Semiclassical singularities from bifurcating orbits

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We investigated numerically, for a generic quantum system (a kicked top), how the singular behavior of classical systems at bifurcations is reflected by their quantum counterpart. Good agreement is found with semiclassical predictions.

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I. INTRODUCTION

The semiclassical approach allows to obtain spectral information about quantum systems from properties of classical periodic orbits. The most famous example of this quantum-classical correspondence is Gutzwiller's trace formula for completely chaotic (hyperbolic) autonomous systems [1], which expresses the density-of-states as a sum of contributions from periodic orbits.

Generic systems are neither hyperbolic nor integrable, but have a mixed phase space in which regions of stability coexist with chaotic dynamics. Characteristic for the mixed phase space is the ubiquity of bifurcations [2,3]. The contributions of Gutzwiller type diverge when orbits bifurcate and have then to be replaced by uniform approximations (collective contributions of the bifurcating orbits) [3–9]. The ensuing semiclassical amplitude is finite at $\hbar \neq 0$, even directly at the bifurcation, and diverges with a power law $\sim \hbar^{-\nu}$ as $\hbar \rightarrow 0$, where ν is called the singularity exponent. This peculiar singularity has been studied recently in the context of spectral fluctuations [10,11]. A full solution has been given in Ref. [11], which accounts also for more complex bifurcations of higher codimension, which are classically nongeneric, but are nevertheless relevant in the quantum realm.

In view of the recent progress, a test of the semiclassical predictions for generic quantum systems close to bifurcations is called for. In this paper we test the predictions for the kicked top [12], a periodically driven system with one degree of freedom, which is also representative for autonomous systems with two freedoms. We devise a filtering technique that allows to extract contributions of individual groups of bifurcating orbits. The singularity exponent ν is found to correspond well to the theoretical predictions.

The paper is organized as follows: In Sec. II we formulate the problem and describe how the singularity exponents ν are derived. In Sec. III we present numerical results for the kicked top. Section IV contains our conclusions.

II. SEMICLASSICAL CONTRIBUTIONS AT BIFURCATIONS

Periodically driven systems are stroboscopically described by a unitary Floquet operator F. Spectral information about this operator is most conveniently extracted from the traces $\mathrm{tr}\, F^n$, where n plays the role of discretized time. The analogue of Gutzwiller's trace formula has been derived by Tabor [13], who found the relation

$$\mathrm{tr}\, F^n = \sum_{\mathrm{p.\,o.}}^{n=rn_0} \frac{n_0}{|2-\mathrm{tr}\, M|^{1/2}} \exp\left(\mathrm{i}\, JS - \mathrm{i}\, \gamma \frac{\pi}{2} \right), \qquad (1)$$

between the traces and the periodic orbits of the corresponding chaotic classical map. For convenience we denote here the inverse Planck's constant by $\hbar^{-1} = J$. The sum is made up of all orbits of primitive (first return) period n_0 with $n = n_0 r$ and r an integer. The rth return of an orbit is characterized by the action $S = rS_0$, trace of the monodromy matrix (linearized map) $M = M_0^r$, and the Maslov index $\gamma = r\gamma_0$ (which for elliptic orbits satisfies a slightly more involved composition law under repetitions).

Equation (1) is valid for completely chaotic systems. In such systems, the Lyapunov exponents λ of all periodic orbits are positive. The eigenvalues of the monodromy matrix M_0 are $e^{\pm\lambda_0}$, hence the semiclassical amplitudes $A \propto (\sinh \lambda/2)^{-1}$ are finite.

The mixed phase space accommodates also elliptic orbits, for which the eigenvalues of M_0 are $e^{\pm i\omega_0}$. This gives the amplitude $A \propto (\sin \omega/2)^{-1}$ for the rth repetition of the orbit. The stability angle $\omega = r\omega_0$ increases linearly with the repetition number, and either by a suitable choice of r or of an external control parameter, the amplitude A can become arbitrarily large. The contribution of an individual orbit eventually diverges when $\omega/2\pi$ is an integer, hence when $\omega_0 = 2\pi n/m$, with n, m integers (taken relatively prime). Normal-form theory [2] shows that this is precisely the condition for a bifurcation, the coalescence of two or more periodic orbits. The type of bifurcation depends on m, with $m = 1$ the tangent bifurcation, $m = 2$ the period-doubling bifurcation, $m = 3$ the period-tripling bifurcation, and so forth.

