_1(\mathbb{T})$-module on X associated with U is defined by the formula

$$U(f)x = fx = \sum_{n\in\Omega} \hat{f}(n)E_n x, \qquad f \in L_1(\mathbb{T}), \quad x \in X. \tag{2.7}$$

Example 2.10. Let $X = \mathcal{F}(\Omega, Y)$ be a Banach space of functions defined on the set $\Omega \subseteq \widehat{\mathbb{G}}$ of positive Haar measure $\mu(\Omega)$ that take values in a Banach space Y which is one of the following spaces. We denote by $L_p(\Omega, Y)$, $p \in [1, \infty]$, the Banach space of functions measurable and integrable together with their pth powers (essentially bounded if $p = \infty$). The norms in these spaces are defined by the formulae

$$\|x\|_p = \left(\int_{\Omega} \|x(g)\|^p\, dg \right)^{1/p}, \qquad p \in [1, \infty),$$

$$\|x\|_\infty = \operatorname*{ess\,sup}_{g\in\Omega} \|x(g)\|, \qquad p = \infty.$$

If $\mathbb{G}$ is a compact group, then $L_p(\Omega, Y)$ is the Banach space of sequences (finite sets, if Ω is finite) of vectors of Y and will be denoted by $l_p(\Omega, Y)$. We denote by $C_b(\Omega, Y)$ and $C_{ub}(\Omega, Y)$ the subspaces of $L_\infty(\Omega, Y)$ formed by the continuous and uniformly continuous functions, respectively. We also consider the space $C_0(\widehat{\mathbb{G}}, Y) \subseteq C_{ub}(\widehat{\mathbb{G}}, Y)$ of continuous functions decaying at infinity (that is, as small as desired outside some compact subset of $\widehat{\mathbb{G}}$) and the space $AP(\widehat{\mathbb{G}}, Y) \subseteq C_{ub}(\widehat{\mathbb{G}}, Y)$ of almost periodic functions. We shall omit Y in the symbols denoting these spaces if $Y = \mathbb{C}$.

In the Banach space $X = \mathcal{F}(\Omega, Y)$ we consider the isometric representation $V\colon \mathbb{G} \to \operatorname{End} X$ defined by the formula

$$\big(V(g)x\big)(\gamma) = \gamma(g)x(\gamma), \qquad \gamma \in \widehat{\mathbb{G}}, \quad x \in X. \tag{2.8}$$

It is strongly continuous in each of the following spaces: $L_p(\Omega, Y)$, $p \in [1, \infty)$, $C_{ub}(\Omega, Y)$ and $AP(\widehat{\mathbb{G}}, Y)$.

The canonical identification of $\widehat{\widehat{\mathbb{G}}}$ with $\mathbb{G}$ (by Pontryagin duality) enables us to define the structure of an $L_1(\mathbb{G})$-module on the X associated with V by the formula

$$\big(V(f)x\big)(\gamma) = \hat{f}(\gamma)x(\gamma), \qquad \gamma \in \Omega, \quad f \in L_1(\mathbb{G}), \quad x \in X. \tag{2.9}$$

In particular, if $\mathbb{G} = \mathbb{T}$, Ω is an arbitrary non-empty subset of $\mathbb{Z} \simeq \widehat{T}$ and $X = \mathcal{F}(\Omega, Y)$ is one of the spaces defined above, then formula (2.9) has the form

$$(fx)(k) = \hat{f}(k)x(k), \qquad k \in \Omega. \tag{2.10}$$

Let us note that in the space of sequences $X = l_p(\Omega, Y)$ for any $p \in [1, \infty)$ there is a resolution $\{E_n, n \in \Omega\}$, $E_n \in \operatorname{End} X$, of the identity defined by the formulae

$$\begin{aligned} (E_n x)(k) &= 0, \qquad n \neq k, \\ E_n x(n) &= x(n). \end{aligned}$$

The construction in Example 2.9 shows that the module structures on X defined by formulae (2.7) and (2.10) coincide.

Example 2.11. Let $X = \mathcal{F}(\mathbb{G}, Y)$ be one of the Banach spaces introduced in Example 2.10. We consider the structure of a Banach $L_1(\mathbb{G})$-module on X defined by the formula

$$(fx)(g) = (f * x)(g) = \int_{\mathbb{G}} f(\tau)x(g - \tau)\, d\tau = \int_{\mathbb{G}} f(\tau)\big(S(-\tau)x\big)(g)\, d\tau. \tag{2.11}$$

Here $S\colon \mathbb{G} \to \operatorname{End} X$ is the isometric representation of $\mathbb{G}$ by shift operators acting on the functions belonging to X, that is,

$$\big(S(\tau)x\big)(g) = x(g + \tau), \qquad g, \tau \in \mathbb{G}, \qquad x \in X. \tag{2.12}$$

Let us note that $X_c = X$, that is, S is a strongly continuous representation in the spaces under consideration, with the exception of $L_\infty(\mathbb{G}, Y)$ if $\mathbb{G}$ is a non-discrete group.

Example 2.12. Let $\mathcal{F}$ be a closed subspace of the Banach space X^* (dual to X) of bounded linear functionals on X that has the following properties:

(1) $\|x\| = \sup_{\xi\in\mathcal{F},\|\xi\|=1}\bigl|\xi(x)\bigr|$ for all $x \in X$,

(2) the convex hull of every relatively $\mathcal{F}$-compact subset of X is relatively $\mathcal{F}$-compact.

Let there also be given an isometric representation $T\colon \mathbb{G} \to \operatorname{End} X$ with the following properties:

(i) for the adjoint representation $T^*\colon \mathbb{G} \to \operatorname{End} X^*$, defined by the formula $\bigl(T^*(g)\xi\bigr)(x) = \xi\bigl(T(g)x\bigr)$, we have $T^*(g)\mathcal{F} \subseteq \mathcal{F}$, $g \in \mathbb{G}$,

(ii) the function $g \mapsto \xi\bigl(T(g)x\bigr)\colon \mathbb{G} \to \mathbb{C}$ is continuous for all $\xi \in \mathcal{F}$ and $x \in X$.

Then for every measure $\mu \in M(\mathbb{G})$ and any $x \in X$ there is precisely one $x_\mu \in X$ such that

$$\int_{\mathbb{G}} \xi\bigl(T(-g)x\bigr)\,\mu(dg) = \xi(x_\mu)$$

for all $\xi \in \mathcal{F}$ (see [12], Ch. IV). We have a homomorphism $T\colon M(\mathbb{G}) \to \operatorname{End} X$ such that $\bigl\|T(\mu)\bigr\| \leqslant \|\mu\|$ and $\mu \in M(\mathbb{G})$. Hence, X is a Banach $M(\mathbb{G})$-module.

The structures of Banach modules in the Banach space $\operatorname{Hom}(X_1, X_2)$ of bounded linear operators acting from the Banach space X_1 to the Banach space X_2 are of special interest. It is with these structures that we deal in § 5.

§ 3. Spectral properties of vectors in Banach modules

We consider a Banach $L_1(\mathbb{G})$-module (X, T), where the representation $T\colon \mathbb{G} \to \operatorname{End} X$ is not assumed to be strongly continuous. We do assume, as always, that Assumption 2.1 holds.

Definition 3.1. The *Beurling spectrum* $\Lambda(M) = \Lambda(M, T)$ of the subset M of X is defined to be the complement of the set

$$\bigl\{\gamma \in \widehat{\mathbb{G}}\colon \exists f \in L_1(\mathbb{G})\colon \hat{f}(\gamma) \neq 0 \text{ and } fx = 0\ \forall x \in M\bigr\}$$

in $\widehat{\mathbb{G}}$. If M consists of a single vector x, then $\Lambda(M)$ is denoted by $\Lambda(x)$ (or $\Lambda(x, T)$) and has the form

$$\Lambda(x) = \bigl\{\gamma \in \widehat{\mathbb{G}}\colon fx \neq 0\ \forall f \in L_1(\mathbb{G}),\ \hat{f}(\gamma) \neq 0\bigr\}.$$

Remark 3.2. $\Lambda(M)$ coincides with the hull of the closed ideal $\operatorname{Im}(M) = \{f \in L_1(\mathbb{G})\colon fx = 0 \text{ for all } x \in M\}$, that is, $\Lambda(M) = \{\gamma \in \widehat{\mathbb{G}}\colon \hat{f}(\gamma) = 0 \text{ for all } f \in \operatorname{Im}(M)\}$ is the set of common zeros of the Fourier transforms of functions belonging to $\operatorname{Im}(M)$.

In the next lemma we state properties of the Beurling spectra of vectors. Some of these properties (for strongly continuous representations) were obtained in [13]. We shall prove the next lemma (see also [10], [12], [14], [15]) using Assumption 2.1 on the Banach $L_1(\mathbb{G})$-modules under consideration.

Lemma 3.3. *Let (X, T) be a Banach $L_1(\mathbb{G})$-module. Then*

(i) *$\Lambda(M)$ is closed for every $M \subseteq X$, and $\Lambda(M) = \varnothing \Leftrightarrow M = \{0\}$,*

(ii) *$\Lambda(Ax + By) \subseteq \Lambda(x) \cup \Lambda(y)$ for all $A, B \in \operatorname{End} X$ commuting with the $T(f)$, $f \in L_1(\mathbb{G})$,*

(iii) *$\Lambda(fx) \subseteq (\operatorname{supp} \hat{f}) \cap \Lambda(x)$ for all $f \in L_1(\mathbb{G})$ and $x \in X$,*

(iv) *$fx = 0$ if $(\operatorname{supp} \hat{f}) \cap \Lambda(x) = \varnothing$, where $f \in L_1(\mathbb{G})$ and $x \in X$,*

(v) *$fx = x$ if $\Lambda(x)$ is a compact set, and $\hat{f} = 1$ in some neighbourhood of $\Lambda(x)$,*

(vi) *if M_0 is dense in $M \subseteq X$, then $\Lambda(M) = \overline{\bigcup_{x \in M_0} \Lambda(x)}$,*

(vii) *$\Lambda(x, T_\gamma) = \Lambda(x, T) + \{\gamma\}$, where $\gamma \in \widehat{\mathbb{G}}$, and $T_\gamma(g) = \gamma(g)T(g)$, $g \in \mathbb{G}$.*

Proof. (i) The set $\Lambda(M)$ is closed (see Remark 3.2). It is clear that $\Lambda(0) = \varnothing$. If $\Lambda(M) = \varnothing$, then for any $\gamma \in \widehat{\mathbb{G}}$ there is an $f \in L_1(\mathbb{G})$ such that $\hat{f}(\gamma) \neq 0$ and $fx = 0$ for all x in M. Definition 3.1 and Remark 3.2 imply that $\operatorname{Im}(M)$ is a closed ideal of the algebra $L_1(\mathbb{G})$ invariant under shifts of functions (here we use condition (ii) in Assumption 2.1). Since the hull $\Lambda(M)$ of $\operatorname{Im}(M)$ is empty, Wiener's theorem [7]–[9] implies that $\operatorname{Im}(M)$ coincides with the whole algebra $L_1(\mathbb{G})$. Therefore, $fx = 0$ for all $f \in L_1(\mathbb{G})$. Since X is a non-degenerate module, we have $x = 0$.

(ii) If $\gamma_0 \notin \Lambda(x) \cup \Lambda(y)$, then there are $f_1, f_2 \in L_1(\mathbb{G})$ such that $\hat{f}_1(\gamma_0)\hat{f}_2(\gamma_0) \neq 0$ and $f_1 x = f_2 y = 0$. Then $\hat{f}(\gamma_0) \neq 0$ and $fx = fy = 0$ for $f = f_1 * f_2$, whence $T(f)(Ax + By) = A(fx) + B(fy) = 0$ (using the fact that $T(f)$ commutes with A and B). Hence, $\gamma_0 \notin \Lambda(Ax + By)$.

(iii) Let $\gamma_0 \notin (\operatorname{supp} \hat{f}) \cap \Lambda(x)$. Let $\varphi \in L_1(\mathbb{G})$ be such that $\widehat{\varphi}(\gamma_0) \neq 0$, $(\operatorname{supp} \widehat{\varphi}) \cap \big((\operatorname{supp} \hat{f}) \cap \Lambda(x)\big) = \varnothing$. Then $\varphi(fx) = (\varphi * f)x = 0$, whence $\gamma_0 \notin \Lambda(fx)$.

(iv) follows immediately from (i) and (iii).

(v) For every $\varphi \in L_1(\mathbb{G})$ we have $\operatorname{supp}\big(\widehat{\varphi}(\hat{f} - 1)\big) \cap \Lambda(x) = \varnothing$. It follows from (iv) that $\varphi(fx - x) = 0$. Since X is a non-degenerate module, we have $fx = x$.

(vi) Let $\Delta = \overline{\bigcup_{x \in M} \Lambda(x)}$. We have $\Delta \subseteq \Lambda(M)$ by the definition of spectrum, since $\Lambda(M)$ is a closed set. Let $\gamma_0 \notin \Delta$. If $f \in L_1(\mathbb{G})$ is such that $\hat{f}(\gamma_0) \neq 0$ and $(\operatorname{supp} \hat{f}) \cap \Delta = \varnothing$, then (iv) implies that $fx = 0$ for all $x \in M$, whence $\gamma_0 \notin \Lambda(M)$.

(vii) follows immediately from Definition 3.1.

The following definition was made independently in [16]–[18]. It plays an important role in various topics in the spectral analysis of representations of Abelian groups (see, for example, [10], [12]).

Definition 3.4. Let σ be a closed subset of $\widehat{\mathbb{G}}$. The submodule

$$X(\sigma) = \big\{x \in X \colon \Lambda(x, T) \subseteq \sigma\big\}$$

of the $L_1(\mathbb{G})$-module (X, T) is called a *spectral submodule*.

Lemma 3.5. *Every spectral submodule $X(\sigma) \subseteq (X, T)$ is closed.*

Proof. The fact that $X(\sigma)$ is a submodule follows from assertions (ii) and (iii) of Lemma 3.3. Consider an arbitrary convergent sequence (x_n) in $X(\sigma)$. Let $x_0 = \lim_{n \to \infty} x_n$ and $\gamma_0 \notin \sigma$. If $f \in L_1(\mathbb{G})$ is such that $\hat{f}(\gamma_0) \neq 0$ and $(\operatorname{supp} \hat{f}) \cap \sigma = \varnothing$, then part (iv) of Lemma 3.3 implies that $fx_0 = \lim_{n\to\infty} fx_n = 0$. Therefore, $\gamma_0 \notin \Lambda(x_0)$, whence $\Lambda(x_0) \subseteq \sigma$. The lemma is proved.

We denote by $\sigma(A)$ and $\varrho(A)$ the spectrum and the resolvent set of an operator $A\colon D(A) \subseteq Z \to Z$, where Z is a Banach space.

Let (X, T) be a Banach $L_1(\mathbb{R})$-module. For any $z \in \mathbb{C} \setminus \mathbb{R}$ we consider the function $f_z \in L_1(\mathbb{R})$ whose Fourier transform is the function $\varphi_z\colon \mathbb{R} \to \mathbb{C}$ defined by the formula $\varphi_z(\lambda) = (\lambda - iz)^{-1}$, $\lambda \in \mathbb{R}$. Hilbert's resolvent identity holds for the operator-valued function

$$R\colon \mathbb{C} \setminus \mathbb{R} \to \operatorname{End} X, \qquad R(z) = T(f_z), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{3.1}$$

Since the $L_1(\mathbb{R})$-module X is non-degenerate, we have $\bigcap_{z\in\mathbb{C}\setminus\mathbb{R}} \operatorname{Ker} R(z) = \{0\}$. Therefore, R is the resolvent of some linear operator $iB\colon D(B) \subseteq X \to X$. The operator B will be called a *generator* of the $L_1(\mathbb{R})$-module (X, T). If $T\colon \mathbb{R} \to \operatorname{End} X$ is a strongly continuous representation, then iB is a generator of the (bounded) strongly continuous group of operators $\bigl\{T(t);\ t \in \mathbb{R}\bigr\}$.

Lemma 3.6. *Let (X, T) be a Banach $L_1(\mathbb{G})$-module. Then*

$$\sigma\bigl(T(f)\bigr) = \hat{f}\bigl(\Lambda(X)\bigr), \qquad \sigma\bigl(T(g)\bigr) = \overline{\bigl\{\gamma(g);\ \gamma \in \Lambda(X)\bigr\}}$$

for all $f \in L_1(\mathbb{G})$ and $g \in \mathbb{G}$. If $\mathbb{G} = \mathbb{R}$, then the following equalities hold for the generator B of the module (X, T):

$$\sigma(B) = \Lambda(X), \qquad \sigma\bigl(T(t)\bigr) = \overline{\bigl\{e^{i\lambda t};\ \lambda \in \sigma(B)\bigr\}}, \qquad t \in \mathbb{R}. \tag{3.2}$$

Such equalities were first obtained in [16] and [17] for bounded strongly continuous representations and, in a more general case (for non-quasianalytic representations), in [19]. The second equality in (3.2) is called the *weak spectral mapping theorem*, and has been rediscovered by many authors (see, for example, [20], [21]).

The assertions in the following theorem were obtained in [19].

Theorem 3.7. *Let (X, T) be a Banach $L_1(\mathbb{G})$-module and σ a compact subset of $\widehat{\mathbb{G}}$. Then*

$$\bigl\|\bigl(T(g) - I\bigr)x\bigr\| \leqslant 2\sqrt{2}\, \sup_{\gamma\in\sigma}\bigl|\gamma(g) - 1\bigr|\, \|x\|, \qquad x \in X(\sigma), \quad g \in \mathbb{G}.$$

In particular, the restriction $T_\sigma\colon \mathbb{G} \to \operatorname{End} X(\sigma)$, $T_\sigma(g) = T(g)\big|_{X(\sigma)}$, $g \in \mathbb{G}$, of the representation T to $X(\sigma)$ is continuous in the uniform operator topology. If $\mathbb{G} = \mathbb{R}$ and $\Lambda(X)$ is a compact subset of $\mathbb{R}$, then the generator B of the Banach $L_1(\mathbb{R})$-module (X, T) belongs to $\operatorname{End} X$ *and* $\|B\| = r(B) = \max_{\lambda\in\sigma(B)} |\lambda|$.

Theorem 3.8. *Let (X, T) be a Banach $L_1(\mathbb{R})$-module whose generator B is a bounded operator. If $\varphi\colon \mathcal{U} \to \mathbb{C}$ is a function holomorphic in a neighbourhood $\mathcal{U}$ of $\sigma(B)$ and $f \in L_1(\mathbb{R})$ is such that $\hat{f}(\lambda) = \varphi(\lambda)$ for all $\lambda \in \mathcal{U} \cap \mathbb{R}$, then $\varphi(B) = T(f)$.*

Proof. We deduce the desired equality from formula (3.1) for the resolvent R of iB, defining the functions of operators by Cauchy's formula (the Riesz–Dunford formula). A more general result can be found in [22].

§ 4. Approximate identities and γ-nets

In this section, (X,T) stands for a Banach $L_1(\mathbb{G})$-module and the results are closely connected with ergodic theorems in Banach modules (see [23]–[25] and the references there) and can be used in the study of causal operators.

Definition 4.1. A bounded net (e_α) in $L_1(\mathbb{G})$ is called a *bounded approximate identity* (b. a. i.) in the algebra $L_1(\mathbb{G})$ if the following two conditions hold:

(i) $\hat{e}_\alpha(0) = 1$ for all α in the net Ω,

(ii) $\lim e_\alpha * f = f$ for all $f \in L_1(\mathbb{G})$.

Remark 4.2. The Banach algebra $L_1(\mathbb{G})$ has a b. a. i which can be constructed as follows. Consider the net Ω of symmetric neighbourhoods of zero in $\widehat{\mathbb{G}}$ having compact closures and measurable with respect to the Haar measure on $\widehat{\mathbb{G}}$ $(\omega_1 \prec \omega_2$ for $\omega_1, \omega_2 \in \Omega$ if $\omega_1 \subset \omega_2)$ and such that every compact subset of $\widehat{\mathbb{G}}$ is contained in some $\omega \in \Omega$. Let $\chi_\omega \colon \widehat{\mathbb{G}} \to \mathbb{R}$ be the characteristic function of the set $\omega \in \Omega$ and $\check{\chi}_\omega \colon \widehat{\mathbb{G}} \to \mathbb{R}$ the function whose Fourier transform is χ_ω. We put $k_\omega = h^{-1}|\check{\chi}_\omega|^2$, where $h = \mu(\omega)$ is the Haar measure of ω. The net (k_ω) thus obtained is a b. a. i. in $L_1(\mathbb{G})$.

Let us note that this net has the following properties:

(i) $k_\omega \in C_0(\mathbb{G})$ and $0 \leqslant k_\omega \leqslant k_\omega(0) = h$, $\omega \in \Omega$,

(ii) $\int_{\mathbb{G}} k_\omega(s)\,ds = \hat{k}_\omega(0) = 1$,

(iii) $\hat{k}_\omega \geqslant 0$ and $\hat{k}_\omega = 0$ outside $\omega^0 = \omega - \omega = \{\gamma_1 - \gamma_2;\ \gamma_1, \gamma_2 \in \omega\}$,

(iv) $\lim \int_U k_\omega(s) ds = 0$ for every open subset U of $\mathbb{G}$ that does not contain the zero of $\mathbb{G}$.

If $\mathbb{G} = \mathbb{R}$ and $\Omega = \bigl\{\bigl(-\frac{h}{2}, \frac{h}{2}\bigr);\ h > 0\bigr\}$ is the net of intervals (the net structure is induced by the increasing order of h), then the above b. a. i. is given by the formula

$$k_h(t) = 4\frac{\bigl(\sin\bigl(\frac{h}{2}\,t\bigr)\bigr)^2}{ht^2}, \qquad h > 0.$$

If $\mathbb{G}$ is a discrete group, then the algebra $L_1(\mathbb{G})$ contains the identity $\delta_0 \in L_1(\mathbb{G})$, where $\delta_0(0) = 1$ and $\delta_0(g) = 0$ for all $g \in \mathbb{G} \setminus \{0\}$.

It is easy to verify that the following family of functions is a b. a. i. in the algebra $L_1(\mathbb{R}^m)$:

$$f_\beta(t) = \frac{1}{\pi^m} \prod_{i=1}^{m} \frac{\beta_i}{t_i^2 + \beta_i^2}, \tag{4.1}$$

where $\beta = (\beta_1, \beta_2, \dots, \beta_m) \in \mathbb{R}^m_+ \setminus \{0\}$, and the net structure on $\mathbb{R}^m_+ \setminus \{0\}$ is defined as follows: $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \prec \beta = (\beta_1, \beta_2, \dots, \beta_m)$ if $\beta_i \leqslant \alpha_i$, $1 \leqslant i \leqslant m$.

We denote by X_{Comp} the submodule of the $L_1(\mathbb{G})$-module (X,T) formed by the vectors with compact Beurling spectrum. We denote by X_Φ the submodule $\bigl\{fx;\ f \in L_1(\mathbb{G}),\ x \in X\bigr\}$.

Lemma 4.3.

$$X_c = X_\Phi = \overline{X}_{\mathrm{Comp}} = \bigl\{x \in X\colon\ \lim e_\alpha x = x \textit{ for every b. a. i. } (e_\alpha) \textit{ in } L_1(\mathbb{G})\bigr\}.$$

Proof. The inclusion $\overline{X}_{\mathrm{Comp}} \subseteq X_c$ follows from Lemma 3.7 since X_c is closed. The submodule X_Φ is closed by the Cohen–Hewitt factorization theorem (Theorem 32.22 in [8]). Since $\lim e_\omega * f = f$ for every $f \in L_1(\mathbb{G})$, where (e_ω) is the approximate identity occurring in Remark 4.2, and $\operatorname{supp} \hat{e}_\omega \subset \omega - \omega$ (see property (iii) in Remark 4.2), part (iii) of Lemma 3.3 implies that $(e_\omega * f)x \in X_{\mathrm{Comp}}$ for every $\omega \in \Omega$. Hence, $X_\Phi \subseteq \overline{X}_{\mathrm{Comp}} \subseteq X_c$.

Now let x be an arbitrary vector in X_c, let $\varepsilon > 0$ and let V be a compact neighbourhood of zero in $\mathbb{G}$ such that

$$\sup_{g \in V} \|T(g)x - x\| < \varepsilon.$$

Consider an $f \in L_1(\mathbb{G})$ such that $f \geqslant 0$, $\hat{f}(0) = 1$ and $\operatorname{supp} \hat{f} \subset V$. Then

$$\|fx - x\| = \left\| \int_{\mathbb{G}} f(g)\big(T(g)x - x\big)\, dg \right\| \leqslant \varepsilon \int_{\mathbb{G}} f(g)\, dg = \varepsilon \hat{f}(0) = \varepsilon.$$

Therefore, $x \in X_\Phi$. Hence, $X_c = X_\Phi = \overline{X}_{\mathrm{Comp}}$.

If (f_α) is an arbitrary b. a. i. in $L_1(\mathbb{G})$, $x \in X$ and $\lim f_\alpha x = x$, then the above results imply that $x \in X_c$. Let y be an arbitrary vector in $X_c = X_\Phi$ and let $f \in L_1(\mathbb{G})$ be such that $fx = y$. Then

$$\lim f_\alpha y = \lim (f_\alpha * f)x = fx = y.$$

The lemma is proved.

