C-SUPPLEMENTED SUBALGEBRAS OF LIE ALGEBRAS

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Abstract

A subalgebra B of a Lie algebra L is *c-supplemented* in L if there is a subalgebra C of L with $L = B + C$ and $B \cap C \leq B_L$, where B_L is the core of B in L. This is analogous to the corresponding concept of a c-supplemented subgroup in a finite group. We say that L is *c-supplemented* if every subalgebra of L is c-supplemented in L. We give here a complete characterisation of c-supplemented Lie algebras over a general field.
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1 Introduction

The concept of a c-supplemented subgroup of a finite group was introduced by Ballester-Bolinches, Wang and Xiuyun in [2] and has since been studied by a number of authors. The purpose of this paper is study the corresponding idea for Lie algebras. As we shall see, stronger results can be obtained in this context.

Throughout L will denote a finite-dimensional Lie algebra over a field F. If B is a subalgebra of L we define B_L, the *core* (with respect to L) of B to be the largest ideal of L contained in B. We say that B is *core-free* in L if $B_L = 0$. A subalgebra B of L is *c-supplemented* in L if there is a subalgebra C of L with $L = B + C$ and $B \cap C \leq B_L$. We say that L is *c-supplemented* if every subalgebra of L is c-supplemented in L. We shall give a complete characterisation of c-supplemented Lie algebras over a general field.

Following [4] we will say that L is *completely factorisable* if for every subalgebra B of L there is a subalgebra C such that $L = B+C$ and $B \cap C = 0$. It turns out that c-supplemented Lie algebras are intimately related to the completely factorisable ones, and our results generalise some of those obtained in [4]. Incidentally, it is claimed in [4] that if F has characteristic zero then L is completely factorisable if and only if the Frattini subalgebra of every subalgebra of L is trivial. We shall see that this is false.

If A and B are subalgebras of L for which $L = A + B$ and $A \cap B = 0$ we will write $L = A \dot{+} B$; if, furthermore, A, B are ideals of L we write $L = A \oplus B$. The notation $A \leq B$ will indicate that A is a subalgebra of B, and $A < B$ will mean that A is a proper subalgebra of B.

2 Preliminary results

First we give some basic properties of c-supplemented subalgebras

Lemma 2.1 *(i) If B is c-supplemented in L and $B \leq K \leq L$ then B is c-supplemented in K.*

(ii) If I is an ideal of L and $I \leq B$ then B is c-supplemented in L if and only if B/I is c-supplemented in L/I.

(iii) If $\mathcal{X}$ is the class of all c-supplemented Lie algebras then $\mathcal{X}$ is subalgebra and factor algebra closed.

Proof.

(i) Suppose that B is c-supplemented in L and $B \leq K \leq L$. Then there is a subalgebra C of L with $L = B + C$ and $B \cap C \leq B_L$. It follows that $K = (B + C) \cap K = B + C \cap K$ and $B \cap C \cap K \leq B_L \cap K \leq B_K$, and so B is c-supplemented in K.

(ii) Suppose first that B/I is c-supplemented in L/I. Then there is a subalgebra C/I of L/I such that $L/I = B/I + C/I$ and $(B/I) \cap (C/I) \leq (B/I)_{L/I} = B_L/I$. It follows that $L = B+C$ and $B \cap C \leq B_L$, whence B is c-supplemented in L.

Suspose conversely that I is an ideal of L with $I \leq B$ such that B is c-supplemented in L. Then there is a subalgebra C of L such that $L = B + C$ and $B \cap C \leq B_L$. Now $L/I = B/I + (C+I)/I$ and $(B/I) \cap (C+I)/I = (B \cap (C+I))/I = (I + B \cap C)/I \leq B_L/I = (B/I)_{L/I}$, and so B/I is c-supplemented in L/I.

(iii) This follows immediately from (i) and (ii).