Catastrophe theory [14] further reveals that the divergence of Eq. (1) at a bifurcation comes from an inadmissible stationary-phase approximation, and provides uniform approximations that regularize the singular behavior [3–9]. Close to a bifurcation one has to replace contributions of individual orbits by collective contributions in the trace formula, of the form

$$A=\frac{J}{2\pi}\int_0^\infty \mathrm{d}I\int_0^{2\pi}\mathrm{d}\phi\Psi(I,\phi)\exp[\mathrm{i}J\Phi(I,\phi)], \quad (2)$$

with the "amplitude function" Ψ and the "phase function" Φ both depending on the type of bifurcation under consideration. Here we have used canonical polar coordinates I, ϕ, which parametrize the phase space of the classical map as

$$p=\sqrt{2I}\sin\phi, \quad q=\sqrt{2I}\cos\phi, \quad (3)$$

giving for the differentials $\mathrm{d}p\,\mathrm{d}q=\mathrm{d}I\,\mathrm{d}\phi$. The phase function is a local approximation to the generating function $S(q',p)$ of the classical map $(q,p)\to(q',p')$. Right at the bifurcation the amplitude function reduces to $\Psi=1$, while the phase function is given by simple normal forms. For generic bifurcations we have [3,6,7]

$$\begin{aligned}
\Phi&=S_0-\varepsilon q-aq^3-bp^2 \quad (m=1),\\
\Phi&=S_0-\varepsilon q^2-aq^4-bp^2 \quad (m=2),\\
\Phi&=S_0-\varepsilon I-aI^{3/2}\cos 3\phi \quad (m=3),\\
\Phi&=S_0-\varepsilon I-aI^2-bI^2\cos 4\phi \quad (m=4).
\end{aligned} \quad (4)$$

Here ε is the bifurcation parameter (bifurcations take place at $\varepsilon=0$), while S_0, a, and b can be regarded as constants. At the bifurcation ($\varepsilon=0$) we can rescale the integration variables q, p for $m=1,2$ or I for $m\geqslant 3$, such that the combination $J\Phi$ appearing in the exponent of Eq. (2) becomes independent of J. What remains is a J-dependent prefactor in front of a J-independent integral. This gives $A\propto J^\nu$ with [6]

$$\begin{aligned}
&\nu=1/6 \ (m=1), \quad \nu=1/4 \ (m=2),\\
&\nu=1/3 \ (m=3), \quad \nu=1/2 \ (m\geqslant 4).
\end{aligned} \quad (5)$$

For $\varepsilon\neq 0$ the integral remains J independent if one also rescales the bifurcation parameter according to $\varepsilon'=\varepsilon J^\mu$ [11], with

$$\begin{aligned}
&\mu=2/3 \ (m=1), \quad \mu=1/2 \ (m=2),\\
&\mu=1/3 \ (m=3), \quad \mu=1/2 \ (m\geqslant 4).
\end{aligned} \quad (6)$$

These exponents determine the semiclassical range of the bifurcations in parameter space.

The case $m=3$ is special in the sense that period-tripling bifurcations are usually accompanied by a tangent bifurcation, so close in parameter space that the semiclassical contribution given above, loses validity for accessible values of J. This allows us to test the predictions for a bifurcation of higher codimension. The normal form is [8,9]

$$\Phi(I,\phi')=S_0-\varepsilon I-aI^{3/2}\cos 3\phi-bI^{3/2}\sin 3\phi-cI^2. \quad (7)$$

The tangent bifurcation takes place at $\varepsilon=9(a^2+b^2)/32c$, while the period-tripling bifurcation occurs at $\varepsilon=0$. For $\varepsilon=a=b=0$ one has to consider an integral of the form

$$J\int_0^\infty \mathrm{d}I\int_0^{2\pi}\mathrm{d}\phi\exp[\mathrm{i}JI^2]\propto J^{1/2}, \quad (8)$$

and obtains $\nu=1/2$. For ε, a, $b\neq 0$ we obtain the two scaling parameters $\mu_\varepsilon=1/2$ and $\mu_a=\mu_b=1/4$ that characterize the semiclassical range of the bifurcation in parameter space.

