

Cosmological solutions of the Einstein-Vlasov-scalar field system

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To my wife Blandine Pulcherie Tamatcho and our child Marlène, Audrey and Brice Tegankong.

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Contents

Abstract	1
Introduction	2
1 Equations and preliminary results	5
1.1 Equations	5
1.2 Auxiliary system and preliminary results	10
1.3 The reduced system	16
2 Local existence and continuation of solutions	26
2.1 Iteration	26
2.2 Local existence	35
2.3 Continuation criteria	41
3 Global existence and asymptotic behaviour of solutions in the past	50
3.1 Global existence in the past	50
3.2 On past asymptotic behaviour	54
4 Global existence and asymptotic behaviour of solutions in the future	62
4.1 Global existence in the future	62
4.2 The future asymptotic behaviour	68
4.2.1 Integration of equations	69
4.2.2 Geodesic completeness	70
Conclusion	75
Appendices	76

Abstract

The aim of this thesis is to obtain as much information as possible, about global solutions of the Cauchy problem for the Einstein-Vlasov-scalar field system with spherical, plane and hyperbolic symmetries written in areal coordinates. The sources of this system are generated by both a distribution function and a linear scalar field subject to the Vlasov and wave equations respectively. This system describes the evolution of self-gravitating collisionless matter and scalar waves within the context of general relativity. We consider the cosmological case. That is spacetimes possess a compact Cauchy hypersurface and then, data are given on a compact 3-manifold.

We extend the local-in-time results obtained by G. Rein for the Einstein-Vlasov system with collisionless matter alone. This extension concerns pointwise estimates for hyperbolic equations by the method of characteristics. This means that the system is transformed to a system of ordinary differential equations which are integrated along characteristics. The constraint equation on the initial data reduced to an ordinary differential equation of first order and is solved. In the past direction, we show global existence results for general data. The proof is based on a change of variables inspired by the work of M. Weaver. The nature of singularity is analyzed. The curvature invariant called Kretschmann scalar blows up as t tends to 0 so that there is a singularity at t equal zero.

We prove that there is no global solution in the future in the spherical symmetry case. In the plane and hyperbolic symmetries, the area radius goes to infinity and so we obtain global solutions in the expanding direction. In the special case of plane symmetry without Vlasov contribution, we show that the asymptotics are Kasner-like at early time. Moreover the spacetime obtained in this case is future geodesically complete.

We conclude the work by showing that the spatially homogeneous solutions of the plane and hyperbolic symmetric Einstein-Vlasov-scalar field system exist globally in the future and the corresponding spacetimes are geodesically complete. Future asymptotics are Kasner-like in the plane symmetric case.

Introduction

The Einstein-Vlasov system governs the time evolution of a self-gravitating collisionless gas in the context of general relativity. In the mathematical study of general relativity, one of the main problems is to establish the existence and properties of global solutions of the Einstein equations coupled to various matter fields such as collisionless matter described by the Vlasov equation (see [1], [21] for reviews) or a scalar field (see [19] for the cosmological case and [6], [7] and references therein for the asymptotically flat case). The aim of the present investigation is to establish in the cosmological case the global in time existence of solutions and their behaviour near the singularity and in the future. In this case the whole universe is modelled and the "particles" in the kinetic description are galaxies or even clusters of galaxies.

In [15] and [16] G. Rein obtained cosmological solutions of the Einstein-Vlasov system with surface symmetry written in areal coordinates. In [27] and [30], these results were generalized to the case of non-vanishing cosmological constant. In the present work, we extend the results of [15] to the case where the source terms of the Einstein equations are generated by both a distribution function f of particles, which is subject to the Vlasov equation, and a massless scalar field ϕ , which is subject to the wave equation. The first result we establish is a local in time existence theorem together with a continuation criterion. With this, we prove global existence in time and study the asymptotic behaviour of solutions when the time coordinate t tends to its limiting values, which might correspond to the approach to the singularity or a phase of unending expansion.

There are several reasons why it is of interest to look at the case of a scalar field. The first is that it is the simplest situation in which wave phenomena can be examined in the context of the Einstein-Vlasov system. In surface symmetry all wave propagation can be eliminated from the Einstein equations by the use of suitable coordinate conditions. This is an analogue of the well-known statement that there are no gravitational waves in spherical symmetry. Mathematically it means that controlling solutions of the Einstein equations can be reduced to controlling solutions of ordinary differential equations in time and in space. The Vlasov equation, being a scalar hyperbolic equation of first order, can also easily be solved in terms of its characteristics. This was the strategy used in [15] and [16]. In the presence of a cosmological constant it is possible to follow the same route.

The inclusion of a scalar field introduces waves into the system which cannot

be eliminated. Mathematically this means that it introduces a non-trivial hyperbolic equation, the wave equation. This work is concerned with symmetric situations where there is a symmetry group acting on two-dimensional spacelike orbits. This means that the wave equation reduces to an effective equation in one space dimension. As a consequence part of the strategy used previously can be carried over. That was based on pointwise estimates and not on integral estimates (energy estimates) as is usual in the theory of hyperbolic equations. Pointwise estimates for solutions of wave equations in terms of data can be obtained in one space dimension but not in higher space dimensions (See e.g. the discussion in [12], p. 14.)

Pointwise estimates for hyperbolic equations in one space dimension can be obtained using the method of characteristics. This means that in fact ordinary differential equations appear once again but this time they are integrated not at a constant value of the spatial or time variable but along characteristics. This method will be applied in the following, the characteristics in this case being null curves of the spacetime geometry.

It should be mentioned that there are global existence results in the literature where the Einstein-Vlasov system is considered in a context where hyperbolic equations play an important role. In fact if we relax the assumptions from surface symmetry, where there are three local Killing vectors, to the case where there are only two local spacelike Killing vectors, then hyperbolic equations necessarily occur. Some relevant papers are [20], [4], [32]. In those references no direct local existence proof was given. Instead an indirect argument was used. First a known local existence theorem for the Einstein-Vlasov equation without symmetry was quoted. Then it was shown that coordinates could be introduced which are well-adapted to making use of the symmetry when proceeding to obtain global results. Apart from the methodological interest of having a direct local existence proof, the direct proof gives stronger results concerning the differentiability required of the initial data and obtained for the solutions. This is very difficult to control in the indirect method and for this reason the latter has only been applied to the case where everything is of infinite differentiability.

The inclusion of a scalar field can be seen as a step towards certain questions of physical interest. In recent years cosmological models with accelerated expansion have become a very active research topic in response to new astronomical observations [25]. The easiest way to obtain models with accelerated expansion is to introduce a positive cosmological constant, a possibility studied mathematically in [28], [30] and [13]. A more sophisticated way is to introduce a scalar field with potential (see [22], section 4.3., [23], [14]). We treat only the case of a linear scalar field but it is likely that the approach developed here will be useful in the nonlinear case. Scalar fields also play a role in theories of gravity generalizing Einstein's theory, such as the Jordan-Brans-Dicke theory. In that case, in contrast to the one considered here, there is a direct coupling between the scalar field and the distribution function. The techniques developed here could serve as a first step towards the study of these more complicated situations, which have hardly been looked at mathematically yet. (See however [2] and [5] where the coupling of the Vlasov equation to a scalar field of the

Jordan-Brans-Dicke type in the absence of Einstein gravity is considered).

A large part of our investigation will focus on the initial value problem for the Einstein-Vlasov-scalar field system with surface symmetry. In the first chapter, we split the wave equation in ϕ into a system of two partial differential equations of first order. This permits us to bound the derivatives of ϕ by the solutions of the field equations and to introduce an auxiliary system. Next, we solve each equation of the auxiliary system when the other unknowns are fixed. Using the results of [15], and under some constraints on the initial data, the full system is equivalent to the auxiliary system and is reduced to a subsystem. This chapter ends with the solvability of the constraint equation on data. By iterating the solutions of the auxiliary system, local in time existence and uniqueness of solutions in both time directions and continuation criteria are established in the second chapter.

The third chapter concentrated on the contracting direction. Solutions of the Einstein-Vlasov-scalar field system with spherical, plane and hyperbolic symmetry exist on the whole interval $]0, 1[$ for general initial data. The proof is based on a change of variables inspired by [32] where the existence up to $t = 0$ for a certain class of T^2 -symmetric solutions of the Einstein-Vlasov system with vanishing cosmological constant was studied. The structure of the initial singularity is analyzed as in [15]. We show that the spacetime has a curvature singularity. Moreover the singularity is crushing for any solution in $]0, 1[$. An idea from [24] which studied the singularity for solutions of the Einstein equations in Gowdy spacetimes, allows us to prove in the special case of the spherical, plane and hyperbolic symmetric Einstein-scalar field system that the singularity is velocity-dominated.

In the expanding direction in chapter four, a global existence result in the cases of plane and hyperbolic symmetry is obtained. But this fails in the spherical case. The spacetime is future geodesically complete in the special case of plane symmetry without Vlasov contribution. The same result holds for spatially homogeneous spacetimes which are solutions of the full system.

The present work is organized as follows : chapter 1 is concerned with the formulation of the surface symmetric Einstein-Vlasov-scalar field system written in areal coordinates and the proof of some preliminary results. The results of chapter 1 are used to obtain in chapter 2, local existence theorems with continuation criteria in both time directions. Chapter 3 focuses on the existence of solutions up to $t = 0$ and their behaviour near the initial singularity. In chapter 4, we prove a global existence result in the future and geodesic completeness.

Chapter 1

Equations and preliminary results

1.1 Equations

Let us recall the formulation of the Einstein-Vlasov-scalar field system ; for the moment we do not assume any symmetry of the spacetime.

We consider a four-dimensional spacetime manifold M , with local coordinates $(x^\alpha) = (t, x^i)$ on which $x^0 = t$ denotes the time and (x^i) the space coordinates. Unless otherwise specified in what follows Greek indices always run from 0 to 3, and Latin ones from 1 to 3. On M , a Lorentzian metric g is given with signature $(-, +, +, +)$. The metric is assumed to be time-orientable, i.e. that the two halves of the light cone at each point of M can be labelled past and future in a way which varies continuously from point to point. With this global direction of time, it is possible to distinguish between future-pointing and past-pointing timelike vectors. The worldline of a particle of non-zero rest mass m is a timelike curve in spacetime. The unit future-pointing tangent vector to this curve is the four-velocity v^α of the particle. Its four-momentum p^α is given by mv^α . Here we assume that all particles have the same mass m , normalized to unity and no distinction need be made between four-velocity and four-momentum. There is also the possibility of considering massless particles, whose worldlines are null curves. In the case $m = 1$ the possible values of the four-momentum are precisely all future-pointing unit timelike vectors. These form a hypersurface (seven-dimensional submanifold)

$$PM := \{g_{\alpha\beta}p^\alpha p^\beta = -1, p^0 > 0\},$$

in the tangent bundle TM called the mass shell and coordinatized by (t, x^i, p^i) . If the coordinates are such that the components g_{0i} vanish then the component p^0 is expressed by the other coordinates via

$$p^0 = \sqrt{-g^{00}} \sqrt{1 + g_{ij}p^i p^j}.$$

The distribution function f , which represents the density of particles with given spacetime position and four-momentum, is a non-negative real-valued function on PM . A basic postulate in general relativity is that a free particle travels along a geodesic. Consider a future-directed timelike geodesic parameterized by proper time. Then its tangent vector at any time is future-pointing unit timelike. Thus this geodesic has a natural lift to a curve on PM , by taking its position and tangent vector together. This defines a flow on PM . Denote the vector field which generates this flow by X . The condition that f represents the distribution of a collection of particles moving freely in the given spacetime is that it should be constant along the flow, i.e. that $Xf = 0$. This is the Vlasov equation. In addition we consider a scalar field ϕ which is a real-valued function on M . The Vlasov equation can be coupled to the Einstein-scalar field equations, giving rise to the Einstein-Vlasov-scalar field system. The unknowns are a 4-manifold, a (time orientable) Lorentz metric g on M , a non-negative real-valued function f on the mass shell defined by g and a real-valued function ϕ on M . The Einstein-Vlasov-scalar field system now reads:

$$\begin{aligned} \partial_t f + \frac{p^i}{p^0} \partial_{x^i} f - \frac{1}{p^0} \Gamma_{\beta\gamma}^i p^\beta p^\gamma \partial_{p^i} f &= 0 \\ G_{\alpha\beta} &= 8\pi T_{\alpha\beta} \\ T_{\alpha\beta} &= - \int_{\mathbb{R}^3} f p_\alpha p_\beta |g|^{\frac{1}{2}} \frac{dp^1 dp^2 dp^3}{p^0} + (\nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} \nabla_\nu \phi \nabla^\nu \phi) \end{aligned}$$

where $p_\alpha = g_{\alpha\beta} p^\beta$, $|g|$ denotes the modulus of determinant of the metric $g_{\alpha\beta}$, $\Gamma_{\alpha\beta}^\lambda$ the Christoffel symbols,

$$G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

the Einstein tensor, and $T_{\alpha\beta}$ the energy-momentum tensor. R is the scalar curvature of g and

$$R_{\alpha\beta} = R^\nu{}_{\alpha,\nu\beta} = \partial_\nu \Gamma_{\alpha\beta}^\nu - \partial_\beta \Gamma_{\alpha\nu}^\nu + \Gamma_{\nu\rho}^\nu \Gamma_{\alpha\beta}^\rho - \Gamma_{\rho\beta}^\nu \Gamma_{\alpha\nu}^\rho$$

the Ricci tensor.

Lemma 1.1 *For any scalar field ϕ of class C^2 defined on M , we have :*

$$\nabla_\alpha (\nabla^\alpha \phi \nabla^\beta \phi - \frac{1}{2} g^{\alpha\beta} \nabla_\nu \phi \nabla^\nu \phi) = \square_g \phi \nabla^\beta \phi$$

Proof: We have, using the properties of g :

$$\begin{aligned} \nabla_\alpha (\nabla^\alpha \phi \nabla^\beta \phi) &= \nabla_\alpha (\nabla^\alpha \phi) \nabla^\beta \phi + \nabla^\alpha \phi \nabla_\alpha \nabla^\beta \phi \\ &= \square_g \phi \nabla^\beta \phi + \nabla^\alpha \phi \nabla_\alpha \nabla^\beta \phi \end{aligned}$$

$$\begin{aligned}
g^{\alpha\beta}\nabla_\alpha(\nabla_\nu\phi\nabla^\nu\phi) &= g^{\alpha\beta}\nabla_\alpha(\nabla_\nu\phi)\nabla^\nu\phi + g^{\alpha\beta}\nabla_\nu\phi(\nabla_\alpha\nabla^\nu\phi) \\
&= (\nabla^\beta\nabla_\nu\phi)\nabla^\nu\phi + \nabla_\nu\phi(\nabla^\beta\nabla^\nu\phi) \\
&= \nabla^\nu\phi\nabla^\beta\nabla_\nu\phi + g_{\lambda\nu}\nabla^\lambda\phi(\nabla^\beta g^{\nu\eta}\nabla_\eta\phi) \\
&= \nabla^\nu\phi\nabla^\beta\nabla_\nu\phi + \delta_\lambda^\eta\nabla^\lambda\phi\nabla^\beta\nabla_\eta\phi \\
&= \nabla^\nu\phi\nabla^\beta\nabla_\nu\phi + \nabla^\lambda\phi\nabla^\beta\nabla_\lambda\phi \\
&= 2\nabla^\nu\phi\nabla^\beta\nabla_\nu\phi
\end{aligned}$$

Therefore

$$\nabla_\alpha(\nabla^\alpha\phi\nabla^\beta\phi - \frac{1}{2}g^{\alpha\beta}\nabla_\nu\phi\nabla^\nu\phi) = \square_g\phi\nabla^\beta\phi + \nabla^\alpha\phi\nabla_\alpha\nabla^\beta\phi - \nabla^\nu\phi\nabla^\beta\nabla_\nu\phi = \square_g\phi\nabla^\beta\phi. \square$$

Remark 1.2 Due to the Bianchi Identities, the Einstein equations imply the conservation law $\nabla_\alpha T^{\alpha\beta} = 0$. Now since the contribution of f to the energy-momentum tensor is divergence-free [8], we deduce that:

$$\nabla_\alpha(\nabla^\alpha\phi\nabla^\beta\phi - \frac{1}{2}g^{\alpha\beta}\nabla_\nu\phi\nabla^\nu\phi) = 0$$

which is equivalent to

$$\square_g\phi\nabla^\beta\phi = 0$$

i.e $\square_g\phi = 0$ or $\nabla^\beta\phi = 0$. Consider the open set $S = \{(t, r) | \square_g\phi(t, r) \neq 0\}$. Suppose that S is non- empty. On S , $\nabla^\beta\phi = 0$ then $\nabla^\beta\nabla_\beta\phi = 0$ i.e $\square_g\phi = 0$, which is a contradiction. Therefore S is empty. This means that

$$\square_g\phi = 0,$$

which is the wave equation for ϕ .

Remark 1.3 The Vlasov equation in a fixed spacetime is a linear hyperbolic equation for a scalar function and hence solving it is equivalent to solving the equations for its characteristics. In coordinate components these are :

$$\begin{cases} \frac{dX^i}{ds} = P^i \\ \frac{dP^i}{ds} = -\Gamma_{\beta\gamma}^i P^\beta P^\gamma \end{cases}$$

Let $X^i(s, x^\alpha, p^i)$, $P^i(s, x^\alpha, p^i)$ be the unique solution of the previous system with initial conditions $X^i(t_0, x^\alpha, p^i) = x^i$ and $P^i(t_0, x^\alpha, p^i) = p^i$. Then the solution of the Vlasov equation can be written as :

$$f(x^\alpha, p^i) = f_0(X^i(t_0, x^\alpha, p^i), P^i(t_0, x^\alpha, p^i))$$

where f_0 is the restriction of f to the hypersurface $t = t_0$. This function f_0 serves as initial datum for the Vlasov equation.

In [18], a definition of spacetimes with spherical, plane and hyperbolic symmetry was given. The spacetime (M, g) is topologically of the form $]0, \infty[\times S^1 \times S$, where S is a 2-sphere, a 2-torus, or a hyperbolic plane, in the case of spherical, plane or hyperbolic symmetry respectively. We now consider a solution of the Einstein-Vlasov-scalar field system where all unknowns are invariant under one of these symmetries and write the Einstein-Vlasov system in areal coordinates. The circumstances under which coordinates of this type exist are discussed in [3]. The metric g takes the form

$$ds^2 = -e^{2\mu(t,r)} dt^2 + e^{2\lambda(t,r)} dr^2 + t^2(d\theta^2 + \sin_k^2 \theta d\varphi^2) \quad (1.1)$$

where

$$\sin_k \theta = \begin{cases} \sin \theta & \text{for } k = 1 \text{ (spherical symmetry);} \\ 1 & \text{for } k = 0 \text{ (plane symmetry);} \\ \sinh \theta & \text{for } k = -1 \text{ (hyperbolic symmetry)} \end{cases}$$

$t > 0$ denotes a time-like coordinate, $r \in \mathbb{R}$ and (θ, φ) range respectively in the domains $[0, \pi] \times [0, 2\pi]$, $[0, 2\pi] \times [0, 2\pi]$, $[0, \infty[\times [0, 2\pi]$ respectively, and stand for angular coordinates. The functions λ and μ are periodic in r with period 1. It has been shown in [15] and [16] that due to the symmetry, f can be written as a function of

$$t, r, w := e^\lambda p^1 \text{ and } F := t^4[(p^2)^2 + \sin_k^2 \theta (p^3)^2],$$

i.e. $f = f(t, r, w, F)$. In these variables, we have $p^0 = e^{-\mu} \sqrt{1 + w^2 + F/t^2}$. The scalar field is a function of t and r which is periodic in r with period 1.

We denote by a dot and by a prime the derivatives of the metric components and of the scalar field with respect to t and r respectively. We specify the regularity properties which we require.

Definition 1.4 Let $I \subseteq]0, \infty[$ be an interval and $(t, r) \in I \times \mathbb{R}$.

a) $f \in C^1(I \times \mathbb{R}^2 \times [0, \infty[)$ is regular if $f(t, r+1, w, F) = f(t, r, w, F)$ for $(t, r, w, F) \in I \times \mathbb{R}^2 \times [0, \infty[$, $f \geq 0$ and $\text{supp} f(t, r, \cdot, \cdot)$ is compact uniformly in r and locally uniformly in t .

b) $\mu \in C^1(I \times \mathbb{R})$ is regular, if $\mu' \in C^1(I \times \mathbb{R})$ and $\mu(t, r+1) = \mu(t, r)$.

c) $\lambda \in C^1(I \times \mathbb{R})$ is regular, if $\lambda' \in C^1(I \times \mathbb{R})$ and $\lambda(t, r+1) = \lambda(t, r)$.

d) $\tilde{\mu}$ (or ϕ_1, ϕ_2) $\in C^1(I \times \mathbb{R})$ is regular, if $\tilde{\mu}(t, r+1) = \tilde{\mu}(t, r)$.

e) ρ (or p, j, q) $\in C^1(I \times \mathbb{R})$ is regular, if $\rho(t, r+1) = \rho(t, r)$.

f) $\phi \in C^2(I \times \mathbb{R})$ is regular, if $\phi(t, r+1) = \phi(t, r)$.

Lemma 1.5 Let $f = f(t, r, w, F)$ be regular on $I \times \mathbb{R}^2 \times [0, \infty[$, $I \subseteq]0, \infty[$ an interval; λ and μ regular on $I \times \mathbb{R}$. Then the nontrivial components of the energy-momentum tensor are:

$$T_{00}(t, r) = e^{2\mu} \frac{\pi}{t^2} \int_{-\infty}^{+\infty} \int_0^{+\infty} \sqrt{1 + w^2 + F/t^2} f(t, r, w, F) dF dw + \frac{1}{2} (\dot{\phi}^2 + e^{2(\mu-\lambda)} \phi'^2)$$

$$T_{11}(t, r) = e^{2\lambda} \frac{\pi}{t^2} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{w^2}{\sqrt{1 + w^2 + F/t^2}} f(t, r, w, F) dF dw + \frac{1}{2} (e^{2(\lambda-\mu)} \dot{\phi}^2 + \phi'^2)$$

$$\begin{aligned}
T_{01}(t, r) &= -e^{\lambda+\mu} \frac{\pi}{t^2} \int_{-\infty}^{+\infty} \int_0^{+\infty} w f(t, r, w, F) dF dw + \dot{\phi} \phi' \\
T_{22}(t, r) &= \frac{\pi}{2t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{F}{\sqrt{1+w^2+F/t^2}} f(t, r, w, F) dF dw + \frac{1}{2} t^2 (e^{-2\mu} \dot{\phi}^2 - e^{-2\lambda} \phi'^2) \\
T_{33}(t, r, \theta) &= T_{22}(t, r) \sin_k^2 \theta.
\end{aligned}$$

Proof: Set $T_{\alpha\beta} = T_{\alpha\beta}^f + T_{\alpha\beta}^\phi$ where $T_{\alpha\beta}^f$ and $T_{\alpha\beta}^\phi$ are respectively the contribution of f and ϕ to the energy-momentum tensor. Concerning the calculation of components $T_{\alpha\beta}^f$, we refer to [15]. We calculate only the components

$T_{\alpha\beta}^\phi := \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} \nabla_\nu \phi \nabla^\nu \phi$. We have $\nabla_\nu \phi \nabla^\nu \phi = g^{\nu\alpha} \nabla_\nu \phi \nabla_\alpha \phi = g^{00} (\nabla_0 \phi)^2 + g^{11} (\nabla_1 \phi)^2 = -e^{-2\mu} \dot{\phi}^2 + e^{-2\lambda} \phi'^2$. Thus

$$\begin{aligned}
T_{00}^\phi &= \dot{\phi}^2 - \frac{1}{2} g_{00} \nabla_\nu \phi \nabla^\nu \phi = \frac{1}{2} (\dot{\phi}^2 + e^{2(\mu-\lambda)} \phi'^2) \\
T_{01}^\phi &= \nabla_0 \phi \nabla_1 \phi = \dot{\phi} \phi' \\
T_{11}^\phi &= \phi'^2 - \frac{1}{2} g_{11} \nabla_\nu \phi \nabla^\nu \phi = \frac{1}{2} (e^{2(\lambda-\mu)} \dot{\phi}^2 + \phi'^2) \\
T_{22}^\phi &= -\frac{1}{2} g_{22} \nabla_\nu \phi \nabla^\nu \phi = \frac{1}{2} t^2 (e^{-2\mu} \dot{\phi}^2 - e^{-2\lambda} \phi'^2) \\
T_{33}^\phi &= T_{22}^\phi \sin_k^2 \theta.
\end{aligned}$$

The remaining components being zero. \square

Following [15], we can now write the complete Einstein-Vlasov-scalar field system as :

$$\partial_t f + \frac{e^{\mu-\lambda} w}{\sqrt{1+w^2+F/t^2}} \partial_r f - (\dot{\lambda} w + e^{\mu-\lambda} \mu' \sqrt{1+w^2+F/t^2}) \partial_w f = 0 \quad (1.2)$$

$$e^{-2\mu} (2t\dot{\lambda} + 1) + k = 8\pi t^2 \rho \quad (1.3)$$

$$e^{-2\mu} (2t\dot{\mu} - 1) - k = 8\pi t^2 p \quad (1.4)$$

$$\mu' = -4\pi t e^{\lambda+\mu} j \quad (1.5)$$

$$e^{-2\lambda} (\mu'' + \mu'(\mu' - \lambda')) - e^{-2\mu} (\ddot{\lambda} + (\dot{\lambda} + \frac{1}{t})(\dot{\lambda} - \dot{\mu})) = 4\pi q \quad (1.6)$$

$$e^{-2\lambda} \phi'' - e^{-2\mu} \ddot{\phi} - e^{-2\mu} (\dot{\lambda} - \dot{\mu} + \frac{2}{t}) \dot{\phi} - e^{-2\lambda} (\lambda' - \mu') \phi' = 0 \quad (1.7)$$

where (1.7) is the wave equation in ϕ and :

$$\begin{aligned}
\rho(t, r) = e^{-2\mu} T_{00}(t, r) &= \frac{\pi}{t^2} \int_{-\infty}^{+\infty} \int_0^{+\infty} \sqrt{1+w^2+F/t^2} f(t, r, w, F) dF dw \\
&\quad + \frac{1}{2} (e^{-2\mu} \dot{\phi}^2 + e^{-2\lambda} \phi'^2)
\end{aligned} \quad (1.8)$$

$$p(t, r) = e^{-2\lambda} T_{11}(t, r) = \frac{\pi}{t^2} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{w^2}{\sqrt{1+w^2+F/t^2}} f(t, r, w, F) dF dw + \frac{1}{2}(e^{-2\mu} \dot{\phi}^2 + e^{-2\lambda} \phi'^2) \quad (1.9)$$

$$j(t, r) = -e^{-(\lambda+\mu)} T_{01}(t, r) = \frac{\pi}{t^2} \int_{-\infty}^{+\infty} \int_0^{+\infty} w f(t, r, w, F) dF dw - e^{-(\lambda+\mu)} \dot{\phi} \phi' \quad (1.10)$$

$$q(t, r) = \frac{2}{t^2} T_{22}(t, r) = \frac{2}{t^2 \sin_k^2 \theta} T_{33}(t, r, \theta) = \frac{\pi}{t^4} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{F}{\sqrt{1+w^2+F/t^2}} f(t, r, w, F) dF dw + e^{-2\mu} \dot{\phi}^2 - e^{-2\lambda} \phi'^2 \quad (1.11)$$

We are going to study the initial value problem corresponding to this system with unknowns f , λ , μ , ϕ and prescribe initial data at time $t = 1$:

$$f(1, r, w, F) = \overset{\circ}{f}(r, w, F), \quad \lambda(1, r) = \overset{\circ}{\lambda}(r), \quad \mu(1, r) = \overset{\circ}{\mu}(r), \\ \phi(1, r) = \overset{\circ}{\phi}(r), \quad \dot{\phi}(1, r) = \psi(r)$$

The choice $t = 1$ is made only for convenience. Analogous results hold in the case that data are prescribed on any hypersurface $t = t_0 > 0$.

1.2 Auxiliary system and preliminary results

Using characteristic derivatives, we show in this section that the first and second derivatives of ϕ can be bounded in terms of λ and μ .

Lemma 1.6 *Let $D^+ = e^{-\mu} \partial_t + e^{-\lambda} \partial_r$; $D^- = e^{-\mu} \partial_t - e^{-\lambda} \partial_r$; $X = \dot{\phi} e^{-\mu} - \phi' e^{-\lambda}$; $Y = \dot{\phi} e^{-\mu} + \phi' e^{-\lambda}$; $a = (-\dot{\lambda} - \frac{1}{t}) e^{-\mu} - \mu' e^{-\lambda}$; $b = -\frac{e^{-\mu}}{t}$; $c = (-\dot{\lambda} - \frac{1}{t}) e^{-\mu} + \mu' e^{-\lambda}$ then X, Y are solutions of the system*

$$D^+ X = aX + bY \quad (1.12)$$

$$D^- Y = bX + cY \quad (1.13)$$

Proof: We have

$$\begin{aligned} D^+ X &= (e^{-\mu} \partial_t + e^{-\lambda} \partial_r)(\dot{\phi} e^{-\mu} - \phi' e^{-\lambda}) \\ &= e^{-\mu}(-\dot{\mu} \dot{\phi} e^{-\mu} + \ddot{\phi} e^{-\mu}) - e^{-\mu}(-\dot{\lambda} \phi' e^{-\lambda} + \dot{\phi}' e^{-\lambda}) \\ &\quad + e^{-\lambda}(-\mu' \dot{\phi} e^{-\mu} + \dot{\phi}' e^{-\mu}) - e^{-\lambda}(-\lambda' \phi' e^{-\lambda} + \phi'' e^{-\lambda}) \\ &= \ddot{\phi} e^{-2\mu} - \phi'' e^{-2\lambda} - \dot{\mu} \dot{\phi} e^{-2\mu} + \lambda' \phi' e^{-2\lambda} + (\dot{\lambda} \phi' - \mu' \dot{\phi}) e^{-\mu-\lambda} \end{aligned}$$

From (1.7), we deduce that

$$\begin{aligned}
D^+ X &= (\mu' \phi' - \lambda' \phi') e^{-2\lambda} + (\dot{\mu} \phi - \dot{\lambda} \phi) e^{-2\mu} \\
&\quad - \frac{2}{t} \dot{\phi} e^{-2\mu} - \dot{\mu} \phi e^{-2\mu} + \lambda' \phi' e^{-2\lambda} + (\dot{\lambda} \phi' - \mu' \dot{\phi}) e^{-\mu-\lambda} \\
&= -\dot{\lambda} \phi e^{-2\mu} + (\dot{\lambda} \phi' - \mu' \dot{\phi}) e^{-\mu-\lambda} - \frac{2}{t} \dot{\phi} e^{-2\mu} \\
&= [(-\dot{\lambda} - \frac{1}{t}) e^{-\mu} - \mu' e^{-\lambda}] (\dot{\phi} e^{-\mu} - \phi' e^{-\lambda}) - \frac{1}{t} (\dot{\phi} e^{-\mu} + \phi' e^{-\lambda}) \\
&= aX + bY
\end{aligned}$$

If we replace, r by $-r$ (i.e ∂_r by $-\partial_r$), the wave equation is invariant, X and Y , a and c , D^+ and D^- interchange respectively; and we can write the equation with D^- to obtain (1.13). \square .

The full system above is overdetermined and we will show that a solution (f, λ, μ, ϕ) of the subsystem consisting of equations (1.2), (1.3), (1.4) and (1.7) solves the remaining equations (1.5) and (1.6). Notice that such a solution determines the right hand side of (1.5) which is then a given function say $\tilde{\mu}$. Then, since (1.4) already provides μ , an idea introduced in [15], and that we will follow here, is to replace μ' in (1.2) and (1.7) by an auxiliary function $\tilde{\mu}$, which is not assumed a priori to be a derivative, and prove later that, under certain conditions, $\tilde{\mu}$ is nothing else than μ' . We then introduce the following auxiliary system obtained by coupling (1.3)-(1.4) to the equations obtained by replacing μ' by $\tilde{\mu}$ in (1.2),(1.5) and (1.12)-(1.13), i.e

$$\partial_t f + \frac{e^{\mu-\lambda} w}{\sqrt{1+w^2+F/t^2}} \partial_r f - (\dot{\lambda} w + e^{\mu-\lambda} \tilde{\mu} \sqrt{1+w^2+F/t^2}) \partial_w f = 0 \quad (1.14)$$

$$\tilde{\mu} = -4\pi t e^{\lambda+\mu} j \quad (1.15)$$

and

$$D^+ X = \tilde{a} X + b Y \quad (1.16)$$

$$D^- Y = b X + \tilde{c} Y \quad (1.17)$$

Where μ' is substituted by $\tilde{\mu}$ in a and c to obtain \tilde{a} and \tilde{c} respectively.

Let us estimate the first and second order derivatives of the scalar field ϕ , using the characteristic curves of (1.7) and system (1.16)-(1.17).