Definition 4.4. Let $\gamma \in \widehat{\mathbb{G}}$. A bounded net (f_α) in the algebra $L_1(\mathbb{G})$ is called a γ-*net* if the following two conditions hold:

(1) $\hat{f}_\alpha(\gamma) = 1$ for all α,

(2) $\lim f_\alpha * f = 0$ for every $f \in L_1(\mathbb{G})$ with $\hat{f}(\gamma) = 0$.

Remark 4.5. Consider the following example of a 0-net in $L_1(\mathbb{G})$: $f_\omega = h^{-1} k_\omega$, $\omega \in \Omega$ (see Remark 4.2), where $\omega_1 \prec \omega_2$ if $\omega_1 \supset \omega_2$. This net has the following properties:

(i) $f_\omega \in C_0(\mathbb{G})$, $\omega \in \Omega$,

(ii) $\hat{f}_\omega(0) = 1$,

(iii) $\hat{f}_\omega \geqslant 0$ and $\hat{f}_\omega = 0$ outside $\omega^0 = \omega - \omega$.

The net $(\gamma f_\omega)(g) = \gamma(g) f_\omega(g)$, $g \in \mathbb{G}$, $\omega \in \Omega$, where $\gamma \in \widehat{\mathbb{G}}$, is a γ-net.

If $\mathbb{G} = \mathbb{R}$ and $\Omega = \big\{\big(-\frac{h}{2}, \frac{h}{2}\big); h > 0\big\}$ is the set of intervals arranged in decreasing order of h, then the above 0-net (f_h) in $L_1(\mathbb{R})$ is given by the formula

$$f_h(t) = 4 \frac{\big(\sin\big(\frac{h}{2}\, t\big)\big)^2}{h^2 t^2}, \qquad h > 0.$$

Here are two more examples of 0-nets in $L_1(\mathbb{R})$:

$$\varphi_T(t) = \begin{cases} (2T)^{-1}, & t \in [-T, T], \\ 0, & t \notin [-T, T], \end{cases} \qquad T > 0,$$

$$\varphi_\varepsilon(t) = \begin{cases} \varepsilon \exp(-\varepsilon t), & t \geqslant 0, \\ 0, & t < 0, \end{cases} \qquad \varepsilon > 0.$$

The first of these corresponds to increasing order of T and the second to decreasing order of ε.

Let us also note that the family of functions (4.1) is a 0-net in $L_1(\mathbb{R}^m)$ as $\beta \to \infty$.

Definition 4.6. A net (f_α) in $L_1(\mathbb{G})$ is called an *invariant integral* if the following two conditions hold:

(i) $\hat{f}_\alpha(0) = 1$ and $\hat{f}_\alpha \geqslant 0$ for all α,

(ii) $\int_{\mathbb{G}} \big| f_\alpha(g+u) - f_\alpha(g) \big| \, dg = 0$ for all $u \in \mathbb{G}$.

Remark 4.7. Every invariant integral (f_α) in $L_1(\mathbb{G})$ is a 0-net. Indeed, Definition 4.6 implies that $\lim(\varphi_u - \varphi) * f_\alpha = 0$ for all $u \in \mathbb{G}$ and $\varphi \in L_1(\mathbb{G})$. Hence, the set $M = \big\{ f \in L_1(\mathbb{G})\colon \lim f_\alpha * f = 0 \big\}$, which is a closed ideal of the algebra $L_1(\mathbb{G})$, contains the ideal $M_0 = \{\varphi_u - \varphi;\ \varphi \in L_1(\mathbb{G}),\ u \in \mathbb{G}\}$. Since the set of common zeros of Fourier transforms of functions belonging to M_0 consists of a single (zero) element, Wiener's theorem [7]–[9] implies that $M = \{f \in L_1(\mathbb{G});\ \hat{f}(0) = 0\}$. Hence, (f_α) is a 0-net.

The net (f_ω), $\omega \in \Omega$, constructed in Remark 4.5 and the family of functions (4.1) are examples of invariant integrals (in what follows, the symbol Ω will usually be omitted).

Remark 4.8. If (f_α) is a γ-net and f is any function belonging to $L_1(\mathbb{G})$ and such that $\hat{f}(\gamma) = 1$, then the net $(f_\alpha * f)$ also is a γ-net. In this case $(f_\alpha * f)$ is an invariant integral if (f_α) is, and $\hat{f}(0) = 1$, $f \geqslant 0$.

Remark 4.9. By definition, every γ-net (f_α) in $L_1(\mathbb{G})$ is such that the net $(\delta_0 - f_\alpha)$ in the algebra $\widetilde{L}_1(\mathbb{G})$ obtained from $L_1(\mathbb{G})$ by adjoining to it the identity δ_0 is an approximate identity in the maximal ideal $\mathcal{I} = \operatorname{Ker}\gamma = \big\{ f \in L_1(\mathbb{G})\colon \hat{f}(\gamma) = 0 \big\}$ of $L_1(\mathbb{G})$ (see Example 6 in [24]).

Definition 4.10. Let (f_α) be a γ-net in $L_1(\mathbb{G})$. We denote by $\mathcal{E}\mathfrak{rg}_\gamma\big(X, (f_\alpha)\big)$ the submodule $\big\{ x \in X\colon \text{there is a} \lim f_\alpha x \big\}$.

Since the net (f_α) is bounded, the submodule $\mathcal{E}\mathfrak{rg}_\gamma\big(X, (f_\alpha)\big)$ is closed in X.

Definition 4.11. A vector x_0 of the Banach $L_1(\mathbb{G})$-module (X, T) is said to be *almost periodic* if its orbit $O(x_0) = \big\{ T(g)x_0;\ g \in \mathbb{G} \big\}$ is precompact in X.

The set of almost periodic vectors in X forms a closed submodule, which we denote by $\mathcal{AP}X$ or $\mathcal{AP}(X, T)$.

Theorem 4.12. $\mathcal{AP} \subset X_c$.

Proof. Let $x \in \mathcal{AP}X$. Then $\varphi_x(g) = T(g)x$, $\varphi_x\colon \mathbb{G}_d \to X$, is a continuous almost periodic function (see Remark 2.5). Hence, it is the uniform limit of some sequence of trigonometric polynomials

$$\sum_{i=1}^{n} \overline{\gamma}_i(g) y_i, \qquad y_i \in X, \quad \overline{\gamma}_i \in \widehat{\mathbb{G}}_d,$$

and $T(g)y_i = \overline{\gamma}_i(g)y_i$, $1 \leqslant i \leqslant n$, $g \in \mathbb{G}$ (see [26]). We claim that $\overline{\gamma}_i \in \widehat{\mathbb{G}}$, $1 \leqslant i \leqslant n$. This will be proved if we can establish that $\overline{\gamma} \in \widehat{\mathbb{G}}$ when $x \in X$ is such that $T(g)x = \overline{\gamma}(g)x$, $g \in \mathbb{G}$. Since the $L_1(\mathbb{G})$-module (X, T) is non-degenerate, there is an $f \in L_1(\mathbb{G})$ such that $fx \neq 0$. We have $fx \in X_c$ and $T(f)T(g)x = T(g)(fx) = \overline{\gamma}(g)(fx)$. Hence, $\overline{\gamma} \in \widehat{\mathbb{G}}$. The theorem is proved.

Corollary 4.13. *If $x \in L_\infty(\mathbb{G}, Y) = X$ and the set of shifts $\{S(g)x, g \in \mathbb{G}\}$ is precompact in X, then x coincides almost everywhere with some function belonging to $AP(\mathbb{G}, Y) \subset (X, S)_c$.*

The following result was obtained in [25].

Theorem 4.14. *A vector x_0 in X_c is almost periodic if the set $\Lambda(x_0)$ is completely imperfect (that is, contains no non-empty perfect subset) and one of the following conditions holds:*

(i) *the Banach space X contains no subspace isomorphic to the Banach space c_0 of numerical sequences converging to zero,*

(ii) *the orbit of x_0 is weakly precompact in X.*

Definition 4.15. A non-zero vector $x_0 \in X$ is called an *eigenvector* of the $L_1(\mathbb{G})$-module (X, T) if there is a character $\gamma_0 \in \widehat{\mathbb{G}}$ such that $T(g)x_0 = \gamma_0(g)x_0$ for all $g \in \mathbb{G}$.

Remark 4.16. It follows from Definition 4.15 that every eigenvector x_0 of (X, T) has a one-point spectrum $\Lambda(x_0) = \{\gamma_0\}$ (the converse assertion also is true: see [25]). Therefore, the eigenvectors belong to $X_{\mathrm{Comp}} \subseteq X_c$.

We denote by $X(\gamma) = X(\{\gamma\})$ the set of vectors in X with the one-point spectrum $\{\gamma\}$. Hence, $X(\gamma) = \{x \in X : T(g)\,x = \gamma(g)\,x\}$. We denote by X_γ the closed submodule $X_\gamma = \overline{\{x \in X;\ \gamma \notin \Lambda(x)\}}$.

Definition 4.17. A point γ in the set $\Lambda(x, T)$ is said to be an *ergodic* point of x if $x \in \mathcal{E}\mathfrak{rg}_\gamma(X, (f_\alpha))$ for some γ-net (f_α).

We denote the set of ergodic points of x by $\Lambda_{\mathrm{erg}}(x)$ or $\Lambda_{\mathrm{erg}}(x, T)$.

Remark 4.18. If $\gamma \in \widehat{\mathbb{G}} \setminus \Lambda(x, T)$, $x \in (X, T)$ and (f_α) is a γ-net in $L_1(\mathbb{G})$, then $\lim f_\alpha x = 0$.

Theorem 4.19. *Let γ be a character in $\widehat{\mathbb{G}}$ and (f_α) a γ-net. Then*

(i) *$\mathcal{E}\mathfrak{rg}_\gamma(X, (f_\alpha)) = X(\gamma) \oplus X_\gamma$, and the operator $P(\gamma)x = \lim f_\alpha x$ in the algebra* $\mathrm{End}(X(\gamma) \oplus X_\gamma)$ *is a projector to $X(\gamma)$ parallel to X_γ, $\|P(\gamma)\| \leqslant 1$, and the limit* $\lim f_\alpha x = x_0 \in X(\gamma)$ *does not depend on the choice of the γ-net (f_α),*

(ii) $\mathcal{AP}X \subset \mathcal{E}\mathfrak{rg}_\gamma(X, (f_\alpha))$,

(iii) *$\mathcal{E}\mathfrak{rg}_\gamma(X, (f_\alpha)) = X$ if X is a reflexive space,*

(iv) *$\mathcal{E}\mathfrak{rg}_\gamma(X, (f_\alpha)) = X$ if and only if the eigenvectors belonging to $X(\gamma)$ separate the functionals belonging to the subspace*

$$X^*(\gamma) = \{\xi \in X^*;\ (T(g))^*\xi = \gamma(g)\,\xi,\ g \in \mathbb{G}\}$$

of the Banach space X^* dual to X,

(v) *$x \in \mathcal{E}\mathfrak{rg}_\gamma(X, (f_\alpha))$ if $\{(f_\alpha)x\}$ is a weakly compact subset of X,*

(vi) *$x \in \mathcal{E}\mathfrak{rg}_\gamma(X, (f_\alpha))$ if γ is an isolated point of $\Lambda(X, T)$, and then $P(\gamma)x = fx$ for all $f \in L_1(\mathbb{G})$ such that $\hat{f}(\gamma) = 1$ and $\operatorname{supp} \hat{f} \cap \Lambda(x, T) = \varnothing$,*

(vii) *for any $x \in X_\gamma$ one can find $y \in X_\gamma$ and $f \in L_1(\mathbb{G})$ such that $\gamma \notin \operatorname{supp} \hat{f}$ and $x = fy$.*

Proof. By Remark 4.9, all these assertions follow from more general results of [24].

By part (i) of Theorem 4.19, the projector $P(\gamma)\,x=\lim f_\alpha x$, $x\in \mathcal{E}\mathfrak{rg}_\gamma\bigl(X,(f_\alpha)\bigr)$, does not depend on the choice of the γ-net (f_α). This enables us to denote $\mathcal{E}\mathfrak{rg}_\gamma\bigl(X,(f_\alpha)\bigr)$ by $\mathcal{E}\mathfrak{rg}_\gamma X$ or $\mathcal{E}\mathfrak{rg}_\gamma(X,T)$. We shall denote by $\mathcal{E}\mathfrak{rg}X$ or $\mathcal{E}\mathfrak{rg}(X,T)$ the closed submodule

$$\mathcal{E}\mathfrak{rg}X=\bigcap_{\gamma\in\widehat{\mathbb{G}}}\mathcal{E}\mathfrak{rg}_\gamma X=\bigcap_{\gamma\in\Lambda(X)}\mathcal{E}\mathfrak{rg}_\gamma X.$$

Corollary 4.20. $\mathcal{AP}(X,T)\subset\mathcal{E}\mathfrak{rg}(X,T)$.

To every almost periodic vector x in $\mathcal{AP}(X,T)$ we assign the Fourier series

$$x\sim\sum x_\gamma, \tag{4.2}$$

where $x_\gamma=P(\gamma)\,x\neq 0$ and $x_\gamma\in X(\gamma)$, whence

$$T(g)\,x_\gamma=\gamma(g)\,x_\gamma,\qquad \gamma\in\widehat{\mathbb{G}},\qquad g\in\mathbb{G}. \tag{4.3}$$

Definition 4.21. The *Bohr spectrum* $\Lambda_b(x)=\Lambda_b(x,T)$ of the vector x is defined to be the set of $\gamma\in\widehat{\mathbb{G}}$ such that $x\in\mathcal{E}\mathfrak{rg}_\gamma X$ and $P(\gamma)\,x=x_\gamma\neq 0$. The *Bohr spectrum* of the module X is defined to be $\Lambda_b(X,T)=\bigcup_{x\in X}\Lambda_b(x,T)$.

Remark 4.22. Since $T(g)\,P(\gamma)\,x=\gamma(g)\,P(\gamma)\,x$ for all $x\in\mathcal{E}\mathfrak{rg}_\gamma X$ and $fP(\gamma)\,x=\hat f(\gamma)\,P(\gamma)\,x$ for all $f\in L_1(\mathbb{G})$, we have $\Lambda_b(x)\subseteq\Lambda(x)$. If $x\in\mathcal{AP}X$, then $\Lambda_b(x)=\{\gamma\in\widehat{\mathbb{G}}\colon x_\gamma\neq 0\}$ (see formula (4.2)), $\overline{\Lambda_b(x)}=\Lambda(x)$, and the set $\Lambda_b(x)$ is at most countable. We deduce the last assertion from the fact that the absolutely convex hull $\operatorname{Co}x=\bigl\{\sum_{i=1}^n c_iT(g_i)\,x;\ \sum_{i=1}^n|c_i|\leqslant 1,\ g_i\in\mathbb{G},\ c_i\in\mathbb{C}\bigr\}$ of the orbit of the almost periodic vector x is precompact using the invariant integral to define x_γ, $\gamma\in\Lambda_b(x)$, which enables us to establish that $x_\gamma\in\overline{\operatorname{Co}x}$ for all $\gamma\in\Lambda_b(x)$. Since $P(\gamma_1)\,P(\gamma_2)=0$ for $\gamma_1\neq\gamma_2$ belonging to $\Lambda_b(x)$ and $\|P(\gamma)\|=1$ for all $\gamma\in\Lambda_b(x)$, the set $\Lambda_b(x)$ is at most countable.

§ 5. Two module structures on the space of operators

In this section we consider two (non-degenerate) Banach $L_1(\mathbb{G})$-modules (X_i,T_i), $i=1,2$, where $T_i\colon\mathbb{G}\to\operatorname{End}X_i$, $i=1,2$, are isometric representations that are not assumed to be strongly continuous.

A special role in this paper is played by two module structures on the space $\operatorname{Hom}(X_1,X_2)$ introduced below and by relations between the Beurling spectra of the operators corresponding to these structures. The first of these (the structure of a Banach $L_1(\mathbb{G}\times\mathbb{G})$-module) is associated with the representation

$$\widetilde{T}\colon\mathbb{G}\times\mathbb{G}\to\operatorname{End}\operatorname{Hom}(X_1,X_2),\qquad \widetilde{T}(g_1,g_2)A=T_2(g_2)\,AT_1(g_1), \tag{5.1}$$

where $A\in\operatorname{Hom}(X_1,X_2)$ and $g_1,g_2\in\mathbb{G}$. Taking into account that the algebra $L_1(\mathbb{G}\times\mathbb{G})$ is isometrically isomorphic to the tensor product $L_1(\mathbb{G})\bigotimes_\pi L_1(\mathbb{G})$ (see, for example, [27]), we see that the formula

$$\widetilde{T}(f_1\otimes f_2)A=T_2(f_2)\,AT_1(f_1),\qquad f_1,f_2\in L_1(\mathbb{G}), \tag{5.2}$$

where $(f_1 \otimes f_2)(g_1, g_2) = f_1(g_1)f_2(g_2)$, $g_1, g_2 \in \mathbb{G}$, enables us to define the structure of a Banach $L_1(\mathbb{G}\times\mathbb{G})$-module $\big(\mathrm{Hom}(X_1, X_2), \widetilde{T}\big)$. An arbitrary $f \in L_1(\mathbb{G}\times\mathbb{G})$ can be written as

$$f = \sum_{k\geqslant 1} f'_k \otimes \varphi'_k \tag{5.3}$$

so that $\sum_{k\geqslant 1} \|f'_k\|_1 \|\varphi'_k\|_1 < \infty$, f'_k, $\varphi'_k \in L_1(\mathbb{G})$, $k \geqslant 1$. We put

$$fA = \widetilde{T}(f)A = \sum_{k\geqslant 1} \widetilde{T}(f'_k \otimes \varphi'_k)A = \sum_{k\geqslant 1} T_2(f'_k)\, AT_1(\varphi'_k). \tag{5.4}$$

Lemma 5.1. *The module structure on* $\mathrm{Hom}(X_1, X_2)$ *associated with the representation* $\widetilde{T}$ *is well defined.*

Proof. We have to establish that the definition of the operator fA in formula (5.4) does not depend on the representation (5.3) of f. First let us note that if the map $(g_1, g_2) \mapsto \widetilde{T}(g_1, g_2)A$ is continuous in the uniform operator topology (that is, $A \in \big(\mathrm{Hom}(X_1, X_2), \widetilde{T}\big)_c$ and $f \in L_1(\mathbb{G}\times\mathbb{G})$, then fA can be determined from $\widetilde{T}$ in the standard way using formula (2.2). The subspace $\big(\mathrm{Hom}(X_1, X_2), \widetilde{T}\big)_c$ contains, in particular, the operators $T_2(\varphi)BT_1(\psi)$, φ, $\psi \in L_1(\mathbb{G})$, $B \in \mathrm{Hom}(X_1, X_2)$. Assume that, along with the representation (5.3), f admits a representation $f = \sum_{k\geqslant 1} f''_k \otimes \varphi''_k$, where $\sum_{k\geqslant 1} \|f''_k\|_1 \|\varphi''_k\|_1 < \infty$. We claim that the operators

$$A_1 = \sum_{k\geqslant 1} T_2(f'_k)\, AT_1(\varphi'_k), \qquad A_2 = \sum_{k\geqslant 1} T_2(f''_k)\, AT_1(\varphi''_k)$$

coincide. Since $A_1, A_2 \in \big(\mathrm{Hom}(X_1, X_2), \widetilde{T}\big)_c$, this will be proved (see formula (2.2) and the paragraph containing it) if we can establish that $(\varphi \otimes \psi)A_1 = (\varphi \otimes \psi)A_2$ for all $\varphi, \psi \in L_1(\mathbb{G})$. Using the equalities

$$(\varphi \otimes \psi)A_1 = \sum_{k\geqslant 1} T_2(f'_k)\big(T_2(\varphi)\, AT_1(\psi)\big)T_1(\varphi'_k) = \widetilde{T}(f)\big(T_2(\varphi)\, AT_1(\psi)\big)$$

and $(\varphi \otimes \psi)A_2 = \widetilde{T}(f)\big(T_2(\varphi)AT_1(\psi)\big)$, we obtain that $A_1 = A_2$. The lemma is proved.

If there is a non-zero operator in $\mathrm{Hom}(X_1, X_2)$ that annihilates the subspace $(X_1, T_1)_c$ of T_1-continuous vectors, then the Banach module $\big(\mathrm{Hom}(X_1, X_2), \widetilde{T}\big)$ thus constructed will not be non-degenerate (in the sense of Assumption 2.1). For this reason we introduce some non-degenerate closed submodules of $\big(\mathrm{Hom}(X_1, X_2), \widetilde{T}\big)$:

$$\begin{aligned}
\mathfrak{U}_{1,\tau}\big(X_1, X_2; (f_\alpha)\big) &= \mathfrak{U}_{1,\tau}(X_1, X_2) = \mathfrak{U}_{1,\tau}\big((f_\alpha)\big) = \mathfrak{U}_{1,\tau} \\
&= \big\{A \in \mathrm{Hom}(X_1, X_2)\colon \tau\text{-}\lim T_2(\psi)\big(AT_1(f_\alpha) - T_2(f_\alpha)A\big) = 0 \\
&\qquad\qquad \text{for every } \psi \in L_1(\mathbb{G})\big\} \\
&= \big\{A \in \mathrm{Hom}(X_1, X_2)\colon \tau\text{-}\lim\big(T_2(\psi)\big)\, A\big(I - T_1(f_\alpha)\big) = 0 \\
&\qquad\qquad \text{for every } \psi \in L_1(\mathbb{G})\big\}, \\
\mathfrak{U}_{2,\tau}\big(X_1, X_2; (f_\alpha)\big) &= \mathfrak{U}_{2,\tau}(X_1, X_2) = \mathfrak{U}_{2,\tau}\big((f_\alpha)\big) = \mathfrak{U}_{2,\tau} \\
&= \big\{A \in \mathrm{Hom}(X_1, X_2)\colon \tau\text{-}\lim\big(AT_1(f_\alpha) - T_2(f_\alpha)A\big) = 0\big\}.
\end{aligned}$$

Here (f_α) is a b. a. i. of the algebra $L_1(\mathbb{G})$ and τ is the uniform (u) or strong (s) operator topology on the space $\mathrm{Hom}(X_1, X_2)$. The symbol τ-lim stands for the limit in the topology τ. The modules $\mathfrak{U}_{i,\tau} = \mathfrak{U}_{i,\tau}(X_1, X_2)$, $i = 1, 2$, $\tau \in \{u, s\}$, are non-degenerate since the module (X_2, T_2) is non-degenerate (see Lemma 5.11).

Lemma 5.2. *The definition of the space $\mathfrak{U}_{1,u}$ does not depend on the choice of approximate identity (f_α). This space coincides with the space*

$$\mathfrak{U}'_{1,u} = \big\{A \in \mathrm{Hom}(X_1, X_2)\colon\ \forall\psi \in L_1(\mathbb{G}),\ \forall\varepsilon > 0\ \exists\varphi \in L_1(\mathbb{G})\colon \\ \big\|T_2(\psi)\, A\big(I - T_1(\varphi)\big)\big\| < \varepsilon\big\}.$$

Proof. The definition of $\mathfrak{U}_{1,u}\big((f_\alpha)\big)$ implies that $\mathfrak{U}_{1,u}\big((f_\alpha)\big) \subseteq \mathfrak{U}'_{1,u}$ for every b. a. i. (f_α).

If $A \in \mathfrak{U}'_{1,u}$, (φ_α) is an arbitrary b. a. i. of $L_1(\mathbb{G})$, ψ is any function belonging to $L_1(\mathbb{G})$ and $\varepsilon > 0$, then there is a $\varphi \in L_1(\mathbb{G})$ such that

$$\big\|T_2(\psi)\, A\big(I - T_1(\varphi)\big)\big\| < \frac{\varepsilon}{2(1 + C)}\,,$$

where $C = \sup\|\varphi_\alpha\|$. Since $\lim \varphi_\alpha * \varphi = \varphi$, there is an α_0 such that $\|\varphi - \varphi_\alpha * \varphi\|\,\|\psi\|\,\|A\| < \frac{\varepsilon}{2}$ for $\alpha \succ \alpha_0$, which implies that

$$\big\|T_2(\psi)\, A\big(I - T_1(\varphi_\alpha)\big)\big\| = \big\|T_2(\psi)\, A\big[\big(I - T_1(\varphi)\big)\big(I - T_1(\varphi_\alpha)\big) - T_1(\varphi - \varphi_\alpha * \varphi)\big]\big\| < \varepsilon$$

for $\alpha \succ \alpha_0$. The lemma is proved.

We wish to consider another closed non-degenerate submodule $\mathfrak{U}_3$ of $\big(\mathrm{Hom}(X_1, X_2), \widetilde{T}\big)$, for which we will need the following definition.

Definition 5.3. A net (x_α) in the Banach $L_1(\mathbb{G})$-module (X, T) is said to be *locally convergent* to $x_0 \in X$ (and we write $x_\alpha \overset{\text{loc}}{\to} x_0$ or $\text{loc-lim}\, x_\alpha = x_0$) if it is bounded and $\lim f(x_\alpha - x_0) = 0$ for every $f \in L_1(\mathbb{G})$.