III. NUMERICAL RESULTS

We now turn to the numerical investigation of the semiclassical singularities at bifurcations for the periodically kicked top [12]. The dynamics consists of a sequence of rotations and torsions, with Floquet operator

$$\begin{aligned}
F=&\exp\left(-\mathrm{i}\frac{k_1}{2j+1}\hat{J}_z^2-\mathrm{i}\,\alpha_1\hat{J}_z\right)\exp(-\mathrm{i}\,\beta\hat{J}_y)\\
&\times\exp\left(-\mathrm{i}\frac{k_2}{2j+1}\hat{J}_x^2-\mathrm{i}\,\alpha_2\hat{J}_x\right).
\end{aligned} \quad (9)$$

The angular momentum operators $\hat{J}_{x,y,z}$ obey the commutator relation $[\hat{J}_i,\hat{J}_j]=\mathrm{i}\,\varepsilon_{ijk}\hat{J}_k$. The phase space is a sphere because the square of the angular momentum $\mathbf{J}^2=j(j+1)$ is conserved. The role of the inverse Planck's constant is played by $J=j+1/2$, which is equal to one-half of the Hilbert space dimension. The semiclassical limit is reached by sending $J\to\infty$. We fix the rotation parameters $\alpha_1=0.8$, $\beta=1$, $\alpha_2=0.3$, and use the torsion strengths $k_1\equiv k$ and $k_2=k/10$ to control the degree of chaos of the classical map. The system is integrable for $k=0$ and displays well-developed chaos from $k\simeq 5$.

The quantum-mechanical evaluation of F is described in Ref. [12]. We computed the traces of the Floquet operator and separated the contributions of different (clusters of) orbits by evaluating the action spectrum (the Fourier transformation of the trace with respect to the inverse of Planck's constant) [15],

$$T^{(n)}(S)=\frac{1}{j_{\max}-j_{\min}+1}\sum_{j=j_{\min}}^{j_{\max}}\mathrm{tr}\,F^n(j)e^{-\mathrm{i}(j+1/2)S}, \quad (10)$$

where the difference $j_{\max}-j_{\min}$ determines the resolution in S ($j_{\min}=1, j_{\max}=100$). The results for parameters close to different types of bifurcations are shown in Fig. 1.

The contribution at given j of orbits pertaining to a given peak can be obtained by an inverse Fourier transformation,

