

On building 4-critical plane and projective plane multiwheels from odd wheels

Dainis Zeps

Institute of Mathematics and Computer Science,
University of Latvia,
Rainis blvd., Riga, Latvia
dainize@mii.lu.lv
<http://www.ltn.lv/~dainize/>

Abstract. We build unbounded classes of plane and projective plane multiwheels that are 4-critical that are received summing odd wheels as edge sums modulo two. These classes can be considered as ascending from a single common graph that can be received as an edge sum modulo two of the octahedron graph O and the minimal wheel W_3 . All graphs of these classes belong to $2n - 2$ -edges-class of graphs, among which are those that quadrangulate projective plane, i.e., graphs from the Grötzsch class received applying Mycielski's Construction to an odd cycle [7].

Keywords: graph coloring, chromatic critical graphs, wheels, planar graphs, projective planar graphs, Grötzsch graph, Mycielski's construction

1 Introduction

We are using a terminology from [3, 6, 8].

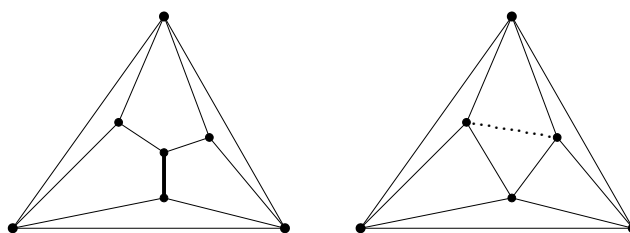


Fig. 1. In the graph w_{1_3} (left) contracting the thick edge we get octahedron graph without an edge (O^- , right). Adding the dotted edge we get the octahedron graph O .

We consider a graph in fig.1 left, that in [9] is denoted G_3 , but in this article gets several denotations due to its particular features, w_{111} or w_{1_3} , g_{111} or g_{1_3} , see lower. In this introduction we consider some simple features of this graph w_{1_3} called the *base graph* in frames of this article, but further in the article we generalize these features to two unbounded classes of graphs, plane and projective plane graphs.

We start from a simple observation that contraction of one of edges (incident to degree 3 vertices) in the base graph (w_{1_3}) turns it into octahedron graph (O) minus an edge (O^-), see fig.1. Now it is obvious that the graph O^- is a minor of w_{1_3} , but the graph O is not.

Further, we may express this fact using terminology from [9], i.e., $\langle O^-, O \rangle$ is minor bracket for the graph w_{1_3} . We remind that $\langle h_1, h_2 \rangle$ is minor bracket for G if $h_1 \prec h_2$, $h_1 \prec G$ and $h_2 \not\prec G$, [9]. We express this as predicate $\langle h_1, h_2 : G \rangle$.

One more observation that the base graph w_{1_3} turns into the wheel graph W_3 after contracting three edges incident to three and four degree vertices.

Further, graph w_{1_3} is 4-critical [9], and this fact might be verified directly for such a small graph. But for further discussion we need to examine this graph more closely.

The base graph w_{1_3} can be considered as an edge sum modulo two of three copies W_3 in the way that each pair overlap just in one edge, and all three wheels have one vertex in common, see fig. 2. Let us assume that overlap either two rim edges, or two spike edges, or spike edge and rim edge, all three ways actually giving the same graph w_{1_3} (see fig. 2): a trivial (automorphism) fact, that should turn out not so for further, see below. Further we are going to use this consideration of w_{1_3} as the sum of three wheels, and for that reason we use this multi-index denotation for it, i.e., w_{111} (or w_{1_3}) with three indices (three ones), not single index, (and second index as factor of equal indices), see lower. Further we are going to use at edge summation modulo two denotations as at simple arithmetic operations, i.e., summation and multiplication, e.g. $w_{1_3} = W_3 + W_3 + W_3 = 3W_3$. Of course, corresponding graph operations behind are indeterministic, i.e., depending on how we configure graphs one against other by edge summation modulo two, i.e., which elements of corresponding graphs we allow to overlap. Besides, w_{1_3} may be expressed in the way $O + W_3$ with rim edges annihilating with a triangle of the octahedron graph, thus having one more equation $w_{1_3} = O + W_3$, becoming an equality under the specified conditions.

