

Arbitrage, Clientele Effects, and the Term Structure of Interest Rates

Eliakim Katz and Eliezer Z. Prisman*

Abstract

This paper derives a new and intuitive estimation procedure for the term structure under potential tax arbitrage. No a priori assumptions regarding the equality of the prices and present values of bonds are made. The data are employed to determine whether this equality holds, and an appropriate estimator is thereby endogenously derived. The suggested estimator is based on the optimizing behavior of an investor in a market with frictions, and emerges directly from the solution of the dual of the no-arbitrage optimization problem. In addition, the proposed estimator benefits from being both theoretically sound and straightforward to apply.

I. Introduction

In recent years, it has been recognized that tax regulations may generate arbitrage opportunities in the bond market. This has brought into doubt the validity of traditional methods used to estimate the term structure of interest rates. The purpose of this paper is to suggest an appropriate method for estimating the term structure in the face of such tax regulations.

Traditionally, it has been assumed that, but for noise, the price of a bond would equal the present value of its cash flow. Regressing bond prices on cash flows was, therefore, considered a correct method of estimating the term structure of interest rates. Schaefer (1982b), however, showed that deviations of the price of a bond from its present value may be due to more than just noise: tax regulations that create arbitrage opportunities may drive a wedge between prices and present values.¹ One implication of such a wedge is that regression analysis is an inappropriate method of estimating the term structure. An alternative estimator of the term structure was proposed by Schaefer (1981),

*Department of Economics, Faculty of Arts and Finance Area, Faculty of Administrative Studies, respectively, York University, Toronto, Ontario, M3J 1P3, Canada. The authors thank the Social Sciences and Humanities Research Council of Canada and the York University Research Administration for partial financial support. They also thank E. Bismuth, Y. Klein, D. Perry, J. Rosenberg, and an anonymous *JFQA* referee for their assistance, and especially *JFQA* Managing Editor Paul Malatesta for his extensive comments and suggestions. The usual caveat applies.

¹The arbitrage referred to in this literature as well as in this paper involves only buy and hold strategies. See Schaefer (1982b), Litzenberger and Rolfo (1984), Jordan (1984), and Dammon and Green (1987).

but this estimator is deficient: it is both arbitrary and based on an unrealistic assumption of market frictions.

This paper derives a new and intuitive estimation procedure for the term structure, which arises from the assumption that arbitrage opportunities are not available in equilibrium. No *a priori* assumption regarding the equality of the prices and present values of bonds is made. Rather, the data determine whether this equality holds and an appropriate estimator is thereby endogenously determined. In addition, the proposed estimation procedure is consistent with utility maximization and benefits from being based on a realistic view of market frictions.

Section II examines tax arbitrage in relation to the potential inequality between bond prices and their present values. In Section III, utility maximization by investors is used to derive the proposed estimator. The compatibility of the estimator with the fundamental assumption that in equilibrium all arbitrage opportunities must be exhausted is considered in Section IV as is the difference between the estimator proposed in this paper and the estimator suggested by Schaefer. Section V offers conclusions.

II. Tax Arbitrage and the Term Structure of Interest Rates

Schaefer (1982b) shows that for individuals in some tax brackets, certain bonds may be overpriced. Therefore, arbitrage opportunities may be present and a bond market equilibrium need not exist. To ensure the existence of an equilibrium, Schaefer assumes that short sales are prohibited, implying that for each tax bracket, the present value of a bond is smaller than or equal to its price. The equilibrium generated by this ban on short sales is characterized by different individuals specializing in the holding of certain subsets of all bonds. A market with such specialized holding of assets is described as having *clienteles effects*.

