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Union Closed Set Conjecture and Maximum Dicut in Connected Digraph

Nana Li

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UNION CLOSED SET CONJECTURE AND MAXIMUM DIRECTED CUT IN CONNECTED DIGRAPH

by

NANA LI

Under the Direction of Dr. Guantao Chen

ABSTRACT

In this dissertation, we study the following two topics, i.e., the union closed set conjecture and the maximum edges cut in connected digraphs.

The union-closed-set-conjecture-topic goes as follows. A finite family of finite sets is *union closed* if it contains the union of any two sets in it. Let $X_{\mathcal{F}} = \cup_{F \in \mathcal{F}} F$. A union closed family of sets is *separating* if for any two distinct elements in $\mathcal{F}$, there is a set in $\mathcal{F}$ containing one of them, but not the other and there does not exist an element which is contained in every set of it. Note that any union closed family $\mathcal{F}$ is a poset with set inclusion as the partial order relation. A separating union closed family $\mathcal{F}$ is *irreducible* (*normalized*) if $|X_{\mathcal{F}}|$ is the minimum (maximum, resp.) with respect to the poset structure of $\mathcal{F}$. In the part of dissertation related to this topic, we develop algorithms to transfer any given separating union closed family to a/an normalized/irreducible family without changing its poset structure. We also study properties of these two extremal union closed families in connection with the *Union Closed Sets Conjecture* of Frankl. Our result may lead to potential full proof of the union closed set conjecture and several other conjectures.

The part of the dissertation related to the maximum edge cuts in connected digraphs goes as follows. In a given digraph D, a set F of edges is defined to be a *directed cut* if there is a nontrivial partition (X, Y) of $V(D)$ such that F consists of all the directed edges from X to Y. The maximum size of a directed cut in a given digraph D is denoted by $\Lambda(D)$, and we let $\mathcal{D}(1,1)$ be the set of all digraphs D such that $d^+(v) = 1$ or $d^-(v) = 1$ for every vertex v in D. In this part of dissertation, we prove that $\Lambda(D) \geq \frac{3}{8}(|E(D)| - 1)$ for any connected digraph $D \in \mathcal{D}(1,1)$, which provides a positive answer to a problem of Lehel, Maffray, and Preissmann. Additionally, we consider triangle-free digraphs in $\mathcal{D}(1,1)$ and answer their another question.

INDEX WORDS: Lattice, Union closed sets, Set theory, Directed cut, Connected digraph

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by

NANA LI

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by

NANA LI

Committee Chair:	Guantao Chen
Committee:	Hendricus van der Holst (Co-chair) Yi Zhao Christian Avart

Electronic Version Approved:

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DEDICATION

This dissertation is dedicated to my parents and my grandparents.

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PART 1

INTRODUCTION

This dissertation mainly include the following two topics.

1.1 Union Closed Set Conjecture

A family of finite sets $\mathcal{F}$ is *union closed* if it contains the union of any two sets in it. Let $X_{\mathcal{F}} = \cup_{A \in \mathcal{F}} A$. Here in this dissertation, we always assume that $|X_{\mathcal{F}}|$ is finite, which implies that $|\mathcal{F}| \leq 2^{|X_{\mathcal{F}}|}$ is also finite. It is commenly believed[1] that Frankl in late 1979 formulated the following conjecture.

Conjecture 1.1.1. *[11] For any union closed family $\mathcal{F}$ of finite sets, in which at least one set is non-empty, there is an element $x \in X_{\mathcal{F}}$ contained in at least half of the sets in $\mathcal{F}$.*

Being simply formulated and hence fantastically interesting as it is, this conjecture has been known to be notoriously difficult and has been widely open for a long time. Closely related to lattice theory, extremal set theory and graph theory, many people from different mathematical areas have made various contributions during the course to attack it. In this part of dissertation, we will present our structural distributions which could potentially fully solve this conjecture and several other related conjectures.