The graph w_{1_3} can be colored in four distinct ways, see fig. 3. We distinct them in way lonely color, D , is applied: the lonely D may color central hub, section hub, rim and two vertices of rim, thus giving four ways. We remind that only in a chromatic critical graph every vertex may receive a lonely color. Thereby, each vertex for w_{1_3} may be colored with a lonely color, thus proving that the graph is 4-critical.

Further, the base graph w_{1_3} may be embedded on the projective plane, quadrangulating it, see a) in fig. 8 lower. Clearly, w_{1_3} belongs to the class of graphs with $2n - 2$ edges by n vertices, where all graphs quadrangulating the projective

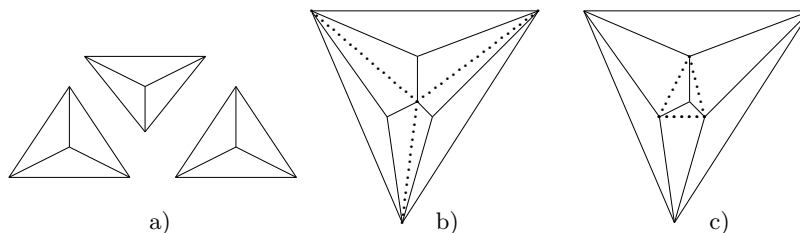


Fig. 2. Forming graph w_{13} (b) by summation modulo two of edges of three graphs W_3 (a): $3W_3 = w_{13}$. Edges that annihilate (one in one pair) by summation modulo two are depicted as dotted (b). In c) w_{13} is received by operation $O + W_3$ where the triangle (depicted dotted) is annihilated in both graphs.

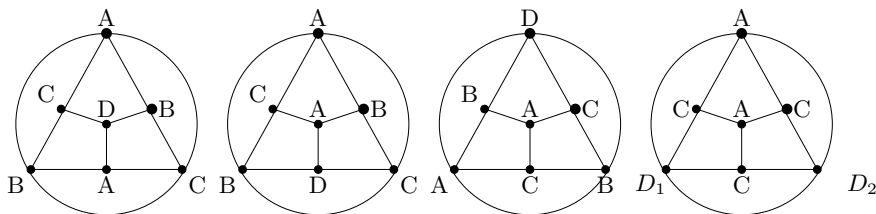


Fig. 3. Four possible colorings of w_{13} . D designates lonely color. Lonely color D may color central hub (first), section hub (second), rim (third), and two lonely colored vertices as two rim vertices (fourth).

plane should belong. Besides, w_{13} is selfdual if considered plane, but of course not such on projective plane. It is remarkable that the dual graph to the octahedron is the cube graph, but the cube graph with one corner cut off becomes just the graph w_{13} , that is selfdual (see [9]).

The aim of this article is to show that these facts concerning this single and very simple graph w_{13} may be extended for unbounded classes of 4-critical graphs, both planar and projective planar. These classes we are going to build from arbitrary odd wheel graphs.

The theorem about octahedron minor bracket is mentioned in [10], but here we give proof of this fact.

2 Building 4-critical plane multiwheels

Let us start with assuming that the wheel graph W_k is built from k simple sections where as simple section we take triangle C_3 with one vertex common

from each triangle for the wheel's hub, and the opposite edges of the triangles as forming wheel's rim. Thus, the wheel W_k is the sum of its k sections (triangles) composing a simple graph (with double-edges turning into simple edges). By the way, if we applied this same summation of the edges modulo two then the sides of triangles that touch each other would annihilate, and remnant or resulting graph would be a simple cycle (rim of the wheel) and an isolated vertex (hub of the wheel). As an obvious observation, let us notice that edge sum modulo two of $k > 3$ triangles gives C_k and an isolated vertex only in case each triangle annihilates two edges, and all triangles have common vertex.