By definition, the submodule $\mathfrak{U}_3 = \mathfrak{U}_3(X_1, X_2)$ consists of those $A \in \mathrm{Hom}(X_1, X_2)$ such that $Ax_\alpha \overset{\text{loc}}{\to} Ax_0$ if $x_\alpha \overset{\text{loc}}{\to} x_0$. The operators belonging to $\mathfrak{U}_3$ are said to be *locally continuous*.

Remark 5.4. The above definitions imply that

$$\mathfrak{U}_{2,u} \subseteq \mathfrak{U}_{1,u} \subseteq \mathfrak{U}_{1,s} \supseteq \mathfrak{U}_{1,u}, \qquad \mathfrak{U}_{2,u} \subseteq \mathfrak{U}_{2,s}.$$

Since $(f_\alpha x) \overset{\text{loc}}{\to} x$ for all $x \in X_1$ and every b. a. i. (f_α) of $L_1(\mathbb{G})$, we have $\mathfrak{U}_{1,s} \supseteq \mathfrak{U}_3$. The operators belonging to one of the submodules $\mathfrak{U}_{2,\tau}$, $\tau \in \{u, s\}$, are said to commute with (f_α) (in the uniform, strong operator topology, respectively).

Remark 5.5. If $(X_i, T_i)_c = X_i$, $i = 1, 2$, then $\mathfrak{U}_{1,s} = \mathfrak{U}_{2,s} = \mathrm{Hom}(X_1, X_2)$.

Under the assumptions of the preceding remark, not all $A \in \mathrm{Hom}(X_1, X_2)$ are locally continuous.

Example 5.6. Let $X = X_1 = X_2 = C_{ub}(\mathbb{R})$ and $T_1 = T_2 = S$ (see Example 2.11). Consider the following linear operator $A \in \operatorname{End} X$:

$$Ax = \xi(x)y, \qquad 0 \neq y \in X, \quad \xi \in X^*, \tag{5.5}$$

where $C_0(\mathbb{R}) \subseteq \operatorname{Ker}\xi$ and $\xi(x_0) = 1$, $x_0 \equiv 1$. The sequence of functions $x_n(t) = e^{-\frac{1}{n}|t|}$, $t \in \mathbb{R}$, $n \geqslant 1$, belongs to $C_0(\mathbb{R})$ and converges locally to x_0 (it also converges locally in the sense of the module structure considered in Example 2.10). Since $\xi(x_n) = 0$, $n \geqslant 1$, we have $Ax_n = 0$, $n \geqslant 1$. On the other hand, $Ax_0 = y \neq 0$, that is, the sequence (Ax_n) does not converge locally to Ax_0. Hence, A is not locally continuous, although the assumptions of Remark 5.5 hold for $\bigl(C_{ub}(\mathbb{R}), S\bigr)$.

Example 5.7. Let $X = X_1 = X_2 = C_b(\mathbb{R})$, $T_1 = T_2 = S$ (see Example 2.11), and let $A \in \operatorname{End} X$ be the operator defined by formula (5.5) using a function $y \in C_b(\mathbb{R}) \setminus C_{ub}(\mathbb{R})$ and a functional ξ of the form

$$\xi(x) = \int_{\mathbb{R}} \alpha(t)\, x(t)\, dt,$$

where $\alpha \in L_1(\mathbb{R})$. Then A is locally continuous but does not commute with any b. a. i. in $L_1(\mathbb{R})$, that is, it does not belong to $\mathfrak{U}_{2,s}$. Since $\bigl(S(\psi)A\bigr)\, x = \xi(x)(\psi * y)$ for all $\psi \in L_1(\mathbb{R})$, we have $A \in \mathfrak{U}_{1,s}$.

Consider the representation

$$T_0 \colon \mathbb{G} \to \operatorname{End}\operatorname{Hom}(X_1, X_2), \qquad T_0(g)A = T_2(g)\, AT_1(-g), \tag{5.6}$$

where $g \in \mathbb{G}$ and $A \in \operatorname{Hom}(X_1, X_2)$. On the spaces $\mathfrak{U}_{0,\tau}$, $\tau \in \{u, s\}$, for which the map $g \mapsto T_0(g)A \colon \mathbb{G} \to \operatorname{Hom}(X_1, X_2)$ is continuous in the topology τ, the formula

$$(fA)\, x = T_0(f)\, Ax = \int_{\mathbb{G}} f(g)\bigl(T_0(-g)\, A\bigr)x\, dg$$

defines the structure of the Banach $L_1(\mathbb{G})$-module associated with T_0.

Remark 5.8. If (f_α) is a b. a. i. in $L_1(\mathbb{G})$, then for $A \in \mathfrak{U}_{0,\tau}$, $\tau \in \{u, s\}$, the net $(f_\alpha A)$ converges to A in the topology τ. Therefore, the $\mathfrak{U}_{0,\tau}$, $\tau \in \{u, s\}$, are non-degenerate $L_1(\mathbb{G})$-modules.

Remark 5.9. If $A \in \mathfrak{U}_{0,s}$, then the boundedness of the representations T_1 and T_2 implies that

$$\lim_{g \to 0}\bigl(T_2(g)Ax - AT_1(g)\, x\bigr) = 0, \qquad x \in X_1.$$

Therefore, $Ax \in (X_1, T_2)_c$ if $x \in (X_1, T_1)_c$. Multiplying this equality by an arbitrary b. a. i. (f_α) and integrating, we obtain that $\lim\bigl(T_2(f_\alpha)Ax - AT_1(f_\alpha)\, x\bigr) = 0$. If T_1 is strongly continuous, then $A \in \mathfrak{U}_{2,s}$, whence $\mathfrak{U}_{0,s} \subseteq \mathfrak{U}_{2,s}$. We likewise show that $\mathfrak{U}_{0,u} \subseteq \mathfrak{U}_{2,u}$.

In what follows (if not otherwise stated), the symbol $\mathfrak{U} = \mathfrak{U}(X_1, X_2)$ stands for one of the following closed subspaces of $\operatorname{Hom}(X_1, X_2)$:

$$\mathfrak{U}_{i,\tau_1} \cap \mathfrak{U}_{0,\tau_2}, \qquad \mathfrak{U}_3 \cap \mathfrak{U}_{0,\tau}, \qquad \tau, \tau_1, \tau_2 \in \{u, s\}, \quad i = 1, 2. \tag{5.7}$$

What has been said implies that $\mathfrak{U}$ is a non-degenerate Banach $L_1(\mathbb{G}\times\mathbb{G})$- and $L_1(\mathbb{G})$-module with the structures associated with the representations $\widetilde{T}$ and T_0, respectively (we retain this notation for the restrictions of $\widetilde{T}$ and T_0 to $\mathfrak{U}$, which are well defined).

Remark 5.10. The classes $\mathfrak{U}(X_1,X_2)$ have the following "physical" property. Their definitions imply that $A=0$ if $A\in\mathfrak{U}(X_1,X_2)$ and $Ax=0$ for all $x\in(X_1,T_1)_c$ (we prove this assertion using Lemma 4.3). Hence, $A=B$ whenever $A,B\in\mathfrak{U}(X_1,X_2)$ coincide on $(X_1,T_1)_c$.

Lemma 5.11. *$\mathfrak{U}$ is both an $L_1(\mathbb{G}\times\mathbb{G})$- and an $L_1(\mathbb{G})$-module.*

Proof. It remains to prove that the module $(\mathfrak{U},\widetilde{T})$ is non-degenerate. We shall do this only for $\mathfrak{U}=\mathfrak{U}_{1,s}\cap\mathfrak{U}_{0,s}$. Let $A\in\mathfrak{U}$ and $\widetilde{T}(f)A=0$ for all $f\in L_1(\mathbb{G}\times\mathbb{G})$. Then $T_2(\psi)AT_1(\varphi)=0$ for all $\varphi,\psi\in L_1(\mathbb{G})$. The definition of $\mathfrak{U}_{1,s}$ implies that $T_2(f_\alpha)Ax=0$ for all $x\in X_1$ and all α ((f_α) is a b. a. i.). Therefore, $T_2(\psi)Ax=0$ for all $\psi\in L_1(\mathbb{G})$. Since the $L_1(\mathbb{G})$-module (X_2,T_2) is non-degenerate, we have $Ax=0$ for all $x\in X_1$, whence $A=0$. The lemma is proved.

Consider the homomorphism $\tau\colon L_1(\mathbb{G})\to\operatorname{End}L_1(\mathbb{G}\times\mathbb{G})$ of Banach algebras defined by the formula

$$\big(\tau(f)\varphi\big)(g_1,g_2)=\int_{\mathbb{G}}f(g)\varphi(g_1-g,g_2+g)\,dg. \tag{5.8}$$

Let us note that the functions $\tau(f)\varphi\in L_1(\mathbb{G}\times\mathbb{G})$, where $f\in L_1(\mathbb{G})$ and $\varphi\in L_1(\mathbb{G}\times\mathbb{G})$, have the Fourier transforms

$$\widehat{\tau(f)\varphi}(\gamma_1,\gamma_2)=\hat{f}(\gamma_1-\gamma_2)\,\widehat{\varphi}(\gamma_1,\gamma_2),\qquad \gamma_1,\gamma_2\in\widehat{\mathbb{G}}. \tag{5.9}$$

Lemma 5.12.

$$\widetilde{T}\big(\tau(f)\varphi\big)=\widetilde{T}(\varphi)\,T_0(f)A=T_0(f)\,\widetilde{T}(\varphi)A \tag{5.10}$$

for all $f\in L_1(\mathbb{G})$, $\varphi\in L_1(\mathbb{G}\times\mathbb{G})$, $A\in\mathfrak{U}=\mathfrak{U}(X_1,X_2)$ and τ occurring in formula (5.8).

Proof. Since the $L_1(\mathbb{G}\times\mathbb{G})$-module $(\mathfrak{U},\widetilde{T})$ is non-degenerate, it is sufficient to prove (5.10) for those $A\in\mathfrak{U}$ such that $A=T_2(\psi)BT_1(\phi)$, $\phi,\psi\in L_1(\mathbb{G})$, $B\in\mathfrak{U}$ (see the proof of Lemma 5.1). Therefore, we can assume that $A\in(\mathfrak{U},\widetilde{T})_c$ and T_1,T_2 are strongly continuous representations. We have

$$\begin{aligned}
\widetilde{T}\big(\tau(f)\varphi\big)A&=\iint_{\mathbb{G}\times\mathbb{G}}\bigg(\int_{\mathbb{G}}f(g)\varphi(g_1-g,g_2+g)\,dg\bigg)T_2(-g_2)\,AT_1(-g_1)\,dg_1\,dg_2\\
&=\int_{\mathbb{G}}f(g)\,T_2(-g)\bigg(\iint_{\mathbb{G}\times\mathbb{G}}\varphi(g_1,g_2)\,T_2(-g_2)\,AT_1(-g_1)\,dg_1\,dg_2\bigg)T_1(g)\,dg\\
&=T_0(f)\widetilde{T}(\varphi)A\\
&=\iint_{\mathbb{G}\times\mathbb{G}}\varphi(g_1,g_2)\,T_2(-g_2)\bigg(\int_{\mathbb{G}}f(g)\,T_2(-g)\,AT_1(g)\,dg\bigg)T_1(-g_1)\,dg_1\,dg_2\\
&=\widetilde{T}(\varphi)\,T_0(f)A,\qquad \varphi\in L_1(\mathbb{G}\times\mathbb{G}),\quad f\in L_1(\mathbb{G})
\end{aligned}$$

(the integrals converge in the strong operator topology). The lemma is proved.

Theorem 5.13.

$$\Lambda(A,T_0)=\overline{\{\gamma_2-\gamma_1\colon (\gamma_1,\gamma_2)\in\Lambda(A,\widetilde{T})\subset\widehat{\mathbb{G}}\times\widehat{\mathbb{G}}\}}\equiv\Delta \tag{5.11}$$

for all $A\in\mathfrak{U}$.

Remark 5.14. It is clear that a similar theorem can be stated for any bimodule with structures associated with representations for which (5.10) holds.

Proof of Theorem 5.13. We claim that $\Lambda(A,T_0)\supseteq\Delta$. Take a $\gamma_0\notin\Lambda(A,T_0)$. There is an $f\in L_1(\mathbb{G})$ such that $\hat{f}(\gamma_0)\neq 0$ and $T_0(f)A=0$. Let σ_0 be a neighbourhood of γ_0 such that $\hat{f}(\gamma)\neq 0$ for all $\gamma\in\sigma_0$. Assume that there is a character (γ_1,γ_2) in $\Lambda(A,\widetilde{T})\subset\widehat{\mathbb{G}}\times\widehat{\mathbb{G}}$ such that $\gamma_2-\gamma_1\in\sigma_0$. Choose a $\varphi\in L_1(\mathbb{G}\times\mathbb{G})$ such that $\widehat{\varphi}(\gamma_1,\gamma_2)\neq 0$. It follows from (5.10) that

$$\widetilde{T}\big(\tau(f)\varphi\big)A=\widetilde{T}(\varphi)\,T_0(f)A=0.$$

Formula (5.9) implies that $\widehat{\tau(f)\varphi}(\gamma_1,\gamma_2)\neq 0$. Therefore, $(\gamma_1,\gamma_2)\notin\Lambda(A,\widetilde{T})$. This is a contradiction. Hence, $\Lambda(A,T_0)\supseteq\Delta$.

To prove the reverse inclusion we take a $\gamma_0\notin\Delta$ and an $f\in L_1(\mathbb{G})$ such that $\hat{f}(\gamma_0)\neq 0$ and $\operatorname{supp}\hat{f}\cap\Delta=\varnothing$. It follows from (5.9) that the function $\psi=\tau(f)\varphi$ is such that $\operatorname{supp}\hat{\psi}\cap\Lambda(A,\widetilde{T})=\varnothing$ for all φ in $L_1(\mathbb{G}\times\mathbb{G})$. Lemma 3.3 and formula (5.10) imply that

$$0=\widetilde{T}(\psi)A=\widetilde{T}(\varphi)\,T_0(f)A,\qquad \varphi\in L_1(\mathbb{G}\times\mathbb{G}).$$

Since the $L_1(\mathbb{G}\times\mathbb{G})$-module $(\mathfrak{U},\widetilde{T})$ is non-degenerate, we have $T_0(f)A=0$, that is, $\gamma_0\notin\Lambda(A,T_0)$, whence $\Delta\subseteq\Lambda(A,T_0)$.

Corollary 5.15.

$$\Lambda(Ax,T_2)\subset\overline{\Lambda(A,T_0)+\Lambda(x,T_1)} \tag{5.12}$$

for all $x\in X_1$ *and* $A\in\mathfrak{U}$.

Proof. Let $\gamma_2\notin\overline{\Lambda(A,T_0)+\Lambda(x,T_1)}$. Consider an $f\in L_1(\mathbb{G})$ such that $\hat{f}(\gamma_2)=1$ and $\operatorname{supp}\hat{f}\cap\overline{\Lambda(A,T_0)+\Lambda(x,T_1)}=\varnothing$. We claim that $T_2(f)Ax=0$. Since $A\in\mathfrak{U}$, it is sufficient to verify this equality for vectors in $(X_1,T_1)_{\mathrm{Comp}}$ with compact Beurling spectra.

Let $x\in(X_1,T_1)_{\mathrm{Comp}}$. Since $\gamma_2\notin\overline{\Lambda(A,T_0)+\Lambda(x,T_1)}$, there is an $h\in L_1(\mathbb{G})$ such that $\hat{h}=1$ in the neighbourhood of $\Lambda(x,T_1)$ and $\gamma_2-\gamma_1$ lies outside some neighbourhood of $\Lambda(A,T_0)$ for every $\gamma_1\in\operatorname{supp}\hat{h}$. Therefore, (γ_1,γ_2) lies outside some neighbourhood of $\Lambda(A,\widetilde{T})$ for every $\gamma_1\in\operatorname{supp}\hat{h}$. By Lemma 3.3, we have

$$T_2(f)\,AT_1(h)\,x=T_2(f)\,Ax=0,$$

as was to be shown.

Example 5.16. For the operator $A \in \mathfrak{U}(X_1, X_2)$ defined by the formula $Ax = \xi(x)y$, where $\xi \in (X_1^*)_c$ and $0 \neq y \in X_2$, we have

$$\Lambda(A, \widetilde{T}) = \Lambda(\xi, T_c^*) \times \Lambda(y, T_2),$$
$$\Lambda(A, T_0) = \overline{\{\gamma_2 - \gamma_1 : \gamma_1 \in \Lambda(\xi, T_c^*),\ \gamma_2 \in \Lambda(y, T_2)\}},$$

where $T_c^* \colon \mathbb{G} \to \operatorname{End}(X_1^*)_c$ is the restriction of the representation $T_1^* \colon \mathbb{G} \to \operatorname{End} X_1^*$ (see Example 2.12) to $(X_1^*)_c$. If $y \in (X_2^*)_c$, then $A \in \mathfrak{U}_{1,u} \cap \mathfrak{U}_{0,u}$.

Remark 5.17. $\mathfrak{U}_{1,s} \cap \mathfrak{U}_{0,u} = \mathfrak{U}_{1,u} \cap \mathfrak{U}_{0,u}$. For by Lemma 4.3, it is sufficient to verify that

$$u\text{-}\lim T_2(\psi)\, A\bigl(I - T_1(f_\alpha)\bigr) = 0$$

for every $A \in \mathfrak{U}_{1,s}$ with compact Beurling spectrum $\Lambda(A, T_0)$, every $\psi \in L_1(\mathbb{G})$ whose Fourier transform has compact support, and every b. a. i. (f_α) in $L_1(\mathbb{G})$. This follows from formula (5.12). Indeed, let φ be a function belonging to $L_1(\mathbb{G})$ such that $\operatorname{supp}\widehat{\varphi}$ is a compact set and $\bigl[\operatorname{supp}\widehat{\varphi} + \Lambda(A, T_0)\bigr] \cap \operatorname{supp}\widehat{\psi} = \varnothing$. Then $\lim_\alpha f_\alpha * \varphi = \varphi$, whence $\lim\bigl(T_1(f_\alpha) - I\bigr)\, T_1(\varphi) = 0$. Formula (5.12) implies that $T_2(\psi)\, AT_1(\varphi) = 0$. Therefore,

$$u\text{-}\lim T_2(\psi)\, A\bigl(I - T_1(f_\alpha)\bigr) = u\text{-}\lim T_2(\psi)\, A\bigl(T_1(\varphi) + \bigl(I - T_1(\varphi)\bigr)\bigr)\bigl(I - T_1(f_\alpha)\bigr) = 0.$$

Remark 5.18. For strongly continuous representations T_1 and T_2, Theorem 5.13 was proved in [28] by another method. We hope that Lemma 5.12 will make it possible to apply Theorem 5.13 to other classes of representations.

§ 6. Causal operators. Examples

Let $\mathfrak{A}$ be a partially ordered set (poset) of indices and let $\widetilde{X}_i = \{X_i^\alpha, \alpha \in \mathfrak{A}\}$, $i = 1, 2$, be two families of closed linear subspaces of Banach spaces X_i, $i = 1, 2$, respectively.

Definition 6.1. An operator $A \in \operatorname{Hom}(X_1, X_2)$ is said to be *causal* with respect to the families of subspaces $\widetilde{X}_i$, $i = 1, 2$, if $AX_1^\alpha \subseteq X_2^\alpha$ for all $\alpha \in \mathfrak{A}$, that is, the ordered pair of subspaces (X_1^α, X_2^α) is invariant under A.

To the best of our knowledge, this is the most general definition of causal operator. However, it is difficult to construct a sufficiently rich theory from it. We shall thus impose certain restrictions on the families $\widetilde{X}_i$, $i = 1, 2$, which will enable us to construct such a theory. We shall give another two definitions of causal operators, which are consistent with familiar ones and equivalent under certain natural conditions.

We consider Banach $L_1(\mathbb{G})$-modules (X_i, T_i), $i = 1, 2$, and a closed semigroup $\mathbb{S} \subset \widehat{\mathbb{G}}$ such that zero belongs to the closure of its interior $\operatorname{Int}\mathbb{S}$ (see Remark 6.14). Sometimes we shall have to impose additional restrictions on $\mathbb{S}$. As before, we assume that Assumption 2.1 holds.

Let the poset $\mathfrak{A}$ be the group $\widehat{\mathbb{G}}$ with the following partial ordering: $\gamma_1 \succ \gamma_2 \Leftrightarrow \gamma_1 \in \gamma_2 + \mathbb{S}$. Consider the families of spectral subspaces

$$X_i^\gamma = X_i(\gamma + \mathbb{S}, T_i), \qquad \gamma \in \widehat{\mathbb{G}}, \quad i = 1, 2.$$

Definition 6.2. An operator $A \in \operatorname{Hom}(X_1, X_2)$ is said to be *causal* with respect to the representations T_i, $i = 1, 2$, and the semigroup $\mathbb{S}$ if $AX_1^\gamma \subseteq X_2^\gamma$ for all $\gamma \in \widehat{\mathbb{G}}$.

For operators belonging to the Banach $L_1(\mathbb{G})$-module $\mathfrak{U}$ (that is, one of the modules defined by formula (5.7)) we make another definition.

Definition 6.3. An operator $A \in \mathfrak{U}$ is said to be *causal* with respect to the representation $T_0 \colon \mathbb{G} \to \operatorname{End}\mathfrak{U}$ and the semigroup $\mathbb{S}$ if $\Lambda(A, T_0) \subseteq \mathbb{S}$.

Theorem 6.4. *Let $A \in \mathfrak{U}$ and let the representation T_0 be* (*as usual*) *defined by formula* (5.6). *Then Definitions* 6.2 *and* 6.3 *are equivalent.*

Proof. Let $A \in \mathfrak{U}$ be causal in the sense of Definition 6.2. We claim that $\Lambda(A, T_0) \subseteq \mathbb{S}$. By Theorem 5.13, it is sufficient to establish that $\gamma_2 - \gamma_1 \in \mathbb{S}$, that is, $\gamma_2 \in \gamma_1 + \mathbb{S}$ if $(\gamma_1, \gamma_2) \in \Lambda(A, \widetilde{T})$ ($\widetilde{T}$ is defined by formula (5.1)). Assume the opposite, that is, assume that there is an ordered pair (γ_1^0, γ_2^0) in $\Lambda(A, \widetilde{T})$ such that $\gamma_2^0 \notin \gamma_1^0 + \mathbb{S}$. Then one can find $f_1, f_2 \in L_1(\mathbb{G})$ such that $\hat{f}_1(\gamma_1^0)\hat{f}_2(\gamma_2^0) \neq 0$, $\operatorname{supp} \hat{f}_2 \cap (\gamma_* + \mathbb{S}) = \varnothing$ and $\operatorname{supp} \hat{f}_1 \subset (\gamma_* + \mathbb{S})$ for some $\gamma_* \in \widehat{\mathbb{G}}$. The existence of such a γ_* follows from the above assumption on the semigroup $\mathbb{S}$. Since A is causal (in the sense of Definition 6.2), Lemma 3.3 implies that

$$\Lambda\big(f_2\big(A(f_1 x)\big)\big) \subseteq \operatorname{supp} \hat{f}_2 \cap \Lambda\big(A(f_1 x)\big) \subseteq \operatorname{supp} \hat{f}_2 \cap (\gamma_* + \mathbb{S}) = \varnothing$$

for all $x \in X_1$. Therefore, $T_2(f_2)\, AT_1(f_1) = 0$, whence $(\gamma_1^0, \gamma_2^0) \notin \Lambda(A, \widetilde{T})$. This is a contradiction.

Now let A be causal in the sense of Definition 6.3. Formula 5.12 implies that for any $\gamma \in \widehat{\mathbb{G}}$ and $x \in X_1^\gamma$ we have

$$\Lambda(Ax, T_2) \subseteq \overline{\Lambda(A, T_0) + \Lambda(x, T_1)} \subseteq \gamma + \mathbb{S} + \mathbb{S} \subseteq \gamma + \mathbb{S},$$

that is, $\Lambda(Ax, T_2) \subseteq X_2^\gamma$. The theorem is proved.

We denote the set of causal operators in $\mathfrak{U} \subseteq \operatorname{Hom}(X_1, X_2)$ by $\mathfrak{Caus}(X_1, X_2; T_0, \mathbb{S})$, or $\mathfrak{Caus}(X_1, X_2)$ if the choice of the semigroup $\mathbb{S}$ and the representations T_1, T_2 occurring in the construction of T_0 is clear.

Definition 6.5. The set $\Lambda(A, T_0) \setminus \{0\} \subseteq \widehat{\mathbb{G}}$ will be called the *memory* of the linear operator $A \in \mathfrak{U}$. An operator $A_0 \in \mathfrak{U}$ will be called an operator *with no memory* if $\Lambda(A_0, T_0) \subseteq \{0\}$.