As a result of the weak inequality between present values and bond prices, the number of feasible term structure estimators (vectors of discount rates) is infinite. Consequently, Schaefer's solution can be viewed as *too* effective: whereas in the absence of frictions no consistent term structure may be feasible, the introduction of an absolute prohibition on short sales generates an infinity of feasible term structures. Schaefer's (1981) choice of an estimator is a term structure that maximizes the present value of an arbitrary prespecified cash flow. As a result, his proposed estimator is similarly arbitrary.²

Litzenberger and Rolfo (1984) argue that tax regulations will not induce a wedge between prices and present values, but their result appears to be based on the assumption that money is not fungible.³ In contrast, Dammon and Green (1987) determine that clientele effects do, in general, emerge in asset markets

²After Schaefer's (1982b) paper, it was accepted that regression is an incorrect method of estimating the term structure. This is because the tax-based wedge between prices and present values causes the difference between the price of a bond and its present value to vary across bonds. Nonetheless, regression continued to be used for its numerical tractability (Jordan (1984)), and also perhaps because of the deficiencies in the alternative estimator suggested by Schaefer (1981).

³See Prisman (1990) for an examination of their model.

with perfectly correlated securities.⁴ Since such securities abound in the bond markets (in contrast with the equity market), Dammon and Green's results confirm the need for a new estimation method for the term structure. Also, Dammon and Green's argument, that arbitrage opportunities need not generate clientele effects if bonds are imperfectly correlated, is partially dependent on the absence of tax-exempt investors. Since a large proportion of investors in the bond market is tax exempt, tax arbitrage is likely to induce clientele effects into the bond market even when bonds are not perfectly correlated.⁵

Let A^ℓ be the net-of-tax payments matrix of an individual in tax bracket ℓ . Element a_{ij}^ℓ of A^ℓ is the after-tax cash flow made to the investor by bond i in period j , $\ell = 1, \dots, r$, $i = 1, \dots, m$, $j = 1, \dots, t$. Let P be the column vector of bond prices $P = (P_1, \dots, P_m)$. Also, let $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ be column vectors such that x_i, y_j are, respectively, the number of units of bond i bought and the number of units of bond j sold short by the investor.

Following Ross (1978), the absence of arbitrage opportunities is defined by the satisfaction of the condition,

$$(1) \quad \min\{x'P - y'P \text{ s.t. } (x - y)'A^\ell \geq 0\} = 0 \quad \text{for all } \ell = 1, \dots, r.$$

It is well known that the no-arbitrage condition defined above holds if and only if for every tax bracket ℓ , there exists a vector of discount factors $d^\ell = (d_1^\ell, \dots, d_t^\ell) > 0$ such that $A^\ell d^\ell = P$. Dammon and Green establish that this condition will be satisfied only if

$$(2) \quad \bigcap_{\ell=1}^r Q^\ell \neq \emptyset \text{ where } Q^\ell = \{q \mid \text{there exists a } d > 0, A^\ell d = q\}.$$

A necessary and sufficient condition for $A^\ell d^\ell$ to equal P for *each* tax bracket is that P belongs to the above intersection. Hence, (2) is only a necessary condition for the equilibrium to be free of clientele effects. The example used by Schaefer (1982b) to demonstrate that tax regulations may lead to the nonexistence of an equilibrium can now be seen as a simple case (two bonds, two time periods, and two tax brackets) of $\bigcap_{\ell=1}^r Q^\ell = \emptyset$. This simple case does not comply even with Dammon and Green's necessary condition for an equilibrium without clientele effects. It is, therefore, very unlikely that the necessary *and* sufficient conditions for the absence of arbitrage opportunities will be satisfied in the typically complex cases encountered in actual markets. Consequently, it is probable that the existence of an equilibrium depends on the presence of frictions that limit arbitrage activity. Thus, a bond market equilibrium is likely to be characterized by clientele effects. This implies that for individuals in certain tax brackets the equilibrium prices of some bonds may be lower than their present values. An estimator of the term structure that relies on the equality

⁴See Dammon and Green (1987), p. 1144, line 31, and p. 1155, last paragraph.

⁵Given progressive taxation, Dammon and Green argue that arbitrage profits will increase and converge the marginal tax rates of investors until no further tax arbitrage opportunities exist. Hence, arbitrage will be self-limiting. The simultaneous presence of tax-exempt investors *and* taxable investors, however, may *raise* arbitrage profits rather than *reduce* tax-rate divergence between investors. Clientele effects are, therefore, likely to emerge even in this case.

between prices and present values may, therefore, be inappropriate. Hence, the type of equilibrium must be determined prior to the execution of the estimation procedure.