If we take wheel W_k to be odd (k is odd) then we get 4-critical graphs. Odd wheels comprise the simplest unbounded class of vertex-3-connected 4-critical graphs. (Let us remember that the odd cycles are the only graphs that are 3-critical.) We are going to generalize this class to 4-critical vertex-3-connected plane and projective plane multiwheels.

Let us go on, and let us replace each section in what we built before in a simple odd wheel with another arbitrary odd wheel in a way that the edge sets of sections are summed modulo two. This new aggregation of wheels M may be expressed as $\sum W_{k_i}$ where the summation is modulo two over index i numbering sections that were replaced by wheels of order k_i in each case. Evidently the summation itself is indeterministic because result of it depends on how wheels overlap each other by summation. Now we ask: under which conditions of overlap the resulting graph is 4-critical. Theorem 4 below says that the number of overlapping edges in each section must be equal to two.

But first we are to prove some lemmas about intersecting wheels without edge losses by summation modulo two. For example, $W_5 + W_5$ may be formed with all five rim vertices of both wheels common and forming subgraph K_5 , but resulting 6-chromatic graph is in no way chromatic critical.

Lemma 1 *Let k odd wheels by summation possibly intersect in vertices but not in edges. The sum of these wheels can't give 4-critical graph except in case $k = 1$.*

Proof. Let by summation of wheels $k > 1$ wheels have become vertex connected components, that are wheels all the same, in the resulting graph. It is obvious that elimination of edge or vertex, or edge contraction in one wheel can't affect coloring of others in four colors in the resulting graph. Thus, the graph can't be chromatic critical.

Let us configure two odd wheels so that they have common two adjacent vertices, and sum their edges modulo two. It is easy to see that the resulting graph is 3-chromatic. It suffices to notice that losing of an edge in both odd wheels allow to color them in 3-colors so that lost edge's ends receive the same color. Further, we may easily apply the use of this fact to *unclosed sequence of wheels* W_{q_1}, \dots, W_{q_i} where two proximal wheels overlap in two adjacent vertices, but next two possibly only in one. Let us formulate it as a lemma.

Lemma 2 *Let summation of edges modulo two be applied to unclosed sequence of wheels. The resulting graph is 3-chromatic.*

We need one more crucial feature of 4-critical graphs. Let graph H is 4-critical and let $H' = H \odot w$ be graph H with vertex w split into two new vertices and edges incident to w be connected either to one or other vertex. We ask whether graph H' can remain to be 4-critical, [see [4] problem 9.20 on page 69]. Of course, it is expectable that H' becomes 3-chromatic. In this paper we leave this fact as hypothetically true without considering further.

We are going to use this hypothesis in the following way. By summation of edge sets of wheels modulo two in order to build new 4-critical graphs we may ignore cases where wheels intersect only in vertices without incident edges, knowing that this can't lead to new 4-critical graphs. Suppose we received a 4-critical graph in this way. Then splitting all vertices that were merged by summation backwards we should receive 3-chromatic graph but it might not be true. Let us formulate this fact as a lemma.

Lemma 3 *Let by summation of edge sets of wheels some wheels intersect in vertices without losing edges. Then resulting graph can't be 4-critical.*

Now we may go over to the main theorem of this chapter.

Theorem 4 *Let $2k + 1$ ($k > 0$) arbitrary odd wheels be summed in a way that edges of wheels are summed modulo two and all wheels have one overlapping vertex. The resulting graph M is 4-critical if and only if each wheel by summation modulo two loses just two of its edges and resulting graph is planar.*

Proof. Let us first observe that two wheels may overlap in one or three edges but not in two, thus, to get two annihilating edges a wheel should overlap with two other wheels with one overlapping edge in each.

Let us first assume that wheel graphs are summed observing two edge loss condition. In this case four configurations of new section are possible, see fig. 4. We denote graphs achieved in this way by $w_{k_1-k_2-...k_q}$ where q is number of sections, where in each section there are $2k_i + 1$ edges, and call them *multiwheels*. In this way of denotation we as if ignore three/four ways of section's configuration, but one could easily elaborate denotation with taking these different types of sections into account, see below.