Proposition 1.7 *Let*

$$\begin{aligned}
K_0 &= 2 \sup\{|\psi(r)| e^{-\hat{\mu}(r)} + |\phi'(r)| e^{-\hat{\lambda}(r)} ; r \in \mathbb{R}\} \\
m(t) &= \sup\{|\dot{\lambda}(t, r)| + \frac{2}{t} + |\tilde{\mu}(t, r)| e^{(\mu-\lambda)(t, r)} ; r \in \mathbb{R}\} \\
K(t) &= \sup\{(|X|^2 + |Y|^2)^{\frac{1}{2}}(t, r) ; r \in \mathbb{R}\}.
\end{aligned}$$

If (X, Y) is a solution of (1.16)-(1.17) with

$$X(1) = e^{-\overset{\circ}{\mu}(r)}\psi(r) - e^{-\overset{\circ}{\lambda}(r)}\overset{\circ}{\phi}'(r)$$

and

$$Y(1) = e^{-\overset{\circ}{\mu}(r)}\psi(r) + e^{-\overset{\circ}{\lambda}(r)}\overset{\circ}{\phi}'(r)$$

then:

1) If $t \in]T, 1]$, $T \geq 0$, we have

$$K(t) \leq K_0 + 2 \int_t^1 m(s)K(s)ds \quad (1.18)$$

2) If $t \geq 1$ the analogous estimate holds with the limits t and 1 exchanged in the integral in (1.18).

Proof: The characteristic curves (t, γ_i) , $i = 1, 2$ of the second order partial differential equation (1.7) satisfy $g(u, u) = 0$ with $u = (1, \dot{\gamma}_i, 0, 0)$. Then, $-e^{2\mu} + e^{2\lambda}\dot{\gamma}_i^2 = 0$ and $\dot{\gamma}_i = \pm e^{\mu-\lambda}$. Therefore, for any function f ,

$$\begin{aligned} \frac{df}{dt}(t, \gamma_i(t)) &= \partial_t f + \dot{\gamma}_i \partial_r f = \partial_t f \pm e^{\mu-\lambda} \partial_r f \\ e^\mu(e^{-\mu} \partial_t f \pm e^{-\lambda} \partial_r f) &= e^\mu D^+ f \quad \text{or} \quad e^\mu D^- f \end{aligned}$$

and we have $D^+ = D^- = e^{-\mu} \frac{d}{dt}$ on the corresponding characteristic. Then (1.16)-(1.17) become

$$\begin{cases} \frac{d}{dt} X(t, \gamma_1(t)) = e^\mu(\tilde{a}X + bY)(t, \gamma_1(t)) \\ \frac{d}{dt} Y(t, \gamma_2(t)) = e^\mu(bX + \tilde{c}Y)(t, \gamma_2(t)) \end{cases}$$

Integrate this system on $[t, 1]$, thus

$$\begin{cases} X(t, \gamma_1(t)) = X(1, \gamma_1(1)) - \int_t^1 e^{\mu(s, \gamma_2(s))}(\tilde{a}X + bY)(s, \gamma_1(s))ds \\ Y(t, \gamma_2(t)) = Y(1, \gamma_2(1)) - \int_t^1 e^{\mu(s, \gamma_2(s))}(bX + \tilde{c}Y)(s, \gamma_2(s))ds \end{cases}$$

Now take the absolute value in each equation and add the two inequalities to obtain:

$$\begin{aligned} |X(t, \gamma_1(t))| + |Y(t, \gamma_2(t))| &\leq |X(1, \gamma_1(1))| + |Y(1, \gamma_2(1))| \\ &+ \int_t^1 e^{\mu(s, \gamma_1(s))}(|\tilde{a}||X| + |b||Y|)(s, \gamma_1(s))ds + \int_t^1 e^{\mu(s, \gamma_2(s))}(|b||X| + |\tilde{c}||Y|)(s, \gamma_2(s))ds \\ &\leq |X(1, \gamma_1(1))| + |Y(1, \gamma_2(1))| + \int_t^1 e^{\mu(s, \gamma_1(s))}[(|\tilde{a}| + |b|)(|X| + |Y|)](s, \gamma_1(s))ds \\ &+ \int_t^1 e^{\mu(s, \gamma_2(s))}[(|b| + |\tilde{c}|)(|X| + |Y|)](s, \gamma_2(s))ds \end{aligned}$$

Take the supremum in space of each term of the above inequality. Now, we have

$$|X(1, \gamma_1(1))| + |Y(1, \gamma_2(1))| \leq K_0 ,$$

$$\begin{cases} X + Y = 2\dot{\phi}e^{-\mu} \\ Y - X = 2\phi'e^{-\lambda} \end{cases}$$

thus

$$\begin{cases} 2|\dot{\phi}|e^{-\mu} \leq |X| + |Y| \\ 2|\phi'|e^{-\lambda} \leq |X| + |Y| \end{cases}$$

which implies

$$4(e^{-2\mu}\dot{\phi}^2 + e^{-2\lambda}\phi'^2) \leq 2(|X| + |Y|)^2, \text{ i.e } K(s) \leq \frac{\sqrt{2}}{2} \sup\{(|X| + |Y|)(s, r); r \in \mathbb{R}\}.$$

Since, $(|X| + |Y|)^2 \leq 2(|X|^2 + |Y|^2)$, we have

$$\sup\{(|X| + |Y|)(s, r); r \in \mathbb{R}\} \leq 2K(s).$$

And (1.18) follows. \square

Corollary 1.8 *If $\square_g \phi = F$ where F is a continuous function of variables t and r , then we have the inequality*

$$K(t) \leq K_0 + 2 \int_t^1 [m(s)K(s) + \sup\{e^{\mu(s,r)}|F(s, r)|; r \in \mathbb{R}\}] ds$$

Lemma 1.9 *Let D^+ and D^- be defined as in Lemma 1.6 and define*

$$X_1 = e^{-\lambda}\partial_r X, \quad Y_1 = e^{-\lambda}\partial_r Y; \quad b_1 = (-2\dot{\lambda} - \frac{1}{t})e^{-\mu} - (\tilde{\mu} + \mu')e^{-\lambda};$$

$$b_2 = -\frac{e^{-\mu}}{t}; \quad b_3 = -\dot{\lambda}'e^{-\mu-\lambda} + (\lambda'\tilde{\mu} - \tilde{\mu}\mu' - \tilde{\mu}')e^{-2\lambda};$$

$$b_4 = (-2\dot{\lambda} - \frac{1}{t})e^{-\mu} + (\tilde{\mu} + \mu')e^{-\lambda}; \quad b_5 = -\dot{\lambda}'e^{-\mu-\lambda} - (\lambda'\tilde{\mu} - \tilde{\mu}\mu' - \tilde{\mu}')e^{-2\lambda};$$

If X and Y satisfy (1.16) and (1.17) then X_1 and Y_1 satisfy

$$D^+ X_1 = b_1 X_1 + b_2 Y_1 + b_3 X \tag{1.19}$$

$$D^- Y_1 = b_4 X_1 + b_5 Y_1 + b_5 Y \tag{1.20}$$

Proof: we have by definition,

$$D^+ X_1 = (\partial_r X)D^+ e^{-\lambda} + e^{-\lambda}D^+(\partial_r X) \tag{1.21}$$

$$(\partial_r X)D^+ e^{-\lambda} = (\partial_r X)(e^{-\mu}\partial_t + e^{-\lambda}\partial_r)e^{-\lambda} = (e^{-\lambda}\partial_r X)(-\dot{\lambda}e^{-\mu} - \lambda'e^{-\lambda})$$

$$= (-\dot{\lambda}e^{-\mu} - \lambda'e^{-\lambda})X_1$$

(1.22)

Next

$$\partial_r D^+ = \partial_r(e^{-\mu}\partial_t + e^{-\lambda}\partial_r) = D^+\partial_r - \mu'e^{-\mu}\partial_t - \lambda'e^{-\lambda}\partial_r,$$

i.e

$$\partial_r(D^+ X) = D^+(\partial_r X) - (\mu'e^{-\mu}\partial_t + \lambda'e^{-\lambda}\partial_r)X,$$

and then

$$D^+(\partial_r X) = \partial_r(D^+ X) + (\mu'e^{-\mu}\partial_t + \lambda'e^{-\lambda}\partial_r)X \tag{1.23}$$

Now we have firstly,

$$\begin{aligned}
(\mu' e^{-\mu} \partial_t + \lambda' e^{-\lambda} \partial_r) X &= \frac{1}{2}(\mu' + \lambda') D^+ X + \frac{1}{2}(\mu' - \lambda') D^- X \\
&= \frac{1}{2}(\mu' + \lambda') D^+ X + \frac{1}{2}(\mu' - \lambda')(D^+ X - 2X_1) \\
&= \mu' D^+ X - (\mu' - \lambda') X_1
\end{aligned}$$

i.e

$$(\mu' e^{-\mu} \partial_t + \lambda' e^{-\lambda} \partial_r) X = \mu'(\tilde{a}X + bY) - (\mu' - \lambda') X_1 \quad (1.24)$$

and secondly,

$$\begin{aligned}
\partial_r(D^+ X) &= (\partial_r \tilde{a}) X + \tilde{a} \partial_r X + (\partial_r b) Y + b \partial_r Y \\
&= (\partial_r \tilde{a}) X + \tilde{a} e^\lambda X_1 + (\partial_r b) Y + b e^\lambda Y_1
\end{aligned}$$

i.e

$$\partial_r(D^+ X) = \tilde{a} e^\lambda X_1 + b e^\lambda Y_1 + [-\dot{\lambda}' e^{-\mu} + \mu'(\dot{\lambda} + \frac{1}{t}) e^{-\mu} - \tilde{\mu}' e^{-\lambda} + \tilde{\mu} \lambda' e^{-\lambda}] X + \frac{\mu'}{t} e^{-\mu} Y \quad (1.25)$$

Substituting (1.25) and (1.24) in (1.23) gives

$$\begin{aligned}
D^+(\partial_r X) &= \tilde{a} e^\lambda X_1 + b e^\lambda Y_1 + [-\dot{\lambda}' e^{-\mu} + \mu'(\dot{\lambda} + \frac{1}{t}) e^{-\mu} - \tilde{\mu}' e^{-\lambda} + \tilde{\mu} \lambda' e^{-\lambda}] X \\
&\quad + \frac{\mu'}{t} e^{-\mu} Y + \mu'(\tilde{a}X + bY) - (\mu' - \lambda') X_1 \\
&= \tilde{a} e^\lambda X_1 + b e^\lambda Y_1 + [-\dot{\lambda}' e^{-\mu} + \mu'(\dot{\lambda} + \frac{1}{t}) e^{-\mu} - \tilde{\mu}' e^{-\lambda} + \tilde{\mu} \lambda' e^{-\lambda}] X \\
&\quad + \frac{\mu'}{t} e^{-\mu} Y + [(-\mu' \dot{\lambda} - \frac{\mu'}{t}) e^{-\mu} - \tilde{\mu} \mu' e^{-\lambda}] X - \frac{\mu'}{t} e^{-\mu} Y - (\mu' - \lambda') X_1
\end{aligned}$$

i.e

$$D^+(\partial_r X) = (\tilde{a} e^\lambda - \mu' + \lambda') X_1 + b e^\lambda Y_1 + (-\dot{\lambda}' e^{-\mu} - \tilde{\mu} \mu' e^{-\lambda} - \tilde{\mu}' e^{-\lambda} + \tilde{\mu} \lambda' e^{-\lambda}) X \quad (1.26)$$

Substituting (1.26) and (1.22) in (1.21) gives

$$\begin{aligned}
D^+ X_1 &= (-\dot{\lambda} e^{-\mu} - \lambda' e^{-\lambda}) X_1 + [\tilde{a} + (-\mu' + \lambda') e^{-\lambda}] X_1 + b Y_1 + (-\dot{\lambda}' e^{-\mu} - \tilde{\mu} \mu' e^{-\lambda} \\
&\quad - \tilde{\mu}' e^{-\lambda} + \tilde{\mu} \lambda' e^{-\lambda}) e^{-\lambda} X \\
&= (-\dot{\lambda} e^{-\mu} + \tilde{a} - \mu' e^{-\lambda}) X_1 + b Y_1 + [-\dot{\lambda}' e^{-\mu-\lambda} + (-\tilde{\mu} \mu' - \tilde{\mu}' + \tilde{\mu} \lambda') e^{-2\lambda}] X
\end{aligned}$$

and equation (1.19) follows. If we replace $e^{-\lambda}$ by $-e^{-\lambda}$, the wave equation is invariant, D^+ and D^- , X_1 and $-Y_1$, b_1 and b_4 , $-b_3$ and b_5 interchange respectively; and we can write equation (1.20).

Proposition 1.10 Let $K(t)$ be defined as in Proposition 1.7 and set:

$$\begin{aligned} A_0 &= 2 \sup\{(|\psi'| + |\mu'| |\psi|)e^{-\dot{\mu}-\dot{\lambda}} + (|\phi''| + |\lambda'| |\phi'|)e^{-2\dot{\lambda}}(r) ; r \in \mathbb{R}\} \\ A(t) &= \sup\{[X_1^2 + Y_1^2]^{1/2}(t, r) ; r \in \mathbb{R}\} \\ v(t) &= \sup\{\frac{2}{t} + 2|\dot{\lambda}| + (|\tilde{\mu}| + |\mu'|)e^{\mu-\lambda}(t, r) ; r \in \mathbb{R}\} \\ h(t) &= \sup\{[|\dot{\lambda}'| e^{-\lambda} + (|\mu'| |\tilde{\mu}| + |\lambda'| |\tilde{\mu}| + |\tilde{\mu}'|)e^{\mu-2\lambda}](t, r) ; r \in \mathbb{R}\} \end{aligned}$$

If in addition to the assumptions of Proposition 1.7 the quantities X_1 and Y_1 satisfy (1.19) and (1.20) and agree with $e^{-\lambda}\partial_r X$ and $e^{-\lambda}\partial_r Y$ respectively for $t = 1$ then

1) If $t \in]T, 1]$, $T \geq 0$, we have the estimate

$$A(t) \leq A_0 + 2 \int_t^1 (v(s)A(s) + h(s)K(s))ds \quad (1.27)$$

2) If $t \geq 1$ the analogous estimate holds with the limits t and 1 exchanged in the integral in (1.27).

Proof: Analogous to the proof of Proposition 1.7, using this time Lemma 1.9. \square

Note that the factor $e^{-\lambda}$ in the definition of X_1 and Y_1 is very important. Without it the above derivation would not work since the derivative λ' would occur in b_1 and b_4 .

Lemma 1.11 Let D^+ and D^- be defined as in lemma 1.6 and define $X_2 = e^\mu X$, $Y_2 = e^\mu Y$. If X and Y hold system (1.12)-(1.13) and the field equations (1.3)-(1.4) are satisfied, then X_2 and Y_2 satisfy

$$D^+ X_2 = e^\mu \left[\frac{k}{t} - 4\pi t(\rho - p) \right] X_2 - \frac{e^{-\mu}}{t} Y_2 \quad (1.28)$$

$$D^- Y_2 = -\frac{e^{-\mu}}{t} X_2 + e^\mu \left[\frac{k}{t} - 4\pi t(\rho - p) \right] Y_2 \quad (1.29)$$

Proof: We have

$$\begin{aligned} D^+ X_2 &= D^+(e^\mu X) = e^\mu D^+ X + (D^+ e^\mu) X \\ &= e^\mu \left[\left(-\dot{\lambda} - \frac{1}{t} \right) e^{-\mu} - \mu' e^{-\lambda} \right] X - \frac{Y}{t} + (\dot{\mu} + \mu' e^{\mu-\lambda}) X \\ &= \left(\dot{\mu} - \dot{\lambda} - \frac{1}{t} \right) X - \frac{Y}{t} \\ &= e^{-\mu} \left(\dot{\mu} - \dot{\lambda} - \frac{1}{t} \right) X_2 - \frac{e^{-\mu}}{t} Y_2 \end{aligned}$$

Subtract the field equations (1.3) and (1.4) to obtain

$$2te^{-2\mu}(\dot{\lambda} - \dot{\mu}) + 2k - 2e^{-2\mu} = 8\pi t^2(\rho - p)$$

i.e

$$\dot{\mu} - \dot{\lambda} = \frac{ke^{2\mu} + 1}{t} - 4\pi te^{2\mu}(\rho - p).$$

Substituting this into the above equation gives (1.28). If we replace ∂_r by $-\partial_r$, the wave equation is invariant, D^+ and D^- , X_2 and Y_2 interchange respectively and we can write equation (1.29).

Remark 1.12 *Let*

$$\begin{aligned} B_0 &= 2 \sup\{(|X_2| + |Y_2|)(1, r) ; r \in \mathbb{R}\} \\ B(t) &= \sup\{(|X_2|^2 + |Y_2|^2)^{1/2}(t, r) ; r \in \mathbb{R}\} \\ l(t) &= \sup\left\{\frac{1}{t} + e^{2\mu}\left[\frac{|k|}{t} + 4\pi t(\rho - p)\right](t, r) ; r \in \mathbb{R}\right\} \end{aligned}$$

If (X_2, Y_2) is solution of (1.28)-(1.29), then we obtain analogously to (1.18), the estimate

$$B(t) \leq B_0 + 2 \int_t^1 l(s)B(s)ds \quad (1.30)$$

with $t \in]T, 1]$, $T > 0$ or $t \geq 1$.

1.3 The reduced system

We first solve each equation of the auxiliary system introduced in the previous section, when the other unknowns are fixed (in the form to be used later for the iteration). In order to clarify our statements, we introduce the notations ϕ_1, ϕ_2 in place of $\dot{\phi}, \phi'$.

Proposition 1.13 1) *Let $\bar{f}, \bar{\lambda}, \bar{\mu}, \bar{\mu}, \bar{\phi}_1, \bar{\phi}_2$ be regular for $(t, r) \in I \times \mathbb{R}$, $I \subset]0, \infty[$. Substitute $f, \lambda, \mu, \dot{\phi}, \phi'$ respectively by $\bar{f}, \bar{\lambda}, \bar{\mu}, \bar{\phi}_1, \bar{\phi}_2$ in ρ and p to define $\bar{\rho}$ and \bar{p} . Suppose that $1 \in I$, $\bar{f} \in C^1(\mathbb{R}^2 \times [0, \infty))$, $\bar{\lambda}, \bar{\mu} \in C^1(\mathbb{R})$ and are periodic of period 1 in r . Assume that:*

$$\frac{e^{-2\bar{\mu}(r)} + k}{t} - k + \frac{8\pi}{t} \int_t^1 s^2 \bar{p}(s, r) ds > 0, \quad (t, r) \in I \times \mathbb{R} \quad (1.31)$$

then the system

$$\partial_t f + \frac{e^{\bar{\mu} - \bar{\lambda}} w}{\sqrt{1 + w^2 + F/t^2}} \partial_r f - (\dot{\bar{\lambda}} w + e^{\bar{\mu} - \bar{\lambda}} \bar{\mu} \sqrt{1 + w^2 + F/t^2}) \partial_w f = 0 \quad (1.32)$$

$$e^{-2\mu}(2t\dot{\lambda} + 1) + k = 8\pi t^2 \bar{\rho} \quad (1.33)$$

$$e^{-2\mu}(2t\dot{\mu} - 1) - k = 8\pi t^2 \bar{p} \quad (1.34)$$

has a unique regular solution (f, λ, μ) on $I \times \mathbb{R}$ with $f(1) = \overset{\circ}{f}$, $\lambda(1) = \overset{\circ}{\lambda}$, $\mu(1) = \overset{\circ}{\mu}$. This solution is given by

$$f(t, r, w, F) = \overset{\circ}{f}((R, W)(1, t, r, w, F), F) \quad (1.35)$$

where (R, W) is the solution of the characteristic system

$$\frac{d}{ds}(r, w) = \left(\frac{e^{\bar{\mu} - \bar{\lambda}} w}{\sqrt{1 + w^2 + F/t^2}}, -\dot{\bar{\lambda}} w - e^{\bar{\mu} - \bar{\lambda}} \bar{\mu} \sqrt{1 + w^2 + F/t^2} \right) \quad (1.36)$$

satisfying $(R, W)(t, t, r, w, F) = (r, w)$;

$$e^{-2\mu(t, r)} = \frac{e^{-2\overset{\circ}{\mu}(r)} + k}{t} - k + \frac{8\pi}{t} \int_t^1 s^2 \bar{p}(s, r) ds \quad (1.37)$$

$$\dot{\lambda}(t, r) = 4\pi t e^{2\mu} \bar{\rho}(t, r) - \frac{1 + k e^{2\mu}(t, r)}{2t} \quad (1.38)$$

$$\lambda(t, r) = \overset{\circ}{\lambda}(r) - \int_t^1 \dot{\lambda}(s, r) ds \quad (1.39)$$

If $I =]T, 1[$ (respectively $I = [1, T[$) with $T \in [0, 1[$ (respectively $T \in]1, \infty[$), then there exists some $T^* \in [T, 1[$ (respectively $T^* \in]1, T[$) such that condition (1.31) holds on $]T^*, 1[\times \mathbb{R}$ (respectively $[1, T^*[\times \mathbb{R}$). T^* is independent of \bar{p} if $I =]T, 1[$, whereas it depends on \bar{p} if $I = [1, T[$.

2) Let $\lambda, \mu, \bar{\lambda}, \bar{\mu}$ be regular; let C, D be regular as $\tilde{\mu}$ (see definition 1.4). Set

$$X = \phi_1 e^{-\mu} - \phi_2 e^{-\lambda}, \quad Y = \phi_1 e^{-\mu} + \phi_2 e^{-\lambda} \quad (1.40)$$

and define the operators \bar{D}^+, \bar{D}^- as D^+, D^- in Lemma 1.6, with λ, μ substituted respectively by $\bar{\lambda}, \bar{\mu}$. Assume that $\psi \in C^1(\mathbb{R})$, $\overset{\circ}{\phi} \in C^2(\mathbb{R})$ are periodic of period 1. Then the system

$$\bar{D}^+ X = C \quad (1.41)$$

$$\bar{D}^- Y = D \quad (1.42)$$

has a unique regular solution (ϕ_1, ϕ_2) such that $(\phi_1, \phi_2)(1) = (\psi, \overset{\circ}{\phi}')$.

Proof: 1) the proof of this point is the same as that of Proposition 2.4 in [15]. The only thing added is the existence of T^* we now prove and that will replace in the case of local existence, the hypothesis $\overset{\circ}{\mu} \leq 0$ if $k = -1$ in [15].

If $I =]T, 1[$; since $\bar{p} \geq 0$, the left hand side of (1.31) is bounded from below by $h(t, r) = \frac{e^{-2\overset{\circ}{\mu}(r)} + k}{t} - k$. If $k \in \{0, 1\}$, we have, since $\frac{1}{t} \geq 1$, $h(t, r) \geq e^{-2\overset{\circ}{\mu}(r)} > 0$ and we can take $T^* = T$. If $k = -1$, since $\overset{\circ}{\mu}$ is bounded, there exists $\beta > 0$,

such that $h(1, r) = e^{-2\overset{\circ}{\mu}(r)} > \beta$. By the continuity of $t \mapsto h(t, r)$ at $t = 1$, we conclude that:

$$\exists T^* \in]T, 1] \text{ such that } e^{-2\mu(t,r)} \geq h(t, r) > \frac{\beta}{2}, \quad t \in]T^*, 1] \text{ and } \bar{p} \geq 0. \quad (1.43)$$

If $I = [1, T[$, define $h(t, r)$ to be all the left hand side of (1.31) and proceed as above in the case $k = -1$, to obtain T^* that depends this time on \bar{p} .

2) The system (1.41)-(1.42) in (X, Y) is a first order linear hyperbolic system, and the existence of a unique solution with the prescribed data $(X, Y)(1) = (e^{-\overset{\circ}{\mu}}\psi - e^{-\overset{\circ}{\lambda}}\overset{\circ}{\phi}', e^{-\overset{\circ}{\mu}}\psi + e^{-\overset{\circ}{\lambda}}\overset{\circ}{\phi}')$ is given by a theorem of Friedrichs [10]. (See also the paper of Douglis [9] where existence of C^1 solutions of hyperbolic systems in one space dimension was proved for C^1 initial data in the more general quasilinear case.) We then deduce from the relations (1.40) that define a bijection $(X, Y) \mapsto (\phi_1, \phi_2)$, the existence of a unique regular solution (ϕ_1, ϕ_2) of (1.41)-(1.42) such that $(\phi_1, \phi_2)(1) = (\psi, \overset{\circ}{\phi}')$. This completes the proof of Proposition 1.13. \square

In fact, since the equations for X and Y are decoupled, it is not necessary to use existence results for hyperbolic systems in the above proof; it suffices to solve a parameter-dependent ordinary differential equation. The above procedure has the advantage that it can easily be generalized to problems where the corresponding equations are coupled as they are, for instance, in the case of a nonlinear scalar field.

Now we show that the solution of the subsystem consisting of equations (1.2), (1.3), (1.4) and (1.7) also satisfies equations (1.5) and (1.6), and that the auxiliary system is equivalent to the full system.

Proposition 1.14 1) *Let (f, λ, μ, ϕ) be a regular solution of (1.2), (1.3), (1.4) and (1.7) on some time interval $I \in]0, \infty[$ with $1 \in I$, and let the initial data satisfy (1.5) for $t = 1$ with $\overset{\circ}{\phi}, \psi \in C^1(\mathbb{R})$, in particular $\overset{\circ}{\mu} \in C^2(\mathbb{R})$. Then (1.5) and (1.6) hold for all $t \in I$, in particular $\mu \in C^2(I \times \mathbb{R})$.*

2) *Let $(f, \lambda, \mu, \tilde{\mu}, \phi_1, \phi_2)$ be a regular solution of equations (1.14), (1.3), (1.4), (1.15), (1.16) and (1.17) with the initial data $(f, \lambda, \mu, \phi_1, \phi_2)(1) = (\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu}, \psi, \overset{\circ}{\phi}')$ that are as in proposition 1.13 and satisfy (1.5) for $t = 1$. Then there exists a unique regular function ϕ satisfying $\phi(1) = \overset{\circ}{\phi}$, $\dot{\phi}(1) = \psi$ such that (f, λ, μ, ϕ) solves the full system (1.2)-(1.11). The function ϕ is given by:*

$$\phi(t, r) = \overset{\circ}{\phi}(r) + \int_1^t \phi_1(s, r) ds.$$

Proof: 1) Firstly, we prove that (1.5) holds. From equations (1.9), (1.2), (1.7) and integration by parts, it follows that, since $\text{supp } f(t, r, \dots)$ is compact :

$$\begin{aligned}
& \int_1^t p'(s, r) s^2 ds = \pi \int_1^t \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{\sqrt{1+w^2+F/s^2}} \partial_r f(s, r, w, F) dF dw ds \\
& + \int_1^t (-\mu' \dot{\phi}^2 e^{-2\mu} - \lambda' \phi'^2 e^{-2\lambda} + \dot{\phi}' \dot{\phi} e^{-2\mu} + \phi' \phi'' e^{-2\lambda}) s^2 ds \\
& = \pi \int_1^t \int_{-\infty}^{\infty} \int_0^{\infty} [-e^{\lambda-\mu} w \partial_t f + (\dot{\lambda} w^2 e^{\lambda-\mu} + \mu' w \sqrt{1+w^2+F/t^2}) \partial_w f] dF dw ds \\
& + \int_1^t [-\mu' \dot{\phi}^2 e^{-2\mu} - \lambda' \phi'^2 e^{-2\lambda} + \dot{\phi}' \dot{\phi} e^{-2\mu} + \phi' \ddot{\phi} e^{-2\mu} + (\dot{\lambda} - \dot{\mu}) \phi' \dot{\phi} e^{-2\mu} \\
& + \frac{2}{s} \phi' \dot{\phi} e^{-2\mu} + (\lambda' - \mu') e^{-2\lambda} \phi'^2] s^2 ds \\
& = \pi \int_{-\infty}^{\infty} \int_0^{\infty} [-e^{\lambda-\mu} w f]_{s=1}^{s=t} dF dw + \pi \int_1^t \int_{-\infty}^{\infty} \int_0^{\infty} (\dot{\lambda} - \dot{\mu}) e^{\lambda-\mu} w f dF dw ds \\
& + \pi \int_1^t \int_0^{\infty} [(\dot{\lambda} w^2 e^{\lambda-\mu} + \mu' w \sqrt{1+w^2+F/t^2}) f]_{-\infty}^{\infty} dF ds \\
& - \pi \int_1^t \int_{-\infty}^{\infty} \int_0^{\infty} (2\dot{\lambda} w e^{\lambda-\mu} + \mu' \sqrt{1+w^2+F/t^2} + \frac{\mu' w^2}{\sqrt{1+w^2+F/s^2}}) f dF dw ds \\
& + \int_1^t [-\mu' \dot{\phi}^2 e^{-2\mu} - \mu' e^{-2\lambda} \phi'^2 + \dot{\phi}' \dot{\phi} e^{-2\mu} + \phi' \ddot{\phi} e^{-2\mu} + (\dot{\lambda} - \dot{\mu}) \phi' \dot{\phi} e^{-2\mu}] ds + \int_1^t 2s \phi' \dot{\phi} e^{-2\mu} ds
\end{aligned}$$

i.e

$$\begin{aligned}
& \int_1^t p'(s, r) s^2 ds = \pi \int_{-\infty}^{\infty} \int_0^{\infty} [-e^{\lambda-\mu} w f]_{s=1}^{s=t} dF dw - \pi \int_1^t \int_{-\infty}^{\infty} \int_0^{\infty} (\dot{\lambda} + \dot{\mu}) e^{\lambda-\mu} w f dF dw ds \\
& - \pi \int_1^t \int_{-\infty}^{\infty} \int_0^{\infty} \mu' (\sqrt{1+w^2+F/t^2} + \frac{w^2}{\sqrt{1+w^2+F/s^2}}) f dF dw ds \\
& + \int_1^t [-\mu' \dot{\phi}^2 e^{-2\mu} - \mu' e^{-2\lambda} \phi'^2 + \dot{\phi}' \dot{\phi} e^{-2\mu} + \phi' \ddot{\phi} e^{-2\mu} + (\dot{\lambda} - \dot{\mu}) \phi' \dot{\phi} e^{-2\mu}] ds + [s^2 \phi' \dot{\phi} e^{-2\mu}]_{s=1}^{s=t} \\
& - \int_1^t s^2 (-2\dot{\mu} \phi' \dot{\phi} e^{-2\mu} + \phi' \ddot{\phi} e^{-2\mu} + \dot{\phi}' \dot{\phi} e^{-2\mu}) ds \\
& = \pi \int_{-\infty}^{\infty} \int_0^{\infty} [-e^{\lambda-\mu} w f]_{s=1}^{s=t} dF dw - \pi \int_1^t \int_{-\infty}^{\infty} \int_0^{\infty} (\dot{\lambda} + \dot{\mu}) e^{\lambda-\mu} w f dF dw ds \\
& - \pi \int_1^t \int_{-\infty}^{\infty} \int_0^{\infty} \mu' (\sqrt{1+w^2+F/t^2} + \frac{w^2}{\sqrt{1+w^2+F/s^2}}) f dF dw ds \\
& + [s^2 \phi' \dot{\phi} e^{-2\mu}]_{s=1}^{s=t} + \int_1^t (\dot{\lambda} + \dot{\mu}) \phi' \dot{\phi} e^{-2\mu} - \int_1^t \mu' (\dot{\phi}^2 e^{-2\mu} + \phi'^2 e^{-2\lambda}) s^2 ds
\end{aligned}$$

Therefore,

$$\int_1^t p'(s, r) s^2 ds = [-e^{\lambda-\mu} j(s, r) s^2]_{s=1}^{s=t} - \int_1^t (\dot{\lambda} + \dot{\mu}) e^{\lambda-\mu} j(s, r) s^2 ds - \int_1^t \mu'(\rho+p)(s, r) s^2 ds \quad (1.44)$$

Adding equations (1.3) and (1.4), we have

$$\dot{\lambda} + \dot{\mu} = 4\pi t e^{2\mu}(\rho + p). \quad (1.45)$$

From equation (1.4), we obtain

$$e^{-2\mu(t, r)} = \frac{e^{-2\mu(r)} + k}{t} - k + \frac{8\pi}{t} \int_t^1 s^2 p(s, r) ds$$

and differentiating this with respect to r , yields :

$$t\mu' e^{-2\mu} = \mu' e^{-2\mu} + 4\pi \int_1^t s^2 p'(s, r) ds \quad (1.46)$$