Hence, an operator $A \in \mathfrak{U} = \mathfrak{U}(X_1, X_2)$ is causal with respect to the semigroup $\mathbb{S}$ if and only if its memory is contained in $\mathbb{S}$. In particular, any operator in $\mathfrak{U}$ with no memory is causal. The set of operators with no memory is a closed linear subspace that coincides with the spectral submodule $\mathfrak{U}\big(\{0\}, T_0\big)$. In what follows it will be denoted by one of the following symbols: $\mathcal{M}(X_1, X_2)$, $\mathcal{M}(X_1, X_2; T_0)$, $\mathcal{M}(\mathfrak{U})$, $\mathcal{M}(\mathfrak{U}, T_0)$. If $A \in \mathcal{E}\mathfrak{rg}_0(\mathfrak{U})$ (see Definition 4.10), then $A_0 = \mathcal{M}(A)$ will stand for its part with no memory, that is, $A_0 = \mathcal{M}(A) = \lim T_0(f_\alpha)A$, where (f_α) is a 0-net in $L_1(\mathbb{G})$.

Lemma 6.6. *The space $\mathcal{M}(\mathfrak{U})$ of operators with no memory has the following two representations*:

$$\mathcal{M}(\mathfrak{U}) = \{A \in \mathfrak{U}\colon T_0(g)A = A \;\forall g \in \mathbb{G}\}, \tag{6.1}$$

$$\mathcal{M}(\mathfrak{U}) = \{A \in \mathfrak{U}\colon AX_1(\bar\sigma, T_1) \subseteq X_2(\bar\sigma, T_2) \;\forall \sigma \in \widehat{\mathbb{G}}\}. \tag{6.2}$$

Proof. The representation (6.1) follows immediately from Remark 4.16, since operators with no memory are, by definition, eigenvectors of the $L_1(\mathbb{G})$-module $(\mathfrak{U}, T_0)$ corresponding to the eigencharacter $\gamma_0 = 0 \in \widehat{\mathbb{G}}$. The representation (6.2) follows from formula (5.12).

Remark 6.7. There are examples of operators $A \in \operatorname{Hom}(X_1, X_2)$ satisfying the equality $T_0(g)A = A$ but lying outside $\mathfrak{U}$ (and hence, outside $\mathcal{M}(\mathfrak{U})$). Such an example can be found in [3], § 5.1.11 for $X_1 = X_2 = l_p$ and $T = S$.

Remark 6.8. Now (and up to Example 6.10) let $X_1 = X_2 = X$ and $T_1 = T_2 = T$. Then it may happen that $\mathfrak{U} = \mathfrak{U}_{1,s} \cap \mathfrak{U}_{0,s}$ is not an algebra, and we denote by $\mathfrak{U}_a$ or $\mathfrak{U}_a(X)$ one of the subspaces defined in (5.7), with the exception of $\mathfrak{U}_{1,s} \cap \mathfrak{U}_{0,s}$, in the case when the spaces of representations and the representations themselves coincide.

Lemma 6.9. *$\mathfrak{U}_a$ is a Banach algebra.*

Proof. First we claim that

$$\mathfrak{U}_a = \mathfrak{U}_{1,s}\big((f_\alpha)\big) \cap \mathfrak{U}_{0,u} = \mathfrak{U}_{1,u} \cap \mathfrak{U}_{0,u}$$

is an algebra (see Remark 5.17). Let $A, B \in \mathfrak{U}_a$. Let us verify that $AB \in \mathfrak{U}_{1,u}$. Let $\varepsilon > 0$ and $\psi \in L_1(\mathbb{G})$. By Remark 5.17, there is a $\varphi \in L_1(\mathbb{G})$ such that

$$\big\|T(\psi)\, A\big(I - T(\varphi)\big)\big\| < \frac{\varepsilon}{2(1+c)\|B\|}\,,$$

where $c = \sup \|f_\alpha\|$. Then

$$C_\alpha = T(\psi)\, AB(I - f_\alpha)\, x = \psi A\big(\varphi + (1-\varphi)\big) B(1 - f_\alpha)\, x$$

for all $x \in X$. Therefore, there is an α_0 such that $\|C_\alpha\| < \varepsilon$ for all $\alpha \succ \alpha_0$.

We have $AB \in \mathfrak{U}_{0,u}$ since $T_0(g)\, AB = \big(T_0(g)\, A\big)\big(T_0(g) B\big)$, $g \in \mathbb{G}$. Hence,

$$AB \in \mathfrak{U}_a = \mathfrak{U}_{1,s}\big((f_\alpha)\big) \cap \mathfrak{U}_{0,u} = \mathfrak{U}_{1,u} \cap \mathfrak{U}_{0,u}.$$

We shall prove the assertion of the lemma only for the submodule $\mathfrak{U}_a = \mathfrak{U}_3 \cap \mathfrak{U}_{0,s}$ (for the others the proof is equally simple). Let $A, B \in \mathfrak{U}_a$. If $x_\alpha \overset{\text{loc}}{\to} x_0$ for the net (x_α) in X, then $Bx_\alpha \overset{\text{loc}}{\to} Bx_0$, whence $ABx_\alpha \overset{\text{loc}}{\to} ABx_0$, that is, $AB \in \mathfrak{U}_3$. It is clear that $AB \in \mathfrak{U}_{0,s}$, whence $AB \in \mathfrak{U}_a = \mathfrak{U}_3 \cap \mathfrak{U}_{0,s}$. The lemma is proved.

The set $\mathfrak{Caus}(X, X)$ of causal operators belonging to $\mathfrak{U}_a$ will be denoted by $\mathfrak{Caus}(X)$.

Here are several examples of causal operators.

Example 6.10. Let X_1 and X_2 be Banach spaces with resolutions of the identity $\mathcal{E}_i = \{E_n^i,\ n \in \Omega_i \subseteq \mathbb{Z}\}$, $i = 1, 2$, equipped with the structures of Banach $L_1(\mathbb{T})$-modules (as in Example 2.9: see formulae (2.6) and (2.7)) associated with the representations U_1 and U_2. We have

$$\Lambda(x, U_i) = \big\{k \in \Omega_i \colon E_k^i x \neq 0\big\} \subseteq \mathbb{Z} \simeq \widehat{T}, \qquad i = 1, 2,$$

which implies that the spectral subspaces $X_i(\sigma_i)$, $\sigma_i \subset \mathbb{Z}$, $i = 1, 2$, have the following representations:

$$X_i(\sigma_i) = \big\{x \in X_i \colon E_k^i x = 0 \ \forall k \in \Omega_i \setminus \sigma_i\big\}, \qquad i = 1, 2. \tag{6.3}$$

The family $\mathcal{A} = \big\{A_{ij} = E_i^2 A E_j^1; i \in \Omega_1, j \in \Omega_2\big\}$ of operators belonging to $\operatorname{Hom}(X_1, X_2)$ is called the *matrix* of A (with respect to the resolutions $\mathcal{E}_i$). Formula (6.3) implies that an operator $A \in \mathfrak{U} = \operatorname{Hom}(X_1, X_2)$ (where $(X_i, U_i) = (X_i, U_i)_c$, $i = 1, 2$) is causal with respect to the representations U_1, U_2 and the semigroup $\mathbb{Z}_+ = \mathbb{N} \cup \{0\} = \mathbb{R}_+ \cap \mathbb{Z}$ if and only if

$$A_{ij} = E_i^2 A E_j^1 = 0 \quad \forall i < j, \qquad i \in \Omega_1, \quad j \in \Omega_2$$

(that is, its matrix is lower-triangular if $A \in \mathfrak{U}_a = \operatorname{End} X$ and $U_1 = U_2 = U$).

Let us note that since the representation $U_0 \colon \mathbb{T} \to \operatorname{End}\operatorname{Hom}(X_1, X_2)$ given, as usual, by the formula $U_0(\gamma)\, A = U_2(\gamma)\, A U_1(\gamma^{-1})$, $\gamma \in \mathbb{T}$, is defined on the compact group $\mathbb{T}$, it has a Fourier series

$$U_0(\gamma) A \sim \sum_{n \in \Omega_2 - \Omega_1} A_n \gamma^n$$

with $A_n \in \operatorname{Hom}(X_1, X_2)$, $n \in \Omega_2 - \Omega_1$, that is, the Fourier coefficients of the function $\gamma \mapsto U_0(\gamma) A$ have the following form (see also formula (4.2) and [29], [30]):

$$A_n = \int_{\mathbb{T}} U_0(\gamma)\, A \gamma^{-n}\, d\gamma = \sum E_i^2 A E_j^1 = \sum A_{ij},$$

where the sum is taken over the $i \in \Omega_2$ and $j \in \Omega_1$ such that $i - j = n$. This is why the A_n will be called the *diagonals* of A. The operator A belongs to $\mathfrak{U}_{0,u}$ if and only if

$$\lim_{n \to \infty} \left\| A - \sum_{|k| \leqslant n} \left(1 - \frac{|k|}{n}\right) A_k \right\| = 0.$$

This follows from the theorem on the approximation of continuous periodic functions by Cesàro means. In particular, $A \in \mathfrak{U}_{0,u}$, if A has absolutely integrable diagonals, that is, $\sum_{n \in \Omega_2 - \Omega_1} \|A_n\| < \infty$.

Let us also note that $\Lambda(A, U_0) = \{n \in \Omega_2 - \Omega_1 \colon A_n \neq 0\}$. Therefore, the no-memory part $\mathcal{M}(A)$ of A coincides with A_0. Hence, $\mathcal{M}(\mathfrak{U}_a) = \mathcal{M}(\operatorname{End} X)$ is the subalgebra of operators with diagonal matrices.

Example 6.11. Let $X_i = L_p(\Omega, Y_i)$, $p \in [1, \infty]$, $i = 1, 2$, be the Banach spaces of functions considered in Example 2.10. They are the spaces of the representations $V_i \colon \mathbb{G} \to \operatorname{End} X_i$ defined by formula (2.8). The $L_1(\mathbb{G})$-module structures on the X_i are defined by formula (2.9), which implies that the spectrum $\Lambda(x, V_i)$ of any $x \in X_i$, $i = 1, 2$, coincides with its essential support $\operatorname{vrai\,supp} x \subseteq \Omega \subseteq \widehat{\mathbb{G}}$. It should be noted that it is this module structure that is used in the conventional definition of causal operators (see [2], [3]) acting on function spaces.

Consider an operator $B = A_0 + A \in \operatorname{Hom}(X_1, X_2)$ of the form

$$(Bx)(\omega) = \mathcal{A}_0(\omega)\, x(\omega) + \int_\Omega \mathcal{A}(\omega, \gamma)\, x(\gamma)\, d\gamma, \qquad \omega \in \Omega, \tag{6.4}$$

where $\mathcal{A}_0 \in L_\infty\bigl(\Omega, \operatorname{Hom}(Y_1, Y_2)\bigr)$ and $\mathcal{A} \colon \Omega \times \Omega \to \operatorname{Hom}(Y_1, Y_2)$ is a strongly measurable operator-valued function such that $\bigl\|\mathcal{A}(\omega, \gamma)\bigr\| \leqslant a(\omega - \gamma)$ for some $a \in L_1(\widehat{\mathbb{G}})$ and almost all $\omega, \gamma \in \Omega$. In this representation A is an integral operator. The operator B is bounded and $\|B\| \leqslant \|a\|_1 + \|\mathcal{A}_0\|_\infty$. The representations $\widetilde{V} \colon \mathbb{G} \times \mathbb{G} \to \operatorname{End} \mathfrak{U}$ and $V_0 \colon \mathbb{G} \to \operatorname{End} \mathfrak{U}$ corresponding to V (see formulae (5.1) and (5.6)) have the following form on B:

$$\bigl(\bigl(\widetilde{V}(g_1, g_2) B\bigr)\, x\bigr)(\omega) = \int_\Omega \omega(g_2) \gamma(g_1)\, \mathcal{A}(\omega, \gamma)\, x(\gamma)\, d\gamma + \omega(g_1) \omega(g_2)\, \mathcal{A}_0(\omega)\, x(\omega), \tag{6.5}$$

$$\bigl(\bigl(V_0(g) B\bigr)\, x\bigr)(\omega) = \int_\Omega \omega(g) \gamma^{-1}(g)\, \mathcal{A}(\omega, \gamma)\, x(\gamma)\, d\gamma + \mathcal{A}_0(\omega)\, x(\omega), \tag{6.6}$$

where $g, g_1, g_2 \in \mathbb{G}$, $x \in X_1$ and V_0 is continuous in the uniform operator topology. Hence,

$$\bigl(\bigl(\widetilde{V}(f) B\bigr)\, x\bigr)(\omega) = \int_\Omega \hat{f}(\omega, \gamma)\, \mathcal{A}(\omega, \gamma)\, x(\gamma)\, d\gamma + \hat{f}(\omega, \omega)\, \mathcal{A}_0(\omega)\, x(\omega), \tag{6.7}$$

$$\bigl(\bigl(V_0(\varphi) B\bigr)\, x\bigr)(\omega) = \int_\Omega \widehat{\varphi}(\gamma - \omega)\, \mathcal{A}(\omega, \gamma)\, x(\gamma)\, d\gamma + \widehat{\varphi}(0)\, \mathcal{A}_0(\omega)\, x(\omega) \tag{6.8}$$

for all $f \in L_1(\mathbb{G} \times \mathbb{G})$, $\varphi \in L_1(\mathbb{G})$ and $x \in X_1$. Formulae (6.7) and (6.8) imply that

$$\Lambda(A, \widetilde{V}) = \operatorname{vrai\,supp} \mathcal{A}, \qquad \Lambda(A_0, \widetilde{V}) = \bigl\{(\omega, \omega) \in \Omega \times \Omega \colon \omega \in \operatorname{vrai\,supp} \mathcal{A}_0\bigr\}, \tag{6.9}$$

$$\Lambda(A, V_0) = \bigl\{\gamma - \omega \in \widehat{\mathbb{G}} \colon (\omega, \gamma) \in \operatorname{vrai\,supp} \mathcal{A}\bigr\} \subset \overline{\Omega - \Omega}, \tag{6.10}$$

$$\Lambda(A_0, V_0) \subseteq \{0\}, \qquad \Lambda(B, V_0) \subseteq \Lambda(A, V_0) \cup \{0\}. \tag{6.11}$$

Let us note that B belongs to each of the classes $\mathfrak{U}$ (see (5.7)).

If (f_α) is the 0-net in Remark 4.5, then formula (6.8) implies that $\bigl\|V_0(f_\alpha)\, A\bigr\| \leqslant \|\hat{f}_\alpha a\|_1 \to 0$, whence $\lim \bigl\|V_0(f_\alpha)\, A\bigr\| = 0$. Hence, $A \in \mathcal{E}\mathfrak{rg}_0(\mathfrak{U})$ and $\mathcal{M}(A) = 0$. Since $V_0(f_\alpha)\, A_0 = \hat{f}_\alpha(0)\, A_0 = A_0$, we obtain that $A_0 \in \mathcal{M}(\mathfrak{U})$, whence $\mathcal{M}(B) = A_0$.

Now let $\mathbb{S}$ be a semigroup in $\widehat{\mathbb{G}}$ satisfying our conditions. Then (6.8), (6.10) and (6.11) imply that $\Lambda(B, V_0) \subseteq \Lambda(A, V_0) \cup \{0\} \subseteq \mathbb{S}$ if and only if the following equality for the kernel $\mathcal{A}$ of A holds almost everywhere:

$$\mathcal{A}(\omega, \gamma) = 0, \qquad \gamma - \omega \notin \mathbb{S}. \tag{6.12}$$

Therefore, B is causal with respect to $\mathbb{S}$ if and only if (6.12) holds for the kernel of A.

Such representations (formulae (6.5)–(6.8) and (6.9)–(6.12)) hold for various classes of singular integral operators.

Example 6.12. Let $X_i = \mathcal{F}(\mathbb{G}, Y_i)$, $i = 1, 2$, be Banach spaces of functions each coinciding with one of the spaces L_∞, C_b, C_0, AP (see Example 2.10). We assume that $X_2 = L_\infty(\mathbb{G}, Y_2)$ if $X_1 = L_\infty(\mathbb{G}, Y_1)$, $X_2 = C_b(\mathbb{G}, Y_2)$ if $X_1 = C_b(\mathbb{G}, Y_1)$, and so on. We equip the Banach spaces under consideration with the structure of a Banach $L_1(\mathbb{G})$-module using formula (2.11) and the representation S (of the group of shifts of functions). Consider a bounded continuous function $\mu\colon \mathbb{G} \to M\big(\mathbb{G}, \mathrm{Hom}(Y_1, Y_2)\big)$, where $M\big(\mathbb{G}, \mathrm{Hom}(Y_1, Y_2)\big)$ is the Banach space of bounded operator-valued measures on $\mathbb{G}$ (see [31]). To this function we assign the operator $A \in \mathrm{Hom}(X_1, X_2)$ defined by the formula

$$(Ax)(g) = \int_{\mathbb{G}} \mu(g)\,(ds)\,x(s+g), \qquad g \in \mathbb{G}, \quad x \in X_1. \tag{6.13}$$

We have $\|A\| \leqslant \sup_{g\in\mathbb{G}} \|\mu(g)\|$. For this definition to be valid, we must impose certain restrictions on μ. If $\mathcal{F} = L_\infty$, then we assume that the values of μ are measures absolutely continuous with respect to the Haar measure on $\mathbb{G}$ (so that we can assume that $\mu(g) \in L_1\big(\mathbb{G}, \mathrm{Hom}(Y_1, Y_2)\big)$ for all $g \in \mathbb{G}$). However, in this case we can put $X_i = L_p(\mathbb{G}, Y_i)$, $p \in [1, \infty]$, $i = 1, 2$. This class contains the integral operators

$$(A_1x)(g) = \int_{\mathbb{G}} \mathcal{K}(g, s)\,x(s)\,ds, \qquad g \in \mathbb{G}, \quad x \in X_1, \tag{6.14}$$

where the kernel $\mathcal{K}\colon \mathbb{G}\times\mathbb{G} \to \mathrm{Hom}(Y_1, Y_2)$ has the following property: the function $s \mapsto \mathcal{K}(s, s+g)\colon \mathbb{G} \to L_1\big(\mathbb{G}, \mathrm{Hom}(Y_1, Y_2)\big)$, which will be denoted by $\widetilde{\mathcal{K}}$, belongs to the space $C_b\big(\mathbb{G}, L_1\big(\mathbb{G}, \mathrm{Hom}(Y_1, Y_2)\big)\big)$.

If $\mathcal{F} = AP$, then we assume that μ is almost periodic. In this case $A\colon AP(\mathbb{G}, Y_1) \to AP(\mathbb{G}, Y_2)$. For C_b and C_0 no supplementary restrictions on μ are needed.

There is another special case of operators of the form (6.13):

$$(A_2x)(g) = \sum_{k\geqslant 1} F_k(g)\,x(g+g_k), \qquad g, g_k \in \mathbb{G}, \quad x \in X_1. \tag{6.15}$$

The functions $F_k\colon \mathbb{G} \to \mathrm{Hom}(Y_1, Y_2)$, $k \geqslant 1$, are assumed to belong to $L_\infty\big(\mathbb{G}, \mathrm{Hom}(Y_1, Y_2)\big)$ if $X_i = L_p(\mathbb{G}, Y_i)$, $i = 1, 2$, $p \in [1, \infty]$. We assume that $F_k \in C_b\big(\mathbb{G}, \mathrm{Hom}(Y_1, Y_2)\big)$, $k \geqslant 1$, if $X_i = C_b(\mathbb{G}, Y_i)$ or $X_i = C_0(\mathbb{G}, Y_i)$,

$i = 1, 2$. The operator is well defined on the space of almost periodic functions (if and) only if the F_k, $k \geqslant 1$, are almost periodic functions.

Formulae (6.13)–(6.15) and (5.6) imply that

$$\big(S_0(u)\, Ax\big)(g) = \big(S(u)\, AS(-u)\, x\big)(g) = \int_{\mathbb{G}} \mu(g+u)\,(ds)\, x(g+s), \quad (6.16)$$

$$\big(S_0(u)\, A_1 x\big)(g) = \int_{\mathbb{G}} \mathcal{K}(g+u, s+u)\, x(s)\, ds, \quad (6.17)$$

$$\big(S_0(u)\, A_2 x\big)(g) = \sum_{k\geqslant 1} F_k(g+u)\, x(g+g_k), \quad (6.18)$$

where $g, u \in \mathbb{G}$ and $x \in X_1$. We deduce from (6.16)–(6.18) that A belongs to $\mathfrak{U}_{0,u} \subset \operatorname{Hom}(X_1, X_2) = \mathfrak{U}$ if μ is uniformly continuous. In particular, $A_i \in \mathfrak{U}_{0,u}$, $i = 1, 2$, if the functions $\widetilde{\mathcal{K}}$ and F_k, $k \geqslant 1$, are uniformly continuous.

For any $f \in L_1(\mathbb{G})$ the operator $S_0(f)\, A = fA \in \operatorname{Hom}(X_1, X_2)$ has the form

$$\big((fA)\, x\big)(g) = \int_{\mathbb{G}} f_\mu(g)\,(ds)\, x(g+s), \qquad x \in X_1, \quad (6.19)$$

where

$$f_\mu(g) = (f * \mu)(g) = \int_{\mathbb{G}} f(\tau)\mu(g-\tau)\,(d\tau). \quad (6.20)$$

Therefore, $\Lambda(A, S_0) = \Lambda(\mu)$ is the Beurling spectrum of μ regarded as an element of the space $C(\mathbb{G}, M\big(\mathbb{G}, \operatorname{Hom}(Y_1, Y_2)\big)$ equipped with the structure of an $L_1(\mathbb{G})$-module via convolution. In particular, $\Lambda(A_2, S_0) = \overline{\bigcup_{k\geqslant 1} \Lambda(F_k)}$.

Hence, A is causal with respect to the representation S_0 and a semigroup $\mathbb{S} \subset \widehat{\mathbb{G}}$ if and only if

$$\Lambda(\mu) \subseteq \mathbb{S}. \quad (6.21)$$

For A_2 this condition can be written as

$$\overline{\bigcup_{k\geqslant 1} \Lambda(F_k)} \subseteq \mathbb{S}. \quad (6.22)$$

Finally, let us note that an A defined by (6.13) is an operator with no memory if and only if μ is constant, that is, A is the operator of convolution with a measure belonging to $M\big(\mathbb{G}, \operatorname{Hom}(Y_1, Y_2)\big)$. A_1 has this property only if $\mathcal{K}(g, s) = \mathcal{K}_0(g-s)$, $g, s \in \mathbb{G}$, for some $\mathcal{K}_0 \in L_1\big(\mathbb{G}, \operatorname{Hom}(Y_1, Y_2)\big)$.

Example 6.13. Let $A\colon D(A) \subset X_2 \to X_2$ be a linear operator that is the generator of a strongly continuous isometric group of operators $\{T_2(t); t \in \mathbb{R}\} \subset \operatorname{End} X_2$, and let $B\colon D(A) \to X_2$ be a linear operator subordinate to A. We denote by X_1 the Banach space $D(A)$ equipped with the graph norm $\|x\|_A = \|x\| + \|Ax\|$, $x \in D(A)$. We denote by $T_1\colon \mathbb{R} \to \operatorname{End} X_1$ the restriction of T_2 to X, which is also a strongly continuous isometric representation. Assume that the function $\mathcal{B}\colon \mathbb{R} \to \operatorname{Hom}(X_1, X_2)$ defined by the formula $\mathcal{B}(t) = T_0(t)B = T_2(t)BT_1(-t)$, $t \in \mathbb{R}$, is continuous (in the uniform operator topology), that is, $B \in \mathfrak{U}_{0,u}$ for

$\mathfrak{U} = \operatorname{Hom}(X_1, X_2)$. The operator $A+B \in \mathfrak{U}$ is causal with respect to $T_0 \colon \mathbb{R} \to \operatorname{End}\mathfrak{U}$ and $\mathbb{S} = \mathbb{R}_+ \subset \mathbb{R}$ in the sense of Definition 6.3 if and only if $\Lambda(A+B, T_0) \subseteq \mathbb{R}_+$. Since $T_0(t)A = A$, that is, $A \in \mathcal{M}(\mathfrak{U})$, this condition is equivalent to the condition $\Lambda(B, T_0) \subseteq \mathbb{R}_+$, which (see Lemma 8.2 below) is in turn equivalent to the existence of a bounded holomorphic continuation of $\mathcal{B}$ to the half-plane $\mathbb{C}_+ = \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}$.

In particular, if $A = d/dt$ is defined on a Banach space X_2 coinciding with one of the Banach spaces $L_p(\mathbb{R}, Y)$, $p \in [1, \infty]$, $C_b(\mathbb{R}, Y)$, $C_0(\mathbb{R}, Y)$, and $B \in \operatorname{End} X_2$, then $A+B$ is causal with respect to $T_0 = S_0 \colon \mathbb{R} \to \operatorname{End}(\operatorname{End} X_2)$ if and only if $\mathcal{B}$ has a bounded holomorphic continuation to $\mathbb{C}_+$. If B is an almost periodic operator with respect to S_0, $B \sim \sum_{j \geqslant 0} B_j$ is its Fourier series and $S_0(t)B_j = e^{i\lambda_j t} B_j$, $j \geqslant 0$, then it is causal if and only if $\lambda_j \geqslant 0$ for all $j \geqslant 0$. In this case $\mathcal{B} = B_0$ if $\lambda_0 = 0$.

Remark 6.14. The condition $0 \in \overline{\operatorname{Int} \mathbb{S}}$ was used only in the proof of the equivalence of Definitions 6.2 and 6.3. In what follows it will be used only in Theorem 7.23.