This paper suggests an estimation procedure that does not *a priori* assume the presence or absence of clientele effects. The procedure determines whether or not clientele effects are present and produces an estimator appropriate to its finding. In addition, Schaefer's assumption of a prohibition on short sales is replaced with the more realistic assumption of limited short sales. This is the norm in many major economies, such as the U.S., Great Britain, and Canada.

III. Utility Analysis

Consider an investor maximizing an intertemporal utility function, $U : R^t \rightarrow R^1$, defined on consumption levels in future periods, $1, \dots, t$. To simplify the analysis without altering the results, the optimization begins with consumption in period 1. The investor is permitted to sell bonds short, but the short part of his portfolio cannot exceed some finite and strictly positive amount, V . Following Schaefer, a short sales constraint may reflect institutional factors as well as tax asymmetries. Hence, V , which might be a fraction of the individual's total portfolio or a fixed dollar amount, is institutionally determined.

A bond makes three types of payments, each of which may be differently treated for tax purposes: nontaxable return of principal, ordinary income, and capital gains. Let B_j be an $m \times 3$ matrix whose i th row is the three-component vector, $(b_{1ij}, b_{2ij}, b_{3ij})$, of the gross cash flow generated by bond i in period j . The three components of this vector correspond to the three types of cash flows. The gross cash flow from bond i in period j is, therefore,

$$b_{ij} = \sum_{k=1}^3 b_{kij}.$$

Let the mapping $T : R^3 \rightarrow R^3$ be the tax function, where T is convex to reflect a progressive tax structure. The argument of T is the investor's cash flow vector, and T assigns a vector consisting of the tax due on each component of this cash flow. Hence, an individual's consumption level in period j , C_j equals $[(x-y)B_j - T((x-y)B_j)](h)$, where h is the column vector $(1, 1, 1)$. A utility maximizing individual solves

$$(3) \quad \begin{aligned} & \max U(C_1, C_2, \dots, C_t) \\ & \text{s.t.} \quad xP - yP \leq W \\ & \quad \quad yP \leq V \\ & \quad \quad y, x \geq 0, \end{aligned}$$

where W is his initial wealth.

Denoting the derivative of a function f with respect to its j th argument by f_j , the first order conditions for a maximum are

$$(4) \quad \sum_{j=1}^t \{(U_j(\cdot)) [b_{1ij}(1-T_1) + b_{2ij}(1-T_2) + b_{3ij}(1-T_3)]\} - \xi P_i \leq 0,$$

$$i = 1, \dots, m,$$

$$(5) \quad \sum_{j=1}^t \{(U_j(\cdot)) [b_{1ij}(1-T_1) + b_{2ij}(1-T_2) + b_{3ij}(1-T_3)]\} - \xi P_i + \lambda P_i \leq 0,$$

$$i = 1, \dots, m,$$

where ξ is the Lagrangian multiplier associated with the wealth constraint, $(x-y)P \leq W$, and λ is the Lagrangian multiplier associated with the short sales constraint.

The contents of the square brackets in (4) and (5) are the after-tax cash flow from bond i in period j , a_{ij} , to an investor in a given tax bracket (the superscript ℓ having been suppressed). Equations (4) and (5) can be written as

$$(4a) \quad \sum_{j=1}^t U_j(\cdot) a_{ij} - \xi P_i \leq 0, \quad i = 1, \dots, m,$$

$$(5a) \quad \sum_{j=1}^t U_j(\cdot) a_{ij} - \xi P_i - \lambda P_i \leq 0, \quad i = 1, \dots, m.$$

Consider the Lagrangian multipliers in (4a) and (5a): ξ and λ are the shadow prices (in terms of utility) of the wealth constraint and the short sales constraint, respectively. Given nonsatiation, ξ must be strictly positive. If utility increases when the constraint on short sales is relaxed, the short sales constraint is binding and λ is strictly positive. Conversely, if utility remains unchanged when the short sales constraint is relaxed, λ is zero.