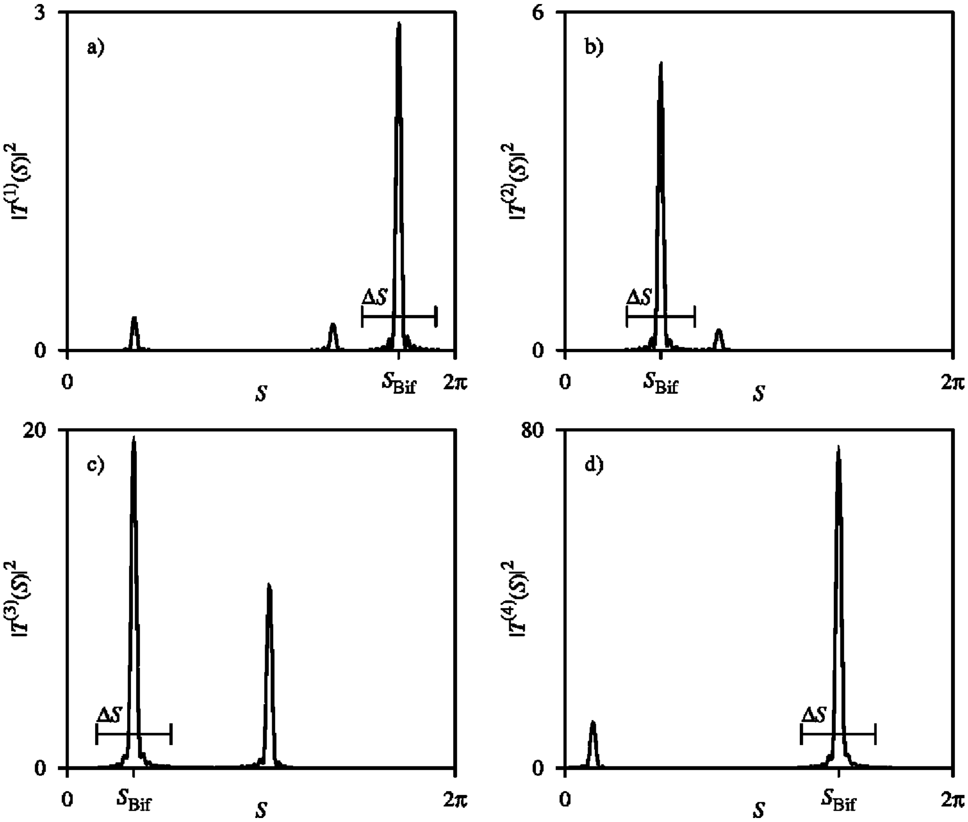

FIG. 1. The action spectrum $|T^{(n)}(S)|^2$ for the kicked top close to different types of bifurcations. For the inverse Fourier transformation (11) we restrict the S integration to the intervals of width ΔS around the centers of the peaks $S_{\rm Bif}$, eliminating in this way the contributions from other periodic orbits. (a) $n=1$, $k=2.6$. The large peak at $S_{\rm Bif}$ comes from orbits that are involved in a tangent bifurcation at $k\simeq 2.5$. (b) $n=2$, $k=2.3$, $\alpha_1=1.39$, close to the period-doubling bifurcation at $k\simeq 2.1$. (c) $n=3$, $k=2$, close to two period-tripling bifurcations at $k\simeq 1.85$ (left peak) and $k\simeq 1.97$ (right peak). (d) $n=4$, $k=1.2$. The large peak arises from orbits involved in a period-quadrupling bifurcation at $k\simeq 1.0$. (The orbits of the smaller peak are also involved in a period-quadrupling bifurcation, at $k\simeq 1.2$.)

$$A\propto\int_{S_{\rm Bif}-\Delta S/2}^{S_{\rm Bif}+\Delta S/2}{\rm d}\,S\left(\sum_{j'=j_{\min}}^{j_{\max}}{\rm tr}\,F^n(j')\exp[-{\rm i}(j'+1/2)S]\right)\times\exp[{\rm i}(j+1/2)S]$$
$$=2\sum_{j'=j_{\min}}^{j_{\max}}{\rm tr}\,F^n(j')\exp[{\rm i}(j-j')S_{\rm Bif}]\frac{\sin[(j-j')\Delta S/2]}{j-j'},\tag{11}$$

where the integral over actions S is restricted to an interval ΔS around the center $S_{\rm Bif}$ of the peak. This eliminates contributions of other periodic orbits.

Our goal is a purely quantum-mechanical test of the semiclassical predictions, which does not require any classical information, as the precise values of control parameters at the bifurcation. In order to achieve this we tune the control parameter k to the value that maximizes the A. The parameter k of the maximum approaches the true bifurcation point with the exponent μ, Eq. (6). The maximal contribution is of the same order of magnitude as the contribution at the bifurcation, and is also less sensitive to changes in the parameters.