There is a great survey paper about the union closed set conjecture by Bruhn, et al [4]. So, we do not aim to give a complete review of the literature on the conjecture. The focus of this part of the dissertation is on the methods employed to attack the conjecture which interest us and the exploration of our contributions. Our selection of the literature is thus not even.

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[1] Some people [2] may call it a "folklore conjecture in 1970's".

1.2 Maximum Directed Cut in Connected Digraph

In a given digraph D, a set F of edges is defined to be a *directed cut* if there is a nontrivial partition (X, Y) of $V(D)$ such that F consists of all the directed edges from X to Y. The maximum size of a directed cut in a given digraph D is denoted by $\Lambda(D)$, and we let $\mathcal{D}(1,1)$ be the set of all digraphs D such that $d^+(v) = 1$ or $d^-(v) = 1$ for every vertex v in D. In this part of dissertation, we prove that $\Lambda(D) \geq \frac{3}{8}(|E(D)| - 1)$ for any connected digraph $D \in \mathcal{D}(1,1)$, which provides a positive answer to a problem of Lehel, Maffray, and Preissmann. Additionally, we consider triangle-free digraphs in $\mathcal{D}(1,1)$ and answer their another question[2].

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[2] This part of dissertation has already been published on Journal of Graph Theory with all copy rights reserved, see [7].

PART 2

UNION CLOSED SET CONJECTURE

2.1 Elementary facts and notations

In this section, we briefly state some notations and elementary facts.

For two distinct families $\mathcal{F}$ and $\mathcal{H}$, we denote by $\mathcal{H} \subseteq \mathcal{F}$ the fact that $\mathcal{H}$ is a sub-family of $\mathcal{F}$, i.e., each set from $\mathcal{H}$ is also contained in $\mathcal{F}$. For any union closed family $\mathcal{F}$ and any sub-family $\mathcal{S}$ of $\mathcal{F}$, we denote by $\cup\mathcal{S}$ the union of all the sets in $\mathcal{S}$, we denote by $\cap\mathcal{S}$ the intersection of all the sets in $\mathcal{S}$, and we denote by $\mathcal{F} - \mathcal{S}$ the family of all the sets in $\mathcal{F}$ not contained in $\mathcal{S}$. For any family $\mathcal{S}$ and any set A (either $A \in \mathcal{S}$ or not), we denote by $\mathcal{S} - \{A\}$ the family of all the sets in $\mathcal{S}$ except the set A, we denote by $\mathcal{S} + \{A\}$ the family of all the sets in $\mathcal{S}$ and the set A, and we denote by $|\mathcal{S}|$ the number of sets in $\mathcal{S}$. For any two sets A and B in $\mathcal{F}$, if either $A \subseteq B$ or $B \subseteq A$, we denote it by $A \sim B$. Otherwise, we denote it by $A \nsim B$.

For a given union closed family $\mathcal{F}$ and a given set Y in $\mathcal{F}$, we denote by $Y \cap \mathcal{F}$ the family of all the sets obtained by the intersection of the set Y and all the sets from $\mathcal{F}$, i.e., $Y \cap \mathcal{F} = \{Y \cap A \mid A \in \mathcal{F}\}$ and we denote by 2^Y the power family of the set Y, i.e., $2^Y = \{A \mid A \subseteq Y\}$, and we denote by $X_{\mathcal{F}}$ the underlying set in $\mathcal{F}$, i.e., $X_{\mathcal{F}} = \cup\mathcal{F}$.