Let ψ be the arbitrage profits (in dollar terms) associated with a one dollar relaxation in the short sales constraint. Arbitrage profits constitute an increase in wealth and, from the above discussion, a one dollar increase in wealth increases utility by ξ . Hence, a one dollar relaxation in the short sales constraint increases utility by $\xi\psi$. But, a one dollar relaxation in the short sales constraint raises utility by λ . Therefore, $\xi\psi = \lambda$ such that $\psi (= \lambda/\xi)$ satisfies $0 \leq \psi \leq 1$.⁶ If at the optimum $\psi > 0$, then $\lambda > 0$ and from Kuhn-Tucker conditions, it follows that $y_i > 0$ for at least one i , $i = 1, \dots, m$.

Nonsatiation ensures that the wealth constraint is always binding, $(x-y)P = W$. Therefore, at the optimum, $x \neq 0$. In contrast, the vector of short sales, y , is non-zero only if there are arbitrage opportunities, in which case $\lambda > 0$ and $\psi > 0$. The Kuhn-Tucker conditions for an optimum determine that the left-hand side of (5a) is equal to zero if y is non-zero. Alternatively, if $y = 0$, both λ and ψ equal zero.

⁶Consider a one dollar relaxation in the short sales constraint. The investor sells short one dollar's worth of the (appropriate) overpriced bond. He then buys a proportion of a correctly priced bond or (portfolio of bonds) that replicates the cash flow of the bond he sold short. The price he has to pay for the proportion of the correctly priced bond must be less than the one dollar paid for the comparable part of the overpriced bond. Hence, his profits are $1 - \theta$ dollars, where $0 \leq \theta \leq 1$ is the price of the proportion of the correctly priced bond. Hence, $0 \leq 1 - \theta = \psi \leq 1$.

$U_j(\cdot)/\xi$ is the marginal utility of consumption in period j normalized by the initial marginal utility of wealth. Hence, it is period j 's discount factor. Denoting $U_j(\cdot)/\xi$ by d_j and the vector (d_1, \dots, d_t) by d , (4a) and (5a) yield

$$(6a) \quad x_i > 0 \rightarrow \sum_{j=1}^t d_j a_{ij} = P_i, \quad i = 1, \dots, m,$$

$$(6b) \quad y_i > 0 \rightarrow \sum_{j=1}^t d_j a_{ij} = P_i(1 - \psi), \quad i = 1, \dots, m,$$

$$(6c) \quad y_i = 0 \rightarrow \sum_{j=1}^t d_j a_{ij} = P_i, \quad i = 1, \dots, m.$$

The intuition behind (6b) is that the prices of those bonds that are sold short must reflect both their cash flow and their potential arbitrage opportunities: the present value of the cash flow of a bond that is sold short captures only part of that bond's value. When an investor sells bond i short, he (a) foregoes the cash flow associated with bond i , and (b) makes arbitrage profits. For this he is paid P_i . To avoid making a loss by short selling the bond, the price of the bond minus the potential arbitrage profits from its short sale must exceed or be equal to the present value of its cash flow. The arbitrage profit associated with one dollar of short sales is ψ and the short sale of bond i is an arbitrage transaction involving P_i dollars. Hence, the arbitrage profit is ψP_i . The price of bond i , P_i , must therefore satisfy

$$(7) \quad P_i - \psi P_i \geq \sum_{j=1}^t d_j a_{ij}.$$

It follows that when $y_i > 0$,

$$(8) \quad \sum d_j a_{ij} = (1 - \psi)P_i.$$

IV. Restricted Arbitrage and the Estimation of the Term Structure

Equilibrium is characterized by the absence of available arbitrage opportunities for every investor: such opportunities will either have been used up to the limit of the constraint or not been available in the first place. There are two cases to consider: either yP is equal to V or yP is equal to zero.