Substituting for the last integral by (1.44), using (1.45) and assuming the fact that the constraint equation (1.5) holds for $t = 1$, relation (1.46) implies

$$-te^{2\mu}(\mu' + 4\pi e^{\lambda+\mu} j t) = 4\pi \int_1^t (\rho + p) s^2 (\mu' + 4\pi e^{\lambda+\mu} j s) ds$$

and since the left hand side is zero at $t = 1$, we obtain

$$\mu' + 4\pi t e^{\lambda+\mu} j = 0$$

on I , i.e (1.5) holds for all $t \in I$. In particular this relation shows that μ is C^2 with respect to r with

$$\mu'' = (\lambda' + \mu')\mu' - 4\pi t e^{\lambda+\mu} j'.$$

Now we prove that equation (1.6) holds. From (1.10), (1.2), (1.7) and integration by parts we obtain the identity

$$\begin{aligned} j'(t, r) &= \frac{\pi}{t^2} e^{\lambda-\mu} \int_{-\infty}^{\infty} \int_0^{\infty} [-\sqrt{1+w^2+F/t^2} \partial_t f \\ &\quad + (\dot{\lambda} w \sqrt{1+w^2+F/t^2} + e^{\mu-\lambda} \mu' (1+w^2+F/t^2)) \partial_w f] dF dw \\ &\quad + (\lambda' + \mu') e^{-(\lambda+\mu)} \dot{\phi} \phi' - e^{-(\lambda+\mu)} (\dot{\phi}' \phi' + \dot{\phi} \phi'') \\ &= -\frac{\pi}{t^2} e^{\lambda-\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_t f dF dw \\ &\quad + \frac{\pi}{t^2} e^{\lambda-\mu} \int_0^{\infty} [(\dot{\lambda} w \sqrt{1+w^2+F/t^2} + e^{\mu-\lambda} \mu' (1+w^2+F/t^2)) f]_{-\infty}^{\infty} \\ &\quad - \frac{\pi}{t^2} e^{\lambda-\mu} \int_{-\infty}^{\infty} \int_0^{\infty} [\dot{\lambda} \sqrt{1+w^2+F/t^2} + \frac{\dot{\lambda} w^2}{\sqrt{1+w^2+F/t^2}} + e^{\mu-\lambda} \mu' 2w] f dF dw \\ &\quad + 2\mu' e^{-(\lambda+\mu)} \dot{\phi} \phi' - e^{-(\lambda+\mu)} \dot{\phi}' \phi' - e^{\lambda-3\mu} [\dot{\phi} \ddot{\phi} + (\dot{\lambda} - \dot{\mu} + \frac{2}{t}) \dot{\phi}^2] \end{aligned}$$

i.e

$$\begin{aligned}
j'(t, r) &= -\frac{\pi}{t^2} e^{\lambda-\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_t f dF dw - 2\mu' \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} w f dF dw \\
&\quad - \dot{\lambda} e^{\lambda-\mu} \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \left(\sqrt{1+w^2+F/t^2} + \frac{w^2}{\sqrt{1+w^2+F/t^2}} \right) f dF dw \\
&\quad + 2\mu' e^{-(\lambda+\mu)} \dot{\phi} \phi' - e^{-(\lambda+\mu)} \dot{\phi}' \phi' - e^{\lambda-3\mu} [\dot{\phi} \ddot{\phi} + (\dot{\lambda} - \dot{\mu} + \frac{2}{t}) \dot{\phi}^2] \\
&= -\frac{\pi}{t^2} e^{\lambda-\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_t f dF dw - 2\mu' \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} w f dF dw \\
&\quad - \dot{\lambda} e^{\lambda-\mu} (\rho + p - \dot{\phi}^2 e^{-2\mu} \\
&\quad - \phi'^2 e^{-2\lambda}) + 2\mu' e^{-(\lambda+\mu)} \dot{\phi} \phi' - e^{-(\lambda+\mu)} \dot{\phi}' \phi' - e^{\lambda-3\mu} [\dot{\phi} \ddot{\phi} + (\dot{\lambda} - \dot{\mu} + \frac{2}{t}) \dot{\phi}^2] \\
&= -\frac{\pi}{t^2} e^{\lambda-\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_t f dF dw - \dot{\lambda} e^{\lambda-\mu} (\rho + p - \dot{\phi}^2 e^{-2\mu} \\
&\quad - \phi'^2 e^{-2\lambda}) - 2\mu' j - e^{-(\lambda+\mu)} \dot{\phi}' \phi' - e^{-3\mu+\lambda} (\dot{\phi} \ddot{\phi} - (\dot{\lambda} - \dot{\mu} + \frac{2}{t}) \dot{\phi}^2) .
\end{aligned}$$

Since by (1.8)

$$\begin{aligned}
\dot{\rho}(t, r) &= -2\frac{\pi}{t^3} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} f dF dw \\
&\quad + \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_t f dF dw \\
&\quad + \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \left(-\frac{F}{t^3}\right) \frac{f}{\sqrt{1+w^2+F/t^2}} dF dw \\
&\quad + e^{-2\mu} (-\dot{\mu} \dot{\phi}^2 + \dot{\phi} \ddot{\phi}) + e^{-2\lambda} (-\dot{\lambda} \phi'^2 + \dot{\phi}' \phi') \\
&= \frac{-2\rho}{t} + \frac{1}{t} (e^{-2\mu} \dot{\phi}^2 + e^{-2\lambda} \phi'^2) - \frac{q}{t} + \frac{1}{t} (e^{-2\mu} \dot{\phi}^2 - e^{-2\lambda} \phi'^2) \\
&\quad + \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_t f dF dw + e^{-2\mu} (-\dot{\mu} \dot{\phi}^2 + \dot{\phi} \ddot{\phi}) + e^{-2\lambda} (-\dot{\lambda} \phi'^2 + \dot{\phi}' \phi') \\
&= \frac{-2\rho}{t} - \frac{q}{t} + \frac{2e^{-2\mu}}{t} \dot{\phi}^2 + e^{-2\mu} (-\dot{\mu} \dot{\phi}^2 + \dot{\phi} \ddot{\phi}) + e^{-2\lambda} (-\dot{\lambda} \phi'^2 + \dot{\phi}' \phi') \\
&\quad + \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_t f dF dw ;
\end{aligned}$$

From (1.3) we obtain

$$\dot{\lambda}(t, r) = 4\pi t e^{2\mu} \rho(t, r) - \frac{1 + k e^{2\mu}(t, r)}{2t}$$

and differentiating with respect to t yields:

$$\begin{aligned}
\ddot{\lambda} &= 4\pi e^{2\mu} \rho + 8\pi t \dot{\mu} e^{2\mu} \rho + \frac{1 + ke^{2\mu}}{2t^2} + 4\pi t e^{2\mu} \dot{\rho} - \frac{k\dot{\mu}e^{2\mu}}{t} \\
&= 4\pi e^{2\mu} \rho + 2\dot{\mu} \left(\dot{\lambda} + \frac{1 + ke^{2\mu}}{2t} \right) - \frac{\dot{\mu}}{t} k e^{2\mu} + \frac{1 + ke^{2\mu}}{2t^2} \\
&\quad + 4\pi t e^{2\mu} \left\{ \frac{-2\rho}{t} - \frac{q}{t} + \frac{2}{t} e^{-2\mu} \dot{\phi}^2 + e^{-2\mu} (-\dot{\mu} \dot{\phi}^2 + \dot{\phi} \ddot{\phi}) + e^{-2\lambda} (-\dot{\lambda} \phi'^2 + \dot{\phi}' \phi') \right\} \\
&\quad + 4 \frac{\pi^2}{t} e^{2\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1 + w^2 + F/t^2} \partial_t f dF dw \\
&= \frac{\dot{\mu} - \dot{\lambda}}{t} - 4\pi q e^{2\mu} + 2\dot{\lambda} \dot{\mu} + 8\pi \dot{\phi}^2 + 4\pi t (-\dot{\mu} \dot{\phi}^2 + \dot{\phi} \ddot{\phi}) + 4\pi t e^{2\mu-2\lambda} (-\dot{\lambda} \phi'^2 + \dot{\phi}' \phi') \\
&\quad + 4 \frac{\pi^2}{t} e^{2\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1 + w^2 + F/t^2} \partial_t f dF dw.
\end{aligned}$$

Combining all these relations gives :

$$\begin{aligned}
e^{-2\lambda} (\mu'' + \mu' (\mu' - \lambda')) - e^{-2\mu} (\ddot{\lambda} + (\dot{\lambda} + \frac{1}{t})(\dot{\lambda} - \dot{\mu})) &= e^{-2\lambda} [\mu' (\mu' + \lambda') - 4\pi t e^{\mu+\lambda} j' + \mu' (\mu' - \lambda')] \\
- e^{-2\mu} \left\{ \frac{\dot{\mu} - \dot{\lambda}}{t} - 4\pi q e^{2\mu} + 2\dot{\lambda} \dot{\mu} + 8\pi \dot{\phi}^2 + 4\pi t (-\dot{\mu} \dot{\phi}^2 + \dot{\phi} \ddot{\phi}) + 4\pi t e^{2\mu-2\lambda} (-\dot{\lambda} \phi'^2 + \dot{\phi}' \phi') \right. \\
&\quad \left. + 4 \frac{\pi^2}{t} e^{2\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1 + w^2 + F/t^2} \partial_t f dF dw + (\dot{\lambda} + \frac{1}{t})(\dot{\lambda} - \dot{\mu}) \right\} \\
&= 2\mu'^2 e^{-2\lambda} - e^{-2\lambda} 4\pi t e^{\lambda+\mu} \left\{ -\frac{\pi}{t^2} e^{\lambda-\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1 + w^2 + F/t^2} \partial_t f dF dw \right. \\
&\quad \left. - e^{\lambda-\mu} \dot{\lambda} (\rho + p) + e^{\lambda-3\mu} \dot{\lambda} \dot{\phi}^2 + e^{-\lambda-\mu} \dot{\lambda} \phi'^2 - 2\mu' j - e^{-(\lambda+\mu)} \dot{\phi}' \phi' - e^{\lambda-3\mu} \dot{\phi} \ddot{\phi} - e^{\lambda-3\mu} (\dot{\lambda} - \dot{\mu} + \frac{2}{t}) \dot{\phi}^2 \right\} \\
&\quad - e^{-2\mu} \left[\frac{-\dot{\lambda}}{t} - 4\pi q e^{2\mu} + 2\dot{\lambda} \dot{\mu} + \frac{\dot{\mu}}{t} + 8\pi \dot{\phi}^2 + 4\pi t (-\dot{\mu} \dot{\phi}^2 + \dot{\phi} \ddot{\phi}) + 4\pi t e^{2\mu-2\lambda} (-\dot{\lambda} \phi'^2 + \dot{\phi}' \phi') \right. \\
&\quad \left. + 4 \frac{\pi^2}{t} e^{2\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1 + w^2 + F/t^2} \partial_t f dF dw \right] - e^{-2\mu} (\dot{\lambda} + \frac{1}{t})(\dot{\lambda} - \dot{\mu}) \\
&= 2\mu'^2 e^{-2\lambda} + \dot{\lambda} (\rho + p) 4\pi t + 8\pi t \mu' e^{\mu-\lambda} j + \frac{\dot{\lambda}}{t} e^{-2\mu} + 4\pi q - 2e^{-2\mu} \dot{\lambda} \dot{\mu} - \frac{\dot{\mu}}{t} e^{-2\mu} - e^{-2\mu} (\dot{\lambda} + \frac{1}{t})(\dot{\lambda} - \dot{\mu}) \\
&= 2\mu'^2 e^{-2\lambda} + \dot{\lambda} (\dot{\lambda} + \dot{\mu}) e^{-2\mu} + 8\pi t \mu' e^{\mu-\lambda} \left(-\frac{1}{4\pi t} e^{-\mu-\lambda} \mu' \right) + \frac{\dot{\lambda}}{t} e^{-2\mu} + 4\pi q - 2e^{-2\mu} \dot{\lambda} \dot{\mu} - \frac{\dot{\mu}}{t} e^{-2\mu} - e^{-2\mu} \dot{\lambda}^2 \\
&\quad + e^{-2\mu} \dot{\lambda} \dot{\mu} - \frac{\dot{\lambda}}{t} e^{-2\mu} + \frac{\dot{\mu}}{t} e^{-2\mu} \\
&= 4\pi q
\end{aligned}$$

2) subtract the two equations (1.16)-(1.17) to obtain

$$\dot{\phi}_2 - \phi'_1 = (\tilde{\mu} - \mu') \phi_1 \quad (1.47)$$

Let us first prove that $\mu' = \tilde{\mu}$. Consider equation (1.44) and write $p = p_1 + p_2$ where p_1 and p_2 are the contributions to p made by f and ϕ respectively. We

obtain from [15] that studies the case $\phi = 0$, and in which p corresponds to p_1 here, using definitions (1.8), (1.9) and (1.10) of ρ, p, j :

$$\begin{aligned} \int_1^t s^2 p_1'(s, r) ds &= - \int_1^t (\dot{\lambda} + \dot{\mu}) e^{\lambda - \mu} s^2 (j + e^{-\lambda - \mu} \phi_1 \phi_2) ds - [e^{\lambda - \mu} s^2 (j + e^{-\lambda - \mu} \phi_1 \phi_2)]_1^t \\ &\quad - \int_1^t \tilde{\mu} [(\rho + p) - (e^{-2\mu} \phi_1^2 + e^{-2\lambda} \phi_2^2)] s^2 ds \end{aligned}$$

We have

$$D^+(\phi_1 e^{-\mu} - \phi_2 e^{-\lambda}) = \dot{\phi}_1 e^{-2\mu} - \dot{\phi}_2 e^{-2\lambda} - \dot{\mu} \phi_1 e^{-2\mu} + \dot{\lambda}' \phi_2 e^{-2\lambda} + (\dot{\lambda} \phi_2 - \mu' \phi_1 + \phi_1' - \dot{\phi}_2) e^{-\mu - \lambda}$$

From (1.16),

$$D^+(\phi_1 e^{-\mu} - \phi_2 e^{-\lambda}) = \left(-\dot{\lambda} - \frac{2}{t}\right) \phi_1 e^{-2\mu} + \dot{\lambda} \phi_2 e^{-\mu - \lambda} - \tilde{\mu} e^{-\mu - \lambda} \phi_1 + \tilde{\mu} e^{-2\lambda} \phi_2$$

We deduce from these two relations that

$$\begin{aligned} \phi_2' e^{-2\lambda} &= \dot{\phi}_1 e^{-2\mu} - \dot{\mu} \phi_1 e^{-2\mu} + \dot{\lambda}' \phi_2 e^{-2\lambda} - \mu' \phi_1 e^{-\mu - \lambda} \\ &\quad + (\phi_1' - \dot{\phi}_2) e^{-\mu - \lambda} - \left(-\dot{\lambda} - \frac{2}{t}\right) \phi_1 e^{-2\mu} + \tilde{\mu} e^{-\mu - \lambda} \phi_1 - \tilde{\mu} e^{-2\lambda} \phi_2 \end{aligned}$$

Then,

$$\begin{aligned} \int_1^t s^2 p_2'(s, r) ds &= \int_1^t s^2 (-\mu' \phi_1^2 e^{-2\mu} - \lambda' \phi_2^2 e^{-2\lambda} + \phi_1' \phi_1 e^{-2\mu} + \phi_2' \phi_2 e^{-2\lambda}) ds \\ &= \int_1^t s^2 [-\mu' \phi_1^2 e^{-2\mu} - \lambda' \phi_2^2 e^{-2\lambda} + \phi_1' \phi_1 e^{-2\mu} \\ &\quad + \phi_2 (\dot{\phi}_1 e^{-2\mu} - \dot{\mu} \phi_1 e^{-2\mu} + \dot{\lambda}' \phi_2 e^{-2\lambda} - \mu' \phi_1 e^{-\mu - \lambda} \\ &\quad + (\phi_1' - \dot{\phi}_2) e^{-\mu - \lambda} - \left(-\dot{\lambda} - \frac{2}{t}\right) \phi_1 e^{-2\mu} + \tilde{\mu} e^{-\mu - \lambda} \phi_1 - \tilde{\mu} e^{-2\lambda} \phi_2)] ds \\ &= \int_1^t s^2 [-\mu' \phi_1^2 e^{-2\mu} + \phi_1' \phi_1 e^{-2\mu} + \phi_2 \dot{\phi}_1 e^{-2\mu} + (\dot{\lambda} - \dot{\mu}) e^{-2\mu} \phi_1 \phi_2 \\ &\quad + (\phi_1' - \dot{\phi}_2) \phi_2 e^{-\mu - \lambda} + (\tilde{\mu} - \mu') \phi_1 \phi_2 e^{-\mu - \lambda} - \tilde{\mu} e^{-2\lambda} \phi_2^2] ds + \int_1^t 2s \phi_1 \phi_2 e^{-2\mu} ds \end{aligned}$$

Integrate the last term of this relation by parts and using (1.47), we obtain

$$\begin{aligned}
\int_1^t s^2 p_2'(s, r) ds &= \int_1^t s^2 [-\mu' \phi_1^2 e^{-2\mu} + \phi_1' \phi_1 e^{-2\mu} + \phi_2 \dot{\phi}_1 e^{-2\mu} + (\dot{\lambda} - \dot{\mu}) e^{-2\mu} \phi_1 \phi_2 \\
&\quad + (\phi_1' - \dot{\phi}_2) \phi_2 e^{-\mu-\lambda} + (\tilde{\mu} - \mu') \phi_1 \phi_2 e^{-\mu-\lambda} - \tilde{\mu} e^{-2\lambda} \phi_2^2] ds + [s^2 \phi_1 \phi_2 e^{-2\mu}]_1^t \\
&\quad - \int_1^t s^2 (-2\dot{\mu} \phi_1 \phi_2 e^{-2\mu} + \dot{\phi}_1 \phi_2 e^{-2\mu} + \phi_1 \dot{\phi}_2 e^{-2\mu}) ds \\
&= [s^2 \phi_1 \phi_2 e^{-2\mu}]_1^t + \int_1^t [(\dot{\lambda} + \dot{\mu}) e^{-2\mu} \phi_1 \phi_2 + (\phi_1' \phi_1 - \phi_1 \dot{\phi}_2) e^{-2\mu} \\
&\quad + (\phi_1' - \dot{\phi}_2) \phi_2 e^{-\mu-\lambda} - \mu' \phi_1^2 e^{-2\mu} - \tilde{\mu} \phi_2^2 e^{-2\lambda} + (\tilde{\mu} - \mu') \phi_1 \phi_2 e^{-\mu-\lambda}] s^2 ds \\
&= [s^2 \phi_1 \phi_2 e^{-2\mu}]_1^t + \int_1^t [(\dot{\lambda} + \dot{\mu}) e^{-2\mu} \phi_1 \phi_2 + (\phi_1' - \dot{\phi}_2)(\phi_1 e^{-2\mu} + \phi_2 e^{-\mu-\lambda}) \\
&\quad - \mu' \phi_1^2 e^{-2\mu} - \tilde{\mu} \phi_2^2 e^{-2\lambda} + (\tilde{\mu} - \mu') \phi_1 \phi_2 e^{-\mu-\lambda}] s^2 ds \\
&= [s^2 \phi_1 \phi_2 e^{-2\mu}]_1^t + \int_1^t [(\dot{\lambda} + \dot{\mu}) e^{-2\mu} \phi_1 \phi_2 - \tilde{\mu}(\phi_1^2 e^{-2\mu} + \phi_2^2 e^{-2\lambda})] s^2 ds
\end{aligned}$$

Then, using (1.45),

$$\int_1^t s^2 p_2'(s, r) ds = [-e^{\lambda-\mu} j s^2]_1^t - \int_1^t 4\pi e^{\lambda+\mu} j s(\rho+p) s^2 ds - \int_1^t \tilde{\mu}(\rho+p) s^2 ds$$

(1.46) becomes :

$$t\mu' e^{-2\mu} = \overset{\circ}{\mu}' e^{-2\overset{\circ}{\mu}} + 4\pi(e^{\overset{\circ}{\lambda}-\overset{\circ}{\mu}} \overset{\circ}{j} - e^{\lambda-\mu} j t^2) + 4\pi \int_1^t (4\pi s e^{\lambda+\mu} j + \tilde{\mu})(\rho+p) s^2 ds$$

and using the definition (1.15) of $\tilde{\mu}$:

$$t\mu' e^{-2\mu} = e^{-2\overset{\circ}{\mu}} (\overset{\circ}{\mu}' + 4\pi e^{\overset{\circ}{\lambda}+\overset{\circ}{\mu}} \overset{\circ}{j}) + t\tilde{\mu} e^{-2\mu} \quad \text{for all } t \in I \subset]0, \infty[$$

Hence, if (1.5) holds for $t = 1$, then, $t\mu' e^{-2\mu} = t\tilde{\mu} e^{-2\mu}$ and $\mu' = \tilde{\mu}$.

Now we prove the existence of ϕ . Define ϕ by : $\phi(t, r) = \overset{\circ}{\phi}(r) + \int_1^t \phi_1(s, r) ds$. Then $\phi(1) = \overset{\circ}{\phi}$, $\dot{\phi}(1) = \phi_1(1) = \psi$ and $\dot{\phi} = \phi_1$. Now (1.47) implies, since $\mu' = \tilde{\mu}$, that $\dot{\phi}_2 = \phi_1'$, hence the relation $\phi_2(1) = \overset{\circ}{\phi}'$ implies $\phi' = \phi_2$. The relation $\mu' = \tilde{\mu}$ also implies that the systems (1.12)-(1.13) and (1.16)-(1.17) are identical. Then a direct calculation, using the fact that (ϕ_1, ϕ_2) satisfies the system (1.12)-(1.13) shows that ϕ satisfies (1.7). \square

We conclude this section with a proposition dealing with the solvability of the constraint equation (1.5) for $t = 1$. Let $\tilde{\psi} = e^{-\mu}\psi$.

Proposition 1.15 *Given a function $\overset{\circ}{\lambda}(r)$, a non-negative function $\bar{f}(r, w, F)$ and functions $\overset{\circ}{\phi}(r)$ and $\tilde{\psi}(r)$, all periodic in r and regular, there exists a function*

$\overset{\circ}{\mu}(r)$, periodic in r and regular, such that the constraint equation

$$\overset{\circ}{\mu}' = -4\pi e^{\overset{\circ}{\lambda} + \overset{\circ}{\mu}} \overset{\circ}{j}$$

holds for a non-negative function f . It can be assumed that $f = \bar{f} + a\Phi$, where $\Phi(r, w, F)$ is a fixed function, independent of the particular choice of input data, and a is a suitable constant.

Proof: This can be proved just as in [27], with Φ chosen as in that reference. \square

This result shows that it is possible to produce a plentiful supply of initial data. It cannot be applied to produce data with $f = 0$. A way of doing that is to adjust $\tilde{\psi} = \bar{\psi} + b\Phi$ (b is a suitable constant) instead of adjusting f . $\Phi(r) \geq 0$ and $\text{supp}\Phi \subset I$, I an interval in which $\partial_r \overset{\circ}{\phi}(r) \neq 0$.

Remark 1.16 1) The two previous propositions show that the coupled system (1.2)-(1.3)-(1.4)-(1.5)-(1.6)-(1.7) reduces to the subsystem (1.14), (1.3), (1.4), (1.15), (1.16), (1.17) on which we concentrate in the next chapter.
 2) In the following, only one norm on function spaces is used, namely the L^∞ -norm, which is denoted by $\|\cdot\|$. For example if $f \in C^k(I \times [0, \infty[)$, we define $\|f(t)\| = \sup\{|f(t, r)|, r \in [0, \infty[$.

Chapter 2

Local existence and continuation of solutions

In this chapter using an iteration we prove the local existence and uniqueness of solutions of the Einstein-Vlasov-scalar field system together with continuation criteria.

2.1 Iteration

Let us first use the solution $(f, \lambda, \mu, \tilde{\mu}, \phi_1, \phi_2)$ of the auxiliary system consisting of the equations (1.14), (1.3), (1.4), (1.15), (1.16) and (1.17), to construct a sequence of iterative solutions as follows. Define $\overset{\circ}{\tilde{\mu}} := \overset{\circ}{\mu}'$, $\lambda_0(t, r) := \overset{\circ}{\lambda}(r)$, $\mu_0(t, r) := \overset{\circ}{\mu}(r)$, $\tilde{\mu}_0(t, r) := \overset{\circ}{\tilde{\mu}}(r)$, $g_0(t, r) = \psi(r)$, $h_0(t, r) = \phi'$ for $t \in]0, 1]$, $r \in \mathbb{R}$. If $\lambda_{n-1}, \mu_{n-1}, \tilde{\mu}_{n-1}$ are already defined and regular on $]T^*, 1] \times \mathbb{R}$ then let

$$G_{n-1}(t, r, w, F) := \left(\frac{we^{\mu_{n-1}-\lambda_{n-1}}}{\sqrt{1+w^2+F/t^2}}, -\dot{\lambda}_{n-1}w - e^{\mu_{n-1}-\lambda_{n-1}}\tilde{\mu}_{n-1}\sqrt{1+w^2+F/t^2} \right) \quad (2.1)$$

and denote by $(R_n, W_n)(s, t, r, w, F)$ the solution of the characteristic system

$$\frac{d}{ds}(R, W) = G_{n-1}(s, R, W, F)$$

with initial data

$$(R_n, W_n)(t, t, r, w, F) = (r, w); \quad (t, r, w, F) \in]0, 1] \times \mathbb{R}^2 \times [0, \infty[;$$

note that F is constant along characteristics. Define

$$f_n(t, r, w, F) := \overset{\circ}{f}((R_n, W_n)(1, t, r, w, F), F), \quad (2.2)$$

that is, f_n is the solution of

$$\partial_t f_n + \frac{we^{\mu_{n-1}-\lambda_{n-1}}}{\sqrt{1+w^2+F/t^2}} \partial_r f_n - (\dot{\lambda}_{n-1}w + e^{\mu_{n-1}-\lambda_{n-1}}\tilde{\mu}_{n-1}\sqrt{1+w^2+F/t^2})\partial_w f_n = 0 \quad (2.3)$$

with $f_n(1) = \overset{\circ}{f}$. Define ρ_n, p_n, j_n, q_n by the formulas (1.8), (1.9), (1.10) and (1.11) with $f, \lambda, \mu, \dot{\phi}, \phi'$ respectively replaced by $f_n, \lambda_{n-1}, \mu_{n-1}, g_{n-1}, h_{n-1}$, ($n \geq 1$). Using Proposition 1.13, 1), define μ_n and λ_n to be the solutions of

$$e^{-2\mu_n(t,r)} = \frac{e^{-2\overset{\circ}{\mu}(r)} + k}{t} - k + \frac{8\pi}{t} \int_t^1 s^2 p_n(s,r) ds \quad (2.4)$$

$$\dot{\lambda}_n(t,r) = 4\pi t e^{2\mu_n(t,r)} \rho_n(t,r) - \frac{1 + ke^{2\mu_n}}{2t} \quad (2.5)$$

$$\lambda_n(t,r) = \overset{\circ}{\lambda}(r) - \int_t^1 \dot{\lambda}_n(s,r) ds \quad (2.6)$$

and set

$$\tilde{\mu}_n(t,r) = -4\pi t e^{(\mu_n+\lambda_n)(t,r)} j_n(t,r) \quad (2.7)$$

Notice that, by Proposition 1.13, the right hand side of (2.4) is positive on $]T^*, 1]$, $\forall n$. Now define g_n and h_n using Proposition 1.13, 2) to satisfy the conditions that the quantities

$$X_n = e^{-\mu_n} g_n - e^{-\lambda_n} h_n, \quad Y_n = e^{-\mu_n} g_n + e^{-\lambda_n} h_n$$

are solutions of the system

$$D_{n-1}^+ X_n = a_{n-1} X_{n-1} + b_{n-1} Y_{n-1} \quad (2.8)$$

$$D_{n-1}^- Y_n = b_{n-1} X_{n-1} + c_{n-1} Y_{n-1} \quad (2.9)$$

where $D_{n-1}^+, D_{n-1}^-, a_{n-1}, b_{n-1}$ and c_{n-1} are defined in the same way as $D^+, D^-, \tilde{a}, b, \tilde{c}$ (see Lemma 1.6), with $\mu, \lambda, \dot{\phi}, \phi', \tilde{a}, b, \tilde{c}$ substituted respectively by $\mu_{n-1}, \lambda_{n-1}, g_{n-1}, h_{n-1}, a_{n-1}, b_{n-1}, c_{n-1}$. Now K_0 and A_0 being defined in Propositions 1.7 and 1.10, we introduce the following quantities that are similar to those defined in those propositions:

$$\left\{ \begin{array}{l} K_n(t) = \sup\{(g_n^2 e^{-2\mu_n} + h_n^2 e^{-2\lambda_n})^{\frac{1}{2}}(t,r) ; r \in \mathbb{R}\} \\ A_n(t) = \sup\{e^{-\lambda_n} [(g_n' - \mu_n' g_n)^2 e^{-2\mu_n} + (h_n' - \lambda_n' h_n)^2 e^{-2\lambda_n}]^{1/2}(t,r) ; r \in \mathbb{R}\} \\ m_{n-1}(t) = \sup\{\frac{2}{t} + (|\dot{\lambda}_{n-1}| + |\tilde{\mu}_{n-1}| e^{\mu_{n-1}-\lambda_{n-1}})(t,r) ; r \in \mathbb{R}\} \\ v_{n-1}(t) = \sup\{\frac{2}{t} + 2|\dot{\lambda}_{n-1}| + (|\tilde{\mu}_{n-1}| + |\mu_{n-1}'|) e^{\mu_{n-1}-\lambda_{n-1}}(t,r) ; r \in \mathbb{R}\} \\ \beta_{n-1}(t) = \sup\{[|\dot{\lambda}_{n-1}'| e^{-\lambda_{n-1}} + (|\mu_{n-1}'| |\tilde{\mu}_{n-1}| + |\lambda_{n-1}'| |\tilde{\mu}_{n-1}| \\ + |\tilde{\mu}_{n-1}'|) e^{\mu_{n-1}-2\lambda_{n-1}}](t,r) ; r \in \mathbb{R}\} \end{array} \right. \quad (2.10)$$

Now we proceed for (2.8)-(2.9) the same way as we did for (1.16)-(1.17) to establish the inequality (1.18) and we obtain the following analogous inequality:

$$K_n(t) \leq K_0 + 2 \int_t^1 m_{n-1}(s)K_{n-1}(s)ds \quad (2.11)$$

We can use (2.8)-(2.9) to establish a system for $(e^{-\lambda_{n-1}}\partial_r X_n, e^{-\lambda_{n-1}}\partial_r Y_n)$ analogous to (1.19)-(1.20) from which we deduce the following inequality which is analogous to (1.27)

$$A_n(t) \leq A_0 + 2 \int_t^1 (v_{n-1}(s)A_{n-1}(s) + \beta_{n-1}(s)K_{n-1}(s))ds \quad (2.12)$$

Throughout the document, we use the fact that by (2.2), $\|f_n(t)\| = \|\overset{\circ}{f}\|$ for $n \in \mathbb{N}$ and $t \in]T^*, 1]$. The numerical constant C may change from line to line and does not depend on n or t or the initial data. In order to prove the local existence theorem, we prove respectively in the next two propositions :

- a uniform bound on the momenta in the support of distribution functions f_n , and a uniform bound of the first derivatives with respect to r of the functions $f_n, \lambda_n, \mu_n, g_n, h_n$;
- the convergence of the iterates.

Proposition 2.1 *We take $\overset{\circ}{f}$ as in proposition 1.13 and such that*

$$\text{supp} \overset{\circ}{f} \subset [0, W_0] \times [0, F_0], \quad W_0 > 0, \quad F_0 > 0. \quad (2.13)$$

then there exist nonnegative constants T_1, T_2 such that the quantities

$$Q_n(t) = \sup\{|w|; (r, w, F) \in \text{supp} f_n(t)\} \text{ for all } t \in [T_1, 1],$$

$$B_n(t) = \sup\{\|\partial_r f_n(s)\| + A_{n-1}^2(s); t \leq s \leq 1\} \text{ for all } t \in [T_2, 1].$$

and $K_n(t)$ for all $t \in [T_1, 1]$ are uniformly bounded in n .