For two distinct sets A and B in a given union closed family $\mathcal{F}$, A is a *parent* of B if $B \subsetneq A$ and for any $C \in \mathcal{F}$ with $B \subseteq C \subseteq A$, either $B = C$ or $A = C$. B is a *child* of A if A is a parent of B. A set A is called a *single-parent-set* if A has only one parent in $\mathcal{F}$. For any given sub-family $\mathcal{S}$ of SPF($\mathcal{F}$) we denote by SPF($\mathcal{S}$) the *single-parent-family* of $\mathcal{S}$, i.e., the family of all single-parent-sets in $\mathcal{S}$. A set G in a union closed family $\mathcal{F}$ is a *generator*, if $G = A \cup B$ for any two sets A and B in $\mathcal{F}$ implies that either $A = G$ or $B = G$. Trivially, the empty set is always a generator in $\mathcal{F}$. Let $G(\mathcal{F})$ be the family of all generators in $\mathcal{F}$. Inspired by [12], we note that $G(\mathcal{F})$ is exactly the family of all the sets in

$\mathcal{F}$ with at most one child. Given a family of sets $\mathcal{B}$ in $\mathcal{F}$, we denote by $< \mathcal{B} >$ the union closed family generated by $\mathcal{B}$, i.e., $< \mathcal{B} >= \{A \mid A = \cup\mathcal{C}$ for some $\mathcal{C} \subseteq \mathcal{B}\}$. Here, $< \mathcal{B} >$ does not necessarily contain $\emptyset$. Moreover, for any set A in a given union closed family $\mathcal{F}$, we define $\mathcal{C}(A) = \{B \mid B \in \text{SPF}(\mathcal{F}), B \supseteq A$ and no single-parent-sets exist between B and $A\}$ to be the cover family of A. Noting that all the second maximal sets in a given union closed family $\mathcal{F}$ has only one parent $X_{\mathcal{F}}$ in $\mathcal{F}$, $\mathcal{C}(A)$ always exists for any given set A in $\mathcal{F} - X_{\mathcal{F}}$. Moreover, it follows readily that $\mathcal{C}(A) = \{A\}$ if $A \in \text{SPF}(\mathcal{F})$. Later in the proof, $X_{\mathcal{F}}$ has the similar role as single-parent-sets of $\mathcal{F}$. Thus, we assume $\mathcal{C}(X_{\mathcal{F}}) = \{X_{\mathcal{F}}\}$.

Following Poonen in [16], the union closed set conjecture does not hold if the union closed family $\mathcal{F}$ is allowed to have an infinite number of sets, i.e., if $|\mathcal{F}|$ is allowed to be infinite. Indeed, the union closed family of sets $\{i + 1, i + 2, i + 3, \cdots\}$ for every positive integer i serves as a counterexample. Consequently, we assume that every union closed family considered in the following contains only finitely many sets.

For a given union closed family $\mathcal{F}$ and a given element x, we denote by $\mathcal{F}_{\bar{x}}$ the family of all the sets in $\mathcal{F}$ not containing element x, i.e., $\mathcal{F}_{\bar{x}} = \{A \mid A \in \mathcal{F}$ and $x \notin A\}$ and we denote by $\mathcal{F}_x$ the family of all the sets in $\mathcal{F}$ containing element x, i.e., $\mathcal{F}_x = \{A \mid A \in \mathcal{F}$ and $x \in A\}$. For a given union closed family $\mathcal{F}$ with a set A in $\mathcal{F}$, we denote by $\mathcal{F}_{\subseteq A}$ the family of all the sets in $\mathcal{F}$ contained in A.

For any given union closed family $\mathcal{F}$ which does not contain $\emptyset$, $\mathcal{F} + \{\emptyset\}$ is also union closed. Clearly, it suffices to consider union closed family which contains $\emptyset$. Note that every family is a poset with the set inclusion as the partial order relation. Moreover, for a given union closed family $\mathcal{F}$, whether $\emptyset \in \mathcal{F}$ or not plays an important role in the partial order relation characterization. This calls for the following definition of *lattice*.