In the figure 2 we see simplest case where summed are three wheels W_3 giving multiwheel w_{1-1-1} . In the place of multiindex $k_1-k_2-...k_q$ with hyphens we equally use a denotation without hyphens in case no confusion might arise, e.g. w_{111} in place of w_{1-1-1} .

It is easy to see that w_{111} is 4-critical. Indeed, reserving central hub vertex as eventual lonely color vertex (receiving color D , see fig. 3) other vertices get forced colors. Further, both hub edge, spike edge and rim edge contractions/eliminations lead to 3-chromatic graph.

Let us assume that the resulting graph is planar. In that case it is convenient to characterize the type of a section by the type (rim or spike edge) of lost edges in the wheel. Then sections are 1) rim-rim-section (rr-section), 2) spike-spike-section (ss-section), 3) spike-rim-section (sr-section) and 4) rim-spike-section (rs-section), see fig.4. A section arisen from w_1 we call *simple section*, which is of

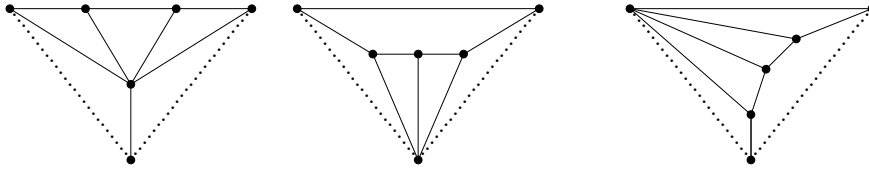


Fig. 4. Four types of sections for planar multiwheels possible. If the third type we take in two oriented ways, leftwards and rightwards, then we get one type of section more. When so numbered first two and two other sections are mutually dual. It is convenient to characterize type of section by type (rim or spike edge) of lost edges in wheel. Then sections are 1) rim-rim-section (rr-section), 2) spike-spike-section (ss-section), 3) spike-rim-section (sr-section) and 4) rim-spike-section (rs-section).

arbitrary type due to automorphisms, i.e., rim edges are spike edges too, and reversely.

Now let us consider the first type of section, rr-section, fig.4. At least one vertex on the rim may receive third color and then the corresponding section hub receives fourth color. Removing a rim edge makes possible to color the rim with two colors, but removing a section's spike edge allows now both previous adjacent vertices color with one color, thus avoiding fourth color.

Let us consider the second type section. Now the same applies for the local rim edges of the section. At least one vertex of the local rim should receive third color, and corresponding local rim edge or spike edge elimination may avoid use of fourth color.

Let us consider the third (and fourth) type of section. Now outer hinges and vertex adjacent to central hub should receive different colors, but removal at least one edge from section violates this condition and allows to color hinge vertices of inner rim with the same color.

Thus, we have proved that multiwheel is 4-critical.

Let us prove the theorem in the other direction.

According lemmas 1,2,3 we are to consider only those sums of wheels where intersections of vertices without incident edges are absent and unclosed sequences of wheels are absent. Even more, if some closed sequences are present, but some wheels as unclosed ends are present, these cases are not to be considered because can't give 4-critical graphs. The only cases are these where only closed sequences of wheels are present. Further, only one closed sequence as cycle is to be considered for further.

Further, let us consider the case of non-planar resulting graph, see fig. 5. In that case we have in the cycle of wheels some with two spikes that are not following one after the other. In these cases such wheel may be as if taken in the cycle in two ways along one or other orientation between two spikes. It is easy to see that this can't give resulting graph as 4-critical, because odd wheel may be

divided only in half of odd wheel and half of even wheel. Taking into cyclic path even part of wheel would spoil odd-times-odd structure necessary for 4-critical multiwheel. (See fig.5, where rights minimal case of non-planar resulting graph is shown to be 3-chromatic.) Thus, we can't afford non-planar spike pairs in edge summation modulo two. We have come to the conclusion that closed sequence should be planar.