Proof: Firstly we bound $Q_n(t)$ and $K_n(t)$. On $\text{supp} f_n(t)$, we have

$$\sqrt{1 + w^2 + F/t^2} \leq \sqrt{1 + Q_n^2 + F_0/t^2} \leq \frac{1}{t}(1 + F_0)(1 + Q_n(t)) \quad (2.14)$$

and thus

$$\begin{aligned} \|\rho_n(t)\| &\leq \frac{\pi}{t^2} \int_{-Q_n(t)}^{Q_n(t)} \int_0^{F_0} \frac{1}{t}(1 + F_0)(1 + Q_n(t)) \|f_n(t)\| dF dw + (K_{n-1}(t))^2 \\ &\leq \frac{C}{t^3} (1 + F_0)^2 (1 + Q_n(t))^2 \|\overset{\circ}{f}\| + (K_{n-1}(t))^2; \end{aligned}$$

$$\begin{aligned} \|\rho_n(t)\|, \|j_n(t)\| &\leq \frac{\pi}{t^2} \int_{-Q_n(t)}^{Q_n(t)} \int_0^{F_0} Q_n(t) \|f_n(t)\| dF dw + (K_{n-1}(t))^2 \\ &\leq \frac{C}{t^2} (1 + F_0)(1 + Q_n(t))^2 \|\overset{\circ}{f}\| + (K_{n-1}(t))^2 \end{aligned}$$

From (1.43), we have, setting $C_0 = \frac{\beta}{2}$,

$$e^{-2\mu_n}(t, r) \geq \frac{C_0}{t} \quad (2.15)$$

Using (2.5)-(2.7) and (2.15) we have

$$\begin{aligned} |\dot{\lambda}_n(s, r)| &\leq 4\pi s e^{\mu_n(s, r)} |\rho_n(s, r)| + \frac{1 + s/C_0}{2s} \\ &\leq \frac{C}{C_0} [(1 + F_0)^2 \frac{(1 + Q_n(s))^2}{s} \| \dot{f} \| + (K_{n-1}(s))^2] + \frac{1 + C_0}{2sC_0}; \end{aligned}$$

and

$$\begin{aligned} |\tilde{\mu}_n e^{\mu_n - \lambda_n}(s, r)| &\leq \frac{4\pi s^2}{C_0} \left[\frac{C}{s^2} (1 + F_0)(1 + Q_n(s))^2 \| \dot{f} \| + K_{n-1}(s)^2 \right] \\ &\leq \frac{C}{C_0} [(1 + F_0)(1 + Q_n(s))^2 \| \dot{f} \| + K_{n-1}(s)^2] \end{aligned}$$

Thus

$$\begin{aligned} |\dot{W}_{n+1}(s)| &\leq |\dot{\lambda}_n(s)| + |\tilde{\mu}_n| e^{\mu_n - \lambda_n}(s) \sqrt{1 + w^2 + F/s^2} \\ &\leq \frac{C}{C_0} [(1 + F_0)^2 \frac{(1 + Q_n(s))^2}{s} \| \dot{f} \| + (K_{n-1}(s))^2 + \frac{1}{s}] |W_{n+1}(s)| \\ &\quad + \frac{C}{C_0} [(1 + F_0)(1 + Q_n(s))^2 \| \dot{f} \| + K_{n-1}(s)^2] \frac{1 + F_0}{s} (1 + |W_{n+1}(s)|) \\ &\leq \frac{C}{C_0} [(1 + F_0)^2 \frac{(1 + Q_n(s))^2}{s} (1 + \| \dot{f} \|)(1 + K_{n-1}(s))^2] (1 + |W_{n+1}(s)|) \\ &\leq C_1 \frac{1}{s} (1 + Q_n(s))^2 (1 + K_{n-1}(s))^2 (1 + |W_{n+1}(s)|) \end{aligned}$$

This implies after integration over $[t, 1]$,

$$W_{n+1}(t) \leq W_0 + C_1 \int_t^1 \frac{1}{s} (1 + Q_n(s))^2 (1 + K_{n-1}(s))^2 (1 + |W_{n+1}(s)|) ds;$$

then

$$Q_{n+1}(t) \leq W_0 + C_1 \int_t^1 \frac{1}{s} (1 + Q_n(s))^2 (1 + K_{n-1}(s))^2 (1 + Q_{n+1}(s)) ds \quad (2.16)$$

with $C_1 = \frac{C}{C_0} (1 + F_0)^2 (1 + \| \dot{f} \|)$. Next we have, using (2.5)-(2.7) and (2.15)

$$|\dot{\lambda}_n(s, r)| + |(\tilde{\mu}_n e^{\mu_n - \lambda_n})(s, r)| \leq C_1 \frac{(1 + Q_n(s))^2}{s} (1 + K_{n-1}(s))^2 \quad (2.17)$$

we then deduce from (2.11) and (2.17) that

$$K_{n+1}(t) \leq K_0 + C_1 \int_t^1 \frac{(1 + Q_n(s))^2}{s} (1 + K_{n-1}(s))^2 K_n(s) ds \quad (2.18)$$

Add (2.16) and (2.18) to obtain

$$Q_{n+1}(t) + K_{n+1}(t) \leq W_0 + K_0 + C_1 \int_t^1 \frac{1}{s} (1 + Q_n(s))^2 (1 + K_{n-1}(s))^2 (1 + Q_{n+1}(s) + K_n(s)) ds \quad (2.19)$$

Now define $H_n(t) := \sup\{Q_m(t) + K_m(t) ; m \leq n\}$. $(H_n)_{n \in \mathbb{N}}$ is an increasing sequence. Then,

$$\begin{aligned} Q_n(t) + K_n(t) &\leq H_n(t) \leq H_{n+1}(t), \\ Q_{n+1}(t) + K_{n+1}(t) &\leq W_0 + K_0 + C_1 \int_t^1 \frac{1}{s} (1 + H_{n+1}(s))^5 ds \quad \text{and} \\ Q_m(t) + K_m(t) &\leq W_0 + K_0 + C_1 \int_t^1 \frac{1}{s} (1 + H_{n+1}(s))^5 ds \quad \text{for } m \leq n; \end{aligned}$$

which imply :

$$H_{n+1}(t) \leq W_0 + K_0 + C_1 \int_t^1 \frac{1}{s} (1 + H_{n+1}(s))^5 ds$$

Let z_1 be the left maximal solution of the equation

$$z_1(t) = W_0 + K_0 + C_1 \int_t^1 \frac{1}{s} (1 + z_1(s))^5 ds$$

which exists on some interval $]T_1, 1]$ with $T_1 \in [T^*, 1[$. By comparing the solution of the integral inequality with that of the corresponding integral equation it follows that

$$H_{n+1}(t) \leq z_1(t), \quad t \in]T_1, 1[, \quad n \in \mathbb{N}.$$

Since $Q_n(t) + K_n(t) \leq H_{n+1}(t)$, we obtain

$$K_n(t), Q_n(t) \leq z_1(t), \quad t \in]T_1, 1[, \quad n \in \mathbb{N}.$$

And all the quantities which were estimated against Q_n and K_n in the above argument are bounded by certain powers of z_1 on $]T_1, 1]$. Namely $|\dot{\lambda}_n(t)|$, $|\rho_n(t)|$, $|p_n(t)|$, $|j_n(t)|$, $|\tilde{\mu}_n e^{\mu_n - \lambda_n}(t)|$, $g_{n-1}^2 e^{-2\mu_{n-1}}(t)$, $h_{n-1}^2 e^{-2\lambda_{n-1}}(t)$ are bounded. (2.4) shows that $e^{-2\mu_n}$ is bounded and by (2.15), $e^{2\mu_n}$ is bounded. (2.6) shows that $|\lambda_n(t)|$ is bounded. Consequently e^{λ_n} and $e^{-\lambda_n}$ are bounded. Then g_n and h_n are bounded. We conclude that, there exists a continuous function $C_2(t)$ which depends only on z_1 as an increasing function, such that

$$\begin{cases} \|\mu_n(t)\|, \|\lambda_n(t)\|, \|\dot{\lambda}_n(t)\|, \|\rho_n(t)\|, \|p_n(t)\|, \\ \|\tilde{\mu}_n e^{\mu_n - \lambda_n}\|, \|g_n(t)\|, \|h_n(t)\| \leq C_2(t) \end{cases} \quad (2.20)$$

Now we bound $B_n(t)$. We have, using (2.20), the estimates

$$\begin{aligned}
\rho'_n(t) &= \frac{\pi^2}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_r f_n dF dw \\
&\quad + (-\mu'_{n-1} g_{n-1}^2 + g_{n-1} g'_{n-1}) e^{-2\mu_{n-1}} + (-\lambda'_{n-1} h_{n-1}^2 + h_{n-1} h'_{n-1}) e^{-2\lambda_{n-1}} \\
&= \frac{\pi^2}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_r f_n dF dw \\
&\quad + (-\mu'_{n-1} g_{n-1} + g'_{n-1}) e^{-\mu_{n-1}} g_{n-1} e^{-\mu_{n-1}} + (-\lambda'_{n-1} h_{n-1} + h'_{n-1}) e^{-\lambda_{n-1}} h_{n-1} e^{-\lambda_{n-1}} \\
&\leq \frac{\pi^2}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_r f_n dF dw + g_{n-1}^2 e^{-2\mu_{n-1}} \\
&\quad + (-\mu'_{n-1} g_{n-1} + g'_{n-1})^2 e^{-2\mu_{n-1}} + (-\lambda'_{n-1} h_{n-1} + h'_{n-1})^2 e^{-2\lambda_{n-1}} + h_{n-1}^2 e^{-2\lambda_{n-1}} \\
&\leq \frac{\pi^2}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_r f_n dF dw + g_{n-1}^2 e^{-2\mu_{n-1}} + h_{n-1}^2 e^{-2\lambda_{n-1}} \\
&\quad + e^{-2\lambda_{n-1}} [(-\mu'_{n-1} g_{n-1} + g'_{n-1})^2 e^{-2\mu_{n-1}} + (-\lambda'_{n-1} h_{n-1} + h'_{n-1})^2 e^{-2\lambda_{n-1}}] e^{2\lambda_{n-1}}
\end{aligned}$$

and

$$\|\rho'_n(t)\| \leq C_2(t)(C_3 + B_n(t))$$

We deduce in the same way as previously :

$$\|\rho'_n(t)\|, \|p'_n(t)\|, \|j'_n(t)\|, \|\mu'_n(t)\|, \|\dot{\lambda}'_n(t)\|, \|\lambda'_n(t)\| \leq C_2(t)(C_3 + B_n(t)) \quad (2.21)$$

$$\|\tilde{\mu}'_n e^{\mu_n - \lambda_n}\| \leq C_2(t)(C_3 + B_n(t)) \quad (2.22)$$

Following step 2 of the proof of theorem 3.1 in [15], we have

$$|\partial_r G_n(s, r, w, F)| \leq C_2(s)(C_3 + B_n(s));$$

$$|\partial_w G_n(s, r, w, F)| \leq C_2(s)$$

and differentiate the characteristic system of the Vlasov equation with respect to r :

$$\begin{aligned}
\left| \frac{d}{ds} \partial_r (R, W)_{n+1}(s, t, r, w, F) \right| &= |\partial_r (R, W)_{n+1}(s, t, r, w, F) \cdot \partial_r G_n(s, R_{n+1}, W_{n+1}, F)| \\
&\leq |\partial_r (R, W)_{n+1}(s, t, r, w, F)| C_2(s)(C_3 + B_n(s))
\end{aligned}$$

therefore for $(r, w, F) \in \text{supp} f_{n+1}(t) \cup \text{supp} f_n(t)$, we obtain by Gronwall's inequality

$$|\partial_r (R, W)_{n+1}(1, t, r, w, F)| \leq \exp \int_t^1 C_2(s)(C_3 + B_n(s)) ds.$$

From the definition of f in (2.2), we obtain

$$\begin{aligned}
\|\partial_r f_{n+1}(t, r, w, F)\| &= \|\partial_r (R, W)_{n+1}(1, t, r, w, F) \cdot \overset{\circ}{\partial}(r, w) f((R, W)_{n+1}(1, t, r, w, F))\| \\
&\leq \|\overset{\circ}{\partial}(r, w) f\| \sup\{\|\partial_r (R, W)_{n+1}(1, t, r, w, F)\|; (r, w, F) \in \text{supp} f_{n+1}(t)\} \\
&\leq \|\overset{\circ}{\partial}(r, w) f\| \exp\left(\int_t^1 C_2(s)(C_3 + B_n(s))ds\right).
\end{aligned} \tag{2.23}$$

with $C_3 = \|\overset{\circ}{\lambda}'\| + \|\overset{\circ}{\mu}'e^{-2\overset{\circ}{\mu}}\| + 1$. We use (2.20), (2.21), (2.22) and estimate (2.12) to obtain

$$A_{n+1}(t) \leq A_0 + \int_t^1 C_2(s)(C_3 + B_n(s))(1 + A_n(s))ds$$

Let $D_n(t) := \sup\{A_m(t) | m \leq n\}$ and $E_n(t) := \sup\{B_m(t) | m \leq n\}$. $\{D_n\}$ and $\{E_n\}$ are increasing sequences. Therefore

$$1 + A_{n+1}(t) \leq A_0 + 1 + \int_t^1 C_2(s)(C_3 + E_n(s))(1 + D_{n+1}(s))ds \tag{2.24}$$

then we deduce by replacing n by any $m \leq n$ in (2.24)

$$1 + D_{n+1}(t) \leq A_0 + 1 + \int_1^t C_2(s)(C_3 + E_n(s))(1 + D_{n+1}(s))ds;$$

which gives:

$$D_{n+1}(t) \leq 2(A_0 + 1) \exp \int_t^1 C_2(s)(C_3 + E_n(s))ds. \tag{2.25}$$

Now add (2.23)-(2.25) to obtain

$$B_{n+1}(t) \leq (2(A_0 + 1) + \|\overset{\circ}{\partial}_{r,w} f\|) \exp \int_t^1 C_2(s)(C_3 + E_n(s))ds$$

and deduce by replacing n by every $m \leq n$ that:

$$E_{n+1}(t) \leq C_4 \exp \int_t^1 C_2(s)(C_3 + E_{n+1}(s))ds$$

where $C_4 = 2(A_0 + 1) + \|\overset{\circ}{\partial}_{(r,w)} f\|$. Let z_2 be the left maximal solution of

$$z_2(t) = C_4 \exp \int_t^1 C_2(s)(C_3 + z_2(s))ds$$

i.e

$$\dot{z}_2(t) = -C_2(t)(C_3 + z_2(t))z_2(t), \quad z_2(1) = C_4;$$

which exists on an interval $]T_2, 1] \subset]T_1, 1]$. Then we have

$$E_{n+1}(t) \leq z_2(t), \quad t \in]T_2, 1], \quad n \in \mathbb{N}$$

and so

$$A_n(t), B_n(t) \leq z_2(t), \quad t \in]T_2, 1], \quad n \in \mathbb{N}$$

and all the quantities estimated against B_n above can be bounded in terms of z_2 on $]T_2, 1]$, uniformly in n . \square

Remark 2.2 *The sequences $\lambda_n, \mu_n, f_n, \tilde{\mu}_n e^{\mu_n - \lambda_n}, \rho_n, p_n, j_n, g_n, h_n, \lambda'_n, \mu'_n, f'_n, g'_n, h'_n, \dot{\lambda}_n, \dot{\mu}_n, \dot{g}_n, \dot{h}_n, \dot{\lambda}'_n, \dot{\rho}'_n, \dot{p}'_n, \dot{j}'_n, \dot{\mu}'_n$ are uniformly bounded in the L^∞ -norm by a function of t on $[T_1^*, 1]$ with $T_1^* = \max(T_1, T_2)$.*

In order to prove the convergence of the iterates in the following proposition, we introduce auxiliary variables \tilde{g}_n and \tilde{h}_n defined by $\tilde{g}_n = g_n e^{-\mu_n}$, $\tilde{h}_n = h_n e^{-\lambda_n}$, for $n \in \mathbb{N}$.

Proposition 2.3 *Let $[T_3, 1] \subset [T_2, 1]$, be an arbitrary compact subset on which the previous estimates hold. Then on such an interval, the iterates converge uniformly.*

Proof: Define for $t \in [T_3, 1]$:

$$\alpha_n(t) := \sup\{\| (f_{n+1} - f_n)(s) \| + \| (\tilde{g}_{n+1} - \tilde{g}_n)(s) \| + \| (\tilde{h}_{n+1} - \tilde{h}_n)(s) \| + \| (\lambda_{n+1} - \lambda_n)(s) \| + \| (\mu_{n+1} - \mu_n)(s) \|; \quad t \leq s \leq 1\}$$

and let C denote a constant which may depend on the functions z_1 and z_2 introduced previously. If we consider the new quantities

$$\tilde{X}_n = (\tilde{g}_{n+1} - \tilde{g}_n) - (\tilde{h}_{n+1} - \tilde{h}_n); \quad \tilde{Y}_n = (\tilde{g}_{n+1} - \tilde{g}_n) + (\tilde{h}_{n+1} - \tilde{h}_n),$$

then we obtain by subtracting the system (2.8)-(2.9) written for $n+1$ and n (see appendix), the new system

$$D_n^+ \tilde{X}_n = a_n \tilde{X}_{n-1} + b_n \tilde{Y}_{n-1} + F_n \quad (2.26)$$

$$D_n^- \tilde{Y}_n = b_n \tilde{X}_{n-1} + c_n \tilde{Y}_{n-1} + G_n \quad (2.27)$$

where

$$F_n = (a_n - a_{n-1} + b_n - b_{n-1})\tilde{g}_{n-1} + (a_{n-1} - a_n + b_n - b_{n-1})\tilde{h}_{n-1} + (e^{-\mu_{n-1}} - e^{-\mu_n})(\dot{g}_n - \dot{h}_n) + (e^{-\lambda_{n-1}} - e^{-\lambda_n})(\tilde{g}'_n - \tilde{h}'_n)$$

and substitute in F_n , \tilde{h}'_n and \tilde{g}'_n respectively by $-\tilde{h}'_n$ and $-\tilde{g}'_n$ to obtain G_n . Now let

$$\theta_n(t) = \sup\{|\tilde{g}_{n+1} - \tilde{g}_n| + |\tilde{h}_{n+1} - \tilde{h}_n|; \quad r \in \mathbb{R}\}$$

Thus similarly to (2.11), we have :

$$\theta_n(t) \leq 2 \int_t^1 (m_n(s)\theta_{n-1}(s) + \sup\{e^{\mu_n}(|F_n(s, r)| + |G_n(s, r)|); r \in \mathbb{R}\})ds \quad (2.28)$$

Using the mean value theorem to express the differences $e^{-\mu_n} - e^{-\mu_{n-1}}$, $e^{-\lambda_n} - e^{-\lambda_{n-1}}$ and remark 2.2, then (2.28) gives

$$|\tilde{g}_{n+1} - \tilde{g}_n| + |\tilde{h}_{n+1} - \tilde{h}_n| \leq C \int_t^1 (\alpha_{n-1} + |\tilde{\mu}_n - \tilde{\mu}_{n-1}| + |\dot{\lambda}_n - \dot{\lambda}_{n-1}|)(s)ds \quad (2.29)$$

The expressions of ρ_n, p_n, j_n yield, using proposition 2.1, that

$$|\rho_{n+1} - \rho_n|(t), |p_{n+1} - p_n|(t), |j_{n+1} - j_n|(t) \leq C\alpha_n(t)$$

From (2.5) and (2.7), we have respectively

$$|\dot{\lambda}_n - \dot{\lambda}_{n-1}|(t) = 4\pi t |(e^{2\mu_n} - e^{2\mu_{n-1}})\rho_n + (\rho_n - \rho_{n-1})e^{2\mu_{n-1}}|(t) \leq C\alpha_{n-1}(t) \quad (2.30)$$

$$|\tilde{\mu}_n - \tilde{\mu}_{n-1}|(t) = 4\pi t |(e^{\lambda_n + \mu_n} - e^{\lambda_{n-1} + \mu_{n-1}})j_n + (j_n - j_{n-1})e^{\lambda_{n-1} + \mu_{n-1}}|(t) \leq C\alpha_{n-1}(t) \quad (2.31)$$

Using the two previous inequalities, (2.29) gives

$$(|\tilde{g}_{n+1} - \tilde{g}_n| + |\tilde{h}_{n+1} - \tilde{h}_n|)(t) \leq C \int_t^1 \alpha_{n-1}(s)ds \quad (2.32)$$

By the mean value theorem, (2.4) gives :

$$|\mu_{n+1} - \mu_n|(t) \leq C \int_t^1 \alpha_n(s)ds \quad (2.33)$$

(2.6) gives :

$$|\lambda_{n+1} - \lambda_n|(t) = \int_t^1 |\dot{\lambda}_{n+1} - \dot{\lambda}_n|ds \leq C \int_t^1 \alpha_n(s)ds \quad (2.34)$$

Now from (2.30)-(2.31), (2.33)-(2.34), the mean value theorem and the fact that $\int_t^1 \alpha_{n-1}(s)ds \leq C\alpha_{n-1}(t)$, we deduce

$$|G_n - G_{n-1}|(s, r, w, F) \leq C\alpha_{n-1}(s)$$

which implies for $(r, w, F) \in \text{supp}f_{n-1}(t) \cup \text{supp}f_n(t)$,

$$|\frac{d}{ds}((R, W)_{n+1} - (R, W)_n)|(s, t, r, w, F) = |G_n - G_{n-1}|(s, r, w, F) \leq C\alpha_{n-1}(s)$$

Integrating over $[t, 1]$ and using the fact that $|(R, W)_{n+1} - (R, W)_n|(t, t, r, w, F) = 0$ gives

$$|(R, W)_{n+1} - (R, W)_n|(1, t, r, w, F) \leq C \int_t^1 \alpha_{n-1}(s) ds$$

This implies using (2.2) and the mean value theorem

$$\begin{aligned} |(f_{n+1} - f_n)(t)| &\leq |\partial_{r,w} \overset{\circ}{f}| |(R, W)_{n+1} - (R, W)_n|(1, t, r, w, F) \\ &\leq C |\partial_{r,w} \overset{\circ}{f}| \int_t^1 \alpha_{n-1}(s) ds \end{aligned} \quad (2.35)$$

Adding (2.32), (2.33), (2.34), (2.35), then

$$\alpha_n(t) \leq C \int_t^1 (\alpha_n(s) + \alpha_{n-1}(s)) ds ; \quad n \geq 1.$$

By Gronwall's inequality $\alpha_n(t) \leq C \int_t^1 \alpha_{n-1}(s) ds$; and by induction

$$\alpha_n(t) \leq C^{n+1} \frac{(1-t)^n}{n!} \leq \frac{C^{n+1}}{n!} \quad \text{for } n \in \mathbb{N}, t \in [T_3, 1]$$

Since the series $\sum \frac{C^{n+1}}{n!}$ converges, we deduce the convergence of $\sum \alpha_n$ which implies that $\alpha_n \rightarrow 0$ for $n \rightarrow \infty$. Every difference term which appears in α_n , converges to zero. We deduce the uniform convergence of

$$f_n, \lambda_n, \mu_n, \tilde{g}_n, \tilde{h}_n, \dot{\lambda}_n, \dot{\mu}_n, \tilde{\mu}_n, \rho_n, \nu_n, j_n. \quad (2.36)$$

And in L^∞ -norm, $\lambda_n \rightarrow \lambda$; $\mu_n \rightarrow \mu$; $\tilde{\mu}_n \rightarrow \tilde{\mu}$; $f_n \rightarrow f$; $\tilde{g}_n \rightarrow \tilde{g}$; $\tilde{h}_n \rightarrow \tilde{h}$. \square

It remains to show that the limits $\tilde{g}, \tilde{h}, f, \lambda, \mu$ are C^1 , f solves the Vlasov equation (1.2), λ, μ solve the field equations (1.3)-(1.4), and to show the existence of a function ϕ that solves the wave equation (1.7). This is the subject of the next section.

2.2 Local existence

Theorem 2.4 (local existence) *Let $\overset{\circ}{f} \in C^1(\mathbb{R}^2 \times [0, \infty[)$ with $\overset{\circ}{f}(r+1, w, F) = \overset{\circ}{f}(r, w, F)$ for $(r, w, F) \in \mathbb{R}^2 \times [0, \infty[, \overset{\circ}{f} \geq 0$, and*

$$W_0 := \sup\{|w|(r, w, F) \in \text{supp} \overset{\circ}{f}\} < \infty$$

$$F_0 := \sup\{F|(r, w, F) \in \text{supp} \overset{\circ}{f}\} < \infty$$

Let $\overset{\circ}{\lambda}, \psi \in C^1(\mathbb{R})$, $\overset{\circ}{\mu}, \overset{\circ}{\phi} \in C^2(\mathbb{R})$ with $\overset{\circ}{\lambda}(r) = \overset{\circ}{\lambda}(r+1)$, $\overset{\circ}{\mu}(r) = \overset{\circ}{\mu}(r+1)$,
 $\overset{\circ}{\phi}(r) = \overset{\circ}{\phi}(r+1)$ and

$$\overset{\circ}{\mu}'(r) = -4\pi e^{\overset{\circ}{\lambda} + \overset{\circ}{\mu}} \overset{\circ}{j}(r), \quad r \in \mathbb{R}$$

Then there exists a unique, left maximal, regular solution (f, λ, μ, ϕ) of system (1.2)-(1.11) with $(f, \lambda, \mu, \phi)(1) = (\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{\phi})$ and $\dot{\phi}(1) = \psi$ on a time interval $]T, 1[$ with $T \in [0, 1[$.

Proof : Consider the sequences of iterates constructed at the beginning of this chapter and the limit obtained in the above proposition. We need the uniform convergence of the derivatives of these iterates.

We know by (2.36) that $(\dot{\lambda}_n)$ and $(\dot{\mu}_n)$ converge uniformly. We must now show that, $\lambda'_n, \mu'_n, \dot{f}_n, \partial_r f_n, \partial_w f_n, \dot{g}_n, \dot{g}'_n, \dot{h}_n, \dot{h}'_n$ also converge uniformly. Using (2.3), the convergence of \dot{f}_n will be a consequence of that of $\lambda_n, \mu_n, \dot{\lambda}_n, \dot{\mu}_n, \partial_r f_n, \partial_w f_n$. Using system (2.8)-(2.9), the convergence of \dot{g}_n, \dot{h}_n will be a consequence of that of $\dot{g}'_n, \dot{h}'_n, \mu_n, \lambda_n, (\dot{\lambda}_n), \dot{\mu}_n$. We then proceed to show the uniform convergence of $\lambda'_n, \mu'_n, \partial_r f_n, \partial_w f_n, \dot{g}'_n, \dot{h}'_n$. In what follows, we fix $T_4 \in [T_2, 1]$, $t \in [T_4, 1]$, $|w| < U$, $F < F_0$, $t \leq s \leq 1$.

Step 1: Convergence of $(\partial_r f_n)$ and $(\partial_w f_n)$.

Following step 4 in the proof of theorem 3.1 in [15], and using (2.36), we can establish with minor changes, using Proposition 2.1, that if we set:

$$\xi_n(s) = e^{(\lambda_n - \mu_n)(s,r)} \partial R_n(s, t, r, w, F) \quad (2.37)$$

$$\eta_n(s) = \partial W_n(s) + (\sqrt{1 + w^2 + F/s^2} e^{\lambda_n - \mu_n} \dot{\lambda}_n) \partial R_n(s) \quad (2.38)$$

in which ∂ stands for ∂_r or ∂_w and $s \mapsto (R_n(s), W_n(s))$ the indicated solution of the characteristic system associated to equation (2.3) in f_n , and then: $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that we have, for $n > N$:

$$(|\xi_{n+1} - \xi_n| + |\eta_{n+1} - \eta_n|)(s) \leq C\epsilon + C \int_s^1 (|\xi_{n+1} - \xi_n| + |\eta_{n+1} - \eta_n|)(\tau) d\tau \quad (2.39)$$

in which $C > 0$ stands, as in what follows, for a constant that may change from line to line. (2.39) implies by Gronwall's lemma, that (ξ_n) and (η_n) converge uniformly. Now, since the transformation $(\partial R_n, \partial W_n) \mapsto (\xi_n, \eta_n)$ defined by (2.37)-(2.38) is invertible with convergent coefficients, this implies the convergence of $\partial_{r,w}(R_n, W_n)$ and, given (2.2), the convergence of $(\partial_r f_n)$ and $(\partial_w f_n)$.

Step 2: convergence of $(\lambda'_n), (\mu'_n), (\tilde{\mu}'_n), (\tilde{g}'_n), (\tilde{h}'_n)$.

We set

$$\begin{aligned} \gamma_n(t) = \sup\{ & |\xi_{n+1} - \xi_n|(s) + |\eta_{n+1} - \eta_n|(s) + \|(\mu'_{n+1} - \mu'_n)(s)\| \\ & + \|(\lambda'_{n+1} - \lambda'_n)(s)\| + \|(\tilde{g}'_{n+1} - \tilde{g}'_n)(s)\| + \|(\tilde{h}'_{n+1} - \tilde{h}'_n)(s)\| ; t \leq s \leq 1 \} \end{aligned} \quad (2.40)$$

Now since $(\mu_n), (\tilde{\mu}_n), (\dot{\lambda}_n), (\tilde{g}_n), (\tilde{h}_n), (\rho_n), (j_n)$ converge uniformly, we take the above integer N sufficiently large so that we have for $n > N$:

$$\begin{aligned} & \|(\mu_{n+1} - \mu_n)(s)\|, \| (j_{n+1} - j_n)(s) \|, \| (\tilde{\mu}_n - \tilde{\mu}_{n-1})(s) \|, \| (\dot{\lambda}_{n+1} - \dot{\lambda}_n)(s) \|, \\ & \| (\tilde{g}_n - \tilde{g}_{n-1})(s) \|, \| (\tilde{h}_n - \tilde{h}_{n-1})(s) \|, \| (\rho_{n+1} - \rho_n)(s) \| \leq \epsilon \end{aligned} \quad (2.41)$$

A) Estimation of $(\lambda'_n), (\mu'_n), (\tilde{\mu}'_n)$. We deduce from (2.37)-(2.38), taking $\partial = \partial_r$ that:

$$\partial R_n(s) = e^{(\mu_n - \lambda_n)(s,r)} \xi_n(s) \quad (2.42)$$

$$\partial W_n(s) = \eta_n(s) - (\sqrt{1+w^2 + F/s^2} \dot{\lambda}_n) \xi_n(s) \quad (2.43)$$

Let us first consider ρ'_n, j'_n, p'_n that involve $\partial_r f_n, (\tilde{g}_n \tilde{h}_n)', \frac{1}{2}(\tilde{g}_n^2 + \tilde{h}_n^2)$. We have, using (2.2), (2.42), (2.43)

$$\begin{aligned} \|(\partial_r f_{n+1} - \partial_r f_n)(s)\| & \leq \| \partial_{r,w} \overset{\circ}{f} \| (|\partial_r R_{n+1} - \partial_r R_n| + |\partial_r W_{n+1} - \partial_r W_n|)(s) \\ & \leq C \| \partial_{r,w} \overset{\circ}{f} \| (|e^{\mu_{n+1} - \lambda_{n+1}} \xi_{n+1} - e^{\mu_n - \lambda_n} \xi_n| \\ & \quad + |\eta_{n+1} - \eta_n| + |\dot{\lambda}_{n+1} \xi_{n+1} - \dot{\lambda}_n \xi_n|)(s). \end{aligned} \quad (2.44)$$

Estimate (2.44) gives, using (2.41) and since $(\lambda_n), (\mu_n), (\xi_n), (\dot{\lambda}_n)$ are bounded,

$$\|(\partial_r f_{n+1} - \partial_r f_n)(s)\| \leq C(|\xi_{n+1} - \xi_n| + |\eta_{n+1} - \eta_n|)(s) + C\epsilon \quad (2.45)$$

Next we have, using (2.41) and remark 2.2 :

$$\begin{aligned} |(\tilde{g}_n \tilde{h}_n)' - (\tilde{g}_{n-1} \tilde{h}_{n-1})'| & = |\tilde{g}'_n \tilde{h}_n - \tilde{g}'_{n-1} \tilde{h}_{n-1} + \tilde{g}_n \tilde{h}'_n - \tilde{g}_{n-1} \tilde{h}'_{n-1}| \\ & \leq C\epsilon + C(|\tilde{g}'_n - \tilde{g}'_{n-1}| + |\tilde{h}'_n - \tilde{h}'_{n-1}|) \end{aligned} \quad (2.46)$$

Now using (2.41) and the fact that $(\tilde{g}_n), (\tilde{h}_n), (\tilde{g}'_n), \tilde{h}'_n$ are bounded, we obtain

$$\begin{aligned} \left| \frac{1}{2}(\tilde{g}_n^2 + \tilde{h}_n^2)' - \frac{1}{2}(\tilde{g}_{n-1}^2 + \tilde{h}_{n-1}^2)' \right| & = |\tilde{g}'_n \tilde{g}_n - \tilde{g}'_{n-1} \tilde{g}_{n-1} + \tilde{h}'_n \tilde{h}_n - \tilde{h}'_{n-1} \tilde{h}_{n-1}| \\ & \leq C(|\tilde{g}'_n - \tilde{g}'_{n-1}| + |\tilde{h}'_n - \tilde{h}'_{n-1}|) + C\epsilon \end{aligned} \quad (2.47)$$

We then deduce, from the expressions of ρ_n, p_n, j_n and using (2.40), (2.45), (2.46), (2.47), (2.41)

$$\|(\rho'_{n+1} - \rho'_n)(s)\|, \| (p'_{n+1} - p'_n)(s) \|, \| (j'_{n+1} - j'_n)(s) \| \leq C\epsilon + C(\gamma_n + \gamma_{n-1})(s) \quad (2.48)$$

Concerning (μ'_n) , we obtain by taking the derivative of (2.4) with respect to r :

$$\mu'_n e^{-2\mu_n} = \frac{\overset{\circ}{\mu}' e^{\overset{\circ}{\mu}}}{t} - \frac{4\pi}{t} \int_t^1 s^2 p'_n(s, r) ds$$

subtracting this relation written for $n+1$ and n , we obtain

$$\mu'_{n+1} - \mu'_n = e^{2\mu_{n+1}} [\mu'_n (e^{-2\mu_n} - e^{-2\mu_{n+1}})] - \frac{4\pi}{t} \int_t^1 s^2 (p'_{n+1} - p'_n)(s, r) ds$$

This gives, using (2.41), (2.48) and the fact that μ_n, μ'_n are bounded :

$$\|(\mu'_{n+1} - \mu'_n)(s)\| \leq C\epsilon + C \int_s^1 (\gamma_{n-1}(\tau) + \gamma_n(\tau)) d\tau \quad (2.49)$$

Concerning (λ'_n) , if we take the derivative of (2.5) with respect to r , we have

$$\dot{\lambda}'_n = (8\pi t \mu'_n \rho_n + 4\pi t \rho'_n) e^{2\mu_n} - k \frac{\mu'_n}{t} e^{2\mu_n} \quad (2.50)$$