To determine the discount factor in equilibrium, the short sales constraint must be relaxed. Let x and y in (9) be the changes that occur in the investor's long and short positions in response to a small relaxation, ϵ , in the short sales constraint. Starting from his equilibrium, an investor's arbitrage profit maximization requires that the changes x and y be determined by solving

$$(9) \quad \max(y - x)'P \text{ s.t. } (y - x)'A \leq 0, \quad yP \leq \epsilon, \quad x, y > 0.$$

The dual of (9) is⁷

$$(10) \quad \min \psi \text{ s.t. } (1 - \psi)P \leq Ad \leq P, \quad d, \psi \geq 0.$$

The constraints in (10) turn out to be identical with the first order conditions for utility maximization, thereby confirming the link between utility and arbitrage maximization. Moreover, since the solution to (10) must satisfy $Ad \leq P$, this solution allows for the presence of clientele effects. \hat{d} , which solves (10), is an estimator of the term structure. Since \hat{d} is based on fully exploited arbitrage, it is consistent with utility maximization and independent of initial wealth, W . Since the elements of A are net of tax cash flows, \hat{d} does, however, depend on the investor's marginal tax bracket.

If the optimal value of (10) is zero, ($\psi = 0$), all bonds are correctly priced and no clientele effects are present for this tax bracket. On the other hand, if ψ is positive, Ad is less than or equal to P , and for some bonds, Ad must be smaller than P . Arbitrage activity is blocked by the short sales constraint, and clientele effects are present.

Problem (10) can be expressed as

$$(11) \quad \begin{aligned} \min \quad & \max \quad Z_i = (P_i - \sum_{j=1}^l a_{ij}d_j)/P_i. \\ \text{s.t. } \quad & d \geq 0, \quad i = 1, \dots, m \\ & Ad \leq P \end{aligned}$$

This formulation of problem (11) highlights an intuitively appealing property of \hat{d} : for a given d , the bond that provides an investor in a given tax bracket with the most profitable arbitrage activity is bond n , where $Z_n = \max_{i=1, \dots, m} Z_i$. The estimator of d , \hat{d} , minimizes the arbitrage return to bond n . It is in the nature of competitive markets that the actions of participants in such markets eliminate arbitrage opportunities. If such elimination is not possible, market forces reduce arbitrage opportunities to their lowest possible level: in a market where, as a result of institutional constraints, arbitrage opportunities *cannot* disappear, they will be minimized both directly and by market pressure on other variables. In the situation discussed in this paper, d will adjust so as to minimize Z_n , reflecting the actions of investors.

⁷Problem (9) may be written in a linear programming canonical form as

$$\begin{aligned} & \max(x, y)(P, -P) \\ \text{s.t. } (y, x) & \begin{bmatrix} A & P \\ -A & \underline{0} \end{bmatrix} \geq \begin{pmatrix} \underline{0} \\ \epsilon \end{pmatrix} \\ & x, y \geq 0, \end{aligned}$$

where $\underline{0}$ is a $1 \times m$ column vector of zeroes. The dual of (7) is, thus,

$$\begin{aligned} & \min \underline{0}'d + \epsilon\psi \\ \text{s.t. } & \begin{bmatrix} -A & P \\ -A & \underline{0} \end{bmatrix} \begin{pmatrix} d \\ \psi \end{pmatrix} \leq \begin{pmatrix} P \\ -P \end{pmatrix} \\ & d, \psi \geq 0. \end{aligned}$$

Some algebraic manipulations show that this dual problem is equivalent to (10) in the text.

In addition to yielding an estimator of the term structure, the methodology developed here facilitates the identification of bonds with clientele effects. If element i in the vector $A\hat{d}$ is smaller than the price of bond i , this bond is identified as having clientele effects. This method of identifying bonds with clientele effects evolves naturally from the model used in this paper.