Note that a lattice is a poset in which every pair of elements has a unique minimal common upper bound and a unique maximal common lower bound; see [11]. For two given elements x and y in a given lattice $\mathcal{L}$, we denote by $x \wedge y$ the maximal common lower bound of x and y in $\mathcal{L}$ and we denote by $x \vee y$ the minimal common upper bound of x and y in $\mathcal{L}$. Note that for two distinct sets A and B in a given union closed family $\mathcal{F}$ which contains $\emptyset$,

$A \vee B = A \cup B$ and $A \wedge B = \cup_{C \subseteq A \cap B} C$, i.e., $A \cup B$ is the unique minimal common upper bound and $\cup_{C \subseteq A \cap B} C$ is the unique maximal common lower bound. Hence, with the set inclusion as a partial order relation, any union closed family which contains $\emptyset$ is a lattice. In this case, for any subfamily $\mathcal{S}$ of $\mathcal{F}$, we denote by $\wedge_{\mathcal{F}} \mathcal{S}$ the unique maximal common lower bound of all the sets from $\mathcal{S}$ and we denote by $\vee_{\mathcal{F}} \mathcal{S}$ the unique minimal common upper bound of all the sets from $\mathcal{S}$. On the other hand, if $\emptyset \notin \mathcal{F}$, then $\vee_{\mathcal{F}} \mathcal{S}$ is the same as $\vee_{\mathcal{F}+\{\emptyset\}} \mathcal{S}$, while $\wedge_{\mathcal{F}} \mathcal{S}$ may not exist. Note that if $\emptyset \notin \mathcal{F}$ and $\wedge_{\mathcal{F}} \mathcal{S}$ exists for a given family $\mathcal{S}$, then the union closed property of $\mathcal{F}$ implies that $\wedge_{\mathcal{F}} \mathcal{S}$ is also unique. In the following, we will omit the subscript $\mathcal{F}$ unless ambiguity occurs.

Generally, two partially ordered sets are isomorphic if they have analogous "structures". Formally, $(\mathcal{L}, \leq)$ and $(\mathcal{K}, \leq')$ are isomorphic to each other if there is a bijective function f from $\mathcal{L}$ to $\mathcal{K}$, such that $x_1 \leq x_2$ if and only if $f(x_1) \leq' f(x_2)$. In this case, we say $\mathcal{L}$ and $\mathcal{K}$ have the same poset structure.

A union closed family $\mathcal{F}$ is *separating* if $\mathcal{F}_x \neq \mathcal{F}_y$ for any two distinct elements x and y in $X_{\mathcal{F}}$ and $\mathcal{F}_i \neq \mathcal{F}$ for any $i \in X_{\mathcal{F}}$. If $\mathcal{F}_x = \mathcal{F}_y$ for two distinct elements in $X_{\mathcal{F}}$ or $\mathcal{F}_i = \mathcal{F}$, then x and y or the element i are *redundant* and one of them can be removed to simplify $\mathcal{F}$. That is the initial intuition to consider separating families. Now, a separating union closed family $\mathcal{F}$ is *irreducible (normalized)* if $|X_{\mathcal{F}}|$ is the minimum (maximum, resp.) with respect to the poset structure of $\mathcal{F}$.

2.2 Historical development of the conjecture

In this section, we will address the historical development of the conjecture and related results. Recall that the conjecture starts with Frankl in 1979.

Conjecture 2.2.1 (Union Closed Set Conjecture [11])**.** *For any union closed family $\mathcal{F}$ of finite sets, in which at least one set is non-empty, there is an element $x \in X_{\mathcal{F}}$ contained in at least half of all sets in $\mathcal{F}$.*

After that, it has traveled all through the world and has brought tremendous interests from various mathematical researchers. In 1990, Poonen [16] proved the conjecture for any union closed family $\mathcal{F}$ with $|X_{\mathcal{F}}| \leq 7$ or $|\mathcal{F}| \leq 28$. More importantly, he also made the following three conjectures, which are rephrased here.