We first deduce that $\dot{\lambda}'_n$ is bounded since $\mu'_n, \rho_n, \rho'_n, \mu_n$ are bounded. Next, subtracting (2.50) written for $n+1$ and n , we obtain :

$$\begin{aligned} \dot{\lambda}'_{n+1} - \dot{\lambda}'_n &= e^{2\mu_{n+1}} [(\mu'_{n+1} - \mu'_n)(8\pi t \rho_{n+1} - \frac{k}{t}) + 8\pi t \mu'_n (\rho_{n+1} - \rho_n) \\ &\quad + 4\pi t (\rho'_{n+1} - \rho'_n)] + (e^{2\mu_{n+1}} - e^{2\mu_n}) (4\pi t \rho'_n - \frac{k}{t} \mu'_n + 8\pi t \mu'_n \rho_n) \end{aligned}$$

using (2.40), (2.41), (2.48), and since $(\mu_n), (\rho_n), (\mu'_n)$ are bounded

$$\|(\dot{\lambda}'_{n+1} - \dot{\lambda}'_n)(s)\| \leq C\epsilon + C(\gamma_{n-1}(s) + \gamma_n(s)) \quad (2.51)$$

Now, (2.6) gives :

$$(\lambda'_{n+1} - \lambda'_n)(t, r) = \int_1^t (\dot{\lambda}'_{n+1} - \dot{\lambda}'_n)(s, r) ds$$

which gives, taking the norms, using (2.51) and integrating over $[s, 1]$, with $\lambda_{n+1}(1, r) - \lambda_n(1, r) = 0$:

$$\|(\lambda'_{n+1} - \lambda'_n)(s)\| \leq C\epsilon + C \int_s^1 (\gamma_{n-1}(\tau) + \gamma_n(\tau)) d\tau \quad (2.52)$$

We will also need to bound $\tilde{\mu}'_{n+1} - \tilde{\mu}'_n$. If we take the derivative of (2.7) with respect to r , we obtain, after subtracting the expressions written for $n+1$ and n :

$$\begin{aligned} \tilde{\mu}'_{n+1} - \tilde{\mu}'_n &= (e^{\lambda_{n+1} + \mu_{n+1}} - e^{\lambda_n + \mu_n})(j'_n + j_n) - 4\pi t [(j'_{n+1} - j'_n) \\ &\quad + (\mu'_{n+1} - \mu'_n + \lambda'_{n+1} - \lambda'_n) j_{n+1} + (\mu'_n + \lambda'_n)(j_{n+1} - j_n)] e^{\lambda_{n+1} + \mu_{n+1}} \end{aligned}$$

We then deduce from (2.40), (2.41), (2.48):

$$\|(\tilde{\mu}'_{n+1} - \tilde{\mu}'_n)(s)\| \leq C\epsilon + C(\gamma_{n-1}(s) + \gamma_n(s)) \quad (2.53)$$

B) Estimation of $(\tilde{g}'_n), (\tilde{h}'_n)$. Recall that

$$\begin{aligned} X_n &= \tilde{g}_n - \tilde{h}_n ; Y_n = \tilde{g}_n + \tilde{h}_n ; a_n = (-\dot{\lambda}_n - \frac{1}{t}) e^{-\mu_n} - \tilde{\mu}_n e^{-\lambda_n} ; \\ b_n &= -\frac{e^{-\mu_n}}{t} ; c_n = (-\dot{\lambda}_n - \frac{1}{t}) e^{-\mu_n} + \tilde{\mu}_n e^{-\lambda_n} \end{aligned} \quad (2.54)$$

Set

$$\begin{aligned} C_n &= a_n X_n + b_n Y_n ; & \tilde{C}_n &= (\lambda'_n - \mu'_n) e^{-\lambda_n} X'_{n+1} + \mu'_n C_n + C'_n \\ D_n &= b_n X_n + c_n Y_n ; & \tilde{D}_n &= (\mu'_n - \lambda'_n) e^{-\lambda_n} Y'_{n+1} + \mu'_n D_n + D'_n \end{aligned} \quad (2.55)$$

Then system (2.8)-(2.9) becomes

$$\begin{cases} D_n^+ X_{n+1} = C_n \\ D_n^- Y_{n+1} = D_n \end{cases}$$

If we take the derivative of the above system with respect to r , a direct calculation shows that (see appendix) :

$$D_n^+ X'_{n+1} = \tilde{C}_n \quad (2.56)$$

$$D_n^- Y'_{n+1} = \tilde{D}_n \quad (2.57)$$

We are going to integrate (2.56)-(2.57) on characteristic curves of the wave operator. Consider the following characteristic curves γ_n^1, γ_n^2 of the wave operator, starting from the point (s, r) , i.e for every n ,

$$\dot{\gamma}_n^1 = e^{\mu_n - \lambda_n}, \dot{\gamma}_n^2 = -e^{\mu_n - \lambda_n}, \gamma_n^1(s) = \gamma_n^2(s) = r \quad (2.58)$$

We have $D_n^+ = e^{-\mu_n} \frac{d}{dt}$ on γ_n^1 and $D_n^- = e^{-\mu_n} \frac{d}{dt}$ on γ_n^2 . We then have, integrating (2.56) over γ_n^1 and (2.57) over γ_n^2 :

$$X'_{n+1}(s) = X'_{n+1}(1) - \int_s^1 e^{\mu_n} \tilde{C}_n(\tau, \gamma_n^1(\tau)) d\tau$$

$$Y'_{n+1}(s) = Y'_{n+1}(1) - \int_s^1 e^{\mu_n} \tilde{D}_n(\tau, \gamma_n^2(\tau)) d\tau$$

Now if we subtract respectively each of this two relations written for $n+1$ and n , we obtain :

$$(X'_{n+1} - X'_n)(s) = \int_s^1 \left[e^{\mu_{n-1}} \tilde{C}_{n-1}(\tau, \gamma_{n-1}^1(\tau)) - e^{\mu_n} \tilde{C}_n(\tau, \gamma_n^1(\tau)) \right] d\tau \quad (2.59)$$

$$(Y'_{n+1} - Y'_n)(s) = \int_s^1 \left[e^{\mu_{n-1}} \tilde{D}_{n-1}(\tau, \gamma_{n-1}^2(\tau)) - e^{\mu_n} \tilde{D}_n(\tau, \gamma_n^2(\tau)) \right] d\tau \quad (2.60)$$

Since e^{μ_n} and \tilde{C}_n are bounded, we have

$$\begin{aligned} & |e^{\mu_n} \tilde{C}_n(\tau, \gamma_n^1(\tau)) - e^{\mu_{n-1}} \tilde{C}_{n-1}(\tau, \gamma_{n-1}^1(\tau))| = \{e^{\mu_n}(\tau, \gamma_n^1(\tau)) - e^{\mu_{n-1}}(\tau, \gamma_{n-1}^1(\tau))\} \tilde{C}_n(\tau, \gamma_n^1(\tau)) \\ & + \{\tilde{C}_n(\tau, \gamma_n^1(\tau)) - \tilde{C}_{n-1}(\tau, \gamma_{n-1}^1(\tau))\} e^{\mu_{n-1}}(\tau, \gamma_{n-1}^1(\tau)) + \{\tilde{C}_{n-1}(\tau, \gamma_{n-1}^1(\tau)) - \tilde{C}_{n-1}(\tau, \gamma_{n-1}^1(\tau))\} \\ & e^{\mu_{n-1}}(\tau, \gamma_{n-1}^1(\tau)) + \{e^{\mu_{n-1}}(\tau, \gamma_n^1(\tau)) - e^{\mu_{n-1}}(\tau, \gamma_{n-1}^1(\tau))\} \tilde{C}_{n-1}(\tau, \gamma_{n-1}^1(\tau)) \\ & \leq C \{ \| (e^{\mu_n} - e^{\mu_{n-1}})(\tau) \| + \| (\tilde{C}_n - \tilde{C}_{n-1})(\tau) \| \\ & + |\tilde{C}_{n-1}(\tau, \gamma_n^1(\tau)) - \tilde{C}_{n-1}(\tau, \gamma_{n-1}^1(\tau))| + |e^{\mu_{n-1}}(\tau, \gamma_n^1(\tau)) - e^{\mu_{n-1}}(\tau, \gamma_{n-1}^1(\tau))| \} \end{aligned} \quad (2.61)$$

Now integrating the relation $\dot{\gamma}_n^1 - \dot{\gamma}_{n-1}^1 = e^{\mu_n - \lambda_n} - e^{\mu_{n-1} - \lambda_{n-1}}$ over $[s, \tau]$ yields

$$|\gamma_n^1 - \gamma_{n-1}^1|(\tau) \leq C \sup\{\|(\lambda_n - \lambda_{n-1})(t)\| + \|(\mu_n - \mu_{n-1})(t)\|, T_4 \leq t \leq 1\} \quad (2.62)$$

we then deduce from (2.36) (the right hand side of (2.62) tends to zero as n tends to ∞), the uniform continuity of $(e^{\mu_{n-1}})$, (\tilde{C}_{n-1}) over the compact set $K = [T_4, 1] \times (\gamma_n^1([T_4, 1]) \cup \gamma_{n-1}^1([T_4, 1]))$ i.e

$$\forall \epsilon > 0, \exists \eta(n, \epsilon) > 0 \text{ such that } |\gamma_n^1 - \gamma_{n-1}^1|(\tau) \leq \eta \Rightarrow |\tilde{C}_{n-1}(\tau, \gamma_n^1(\tau)) - \tilde{C}_{n-1}(\tau, \gamma_{n-1}^1(\tau))| + |e^{\mu_{n-1}}(\tau, \gamma_n^1(\tau)) - e^{\mu_{n-1}}(\tau, \gamma_{n-1}^1(\tau))| < \epsilon$$

And from (2.61), (2.59)-(2.60), we deduce that

$$|X'_{n+1} - X'_n|(s) \leq C\epsilon + C \int_s^1 \|(\tilde{C}_n - \tilde{C}_{n-1})(\tau)\| d\tau \quad (2.63)$$

$$|Y'_{n+1} - Y'_n|(s) \leq C\epsilon + C \int_s^1 \|(\tilde{D}_n - \tilde{D}_{n-1})(\tau)\| d\tau \quad (2.64)$$

Therefore, for n sufficiently large, (2.63)- (2.64) implies respectively :

$$-C\epsilon - C \int_s^1 \|(\tilde{C}_n - \tilde{C}_{n-1})(\tau)\| \leq X'_{n+1} - X'_n \leq C\epsilon + C \int_s^1 \|(\tilde{C}_n - \tilde{C}_{n-1})(\tau)\| d\tau$$

$$-C\epsilon - C \int_s^1 \|(\tilde{D}_n - \tilde{D}_{n-1})(\tau)\| \leq Y'_{n+1} - Y'_n \leq C\epsilon + C \int_s^1 \|(\tilde{D}_n - \tilde{D}_{n-1})(\tau)\| d\tau$$

We deduce from (2.54) (definition of X_n and Y_n), by adding and subtracting the previous inequalities, that

$$\begin{aligned} \|(\tilde{g}'_{n+1} - \tilde{g}'_n)(s)\|, \|(\tilde{h}'_{n+1} - \tilde{h}'_n)(s)\| &\leq C\epsilon + C \int_s^1 [\|(\tilde{C}_n - \tilde{C}_{n-1})(\tau)\| \\ &+ \|(\tilde{D}_n - \tilde{D}_{n-1})(\tau)\|] d\tau \end{aligned} \quad (2.65)$$

Now from (2.55), (2.41) and the fact that the sequences $(\dot{\lambda}_n)$, (λ_n) , (μ_n) , $(\dot{\mu}_n)$ are bounded together with their first derivatives, we have (see appendix)

$$\|(\tilde{C}_n - \tilde{C}_{n-1})(\tau)\| \leq C\epsilon + C(\gamma_{n-1}(\tau) + \gamma_n(\tau)) \quad (2.66)$$

and

$$\|(\tilde{D}_n - \tilde{D}_{n-1})(\tau)\| \leq C\epsilon + C(\gamma_{n-1}(\tau) + \gamma_n(\tau)) \quad (2.67)$$

Therefore, we deduce from (2.65), (2.66)-(2.67) that

$$\|(\tilde{g}'_{n+1} - \tilde{g}'_n)(s)\|, \|(\tilde{h}'_{n+1} - \tilde{h}'_n)(s)\| \leq C\epsilon + C \int_s^1 (\gamma_{n-1}(\tau) + \gamma_n(\tau)) d\tau \quad (2.68)$$

C) Convergence of $(\lambda'_n), (\mu'_n), (\tilde{\mu}'_n), (\tilde{g}'_n), (\tilde{h}'_n)$. Add inequalities (2.39), (2.49), (2.52) and (2.68) and take the supremum over $s \in [t, 1]$ to obtain using (2.40)

$$\gamma_n(t) \leq C\epsilon + C \int_t^1 (\gamma_{n-1}(s) + \gamma_n(s)) ds \quad (2.69)$$

Define $\Gamma_n(t) = \sup\{\gamma_m, m \leq n\}$; then (Γ_n) is an increasing sequence and (2.69) gives

$$\Gamma_n(t) \leq C\epsilon + C \int_t^1 \Gamma_n(s) ds$$

And by Gronwall's lemma,

$$\Gamma_n(t) \leq C\epsilon, \quad t \in [T_4, 1], \quad n \text{ sufficiently large.}$$

We then deduce that (Γ_n) converges uniformly to 0, and from (2.40)-(2.53), $(\lambda'_n), (\mu'_n), (\tilde{g}'_n), (\tilde{h}'_n), (\tilde{\mu}'_n)$ converge uniformly on $[T_4, 1]$. We deduce from system (2.8)-(2.9), the uniform convergence of (\tilde{g}_n) and (\tilde{h}_n) . The regularity of $f, \lambda, \mu, \tilde{g}, \tilde{h}$ (and $\tilde{\mu}$) follows from step 1 and step 2. Therefore $g = e^\mu \tilde{g}$ and $h = e^\lambda \tilde{h}$ are regular. Note that, using the convergence of the derivatives, we can prove that the limit $(f, \lambda, \mu, \tilde{\mu}, g, h)$ is a regular solution of (1.14), (1.3), (1.4), (1.15), (1.16), (1.17) and by Proposition 1.14, we conclude the existence of a regular function ϕ such that (f, λ, μ, ϕ) is a solution of the full system (1.2)-(1.11).

To end this theorem, we prove the uniqueness of the solution. Let $u_i = (f_i, \lambda_i, \mu_i, \phi_i)$, $i = 1, 2$ be two regular solutions of the Cauchy problem for the same initial data $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{\phi}, \psi)$ at $t = 1$. Using the fact that u_i is a solution of the system, one proceeds similarly as to prove the convergence of iterates to obtain

$$\alpha(t) \leq C \int_t^1 \alpha(s) ds$$

where

$$\alpha(t) = \sup\{\|f_1(s) - f_2(s)\| + \|\lambda_1(s) - \lambda_2(s)\| + \|\mu_1(s) - \mu_2(s)\| + \|\tilde{g}_1(s) - \tilde{g}_2(s)\| + \|\tilde{h}_1(s) - \tilde{h}_2(s)\|; s \in [t, 1]\},$$

with $\tilde{g}_1 = \dot{\phi}_1 e^{\mu_1}$; $\tilde{g}_2 = \dot{\phi}_2 e^{\mu_2}$; $\tilde{h}_1 = \phi'_1 e^{\lambda_1}$; $\tilde{h}_2 = \phi'_2 e^{\lambda_2}$. We deduce that $\alpha(t) = 0$, for $t \in]0, 1]$. This implies that $f_1 = f_2$, $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$, $\tilde{g}_1 = \tilde{g}_2$ and $\tilde{h}_1 = \tilde{h}_2$. By Proposition 1.14, we conclude that $\phi_1 = \phi_2$. \square

2.3 Continuation criteria

Let us now prove continuation criteria for t decreasing.

Theorem 2.5 *Let $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu}, \overset{\circ}{\phi}, \psi)$ be initial data as in theorem 2.4. Let (f, λ, μ, ϕ) be a solution of (1.2)-(1.11) on a left maximal interval of existence $]T, 1]$ for*

which :

1. $\sup\{|w|(t, r, w, F) \in \text{supp}f\} < \infty$,
2. $\sup\{(e^{-2\mu}\dot{\phi}^2 + e^{-2\lambda}\phi'^2)(t, r); t \in]T, 1]\} < \infty$,
3. μ is bounded.

Then $T = 0$.

If $k \geq 0$ or $\mu \leq 0$ then condition 3 is automatically satisfied.

Proof : Let $(f, \lambda, \mu, \tilde{\mu}, g, h)$ be a left maximal solution of the auxiliary system (1.14), (1.3), (1.4), (1.15), (1.16), (1.17) with existence interval $]T, 1[$. By Proposition 1.14, there exists ϕ such that (f, λ, μ, ϕ) solves (1.2)-(1.11). By assumption

$$Q^* := \sup\{|w|(r, w, F) \in \text{supp}f(t), t \in]T, 1]\} < \infty.$$

We want to show that $T = 0$, so let us assume that $T > 0$ and take $t_1 \in]T, 1[$. We will show that the system has a solution with initial data $(f(t_1), \lambda(t_1), \mu(t_1), \phi(t_1), \dot{\phi}(t_1))$ prescribed at $t = t_1$ which exists on an interval $[t_1 - \delta, t_1]$ with $\delta > 0$ independent of t_1 . By moving t_1 close enough to T this would extend our initial solution beyond T , a contradiction to the initial solution being left maximal. We have proved in Proposition 2.1 that such a solution exists at least on the left maximal existence interval of the solutions (z_1, z_2) of

$$z_1(t) = W(t_1) + K(t_1) + C_1 \int_t^{t_1} \frac{1}{s} (1 + z_1(s))^5 ds$$

$$z_2(t) = C_4 \exp \left[\int_t^{t_1} C_2(s)(C_3 + z_2(s)) ds \right],$$

where

$$W(t_1) := \sup\{|w|(r, w, F) \in \text{supp}f(t_1)\},$$

$$K(t_1) := \sup\{(|\dot{\phi}^2|e^{-2\mu} + |\phi'^2|e^{-2\lambda})^{\frac{1}{2}}(t_1, r) ; r \in \mathbb{R}\},$$

$$A(t_1) := \sup\{e^{-\lambda}[(\dot{\phi}' - \mu'\dot{\phi})^2 e^{-2\mu} + (\phi'' - \lambda'\phi')^2 e^{-2\lambda}]^{1/2}(t_1, r) ; r \in \mathbb{R}\},$$

$$C_1 = \frac{C}{C_0} (1 + F_0)^2 (1 + \|f(t_1)\|) ; C_0 = \inf\{e^{-\mu(t_1, r)} ; r \in \mathbb{R}\}$$

$$C_3 := \|e^{-2\mu(t_1)} \mu'(t_1)\| + \|\lambda'(t_1)\| + 1 ; C_4 := 2 + 2A(t_1) + \|\partial_{(r, w)} f(t_1)\|$$

and C_2 is an increasing function of z_1 . Now $W(t_1) \leq Q^*$, $\|f(t_1)\| = \|\dot{f}\|$, F_0 is unchanged since F is constant along characteristics, and since $t_1 < 1$, taking

$t = t_1$ in (2.4) shows that: $C_0(\mu(t_1)) \geq C_0(\overset{\circ}{\mu})$. Thus there exists a constant $M_1 > 0$ such that

$$\frac{1}{T}C_1(f(t_1), F_0, \mu(t_1)) \leq M_1 \text{ for } t_1 \in]T, 1].$$

The expressions of $\rho, p, j, \dot{\lambda}, \tilde{\mu}$ show since $|w| \leq Q^*$, that

$$|\rho(s, r)|, |p(s, r)|, |j(s, r)|, |\dot{\lambda}(s, r)|, |\tilde{\mu}e^{\mu-\lambda}(s, r)| \leq C + (K(s))^2$$

$K(s)$ is bounded on $]T, 1]$. We proceed as in step 6 of the proof of theorem 3.1 in [15] to prove that $\partial_{(r,w)}f$ is uniformly bounded on $]T, 1]$. Let $\mu' = \tilde{\mu}$; consider the relations

$$\mu' = -4\pi t e^{\lambda+\mu} j; \quad \mu'' = -[(\lambda' + \mu')j + j']4\pi t e^{\lambda+\mu} \quad (2.70)$$

$$\dot{\lambda}' = e^{2\mu}(8\pi t \mu' \rho + \frac{k\mu'}{t} + 4\pi t \rho'); \quad \lambda' = \overset{\circ}{\lambda}' + \int_1^t \dot{\lambda}'(s, r) ds \quad (2.71)$$

We bound ρ', j' by quantities which depend on $\partial_r f$ and $A(s)$. Since $\partial_r f, \rho$ and j are bounded, we deduce from the above relations that μ'', λ' and consequently $v(s)$ and $h(s)$ (see Proposition 1.10) are bounded each by $A(s)$. Using inequality (1.27) and the fact that $K(s)$ is bounded, we obtain

$$A(t) \leq A_0 + C \int_t^1 (A(s) + 1) ds$$

And we deduce that $A(t) \leq A^* = (1 + A_0)e^C; t \in]T, 1]$. Therefore

$$D := \sup\{\|\partial_{(r,w)}f(t)\| + A(t) \mid t \in]T, 1]\} < \infty.$$

From

$$\mu'(t, r) = \frac{e^{2\mu}}{t} \left(\overset{\circ}{\mu}'(r) e^{-2\overset{\circ}{\mu}} + 4\pi \int_1^t p'(s, r) s^2 ds \right),$$

(2.70), (2.71) and since $K(t)$ and $A(t)$ are bounded, p' is bounded and we obtain a uniform bound $C_3(\mu(t_1), \lambda(t_1)) \leq M_3$. Let y_2 be the left maximal solution of

$$y_2(t) = D \exp \left[\int_t^{t_1} C_2^*(s)(M_3 + y_2(s)) ds \right],$$

where C_2^* depends on y_1 in the same way as C_1 depends on z_1 . y_2 exists on an interval $[t_1 - \alpha, t_1]$ with $\alpha > 0$ independent of t_1 . If we choose t_1 such that $T > t_1 - \alpha$ then z_1 is bounded; $z_2 \leq y_2$ by construction. In particular, z_2 exists on $I \subset [t_1 - \alpha, t_1]$.

From (1.37), we deduce for $k \geq 0$ that $e^{-2\mu} \geq \frac{e^{-2\overset{\circ}{\mu}} + k}{t} - k \geq e^{-2\overset{\circ}{\mu}}$ since $\frac{1}{t} \geq 1$.

If $k = -1$ and $\overset{\circ}{\mu} \leq 0$, then $e^{-2\mu} \geq \frac{e^{-2\overset{\circ}{\mu}} - 1}{t} + 1 \geq 1$. Either, μ is bounded above. Then condition 3 holds. This complete the proof of the theorem. \square

Theorem 2.6 Let (f, λ, μ, ϕ) be a solution of (1.2)-(1.11) on a left maximal interval of existence $]T, 1]$, $T > 0$, with initial data as in Theorem 2.5. If

1. $Q(t) = \sup\{|w|(r, w, f) \in \text{supp}f(t), t \in]T, 1]\} < \infty$

2. μ is bounded.

then $T = 0$.

If $k \geq 0$ or $\mu \leq 0$ then condition 2 is automatically satisfied.

Proof: We need to prove that $K(t)$ is bounded for all $t \in]T, 1]$. Unless otherwise specified in what follows constants denoted by C will be positive, may depend on the initial data and may change their value from line to line.

We deduce from system (1.28)-(1.29) :

$$D^+ X_2^2 = 2e^\mu \left[\frac{k}{t} - 4\pi t(\rho - p) \right] X_2^2 - 2 \frac{e^{-\mu}}{t} X_2 Y_2$$

$$D^- Y_2^2 = -2 \frac{e^{-\mu}}{t} X_2 Y_2 + 2e^\mu \left[\frac{k}{t} - 4\pi t(\rho - p) \right] Y_2^2$$

On the corresponding characteristic curves of the wave equation, $D^+ = D^- = e^{-\mu} \frac{d}{dt}$ and then

$$\frac{d}{dt} X_2^2(t, \gamma_1(t)) = 2e^{2\mu} \left[\frac{k}{t} - 4\pi t(\rho - p) \right] X_2^2(t, \gamma_1(t)) - \frac{2}{t} X_2 Y_2(t, \gamma_1(t))$$

$$\frac{d}{dt} Y_2^2(t, \gamma_2(t)) = -\frac{2}{t} X_2 Y_2(t, \gamma_2(t)) + 2e^{2\mu} \left[\frac{k}{t} - 4\pi t(\rho - p) \right] Y_2^2(t, \gamma_2(t))$$

Integrate the first of the two previous equations on $[t, 1]$:

$$\begin{aligned} X_2^2(t, \gamma_1(t)) &= X_2^2(1, \gamma_1(1)) + 2 \int_t^1 \{e^{2\mu} [-\frac{k}{s} + 4\pi s(\rho - p)] X_2^2 + \frac{1}{s} X_2 Y_2\}(s, \gamma_1(s)) ds \\ &\leq X_2^2(1, \gamma_1(1)) + 2 \int_t^1 \{e^{2\mu} [-\frac{k}{s} + 4\pi s(\rho - p)] X_2^2 + \frac{1}{2s} (X_2^2 + Y_2^2)\}(s, \gamma_1(s)) ds \\ &\leq X_2^2(1, \gamma_1(1)) + 2 \int_t^1 \{[\frac{1}{2s} + e^{2\mu} (-\frac{k}{s} + 4\pi s(\rho - p))] X_2^2 + \frac{1}{2s} Y_2^2\}(s, \gamma_1(s)) ds \end{aligned}$$

In a similar way,

$$Y_2^2(t, \gamma_2(t)) \leq Y_2^2(1, \gamma_2(1)) + 2 \int_t^1 \{ \frac{1}{2s} X_2^2 + [\frac{1}{2s} + e^{2\mu} (-\frac{k}{s} + 4\pi s(\rho - p))] Y_2^2 \}(s, \gamma_2(s)) ds$$

Add the two previous inequalities and take the supremum over space :

$$B(t)^2 \leq B(1)^2 + 2 \int_t^1 l(s) B(s)^2 ds \quad (2.72)$$

where

$$B(t) = \sup\{(|X_2|^2 + |Y_2|^2)^{1/2}(t, r) ; r \in \mathbb{R}\}$$

$$l(t) = \sup\{\frac{1}{t} + e^{2\mu} [\frac{|k|}{t} + 4\pi t(\rho - p)](t, r) ; r \in \mathbb{R}\}$$

Now subtract the two equations (1.8)-(1.9) to obtain :

$$\begin{aligned}
\rho - p &= \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \left(\sqrt{1 + w^2 + F/t^2} - \frac{w^2}{\sqrt{1 + w^2 + F/t^2}} \right) f dF dw \\
&= \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{1 + F/t^2}{\sqrt{1 + w^2 + F/t^2}} f dF dw \\
&\leq \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{1 + F/t^2}{\sqrt{1 + F/t^2}} f dF dw \\
&\leq \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1 + F/t^2} f dF dw \\
&\leq \frac{C}{t^3} \quad ; \quad \text{since } Q(t) < \infty
\end{aligned}$$

Using (2.15) for the bound of e^μ , we obtain :

$$l(t) \leq \frac{1}{t} + \frac{|k|}{C_0} + \frac{C}{tC_0}$$

Then (2.72) implies :

$$B(t)^2 \leq B(1)^2 + 2 \int_t^1 \left(C \left(1 + \frac{1}{s} \right) + \frac{1}{s} \right) B(s)^2 ds$$

And by Gronwall's lemma,

$$B(t)^2 \leq B(1)^2 t^{-2} e^{C(1-t)} t^{-C} \leq C t^{-C-2}$$

Now, we have from (1.9),

$$\begin{aligned}
p(s, r) &= \frac{\pi}{s^2} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{w^2}{\sqrt{1 + w^2 + F/s^2}} f(s, r, w, F) dF dw + \frac{1}{2} (X^2 + Y^2)(s, r) \\
&\leq \frac{\pi}{s^2} \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{w^2}{|w|} f(s, r, w, F) dF dw + \frac{1}{2} e^{-2\mu} (X_2^2 + Y_2^2)(s, r) \\
&\leq \frac{C}{s^2} + \frac{1}{2} e^{-2\mu} B(s)^2 \\
&\leq \frac{C}{s^2} + C s^{-C-2} e^{-2\mu}
\end{aligned}$$

Then using (1.37) (where \bar{p} is replaced by p), we obtain the estimate :

$$\begin{aligned}
e^{-2\mu(t,r)} &\leq \frac{e^{-2\mathring{\mu}(r)} + k}{t} - k + \frac{8\pi}{t} \int_t^1 s^2 \left(\frac{C}{s^2} + Cs^{-C-2}e^{-2\mu} \right) ds \\
&\leq \frac{e^{-2\mathring{\mu}(r)} + k}{t} - k + \frac{C}{t} \int_t^1 (1 + s^{-C}e^{-2\mu}) ds \\
&\leq \frac{e^{-2\mathring{\mu}(r)} + |k|}{t} + \frac{C}{t} \int_t^1 (1 + s^{-C}e^{-2\mu}) ds \\
&\leq \frac{e^{-2\mathring{\mu}(r)} + |k|}{t} + \frac{C}{t} \left(1 + \int_t^1 s^{-C}e^{-2\mu} ds \right) \\
&\leq \frac{C}{t} \left(1 + \int_t^1 s^{-C}e^{-2\mu} ds \right).
\end{aligned}$$

By Gronwall's lemma, we deduce that $e^{-2\mu} \leq Ct^{-1} \exp[\frac{C}{1-C}(1-t^{1-C})]$. Therefore,

$$(X^2 + Y^2)(t, r) = e^{-2\mu}(X_2^2 + Y_2^2)(t, r) \leq Ct^{-3-C} \exp[\frac{C}{1-C}(1-t^{1-C})]$$

i.e $K(t)$ is bounded. And we conclude by Theorem 2.5 that $T = 0$. \square

We prove in the next theorem, the analogue of theorems 2.4 and 2.5 for $t \geq 1$.

Theorem 2.7 *Let $(f, \mathring{\lambda}, \mathring{\mu}, \mathring{\phi}, \psi)$ be initial data as in Theorem 2.4. Then there exists a unique, right maximal, regular solution (f, λ, μ, ϕ) of (1.2)-(1.11) with $(f, \lambda, \mu, \phi, \dot{\phi})(1) = (\mathring{f}, \mathring{\lambda}, \mathring{\mu}, \mathring{\phi}, \mathring{\psi})$ on a time interval $[1, T[$ with $T \in]1, \infty]$. If*

$$\sup\{|w| \mid (t, r, w, F) \in \text{supp}f\} < \infty;$$

$$\sup\{e^{2\mu(t,r)} \mid r \in \mathbb{R}, t \in [1, T[\} < \infty$$

and

$$K(t) < \infty$$

then $T = \infty$.

Proof : We give only those parts of the proof which differ from the proof of Theorem 2.4 for $t \leq 1$. The iterates are defined in the same way as before, except that now (2.4) is used to define μ_n only on the interval $[1, T_n[$, where

$$T_n := \sup \left\{ \tau \in]1, T_{n-1}[\mid \frac{e^{-2\mathring{\mu}(r)} + k}{t} - k - \frac{8\pi}{t} \int_1^t s^2 p_n(s, r) ds > 0, r \in \mathbb{R}, t \in [1, \tau] \right\},$$

$[1, T_{n-1}[$ being the existence interval of the previous iterates and $T_0 = \infty$. Define:

$$Q_n(t) := \sup \{|w|, (r, w, F) \in \text{supp}f_n(t)\}, \quad t \in [1, T_n[$$

$$E_n(t) := \sup \left\{ s e^{2\mu_n(s,r)} \mid r \in \mathbb{R}, 1 \leq s \leq t \right\}$$

we obtain the estimates

$$\begin{aligned} \sqrt{1 + w^2 + F/t^2} &\leq \sqrt{1 + (Q_n(t))^2 + F_0} \leq (1 + F_0)(1 + Q_n(t)); \\ \|\rho_n(t)\|, \|p_n(t)\|, \|j_n(t)\| &\leq \frac{C^*}{t} (1 + Q_n(t))^2 + (K_{n-1}(t))^2; \end{aligned} \quad (2.73)$$

and

$$|e^{\mu_n - \lambda_n} \tilde{\mu}_n(t, r)| + |\dot{\lambda}_n(t, r)| \leq C^* (1 + Q_n(t))^2 + (1 + K_{n-1}(t))^2 (1 + E_n(t)).$$

where $C^* = C(1 + F_0)^2(1 + \|\dot{f}\|)$. Thus, we have similarly to (2.16):

$$Q_{n+1}(t) \leq W_0 + C^* \int_1^t (1 + Q_n(s))^2 (1 + E_n(s)) (1 + K_{n-1}(s))^2 (1 + Q_{n+1}(s)) ds. \quad (2.74)$$

and similarly to (2.18):

$$K_{n+1}(t) \leq K_0 + C^* \int_1^t (1 + Q_n(s))^2 (1 + K_{n-1}(s))^2 K_n(s) (1 + E_n(s)) ds. \quad (2.75)$$

We deduce from the field equation (1.4) that

$$(2\dot{\mu}_n e^{2\mu_n})t = e^{2\mu_n} + k e^{4\mu_n} + 8\pi (t e^{2\mu_n})^2 p_n$$

Integrating over $[1, s]$ and using integration by parts for the left hand side yields the following estimate, since (2.73) holds,

$$E_n(t) \leq \|e^{2\dot{\mu}}\| + C^* \int_1^t (1 + Q_n(s))^2 (1 + E_n(s))^2 (1 + K_{n-1}(s))^2 ds. \quad (2.76)$$

Reasoning in the same way as in the proof of Proposition 2.1, we can say the differential inequalities (2.74), (2.75), (2.76) allow us to estimate Q_n , K_n and E_n against the solution z_1 , z_2 and z_3 of the system

$$\begin{aligned} z_1(t) &= W_0 + C^* \int_1^t (1 + z_1(s))^3 (1 + z_2(s))^2 (1 + z_3(s)) ds, \\ z_2(t) &= K_0 + C^* \int_1^t (1 + z_1(s))^2 (1 + z_2(s))^3 (1 + z_3(s)) ds, \\ z_3(t) &= \|e^{2\dot{\mu}}\| + C^* \int_1^t (1 + z_1(s))^2 (1 + z_2(s))^2 (1 + z_3(s))^2 ds, \end{aligned}$$

and in particular $T_n \geq T$ where $[1, T[$ is the right maximal existence interval of (z_1, z_2, z_3) . One can now establish a bound on first order derivatives of the iterates in the same way as in the proof of Proposition 2.1 and obtains a local solution on a right maximal existence interval which is extendible if the quantities $Q(t)$, $E(t) = \|e^{2\mu(t)}\|$ and $K(t)$ can be bounded. \square

Theorem 2.8 Let (f, λ, μ, ϕ) be a right maximal regular solution obtained in Theorem 2.7. Assume that

$$\sup\{|w| \mid (t, r, w, F) \in \text{supp}f\} < \infty;$$

and

$$\sup\{e^{2\mu(t,r)} \mid r \in \mathbb{R}, t \in [1, T[] < C < \infty;$$

then $T = \infty$.