The *Frattini ideal* of L, $\phi(L)$, is the largest ideal of L contained in all maximal subalgebras of L. We say that L is *ϕ-free* if $\phi(L) = 0$. The next result shows that subalgebras of the Frattini ideal of a c-supplemented Lie algebra L are necessarily ideals of L.

Proposition 2.2 *Let B, D be subalgebras of L with $B \leq \phi(D)$. If B is c-supplemented in L then B is an ideal of L and $B \leq \phi(L)$.*

Proof. Suppose that $L = B + C$ and $B \cap C \leq B_L$. Then $D = D \cap L = D \cap (B + C) = B + D \cap C = D \cap C$ since $B \leq \phi(D)$. Hence $B \leq D \leq C$, giving $B = B \cap C \leq B_L$ and B is an ideal of L. It then follows from [6, Lemma 4.1] that $B \leq \phi(L)$.

The Lie algebra L is called *elementary* if $\phi(B) = 0$ for every subalgebra B of L; it is an *E-algebra* if $\phi(B) \leq \phi(L)$ for all subalgebras B of L. Then we have the following useful corollary.

Corollary 2.3 *If L is c-supplemented then L is an E-algebra.*

Proof. Simply put $B = \phi(D)$ in Proposition 2.2.

It is clear that if L is completely factorisable then it is c-supplemented. However, the converse is false. Every completely factorisable Lie algebra must be ϕ-free, whereas the same is not true for c-supplemented algebras. For example, the three-dimensional Heisenberg algebra is c-supplemented, as will be clear from the next result which gives the true relationship between these two classes of algebras.

Proposition 2.4 *Let L be a Lie algebra. Then the following are equivalent:*

(i) L *is c-supplemented.*

(ii) $L/\phi(L)$ *is completely factorisable and every subalgebra of* $\phi(L)$ *is an ideal of* L*.*

Proof. (i) $\Rightarrow$ (ii): Suppose first that L is ϕ-free and c-supplemented, and let B be a subalgebra of L. Then there is a subalgebra C of L such that $L = B + C$. Choose D to be a subalgebra of L minimal with respect to $L = B + D$. Then $B \cap D \leq \phi(D)$, by [6, Lemma 7.1], whence $B \cap D = 0$ since L is elementary, by Corollary 2.3. Hence L is completely factorisable, and (ii) follows from Lemma 2.1(iii) and Proposition 2.2.

(ii) $\Rightarrow$ (i): Suppose that (ii) holds and let B be a subalgebra of L. Then there is a subalgebra $C/\phi(L)$ of $L/\phi(L)$ such that $L/\phi(L) = ((B + \phi(L))/\phi(L)) + (C/\phi(L))$ and $0 = ((B+\phi(L))/\phi(L)) \cap (C/\phi(L)) = (B \cap C + \phi(L))/\phi(L)$. Hence $L = B + C$ and $B \cap C \leq \phi(L)$, so $B \cap C$ is an ideal of L and $B \cap C \leq B_L$; that is, L is c-supplemented.

Note that if L is the three-dimensional Heisenberg algebra, then condition (ii) in the above result holds, since $\phi(L) = L^2$ is one dimensional and $L/\phi(L)$ is abelian. Finally we shall need the following result concerning direct sums of of completely factorisable Lie algebras.

Lemma 2.5 *If* A *and* B *are completely factorisable, then so is* $L = A \oplus B$*.*

Proof. Suppose that A, B are completely factorisable and put $L = A \oplus B$. Let U be a subalgebra of L. If $A \leq U$, then $U = A \oplus (B \cap U)$. Since B is completely factorisable there is a subalgebra C of B such that $B = B \cap U + C$ and $U \cap C = B \cap U \cap C = 0$. Hence $L = U \dot{+} C$.