We extract the exponents ν from logarithmic plots of the maximal $|A|$ versus J, shown in Fig. 2. In all cases we find good agreement with the theoretical predictions. For a tangent bifurcation at $k\simeq 2.5$, the observed exponent is $\nu\simeq 0.1866$ (theoretically, $\nu=1/6$). Two different period-doubling bifurcations appear at $k\simeq 2.8$ and produce overlapping peaks in the action spectrum. We separated them by changing α_1 to $\alpha_1=1.39$, moving in that way one of the period-doubling bifurcations to $k\simeq 2.1$. The exponent for this bifurcation is $\nu\simeq 0.2636$ (theoretically, $\nu=1/4$). Back to the original value $\alpha_1=0.8$, we find for the period-quadrupling bifurcation at $k\simeq 1.0$ the exponent $\nu\simeq 0.5734$ (theoretically, $\nu=1/2$).

Now we turn to period-tripling bifurcations, which are typically accompanied by a tangent bifurcation, so close in parameter space, that one has to treat the situation as a bifurcation of higher codimension. For the kicked top, an angular momentum of about $J\simeq 10^5$ would be needed for separating

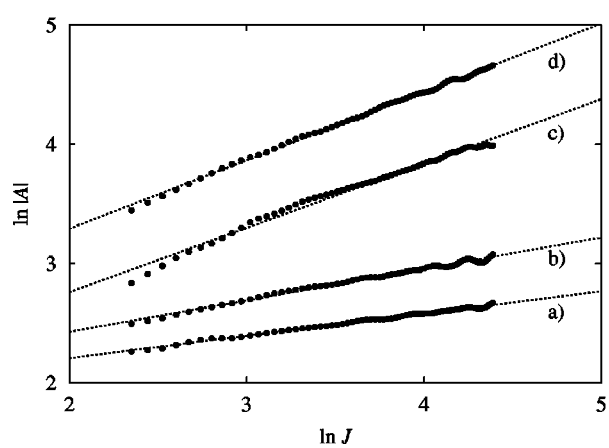

FIG. 2. Logarithmic plots versus inverse Planck's constant J of the maximal (in parameter space) contributions $|A|$ of orbits involved in the bifurcations of Fig. 1, calculated by Eq. (11). (a) Tangent bifurcation. The average (dotted line) gives the exponent $\nu\simeq 0.1866$. (b) Period-doubling bifurcation, $\nu\simeq 0.2636$. (c) Period-tripling bifurcation of higher codimension, $\nu\simeq 0.5327$. (d) Period-quadrupling bifurcation, $\nu\simeq 0.5734$.

the orbits in the ''period-tripling plus tangent'' bifurcation at $k \simeq 1.85$. The same is true for a similar sequence of bifurcations at $k \simeq 1.97$. For the much smaller values of J that we use here, we have the unique opportunity to test the exponent for a case of higher codimension. As before, the result $\nu \simeq 0.5327$ (for the bifurcations at $k \simeq 1.85$) is close to the theoretical expectation $\nu = 1/2$.

IV. CONCLUSIONS

We have studied the asymptotic behavior for $\hbar \rightarrow 0$ of periodic-orbit contributions to semiclassical trace formulas, around points in parameter space where orbits bifurcate. For the most common types of bifurcations the theoretically predicted power-law divergence $\propto \hbar^{-\nu}$ was tested numerically for a representative dynamical system, the kicked top, giving good agreement for the exponents ν.

In the semiclassical limit the contribution of nonbifurcating orbits reaches a constant value $|A| = \mathcal{O}(\hbar^0)$, corresponding to $\nu = 0$. It follows from Eq. (2) that the exponent for bifurcating orbits falls into the range $0 < \nu < 1$. As a consequence, the semiclassical contribution of bifurcating orbits is dominant when parameters are close enough to the bifurcation point. On first sight this seems to require a careful tuning of the parameters. From the perspective of spectral statistics, however, a careful tuning often turns out to be unnecessary [10,11]. Some quantities are dominated by bifurcating orbits as the consequence of a competition between different sort of bifurcations, in which each bifurcationenters with weight given by the exponents ν and μ. A numerical investigation of this competition is challenging.

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