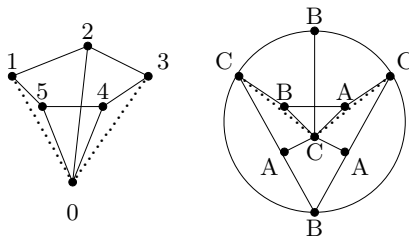


Fig. 5. Illustration to the proof. The minimal possible case of a nonplanar section for as if eventual nonplanar multiwheel w_{115} . Left, we see non-planar section, where annihilated spikes (dotted) are not sequencing ones, and section falls into as if two subwheels, one even subwheel with subrim 1-2-3 and one odd subwheel with subrim 3-4-5-1. Right, we color this nonplanar multiwheel into three colors, i.e., it isn't 4-critical, even not 4-chromatic.

Thus, under assumption that all summing wheels have in common one vertex each in each wheel with two incident edges that annihilate by summation modulo two we have proved what was necessary. We have come to the 4-critical graph only by specified conditions.

We have proved the theorem.

2.1 Denotations for plane multiwheels. Some characteristics

Let us introduce some notational conventions for wheels and multiwheels. Along with the traditional denotation for wheels with capital letter W with index k , i.e. W_k , for odd wheels of order $k = 2q + 1$ we use denotation w_q . For plane multiwheels we use the letter w with odd ($k \geq 3$) indices, w_{q_1, \dots, q_k} where w_{q_i} was i -th wheel in summation. Let us introduce a quantity $Q = \sum q_i$ and sometimes use the denotation w_Q for this multiwheel.

If odd wheel has order $k = 2q + 1$, it has $n = 2q + 2$ vertices and $m = 2n - 2 = 4q + 2$ edges. Similar expressions hold for multiwheels, where in place of q stands Q . Indeed, multiwheel has $2Q + 1$ vertices and $4Q$ or $2n - 2$ edges. To get analogue expressions both for wheels and multiwheels we had to use for wheels in place of quantity q quantity $s = k/2$, giving fractional numbers for

odd wheels. It would be interesting to ask then what pre-wheel stands behind $w_{1/2}$. It might be multiedge or edge adjacent to loop.

Both classes, wheels and multiwheels have $m = 2n - 2$ edges. Let us notice that to the class of graphs $\mathbf{G}_{n,2n-2}$ belong these quadrangulating projective plane. In next section we show how this fact turns crucial for multiwheels in generalizing them for projective plane.

Another question would be how to designate indices in multiwheel w_{q_1, \dots, q_k} if we wanted to take into account type of sections standing behind corresponding indices. We have four types of sections, therefore we have to equip this index with this additional information. One way would be to use four colors for indices. Other way would be to supply index with diacritic sign, say, $w_{1-\grave{\text{a}}-\grave{\text{a}}-\grave{\text{a}}-\grave{\text{a}}}$ or $w_{1\grave{\text{a}}\grave{\text{a}}\grave{\text{a}}\grave{\text{a}}}$ for w_{13333} with types of sections in augmented order.

3 Grötzsch graph, Mycielski's Construction and 4-critical projective plane multiwheels

We start with the observation that Grötzsch graph [7] may be considered as an edge sum modulo two of five wheels w_1 and one wheel w_2 , see fig. 6.

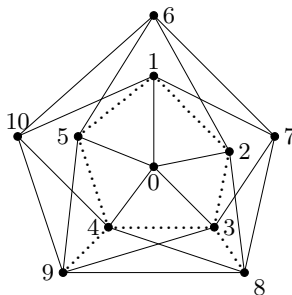


Fig. 6. An example of a non-planar graph that is 4-critical: Grötzsch graph. It may be built as an edge sum $5W_3 + W_5$ modulo two. Annihilated edges are depicted as dotted for one section standing for W_3 and for W_5 .