§ 7. Causal invertibility. The algebra of causal operators and its radical

We consider Banach $L_1(\mathbb{G})$-modules (X_i, T_i), $i = 1, 2$, a semigroup $\mathbb{S} \subset \widehat{\mathbb{G}}$ and the set $\mathfrak{Caus}(X_1, X_2) \subset \mathfrak{U} = \mathfrak{U}(X_1, X_2) \subset \operatorname{Hom}(X_1, X_2)$ of operators causal with respect to T_0 and $\mathbb{S}$. As before, $\mathfrak{U}$ is one of subspaces (5.7) and $\mathfrak{U}_a = \mathfrak{U}_a(X)$ is a closed subalgebra of $\operatorname{End} X$ that coincides with $\mathfrak{U}(X, X)$, with the exception of the case when $\mathfrak{U} = \mathfrak{U}_{1,s} \cap \mathfrak{U}_{0,s}$. To say that $A \in \mathfrak{Caus}(X)$ means that $X_1 = X_2 = X$ and $T_1 = T_2 = T$.

Definition 7.1. An operator $A \in \mathfrak{Caus}(X_1, X_2)$ is said to be *hypercausal* if $0 \notin \Lambda(A, T_0)$. It is said to be *uniformly* (or *strongly*) *causal* if $u\text{-}\lim f_\alpha A = 0$ (or $s\text{-}\lim f_\alpha A = 0$) for some 0-net (f_α) in $L_1(\mathbb{G})$ (and then the same is true for all 0-nets (f_α) in $L_1(\mathbb{G})$).

We denote the set of hypercausal operators belonging to $\mathfrak{Caus}(X_1, X_2)$ by $\mathcal{HC}(X_1, X_2)$ (or $\mathcal{HC}(X)$, if $X_1 = X_2 = X$). The set of uniformly causal operators will be denoted by $\mathcal{UC}(X_1, X_2)$ (or $\mathcal{UC}(X)$, if $X_1 = X_2 = X$).

The results of §§ 3, 4 (see Theorem 4.19) imply the following theorem.

Theorem 7.2. *$\mathcal{UC}(X_1, X_2)$ is a closed submodule of the $L_1(\mathbb{G})$-module $(\mathfrak{Caus}(X_1, X_2), T_0)$ and of the $L_1(\mathbb{G} \times \mathbb{G})$-module $(\mathfrak{Caus}(X_1, X_2), \widetilde{T})$. It has the following properties*:

(i) *any operator belonging to $\mathcal{UC}(X_1, X_2)$ is the limit* (*in the operator norm*) *of a sequence* (*a net*) *of hypercausal operators*,

(ii) $\mathcal{E}\mathfrak{rg}_0(\mathfrak{Caus}(X_1, X_2)) = \mathcal{M}(\mathfrak{U}) \oplus \mathcal{UC}(X_1, X_2)$,

(iii) *for any $A \in \mathcal{UC}(X_1, X_2)$ there is an $f \in L_1(\mathbb{G})$, $\hat{f}(0) = 0$, such that $A = fA = T_0(f)A$*,

(iv) $\mathcal{AP}((\mathfrak{Caus}(X_1, X_2), T_0)) \subseteq \mathcal{M}(\mathfrak{U}) \oplus \mathcal{UC}(X_1, X_2)$, *and the almost periodic operator $A \in \mathfrak{Caus}(X_1, X_2)$ belongs to $\mathcal{UC}(X_1, X_2)$ if and only if $0 \notin \Lambda_b(A, T_0)$*,

(v) *$\mathcal{UC}(X)$ is a closed two-sided ideal of the Banach algebra $\mathfrak{Caus}(X)$ of causal operators.*

Remark 7.3. $A \in \mathfrak{Caus}(X_1, X_2)$ is uniformly causal if and only if 0 is an ergodic point of $\Lambda(A, T_0)$ that does not belong to the Bohr spectrum of A (regarded as an element of the $L_1(\mathbb{G})$-module $(\mathfrak{U}, T_0)$).

In the case when $X_1 = X_2 = H$ is a Hilbert space, a definition similar to that of uniformly causal operators was made in [1] in a somewhat different way (in what follows we use the notation of Example 2.8). An operator $A \in \mathfrak{Caus} H$ is said to be *strictly causal* (see [1]) if for any $\varepsilon > 0$ there is a partition $\mathbb{R} = \bigcup_{i=1}^n \Delta_i$, $\Delta_i \in \Sigma$, $1 \leqslant i \leqslant n$, such that $\|E(\Delta_i) A E(\Delta_i)\| < \varepsilon$.

Remark 7.4. Let us return to the examples of causal operators considered in § 6. The integral operators in Example 6.12 (in this case $A_0 = 0$) are uniformly causal. For an operator A defined by formula (6.13) to be uniformly causal it is necessary (if $\dim E, \dim F < \infty$) and sufficient that $\mu \in \mathcal{E}\mathfrak{rg}_0\bigl(C_b\bigl(\mathbb{G}, M\bigl(\mathbb{G}, \operatorname{Hom}(E, F)\bigr)\bigr), S_0\bigr)$ and $0 \notin \Lambda_b(\mu)$. This condition is equivalent to the following: 0 is an ergodic point of A and zero does not belong to the Bohr spectrum of μ. Hence, A is uniformly causal if $\lim f_\alpha * \mu = 0$ for at least one 0-net in $L_1(\mathbb{G})$.

If $A \in \mathfrak{U} = \operatorname{Hom}(X_1, X_2)$ is the $L_1(\mathbb{T})$-module considered in Example 6.10, then the inclusion $\Lambda(A, T_0) \subseteq \Omega_2 - \Omega_1 \subseteq \mathbb{Z}$ implies that zero (if it belongs to $\Omega_2 - \Omega_1$) can only be an isolated point of $\Lambda(A)$, whence $A \in \mathcal{M}(\mathfrak{U}) \oplus \mathcal{UC}(X_1, X_2)$.

The majority of the results stated below hold under the following assumption.

Assumption 7.5. $\mathfrak{U}(X_1, X_2) \subseteq \mathfrak{U}_{0,u}(X_1, X_2)$ *and* $\mathbb{S} \cap (-\mathbb{S}) = \{0\}$.

Lemma 7.6. *Let* (X_i, T_i), $i = 1, 2, 3$, *be Banach* $L_1(\mathbb{G})$*-modules, let* $A \in \mathfrak{Caus}(X_1, X_2; T_0) \subset \mathfrak{U}(X_1, X_2)$, $B \in \mathfrak{Caus}(X_2, X_3; T_0') \subset \mathfrak{U}(X_2, X_3)$, *where* T_0, *as always, is defined by formula* (5.6), *and let* $T_0' \colon \mathbb{G} \to \operatorname{End}\operatorname{Hom}(X_2, X_3)$, $T_0'(g)\, C = T_3(g)\, C T_2(-g)$, $g \in \mathbb{G}$, $C \in \mathfrak{U}(X_2, X_3)$. *Then* $BA \in \mathfrak{Caus}(X_1, X_3; T_0'')$, *where* $T_0'' \colon \mathbb{G} \to \operatorname{End}\operatorname{Hom}(X_1, X_3)$, $T_0'(g) D = T_3(g)\, C T_1(-g)$, $g \in \mathbb{G}$, $D \in \mathfrak{U}(X_1, X_3)$, *and* $BA \in \mathcal{UC}(X_1, X_3)$ *if one of the operators* A, B *is uniformly causal and Assumption* 7.5 *holds.*

Proof. First we make the following comment to the assumptions of the lemma: the three subspaces of operators occurring in the statement of the lemma are assumed to belong to the same class. For example, if $\mathfrak{U}(X_1, X_2) = \mathfrak{U}_{2,u}(X_1, X_2) \cap \mathfrak{U}_{0,u}(X_1, X_2)$, then $\mathfrak{U}(X_i, X_3) = \mathfrak{U}_{2,u}(X_i, X_3) \cap \mathfrak{U}_{0,u}(X_i, X_3)$, $i = 1, 2$. The assertion that BA belongs to $\mathfrak{U}(X_1, X_3)$ can be proved in the same way as in Lemma 6.9.

Let σ be an arbitrary closed subset of $\widehat{\mathbb{G}}$ and let x belong to $X_1(\sigma, T_1)$. Corollary 5.13 implies that

$$\Lambda(BAx, T_3) \subseteq \overline{\Lambda(B, T_0') + \Lambda(Ax, T_2)} \subseteq \overline{\Lambda(B, T_0') + \Lambda(A, T_0) + \Lambda(x, T_1)} \subseteq \overline{\sigma + \mathbb{S}}. \tag{7.1}$$

Since the semigroup $\mathbb{S}$ is closed, Definition 6.2 implies that BA is causal (with respect to T_0'' and $\mathbb{S}$).

Now assume that one of the operators under investigation is uniformly causal. We can assume without loss of generality that A is uniformly causal, $0 \notin \Lambda(A, T_0)$ and $\Lambda(A, T_0)$ is a compact set. The above inclusions imply that $\Lambda(BA) \subseteq \overline{\Lambda(A) + \mathbb{S}} = \Lambda(A) + \mathbb{S}$. Since $\Lambda(A, T_0)$ is compact, the assumptions on $\mathbb{S}$ imply that $0 \notin \Lambda(BA, T_0'')$. The lemma is proved.

Corollary 7.7. *$\mathfrak{Caus}X$ is a closed subalgebra of $\mathfrak{U}_a \subseteq \operatorname{End} X$, and $\mathcal{UC}(X)$ is a closed two-sided ideal of $\mathfrak{Caus}X$ if Assumption 7.5 holds.*

In the proof of (7.1) we did not use the assumption that A and B are causal. Therefore, the following corollary holds.

Corollary 7.8.

$$\Lambda(BA) \subseteq \overline{\Lambda(A) + \Lambda(B)} \tag{7.2}$$

for all $A \in \mathfrak{U}(X_1, X_2)$ and $B \in \mathfrak{U}(X_2, X_3)$.

Definition 7.9. An $A \in \mathfrak{Caus}(X_1, X_2)$ is said to be *causally invertible* if it is invertible and $A^{-1} \in \operatorname{Hom}(X_2, X_1)$ is causal with respect to $T_0^{-1}\colon \mathbb{G} \to \operatorname{End}\operatorname{Hom}(X_2, X_1)$, $T_0^{-1}(g)B = T_1(g)BT_2(-g)$, $g \in \mathbb{G}$, $B \in \mathfrak{U}(X_2, X_1)$.

We denote by $\sigma_{\mathfrak{Caus}}(A)$ the spectrum of $A \in \mathfrak{Caus}X$ in the algebra $\mathfrak{Caus}X$. Conditions for causal invertibility are of importance in the study of the stability of solutions of differential equations (see [2], § 5.1 and [3], Ch. III).

Lemma 7.10. *Let Assumption 7.5 hold and let $A \in \mathcal{UC}(X_1, X_2)$. Then A is not causally invertible.*

The proof follows immediately from Lemma 7.6 and the fact that the identity operator is not uniformly causal.

We need the following definition.

Definition 7.11 [32]. A subalgebra $\mathcal{A}$ of a Banach algebra $\mathcal{B}$ is said to be *full* (in $\mathcal{B}$) if every $a \in \mathcal{A}$ that is invertible in $\mathcal{B}$ is invertible in $\mathcal{A}$.

The definitions imply that a subalgebra $\mathfrak{U}_a(X)$ is full in $\operatorname{End} X$ if it coincides with one of the subalgebras $\mathfrak{U}_{2,s} \cap \mathfrak{U}_{0,s}$, $\mathfrak{U}_{2,u} \cap \mathfrak{U}_{0,u}$.

Theorem 7.12. *Let $A \in \mathcal{M}\big(\mathfrak{U}(X_1, X_2)\big)$. Then $T_2(f)\,A = AT_1(f)$ for all $f \in L_1(\mathbb{G})$, and $A^{-1} \in \mathcal{M}\big(\mathfrak{U}(X_2, X_1), T_0^{-1}\big)$ if A is invertible. In particular, $\mathcal{M}(\mathfrak{U}_a)$ is a full subalgebra of $\operatorname{End} X$.*

Proof. Since $T_2(f)\,A - AT_1(f)$, $f \in L_1(\mathbb{G})$, belongs to $\mathfrak{U}(X_1, X_2)$, it is sufficient (see Remark 5.10) to prove that its restriction to $(X_1, T_1)_c$ is the zero operator. If $x \in (X_1, T_1)_c$, then we have

$$\big(T_2(f)\,A - AT_1(f)\big)\,x = T_2(f)\,Ax - A\int_{\mathbb{G}} f(g)\,T_1(-g)\,x\,dg = T_2(f)\,Ax - T_2(f)Ax = 0$$

(see Lemma 2.7). The other assertions of the theorem follow immediately from Lemma 6.6.

Taking into account Theorem 7.12, it is natural to call operators belonging to $\mathcal{M}(X)$ *multipliers* of the $L_1(\mathbb{G})$-module X.

Theorem 7.13. *Let Assumption 7.5 hold, let $A \in \mathfrak{Caus}(X_1, X_2)$ be a causally invertible operator and let $0 \in \Lambda_{\mathrm{erg}}(A, T_0)$, that is, $A \in \mathcal{M}(X_1, X_2) \oplus \mathcal{UC}(X_1, X_2)$.*

Then $0 \in \Lambda_{\mathrm{erg}}(A^{-1}, T_0^{-1})$, *the operator* $A_0 = \mathcal{M}(A)$ *is invertible and* $\big(\mathcal{M}(A)\big)^{-1} = \mathcal{M}(A^{-1})$. *If* $A \in \mathfrak{Caus}X$ *and* $0 \in \Lambda_{\mathrm{erg}}(A, T_0)$, *then* $\sigma_{\mathfrak{Caus}}(A) \supseteq \sigma(A_0)$.

Proof. Assertion (i) of Theorem 4.19 implies that A can be represented in the form $A = A_0 + A_1$, where $A_1 \in \mathcal{UC}(X_1, X_2)$. By Lemma 7.6, $B_1 = A^{-1}A_1$ and $B_2 = A_1 A^{-1}$ belong to the ideals $\mathcal{UC}(X_1)$ and $\mathcal{UC}(X_2)$ of uniformly causal operators, respectively. Then for any 0-net (f_α) in $L_1(\mathbb{G})$ we have $u\text{-}\lim T_0^i(f_\alpha)B_i = 0$, $i = 1, 2$, where $T_0^i(g)\, C_i = T_i(g)\, C_i T_i(-g)$, $C_i \in \operatorname{End} X_i$, $g \in \mathbb{G}$, $i = 1, 2$.

Therefore, there is an $\omega \in \Omega$ such that $\big\|T_0^i(f_\omega)B_i\big\| < 1/2$, $i = 1, 2$. By Theorem 7.12,

$$\begin{gathered} I = A^{-1}A = T_0^1(f_\alpha)\big(A^{-1}(A_0 + A_1)\big) = \big(T_0^{-1}(f_\alpha)A^{-1}\big)A_0 + T_0^1(f_\alpha)B_1, \\ I = AA^{-1} = A_0\big(T_0^{-1}(f_\alpha)A^{-1}\big) + T_0^2(f_\alpha)B_2 \end{gathered} \tag{7.3}$$

for all $\alpha \in \Omega$. Hence, $\|A_\omega A_0 - I\| < 1/2$ and $\|A_0 A_\omega - I\| < 1/2$ for $A_\omega = T_0^{-1}(f_\omega)A^{-1}$. These inequalities imply that $A_\omega A_0$ and $A_0 A_\omega$ are invertible operators, which implies that A_0 has a left $\big((A_\omega A_0)^{-1}A_\omega\big)$ and a right $\big(A_\omega(A_0A_\omega)^{-1}\big)$ inverse operator. Hence, A_0 is an invertible operator and we can pass to the limit in (7.3). Therefore, $u\text{-}\lim T_0^{-1}(f_\alpha)A^{-1}$ exists. Hence, $0 \in \Lambda'_{\mathrm{erg}}(A^{-1}, T_0^{-1})$, $A_0\mathcal{M}(A^{-1}) = I$ and $\mathcal{M}(A^{-1})A_0 = I$. Therefore, $\mathcal{M}(A^{-1}) = \big(\mathcal{M}(A)\big)^{-1}$ and $0 \in \Lambda_{\mathrm{erg}}(A^{-1}, T_0^{-1})$.

The assertion concerning the causal spectrum follows from these assertions.

We equip $\operatorname{Hom}(X_1, X_2)$ with two further Banach $L_1(\mathbb{G})$-module structures which will be needed later in the study of $\mathfrak{Caus}(X)$. These structures are associated with the representations

$$\begin{aligned} T_1^r &: \mathbb{G} \to \operatorname{End}\big(\operatorname{Hom}(X_1, X_2)\big), \qquad T_1^r(g)A = AT_1(g), \\ T_2^l &: \mathbb{G} \to \operatorname{End}\big(\operatorname{Hom}(X_1, X_2)\big), \qquad T_2^l(g)A = T_2(g)A, \end{aligned}$$

where $A \in \operatorname{Hom}(X_1, X_2)$ and $g \in \mathbb{G}$. Hence,

$$T_1^r(f)A = AT_1(f), \qquad T_2^l(f)A = T_2(f)A, \qquad f \in L_1(\mathbb{G}), \quad A \in \operatorname{Hom}(X_1, X_2).$$

It is clear that T_1^r and T_2^l commute and $T_0(g) = T_2^l(g)T_1^r(-g)$, $g \in \mathbb{G}$. Let us note that the module structures thus defined are non-degenerate.

We conclude this section with a study of the Banach algebra $\mathfrak{Caus}X$ of causal operators, which is a subalgebra of $\mathfrak{U}_a(X)$ under the conventional assumption concerning the coincidence of the latter algebra with one of the above classes of operators belonging to $\operatorname{End} X$. We denote by $\mathfrak{Rad}(\mathfrak{Caus}X)$ the radical (see [33]) of $\mathfrak{Caus}X$.

It may happen that the radical of $\mathfrak{Caus}X$ does not contain the two-sided ideal $\mathcal{UC}(X)$ of uniformly causal operators. This is why we need some supplementary conditions describing the set $\mathfrak{Rad}(\mathfrak{Caus}X)$.

Definition 7.14. The operators belonging to the subspace

$$\mathcal{RC}(X) = \big((\mathfrak{U}_a, T^l)_c \cup (\mathfrak{U}_a, T^r)_c\big) \cap \mathcal{UC}(X)$$

will be called *radically causal.*

Definition 7.14 implies that $A \in \mathcal{UC}(X)$ is radically causal if and only if one of the following conditions holds:

(i) u-$\lim T(f_\alpha)A = A$ or u-$\lim AT(f_\alpha) = A$ for some b. a. i. (f_α) in $L_1(\mathbb{G})$,

(ii) for any $\varepsilon > 0$ there is a $\varphi \in L_1(\mathbb{G})$ such that $\|T(\varphi)A - A\| < \varepsilon$ or $\|AT(\varphi) - A\| < \varepsilon$,

(iii) one of the functions $g \mapsto T(g)A$, $g \mapsto AT(g) \colon \mathbb{G} \to \operatorname{End} X$ is continuous (in the uniform operator topology).

Let us note that $\mathcal{RC} = \mathcal{UC}$ if $\Lambda(X,T)$ is compact.

Remark 7.15. If $\mathfrak{U}_a$ is contained in one of the algebras $\mathfrak{U}_{2,u}$, $\mathfrak{U}_{0,u}$, then $(\mathfrak{U}_a, T^l)_c = (\mathfrak{U}_a, T^r)_c$. This implies that $\lim T(f_\alpha)\, A = A$ if and only if $\lim AT(f_\alpha) = A$.

Assumption 7.16. *A semigroup $\mathbb{S} \subset \widehat{\mathbb{G}}$ satisfies the following condition: for any compact sets $K_1 \subset \mathbb{S} \setminus \{0\}$ and $K_2 \subset \mathbb{S}$ there is a positive integer m such that $(mK_1) \cap K_2 = \varnothing$, where $mK_1 = \underbrace{K_1 + K_1 + \dots + K_1}_{m}$.*

For example, Assumption 7.16 holds for the semigroup $\mathbb{S} = \mathbb{R}^n_+$ contained in $\mathbb{R}^n$, but does not hold for the semigroup $\mathbb{S} = \{(x,y) \in \mathbb{R}^2 \colon y \geqslant 0\} \subset \widehat{\mathbb{G}} = \mathbb{R}^2$. Let us also note that $-\mathbb{S} \cap \mathbb{S} = \{0\}$ if Assumption 7.16 holds for $\mathbb{S}$.

Theorem 7.17. *If Assumption* 7.16 *holds for a semigroup* $\mathbb{S}$, *then* $\mathcal{RC}(X) \subseteq \mathfrak{Rad}(\mathfrak{Caus}X)$.

Proof. Let $A \in \mathcal{RC}(X)$. Then A is the uniform limit of the sequence of hypercausal operators defined by the formula $A_n = A - T_0(f_n)A$, $n \geqslant 1$, where (f_n) is a 0-net in $L_1(\mathbb{G})$ such that $\hat{f}_n = 1$ in some neighbourhood of zero. It is obvious that $A_n \in \mathcal{RC}(X)$, $n \geqslant 1$. Taking into account that the radical of $\mathfrak{Caus}X$ is closed, it is sufficient to consider the case when $A \in \mathcal{HC}(X)$, that is, $0 \notin \Lambda(A, T_0)$.

Let $f \in L_1(\mathbb{G})$ be an arbitrary function with compact support $\operatorname{supp} \hat{f}$ and B any operator in $\mathfrak{Caus}X$. Corollaries 5.15 and 7.8 imply that

$$\Lambda\bigl(\bigl(T(f)\, AB\bigr)^k, T_0\bigr) \subseteq \operatorname{supp} \hat{f} \cap (k-1)\bigl(\operatorname{supp} \hat{f} \cap \Lambda(A, T_0)\bigr)$$

for all positive integers k. By Assumption 7.16, the right-hand side of this inclusion is empty if $k \in \mathbb{N}$ is sufficiently large. Therefore, $\bigl(T(f)\, AB\bigr)^k = 0$, that is, $T(f)\, AB$ is a nilpotent operator (which implies that $BT(f)\, A$ is a nilpotent operator). This means that $T(f)\, A$ belongs to $\mathfrak{Rad}(\mathfrak{Caus}X)$. We prove likewise that $AT(f) \in \mathfrak{Rad}(\mathfrak{Caus}X)$. By Definition 7.14, the closure of one of the sets $\{T(f)\, A, f \in L_1(\mathbb{G})\}$, $\{AT(f), f \in L_1(\mathbb{G})\}$ contains A, whence $A \in \mathfrak{Rad}(\mathfrak{Caus}X)$.

Corollary 7.18. $\sigma_{\mathfrak{Caus}}(A + B) = \sigma_{\mathfrak{Caus}}(A)$ *for all* $A \in \mathfrak{Caus}X$ *and* $B \in \mathcal{RC}(X)$.

Theorem 7.19. *Let $A \in \mathcal{UC}(X)$ be a compact operator, let T be a strongly continuous representation and let Assumption* 7.16 *hold for a semigroup* $\mathbb{S}$. *Then* $A \in \mathfrak{Rad}(\mathfrak{Caus}X)$.

Proof. Since A is compact and T is strongly continuous, the function $g \mapsto T(g)A \colon \mathbb{G} \to \operatorname{End} X$ is continuous in the strong operator topology, whence $A \in \mathcal{RC}(X)$. It remains to use Theorem 7.17.

We can state other conditions on compact operators under which they belong to the radical of $\mathfrak{Caus}X$. Let us begin with compact operators with no memory. We shall reinforce the corresponding results obtained in [2], [3], [5]. The two-sided ideal of compact operators will be denoted by $\mathfrak{K}(X)$. The ideal of compact causal operators will be denoted by $\mathfrak{KC}(X)$.

Theorem 7.20. *Let $A \in \mathfrak{K}(X) \cap \mathcal{M}(\mathfrak{U}_a)$. Then its image* $\operatorname{Im} A$ *is contained in the closure of the linear span of the set of eigenvectors of the $L_1(\mathbb{G})$-module (X, T).*

Proof. Since $A \in \mathfrak{K}(X)$, the set $\{AT(g)\,x,\ g \in \mathbb{G}\}$ is precompact in X for every fixed $x \in X$. The equalities $T(g)Ax = AT(g)\,x,\ g \in \mathbb{G}$, imply that Ax is an almost periodic vector in X_c (see Definition 4.11 and Theorem 4.12). Therefore, Ax is the limit of a linear combination of eigenvectors of the $L_1(\mathbb{G})$-module (X, T).

Corollary 7.21. *If $\Lambda_b(X,T) = \varnothing$, that is, the Bohr spectrum of every $x \in (X,T)$ is empty, then $\mathfrak{K}(X) \cap \mathcal{M}(\mathfrak{U}_a) = \{0\}$.*

Corollary 7.22. *Let $\Lambda_b(X,T) = \varnothing$, $X = X_c$, $A \in \mathfrak{KC}(X)$, and let 0 be an ergodic point, that is, $A \in \mathcal{E}\mathfrak{rg}_0(\mathfrak{Caus}X, T_0)$. Then $A \in \mathfrak{Rad}(\mathfrak{Caus}X)$.*

Proof. We can represent A in the form $A = A_0 + A_1$, where $A_0 \in \mathcal{M}(\mathfrak{U}_a)$ and $A_1 \in \mathcal{UC}(X)$. By Corollary 7.21, we have $A_0 = 0$. Therefore, $A = A_1 \in \mathcal{UC}(X)$. It remains to use Theorem 7.19.