Now $A \leq A + U$ so, by the above, there is a subalgebra C of B with $L = A + U + C$ and $(A + U) \cap C = 0$. Moreover, since A is completely factorisable, there is a subalgebra D of A such that $A = A \cap U + D$ and $U \cap D = A \cap U \cap D = 0$. It follows that $L = U + (D \oplus C)$ and $U \cap (D + C) \leq U \cap [(A + U) \cap (D + C)] = U \cap [D + (A + U) \cap C] = U \cap D = 0$. It follows that L is completely factorisable.

Note that the corresponding result for c-supplemented Lie algebras is false. For, let $L_1 = Fx + Fy + Fz$ with $[x, y] = -[y, x] = y + z, [x, z] = -[z, x] = z$ and all others products equal to zero. Then it is straightforward to check that $\phi(L_1) = Fz$ and that L_1 is c-supplemented. Now take L to be a direct sum of two copies of L_1: say, $L = A \oplus B$ where $A = Fx + Fy + Fz$, $B = Fa + Fb + Fc$, $[x, y] = -[y, x] = y + z, [x, z] = -[z, x] = z$, $[a, b] = -[b, a] = b + c, [a, c] = -[c, a] = c$ and all others products equal to zero. Suppose that $F(z + c)$ is c-supplemented in L. Then there is a subalgebra M of L with $L = F(z + c) + M$ and $F(z + c) \cap M \leq (F(z + c))_L$. If $z + c \notin M$ then M is a maximal subalgebra of L, contradicting the fact that $z + c \in (\phi(A) \oplus \phi(B)) = \phi(L)$, by [6, Theorem 4.8]. It follows that $z + c \in M$, whence $F(z + c)$ is an ideal of L. But $[x, z + c] = z \notin F(z + c)$, a contradiction. Thus L is not c-supplemented in L.

3 The structure theorems

We can now give the main structure theorems for c-supplemented Lie algebras. First we determine the solvable ones.

Theorem 3.1 *Let L be a solvable Lie algebra. Then the following are equivalent:*

(i) L is c-supplemented.

(ii) L is supersolvable and every subalgebra of $\phi(L)$ is an ideal of L.

Proof. (i) $\Rightarrow$ (ii): We have that every subalgebra of $\phi(L)$ is an ideal of L by Proposition 2.4, so we have only to show that L is supersolvable. Let L be a minimal counter-example. Then all proper subalgebras and factor algebras of L are supersolvable, by Lemma 2.1(iii). If we can show that all maximal subalgebras have codimension one in L, we shall have the desired contradiction, by [3, Theorem 7]; so let M be any maximal subalgebra of L. Since the result is clear if $M_L \neq 0$, we may assume that $M_L = 0$.

Pick a minimal ideal A of L. Then $L = A \dot{+} M$ and A is the unique minimal ideal of L, by [7, Lemma 1.4]. Let $a \in A$. Then Fa is c-supplemented in L, and so there is a subalgebra B of L such that $L = Fa + B$ and $Fa \cap B \leq (Fa)_L$. If $a \in B$ then Fa is an ideal of L, whence $A = Fa$ and M has codimension one in L.

So suppose that $L = Fa \dot{+} B$. Since $A \not\leq B$ we have $B_L = 0$. But then $L = A \dot{+} B$ by [7, Lemma 1.4] again. It follows that dim $A = 1$ and M has codimension one in L.

(ii) $\Rightarrow$ (i): By Proposition 2.4, it suffices to show that if L is supersolvable and ϕ-free then it is completely factorisable. Let L be a minimal counter-example. Then L is elementary, by [5, Theorem 1], and so every proper subalgebra of L is completely factorisable. Also $L = A\dot{+}B$ where $A = Fa_1 \oplus \ldots \oplus Fa_n$ is the abelian socle of L and B is abelian, by [7, Theorem 7.3]. Let U be a subalgebra of L. If $A \leq U$ it is clear that there is a subalgebra C of L such that $L = U + C$ and $U \cap C = 0$. So suppose that $a_i \notin U$ for some $1 \leq i \leq n$; we may as well assume that $i = 1$. Then $L/Fa_1 \cong (Fa_2 \oplus \ldots \oplus Fa_n)\dot{+}B$, which is a proper subalgebra of L and so is completely factorisable. Hence there is a subalgebra C of L such that $L/Fa_1 = ((U + Fa_1)/Fa_1) + (C/Fa_1)$ and $Fa_1 = (U + Fa_1) \cap C = U \cap C + Fa_1$. It follows that $L = U + C$ and $U \cap C \leq Fa_1$. But $a_1 \notin U \cap C$ so $U \cap C = 0$ and L is completely factorisable, a contradiction.