Proof : We deduce from system (1.12)-(1.13):

$$D^+ X^2 = 2aX^2 + 2bXY$$

$$D^- Y^2 = 2bXY + 2cY^2$$

On the respectively characteristic curves of the wave equation, $D^+ = D^- = e^{-\mu} \frac{d}{dt}$ and then we obtain :

$$\frac{d}{dt} X^2(t, \gamma_1(t)) = 2e^\mu (aX^2 + bXY)(t, \gamma_1(t)) \quad (2.77)$$

$$\frac{d}{dt} Y^2(t, \gamma_2(t)) = 2e^\mu (bXY + cY^2)(t, \gamma_2(t)) \quad (2.78)$$

From (2.77), we have:

$$\begin{aligned} \frac{d}{dt} X^2(t, \gamma_1(t)) &= 2(-\dot{\lambda} - \mu' e^{\mu-\lambda} - \frac{1}{t}) X^2 - \frac{2XY}{t} \\ &= 8\pi t e^{2\mu} (j - \rho) X^2 + \frac{1 + k e^{2\mu}}{t} X^2 - \frac{2}{t} X^2 - \frac{2XY}{t} \\ &\leq (\frac{-1}{t} + \frac{k e^{2\mu}}{t}) X^2 + \frac{X^2 + Y^2}{t} \quad \text{since } j - \rho < 0; \end{aligned}$$

If $k = 0$ or -1 then

$$\frac{d}{dt} X^2(t, \gamma_1(t)) \leq \frac{1}{t} Y^2(t, \gamma_1(t)).$$

If $k = 1$ then

$$\frac{d}{dt} X^2(t, \gamma_1(t)) \leq \frac{1}{t} (CX^2 + Y^2)(t, \gamma_1(t)).$$

Since $j + \rho > 0$, we deduce as above, from (2.78), the estimates

$$\frac{d}{dt} Y^2(t, \gamma_2(t)) \leq \frac{1}{t} X^2(t, \gamma_2(t)) \quad \text{for } k = 0 \text{ or } k = -1;$$

$$\frac{d}{dt} Y^2(t, \gamma_2(t)) \leq \frac{1}{t} (X^2 + CY^2)(t, \gamma_2(t)) \quad \text{for } k = 1.$$

After integrating these two inequalities over $[1, t]$ and taking the maximum over space, we obtain :

$$K(t)^2 \leq K(1)^2 + \int_1^t \frac{1}{s} K(s)^2 ds \quad \text{for } k = 0 \text{ or } k = -1;$$

$$K(t)^2 \leq K(1)^2 + \int_1^t \frac{1+C}{s} K(s)^2 ds \quad \text{for } k = 1.$$

We deduce by Gronwall's lemma that:

$$K(t)^2 \leq K(1)^2 t, \quad \text{for } k = 0 \text{ or } k = -1 \text{ and } t \in [1, T[\quad (2.79)$$

$$K(t)^2 \leq K(1)^2 t^{1+C}, \quad \text{for } k = 1 \text{ and } t \in [1, T[\quad (2.80)$$

And we conclude by Theorem 2.7 that $T = \infty$. \square

Chapter 3

Global existence and asymptotic behaviour of solutions in the past

3.1 Global existence in the past

We prove that the solutions obtained in Theorem 2.4 exist on the whole interval $]0, 1[$.

Theorem 3.1 *Consider a solution of the Einstein-Vlasov system with $k \geq 0$ and initial data given for $t = 1$. Then this solution exists on the whole interval $]0, 1[$. If $k < 0$ and $\dot{\mu} \leq 0$, the same result holds.*

Proof : we follow the works of M. Weaver [32] and S.B. Tchapnda [29]. The strategy of the proof is the following : suppose we have a solution on an interval $]T, 1[$ with $T > 0$. We want to show that the solution can be extended to the past. By consideration of the maximal interval of existence this will prove the assertion.

Firstly let us prove that under the hypotheses of the theorem, μ is bounded above.

From the field equation (1.4) we have for $k \geq 0$,

$$\frac{d}{dt}(te^{-2\mu}) = -k - 8\pi t^2 p \leq 0. \quad (3.1)$$

So $te^{-2\mu}$ cannot increase towards the future, i.e. it cannot decrease towards the past. Thus on $]T, 1[$, $te^{-2\mu}$ must remain bounded away from zero and hence μ is bounded above.

For the case $k = -1$, since $p(s, r) \geq 0$, we get from (1.37), $e^{-2\mu} \geq \frac{e^{-2\dot{\mu}} - 1}{t} + 1 \geq 1$ which gives the upper bound of μ for $\dot{\mu} \leq 0$.

Now, let us prove that w is bounded.
 Consider the following rescaled version of w , called u_1 , which has been inspired by the works of [32] (p. 1090) and [30] (p. 5):

$$u_1 = \frac{e^\mu}{2t}w.$$

If we prove that μ is bounded below then the boundedness of u_1 will imply the boundedness of w . So let us show that μ is bounded below under the assumption that u_1 is bounded.

We have

$$\frac{d}{dt}(te^{-2\mu}) = -k - 8\pi t^2 p. \quad (3.2)$$

Transforming the integral term defining p to u_1 as an integration variable instead of w yields

$$p = \int_{-\infty}^{\infty} \int_0^{\infty} \frac{8\pi t e^{-3\mu} u_1^2}{\sqrt{1 + 4t^2 e^{-2\mu} u_1^2 + F/t^2}} f dF du_1 + \frac{1}{4}(X^2 + Y^2);$$

where X and Y are defined in lemma 1.5. The integrand term in p can then be estimated by $4\pi e^{-2\mu}|u_1|$. We have

$$\begin{aligned} e^{2\mu}(\rho - p) &= \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{1 + F/t^2}{\sqrt{1 + w^2 + F/t^2}} e^{2\mu} f dF dw \\ &= \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{(1 + F/t^2)2te^\mu}{\sqrt{1 + 4t^2 e^{-2\mu} u_1^2 + F/t^2}} f dF du_1 \\ &\leq \frac{2\pi}{t} \int_{-\infty}^{\infty} \int_0^{\infty} (1 + F/t^2) e^\mu f dF du_1 \\ &\leq \frac{2\pi}{t^3} e^\mu \int_{-\bar{u}_1}^{\bar{u}_1} \int_0^{F_0} (1 + F) f dF du_1 \\ &\leq Ct^{-3} \bar{u}_1 e^\mu \end{aligned}$$

where \bar{u}_1 is the maximum modulus of u_1 on the support of f at a given time. We can then estimate from (2.72) $l(s)$ by $h(s) = C \sup\{1 + s^{-1} + s^{-2}e^\mu \bar{u}_1(s, r); r \in \mathbb{R}\}$ and $B(t)^2$ by $B(1)^2 \exp(\int_t^1 h(s) ds)$. Thus

$$X^2 + Y^2 \leq e^{-2\mu} B(t)^2 \leq e^{-2\mu} B(1)^2 \exp\left(\int_t^1 h(s) ds\right).$$

Therefore, using the bound for μ and u_1 , p can be estimated by $Ce^{-2\mu}$ and so (3.2) implies that

$$\left| \frac{d}{dt}(te^{-2\mu}) \right| \leq C(1 + te^{-2\mu}),$$

integrating this with respect to t over $[t, 1]$ yields

$$te^{-2\mu}(t, r) \leq e^{-2\mu(1, r)} + \int_t^1 C \left(1 + se^{-2\mu(s, r)}\right) ds,$$

which implies by the Gronwall inequality that $te^{-2\mu}$ is bounded on $]T, 1]$; that is μ is bounded below on the given time interval.

The next step is to prove that u_1 is bounded. To this end, it suffices to get a suitable integral inequality for \bar{u}_1 . Since $u_1 = u_1(t, r(t))$, we can compute \dot{u}_1 :

$$\dot{u}_1 = -\frac{e^\mu}{2t^2}w + \frac{e^\mu}{2t}w(\dot{\mu} + \dot{r}\mu') + \frac{e^\mu}{2t}\dot{w}$$

i.e.

$$\dot{u}_1 = \left(\dot{\mu} + \dot{r}\mu' - \frac{1}{t} \right) u_1 + \frac{e^\mu}{2t}\dot{w} \quad (3.3)$$

We have

$$\mu' = -4\pi te^{\mu+\lambda}j, \quad \dot{r} = \frac{e^{\mu-\lambda}w}{\sqrt{1+w^2+F/t^2}}$$

and

$$\dot{w} = 4\pi te^{2\mu}(j\sqrt{1+w^2+F/t^2} - \rho w) + \frac{1+ke^{2\mu}}{2t}w$$

so that (3.3) implies the following :

$$\dot{u}_1 = e^{2\mu} \left[-4\pi t(\rho - p) + \frac{k}{t} \right] u_1 + 2\pi e^{3\mu} j \frac{1 + F/t^2}{\sqrt{1 + 4t^2 e^{-2\mu} u_1^2 + F/t^2}}.$$

i.e

$$\dot{u}_1 |u_1| = e^{2\mu} \left[-4\pi t(\rho - p) + \frac{k}{t} \right] u_1 |u_1| + 2\pi e^{3\mu} j \frac{(1 + F/t^2)|u_1|}{\sqrt{1 + 4t^2 e^{-2\mu} u_1^2 + F/t^2}} \quad (3.4)$$

In order to estimate the modulus of the first term on the right hand side of equation (3.4), we need the estimate of $e^{2\mu}(\rho - p)\bar{u}_1^2$. For convenience let \log_+ be defined by $\log_+(x) = \log x$ when $\log x$ is positive and $\log_+(x) = 0$, otherwise. Then estimating the integral defining $\rho - p$ shows that

$$\rho - p \leq C(1 + \log_+(\bar{w})),$$

i.e.

$$\rho - p \leq C(1 + \log_+(\bar{u}_1) - \mu).$$

The expression $-\mu$ is not under control ; however the expression we wish to estimate contains a factor $e^{2\mu}$. The function $\mu \mapsto -\mu e^{2\mu}$ has an absolute maximum at $-1/2$ which is $(1/2)e^{-1}$. Thus the first term on the right hand side of equation (3.4) can be estimated by $C\bar{u}_1^2(1 + \log_+(\bar{u}_1))$.

Next the second term on the right hand side of equation (3.4) will be estimated.

By definition

$$j = \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} w f(t, r, w, F) dF dw - \dot{\phi}\phi' e^{-\mu-\lambda} = j_1 + j_2$$

The first term of j can be estimated by $C\bar{w}^2$, i.e.

$$|j_1| \leq C\bar{u}_1^2 e^{-2\mu}$$

and the second term $|j_2| \leq \frac{1}{2}e^{-2\mu}B(t)^2$; so that it suffices to estimate the quantity

$$\frac{(\bar{u}_1^2 + B(t)^2)(1 + F/t^2)}{\sqrt{1 + 4t^2 e^{-2\mu}u_1^2 + F/t^2}}|u_1| \quad (3.5)$$

in order to estimate the second term on the right hand side of equation (3.4). But since μ and t^{-1} are bounded on the interval being considered, the quantity (3.5) can be estimated by $C(\bar{u}_1^2 + B(t)^2)$. Thus adding the estimates for the first and second terms on the right hand side of (3.4) allows us to deduce from (3.4) that

$$|\dot{u}_1||u_1| \leq C\bar{u}_1^2(1 + \log_+(\bar{u}_1)) + C(\bar{u}_1^2 + B(t)^2)$$

i.e

$$\left| \frac{d}{dt}|u_1|^2 \right| \leq C(\bar{u}_1)^2(1 + \log_+(\bar{u}_1^2)) + CB(t)^2$$

Integrating over $[t, 1]$ gives :

$$\bar{u}_1^2(t) \leq \bar{u}_1^2(1) + C \int_t^1 [\bar{u}_1^2(s)(1 + \log_+(\bar{u}_1^2(s)) + B(s)^2] ds \quad (3.6)$$

We deduce from the estimate of $\rho - p$ and from inequality (2.72), that $l(s)$ can be estimated by $C(1 + \log_+(\bar{u}_1))$. We then obtain

$$B(t)^2 \leq B(1)^2 + C \int_t^1 (1 + \log_+(\bar{u}_1^2(s))B(s)^2) ds \quad (3.7)$$

Adding (3.6) and (3.7) gives estimate :

$$\bar{u}_1^2(t) + B(t)^2 \leq \bar{u}_1^2(1) + B(1)^2 + C \int_t^1 (1 + \bar{u}_1^2 + B(s)^2) [1 + \log_+(1 + \bar{u}_1^2(s) + B(s)^2)] ds \quad (3.8)$$

Set $v(t) = \bar{u}_1^2(t) + B(t)^2$, then (3.8) can be written

$$v(t) \leq v(1) + C \int_t^1 (1 + v(s)) [1 + \log_+(1 + v(s))] ds \quad (3.9)$$

By the comparison principle for solutions of integral equations, it is enough to show that the solution of the integral equation

$$a(t) = a(1) + C \int_t^1 (1 + a(s)) (1 + \log_+(1 + a(s))) ds$$

is bounded. The solution $a(t)$ is a non-increasing function. Thus either $a(t) \leq e$ everywhere, in which case the desire conclusion is immediate. Or there is some

T_1 in $]T, 1]$ such that $e \leq a(t)$ on $]T, T_1]$. We take T_1 maximal with that property. Then it follows on $]T, T_1]$ that inequality

$$a(t) \leq C \left(1 + \int_t^{T_1} a(s)(1 + \log a(s))ds \right)$$

holds for a constant C . The boundedness of $a(t)$ follows from that of the solution of the differential equation $\dot{x} = Cx(1 + \log x)$ which is $\exp(\exp(Ct)-1)$. In either case $a(t)$ is bounded. Thus \bar{u}_1^2 and $B(t)^2$ are bounded i.e w and $K(t)$ are bounded. The proof of the theorem is complete using theorem 2.6. \square

3.2 On past asymptotic behaviour

In this section we examine the behaviour of solutions as $t \rightarrow 0$.

Firstly we follow the work of Ringstrom [24](P. S310-S311) to bound the quantity $|\phi'|e^{\mu-\lambda}$ by $C|t \log t|^{-1}$, where C is a positive constant.

Lemma 3.2 *Let $A_1 = \frac{1}{8}(-\dot{\phi} + \frac{\phi}{t \log t} + \phi'e^{\mu-\lambda})^2$ and $A_2 = \frac{1}{8}(-\dot{\phi} + \frac{\phi}{t \log t} - \phi'e^{\mu-\lambda})^2$ with $t \in]0, 1[$. If ϕ satisfies the wave equation, then*

$$\begin{aligned} (\partial_t + e^{-\mu-\lambda}\partial_r)A_1 &= -\frac{1}{4t}\left(1 + \frac{1}{\log t}\right)\left[\left(-\dot{\phi} + \frac{\phi}{t \log t}\right)^2 + \phi'^2 e^{2\mu-2\lambda}\right] \\ &+ \frac{1}{2t}\left(1 + \frac{1}{\log t}\right)\phi'^2 e^{2\mu-2\lambda} + \frac{1}{4}\left(\dot{\lambda} - \dot{\mu} + \frac{1}{t}\right)\left(\dot{\phi} - \phi'e^{\mu-\lambda}\right)\left(-\dot{\phi} + \frac{\phi}{t \log t} + \phi'e^{\mu-\lambda}\right) \end{aligned} \quad (3.10)$$

$$\begin{aligned} (\partial_t + e^{-\mu-\lambda}\partial_r)A_2 &= -\frac{1}{4t}\left(1 + \frac{1}{\log t}\right)\left[\left(-\dot{\phi} + \frac{\phi}{t \log t}\right)^2 + \phi'^2 e^{2\mu-2\lambda}\right] \\ &+ \frac{1}{2t}\left(1 + \frac{1}{\log t}\right)\phi'^2 e^{2\mu-2\lambda} + \frac{1}{4}\left(\dot{\lambda} - \dot{\mu} + \frac{1}{t}\right)\left(\dot{\phi} + \phi'e^{\mu-\lambda}\right)\left(-\dot{\phi} + \frac{\phi}{t \log t} - \phi'e^{\mu-\lambda}\right) \end{aligned} \quad (3.11)$$

Proof : We have,

$$\begin{aligned} 8(\partial_t + e^{-\mu-\lambda}\partial_r)A_1 &= 2\left[-\ddot{\phi} + \frac{\dot{\phi}}{t \log t} - \frac{\phi(1 + \log t)}{(t \log t)^2} + (\dot{\mu} - \dot{\lambda})\phi'e^{\mu-\lambda} + \dot{\phi}'e^{\mu-\lambda}\right. \\ &+ \left. e^{\mu-\lambda}\left(-\dot{\phi}' + \frac{\phi'}{t \log t} + (\mu' - \lambda')\phi'e^{\mu-\lambda} + \phi''e^{\mu-\lambda}\right)\right]\left(-\dot{\phi} + \frac{\phi}{t \log t} + \phi'e^{\mu-\lambda}\right) \\ &= 2\left[-\ddot{\phi} + \phi''e^{2\mu-2\lambda} + \frac{\dot{\phi}}{t \log t} - \frac{\phi(1 + \log t)}{(t \log t)^2} + (\dot{\mu} - \dot{\lambda})\phi'e^{\mu-\lambda}\right. \\ &+ \left. \frac{\phi'e^{\mu-\lambda}}{t \log t} + (\mu' - \lambda')\phi'e^{2\mu-2\lambda}\right]\left(-\dot{\phi} + \frac{\phi}{t \log t} + \phi'e^{\mu-\lambda}\right) \end{aligned}$$

i.e

$$\begin{aligned}
8(\partial_t + e^{-\mu-\lambda}\partial_r)A_1 &= 2[(\dot{\lambda} - \dot{\mu} + \frac{1}{t})\dot{\phi} + (\mu' - \lambda')\phi'e^{2\mu-2\lambda} + \frac{\dot{\phi}}{t \log t} - \frac{\phi(1 + \log t)}{(t \log t)^2}] \\
&+ (\dot{\mu} - \dot{\lambda})\phi'e^{\mu-\lambda} + \frac{\phi'e^{\mu-\lambda}}{t \log t} + (\mu' - \lambda')\phi'e^{2\mu-2\lambda}](-\dot{\phi} + \frac{\phi}{t \log t} + \phi'e^{\mu-\lambda}) \\
&= 2[(\dot{\lambda} - \dot{\mu} + \frac{1}{t})\dot{\phi} + \frac{\dot{\phi}}{t \log t} - \frac{\phi}{(t \log t)^2} - \frac{\phi}{t^2 \log t} \\
&+ (\dot{\mu} - \dot{\lambda})\phi'e^{\mu-\lambda} + \frac{\phi'e^{\mu-\lambda}}{t \log t}](-\dot{\phi} + \frac{\phi}{t \log t} + \phi'e^{\mu-\lambda}) \\
&= 2[(\dot{\lambda} - \dot{\mu} + \frac{1}{t})(\dot{\phi} - \phi'e^{\mu-\lambda}) + \frac{1}{t}\phi'e^{\mu-\lambda} + \frac{1}{t \log t}\phi'e^{\mu-\lambda} \\
&+ \frac{1}{t \log t}(\dot{\phi} - \frac{\phi}{t \log t}) + \frac{1}{t}(\dot{\phi} - \frac{\phi}{t \log t})](-\dot{\phi} + \frac{\phi}{t \log t} + \phi'e^{\mu-\lambda}) \\
&= 2(-\dot{\phi} + \frac{\phi}{t \log t} + \phi'e^{\mu-\lambda})(-\frac{1}{t \log t} - \frac{1}{t})(-\dot{\phi} + \frac{\phi}{t \log t} - \phi'e^{\mu-\lambda}) \\
&+ 2(\dot{\lambda} - \dot{\mu} + \frac{1}{t})(\dot{\phi} - \phi'e^{\mu-\lambda})(-\dot{\phi} + \frac{\phi}{t \log t} + \phi'e^{\mu-\lambda}) \\
&= -\frac{2}{t}(1 + \frac{1}{\log t}) \left[(-\dot{\phi} + \frac{\phi}{t \log t})^2 - \phi'^2 e^{2\mu-2\lambda} \right] \\
&+ 2(\dot{\lambda} - \dot{\mu} + \frac{1}{t})(\dot{\phi} - \phi'e^{\mu-\lambda})(-\dot{\phi} + \frac{\phi}{t \log t} + \phi'e^{\mu-\lambda}) \\
&= -\frac{2}{t}(1 + \frac{1}{\log t}) \left[(-\dot{\phi} + \frac{\phi}{t \log t})^2 + \phi'^2 e^{2\mu-2\lambda} \right] + \frac{4}{t}(1 + \frac{1}{\log t})\phi'^2 e^{2\mu-2\lambda} \\
&+ 2(\dot{\lambda} - \dot{\mu} + \frac{1}{t})(\dot{\phi} - \phi'e^{\mu-\lambda})(-\dot{\phi} + \frac{\phi}{t \log t} + \phi'e^{\mu-\lambda});
\end{aligned}$$

and (3.10) follows. If we replace r by $-r$, the wave equation is invariant, $\partial_t + e^{-\mu-\lambda}\partial_r$ and $\partial_t - e^{-\mu-\lambda}\partial_r$, A_1 and A_2 interchange; and we can write (3.11). \square

Lemma 3.3 *Let (f, λ, μ, ϕ) be a left maximal solution of the Einstein-Vlasov-scalar field system on the interval $]T, 1]$, $0 \leq T < e^{-1}$. Assume that*

$$Q(t) = \sup\{|w| \mid (r, w, F) \in \text{supp} f(t)\} \leq Ct^\alpha$$

for some positive constants C, α and for some $t \in]T, e^{-1}]$. Then

$$(-\dot{\phi} + \frac{\phi}{t \log t})^2 + \phi'^2 e^{2\mu-2\lambda} \leq C(t \log t)^{-2} \quad (3.12)$$

Proof : Consider the two characteristic curves $(t, \gamma_1(t))$ and $(t, \gamma_2(t))$ of the wave operator. Since $t \in]0, e^{-1}]$, the term $(1 + \frac{1}{\log t})\phi'^2 e^{2\mu-2\lambda}$ is nonnegative

and $(-\dot{\phi} + \frac{\phi}{t \log t})^2 + \phi'^2 e^{2\mu-2\lambda} = 4(A_1 + A_2)$, then from (3.10) :

$$\begin{aligned} (\partial_t + e^{-\mu-\lambda} \partial_r) A_1(t, \gamma_1(t)) &\geq -\frac{1}{t} \left(1 + \frac{1}{\log t}\right) (A_1 + A_2)(t, \gamma_1(t)) \\ &\quad - \frac{1}{4} \left(\dot{\lambda} - \dot{\mu} + \frac{1}{t}\right) (-\dot{\phi} + \phi' e^{\mu-\lambda}) \left(-\dot{\phi} + \frac{\phi}{t \log t} + \phi' e^{\mu-\lambda}\right)(t, \gamma_1(t)) \end{aligned} \quad (3.13)$$

Similarly, we deduce from (3.11) that :

$$\begin{aligned} (\partial_t - e^{-\mu-\lambda} \partial_r) A_2(t, \gamma_2(t)) &\geq -\frac{1}{t} \left(1 + \frac{1}{\log t}\right) (A_1 + A_2)(t, \gamma_2(t)) \\ &\quad - \frac{1}{4} \left(\dot{\lambda} - \dot{\mu} + \frac{1}{t}\right) (-\dot{\phi} - \phi' e^{\mu-\lambda}) \left(-\dot{\phi} + \frac{\phi}{t \log t} - \phi' e^{\mu-\lambda}\right)(t, \gamma_2(t)) \end{aligned} \quad (3.14)$$

Since $Q(t) \leq Ct^\alpha$, we can bound $\rho - p$ by $Ct^{-3+\alpha}$ (see the proof of Theorem 2.6). $e^{2\mu} \leq Ct$; then

$(\dot{\lambda} - \dot{\mu})(t) + \frac{1}{t} = -\frac{k e^{2\mu}}{t} + 4\pi t e^{2\mu}(\rho - p) \leq C(1 + t^{-1+\alpha})$; and from (2.72), $l(s)$ can be bounded by $s^{-1} + C + Cs^{-1+\alpha}$. We deduce from (2.72) (consider the integral term in the interval $[t, e^{-1}]$) that

$B(t)^2 \leq B(e^{-1})^2 \exp[2 \int_t^{e^{-1}} (s^{-1} + C + Cs^{-1+\alpha}) ds]$ i.e. $B(t)^2 \leq Ct^{-2}$. Therefore $|\dot{\phi}(t)|$ and $|\phi'|e^{\mu-\lambda}(t)$ are bounded each by Ct^{-1} . We can then have a lower bound of the second term of the right hand side of each inequality (3.13) and (3.14) which is $-C(t^{-2} + t^{-3+\alpha})$. Then

$$(\partial_t + e^{-\mu-\lambda} \partial_r) A_1(t, \gamma_1(t)) \geq -\frac{1}{t} \left(1 + \frac{1}{\log t}\right) (A_1 + A_2)(t, \gamma_1(t)) - C(t^{-2} + t^{-3+\alpha})$$

and

$$(\partial_t + e^{-\mu-\lambda} \partial_r) A_2(t, \gamma_2(t)) \geq -\frac{1}{t} \left(1 + \frac{1}{\log t}\right) (A_1 + A_2)(t, \gamma_2(t)) - C(t^{-2} + t^{-3+\alpha})$$

On the corresponding characteristic, we have $\partial_t + e^{-\mu-\lambda} \partial_r = \partial_t - e^{-\mu-\lambda} \partial_r = \frac{d}{dt}$. Take the supremum in the space of each of the above two inequalities and add them. Then

$$\frac{d}{dt} (A_1 + A_2)(t, r) \geq -\frac{2}{t} \left(1 + \frac{1}{\log t}\right) (A_1 + A_2)(t, r) - C(t^{-2} + t^{-3+\alpha})$$

Set $u(t) = (A_1 + A_2)(t)$ and $v(t) = (t \log t)^2 (A_1 + A_2)(t)$. If $v(t)$ is bounded, then we conclude that $u(t)$ is bounded by $C(t \log t)^{-2}$. Let us prove that $v(t)$ is bounded. We have :

$$\begin{aligned} \frac{dv}{dt} &= 2t(\log t)u + 2t(\log t)^2 u + (t \log t)^2 \frac{du}{dt} \\ &= (t \log t)^2 \left(\frac{du}{dt} + \frac{2}{t} \left(1 + \frac{1}{\log t}\right) u \right) \\ &\geq (t \log t)^2 (-Ct^{-2} - Ct^{-3+\alpha}) = -C(\log t)^2 (1 + t^{-1+\alpha}) \end{aligned}$$

Then, $v(t) \leq v(e^{-1}) + C \int_t^{e^{-1}} (1 + s^{-1+\alpha})(\log s)^2 ds \leq v(e^{-1}) + C$. We obtain the desired conclusion of the lemma. \square

Remark 3.4 In the case $f = 0$, we obtain from previous theorem and theorem 2.5, the global existence of solutions and the above estimates hold on the whole interval $]0, e^{-1}]$.

Remark 3.5 We have from the field equation (1.4),

$$e^{-2\mu} = \frac{e^{-2\overset{\circ}{\mu}} + k}{t} - k + \frac{8\pi}{t} \int_t^1 s^2 p(s, r) ds \geq \frac{e^{-2\overset{\circ}{\mu}} + k}{t} - k$$

and then, for $k = 0$, $e^{-2\mu} \geq \frac{e^{-2\overset{\circ}{\mu}}}{t} \geq \frac{C_0}{t}$;
for $k = 1$, $e^{-2\mu} \geq \frac{e^{-2\overset{\circ}{\mu}} + 1}{t} - 1 \geq \frac{e^{-2\overset{\circ}{\mu}}}{t} \geq \frac{C_0}{t}$;
for $k = -1$, $e^{-2\mu} \geq \frac{e^{-2\overset{\circ}{\mu}} - 1}{t} + 1 \geq \frac{C_0}{t} + 1$;
where

$$C_0 = \begin{cases} \inf e^{-2\overset{\circ}{\mu}} & \text{for } k = 0 \text{ or } 1 \\ \inf e^{-2\overset{\circ}{\mu}} - 1 & \text{for } k = -1 \end{cases}$$

Thus
 $e^{2\mu} \leq \frac{t}{C_0}$ for $k = 0$ or $k = 1$;
and $e^{2\mu} \leq \frac{t}{C_0 + t}$ for $k = -1$.

Remark 3.6 Let us prove that $\inf e^{-2\overset{\circ}{\mu}(r)} = \frac{1}{\|e^{2\overset{\circ}{\mu}}\|}$.

For all $r \in \mathbb{R}$, $e^{2\overset{\circ}{\mu}(r)} \leq \|e^{2\overset{\circ}{\mu}}\|$, hence $e^{-2\overset{\circ}{\mu}(r)} \geq \frac{1}{\|e^{2\overset{\circ}{\mu}}\|}$; and

$$\inf e^{-2\overset{\circ}{\mu}(r)} \geq \frac{1}{\|e^{2\overset{\circ}{\mu}}\|}.$$

For all $r \in \mathbb{R}$, $e^{-2\overset{\circ}{\mu}(r)} \geq \inf e^{-2\overset{\circ}{\mu}(r)}$, i.e. $e^{2\overset{\circ}{\mu}(r)} \leq \frac{1}{\inf e^{-2\overset{\circ}{\mu}(r)}}$ and

$$\|e^{2\overset{\circ}{\mu}}\| \leq \frac{1}{\inf e^{-2\overset{\circ}{\mu}(r)}} \quad \text{i.e.} \quad \inf e^{-2\overset{\circ}{\mu}(r)} \leq \frac{1}{\|e^{2\overset{\circ}{\mu}}\|}. \quad \text{This completes the proof.}$$

First we analyze the curvature invariant $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ called the Kretschman scalar in order to prove that there is a spacetime singularity. Thus the spacetime cannot be extended further.

Theorem 3.7 Let (f, λ, μ, ϕ) be a regular solution of the surface-symmetric Einstein-Vlasov-scalar field system on the interval $]0, 1]$ with data given for $t = 1$. Then

$$(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})(t, r) \geq \frac{4}{t^6} \left(\inf e^{-2\overset{\circ}{\mu}} + k \right)^2,$$

with $r \in \mathbb{R}$.