Theorem 7.23. *Let $A \in \mathfrak{KC}(X)$, $\Lambda_b(X,T) = \varnothing$, and let Assumption* 7.5 *hold. Then $A \in \mathfrak{Rad}(\mathfrak{Caus}X)$.*

Proof. Assume the opposite. Take an α such that $0 \neq \alpha \in \sigma(A)$. The Riesz projector P corresponding to the one-point set $\{\alpha\}$ is defined by the formula

$$P = \frac{1}{2\pi i} \int_\Gamma (A - \lambda I)^{-1}\, d\lambda, \tag{7.4}$$

where Γ is a closed Jordan contour encircling $\{\alpha\}$, separating $\{\alpha\}$ from $\sigma(A) \setminus \{\alpha\}$ and positively oriented. Since the set $\sigma(A)$ is at most countable, we have $\sigma_{\mathfrak{Caus}}(A) = \sigma(A)$ (see [3], [32]). Therefore, the $(A - \lambda I)^{-1}$, $\lambda \in \Gamma$, are causal operators. Hence, P is a causal operator, that is, $P \in \mathfrak{Caus}X \subseteq \mathfrak{U}_{0,s}$. Therefore, one can find a $\gamma \in \widehat{\mathbb{G}}$ such that $PX^\gamma \neq \{0\}$. Moreover, the condition imposed on $\mathbb{S}$ and the fact that (the image of) P is finite-dimensional enable us to assume that $PX^{\gamma+\varepsilon} = \{0\}$ for all $\varepsilon \in \mathbb{S} \setminus \{0\}$. Passing to the restrictions P^γ and T_0^γ of P and T_0 to the spectral subspace X^γ and using formula (5.11), we obtain that $\Lambda(P^\gamma, T_0^\gamma) = \{0\}$. Corollary 7.21 now implies that $P^\gamma = 0$. This is a contradiction.

Remark 7.24. Consider the following properties of an $A \in \mathfrak{Caus}X$:

(i) $A - T_0(f)\,A \in \mathfrak{K}(X)$ for all $f \in M_0 = \bigl\{\varphi \in L_1(\mathbb{G})\colon \widehat{\varphi}(0) = 1\bigr\}$,

(ii) $A - T_0(f)\,A \in \mathcal{RC}(X)$ for all $f \in M_0$.

If $A \in \mathfrak{U}_{0,u}$, then (i) is equivalent to

(i′) $T(g)\,AT(-g) - A \in \mathfrak{K}(X)$ for all $g \in \mathbb{G}$,

and (ii) is equivalent to

(ii′) $T(g)\,AT(-g) - A \in \mathcal{RC}(X)$ for all $g \in \mathbb{G}$.

The operators $A - T_0(f)A$, $f \in M_0$, are uniformly causal and, under certain conditions, belong to the radical $\mathfrak{Rad}(\mathfrak{Caus}X)$ of $\mathfrak{Caus}X$, as was proved in a series of assertions in this section. For example, if (i′) holds and $X = X_c$, then Theorem 7.19 implies that $A \in \mathfrak{Rad}(\mathfrak{Caus}X)$. Since the $A - T_0(f)\,A$, $f \in M_0$, belong to the radical, we have the estimate

$$r(A) \leqslant \inf_{f\in M_0} \|T_0(f)\,A\|$$

for the spectral radius $r(A)$ of A. It should be noted that $A = A_0 + A_1$ if $A \in \mathcal{E}\mathfrak{rg}_0(\mathfrak{Caus}X, T_0)$, where $A_0 \in \mathcal{M}(\mathfrak{U}_a)$ and $A_1 \in \mathcal{UC}(X)$. Moreover, $\sigma_{\mathfrak{Caus}}(A) = \sigma(A_0)$. In particular, $r(A) = r(A_0)$.

This remark implies the following theorem.

Theorem 7.25. *Let $A \in \mathfrak{Caus}X \cap \mathcal{E}\mathfrak{rg}_0(\mathfrak{Caus}X, T_0)$ and $A = A_0 + A_1$, where $A_0 \in \mathcal{M}(\mathfrak{U}_a)$ and $A_1 \in \mathcal{UC}(X)$. If the assumptions of any of Theorems* 7.17, 7.19, 7.22, 7.23 *hold for A_1, then $\sigma_{\mathfrak{Caus}}(A) = \sigma(A_0)$.*

Other approaches to the estimation of the spectral radii of causal operators can be found in [5], § 2.4, and [34]. Special attention should be paid to [2] and [3], where one can find conditions sufficient for an operator to belong to the radical of the algebra of causal operators. These conditions are stated in other terms, and other approaches are used. For example, instead of the Bohr spectrum of a vector in a Banach $L_1(\mathbb{G})$-module, continuous chains of subspaces indexed by points of $\mathbb{R}$ are used (that is, $\mathbb{G} = \mathbb{R}$ and $\mathbb{S} = \mathbb{R}_+$). As a rule, these chains coincide with the spectral submodules $X^t = X\big((-\infty, t], V\big)$, $t \in \mathbb{R}$ (see Example 2.10 with $\Omega = \mathbb{G} = \mathbb{R}$). In this case, simple examples show that Theorem 7.23 has a wider scope of application.

Example 7.26. Let $\mathfrak{Caus}X$ be the algebra of causal operators considered in Example 6.10, where $X_1 = X_2 = X$, $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$, and $E_k^1 = E_k^2 = E_k$, $k \in \Omega_1 = \Omega_2 = \Omega$. We assume for definiteness that $\Omega = \mathbb{Z}$. A causal operator $A \in \mathfrak{Caus}X$ (whose matrix A_{ij} is lower-triangular: $A_{ij} = 0$ if $i < j$) belongs to $\mathcal{UC}(X)$ if its diagonal A_0 is zero. Such an operator is radically causal (which implies that $A \in \mathfrak{Rad}(\mathfrak{Caus}X)$) if $A \in \mathfrak{U}_{0,u}$ and

$$\lim_{k\to\infty} \bigg\| \sum_{i-j=n,\, i\geqslant k} E_i A E_j \bigg\| = 0$$

for all $n \in \mathbb{Z}_+$ (see Theorem 7.23).

Example 7.27. Consider the operator $B = A_0 + A$ defined by formula (6.4) in Example 6.11, where $X_1 = X_2 = X = L_p(\Omega, Y)$, $p \in [1, \infty]$. We assume that condition (6.12) holds for the kernel $\mathcal{A}$ of the integral operator A, that is, B is a causal operator. It was mentioned in Example 6.11 that $\mathcal{M}(A) = 0$ and $\mathcal{M}(A_0) = A_0$. Since the Bohr spectrum $\Lambda_b(\varphi) = \Lambda_b(\varphi, V)$ is empty for every $\varphi \in X$ if $\widehat{\mathbb{G}}$ is a non-discrete group, every compact integral operator A (these conditions hold if $\overline{\Omega}$ is a compact subset of $\widehat{\mathbb{G}}$ and $\dim Y < \infty$) belongs to $\mathfrak{Rad}(\mathfrak{Caus}X)$. Therefore, $\sigma_{\mathfrak{Caus}}(B) = \sigma(A_0)$. If Ω is precompact, then $\Lambda(A, \widetilde{V})$ is contained in the compact set $\overline{\Omega - \Omega}$. Hence, A is radically causal, and Theorem 7.23 implies that $A \in \mathfrak{Rad}(\mathfrak{Caus}X)$.

§ 8. Causal operators and methods of the theory of holomorphic functions. A theorem on the spectral components of causal operators

The fact that causal operators have their Beurling spectra in a semigroup $\mathbb{S}$ enables us to use the theory of holomorphic functions (for a certain class of semigroups in $\mathbb{G}$). To avoid technical difficulties, which would considerably increase the length of this paper, we restrict ourselves to the study of the groups $\mathbb{R}$, $\mathbb{T}$ and the corresponding semigroups $\mathbb{R}_+ \subset \mathbb{R} \simeq \widehat{\mathbb{R}}$ and $\mathbb{Z}_+ \subset \mathbb{Z} \simeq \widehat{\mathbb{T}}$.

An important role in this section is played by the family of functions $\{f_z;\, z \in \mathbb{C}_+\}$ in $L_1(\mathbb{R})$ of the form

$$f_z(t) = \frac{\beta}{\pi}\,\frac{1}{(t+\alpha)^2+\beta^2}\,, \qquad z = \alpha + i\beta, \quad \beta > 0, \tag{8.1}$$

where $\mathbb{C}_+ = \{z \in \mathbb{C}\colon z = \alpha + i\beta,\ \beta > 0\}$. These functions have Fourier transforms of the form

$$\hat{f}_z(\lambda) = e^{-\beta|\lambda| + i\alpha\lambda}, \qquad \lambda \in \mathbb{R}, \quad z = \alpha + i\beta, \quad \beta > 0. \tag{8.2}$$

In the following lemma we denote by $\mathbb{R}_+^{\downarrow}$ the set $\mathbb{R}_+ \setminus \{0\}$ arranged in decreasing order. The symbol $\mathbb{R}_+^{\uparrow}$ stands for the set $\mathbb{R}_+ \setminus \{0\}$ arranged in increasing order.

Lemma 8.1. *The following assertions hold for* $\{f_z;\, z \in \mathbb{C}_+\}$:

(i) *the map*

$$\Phi\colon \mathbb{C}_+ \to L_1(\mathbb{R}), \qquad \Phi(z) = f_z, \quad z \in \mathbb{C}_+, \tag{8.3}$$

from the open half-plane $\mathbb{C}_+$ *to the Banach algebra* $L_1(\mathbb{R})$ *is continuous, and*

$$\|\Phi(z)\| = 1, \qquad z \in \mathbb{C}_+, \tag{8.4}$$

(ii) $\Phi(z_1 + z_2) = \Phi(z_1) * \Phi(z_2)$ *for all* $z_1, z_2 \in \mathbb{C}_+$ *(that is,* Φ *is a homomorphism from the semigroup* $\mathbb{C}_+ \subset \mathbb{C}$ *to the algebra* $L_1(\mathbb{R})$*)*,

(iii) *the net* $(f_{i\beta})$, $\beta \in \mathbb{R}_+^{\downarrow}$, *is a b. a. i. in* $L_1(\mathbb{R})$,

(iv) *for any* $\alpha \in \mathbb{R}$ *the net* $(f_{\alpha+i\beta})$, $\beta \in \mathbb{R}_+^{\uparrow}$, *is an* α*-net in* $L_1(\mathbb{R})$.

Proof. It is obvious that Φ is continuous.

Let us note that every f_z, $z = \alpha + i\beta \in \mathbb{C}_+$, can be written as

$$f_z(t) = f_{i\beta}(t+\alpha), \qquad t \in \mathbb{R},$$

whence $\|f_z\| = \|f_{i\beta}\|$. The $f_{i\beta}$, $\beta > 0$, are positive functions, whence $\|f_z\| = \|f_{i\beta}\| = \bigl|\hat{f}_{i\beta}(0)\bigr| = 1$. Therefore, (8.4) holds.

Assertion (ii) follows from (8.2) and the equalities

$$\hat{f}_{z_1+z_2}(\lambda) = \hat{f}_{z_1}(\lambda)\hat{f}_{z_2}(\lambda), \qquad \lambda \in \mathbb{R}, \quad z_1, z_2 \in \mathbb{C}_+.$$

A simple verification shows that assertions (iii) and (iv) follow from the definitions of b. a. i. and γ-nets (see Definitions 4.1, 4.4 and 4.6).

Lemma 8.2. *Let (X,T) be a Banach $L_1(\mathbb{R})$-module, and let $X_c = X$ and $x \in X$. The inclusion*

$$\Lambda(x,T) \subset \mathbb{R}_+ \tag{8.5}$$

holds if and only if the function

$$\varphi\colon \mathbb{R} \to X, \qquad \varphi(t) = T(t)\,x, \quad t \in \mathbb{R}, \tag{8.6}$$

has precisely one bounded holomorphic continuation $\bar{\varphi}$ to the half-plane $\mathbb{C}_+$. If (8.5) holds, then $\bar{\varphi}$ has the form

$$\varphi(z) = T(f_z)\,x, \qquad z \in \mathbb{C}_+. \tag{8.7}$$

Proof. We denote by B the generator of the $L_1(\mathbb{R})$-module (X,T), that is, iB is the generator of the strongly continuous group of operators $\{T(t);\ t \in \mathbb{R}\}$. If $\Lambda(X)$ is compact, then $B \in \operatorname{End} X$, and there is a holomorphic continuation $\bar{\varphi}$ defined by the formula $\bar{\varphi}(z) = e^{izB}$, $z \in \mathbb{C}_+$ (in fact, it is an entire function). Hence, the problem is reduced to that of finding conditions under which $\bar{\varphi}$ is bounded.

Let $\Lambda(x,T) \in \mathbb{R}_+$. First we assume that $\sigma = \Lambda(x)$ is compact. We assume without loss of generality that $\sigma = \Lambda(X)$ is a compact subset of $\widehat{\mathbb{G}}$ (otherwise we must consider the restriction of T to the spectral submodule $X(\sigma)$). Then $\bar{\varphi}(z) = e^{izB}x$, $z \in \mathbb{C}$, is a holomorphic extension of φ to $\mathbb{C}$. We claim that $\bar{\varphi}$ is bounded in $\mathbb{C}_+$. Let us fix a $z \in \mathbb{C}_+$ and a sequence (ε_n) of positive numbers converging to zero. Consider a sequence of functions $f_{z,n}$, $n \geqslant 1$, belonging to $L_1(\mathbb{R})$ and defined by the formula $f_{z,n}(t) = f_z(t)\exp\bigl((\beta - it)\varepsilon_n\bigr)$, $t \in \mathbb{R}$. It is clear that $\hat{f}_{z,n}(\lambda) = \exp(i\lambda z)$ for $\lambda \in (-\varepsilon_n, \infty)$ and $\lim_{n\to\infty} f_{z,n} = f_z$ (in $L_1(\mathbb{R})$) for every $z \in \mathbb{C}_+$. Theorem 3.8 implies that $\psi_z(B) = \exp(izB) = T(f_{z,n})$, $n \geqslant 1$, for $\psi_z(\lambda) = \exp(\lambda z)$, $\lambda \in \mathbb{C}$, whence

$$e^{izB} = T(f_z) = \lim_{n\to\infty} T(f_{z,n}). \tag{8.8}$$

We thus obtain formula (8.7) and the estimates

$$\|e^{izB}\| = \|T(f_z)\| \leqslant \|f_z\| = 1, \qquad z \in \mathbb{C}_+.$$

If x is an arbitrary vector in X_c, then $x = \lim_{n\to\infty} x_n$, where the $\Lambda(x_n)$, $n \geqslant 1$, are compact sets. Combining these results with part (i) of Lemma 8.1, we obtain that the sequence of holomorphic functions

$$\bar{\varphi}_n(z) = T(f_z)\,x_n, \qquad z \in \mathbb{C}, \qquad n \geqslant 1,$$

converges uniformly in $\mathbb{C}_+$ to the holomorphic function $\bar{\varphi}\colon \mathbb{C}_+ \to X$ given by (8.7). Hence, the desired continuation is unique, which completes the proof of necessity.

Now assume that φ has a bounded holomorphic extension $\bar{\varphi}\colon \mathbb{C}_+ \to X$, and that $\Lambda(x) \cap \mathbb{R}_- \neq \varnothing$, where $\mathbb{R}_- = \mathbb{R} \setminus \mathbb{R}_+ = (-\infty, 0)$. Then there is a non-zero function $\psi \in L_1(\mathbb{R})$ such that $\operatorname{supp}\widehat{\psi}$ is a compact set in $\mathbb{R}_-$ and $y = \psi x$ is a non-zero vector. Hence, $\Delta = \Lambda(y) \subseteq \operatorname{supp}\widehat{\psi} \cap \Lambda(x)$ is a compact set in $\mathbb{R}_-$.

We can again (as in the proof of necessity) assume (by passing to the restriction of T to $X(\Delta)$) that $\Lambda(X) = \Delta$. The function $\varphi_1(t) = T(t)y$, $t \in \mathbb{R}$, (as well as φ) has a holomorphic extension to $\mathbb{C}_+$, which coincides with $\bar{\varphi}_1(z) = T(\psi)\bar{\varphi}(z)$, $z \in \mathbb{C}_+$. This extension in turn coincides with the function $z \mapsto e^{izB}y$, $z \in \mathbb{C}_+$. Lemma 3.6 implies that $\sigma(B) = \Lambda(X) = \Delta \subset (-\infty, -\varepsilon]$ for some $\varepsilon < 0$. Hence, $\|\exp(\beta B)\| \leqslant C\exp(-\varepsilon\beta/2)$ for all $\beta > 0$, where $C \geqslant 1$. Therefore,

$$\|y\| = \|\exp(\beta B)\exp(-\beta B)y\| \leqslant C\exp\left(-\frac{\varepsilon\beta}{2}\right)\|\bar{\varphi}_1(i\beta)\|, \qquad \beta > 0.$$

These estimates imply that $\|\bar{\varphi}_1(i\beta)\| \geqslant C\exp(-\varepsilon\beta/2)$, $\beta > 0$, which contradicts the fact that $\bar{\varphi}$ is bounded in $\mathbb{C}_+$. This contradiction completes the proof.

Combining the proof of sufficiency in the proof of Lemma 8.2 with theorems of Phragmén–Lindelöf type [35], we obtain the following corollary.

Corollary 8.3. *If the function φ (see (8.6)) has (precisely) one holomorphic extension $\varphi\colon \mathbb{C}_+ \to X$ such that*

$$\lim_{0<\beta\to\infty} \frac{\ln\|\varphi(i\beta)\|}{\beta} = 0, \tag{8.9}$$

then $\Lambda(x) \subseteq \mathbb{R}_+$.

Corollary 8.4. *If $\Lambda(x) \subseteq \mathbb{R}_+$, then $T\colon \mathbb{R} \to \operatorname{End} X$ has precisely one bounded holomorphic extension $\overline{T}\colon \mathbb{C}_+ \to \operatorname{End} X$, namely,*

$$\overline{T}(z) = T(\alpha)\exp(-\beta B), \qquad z = \alpha + i\beta \in \mathbb{C}_+, \tag{8.10}$$

where B is the generator of the $L_1(\mathbb{R})$-module $(X,T) = (X,T)_c$. Moreover, $\|\overline{T}(z)\| \leqslant 1$ for all $z \in \mathbb{C}_+$.

Since B is sectorial (see [36]), the operators $\exp(-\beta B)$, $\beta > 0$, are defined as usual in terms of integrals of Dunford–Riesz type [11]. We can prove that the extension of the function φ defined by (8.6) is holomorphic when (8.5) holds using the fact that the restriction of B is sectorial.

Remark 8.5. Assumption (8.9) can be replaced by the assumption that

$$\lim_{n\to\infty} \frac{\ln\|\varphi(i\beta_n)\|}{\beta_n} = 0$$

for some sequence $0 < \beta_n \to \infty$, $n \geqslant 1$. However, in this case we must impose further conditions on $\bar{\varphi}$ to guarantee the uniqueness of $\bar{\varphi}$. For example, it is sufficient to assume that $\sup_{0<\operatorname{Im} z\leqslant\beta} \|\bar{\varphi}(z)\| = \Phi(\beta) < \infty$ for all $\beta > 0$.

Now let us return to the study of the Banach $L_1(\mathbb{R})$-modules (X_i, T_i), $i = 1, 2$, and the Banach $L_1(\mathbb{R})$-module of operators $(\mathfrak{U}, T_0) = (\mathfrak{U}(X_1, X_2), T_0)$. Our assumptions imply that $\mathfrak{U} \subseteq \mathfrak{U}_{0,s}$. Hence, any $A \in \mathfrak{U}$ is a strong limit of the sequence of operators $A_n = T_0(f_n)\,A$, $n \geqslant 1$, where (f_n) is any b. a. i. of $L_1(\mathbb{R})$. If the $\operatorname{supp}\hat{f}_n$, $n \geqslant 1$, are compact sets, then so are the $\Lambda(A_n, T_0)$, $n \geqslant 1$. This enables us to repeat almost verbatim the proof of Lemma 8.2 for the $L_1(\mathbb{R})$-module $(\mathfrak{U}, T_0)$. One need only observe that the operator-valued function holomorphic (on $\mathbb{C}_+$) in the strong operator topology which was obtained in this proof is holomorphic in the uniform operator topology as well. In this way we obtain the following theorem.

Theorem 8.6. *$C \in (\mathfrak{U}, T_0)$ is causal if and only if the function*

$$\varphi_C\colon \mathbb{R} \to \mathfrak{U}, \qquad \varphi_C(t) = T_0(t)C = T_2(t)\, CT_1(-t), \qquad t \in \mathbb{R},$$

has precisely one bounded holomorphic continuation $\bar{\varphi}_C$: $\mathbb{C}_+ \to \mathfrak{U}$. If $C \in \mathfrak{Caus}(X_1, X_2)$, then $\bar{\varphi}_C$ is given by the formula

$$\bar{\varphi}_C(z) = T_0(f_z)C, \qquad z \in \mathbb{C}_+, \tag{8.11}$$

and $T_0(f_z)C \in \mathfrak{Caus}(X_1, X_2)$ for all $z \in \mathbb{C}_+$.

Corollary 8.7. *An operator C is causal if and only if φ_C has a holomorphic continuation to $\mathbb{C}_+$ such that one of the following conditions holds*:

(i) $\lim_{0<\beta\to\infty} \frac{1}{\beta} \ln \|T_0(f_{i\beta})C\| = 0$,

(ii) $\sup_{0<\operatorname{Im} z\leqslant\beta} \big\|T_0(f_z)C\big\| = M(\beta) < \infty$ *for all* $\beta > 0$ *and there is a sequence* $0 < \beta_n \to \infty$, $n \geqslant 1$, *such that* $\lim_{n\to\infty} \frac{1}{\beta_n} \ln\big\|T_0(f_{i\beta_n})C\big\| = 0$.

Theorem 8.8. *Let $A \in \mathfrak{Caus}(X_1, X_2)$. The following assertions are equivalent*:

(i) *A is causally invertible*,

(ii) *A is invertible, and the assumptions of Theorem* 8.6 *hold for $C = A^{-1}$* ,

(iii) *A is invertible, and one of the conditions in Corollary* 8.7 *holds for $C = A^{-1}$.*

This follows from Theorem 8.6 and Corollary 8.7 applied to $C = A^{-1} \in \operatorname{Hom}(X_2, X_1)$. It need only be noted that in this case we use the representation of T_0^{-1} given in Definition 7.9.

Remark 8.9. Let $(X_i, T_i) = (X_i, T_i)_c$, $i = 1, 2, 3$, be three Banach $L_1(\mathbb{R})$-modules. Consider the representations T_0, T_0' and T_0'' introduced in Lemma 7.6 and two causal operators $A \in \mathfrak{Caus}(X_1, X_2)$ and $B \in \mathfrak{Caus}(X_2, X_3)$. By Lemma 7.6, $C = BA$ is also causal, that is, $C \in \mathfrak{Caus}(X_1, X_3)$. Theorem 8.6 implies that the functions $\varphi_A(t) = T_0(t)\, A$, $\varphi_B(t) = T_0'(t)B$ and $\varphi_C(t) = T_0''(t)C$ have holomorphic continuations $\bar{\varphi}_A$, $\bar{\varphi}_B$ and $\bar{\varphi}_C$ to the half-plane $\mathbb{C}_+$ (see formula (8.11)). Since $\varphi_C(t) = \varphi_B(t)\, \varphi_A(t)$ for all $t \in \mathbb{R}$, $\bar{\varphi}_C$ and $\bar{\varphi}_B \bar{\varphi}_A$ are two bounded holomorphic continuations of φ_C to $\mathbb{C}_+$. Then uniqueness of continuation implies that

$$\bar{\varphi}_C(z) = \bar{\varphi}_B(z)\, \bar{\varphi}_A(z), \qquad z \in \mathbb{C}_+. \tag{8.12}$$

Lemma 8.10. *Let $A \in \mathfrak{Caus}(X_1, X_2)$ be causally invertible. Then the $T_0(f_z)(A) = \bar{\varphi}_A(z)$, $z \in \mathbb{C}_+$, are causally invertible, and*

$$\big(T_0(f_z)\, A\big)^{-1} = T_0^{-1}(f_z)\, A^{-1}, \qquad z \in \mathbb{C}_+. \tag{8.13}$$

This follows from Remark 8.9. In this case $X_3 = X_1$, $C = I$, $B = A^{-1}$, $T_0' = T_0^{-1}$ and $T_0''(t)L = T_1(t)LT_1(-t)$, $t \in \mathbb{R}$, $L \in \operatorname{End} X_1$. Formula (8.13) follows from (8.12).