We shall need the following classification of Lie algebras with core-free subalgebras of codimension one which is given by Amayo in [1].

Theorem 3.2 *([1, Theorem 3.1]) Let L have a core-free subalgebra of codimension one. Then either (i) dim L ≤ 2, or else (ii) L $\cong L_m(\Gamma)$ for some m and Γ satisfying certain conditions (see [1] for details).*

We shall also need the following properties of $L_m(\Gamma)$ which are given by Amayo in [1].

Theorem 3.3 *([1, Theorem 3.2])*

(i) If $m > 1$ and m is odd, then $L_m(\Gamma)$ is simple and has only one subalgebra of codimension one.

(ii) If $m > 1$ and m is even, then $L_m(\Gamma)$ has a unique proper ideal of codimension one, which is simple, and precisely one other subalgebra of codimension one.

(iii) $L_1(\Gamma)$ has a basis $\{u_{-1}, u_0, u_1\}$ with multiplication $[u_{-1}, u_0] = u_{-1} + \gamma_0 u_1$ ($\gamma_0 \in F, \gamma_0 = 0$ if $\Gamma = \{0\}$), $[u_{-1}, u_1] = u_0, [u_0, u_1] = u_1$.

(iv) If F has characteristic different from two then $L_1(\Gamma) \cong L_1(0) \cong sl_2(F)$.

(v) If F has characteristic two then $L_1(\Gamma) \cong L_1(0)$ if and only if γ_0 is a square in F.

The above properties enable us to determine which of the algebras $L_m(\Gamma)$ are c-supplemented.

Proposition 3.4 *If $L \cong L_m(\Gamma)$ then L is c-supplemented if and only $L \cong L_1(0)$ and F has characteristic different from two.*

Proof. Suppose that $L \cong L_m(\Gamma)$ and L is c-supplemented, and let $x \in L$. Then there is a subalgebra M_1 of L such that $L = Fx + M_1$, and $Fx \cap M_1 \leq (Fx)_L = 0$, since $L_m(\Gamma)$ has no one-dimensional ideals. Choose $y \in M_1$. Then, similarly, there is a subalgebra M_2 of codimension one in L such that $L = Fy + M_2$ and $M_1 \neq M_2$. Since $L = M_1 + M_2$ we have that $M_1 \cap M_2 \neq 0$. Let $z \in M_1 \cap M_2$. Then there is a subalgebra M_3 of codimension one in L such that $L = Fz + M_3$, so L has at least three subalgebras of codimension one in L. It follows from Theorem 3.3 that $m = 1$.

Suppose that $L \ncong L_1(0)$. Then F has characteristic two and γ_0 is not a square in F. Since L is completely factorisable there is a two-dimensional subalgebra M of L such that $L = Fu_1 + M$. It follows that $M = F(u_{-1} + \alpha u_1) + F(u_0 + \beta u_1)$ for some $\alpha, \beta \in F$. But then $[u_{-1} + \alpha u_1, u_0 + \beta u_1] \in M$ shows that $\gamma_0 = \beta^2$, a contradiction. A further straightforward calculation shows that if $L \cong L_1(0)$ and F has characteristic two, then Fu_1 is contained in every maximal subalgebra of L, and so has no c-supplement in L.