Proof We can compute the Kretschman scalar (see [15]) and obtain

$$\begin{aligned} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} &= 4[e^{-2\lambda}(\mu'' + \mu'(\mu' - \lambda')) - e^{-2\mu}(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu}))]^2 \\ &\quad + \frac{8}{t^2}[e^{-4\mu}\dot{\lambda}^2 + e^{-4\mu}\dot{\mu}^2 - 2e^{-2(\lambda+\mu)}(\mu')^2] \\ &\quad + \frac{4}{t^4}(e^{-2\mu} + k)^2 \\ &=: K_1 + K_2 + K_3 \end{aligned}$$

The first term K_1 is nonnegative and can be dropped.
 Inserting the expressions

$$e^{-2\mu}\dot{\lambda} = 4\pi t\rho - \frac{k + e^{-2\mu}}{2t} ; \quad e^{-2\mu}\dot{\mu} = 4\pi tp + \frac{k + e^{-2\mu}}{2t} ; \quad e^{-\lambda-\mu}\mu' = -4\pi tj$$

into the formula for K_2 yields

$$\begin{aligned} K_2 &= \frac{8}{t^2} \left[\left(4\pi t\rho - \frac{k + e^{-2\mu}}{2t}\right)^2 + \left(4\pi tp + \frac{k + e^{-2\mu}}{2t}\right)^2 - 2(-4\pi tj)^2 \right] \\ &= \frac{8}{t^2} \left[16\pi^2 t^2 (\rho^2 + p^2 - 2j^2) - 4\pi t(\rho - p) \frac{k + e^{-2\mu}}{t} + \frac{(k + e^{-2\mu})^2}{2t^2} \right]. \end{aligned}$$

Now

$$\begin{aligned} |j(t, r)| &\leq \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} |w| f dF dw + |\dot{\phi}| e^{-\mu} |\phi'| e^{-\lambda} \\ &\leq \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} (1 + w^2 + F/t^2)^{1/4} f^{1/2} \frac{|w|}{(1 + w^2 + F/t^2)^{1/4}} f^{1/2} dF dw \\ &\quad + \frac{1}{2} (\dot{\phi}^2 e^{-2\mu} + \phi'^2 e^{-2\lambda}) \\ &\leq \frac{\pi}{t^2} \left[\int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1 + w^2 + F/t^2} f dF dw \right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{\sqrt{1 + w^2 + F/t^2}} f dF dw \right]^{\frac{1}{2}} \\ &\quad + \frac{1}{2} (\dot{\phi}^2 e^{-2\mu} + \phi'^2 e^{-2\lambda}) \quad \text{by the Cauchy-Schwarz inequality.} \\ &\leq \frac{\pi}{2t^2} \left[\int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1 + w^2 + F/t^2} f dF dw + \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{\sqrt{1 + w^2 + F/t^2}} f dF dw \right] \\ &\quad + \frac{1}{2} (\dot{\phi}^2 e^{-2\mu} + \phi'^2 e^{-2\lambda}) \\ &\leq \frac{1}{2} (\rho + p)(t, r). \end{aligned}$$

In fact the above inequality holds in general for all choice of matter satisfying the dominant energy condition. Therefore

$$\rho^2 + p^2 - 2j^2 \geq \rho^2 + p^2 - \frac{1}{2}(\rho + p)^2 = \frac{1}{2}\rho^2 + \frac{1}{2}p^2 - \rho p = \frac{1}{2}(\rho - p)^2$$

and

$$\begin{aligned} K_2 &\geq \frac{8}{t^2} \left[8\pi t^2 (\rho - p)^2 - 4\pi t(\rho - p) \frac{k + e^{-2\mu}}{t} + \frac{(k + e^{-2\mu})^2}{2t^2} \right] \\ &\geq \frac{4}{t^2} \left[4\pi t(\rho - p) - \frac{k + e^{-2\mu}}{t} \right]^2 \geq 0. \end{aligned}$$

Recalling the expression for $e^{-2\mu}$ we get

$$\begin{aligned} e^{-2\mu} + k &= \frac{e^{-2\hat{\mu}(r)} + k}{t} + \frac{8\pi}{t} \int_t^1 s^2 p(s, r) ds \\ &\geq \frac{e^{-2\hat{\mu}} + k}{t} \geq \frac{\inf e^{-2\hat{\mu}} + k}{t} \end{aligned}$$

thus

$$K_3 = \frac{4}{t^4} (e^{-2\mu} + k)^2 \geq \frac{4}{t^6} \left(\inf e^{-2\hat{\mu}} + k \right)^2$$

and so we deduce from (3.15) that

$$(R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta})(t, r) \geq \frac{4}{t^6} \left(\inf e^{-2\hat{\mu}(r)} + k \right)^2, \quad t \in]0, 1], \quad r \in \mathbb{R},$$

and the proof is complete. \square

Remark 3.8 We deduce from the above inequality that

$$\lim_{t \rightarrow 0} (R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta})(t, r) = \infty,$$

uniformly in $r \in \mathbb{R}$.

Next we prove that the singularity at $t = 0$ is a crushing singularity i.e. the mean curvature of the surfaces of constant t blows up. In the case where there is only a scalar field and no Vlasov contribution this singularity is velocity dominated i.e the generalized Kasner exponents have limits as $t \rightarrow 0$.

Theorem 3.9 Let (f, λ, μ, ϕ) be a regular solution of the surface-symmetric Einstein-Vlasov-scalar field system on the interval $]0, 1]$ with initial data given on $t = 1$. Let

$$K(t, r) := -e^{-\mu} \left(\dot{\lambda}(t, r) + \frac{2}{t} \right)$$

denotes the mean curvature of the hypersurfaces of constant t . Then

$$K(t, r) \leq -Ct^{-3/2},$$

where C is a positive constant.

Proof For a metric of the form $ds^2 = -\alpha(t, r)^2 dt^2 + g_{ij} dx^i dx^j$, the second fundamental form of the hypersurfaces of constant t is given by $K_{ij} = -(2\alpha)^{-1} \partial_t g_{ij}$ and its trace $K(t, r) = g^{ij} K_{ij}$ is the mean curvature of that hypersurfaces. For g defined by (1.1), we have

$$\begin{aligned} K_{ij} &= -\frac{e^{-\mu}}{2} \partial_t g_{ij}. \quad \text{Thus} \\ K_{11} &= -\frac{e^{-\mu}}{2} \partial_t g_{11} = -\dot{\lambda} e^{2\lambda - \mu} \quad ; \quad K_{22} = -\frac{e^{-\mu}}{2} \partial_t g_{22} = -te^{-\mu} \quad ; \\ K_{33} &= -te^{-\mu} \sin_k^2 \theta. \end{aligned}$$

And $K(t, r) = -\left(\dot{\lambda} + \frac{2}{t}\right)e^{-\mu}$. We have

$$\dot{\lambda} = e^{2\mu} \left(4\pi t \rho - \frac{k + e^{-2\mu}}{2t} \right) \geq -e^{2\mu} \left(\frac{k + e^{-2\mu}}{2t} \right). \quad (3.15)$$

and

$$K(t, r) \leq \frac{k - 3e^{-2\mu}}{2t} e^\mu.$$

For $k = 0$ or $k = -1$,

$$K(t, r) \leq -\frac{3}{2t} e^{-\mu}.$$

and the estimate

$$e^{-2\mu} \geq \frac{e^{-2\mu} + k}{t}$$

implies

$$K(t, r) \leq -\frac{3}{2t} \left(\frac{\inf e^{-2\mu} + k}{t} \right)^{1/2} \leq -Ct^{-3/2} \text{ where } C = \frac{3}{2} (\inf e^{-2\mu} + k)^{1/2}.$$

For $k = 1$ we have

$$e^{-2\mu} \geq \frac{e^{-2\mu}}{t} > 1 = k \text{ (since } t < 1)$$

thus

$$\begin{aligned} K(t, r) &\leq e^\mu \frac{1 - 3e^{-2\mu}}{2t} \leq \left(\frac{e^{2\mu} - 3}{2} \right) \frac{e^{-\mu}}{t} \\ &\leq -\frac{e^{-\mu}}{t} \leq -\frac{\inf e^{-\mu}}{t^{3/2}} \\ &\leq -Ct^{-3/2} \text{ where } C = \inf e^{-\mu}. \end{aligned}$$

□

Remark 3.10 We deduce from above that

$$\lim_{t \rightarrow 0} K(t, r) = -\infty,$$

uniformly in $r \in \mathbb{R}$.

Theorem 3.11 Let (λ, μ, ϕ) be a regular solution of the spherical, plane and hyperbolic symmetric Einstein-scalar field system on the interval $]0, 1]$ with initial data given at $t = 1$. Then

$$\lim_{t \rightarrow 0} \frac{K_1^1(t, r)}{K(t, r)} = a(r) ; \lim_{t \rightarrow 0} \frac{K_2^2(t, r)}{K(t, r)} = \lim_{t \rightarrow 0} \frac{K_3^3(t, r)}{K(t, r)} = \frac{1}{2}(1 - a(r)),$$

uniformly in $r \in \mathbb{R}$,

where

$$\frac{K_1^1(t, r)}{K(t, r)}, \frac{K_2^2(t, r)}{K(t, r)}, \frac{K_3^3(t, r)}{K(t, r)}$$

are the generalized Kasner exponents and $a(r)$ a function of r .

Proof We have

$$\frac{K_1^1(t, r)}{K(t, r)} = \frac{t\dot{\lambda}(t, r)}{t\dot{\lambda}(t, r) + 2}; \quad \frac{K_2^2(t, r)}{K(t, r)} = \frac{K_3^3(t, r)}{K(t, r)} = \frac{1}{t\dot{\lambda}(t, r) + 2}.$$

with

$$t\dot{\lambda} = 4\pi t^2 e^{2\mu} \rho - \frac{k}{2} e^{2\mu} - \frac{1}{2}.$$

As we have seen previously

$$e^{2\mu(t, r)} \leq Ct$$

which implies that

$$e^{2\mu(t, r)} \rightarrow 0 \quad \text{as } t \rightarrow 0$$

Let $t_0 \in]0, e^{-1}]$ and $t \in]0, t_0]$. From (3.11),

$$\partial_t \left(\frac{\phi}{\log t} \right) = \frac{1}{\log t} \left(\dot{\phi} - \frac{\phi}{t \log t} \right) = O(t^{-1} (\log t)^{-2})$$

so that

$$\frac{\phi(t, r)}{\log t} = \frac{\phi(t_0, r)}{\log t_0} - \int_t^{t_0} s^{-1} (\log s)^{-2} ds.$$

The integral term of the above relation converges as $t \rightarrow 0$. Set

$$A(r) = \lim_{t \rightarrow 0} \frac{\phi(t, r)}{\log t} = \frac{\phi(t_0, r)}{\log t_0} - \int_0^{t_0} s^{-1} (\log s)^{-2} ds$$

Since from (3.11), $(\dot{\phi} - \frac{\phi}{t \log t}) = O((t |\log t|)^{-1})$, we have

$$t\dot{\phi} = \frac{\phi}{\log t} + O((|\log t|)^{-1})$$

so that $t\dot{\phi} \rightarrow A(r)$ as $t \rightarrow 0$. Inequality (3.11) shows also that $\phi'^2 e^{2\mu-2\lambda} = O((t |\log t|)^{-2})$. Using these limits, we have

$$t\dot{\lambda}(t, r) = 2\pi(t^2 \dot{\phi}^2 + t^2 \phi'^2 e^{2\mu-2\lambda}) - \frac{k}{2} e^{2\mu} - \frac{1}{2} \rightarrow 2\pi A(r)^2 - \frac{1}{2} \quad \text{as } t \rightarrow 0, \text{ uniformly in } r.$$

We take $a(r) = \frac{4\pi A(r)^2 - 1}{4\pi A(r)^2 + 3}$ to complete the proof. \square

Chapter 4

Global existence and asymptotic behaviour of solutions in the future

4.1 Global existence in the future

We prove the global existence in the future in the cases of plane and hyperbolic symmetries.

Theorem 4.1 *Assume that (f, λ, μ, ϕ) is a right maximal regular solution of the Einstein-Vlasov system with scalar field obtained in Theorem 2.7. Then for $k = 0$ or $k = -1$,*

$$\sup\{e^{2\mu(t,r)} | r \in \mathbb{R}, t \in [1, T[]\} < \infty.$$

Proof : We now establish a series of estimates which will result in an upper bound on μ . Similar estimates were used in [4] and [3]. Unless otherwise specified in what follows constants denoted by C will be positive, may depend on the initial data and may change their value from line to line.

Firstly, integration of (1.4) with respect to t and the fact that p is non-negative imply that

$$e^{2\mu(t,r)} = \left[\frac{e^{-2\hat{\mu}(r)} + k}{t} - k - \frac{8\pi}{t} \int_1^t s^2 p(s, r) ds \right]^{-1} \geq \frac{t}{C - kt}, \quad t \in [1, T[\quad (4.1)$$

Next let us show that

$$\int_0^1 e^{\mu+\lambda} \rho(t, r) dr \leq Ct, \quad t \in [1, T[\quad (4.2)$$

We have

$$\frac{d}{dt} \int_0^1 e^{\mu+\lambda} \rho(t, r) dr = \int_0^1 [(\dot{\lambda} + \dot{\mu}) e^{\mu+\lambda} \rho + e^{\mu+\lambda} \dot{\rho}] dr$$

with

$$\begin{aligned} \dot{\rho} &= \frac{-2\pi}{t^3} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} f dF dw + \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_t f dF dw \\ &+ \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \left(\frac{-F}{t^3} \right) \frac{f}{\sqrt{1+w^2+F/t^2}} dF dw + e^{-2\mu} (-\dot{\mu} \dot{\phi}^2 + \dot{\phi} \ddot{\phi}) + e^{-2\lambda} (-\dot{\lambda} \phi'^2 + \dot{\phi}' \phi') \\ &= -\frac{2}{t} \rho - \frac{1}{t} q + \frac{1}{t} (e^{-2\mu} \dot{\phi}^2 + e^{-2\lambda} \phi'^2) + \frac{1}{t} (e^{-2\mu} \dot{\phi}^2 - e^{-2\lambda} \phi'^2) \\ &+ \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_t f dF dw + e^{-2\mu} (-\dot{\mu} \dot{\phi}^2 + \dot{\phi} \ddot{\phi}) + e^{-2\lambda} (-\dot{\lambda} \phi'^2 + \dot{\phi}' \phi') \\ &= -\frac{2}{t} \rho - \frac{1}{t} q + \frac{2}{t} e^{-2\mu} \dot{\phi}^2 + e^{-2\mu} (-\dot{\mu} \dot{\phi}^2 + \dot{\phi} \ddot{\phi}) \\ &+ e^{-2\lambda} (-\dot{\lambda} \phi'^2 + \dot{\phi}' \phi') + \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_t f dF dw. \end{aligned}$$

The Vlasov equation and integration by parts imply,

$$\begin{aligned} &\frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_t f dF dw \\ &= \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \left[(\dot{\lambda} w + e^{\mu-\lambda} \mu' \sqrt{1+w^2+F/t^2}) \partial_w f - \frac{e^{\mu-\lambda} w}{\sqrt{1+w^2+F/t^2}} \partial_r f \right] dF dw \\ &= \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \left[\dot{\lambda} w \sqrt{1+w^2+F/t^2} + e^{\mu-\lambda} \mu' (1+w^2+F/t^2) \right] \partial_w f dF dw \\ &- \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} e^{\mu-\lambda} w \partial_r f dF dw \\ &= -e^{\mu-\lambda} \left[j' + e^{-\mu} (-\mu' \dot{\phi} + \dot{\phi}') \phi' e^{-\lambda} + e^{-\lambda} (-\lambda' \phi' + \phi'') \dot{\phi} e^{-\mu} \right] \\ &- \dot{\lambda} \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \left(\sqrt{1+w^2+F/t^2} + \frac{w^2}{\sqrt{1+w^2+F/t^2}} \right) f dF dw \\ &- \mu' e^{\mu-\lambda} \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} 2w f dF dw \end{aligned}$$

And so

$$\begin{aligned}
\dot{\rho} &= -\frac{2}{t}\rho - \frac{1}{t}q - e^{\mu-\lambda}j' - 2\mu'je^{\mu-\lambda} - \dot{\lambda}(\rho+p) + \frac{2}{t}e^{-2\mu}\dot{\phi}^2 \\
&\quad + e^{-2\mu}(-\dot{\mu}\dot{\phi}^2 + \dot{\phi}\ddot{\phi}) + e^{-2\lambda}(-\dot{\lambda}\phi'^2 + \dot{\phi}'\phi') + (\lambda' + \mu')e^{-2\lambda}\dot{\phi}\phi' \\
&\quad - e^{-2\lambda}(\dot{\phi}'\phi' + \dot{\phi}\phi'') - 2\mu'\dot{\phi}\phi'e^{-2\lambda} + \dot{\lambda}(e^{-2\mu}\dot{\phi}^2 + e^{-2\lambda}\phi'^2) \\
&= -\frac{2}{t}\rho - \frac{1}{t}q - e^{\mu-\lambda}(j' + 2\mu'j) - \dot{\lambda}(\rho+p) + (\dot{\lambda} - \dot{\mu} + \frac{2}{t})\dot{\phi}^2e^{-2\mu} \\
&\quad + (\lambda' - \mu')\dot{\phi}\phi'e^{-2\lambda} + \dot{\phi}(e^{-2\mu}\ddot{\phi} - e^{-2\lambda}\phi'') \\
&= -\frac{2}{t}\rho - \frac{1}{t}q - e^{\mu-\lambda}(j' + 2\mu'j) - (4\pi t\rho e^{2\mu} - \frac{1+ke^{2\mu}}{2t})(\rho+p);
\end{aligned}$$

where we use the wave equation to substitute the term $e^{-2\mu}\ddot{\phi} - e^{-2\lambda}\phi''$ and also the expression of $\dot{\lambda}$. Therefore,

$$\begin{aligned}
\frac{d}{dt} \int_0^1 e^{\mu+\lambda}\rho(t,r)dr &= -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[2\rho + q + e^{\mu-\lambda}(j' + 2\mu'j) - \frac{\rho+p}{2}(1+ke^{2\mu}) \right] dr \\
&= -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[2\rho + q - \frac{\rho+p}{2}(1+ke^{2\mu}) \right] - \int_0^1 (e^{2\mu}j)'dr
\end{aligned} \tag{4.3}$$

and since μ and j are periodic with respect to r , we deduce that:

$$\frac{d}{dt} \int_0^1 e^{\mu+\lambda}\rho(t,r)dr = -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[2\rho + q - \frac{\rho+p}{2}(1+ke^{2\mu}) \right] dr$$

For $k=0$, since $q \geq -2\rho$, we have $q + \frac{3\rho-p}{2} \geq q + \rho \geq -\rho$; we deduce that:

$$\frac{d}{dt} \int_0^1 e^{\mu+\lambda}\rho(t,r)dr \leq \frac{1}{t} \int_0^1 e^{\mu+\lambda}\rho(t,r)dr$$

Integrating this inequality with respect to t yields (4.2) for $k=0$. For $k=-1$, we have, using (4.1):

$$\begin{aligned}
\frac{d}{dt} \int_0^1 e^{\mu+\lambda}\rho(t,r)dr &= -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left[2\rho + q - \frac{\rho+p}{2}(1-e^{2\mu}) \right] dr \\
&\leq -\frac{1}{t} \int_0^1 e^{\mu+\lambda} \left(2\rho + q - \frac{\rho+p}{2} \right) dr - \frac{1}{C+t} \int_0^1 e^{\mu+\lambda} \frac{\rho+p}{2} dr \\
&\leq -\frac{1}{t} \int_0^1 e^{\mu+\lambda} (2\rho + q) dr + \left(\frac{1}{t} - \frac{1}{C+t} \right) \int_0^1 e^{\mu+\lambda} \frac{\rho+p}{2} dr \\
&\leq \frac{1}{t} \int_0^1 e^{\mu+\lambda} \rho dr + \left(\frac{1}{t} - \frac{1}{C+t} \right) \int_0^1 e^{\mu+\lambda} \rho dr \\
&\leq \left(\frac{2}{t} - \frac{1}{C+t} \right) \int_0^1 e^{\mu+\lambda} \rho dr
\end{aligned}$$

where we use the fact that $2\rho + q \geq -\rho$ and $\frac{\rho+p}{2} \leq \rho$. Integrating the above inequality with respect to t yields (4.2) for $k = -1$. Using equation $\mu' = -4\pi t e^{\mu+\lambda} j$, the fact that $|j| \leq \rho$ and (4.2) we find

$$\begin{aligned} |\mu(t, r) - \int_0^1 \mu(t, \sigma) d\sigma| &= \left| \int_0^1 \int_\sigma^r \mu'(t, \tau) d\tau d\sigma \right| \leq \int_0^1 \int_0^1 |\mu'(t, \tau)| d\tau d\sigma \\ &\leq 4\pi t \int_0^1 e^{\mu+\lambda} |j(t, \tau)| d\tau \leq 4\pi t \int_0^1 e^{\mu+\lambda} \rho(t, \tau) d\tau \end{aligned}$$

that is

$$|\mu(t, r) - \int_0^1 \mu(t, \sigma) d\sigma| \leq Ct^2, \quad t \in [1, T[, \quad r \in [0, 1] \quad (4.4)$$

Next we show that

$$e^{\mu(t, r) - \lambda(t, r)} \leq Ct, \quad t \in [1, T[, \quad r \in [0, 1]. \quad (4.5)$$

To see this observe that : relation $\dot{\mu} - \dot{\lambda} = 4\pi t e^{2\mu}(p - \rho) + \frac{1+ke^{2\mu}}{t}$, $p - \rho \leq 0$ and (4.1) imply that

$$\begin{aligned} \frac{\partial}{\partial t} e^{\mu-\lambda} &= (\dot{\mu} - \dot{\lambda}) e^{\mu-\lambda} = e^{\mu-\lambda} \left[4\pi t e^{2\mu}(p - \rho) + \frac{1+ke^{2\mu}}{t} \right] \\ &\leq \frac{1+ke^{2\mu}}{t} e^{\mu-\lambda} \leq \left(\frac{1}{t} + \frac{k}{C-kt} \right) e^{\mu-\lambda}; \end{aligned}$$

and integrating this inequality with respect to t yields

$$e^{\mu-\lambda} \leq C \frac{t}{C-kt} \leq Ct,$$

i.e. (4.5).

We now estimate the average of $\mu(t)$ over the interval $[0, 1]$ which in combination with (4.4) will yield the desired upper bound on μ . We use (1.4), (4.2), (4.5) and the fact that $p \leq \rho$, $ke^{2\mu} \leq 0$:

$$\begin{aligned} \int_0^1 \mu(t, r) dr &= \int_0^1 \overset{\circ}{\mu}(r) dr + \int_1^t \int_0^1 \dot{\mu}(s, r) dr ds \\ &\leq C + \int_1^t \frac{1}{2s} \int_0^1 [e^{2\mu}(8\pi s^2 p + k) + 1] dr ds \\ &= C + \frac{1}{2} \ln t + 4\pi \int_1^t \int_0^1 s e^{2\mu} p dr ds + \int_1^t \int_0^1 \frac{ke^{2\mu}}{2s} dr ds \\ &\leq C + \frac{1}{2} \ln t + 4\pi \int_1^t \int_0^1 s e^{\mu-\lambda} e^{\mu+\lambda} \rho dr ds + \int_1^t \int_0^1 \frac{ke^{2\mu}}{2s} dr ds \\ &\leq C + \frac{1}{2} \ln t + C \int_1^t s^3 ds \\ &= C + \frac{1}{2} \ln t + Ct^4 \end{aligned}$$

with (4.4) this implies

$$\mu(t, r) \leq C(1 + \ln t + t^4 + t^2) \leq Ct^4, \quad t \in [1, T[, \quad r \in [0, 1]. \quad (4.6)$$

Remark 4.2 *we have proven that for initial data as in Theorem 2.4 (local existence), the right maximal regular solution of the Einstein-Vlasov-scalar field satisfies estimates (4.2)-(4.5)-(4.6).*

In the next theorem, we prove that this solution exists on the full interval $[1, \infty[$.

Theorem 4.3 *Assume that (f, λ, μ, ϕ) is a solution of the full system on a right maximal interval of existence $[1, T[$, then $T = \infty$.*

Proof: Assume that $T < \infty$. We show that under the bound of μ , we obtain the bound of $\sup\{|w|(t, r, w, F) \in \text{supp}f\}$ which is in contradiction to Theorem 2.8. The proof of the bound on w is similar to the proof of [see [15], Theorem 6.2]. Let

$$W_0 := \sup\{|w|(r, w, F) \in \text{supp}f\} < \infty,$$

$$F_0 := \sup\{F|(r, w, F) \in \text{supp}f\} < \infty.$$

Except in the vacuum case we have $W_0 > 0$ and $F_0 > 0$. For $t \geq 1$ define

$$P_+(t) := \max\{0, \max\{w|(r, w, F) \in \text{supp}f(t)\}\},$$

$$P_-(t) := \min\{0, \min\{w|(r, w, F) \in \text{supp}f(t)\}\}.$$

Constants denoted by C will be positive, may depend on the initial data and may change their value from line to line. Let $(r(s), w(s), F)$ be a characteristic in the support of f . Assume that $P_+(t) > 0$ for some $t \in [1, T[$, and let $w(t) = w > 0$. We have

$$\begin{aligned} \dot{w} &= -\dot{\lambda}w - e^{\mu-\lambda}\mu'\sqrt{1+w^2+F/s^2} \\ &= (-4\pi se^{2\mu}\rho + \frac{1+ke^{2\mu}}{2s})w + 4\pi se^{2\mu}j\sqrt{1+w^2+F/s^2} \\ &= 4\pi se^{2\mu}(j\sqrt{1+w^2+F/s^2} - \rho w) + \frac{1+ke^{2\mu}}{2s}w \\ &= \frac{4\pi^2}{s}e^{2\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \left(\tilde{w}\sqrt{1+w^2+F/s^2} - w\sqrt{1+\tilde{w}^2+\tilde{F}/s^2} \right) f d\tilde{F} d\tilde{w} \\ &+ \frac{1+ke^{2\mu}}{2s}w - 4\pi se^{2\mu} \left[\frac{1}{2}w(\dot{\phi}^2 e^{-2\mu} + \phi'^2 e^{-2\lambda}) + \sqrt{1+w^2+F/s^2} \dot{\phi}\phi' e^{-\mu-\lambda} \right] \end{aligned} \quad (4.7)$$

Using the fact that $K(s)^2 \leq K(1)^2 s$ for $s \in [1, T[$ (see (2.78)), we have :

$$\begin{aligned}
& \frac{1}{2} w (\dot{\phi}^2 e^{-2\mu} + \phi'^2 e^{-2\lambda}) + \sqrt{1 + w^2 + F/s^2} \dot{\phi} \phi' e^{-\mu-\lambda} \\
& \geq (w + \sqrt{1 + w^2 + F/s^2}) \dot{\phi} \phi' e^{-\mu-\lambda} \\
& \geq -\sqrt{1 + w^2 + F/s^2} K(s)^2 \\
& \geq -C(1 + P_+(s))s
\end{aligned} \tag{4.8}$$

Set $\gamma = \tilde{w} \sqrt{1 + w^2 + F/s^2} - w \sqrt{1 + \tilde{w}^2 + \tilde{F}/s^2}$. As long as $w(s) > 0$, we have the following estimates: if $\tilde{w} \leq 0$ then $\gamma \leq 0$. If $\tilde{w} > 0$ then

$$\begin{aligned}
& \frac{4\pi^2}{s} e^{2\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \gamma f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w \leq \frac{1 + ke^{2\mu}}{2s} w \\
& + \frac{4\pi^2}{s} e^{2\mu} \int_0^{P_+(s)} \int_0^{F_0} \frac{\tilde{w}^2(1 + w^2 + F/s^2) - w^2(1 + \tilde{w}^2 + \tilde{F}/s^2)}{\tilde{w} \sqrt{1 + w^2 + F/s^2} + w \sqrt{1 + \tilde{w}^2 + \tilde{F}/s^2}} f d\tilde{F} d\tilde{w} \\
& \leq \frac{4\pi^2}{s} e^{2\mu} \int_0^{P_+(s)} \int_0^{F_0} \frac{\tilde{w}^2(1 + F/s^2)}{2\tilde{w}w} f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w \\
& \leq \frac{4\pi^2}{s} e^{2\mu} \int_0^{P_+(s)} \int_0^{F_0} \frac{\tilde{w}(1 + F)}{w} f d\tilde{F} d\tilde{w} + \frac{1 + ke^{2\mu}}{2s} w \\
& \leq 4\pi^2 F_0(1 + F_0) \| \overset{\circ}{f} \| \frac{e^{2\mu}}{2s} (P_+(s))^2 \frac{1}{w} + \frac{1 + ke^{2\mu}}{2s} w \\
& \leq \frac{C}{s} \left[\frac{(P_+(s))^2}{w} + w \right]
\end{aligned} \tag{4.9}$$

Therefore, using (4.8) and (4.9), (4.7) yields:

$$\dot{w}(s) \leq \frac{C}{s} \left[\frac{(P_+(s))^2}{w(s)} + w(s) \right] + C(1 + P_+(s))s^2$$

i.e.

$$\begin{aligned}
\dot{w}(s)w(s) & \leq \frac{C}{s} (P_+(s))^2 + CP_+(s)(1 + P_+(s))s^2 \\
\frac{d}{ds} w(s)^2 & \leq C(s^{-1} + s^2)(P_+(s))^2 + CP_+(s)s^2
\end{aligned}$$

as long as $w(s) > 0$. Let $t_1 \in [1, t[$ be defined minimal such that $w(s) > 0$ for $s \in [t_1, t[$, then

$$w(t)^2 \leq w(t_1)^2 + C \int_{t_1}^t [(s^{-1} + s^2)(P_+(s))^2 + s^2 P_+(s)] ds$$

If $t_1 = 1$ then $w(t_1) \leq w_0$ and

$$w(t)^2 \leq w_0^2 + C \int_1^t [(s^{-1} + s^2)(P_+(s))^2 + s^2 P_+(s)] ds.$$

If $t_1 > 1$ then $w(t_1) = 0$ (t_1 is the minimal) and

$$\begin{aligned} w(t)^2 &\leq C \int_{t_1}^t [(s^{-1} + s^2)(P_+(s))^2 + s^2 P_+(s)] ds \\ &\leq w_0^2 + C \int_1^t [(s^{-1} + s^2)(P_+(s))^2 + s^2 P_+(s)] ds. \end{aligned}$$

Thus

$$(P_+(t))^2 \leq w_0^2 + C \int_1^t [(s^{-1} + s^2)(P_+(s))^2 + s^2 P_+(s)] ds; \text{ for all } t \in [1, T[$$

Now we use the fact that $P_+ \leq \frac{1}{2}(1 + P_+^2)$ to obtain

$$\begin{aligned} (P_+(t))^2 &\leq w_0^2 + C \int_1^t [(s^{-1} + s^2)(P_+(s))^2 + s^2] ds \\ &\leq (w_0^2 + Ct^3) + C \int_1^t (s^{-1} + s^2)(P_+(s))^2 ds; \text{ for all } t \in [1, T[\end{aligned}$$

If $t < \infty$, applying Gronwall's inequality to this estimate implies that P_+ is bounded on $[1, T[$.

Estimating $\dot{w}(s)$ from below in the case $w(s) < 0$ along the same lines shows that P_- is bounded as well and the proof is complete.

Remark 4.4 *In the case of spherical symmetry ($k = 1$), there is no global existence in the future, regardless of the size of initial data. For any solution (f, λ, μ, ϕ) , the estimate*

$$e^{-2\mu(t,r)} = \frac{e^{-2\hat{\mu}(r)} + 1}{t} - 1 - \frac{8\pi}{t} \int_1^t s^2 p(s, r) ds \leq \frac{e^{-2\hat{\mu}(r)} + 1}{t} - 1$$

has to hold on the interval $[1, T[$. Since the right hand side of this inequality tends to -1 for $t \rightarrow \infty$, it follows that $T < \infty$. And we deduce from the previous Theorem ($|w| < \infty$) that $\|e^{2\mu}\| \rightarrow \infty$ for $t \rightarrow T$.

4.2 The future asymptotic behaviour

In this section we prove that the spacetime obtained in Theorem 4.3 (with $f = 0$) in the plane symmetric case is timelike and null geodesically complete in the expanding direction. Later on, we prove that this result holds for homogeneous solutions of the Einstein-Vlasov-scalar field system in plane and hyperbolic symmetry.

Let us determine first the explicit solutions ϕ, μ, λ in the case $f = 0$ and $k = 0$.

4.2.1 Integration of equations

In the case $k = 0$ the wave equation can be reduced to a simple linear equation as observed in [19]. This reduction goes as follows. In that case the field equations imply that $\lambda - \mu + \log t$ is constant in time. It may, however, be dependent on r . Suppose that r is replaced by a new coordinate s on the initial hypersurface. Choosing s appropriately makes the transformed quantity $\lambda - \mu + \log t$ constant on the initial hypersurface and hence everywhere. Let us call this constant η . Then $\lambda - \mu + \log t = \eta$ and the wave equation (1.7) simplifies to

$$\partial_{tt}\phi + t^{-1}\partial_t\phi = t^2 e^{-2\eta}\partial_{ss}\phi. \quad (4.10)$$

Now with the change of coordinate r by s , $\sqrt{g_{rr}}dr = e^\lambda dr = e^{\tilde{\lambda}}ds = \sqrt{g_{ss}}ds$ with $\tilde{\lambda} - \mu + \log t = \eta$ i.e. $ds = e^{\lambda - \tilde{\lambda}}dr = te^{-\eta}e^{\lambda - \mu}dr$ and $1 = e^{-\eta} \int_0^1 te^{(\lambda - \mu)(t,r)}dr$ for all $t \geq 1$. In particular for $t = 1$, $1 = Ae^{-\eta}$ is the period of variables r and s for $A = \int_0^1 e^{(\lambda - \mu)(1,r)}dr$.