In contrast, Schaefer's model does not provide an estimator based on the investor's optimization. Schaefer suggests that d can be estimated by solving

$$(12) \quad \max \sum_{j=1}^t d_j s_j \quad \text{s.t.} \quad Ad \leq P, \quad d \geq 0,$$

where $s = (s_1, \dots, s_t)$ is an arbitrary, prespecified cash flow. The arbitrariness of s implies that the term structure estimator, $\hat{d}(s)$, is similarly arbitrary. Schaefer appears to recognize the weakness of this estimator and does not attempt to identify overpriced bonds by using $\hat{d}(s)$. Instead, he suggests the following technique to identify overpriced bonds. For each tax bracket, determine the maximum value of bond i ($i = 1, \dots, m$) over all feasible vectors, $d \in \{d | Ad \leq P, d > 0\}$. Define this maximum value as

$$(13) \quad R_i \equiv \max \sum_{j=1}^t a_{ij} d_j \quad \text{s.t.} \quad Ad \leq P, \quad d \geq 0,$$

and the optimal solution of (13) by \bar{d}^i .

If the price of bond i exceeds its present value, the bond is overpriced: duality theory can be used to prove the existence of a lower priced bond or portfolio generating the same after-tax cash flow as bond i . Bond i is, therefore, incorrectly priced for the tax bracket in question. Furthermore, in (10) and (12), the i th constraint, $\sum_{j=1}^t a_{ij} d_j \leq P_i$, is superfluous and can be removed without altering the shape of the feasible set, $Ad \leq P$. Also, there is no feasible d that satisfies $\sum_{j=1}^t a_{ij} d_j = P_i$.

The solution of (13), \bar{d}^i , developed by Schaefer, maximizes the value of each bond, i , whereas the estimator developed in this paper does not. Hence,

$$(14) \quad \sum_{j=1}^t a_{ij} \bar{d}_{ij} \geq \sum_{j=1}^t a_{ij} \hat{d}_j, \quad i = 1, \dots, m.$$

Hence, as acknowledged by Schaefer, his condition for a bond to display clientele effects is sufficient rather than necessary. Schaefer's method provides a lower bound on the number of bonds that have clientele effects. The method developed here generates more such bonds with a single estimator of d .

V. Conclusions

Tax regulations have recently been shown to be capable of driving a wedge between the prices of certain bonds and their present values. In the presence of such a wedge, the use of regression analysis to estimate the term structure is incorrect. There is, therefore, a need for a method that identifies bonds

characterized by clientele effects, as well as for an appropriate procedure for estimating the term structure in the presence of such effects. This paper satisfies these needs by developing a procedure that simultaneously determines the presence of clientele effects and provides an appropriate estimator of the term structure.

The proposed means of identifying bonds characterized by clientele effects is endogenous to the model rather than based on extraneous considerations. The suggested estimator is based on the optimizing behavior of an investor in a market with frictions, and it emerges directly from the solution of the dual of the no-arbitrage optimization problem. The estimator reduces perceived arbitrage opportunities in the market, emulating the effect of competition. Finally, the proposed estimator bypasses the typical trade-off between theoretical validity and pragmatic considerations. While being theoretically sound, the estimator is straightforward to apply.

References

- Dammon, M. R., and C. R. Green. "Tax Arbitrage and the Existence of Equilibrium Prices for Financial Assets." *Journal of Finance*, 42 (Dec. 1987), 1143-1166.
- Jordan, J. V. "Tax Effects in Term Structure Estimation." *Journal of Finance*, 39 (June 1984), 393-406.
- Litzenberger, R. H., and J. Rolfo. "An International Study of Tax Effects on Government Bonds." *Journal of Finance*, 39 (March 1984), 1-22.
- Prisman, E. Z. "Bond Pricing in Markets with Taxes: the Clientele Model vs. the Non-Clientele Model." *Journal of Banking and Finance*, 14 (March 1990), 33-39.
- Schaefer, S. M. "Measuring a Tax Specific Term Structure of Interest Rates in the Market for British Government Securities." *The Economic Journal*, 91 (June 1981), 415-438.
- _____. "Tax Induced Clientele Effects in the Market for British Government Securities." *Journal of Financial Economics*, 10 (July 1982a), 121-159.
- _____. "Taxes and Security Market Equilibrium." In *Financial Economics: Essays in Honor of Paul H. Cootner*, W. F. Sharpe and C. M. Cootner, eds. Englewood Cliffs, NJ: Prentice-Hall (1982b), 159-178.