The Grötzsch graph is 4-critical and it quadrangulates projective plane, see fig. 7. Indeed, it has 11 vertices and 20 edges, i.e., it belongs to the $2n - 2$ -edges-class of graphs, and fig. 7 shows how this embedding on the projective plane is performed.

If in place of the Grötzsch graph formed as $5w_1 + w_2$ we take only three plus one wheel, we get graph that is isomorphic to the base graph w_{1_3} . Taking this fact into account, we designate this graph g_{111} or g_{1_3} and the traditional

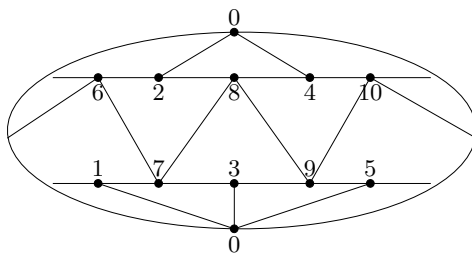


Fig. 7. The Grötzsch graph on the projective plane. It quadrangulates projective plane. Compare [5].

Grötzsch graph as g_{111111} or g_{15} . First, let us notice that both graphs are 4-critical, both quadrangulate projective plane, first being planar, bet second - projective planar. We might ask - are all graphs $qw_1 + w_q$ summed according multiwheel summation pattern 4-critical? The answer is quite obviously positive, and we express the fact in the lemma what follows. We say that sum $qw_1 + w_q$, $k = 2q + 1$, $q > 0$, modulo two is got according the multiwheel pattern if k wheels w_1 each looses two edges and w_q looses k edges. This class of graphs we call the Grötzsch class.

Lemma 5 *For $k = 2q + 1$, $q > 0$, resulting graphs from $kw_1 + w_q$ summed according the multiwheel pattern are 4-critical and quadrangulate the projective plane.*

Of course, for $k = 1, 2$, we get the base graph and the Grötzsch graph, which are 4-critical, and further graphs are 4-critical due to symmetry.

This class $kw_1 + w_q$, extending the base graph and the Grötzsch graph may be received by the Mycielski's Construction [7]. For that we are to take 3-critical graph, i.e., arbitrary odd cycle C_k , and apply Mycielski's Construction. Thus we see that if in Mycielski's Construction we replace each new got k-critical graph with arbitrary k-critical graph then we should receive $k + 1$ -critical graph, which fact follows from the proof of the Mycielski's Construction's applicability to get k-critical graphs, see [7]. Natural question would arise does there exist Mycielski's Construction's generalization that works backwards too, i.e., that each $k + 1$ -critical graph has as antecedent k-critical graphs in terms of this or similar construction. In order to include in Mycielski's Construction previous planar class we are to allow to match previous graphs m edges with m new vertices plus extra vertex. This would work for the step from 3-critical to 4-critical graphs, and give just our plane class of multiwheels.

Further we are going to build more multiwheels, but the previous class should be the only that were quadrangulating projective plane.

Further we generalize the projective planar multiwheels similarly as in the case of the plane multiwheels, i.e., sections of w_1 may be replaced with arbitrary odd wheels. Fig. 8 shows simplest properly projective plane multiwheel q_{112} .

Construction 6 *Let us take odd in number ($k = 2q + 1$) odd wheels and one wheel w_q . Let us take in each of the first wheels two proximal spikes and rim edge so that they do not form triangle, and the middle spike edge matches with the central wheel w_q , and other two chosen edges (spike and rim edge) match in cyclical sequence of wheels.*

The resulting graph built according construction 6 belongs to $2n - 2$ edges class and is 4-critical. We call the resulting graph *multiwheel* similarly to those planar ones.

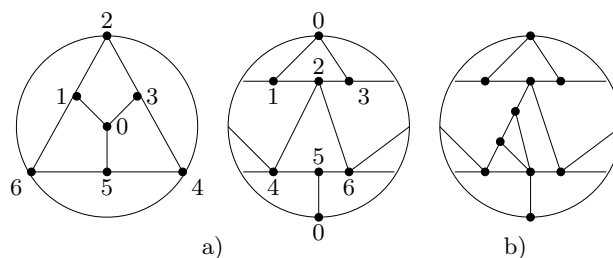


Fig. 8. a) The graphs w_{111} and g_{111} are isomorphic; b) simplest properly projective plane multiwheel with minimal edge number q_{112} .