Conjecture 2.2.2. *[16] Let $\mathcal{F}$ be a union closed family with $\mathcal{F}_x \neq \mathcal{F}_y$ for any two distinct elements x and y in $X_{\mathcal{F}}$. If $\mathcal{F}$ is not a power family, then there exists an element $x \in X_{\mathcal{F}}$, with $|\mathcal{F}_x| > \frac{|\mathcal{F}|}{2}$.*

Conjecture 2.2.3. *[16] For any union closed family $\mathcal{F}$, if there is only one element x with $|\mathcal{F}_x| \geq \frac{|\mathcal{F}|}{2}$, then x is in every nonempty set of $\mathcal{F}$.*

Conjecture 2.2.4. *[16] For any union closed family $\mathcal{F}$, if there is only one element x in $X_{\mathcal{F}}$ with $|\mathcal{F}_x| \geq \frac{|\mathcal{F}|}{2}$, then $\mathcal{F} = \{\{x\}\}$ or $\mathcal{F} = \{\emptyset\} + (\{\{x\}\} \bar{\cup}\, 2^{X_{\mathcal{F}}-x})$. Here, $\{\{x\}\} \bar{\cup}\, 2^{X_{\mathcal{F}}-x} := \{S \cup T \mid S \in \{\{x\}\}, T \in 2^{X_{\mathcal{F}}-x}\}$ and $X_{\mathcal{F}} - x$ is the set of all the elements in $X_{\mathcal{F}}$ except the element x.*

In the same year, Wójcik [19] proposed the following another conjecture.

Conjecture 2.2.5. *[19] In any union closed family $\mathcal{F}$, either there is an element which is contained in more than half of all the sets in $\mathcal{F}$, or each element is contained in exactly half of all the sets in $\mathcal{F}$.*

Note that it follows readily that Conjecture 2.2.2 implies Conjecture 2.2.5. In 1993, Knill [13][1] observed that if Y is defined to be the minimal subset of $X_{\mathcal{F}}$ such that $Y \cap A \neq \emptyset$ for any $A \in \mathcal{F} - \{\emptyset\}$, then $Y \cap (\mathcal{F} - \{\emptyset\}) := \{Y \cap A \mid A \in \mathcal{F}\} = 2^Y - \{\emptyset\}$. Based on this, he deduced that for any union closed family $\mathcal{F}$, there is some element contained in at least $\frac{|\mathcal{F}|}{\log_2 |\mathcal{F}|}$ members of $\mathcal{F}$. This is until now the best known result of the union closed set conjecture with respect to magnitude.

Then, four years later, Johunson and Vaughan [12] introduced the *dual* of a given union closed family $\mathcal{F}$ which contains $\emptyset$, and proved that the union closed set conjecture is true either for $\mathcal{F}$ or for the dual of $\mathcal{F}$. In 1999, Wójcik [20] improved Knill's result by a

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[1] Based on [20], it comes from a manuscript version of [13].

multiplicative constant. Moreover, he defined a given union closed family $\mathcal{F}$ to be *normalized* if $\mathcal{F}_x \neq \mathcal{F}_y$ for any two distinct elements x and y in $X_{\mathcal{F}}$, $\emptyset \in \mathcal{F}$ and $|X_{\mathcal{F}}| = |\mathcal{F}| - 1$, [2] and proved the equivalence of the following conjecture and the union closed set conjecture.

Conjecture 2.2.6. *[20] In any normalized union closed family $\mathcal{F}$, there is a generator G in $\mathcal{F}$ with $|G| \geq \frac{|\mathcal{F}|}{2}$.*