Corollary 8.11. *If $X_1 = X_2 = X$ and $T_1 = T_2 = T$, then*

$$T_0(f_z)(AB) = \big(T_0(f_z)\,A\big)\big(T_0(f_z)B\big)$$

for every $T_0(f_z)$, $z \in \mathbb{C}_+$, for all $A, B \in \mathfrak{Caus}X$, that is, the $T_0(f_z)$, $z \in \mathbb{C}_+$, are homomorphisms of the algebra $\mathfrak{Caus}X$ that preserve the identity element.

Now let $\mathbb{G} = \mathbb{T}$, let (X, T) be a Banach $L_1(\mathbb{T})$-module, let $\widehat{\mathbb{G}} \simeq \mathbb{Z}$ and let $\mathbb{S} = \mathbb{Z}_+$. In what follows we assume that $X_c = X$. Hence, the functions

$$\varphi_x \colon \mathbb{T} \to X, \qquad x \in X, \quad \varphi_x(\theta) = T(\theta)\,x, \quad \theta \in \mathbb{T},$$

are continuous, and we can consider their Fourier series

$$\varphi_x \sim \sum_{n\in\mathbb{Z}} \theta^n P_n x, \qquad \theta \in \mathbb{T}, \quad x \in X, \tag{8.14}$$

where $\{P_n, n \in \mathbb{Z}\}$ is a bounded sequence of projectors.

We state below several assertions similar to 8.2, 8.6, 8.8 and 8.10. We omit the proofs as they are simple and similar to those of the assertions cited (corresponding to the representations of the group $\mathbb{G} = \mathbb{R}$). It is possible to reduce these to results obtained above by considering the (periodic) representation $\overline{T}(t) = T(e^{it})$, $t \in \mathbb{R}$, $\overline{T} \colon \mathbb{R} \to \operatorname{End} X$ (X is regarded as an $L_1(\mathbb{R})$-module). In this case $\Lambda(X, \overline{T}) = \Lambda(X, T) \subseteq \mathbb{Z}$.

Lemma 8.12. *$\Lambda(x, T) \subseteq \mathbb{Z}_+$ for $x \in X$ if and only if the function φ_x given by* (8.14) *has a holomorphic continuation $\bar\varphi_x$ to the disc $\mathbb{D} = \big\{z \in \mathbb{C} \colon |z| \leqslant 1\big\}$. If $\Lambda(x, T) \subseteq \mathbb{Z}_+$, then this holomorphic continuation is given by*

$$\bar\varphi_x(z) = \sum_{n\geqslant 0} z^n P_n x = T(g_z)\,x, \qquad |z| < 1,$$

where $g_z \in L_1(\mathbb{T})$ is defined by the formula

$$g_z(\theta) = \frac{1 - z^2}{(1 - z\theta)(1 - z\bar\theta\,)}, \qquad \theta \in \mathbb{T}, \quad |z| < 1. \tag{8.15}$$

Consider the Banach $L_1(\mathbb{T})$-modules (X_i, T_i), $i = 1, 2$, and the Banach $L_1(\mathbb{T})$-module $(\mathfrak{U}, T_0) = \big(\mathfrak{U}(X_1, X_2), T_0\big)$. We assume that the T_i, $i = 1, 2$, are strongly continuous.

Theorem 8.13. *$C \in (\mathfrak{U}, T_0)$ is causal if and only if the function*

$$\varphi_C \colon \mathbb{T} \to \mathfrak{U}, \qquad \varphi_C(\theta) = T_0(\theta)\,C = T_2(\theta)\,CT_1(\theta^{-1}), \qquad \theta \in \mathbb{T},$$

has a holomorphic extension $\bar\varphi_C \colon \mathbb{D} \to \mathfrak{U}$. If $C \in \mathfrak{Caus}(X_1, X_2)$, then $\bar\varphi_C$ is given by $\bar\varphi_C(z) = T_0(g_z)\,C$, $z \in \mathbb{D}$, where g_z is defined by formula (8.15), *and $T_0(g_z)\,C \in \mathfrak{Caus}(X_1, X_2)$ for all $z \in \mathbb{D}$.*

Theorem 8.14. *Let* $A \in \mathfrak{Caus}(X_1, X_2)$. *The following two conditions are equivalent*:

(i) *A is causally invertible*,

(ii) *the function* $\psi_{A^{-1}}: \mathbb{T} \to \operatorname{Hom}(X_2, X_1)$, $\psi_{A^{-1}}(\theta) = T_1(\theta)\, A^{-1} T_1(\theta)$, $\theta \in \mathbb{T}$, *has a holomorphic extension to* $\mathbb{D}$.

Lemma 8.15. *Let* $A \in \mathfrak{Caus}(X_1, X_2)$ *be causally invertible. Then the* $T_0(g_z)\, A = \bar{\varphi}_A(z)$, $z \in \mathbb{D}$, *are invertible operators*, *and*

$$\bigl(T_0(g_z)\, A\bigr)^{-1} = T_0^{-1}(g_z)\, A^{-1}, \qquad z \in \mathbb{D}. \tag{8.16}$$

Corollary 8.16. *The* $T_0(g_z)\colon \mathfrak{Caus}X \to \mathfrak{Caus}X$, $z \in \mathbb{D}$, *are homomorphisms of the Banach algebra* $(\mathfrak{Caus}X, T_0)$, *and* $\|T_0(g_z)\| \leqslant 1$ *for all* $z \in \mathbb{D}$.

One of the main results stated in this section is the theorem on spectral components of causal operators stated below. We know of no close analogues of this theorem, even for the causal operators considered in [1]–[6].

In what follows we consider a Banach $L_1(\mathbb{G})$-module (X, T). We assume that T is strongly continuous and $\mathbb{G} \in \{\mathbb{R}, \mathbb{T}\}$. A spectral component of an element a of the Banach algebra $\mathcal{B}$ is defined to be a non-empty closed subset σ_1 of the spectrum $\sigma(a)$ of a such that $\sigma_2 = \sigma(a) \setminus \sigma_1$ is a non-empty closed set and $\sigma_1 \cap \sigma_2 = \varnothing$, that is, σ_1 is both open and closed in the topology on $\sigma(a)$ induced from $\mathbb{C}$.

Theorem 8.17. *Let* $A \in \mathfrak{Caus}X$ *be a causal operator belonging to* $\mathfrak{U}_{0,u} \cap \mathcal{E}\mathfrak{rg}_0(\mathfrak{U}, T_0)$. *Then* $\sigma_{\mathfrak{Caus}}(A) \supseteq \sigma(A_0)$, *where* $A_0 = \mathcal{M}(A)$, *and every spectral component of* $\sigma_{\mathfrak{Caus}}(A)$ *contains at least one spectral component of* $\sigma(A_0) = \sigma_{\mathfrak{Caus}}(A_0)$.

Proof. Let σ_0 be a spectral component of $\sigma_{\mathfrak{Caus}}(A)$, that is, $\sigma_{\mathfrak{Caus}}(A) = \sigma_0 \cup \sigma_1$, where $\bar{\sigma}_1 = \sigma_1$ and $\sigma_0 \cap \sigma_1 = \varnothing$. We denote by P_0 the Riesz projector corresponding to σ_0 (see formula (7.4), where Γ is a contour that encircles σ_0 and separates σ_0 from σ_1).

Assuming for definiteness that $\mathbb{G} = \mathbb{R}$, we consider the family of homomorphisms $T_0(f_{i\beta}) \in \operatorname{End}(\mathfrak{Caus}X)$ for $\beta > 0$ (see Corollary 8.11) and the family of operators $A_\beta = T_0(f_{i\beta})\, A$, $\beta > 0$. Since each of the $T_0(f_{i\beta})$ is a homomorphism of the Banach algebra, we have $\sigma_{\mathfrak{Caus}}(A_\beta) \subseteq \sigma_{\mathfrak{Caus}}(A)$, which yields the corresponding partition of the causal spectrum of A_β:

$$\sigma_{\mathfrak{Caus}}(A_\beta) = \sigma_{0,\beta} \cup \sigma_{1,\beta}, \qquad \beta > 0, \tag{8.17}$$

where $\sigma_{i,\beta} = \sigma_i \cap \sigma_{\mathfrak{Caus}}(A_\beta)$, $i = 0, 1$. Hence, the contour Γ (see formula (7.4)) encircles $\sigma_{0,\beta}$ and separates $\sigma_{0,\beta}$ from $\sigma_{1,\beta}$. Making $T_0(f_{i\beta})$ act on both sides of formula (7.4), we obtain the formula

$$P_\beta = T_0(f_{i\beta}) P_0 = \frac{1}{2\pi i} \int_\Gamma (A_\beta - \lambda I)^{-1}\, d\lambda$$

for the Riesz projector P_β, $\beta > 0$, corresponding to the spectral component $\sigma_{0,\beta}$.

Now let us use assertions (iii) and (iv) of Lemma 8.1. Since $A \in \mathfrak{U}_{0,u}$ and $(f_{i\beta})$, $\beta \in \mathbb{R}_+^{\downarrow}$, is a b. a. p. in $L_1(\mathbb{R})$, we have $\lim_{\beta \to 0} T_0(f_{i\beta})\, A = A$ (in this proof all limits are understood in the sense of the uniform operator topology). Hence, $P_0 = \lim_{\beta\to 0} P_\beta$. Since $A \in \mathcal{E}\mathfrak{rg}_0(\mathfrak{U}, T_0)$ and $(f_{i\beta})$, $\beta \in \mathbb{R}_+^{\uparrow}$, is a 0-net, we have $\lim_{\beta\to\infty} T_0(f_{i\beta})\, A = A_0 = \mathcal{M}(A)$. Hence, the following limit exists:

$$\lim_{\beta\to\infty} P_\beta = \lim_{\beta\to\infty} T_0(f_{i\beta}) P_0 = \frac{1}{2\pi i}\int_\Gamma (A_0 - \lambda I)^{-1}\, d\lambda = P_\infty.$$

Let us note that P_∞ is the Riesz projector corresponding to the spectral set $\sigma_{0,\infty}$ occurring in the partition $\sigma_{\mathfrak{Caus}}(A_0) = \sigma(A_0) = \sigma_{0,\infty} \cup \sigma_{1,\infty}$, $\sigma_{0,\infty} = \sigma_0 \cap \sigma(A_0)$, $\sigma_{1,\infty} = \sigma_1 \cap \sigma(A_0)$, as follows from the inclusion $\sigma(A_0) \subseteq \sigma_{\mathfrak{Caus}}(A)$ (see Theorem 7.13).

Hence, $\{P_\beta\}$, $\beta \in [0,\infty] = \overline{\mathbb{R}}_+$, is a family of projectors defined on the Aleksandrov one-point compactification $\overline{\mathbb{R}}_+$ of $\mathbb{R}_+$ and continuous in the operator norm. Since this family is continuous, P_∞ is different from the zero operator and from the identity operator. Hence, $\sigma_{0,\infty}$ is the spectral component of $\sigma(A_0)$ contained in σ_0.

In the case when $\mathbb{G} = \mathbb{T}$ we use the family of homomorphisms $T_0(g_{i\alpha})$, $\alpha \in (0,1)$ (see formula (8.15)) and Assertions 8.12–8.16.

Corollary 8.18. *If σ_0 is a finite set and the Riesz projector P_0 is finite-dimensional, then the spectral component σ_0 lies in the spectrum of A_0 and the dimensions of the images of the P_β, $\beta \in \overline{\mathbb{R}}_+$, coincide.*

Corollary 8.19. *If $A \in \mathcal{UC}(X)$, then $\sigma_{\mathfrak{Caus}}(A)$ is a connected set containing* 0.

Proof. This corollary is obvious under the assumptions of the theorem. However, it also holds under certain very weak restrictions imposed on the uniformly causal operator. The inclusion $0 \in \sigma_{\mathfrak{Caus}}(A)$ follows immediately from Corollary 7.7. We prove by contradiction that the set under consideration is connected by applying Corollary 7.7 to the restriction of A to the image of the corresponding Riesz projector.

Corollary 8.20. *The number of connected components of $\sigma_{\mathfrak{Caus}}(A)$ does not exceed the number of connected components of $\sigma(A_0)$.*

Corollary 8.21. *If A is a compact operator, then $\sigma_{\mathfrak{Caus}}(A) = \sigma(A) = \sigma(A_0)$.*

In all these corollaries (with the exception of Corollary 8.19) it is assumed that the assumptions of Theorem 8.17 hold for A.

We now state several corollaries concerning the causal operators considered in the examples given in § 6. We keep to the notation used there.

Let $X_1 = X_2 = X$ and $\mathcal{E}_1 = \mathcal{E}_2$, and let $\Omega_1 = \Omega_2 = \Omega$ and $E_n^1 = E_n^2 = E_n$, $n \geqslant 1$ (see Examples 6.10 and 2.9). Let $A = \sum_{n\geqslant 0} A_n$, $\sum_{n\geqslant 0} \|A_n\| < \infty$, be a causal operator belonging to $\operatorname{End} X$. Since its diagonals are absolutely integrable, A belongs to $\mathfrak{U}_{0,u}$. Since 0 is an isolated point of $\Lambda(A, U_0) \subseteq \mathbb{Z}_+$, we have $A \in \mathcal{E}\mathfrak{rg}_0(\mathfrak{U}, U_0)$.

Corollary 8.22.

$$\overline{\bigcup_{i\in\Omega} \sigma(A_i^0)} \subseteq \sigma_{\mathfrak{Caus}}(A),$$

where A_i^0 is the restriction of A_{ii} to $X_i = \operatorname{Im} E_i = E_i X$. Moreover, every spectral component of $\sigma_{\mathfrak{Caus}}(A)$ contains a spectral component of A_0 and a spectral component of one of the A_i^0, $i \in \Omega$. If the A_i^0, $i \in \Omega$, are zero operators (that is, $A_0 = 0$), then $0 \in \sigma_{\mathfrak{Caus}}(A)$ and $\sigma_{\mathfrak{Caus}}(A)$ is a connected set.

In the next corollary we use the notation of Examples 2.10 and 6.11, where $X_1 = X_2 = X$, $\mathbb{G} = \mathbb{R}$ (whence $\Omega \subseteq \mathbb{R}$).

Corollary 8.23. *Let $B \in \operatorname{End} L_p$, $p \in [1, \infty]$, be the operator defined by formula (6.4) and assume that it is causal with respect to the representation V_0 and the semigroup $\mathbb{R}_+$ (that is, (6.12) holds). Then*

(i) *$\sigma_{\mathfrak{Caus}}(B)$ is a connected set and $\sigma(B) \supseteq \overline{\bigcup_{\omega\in\Omega} \sigma\big(\mathcal{A}_0(\omega)\big)} = \Delta_0$ if Ω is a connected set and $\mathcal{A}_0$ is a continuous function,*

(ii) *$\sigma_{\mathfrak{Caus}}(B) = \Delta_0$ if Ω is a compact set, $p \neq \infty$ and $\mathcal{A}_0$ has only finitely many discontinuities of the first kind.*

Now consider Example 6.12 with $\mathbb{G} \in \{\mathbb{R}, \mathbb{T}\}$ and $Y_1 = Y_2$. Assume that the operator is causal with respect to the representation S_0 and the semigroup $\mathbb{R}_+$ ($\mathbb{Z}_+$), and assume that the function μ is almost periodic, that is, it has a Fourier series

$$\mu(t) \sim \sum_{j \geqslant 0} \mu_j e^{it\lambda_j}, \qquad t \in \mathbb{R}, \quad \lambda_j \geqslant 0$$

($\mu(\theta) \sim \sum_{j\in\mathbb{Z}_+} \mu_j \theta^j$, $\theta \in \mathbb{T}$). Under these assumptions we have $B \in \mathfrak{U}_{0,u} \cap \mathcal{E}\mathfrak{rg}_0(\mathfrak{U}, S_0)$. Hence, Theorem 8.17 is applicable to B. In this case $\mathcal{M}(B) = B_0$, where $(B_0 x)(g) = \int_{\mathbb{G}} \mu_0(ds)\, x(g+s)$, $x \in L_p$.

Corollary 8.24. *If μ_0 is the sum of two measures one of which is absolutely continuous and the other discrete, then*

$$\sigma_{\mathfrak{Caus}}(B) \supseteq \sigma(B_0) = \overline{\bigcup_{\gamma\in\widehat{\mathbb{G}}} \sigma\big(\hat{\mu}_0(\gamma)\big)}$$

and $\sigma_{\mathfrak{Caus}}(B)$ is a connected set when $\mathbb{G} = \mathbb{R}$.

§ 9. Operators with two-point Bohr spectrum

Let us return to the study of the Banach $L_1(\mathbb{G})$-modules (X_i, T_i), $i = 1, 2$, and the Banach $L_1(\mathbb{G})$-module $\mathfrak{U} = \mathfrak{U}(X_1, X_2)$ in $\operatorname{Hom}(X_1, X_2)$.

Definition 9.1. An $A \in \mathfrak{U}$ is called an *operator with two-point Bohr spectrum* (with respect to the representation T_0) if $\Lambda(A, T_0) = \{\gamma_1, \gamma_2\}$ is a two-point subset of $\widehat{\mathbb{G}}$. If $\Lambda(A, T_0) = \{\gamma_0\}$ is a one-point set and $\gamma_0 \neq 0$, then A will be called a *circular operator* or an *abstract weighted shift operator*.

Remark 9.2. Let $A \in \mathfrak{U}$ be an operator with two-point spectrum $\Lambda(A, T_0) = \{\gamma_1, \gamma_2\} \subset \widehat{\mathbb{G}}$. Consider functions f_i, $i = 1, 2$, belonging to $L_1(\mathbb{G})$ and such that $\hat{f}_i = 1$ in some neighbourhood of γ_i, $i = 1, 2$, $\gamma_1 \notin \operatorname{supp} \hat{f}_2$ and $\gamma_2 \notin \operatorname{supp} \hat{f}_1$. Assertions (iii) and (v) of Lemma 3.3 imply that

$$A = A_1 + A_2, \tag{9.1}$$

where $A_i = T_0(f_i)\,A$ and $\Lambda(A_i, T_0) = \{\gamma_i\}$, $i = 1,2$. Therefore (see Remark 4.16), we have

$$T_0(g)\,A_i = T_2(g)\,A_i T_1(g) = \gamma_i(g)\,A_i, \qquad i = 1,2, \quad g \in \mathbb{G},$$

that is, each A_i, $i = 1,2$, has a one-point Beurling spectrum. Hence, the A_i, $i = 1,2$, are circular operators (if $\gamma_i \neq 0$) that are eigenvectors of the $L_1(\mathbb{G})$-module $(\mathfrak{U}, T_0)$. The equalities

$$T_0(g)\,A = \gamma_1(g)\,A_1 + \gamma_2(g)\,A_2, \qquad g \in \mathbb{G}, \tag{9.2}$$

imply that A (regarded as an element of the $L_1(\mathbb{G})$-module $(\mathfrak{U}, T_0)$; see Definition 4.11) is almost periodic. We obtain from (9.2) that the Bohr spectrum $\Lambda_b(A, T_0)$ of A coincides with its Beurling spectrum $\Lambda_b(A, T_0) = \{\gamma_1, \gamma_2\}$, which justifies the terminology in Definition 9.1.

Here are some examples of operators with two-point Bohr spectrum.

Example 9.3. Let X_1 and X_2 be the Banach spaces considered in Example 6.10. Any $A \in \mathrm{Hom}(X_1, X_2) = \mathfrak{U}$ whose matrix is bidiagonal is an operator with a two-point Bohr spectrum contained in $\Omega_2 - \Omega_1 \subseteq \mathbb{Z}$. The causal operators in $\mathfrak{U}$ whose matrices are lower-triangular with only finitely many non-zero diagonals can be regarded as operators with two-point Bohr spectrum with respect to a slightly altered representation obtained by "lumping together" the partitions of the identities in X_1 and X_2 (this approach was described in detail in [29]).

Example 9.4. Let X_i, $i = 1,2$, be the Banach spaces considered in Example 6.12. The operator $B \in \mathfrak{U}$ defined by the formula

$$(Bx)(g) = \sum_{j=1,2} \left(C_j \gamma_j(g)\,x(g) + \int_{\mathbb{G}} \gamma_i(g)\,\mathcal{A}_j(g-s)\,x(s)\,ds \right),$$

where $\gamma_j \in \widehat{\mathbb{G}}$, $C_j \in \mathrm{Hom}(Y_1, Y_2)$ and $\mathcal{A}_j \in L_1\big(\mathbb{G}, \mathrm{Hom}(Y_1, Y_2)\big)$, $j = 1,2$, has the two-point Bohr spectrum $\Lambda_b(B, S_0) = \{\gamma_1, \gamma_2\}$ with respect to S_0.

Example 9.5. Let X_1 and X_2 be the Banach spaces considered in Example 6.11 with $\Omega = \widehat{\mathbb{G}}$. The difference operator $A \in \mathrm{Hom}(X_1, X_2)$ defined by the formula

$$(Ax)(\gamma) = U_1(\gamma)\,x(\gamma - \gamma_1) + U_2(\gamma)\,x(\gamma - \gamma_2), \qquad x \in X_1, \quad \gamma \in \widehat{\mathbb{G}},$$

where $U_i \in C_b\big(\widehat{\mathbb{G}}, \mathrm{Hom}(Y_1, Y_2)\big)$ and $\gamma_i \in \widehat{\mathbb{G}}$, $i = 1,2$, has the two-point Bohr spectrum $\Lambda_b(A) = \Lambda_b(A, V_0) = \{\gamma_1, \gamma_2\}$. In particular, if $\mathbb{G} = \mathbb{T}$, then the difference operator $D \in \mathrm{End}\,X$, $X = L_p(\mathbb{Z}, Y) = l_p(\mathbb{Z}, Y)$, $p \in [1, \infty]$, defined by the formula

$$(Dx)(n) = x(n) - U(n)\,x(n-1), \qquad n \in \mathbb{Z}, \quad x \in l_p(\mathbb{Z}, Y), \tag{9.3}$$

where $U \in l_\infty(\mathbb{Z}, \mathrm{End}\,Y)$, has the two-point Bohr spectrum $\Lambda_b(D, V_0) = \{0, 1\} \subset \mathbb{Z} \simeq \mathbb{T}$. $A = I - D$ is a weighted shift operator (circular operator), whose spectral properties play an important role in the study of abstract parabolic operators (see [37], [38]).

Mlak [39] studied the properties of circular operators acting on a Hilbert space H: an $A \in \operatorname{End} H$ was said to be circular if

$$U(t)\,A = \exp(i\alpha t)\,AU(t), \qquad t \in \mathbb{R},$$

for some group of unitary operators $U(t)$, $t \in \mathbb{R}$, belonging to $\operatorname{End} H$ and some $\alpha \in \mathbb{R}$. It is clear that $\Lambda(A, U_0) = \{\alpha\}$, that is, A is a circular operator in the sense of Definition 9.1 (if $\alpha \neq 0$).

If A is a circular operator belonging to $\mathfrak{U}_a = \mathfrak{U}_a(X)$ and $\Lambda(A, T_0) = \{\gamma_0\} \subset \widehat{\mathbb{G}}$, then the $A - \lambda I$, $\lambda \in \mathbb{C} \setminus \{0\}$, are operators with the two-point Bohr spectrum $\Lambda(A - \lambda I, T_0) = \{0, \gamma_0\} \subset \widehat{\mathbb{G}}$. Hence, we have to consider operators with two-point Bohr spectrum when using the resolvents of circular operators.

Let $A \in \big(\mathfrak{U}(X_1, X_2), T_0\big)$ and $\Lambda(A, T_0) = \{\gamma_1, \gamma_2\}$. If $\gamma_1 = 0$, then A is causal (by Definition 6.3) with respect to every semigroup $\mathbb{S}$ containing γ_2. In the results stated below we do not assume that A is a causal operator. However, these results can be of use in the study of causal operators with two-point Bohr spectrum.

We shall study operators with two-point Bohr spectrum using the spectral theory of ordered pairs of linear operators.

Consider an ordered pair (A_1, A_2) of operators belonging to the Banach space $\operatorname{Hom}(X_1, X_2)$, where X_1 and X_2 are complex Banach spaces. Let us recall some definitions and results in the spectral theory of ordered pairs of operators following [40], § 6, where one can find many references concerning the problems considered in this paper.

Definition 9.6. The *resolvent set* $\varrho(A_1, A_2)$ of the pair (A_1, A_2) is defined to be the set of all $\lambda \in \mathbb{C}$ for which the operator $A_1 - \lambda A_2 \in \operatorname{Hom}(X_1, X_2)$ has a continuous inverse. The set $\sigma(A_1, A_2) = \mathbb{C} \setminus \varrho(A_1, A_2)$ is called the *spectrum* of this pair.

The operator-valued function $R = R(\,\cdot\,; A_1, A_2)\colon \varrho(A_1, A_2) \to \operatorname{Hom}(X_1, X_2)$, $R(\lambda) = (A_1 - \lambda A_2)^{-1}$, $\lambda \in \varrho(A_1, A_2)$, is called the *resolvent* of (A_1, A_2). The functions $R_l = R_l(\,\cdot\,; A_1, A_2)\colon \varrho(A_1, A_2) \to \operatorname{End} X_1$, $R(\lambda) = (A_1 - \lambda A_2)^{-1} A_2$, $\lambda \in \varrho(A_1, A_2)$, and $R_r = R_r(\,\cdot\,; A_1, A_2)\colon \varrho(A_1, A_2) \to \operatorname{End} X_2$, $R(\lambda) = A_2 (A_1 - \lambda A_2)^{-1}$, $\lambda \in \varrho(A_1, A_2)$, are pseudoresolvents (they satisfy Hilbert's resolvent identity). They are called the *left* and *right pseudoresolvents* of (A_1, A_2).