Theorem 7 *Multiwheels built according the construction 6 are 4-critical.*

Proof. Let us use the fact that the base graph belongs to Grötzsch class, and the construction for the base graph extended with non-planar section (see fig.5) may be used for Grötzsch class in the whole.

Let us end this section with one more theorem.

Theorem 8 *Multiwheel quadrangulates projective plane only if it belongs to the Grötzsch class.*

Proof. The only subclass to be considered is plane multiwheels with simple sections, excluding the base graph, i.e., w_{1_q} , $q > 1$. It suffices to consider the minimal graph from the class w_{1_2} . For a graph to quadrangulate a surface it is necessary that every edge goes into at least two square cycles. But the edge of w_{1_2} that is incident to vertices of degree three and four doesn't fulfil this condition.

4 Octahedral theorem

Let us formulate what we call octahedral theorem for the plane multiwheels.

As was told in the introduction, a minor bracket works for the base graph, i.e., $\langle O^-, O; w_{1_3} \rangle$ is true: $O^- \prec w_{1_3}$, $O \not\prec w_{1_3}$ and $O^- \prec O$. It easily follows from the facts that the base graph is only vertex 3-connected, i.e., it has triples of separating vertices, as long as octahedron graph doesn't have. This argument directly applies to the plane multiwheels in general, because they are built allowing triples of separating vertices for each section, that excludes possibility for O to be minor. Both the plane and the projective plane multiwheels have the base graph as their minor. Besides, the Grötzsch graph doesn't have O as minor. Indeed, it has 5 cubic vertices, which may be separated with a triple of vertices, and adjacent to central hub vertex, and remaining 5 vertices aren't sufficient to hold O as minor. This argument easily generalizes to the Grötzsch class in the whole. It only remains to persuade oneself that it works for the projective plane multiwheel in general. And again, sections that are differing from simple ones can be separated by triples of vertices, see fig.8, b. Thus, we have proved the theorem.

Let us formulate the fact for arbitrary multiwheels as the theorem.

Theorem 9 *The minor bracket $\langle O^-, O \rangle$ works for both the plane and the projective plane multiwheel graph classes.*

References

1. Jensen Tommy R., Toft Bjarne. *Graph Coloring Problems*, John Willey and Sons, 1995, 295pp.
2. Koester G. *4-critical 4-valent planar graphs constructed with crowns*, Math Scand.67, 1990, 15-22.
3. Kratochvíl J. *About minor closed classes and the generalization of the notion of free-planar graphs*, personal communication, 1994, 2pp.
4. Lovász László. *Combinatorial Problems and Exercises*, Elsevier Science Publishers, 2nd edition, 1993, 635pp.
5. Mohar Bojan. *Quadrangulations and 5-critical Graphs on the Projective Plane*, 565-580, in Topics in Discrete Mathematics, Dedicated to Jarik Nešetřil on the Occasion of his 60th Birthday, Springer, 2006.
6. Mohar Bojan, Thomassen Carsten. *Graphs on Surfaces*, J. Hopkins Univ. Press, 2001.
7. Youngs D.A. *4-Chromatic Projective Graphs*, Journal of Graph Theory, Vol. 21, No. 2, 1996, 219-227
8. Zeps D. *Free Minor Closed Classes and the Kuratowski Theorem*, KAM Series, 98-409, Prague, 1998, 10 pp.
9. Zeps D. *4-critical wheel graphs of higher order*, arXiv:1106.1336v1, 2007, 4 pp, congreso.us.es/eurocomb07/program_thursday.html
10. Zeps D. *Application of the Free Minor Closed Classes in the Context of the Four Color Theorem*, hal-00408145, 2008, 19pp.