Now if we set $T = \beta t^2$ (β a positive constant), then (4.10) reads $\partial_{TT}\phi + T^{-1}\partial_T\phi = \frac{1}{4\beta}e^{-2\eta}\partial_{ss}\phi$. Setting $Ae^{-\eta} = 2\pi$ and $4\beta e^{2\eta} = 1$, we can use the results of Jurke (cf. [11]) obtained in the case of polarized Gowdy T^3 -models (here W is replaced by ϕ , t by T , the period of the space coordinate is 2π). The equations are similar with the same boundary conditions. Following this, for all $(t, r) \in [1, \infty[\times \mathbb{R}$ and $T = t^2$, ϕ must be in this form :

$$\phi(t, r) = \begin{cases} a_1 + 2b \log t & \text{(homogeneous case)} \\ a_1 + 2b \log t + t^{-1}\alpha(t, r) + \beta(t, r) & \text{(non-homogeneous case)} \end{cases} \quad (4.11)$$

where a_1 and b are constants fixed by initial values of ϕ and $\dot{\phi}$, α and β real-valued C^2 -functions with $|\alpha|, |\alpha'| \leq C$, $|\dot{\alpha}| \leq Ct$, $|\beta|, |\beta'| \leq Ct^{-3}$, $|\dot{\beta}| \leq Ct^{-2}$, C a positive constant and $\frac{1}{4}(t^{-2}\partial_{tt} - t^{-3}\partial_t)\alpha(t, r) = \partial_{rr}\alpha(t, r)$. We deduce that $|\dot{\phi}\phi'| \leq Ct^{-1}$.

From the field equation (1.4), we have :

$$2t\dot{\mu} = 1 + 8\pi t^2 e^{2\mu} p$$

i.e

$$\dot{\mu} = \frac{1}{2t} + 2\pi t(\dot{\phi}^2 + e^{2\mu - 2\lambda}\phi'^2) = \frac{1}{2t} + 2\pi t(\dot{\phi}^2 + 4t^2\phi'^2)$$

From (1.3)

$$2t\dot{\lambda} = -1 + 8\pi t^2 e^{2\mu} \rho$$

i.e

$$\dot{\lambda} = \frac{-1}{2t} + 2\pi t(\dot{\phi}^2 + e^{2\mu - 2\lambda}\phi'^2) = \frac{-1}{2t} + 2\pi t(\dot{\phi}^2 + 4t^2\phi'^2)$$

We deduce from the expression of ϕ , that $\dot{\mu}$ and $\dot{\lambda}$ are bounded each by a positive constant C . From Theorem 15 of [11] (the hypothesis of this theorem

are satisfied : a_t is replaced here by $\dot{\mu}$, μ can be cast for all $(t, r) \in [1, \infty[\times \mathbb{R}$ into the form

$$\mu(t, r) = \begin{cases} \frac{1}{2}(16\pi b^2 + 1) \log t + \gamma & \text{if } \mu' = 0 \\ \nu t^2 + \delta(t, r) & \text{(non-homogeneous case)} \end{cases} \quad (4.12)$$

where ν is a positive constant, γ a constant fixed by initial value of μ and the function δ satisfies the inequalities, $|\delta(t, r)| \leq C(1+t)$, $|\dot{\delta}| \leq Ct$ with a positive constant C . Note that if $\phi' = 0$ then from equation (1.5), $\mu' = 0$ and $\lambda' = 0$; the solutions are independent of r and we are in the homogeneous case. Since $\dot{\lambda} - \dot{\mu} = -t^{-1}$, we deduce that $\lambda(t, r) = \mu(t, r) - \log t + \overset{\circ}{\lambda}(r) - \overset{\circ}{\mu}(r)$.

4.2.2 Geodesic completeness

Let $] \tau_-, \tau_+[\ni \tau \mapsto (x^\alpha(\tau), p^\beta(\tau))$ be a geodesic whose existence interval is maximally extended and such that $x^0(\tau_0) = t(\tau_0) = 1$ for some $\tau_0 \in] \tau_-, \tau_+[$. We want to show that for future-directed timelike and null geodesics, $\tau_+ = +\infty$. Consider first the case of a timelike geodesic, i.e.,

$$g_{\alpha\beta} p^\alpha p^\beta = -m^2 ; p^0 > 0$$

with $m > 0$. Since $dt/d\tau = p^0 > 0$, the geodesic can be parameterized by the coordinate time t . Recall that along the geodesics the variables $t, r, p^0, w := e^\lambda p^1, F := t^4 [(p^2)^2 + \sin_k^2 \theta (p^3)^2]$ satisfy the following system of differential equations :

$$\frac{dr}{d\tau} = e^{-\lambda} w, \quad \frac{dw}{d\tau} = -\dot{\lambda} p^0 w - e^{2\mu-\lambda} \mu' (p^0)^2, \quad \frac{dF}{d\tau} = 0 \quad (4.13)$$

$$\frac{dt}{d\tau} = p^0, \quad \frac{dp^0}{d\tau} = -\dot{\mu} (p^0)^2 - 2e^{-\lambda} \mu' p^0 w - e^{-2\mu} \dot{\lambda} w^2 - e^{-2\mu} t^{-3} F. \quad (4.14)$$

With respect to coordinate time the geodesic exists on the interval $[1, \infty[$ since on bounded t -intervals the Christoffel symbols are bounded and the right hand sides of the geodesic equations written in coordinate time are linearly bounded in p^1, p^2, p^3 . Along the geodesic we define w and F as above. Then the relation between coordinate time and proper time along the geodesic is given by

$$\frac{dt}{d\tau} = p^0 = e^{-\mu} \sqrt{m^2 + w^2 + F/t^2},$$

and to control this we need to control w as a function of coordinate time.

The plane symmetric case without Vlasov

Assume that $w(t) > 0$ for some $t \geq 1$. By (4.13) and the fact that $|\dot{\phi}||\phi'| \leq Ct^{-1}$, $e^{(\mu-\lambda)t} = e^{-\eta t}$, we have as long as $w(s) > 0$

$$\begin{aligned}
\frac{d}{ds}w(s) &= -\dot{\lambda}w - e^{2\mu-\lambda}\mu'p^0 = 4\pi se^{2\mu}(j\sqrt{m^2+w^2+F/s^2} - \rho w) + \frac{1}{2s}w \\
&\leq \frac{1}{2s}w - 4\pi se^{2\mu}\rho w + |4\pi se^{2\mu}j(w + \sqrt{m^2+F/s^2})| \\
&\leq \frac{1}{2s}w + 4\pi se^{2\mu}|j|\sqrt{m^2+F/s^2} \quad \text{since } |j| \leq \rho \\
&\leq \frac{1}{2s}w + 4\pi se^{\mu-\lambda}|\dot{\phi}||\phi'|\sqrt{m^2+F/s^2} \\
&\leq \frac{1}{2s}w + Cs
\end{aligned} \tag{4.15}$$

Let $t_0 \in [1, t[$ be defined minimal such that $w(s) > 0$ for $s \in [t_0, t[$. Then Gronwall's inequality shows that

$$w(t) \leq \left[w(t_0) + C \int_{t_0}^t s \exp\left(\int_s^{t_0} \frac{d\tau}{2\tau}\right) ds \right] \exp\left(\int_{t_0}^t \frac{d\tau}{2\tau}\right).$$

Now either $t_0 = 1$ and $w(t_0) = w(1)$ or $t_0 > 1$ and $w(t_0) = 0$ (t_0 is the minimal). Thus

$$w(t) \leq \left[|w(1)| + C \int_1^t s \exp\left(\int_s^1 \frac{d\tau}{2\tau}\right) ds \right] \exp\left(\int_1^t \frac{d\tau}{2\tau}\right) \leq Ct^2.$$

Estimating $\dot{w}(s)$ from below in the case $w(s) < 0$ along the same lines yields the upper bound of $-w(t)$.

Since $|\delta| \leq C(1+t)$, $\nu t^2 + \delta > \nu t^2 - C(1+t) = \nu t^2(1 - \frac{C(1+t)}{\nu t^2})$. For t large, we choose $\frac{C(1+t)}{\nu t^2} < \frac{1}{2}$ i.e $\nu t^2 + \delta > \frac{1}{2}\nu t^2$. Then along the geodesic we have :

$$\begin{aligned}
\frac{d\tau}{dt} &= \frac{e^\mu}{\sqrt{m^2+w^2+F/t^2}} = \frac{e^{\nu t^2+\delta}}{\sqrt{m^2+Ct^4+F/t^2}} \\
&\geq \frac{e^{\frac{1}{2}\nu t^2}}{t^2\sqrt{m^2+C+F}} \geq \frac{\nu t^2}{2t^2\sqrt{m^2+C+F}} = \text{constant}.
\end{aligned}$$

Since the left hand side is constant, the integral over $[1, \infty[$ diverges. In the homogeneous case ($\nu = 0$, $j = 0$, $\rho \geq 0$), (4.15) becomes

$$\frac{d}{ds}w(s) \leq \frac{1}{2s}w$$

and we can prove similarly as above that $w(t) \leq Ct^{1/2}$ for $t \geq 1$. Therefore

$$\frac{d\tau}{dt} = \frac{e^{\frac{1}{2}(16\pi b^2+1)\log t+\gamma}}{\sqrt{m^2+Ct+F/t^2}} \geq \frac{t^{\frac{1}{2}(16\pi b^2+1)}e^\gamma}{t^{1/2}\sqrt{m^2+C+F}} \geq Ct^{8\pi b^2}$$

The integral on the right hand side of the above inequality over $[1, \infty[$ diverges. In either case, we conclude that $\tau_+ = +\infty$ as desired.

In the case of a future-directed null geodesic, i.e. $m = 0$ and $p^0(\tau_0) > 0$, p^0 is everywhere positive and the quantity F is again conserved. The argument can now be carried out exactly as in the timelike case, implying that $\tau_+ = +\infty$. We have proven :

Theorem 4.5 *Consider initial data with plane symmetry for the Einstein-scalar field system written in areal coordinates. Then the corresponding spacetime is timelike and null geodesically complete in the expanding direction.*

Remark 4.6 *In the homogeneous plane symmetric case with only a scalar field, $\rho(t) = p(t) = \frac{1}{2}e^{-2\mu}\dot{\phi}^2$, $(\dot{\lambda} - \dot{\mu})(t) = -t^{-1}$ and solving the wave equation, we obtain $|\dot{\phi}(t)| = |\psi|t^{-1}$. From the field equations, $\dot{\lambda}(t) = (2\pi\psi^2 - \frac{1}{2})t^{-1}$ and*

$$\lambda(t) = \overset{\circ}{\lambda} + (2\pi\psi^2 - \frac{1}{2}) \log t ; \mu(t) = \overset{\circ}{\mu} + (2\pi\psi^2 + \frac{1}{2}) \log t ;$$

$$\rho(t) = p(t) = \frac{1}{2}\psi^2 e^{-2\overset{\circ}{\mu}} t^{-3-4\pi\psi^2} ; j(t) = 0 ; q(t) = 2p(t).$$

We can compute the limiting values of the generalized Kasner exponents :

$$t\dot{\lambda} = 2\pi\psi^2 - \frac{1}{2} ;$$

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{K_1^1(t, r)}{K(t, r)} &= \lim_{t \rightarrow \infty} \frac{t\dot{\lambda}(t, r)}{t\dot{\lambda} + 2} = \frac{4\pi\psi^2 - 1}{4\pi\psi^2 + 3} ; \\ \lim_{t \rightarrow \infty} \frac{K_2^2(t, r)}{K(t, r)} &= \lim_{t \rightarrow \infty} \frac{K_3^3(t, r)}{K(t, r)} = \lim_{t \rightarrow \infty} \frac{1}{t\dot{\lambda} + 2} = \frac{2}{4\pi\psi^2 + 3}, \end{aligned}$$

The spatially homogeneous case

Consider now the Einstein-Vlasov-scalar field system for $k \leq 0$. We study the future behaviour for solutions which are independent of the space coordinate r . In this case $\mu' = 0$, $\phi' = 0$ and equation (1.5) shows that $j = 0$.

We prove that $w(t)$ is bounded for finite time t and the spacetime obtained is geodesically complete. Since $\rho \geq 0$, $j = 0$ and $k \leq 0$, we have from (4.13) as long as $w(s) > 0$,

$$\begin{aligned} \frac{d}{ds} w(s) &= -\dot{\lambda}w - e^{2\mu-\lambda}\mu'p^0 = 4\pi s e^{2\mu} (j\sqrt{m^2 + w^2 + F/s^2} - \rho w) + \frac{1 + ke^{2\mu}}{2s} w \\ &= -4\pi s e^{2\mu} \rho w + \frac{1 + ke^{2\mu}}{2s} w \\ &\leq \frac{1}{2s} w \end{aligned}$$

and we can prove as above that $w(t) \leq Ct^{1/2}$ for $t \geq 1$. Using (4.1), $e^{2\mu} \geq \frac{t}{C} \geq \frac{1}{C}$ for $k = 0$; and $e^{2\mu} \geq \frac{t}{C+t} = \frac{1}{1+C/t} \geq \frac{1}{1+C}$ for $k = -1$. In each case, $e^\mu \geq C$.

Then along the geodesic,

$$\begin{aligned}\frac{d\tau}{dt} &= \frac{e^\mu}{\sqrt{m^2 + w^2 + F/t^2}} \geq \frac{C}{\sqrt{m^2 + Ct + F/t^2}} \\ &\geq \frac{C}{t^{1/2}\sqrt{m^2 + C + F}} \geq Ct^{-1/2}.\end{aligned}$$

The integral on the right hand side over $[1, \infty[$ diverges. Therefore $\tau_+ = +\infty$. In the case of a future-directed null geodesic, the argument is the same as what we done in the previous subsection.

In the homogeneous case, a number of further estimates can also be obtained. We follow the proof of Theorem 4.1 and [17] to obtain a better bound of μ which depends explicitly on the data. Recall (4.3)

$$\frac{d}{dt}(e^{\mu+\lambda}\rho)(t) = -\frac{1}{t}e^{\mu+\lambda} \left[2\rho + q - \frac{\rho+p}{2}(1 + ke^{2\mu}) \right].$$

Since $q \geq 0$, we obtain :
for $k = 0$,

$$\frac{d}{dt}(e^{\mu+\lambda}\rho)(t) \leq -\frac{1}{t}e^{\mu+\lambda}\rho(t);$$

thus

$$e^{\mu+\lambda}\rho(t) \leq e^{\overset{\circ}{\mu}+\overset{\circ}{\lambda}}\overset{\circ}{\rho}t^{-1}; \quad (4.16)$$

for $k = -1$

$$\begin{aligned}\frac{d}{dt}(e^{\mu+\lambda}\rho)(t) &\leq -\frac{1}{t}e^{\mu+\lambda} \left[2\rho - \frac{\rho+p}{2} - \frac{1}{C_0+t} \frac{\rho+p}{2} \right] \\ &\leq e^{\mu+\lambda}\rho \left(-\frac{1}{t} - \frac{1}{C_0+t} \right) \\ &\leq -\frac{2}{C_1+t}e^{\mu+\lambda}\rho(t)\end{aligned}$$

where $C_0 = k + e^{-2\overset{\circ}{\mu}}$ (see (4.1)) and $C_1 = \max(0, C_0)$. Thus

$$e^{\mu+\lambda}\rho(t) \leq (C_1 + 1)^2 e^{\overset{\circ}{\mu}+\overset{\circ}{\lambda}}\overset{\circ}{\rho}t^{-2}. \quad (4.17)$$

Next we have similarly to (4.5),

$$e^{\mu-\lambda} \leq e^{\overset{\circ}{\mu}-\overset{\circ}{\lambda}}t \quad \text{for } k = 0 \quad (4.18)$$

and

$$e^{\mu-\lambda} \leq e^{\overset{\circ}{\mu}-\overset{\circ}{\lambda}} \frac{e^{-2\overset{\circ}{\mu}}t}{e^{-2\overset{\circ}{\mu}} - 1 + t} \quad \text{for } k = -1$$

If $\overset{\circ}{\mu} \leq 0$ then $\frac{t}{e^{-2\overset{\circ}{\mu}} - 1 + t} \leq 1$. If $\overset{\circ}{\mu} \geq 0$ then $\frac{e^{-2\overset{\circ}{\mu}}t}{e^{-2\overset{\circ}{\mu}} - 1 + t} \leq 1$. Thus in either case,

$$e^{\mu-\lambda} \leq e^{|\overset{\circ}{\mu}|-\overset{\circ}{\lambda}} \quad \text{for } k = -1 \quad (4.19)$$

Now

$$\mu(t) = \overset{\circ}{\mu} + \int_1^t \dot{\mu}(s) ds = \overset{\circ}{\mu} + \int_1^t \frac{1}{2s} [e^{2\mu}(8\pi s^2 p + k) + 1] ds.$$

For $k = 0$,

$$\begin{aligned} \mu(t) &\leq \overset{\circ}{\mu} + \frac{1}{2} \log t + 4\pi \int_1^t s e^{2\mu} \rho ds = \overset{\circ}{\mu} + \frac{1}{2} \log t + 4\pi \int_1^t s e^{\mu-\lambda} e^{\mu+\lambda} \rho ds \\ &\leq \overset{\circ}{\mu} + \frac{1}{2} \log t + 4\pi e^{\overset{\circ}{\mu}+\overset{\circ}{\lambda}} \overset{\circ}{\rho} e^{\overset{\circ}{\mu}-\overset{\circ}{\lambda}} \int_1^t s ds \\ &\leq \overset{\circ}{\mu} + \frac{1}{2} \log t + 2\pi e^{2\overset{\circ}{\mu}} \overset{\circ}{\rho} t^2 \end{aligned}$$

i.e

$$\mu(t) \leq \overset{\circ}{\mu} + 2\pi e^{2\overset{\circ}{\mu}} \overset{\circ}{\rho} t^2. \quad (4.20)$$

For $k = -1$,

$$\begin{aligned} \mu(t) &\leq \overset{\circ}{\mu} + 4\pi \int_1^t s e^{2\mu} \rho ds + \int_1^t \frac{1}{2s} (1 - e^{2\mu}) ds \\ &\leq \overset{\circ}{\mu} + 4\pi \int_1^t s e^{\mu-\lambda} e^{\mu+\lambda} \rho ds + \int_1^t \frac{1}{2s} \left(1 - \frac{s}{C_0 + s}\right) ds \\ &\leq \overset{\circ}{\mu} + 4\pi (C_1 + 1)^2 e^{|\overset{\circ}{\mu}| - \overset{\circ}{\lambda}} \overset{\circ}{\rho} e^{\overset{\circ}{\mu} + \overset{\circ}{\lambda}} \int_1^t s^{-1} ds + \int_1^t \frac{1}{2} \left(\frac{1}{s} - \frac{1}{C_0 + s}\right) ds \\ &\leq \overset{\circ}{\mu} + \frac{1}{2} \log(C_0 + 1) + 4\pi (C_1 + 1)^2 e^{|\overset{\circ}{\mu}| + \overset{\circ}{\mu}} \overset{\circ}{\rho} \int_1^t s^{-1} ds \end{aligned}$$

i.e

$$\mu(t) \leq \overset{\circ}{\mu} + \frac{1}{2} \log(C_0 + 1) + 4\pi e^{|\overset{\circ}{\mu}| + \overset{\circ}{\mu}} \overset{\circ}{\rho} (C_1 + 1)^2 \log t. \quad (4.21)$$

We have proven :

Theorem 4.7 *Consider spatially homogeneous solutions of plane and hyperbolic symmetric Einstein-Vlasov-scalar field system written in areal coordinates and initial data given for $t = 1$. Then these solutions exist on the whole interval $[1, \infty[$. The corresponding spacetimes are timelike and null geodesically complete in the expanding direction and estimates (4.16)-(4.17)-(4.18)-(4.19)-(4.20)-(4.21) hold.*

Conclusion

We have investigated in spherical, plane and hyperbolic symmetry, cosmological solutions of the Einstein-Vlasov-scalar field system. But there are some remaining problems listed as follows :

- past global existence for $k = -1$ without any restriction on the data;
- asymptotics in the past direction with Vlasov matter. Maybe the idea of [24] could be used here?
- future geodesic completeness in general;
- future asymptotics. Maybe one can use the results of [26] which study the future asymptotic behavior of massless scalar fields in a class of cosmological vacuum spacetimes;
- in the case of a non-linear scalar field, using our approach can we obtain similar results as in the case of non-vanishing cosmological constant?

Thus it appears that the work of this thesis can serve as a useful starting point for future investigations.

Appendices

Appendix A

Here we establish relations (2.26) and (2.27). Relation (2.8) gives :

$$e^{-\mu_{n-1}}(\dot{g}_n - \dot{h}_n) + e^{-\lambda_{n-1}}(\tilde{g}'_n - \tilde{h}'_n) = a_{n-1}(\tilde{g}_{n-1} - \tilde{h}_{n-1}) + b_{n-1}(\tilde{g}_{n-1} + \tilde{h}_{n-1})$$

i.e

$$e^{-\mu_{n-1}}\dot{g}_n - e^{-\lambda_{n-1}}\tilde{h}'_n = e^{-\mu_{n-1}}\dot{h}_n - e^{-\lambda_{n-1}}\tilde{g}'_n + (a_{n-1} + b_{n-1})\tilde{g}_{n-1} + (b_{n-1} - a_{n-1})\tilde{h}_{n-1}$$

Therefore,

$$\begin{aligned} D_n^+ \tilde{X}_n &= D_n^+(\tilde{g}_{n+1} - \tilde{h}_{n+1}) - D_n^+(\tilde{g}_n - \tilde{h}_n) \\ &= a_n(\tilde{g}_n - \tilde{h}_n) + b_n(\tilde{g}_n + \tilde{h}_n) - e^{-\mu_n}(\dot{g}_n - \dot{h}_n) - e^{-\lambda_n}(g'_n - h'_n) \\ &= (a_n + b_n)\tilde{g}_n + (b_n - a_n)\tilde{h}_n - \dot{g}_n(e^{-\mu_n} - e^{-\mu_{n-1}}) + \tilde{h}'_n(e^{-\lambda_n} - e^{-\lambda_{n-1}}) \\ &\quad + \dot{h}_n e^{-\mu_n} - \tilde{g}'_n e^{-\lambda_n} - \dot{g}_n e^{-\mu_{n-1}} + \tilde{h}'_n e^{-\lambda_{n-1}} \\ &= (a_n + b_n)\tilde{g}_n + (b_n - a_n)\tilde{h}_n - \dot{g}_n(e^{-\mu_n} - e^{-\mu_{n-1}}) + \tilde{h}'_n(e^{-\lambda_n} - e^{-\lambda_{n-1}}) \\ &\quad + \dot{h}_n e^{-\mu_n} - \tilde{g}'_n e^{-\lambda_n} - e^{-\mu_{n-1}}\dot{h}_n + e^{-\lambda_{n-1}}\tilde{g}'_n - (a_{n-1} + b_{n-1})\tilde{g}_{n-1} - (b_{n-1} - a_{n-1})\tilde{h}_{n-1} \\ &= (a_n + b_n)(\tilde{g}_n - \tilde{g}_{n-1}) + (b_n - a_n)(\tilde{h}_n - \tilde{h}_{n-1}) + (a_n + b_n - a_{n-1} - b_{n-1})\tilde{g}_{n-1} \\ &\quad + (b_n - a_n + a_{n-1} - b_{n-1})\tilde{h}_{n-1} + (e^{-\mu_n} - e^{-\mu_{n-1}})(\dot{h}_n - \dot{g}_n) + (e^{-\lambda_n} - e^{-\lambda_{n-1}})(\tilde{h}'_n - \tilde{g}'_n) \\ &= a_n[(\tilde{g}_n - \tilde{g}_{n-1}) - (\tilde{h}_n - \tilde{h}_{n-1})] + b_n[(\tilde{g}_n - \tilde{g}_{n-1}) + (\tilde{h}_n - \tilde{h}_{n-1})] \\ &\quad + (a_n + b_n - a_{n-1} - b_{n-1})\tilde{g}_{n-1} + (b_n - a_n + a_{n-1} - b_{n-1})\tilde{h}_{n-1} \\ &\quad + (e^{-\mu_n} - e^{-\mu_{n-1}})(\dot{h}_n - \dot{g}_n) + (e^{-\lambda_n} - e^{-\lambda_{n-1}})(\tilde{h}'_n - \tilde{g}'_n) \\ &= a_n \tilde{X}_{n-1} + b_n \tilde{Y}_{n-1} + F_n \end{aligned}$$

If we replace \tilde{h}'_n and \tilde{g}'_n respectively by $-\tilde{h}'_n$ and $-\tilde{g}'_n$, (2.8) and (2.9), D_n^+ and D_n^- , \tilde{X}_n and \tilde{Y}_n interchange respectively and we can write (2.27).

Appendix B

Let us prove relations (2.56) and (2.57). We have

$$\partial_r(D_n^+ X_{n+1}) = (\partial_r D_n^+) X_{n+1} + D_n^+ X'_{n+1}$$

i.e

$$\begin{aligned} D_n^+ X'_{n+1} &= \partial_r(D_n^+ X_{n+1}) - (\partial_r D_n^+) X_{n+1} \\ &= C'_n + \mu'_n e^{-\mu_n} \partial_t X_{n+1} + \lambda'_n e^{-\lambda_n} X'_{n+1} \end{aligned}$$

We deduce from relation $(e^{-\mu_n} \partial_t + e^{-\lambda_n} \partial_r) X_{n+1} = C_n$, that

$$e^{-\mu_n} \partial_t X_{n+1} = C_n - e^{-\lambda_n} X'_{n+1}$$

Therefore

$$\begin{aligned} D_n^+ X'_{n+1} &= C'_n + \mu'_n C_n + \lambda'_n e^{-\lambda_n} X'_{n+1} - \mu'_n e^{-\lambda_n} X'_{n+1} \\ &= C'_n + (\lambda'_n - \mu'_n) e^{-\lambda_n} X'_{n+1} + \mu'_n C_n = \tilde{C}_n \end{aligned}$$

If we replace r by $-r$ (i.e ∂_r by $-\partial_r$), D_n^+ and D_n^- , X_{n+1} and Y_{n+1} interchange respectively and we can write (2.57).

Appendix C

Let us prove inequalities (2.66)-(2.67). we have from (2.55)

$$\begin{aligned} \tilde{C}_n + \tilde{C}_{n-1} &= (\lambda'_n - \mu'_n) e^{-\lambda_n} X'_{n+1} - (\lambda'_{n-1} - \mu'_{n-1}) e^{-\lambda_{n-1}} X'_n \\ &\quad + \mu'_n C_n - \mu'_{n-1} C_{n-1} + \tilde{C}'_n - \tilde{C}'_{n-1} \end{aligned}$$

Now we obtain the following relations :

$$\begin{aligned} i) \quad &(\lambda'_n - \mu'_n) e^{-\lambda_n} X'_{n+1} - (\lambda'_{n-1} - \mu'_{n-1}) e^{-\lambda_{n-1}} X'_n = (\lambda'_n - \mu'_n) e^{-\lambda_n} (X'_{n+1} - X'_n) \\ &\quad + [(\lambda'_n - \mu'_n) e^{-\lambda_n} - (\lambda'_{n-1} - \mu'_{n-1}) e^{-\lambda_{n-1}}] X'_n \\ &= (\lambda'_n - \mu'_n) e^{-\lambda_n} (\tilde{g}'_{n+1} - \tilde{g}'_n + \tilde{h}'_n - \tilde{h}'_{n+1}) + [(\lambda'_n - \lambda'_{n-1}) e^{-\lambda_n} \\ &\quad - (\mu'_n - \mu'_{n-1}) e^{-\lambda_n} + (\lambda'_{n-1} - \mu'_{n-1}) (e^{-\lambda_n} - e^{-\lambda_{n-1}})] X'_n \end{aligned}$$

$$ii) \quad \mu'_n C_n - \mu'_{n-1} C_{n-1} = (\mu'_n - \mu'_{n-1}) C_n + \mu'_{n-1} (C_n - C_{n-1})$$

$$\begin{aligned} iii) \quad &\tilde{C}'_n + \tilde{C}'_{n-1} = a'_n X_n + a_n X'_n + b'_n Y_n + b_n Y'_n - a'_{n-1} X_{n-1} - a_{n-1} X'_{n-1} - b'_{n-1} Y_{n-1} - b_{n-1} Y'_{n-1} \\ &= (a'_n - a'_{n-1}) X_n + a'_{n-1} (X_n - X_{n-1}) + (a_n - a_{n-1}) X'_n + a_{n-1} (X'_n - X'_{n-1}) \\ &\quad + (b'_n - b'_{n-1}) Y_n + b'_{n-1} (Y_n - Y_{n-1}) + (b_n - b_{n-1}) Y'_n + b_{n-1} (Y'_n - Y'_{n-1}) \end{aligned}$$

$$\begin{aligned}
iv) \quad a'_n - a'_{n-1} &= -\mu'_n(-\dot{\lambda}_n - \frac{1}{t} - \dot{\lambda}'_n)e^{-\mu_n} + (-\tilde{\mu}'_n + \tilde{\mu}'_n \lambda'_n)e^{-\lambda_n} \\
&+ \mu'_{n-1}(-\dot{\lambda}_{n-1} - \frac{1}{t} - \dot{\lambda}'_{n-1})e^{-\mu_{n-1}} + (-\tilde{\mu}'_{n-1} + \tilde{\mu}'_{n-1} \lambda'_{n-1})e^{-\lambda_{n-1}} \\
&= (\mu'_{n-1} - \mu'_n)(-\dot{\lambda}_n - \frac{1}{t})e^{-\mu_n} - \mu'_{n-1}[(-\dot{\lambda}_n - \frac{1}{t})e^{-\mu_n} + (-\dot{\lambda}_{n-1} - \frac{1}{t})e^{-\mu_{n-1}}] \\
&+ (\dot{\lambda}'_{n-1} - \dot{\lambda}'_n)e^{-\mu_n} - \dot{\lambda}'_{n-1}(e^{-\mu_n} - e^{-\mu_{n-1}}) + (\tilde{\mu}'_{n-1} - \tilde{\mu}'_n)e^{-\lambda_n} \\
&- \tilde{\mu}'_{n-1}(e^{-\lambda_n} - e^{-\lambda_{n-1}}) + (\lambda'_n - \lambda'_{n-1})\tilde{\mu}_n e^{-\lambda_n} + \lambda'_{n-1}(\tilde{\mu}_n e^{-\lambda_n} - \tilde{\mu}_{n-1} e^{-\lambda_{n-1}})
\end{aligned}$$

$$\begin{aligned}
v) \quad b'_n - b'_{n-1} &= \frac{\mu'_n}{t}e^{-\mu_n} - \frac{\mu'_{n-1}}{t}e^{-\mu_{n-1}} \\
&= \frac{1}{t}(\mu'_n - \mu'_{n-1})e^{-\mu_n} + \frac{\mu'_{n-1}}{t}(e^{-\mu_n} - e^{-\mu_{n-1}})
\end{aligned}$$

Using remark 2.2, (2.36) and (2.40), we obtain respectively from i), ii), iv) and v) the following estimations :

$$\begin{aligned}
|(\lambda'_n - \mu'_n)e^{-\lambda_n} X'_{n+1} - (\lambda'_{n-1} - \mu'_{n-1})e^{-\lambda_{n-1}} X'_n| &\leq C + C(|\tilde{g}'_{n+1} - \tilde{g}'_n| + |\tilde{h}'_{n+1} - \tilde{h}'_n| \\
&+ |\mu'_n - \mu'_{n-1}| + |\lambda'_n - \lambda'_{n-1}|) \\
&\leq C\epsilon + C(\gamma_n(\tau) + \gamma_{n-1}(\tau))
\end{aligned}$$

$$\begin{aligned}
|\mu'_n C_n - \mu'_{n-1} C_{n-1}| &\leq |\mu'_n - \mu'_{n-1}| |C_n| + |\mu'_{n-1}| |C_n - C_{n-1}| \\
&\leq C\epsilon + C\gamma_{n-1}(\tau)
\end{aligned}$$

$$\begin{aligned}
|a'_n - a'_{n-1}| &\leq C + (|\mu'_{n-1} - \mu'_n| + |\dot{\lambda}'_{n-1} - \dot{\lambda}'_n| + |\tilde{\mu}'_{n-1} - \tilde{\mu}'_n| + |\lambda'_n - \lambda'_{n-1}|) \\
&\leq C\epsilon + C\gamma_{n-1}(\tau)
\end{aligned}$$

$$|b'_n - b'_{n-1}| \leq C\epsilon + C|\mu'_n - \mu'_{n-1}| \leq C\epsilon + C\gamma_{n-1}(\tau)$$

$$|X'_n - X'_{n-1}|, |Y'_n - Y'_{n-1}| \leq C(|\tilde{g}'_{n-1} - \tilde{g}'_n| + |\tilde{h}'_{n-1} - \tilde{h}'_n|) \leq C\gamma_{n-1}(\tau)$$

Using this, we deduce from iii) that

$$|\tilde{C}'_n + \tilde{C}'_{n-1}| \leq C\epsilon + C(\gamma_n(\tau) + \gamma_{n-1}(\tau))$$

Combining all the necessary estimates, (2.66) follows. Similarly, we prove inequality (2.67).

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