Conversely, suppose that $L \cong L_1(0)$ and F has characteristic different from two. Then $L \cong sl_2(F)$, by Theorem 3.3 (iv) and it is easy to check that L is c-supplemented.

We can now determine the simple and semisimple c-supplemented Lie algebras.

Corollary 3.5 *If L is simple then L is c-supplemented if and only $L \cong L_1(0)$ and F has characteristic different from two.*

Proof. Let L be simple and c-supplemented. Then L has a core-free maximal subalgebra of codimension one in L and so $L \cong L_m(\Gamma)$, by Theorem 3.2. The result now follows from Proposition 3.4.

Notice, in particular, that $sl_2(F)$ is the only simple completely factorisable Lie algebra over any field. However, this is not the only simple elementary Lie algebra, even over a field of characteristic zero: over the real field every compact simple Lie algebra, and $so(n, 1)$ for $n > 3$, for example, are elementary, as is shown in [8, Theorem 5.1]. This justifies the assertion made at the end of the third paragraph of the introduction.

Proposition 3.6 *Let L be a semisimple Lie algebra over a field F. Then the following are equivalent:*

(i) L is c-supplemented.

(ii) $L = S_1 \oplus \ldots \oplus S_n$ where $S_i \cong sl_2(F)$ for $1 \leq i \leq n$ and F has characteristic different from two.

Proof. (i) $\Rightarrow$ (ii): Let L be semisimple and c-supplemented and suppose the result holds for all such algebras of dimension less than dim L. Then $\phi(L) = 0$, since $\phi(L)$ is nilpotent, and so L is completely factorisable. Let A be a minimal ideal of L and pick $a \in A$. Let M be a subalgebra of L such that $L = Fa\dot{+}M$ and put $B = A + M_L$. Then $M_L < B$ and $A \cap M_L = 0$, since $a \notin M_L$. If dim $L/M_L \leq 2$ then A is abelian, contradicting the fact that L is semisimple. It follows from Theorem 3.2 and Proposition 3.4 that $L/M_L \cong L_1(0)$, whence $B = L$ and $L = A \oplus M_L$. Since A, M_L are semisimple and c-supplemented the result follows.

(ii) $\Rightarrow$ (i):The converse follows from Corollary 3.5 and Lemma 2.5.

Finally we have the main classification theorem.

Theorem 3.7 *Let L be Lie algebra. Then the following are equivalent:*

(i) L is c-supplemented.

(ii) $L/\phi(L) = R \oplus S$ where R is supersolvable and ϕ-free, S is given by Proposition 3.6, and every subalgebra of $\phi(L)$ is an ideal of L.

Proof. (i) $\Rightarrow$ (ii): Factor out $\phi(L)$ so that L is ϕ-free and c-supplemented and hence completely factorisable, by Proposition 2.4. Then $L = R\dot{+}S$ where R is the radical of L and S is semisimple. It suffices to show that $SR = 0$; the rest follows from Lemma 2.1, Corollary 2.3, Proposition 2.4, Theorem

3.1 and Proposition 3.6. Suppose there is $0 \neq x \in L^{(3)} \cap R$. Then there is a subalgebra M of L such that $L = Fx \dot{+} M$ and L/M_L is given by Theorem 3.2. If $L/M_L \cong L_m(\Gamma)$ then L/M_L is simple, by Proposition 3.4, and $M_L < R + M_L$, so $L = R + M_L$. But then L/M_L is solvable, a contradiction. It follows that dim $L/M_L \leq 2$, whence $x \in L^{(3)} \cap R \leq L^{(3)} \leq M_L \leq M$, a contradiction. Hence $L^{(3)} \cap R = 0$. But $SR = S^2R \leq S(SR) = S^2(SR) \leq L^{(3)} \cap R = 0$, as required.

(ii) $\Rightarrow$ (i): This follows from Proposition 2.4, Lemma 2.5, Theorem 3.1 and Proposition 3.6.

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