Definition 9.7. The subset $\tilde\sigma(A_1, A_2)$ of the extended complex plane $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which coincides with $\sigma(A_1, A_2)$ if the function $R(\,\cdot\,; A_1, A_2)$ has a holomorphic continuation to ∞ and $\lim_{|\lambda| \to \infty} R(\lambda; A_1, A_2) = 0$ (otherwise $\tilde\sigma(A_1, A_2) = \sigma(A_1, A_2) \cup \{\infty\}$), will be called the *extended spectrum* of (A_1, A_2). The set $\tilde\varrho(A_1, A_2) = \widetilde{\mathbb{C}} \setminus \tilde\sigma(A_1, A_2)$ will be called the *extended resolvent set* of (A_1, A_2).

Theorem 9.8 [40]. *The following formula holds for the extended spectra of* (A_1, A_2) *and* (A_2, A_1):

$$\tilde\sigma(A_2, A_1) = \big\{1/\lambda \colon \lambda \in \tilde\sigma(A_1, A_2)\big\}.$$

Corollary 9.9. $\infty \notin \tilde\sigma(A_1, A_2)$ *if and only if* A_2 *has a continuous inverse.*

Definition 9.10. An ordered pair of closed subspaces (X_1^0, X_2^0), where $X_1^0 \subset X_1$ and $X_2^0 \subset X_2$, is said to be *invariant* for the pair (A_1, A_2) if $A_1 X_1^0 \subset X_2^0$ and $A_2 X_1^0 \subset X_2^0$.

Definition 9.11. Let $X_1 = X_1^0 \oplus X_1^1$ and $X_2 = X_2^0 \oplus X_2^1$ be direct sums of closed subspaces, where (X_1^0, X_2^0) and (X_1^1, X_2^1) are invariant pairs of subspaces for (A_1, A_2). Let $A_i^{(k)}\colon X_1^k \to X_2^k$, $i = 1, 2$, $k = 0, 1$, be the restrictions of A_i, $i = 1, 2$, to X_1^k, $k = 0, 1$. Then we write

$$(A_1, A_2) = \bigl(A_1^{(0)}, A_2^{(0)}\bigr) \oplus \bigl(A_1^{(1)}, A_2^{(1)}\bigr) \tag{9.4}$$

and say that (A_1, A_2) admits the representation (9.4) with respect to the decomposition of X_1 and X_2 under consideration and is the *direct sum* of $\bigl(A_1^{(0)}, A_2^{(0)}\bigr)$ and $\bigl(A_1^{(1)}, A_2^{(1)}\bigr)$ (the parts of A_1 and A_2). In this case we shall also write $A_i = A_i^{(0)} \oplus A_i^{(1)}$, $i = 1, 2$ (the operators A_1 and A_2 are the direct sums of their parts).

Again consider an operator $A \in \mathfrak{U}(X_1, X_2)$ with two-point Bohr spectrum $\{\gamma_1, \gamma_2\} \subset \widehat{\mathbb{G}}$. By Remark 9.2, formula (9.1) holds for A with $\Lambda(A_i, T_0) = \{\gamma_i\}$, $i = 1, 2$, and

$$T_0(g)\, A_i = \gamma_i(g)\, A_i, \qquad i = 1, 2, \quad g \in \mathbb{G}. \tag{9.5}$$

From now on we shall assume that the following two assumptions hold.

Assumption 9.12. *For the character $\gamma_0 = \gamma_2\gamma_1^{-1} \in \widehat{\mathbb{G}}$ the set $\{\gamma_0(g);\ g \in \mathbb{G}\}$ is dense in $\mathbb{T} \subset \mathbb{C}$ (here and below we use the multiplicative form for the binary operation in $\widehat{\mathbb{G}}$).*

Assumption 9.13. *The operator A has a (continuous) inverse $B = A^{-1}$ in $\bigl(\mathfrak{U}(X_2, X_1), T_0^{-1}\bigr)$.*

Let us note that Assumption 9.12 holds if $\mathbb{G}$ is a connected group (and γ_0 is a non-zero character).

Lemma 9.14. $\sigma(A_1, A_2) \cap \mathbb{T} = \varnothing$.

Proof. It follows from (9.2) that

$$T_0(g)\, A = T_2(g)\, A T_1(-g) = \gamma_1(g)\bigl(A_1 - \gamma_0(g)\, A_2\bigr), \qquad g \in \mathbb{G}.$$

Since A is an invertible operator, the $\gamma_0(g)$, $g \in \mathbb{G}$, are contained in the resolvent set $\varrho(A_1, A_2)$ of (A_1, A_2) in $\mathfrak{U}(X_1, X_2)$. Since the set $\sigma(A_1, A_2)$ is closed, Assumption 9.12 implies that $\mathbb{T} \subset \varrho(A_1, A_2)$. The lemma is proved.

Corollary 9.15. *The set $\tilde\sigma(A_1, A_2)$ can be represented in the form*

$$\tilde\sigma(A_1, A_2) = \sigma_0 \cup \sigma_1, \tag{9.6}$$

where $\sigma_0 = \{\lambda \in \sigma(A_1, A_2);\ |\lambda| < 1\}$ and $\sigma_1 \subseteq \{\lambda \in \sigma(A_1, A_2);\ |\lambda| > 1\} \cup \{\infty\}$.

Corollary 9.16. *The set $\sigma(A_1, A_2)$ is invariant under rotations about zero in $\mathbb{C}$.*

This implies that the usual spectrum of every circular operator belonging to $\mathfrak{U}_a(X)$ is invariant under rotation.

Integrating over the circle $\mathbb{T}$, regarded as a positively oriented closed contour encircling σ_0, we obtain the formulae

$$P_0 = -\frac{1}{2\pi i}\int_{\mathbb{T}} R_l(\lambda; A_1, A_2)\, d\lambda = -\frac{1}{2\pi i}\int_{\mathbb{T}} (A_1 - \lambda A_2)^{-1} A_2\, d\lambda, \tag{9.7}$$

$$Q_0 = -\frac{1}{2\pi i}\int_{\mathbb{T}} R_r(\lambda; A_1, A_2)\, d\lambda = -\frac{1}{2\pi i}\int_{\mathbb{T}} A_2(A_1 - \lambda A_2)^{-1}\, d\lambda \tag{9.8}$$

for the projectors P_0, $P_1 = I - P_0 \in \operatorname{End} X_1$ and Q_0, $Q_1 = I - Q_0 \in \operatorname{End} X_2$.

Consider the closed subspaces (images of projectors)

$$X_1^0 = \operatorname{Im} P_0 = P_0 X_1, \qquad X_1^1 = \operatorname{Im} P_1, \qquad X_2^0 = \operatorname{Im} Q_0, \qquad X_2^1 = \operatorname{Im} P_1.$$

We have

$$X_1 = X_1^0 \oplus X_1^1, \qquad X_2 = X_2^0 \oplus X_2^1. \tag{9.9}$$

Formulae (9.7) and (9.8) imply that

$$A_1 P_0 = Q_0 A_1, \qquad A_2 P_0 = Q_0 A_2, \tag{9.10}$$

whence (X_1^0, X_2^0) and (X_1^1, X_2^1) are invariant for (A_1, A_2). Thus

$$(A_1, A_2) = \big(A_1^{(0)}, A_2^{(0)}\big) \oplus \big(A_1^{(1)}, A_2^{(1)}\big)$$

(see formula (9.4)), where $A_1^{(0)}, A_2^{(0)} \in \operatorname{Hom}(X_1^0, X_2^0)$ and $A_1^{(1)}, A_2^{(1)} \in \operatorname{Hom}(X_1^1, X_2^1)$ are the restrictions of A_1 and A_2 to X_1^0 and X_1^1.

Lemma 9.17. *The projectors P_0 and Q_0 commute with the operators of the representations T_1 and T_2, respectively.*

Proof. We have

$$\begin{aligned} T_1(g) R_l(\lambda; A_1, A_2)\, T_1(-g) &= T_1(g)(A_1 - \lambda A_2)^{-1} T_2(-g)\big(T_0(g)\, A_2\big) \\ &= \big(\gamma_1(g)\, A_1 - \lambda\gamma_2(g)\, A_2\big)^{-1}\gamma_2(g)\, A_2 = \gamma_0(g) R_l\big(\gamma_0(g)\lambda; A_1, A_2\big), \end{aligned}$$

where $\lambda \in \mathbb{T}$ and $g \in \mathbb{G}$. Formula (9.7) implies that $T_1(g) P_0 T_1(-g) = P_0$ for all $g \in \mathbb{G}$. We likewise establish that Q_0 commutes with the operators of the representation T_2.

Lemma 9.17 implies that the X_i^k are invariant under T_i, $i = 1, 2$, $k = 0, 1$ (respectively). This enables us to consider the following representations on $\operatorname{Hom}(X_1^k, X_2^k)$:

$$T_0^k(g) C_k = T_2^0(g) C_k T_1^0(-g), \qquad C_k \in \operatorname{Hom}(X_1^k, X_2^k), \quad k = 0, 1, \tag{9.11}$$

where T_i^k is the restriction of T_i to X_i^k, $k = 0, 1$, $i = 1, 2$.

Theorem 9.18. *Let Assumption* 9.13 *hold for* $A \in \mathfrak{U}(X_1, X_2)$. *Then*

(i) *the operator* $A^{-1} \in \mathfrak{U}(X_2, X_1)$ *is almost periodic with respect to* $T_0^{-1}\colon \mathbb{G} \to \operatorname{End}\mathfrak{U}(X_2, X_1)$ *and its Bohr spectrum* $\Lambda_b(A^{-1}, T_0^{-1})$ *is contained in* $\{\gamma_2^{-1}\gamma_0^n;\ n \geqslant 0\} \cup \{\gamma_1^{-1}\gamma_0^n;\ n \leqslant 0\}$,

(ii) $\sigma\big(A_1^{(0)}, A_2^{(0)}\big) = \tilde\sigma\big(A_1^{(0)}, A_2^{(0)}\big) = \sigma_0$, *the operator* $A_2^{(0)} \in \operatorname{Hom}(X_1^0, X_2^0)$ *is invertible*, $\sigma\big(\big(A_2^{(0)}\big)^{-1}A_1^{(0)}\big) = \sigma_0$ $\big($*hence, the spectral radii of* $\big(A_2^{(0)}\big)^{-1}A_1^{(0)} \in \operatorname{End} X_1^0$ *and* $A_1^{(0)}\big(A_2^{(0)}\big)^{-1} \in \operatorname{End} X_2^0$ *are such that* $r\big(\big(A_2^{(0)}\big)^{-1}A_1^{(0)}\big) = r\big(A_1^{(0)}\big(A_2^{(0)}\big)^{-1}\big) < 1\big)$, *and* $\sigma\big(A_1^{(0)}\big(A_2^{(0)}\big)^{-1}\big) \setminus \{0\} = \sigma_0 \setminus \{0\}$,

(iii) $\tilde\sigma\big(A_1^{(1)}, A_2^{(1)}\big) = \sigma_1$, *the operator* $A_1^{(1)} \in \operatorname{Hom}(X_1^1, X_2^1)$ *is invertible, and* $\sigma\big(\big(A_1^{(1)}\big)^{-1}A_2^{(1)}\big) = \sigma\big(A_2^{(1)}\big(A_1^{(1)}\big)^{-1}\big) = \{\lambda^{-1};\ \lambda \in \sigma_1\}$ $\big($*whence* $r\big(\big(A_1^{(1)}\big)^{-1}A_2^{(1)}\big) = r\big(A_2^{(1)}\big(A_1^{(1)}\big)^{-1}\big) < 1\big)$,

(iv) $\Lambda_b\big(A_1^{(0)}, T_0^0\big) \cup \Lambda_b\big(A_1^{(1)}, T_0^1\big) = \{\gamma_1\}$, $\Lambda_b\big(A_2^{(0)}, T_0^0\big) \cup \Lambda_b\big(A_2^{(1)}, T_0^1\big) = \{\gamma_2\}$,

(v) *with respect to the decomposition* (9.9) A^{-1} *is a direct sum* $A^{-1} = \widetilde{A}_1^{-1} \oplus \widetilde{A}_2^{-1}$, *where* $\widetilde{A}_1 = A_1^{(0)} + A_2^{(0)}$, $\widetilde{A}_2 = A_1^{(1)} + A_2^{(1)}$, *and*

$$\begin{aligned}\widetilde{A}_1^{-1} &= \big(A_2^{(0)}\big)^{-1}\sum_{n\geqslant 0}(-1)^n\big(A_1^{(0)}\big(A_2^{(0)}\big)^{-1}\big)^n \\ &= \sum_{n\geqslant 0}(-1)^n\big(\big(A_2^{(0)}\big)^{-1}A_1^{(0)}\big)^n\big(A_2^{(0)}\big)^{-1}, \end{aligned}\tag{9.12}$$

$$\begin{aligned}\widetilde{A}_2^{-1} &= \big(A_1^{(1)}\big)^{-1}\sum_{n\geqslant 0}(-1)^n\big(A_2^{(1)}\big(A_1^{(1)}\big)^{-1}\big)^n \\ &= \sum_{n\geqslant 0}(-1)^n\big(\big(A_1^{(1)}\big)^{-1}A_2^{(1)}\big)^n\big(A_1^{(1)}\big)^{-1}. \end{aligned}\tag{9.13}$$

Proof. The equalities in assertions (ii) and (iii) were established in [40], Theorem 6.3. Using the inclusions

$$\sigma_0 \cup \{1/\lambda;\ \lambda \in \sigma_1\} \subset \big\{\lambda \in \mathbb{C}\colon |\lambda| < 1\big\},$$

we obtain that $A_2^{(0)}$ and $A_1^{(1)}$ are invertible operators and the estimates (for spectral radii) in (iii) hold.

We deduce (v) from (ii) and (iii) using the decomposition $A = \widetilde{A}_1 \oplus \widetilde{A}_2$.

Assertion (iv) follows using the definitions of T_0^0, T_0^1 and the equalities $\Lambda_b(A_i, T_0) = \{\gamma_i\}$, $i = 1, 2$.

To prove that A^{-1} is almost periodic with respect to T_0^{-1}, it is sufficient to observe that the function $\psi(g) = T_0^{-1}(g)\,A^{-1} = T_1(g)\,A^{-1}T_2(-g)$, $g \in \mathbb{G}$, is almost periodic, since it is related to $\varphi(g) = T_0(g)\,A$ by the formula $\psi(g) = \varphi(g)^{-1}$, $g \in \mathbb{G}$. Hence, $\Lambda_b(A^{-1}, T_0^{-1}) \subseteq \{\gamma_1^n\gamma_2^m; m, n\mathbb{Z}\}$. More detailed information on the Bohr spectrum of A^{-1} can be obtained from the representation $A^{-1} = \widetilde{A}_1^{-1} \oplus \widetilde{A}_2^{-1}$ and formulae (9.12), (9.13) using the representations T_0^0, T_0^1 (see formula (9.11)) and their inverses $(T_0^0)^{-1}$, $(T_0^1)^{-1}$. For example, we obtain from (9.12) that

$$\big(T_0^0\big)^{-1}(g)\big(\widetilde{A}_1\big)^{-1} = \sum_{n\geqslant 0}(-1)^n\big(\gamma_1\gamma_2^{-1}\big)^n(g)\gamma_2^{-1}(g)\big(\big(A_2^{(0)}\big)^{-1}A_1^{(0)}\big)^n\big(A_2^{(0)}\big)^{-1}.$$

Hence, $\Lambda_b\big(\widetilde{A}_1^{-1},(T_0^0)^{-1}\big) \subseteq \{\gamma_1^{n-1}\gamma_2^n;\ n\geqslant 0\} = \{\gamma_1^{-1}\gamma_0^n;\ n\leqslant 0\}$.

We prove likewise that

$$\Lambda_b\big(\widetilde{A}_2^{-1},(T_0^1)^{-1}\big) \subseteq \{\gamma_2^{-n-1}\gamma_1^n;\ n\geqslant 0\} = \{\gamma_2^{-1}\gamma_0^n;\ n\leqslant 0\}.$$

It remains to observe that $T_0^{-1}A^{-1} = (T_0^0)^{-1}(g)\,\widetilde{A}_1^{-1} \oplus (T_0^1)^{-1}(g)\,\widetilde{A}_2^{-1}$.

Theorem 9.19. *Let $A \in \mathfrak{U}(X_1,X_2)$ be an operator with the two-point Bohr spectrum $\Lambda_b(A,T_0) = \{e_0,\gamma_0\} \subset \widehat{\mathbb{G}}$, where e_0 is the identity in $\widehat{\mathbb{G}}$, and let Assumption* 9.13 *hold. The operator A is causally invertible with respect to a semigroup $\mathbb{S}$ containing γ_0 if and only if the operator A_1 is invertible and $r(A_2A_1^{-1}) = r(A_1^{-1}A_2) < 1$.*

Proof. Assertion (i) of Theorem 9.18 implies that A is causally invertible if and only if $\Lambda_b(A^{-1},T_0^{-1}) \subset \{\gamma_0^n;\ n\geqslant 0\}$. This inclusion implies that $\tilde{\sigma}(A_1,A_2) = \sigma_1$. By assertion (iii) of Theorem 9.18, this equality holds if and only if A_1 is invertible and $r(A_2A_1^{-1}) = r(A_1^{-1}A_2) < 1$.

Theorem 9.18 can be used in the study of the structure of operators inverse to operators belonging to certain classes. For example, for difference operators of the form (9.3), this theorem implies that the family of evolutionary operators corresponding to U is exponentially dichotomous if the operator D is invertible (see [37] for more details).

Bibliography

[1] A. Feintuch and R. Saeks, *System theory. A Hilbert space approach*, Academic press, New York–London 1982.

[2] V. G. Kurbatov, *Linear differential-difference equations*, Voronezh Univ. Press, Voronezh 1990. (Russian)

[3] V. G. Kurbatov, *Functional differential operators and equations*, Kluwer, Dordrecht–Boston–London 1999.

[4] V. Kolmanovskii and A. Myshkis, *Introduction to the theory and applications of functional differential equations*, Kluwer, Dordrecht–Boston–London 1999.

[5] N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, *Methods of the contemporary theory of linear functional differential equations*, Scientific Research Centre for Regular and Chaotic Dynamics, Izhevsk 2000. (Russian)

[6] I. C. Gohberg and M. G. Krein, *Theory and applications of Volterra operators in Hilbert space*, Nauka, Moscow 1967; English transl., Amer. Math. Soc., Providence, RI 1970.

[7] W. Rudin, *Fourier analysis on groups*, Int. Publ., New York 1962.

[8] E. Hewitt and K.-A. Ross, *Abstract harmonic analysis*, vol. II, Springer-Verlag, Berlin–New York 1977.

[9] I. M. Gel'fand, D. A. Raikov, and G. E. Shilov, *Commutative normed rings*, Fizmatgiz, Moscow 1960; English transl., Chelsea, New York 1964.

[10] A. G. Baskakov, *Harmonic analysis of linear operators*, Voronezh Univ. Press, Voronezh 1987. (Russian)

[11] N. Dunford and J. T. Schwartz, *Linear operators. Part* III: *Spectral operators*, Wiley–Interscience, New York 1988; Russian transl. of earlier ed., Mir, Moscow 1966.

[12] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics.* I: *C*- and W*-algebras, symmetry groups, decomposition of states*, Texts and Monographs in Physics, Springer-Verlag, New York–Heidelberg–Berlin 1979; Russian transl., Mir, Moscow 1982.

[13] Y. Domar, "Harmonic analysis in certain commutative Banach algebras", *Acta Math.* **96** (1956), 1–66.

[14] Y. Domar and L. A. Lindahl, "Three spectral notions for representations of commutative Banach algebras", *Ann. Inst. Fourier* **25**:2 (1975), 1–32.

[15] Yu. I. Lyubich, V. I. Machaev, and G. M. Fel'dman, "On representations with a separable spectrum", *Funktsional. Anal. i Prilozhen.* **7**:2 (1973), 52–61; English transl., *Functional Anal. Appl.* **7** (1973), 129–136.

[16] A. G. Baskakov, "On the spectral analysis of bounded representations of locally compact Abelian groups", *Trudy Mat. Fak. Voronezhsk. Gos. Univ.* **1973**, no. 10, 103–108. (Russian)

[17] A. G. Baskakov, *Problems of the theory of vector-valued almost periodic functions*, Diss. Candidate Phys.-Math. Sci., Voronezh State Univ., Voronezh 1973. (Russian)

[18] W. Arveson, "On group of automorphisms of operator algebras", *J. Functional Anal.* **15** (1974), 217–243.

[19] A. G. Baskakov, "Inequalities of Bernshtein type in abstract harmonic analysis", *Sibirsk. Mat. Zh.* **20** (1979), 942–952; English transl., *Siberian Math. J.* **20** (1980), 665–672.

[20] E. Marshall, "On the functional calculus of non-quasianalytic groups of operators and cosine functions", *Rend. Circ. Mat. Palermo* **35** (1986), 58–81.

[21] R. Nagel and S.-Z. Huang, "Spectral mapping theorems for C_0-groups satisfying non-quasi-analytic growth conditions", *Math. Nachr.* **169** (1994), 207–218.

[22] A. G. Baskakov, *Harmonic analysis of linear operators*, Diss. Doctor Phys.-Math. Sci., Inst. Math., Kiev 1987. (Russian)

[23] A. G. Baskakov, "General ergodic theorems in Banach modules", *Funktsional. Anal. i Prilozhen.* **14**:3 (1980), 63–64; English transl., *Functional Anal. Appl.* **14** (1981), 215–217.

[24] A. G. Baskakov, "Operator ergodic theorems and complementability of subspaces of Banach spaces", *Izv. Vyssh. Uchebn. Zaved. Mat.* **1988**, no. 11, 3–11; English transl., *Soviet Math.* **32**:11 (1988), 1–14.

[25] A. G. Baskakov, "Spectral criteria for almost periodicity of solutions of functional equations", *Mat. Zametki* **24**:2 (1978), 195–206; English transl., *Math. Notes* **24** (1979), 606–612.

[26] Yu. I. Lyubich, *Introduction to the theory of Banach representations*, Vishcha Shkola, Khar'kov 1985. (Russian)

[27] H. H. Schaefer, *Topological vector spaces*, Macmillan, New York 1966; Russian transl., Mir, Moscow 1971.

[28] A. G. Baskakov, "Methods of abstract harmonic analysis in the perturbation of linear operators", *Sibirsk. Mat. Zh.* **24**:1 (1983), 21–39; English transl., *Siberian Math. J.* **24** (1983), 17–32.

[29] A. G. Baskakov, "Estimates for the entries of inverse matrices and the spectral analysis of linear operators", *Izv. Ross. Akad. Nauk Ser. Mat.* **61**:6 (1997), 3–26; English transl., *Izv. Math.* **61** (1997), 1113–1135.

[30] A. G. Baskakov, "Asymptotic estimates for the entries of the matrices of inverse operators and harmonic analysis", *Sibirsk. Mat. Zh.* **38**:1 (1997), 14–28; English transl., *Siberian Math. J.* **38** (1997), 10–22.

[31] I. Diestal and J. J. Uhl, *Vector measures*, Mathematical Surveys and Monographs, vol. 15, Amer. Math. Soc., Providence, RI 1977.

[32] N. Bourbaki, *Théories spectrales*, Hermann, Paris 1967; Russian transl., Mir, Moscow 1972.

[33] M. A. Naimark, *Normed rings*, Nauka, Moscow 1968; English transl., Wolters-Noordhoff, Groningen 1970.

[34] P. P. Zabreiko, "On the spectral radius of the Volterra operator", *Matematikos Rinkinus* **7**:2 (1967), 281–287. (Russian)

[35] M. A. Evgrafov, *Asymptotic estimates and entire functions*, Fizmatlit, Moscow 1962; English transl., Gordon and Breach, New York 1961.

[36] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math., vol. 840, Springer-Verlag, Berlin–Heidelberg–New York 1981; Russian transl., Mir, Moscow 1985.

[37] A. G. Baskakov, "Semigroups of difference operators in spectral analysis of linear differential operators", *Funktsional. Anal. i Prilozhen.* **30**:3 (1996), 1–11; English transl., *Functional Anal. Appl.* **30**:3 (1996), 149–157.

[38] C. Chicone and Y. Latushkin, *Evolution semigroups in dynamical systems and differential equations*, Mathematical Surveys and Monographs, vol. 70, Amer. Math. Soc., Providence, RI 1999.

[39] W. Mlak, "Notes on circular operators. I–IV", *Univ. Iagel. Acta Math.* **29** (1992), 145–175.

[40] A. G. Baskakov and K. I. Chernyshov, "Spectral analysis of linear relations, and degenerate semigroups of operators", *Mat. Sb.* **193**:11 (2002), 3–42; English transl., *Sb. Math.* **193** (2002), 1573–1610.

Received 22/JAN/04

Translated by V. M. MILLIONSHCHIKOV

Typeset by $\mathcal{AMS}$-TEX