Shifting or compression is a common technique in extremal combinatorics. In 2003, Reimer [17] used the up compression method to prove that the average set size of any union closed family $\mathcal{F}$ is at least $\frac{\log_2 |\mathcal{F}|}{2}$, i.e, for any union closed family $\mathcal{F}$, $\frac{1}{|\mathcal{F}|}\sum_{A\in\mathcal{F}} |A| \geq \frac{\log_2 |\mathcal{F}|}{2}$. Here, the interesting part is the up compression method, which goes as follows. For a fixed element i in $X_{\mathcal{F}}$, let $f_i(A) = A + i$ if $A + i \notin \mathcal{F}$ and $f_i(A) = A$ otherwise for every $A \in \mathcal{F}$. It turns out that this up-compressed family $f_i(\mathcal{F}) := \{f_i(A) \mid A \in \mathcal{F}\}$ is also union closed. Moreover, it has a "good" property, i.e., for any $A \in \mathcal{F}$ with $i \notin A$, $A \cup \{i\} \in \mathcal{F}$. Note that a given family $\mathcal{F}$ with an underlying set $X_{\mathcal{F}}$ is an up set, if for any given set A in $\mathcal{F}$, all the sets between A and $X_{\mathcal{F}}$ are also in $\mathcal{F}$. Then, after all the elements in $X_{\mathcal{F}}$ has been up compressed, the family obtained in the end of this process is an up set. Let us look at the following excerpt from [4], "...Compression subjects the given initial object (the union-closed family), to small incremental changes until a simpler object is reached (an up-set), while maintaining the essential properties of the initial object..." Being able to maintain the essential properties of the initial object, this up compression method has played an important role in the potential full proof for the union closed set conjecture.

In 2010, Roberts and Simpson [18] showed that if $\mathcal{F}$ is a counterexample with $|X_{\mathcal{F}}|$ minimum, then $|\mathcal{F}| \geq 4|X_{\mathcal{F}}| - 1$. One year later, Falgas-Ravry [10] improved Reimer's bound by showing that the average set size is at least $\frac{\binom{|X_{\mathcal{F}}|}{2}}{|\mathcal{F}|}$.

In 2013, Bruhn et al [6] showed that Frankl's conjecture is equivalent to the conjecture that in a finite non-trivial bipartite graph there are two adjacent vertices each belonging to at most half of the maximal stable sets. Soon after that, Bruhn and Schaudt [5] showed that

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[2] Our definition is more broader than theirs.

for every fixed edge-probability, almost every random bipartite graph almost surely satisfies Frankl's conjecture. At the same year, Balla et al [2] determined the minimum possible average size of a set among all union closed families of a given size precisely, characterized their corresponding structures, and verified the union closed set conjecture for any union closed family $\mathcal{F}$ with $|\mathcal{F}| \geq \frac{2}{3}2^{|X_{\mathcal{F}}|}$. After that, in [9], Eccles proved that the union closed set conjecture holds for families $\mathcal{F}$ with $|\mathcal{F}| \geq (\frac{2}{3} - c)2^{|X_{\mathcal{F}}|}$ for a positive constant c. We refer to Bruhn and Schaudt [4] for literatures and recent developments for the union closed set conjecture.

2.3 Our results

Our main contribution to the union closed set conjecture is the following theorem, which implies that the irreducible family and the normalized family essentially share the same poset structure.

Theorem 2.3.1. *Any separating family $\mathcal{F}$ can be normalized to a normalized family $\mathcal{F}^N$ which has the same poset structure as $\mathcal{F}$. On the other hand, any separating family $\mathcal{F}$ can be reduced to an irreducible family $\mathcal{F}^I$ which has the same poset structure as $\mathcal{F}$.*

2.3.1 Normalization and reduction algorithms

In this section, we give characterizations of normalized union closed family based on a bijective function and the duality of a given union closed family. Then the corresponding normalization and reduction algorithms are developed, and the properties of irreducible family and normalized family are investigated. We have reasons to believe that both of them play a very important role in the potential proof for the union closed set conjecture.

Let us get started with the following definitions and lemmas. Recall that a union closed family $\mathcal{F}$ is separating if $\mathcal{F}_x \neq \mathcal{F}_y$ for any two distinct elements x and y in $\mathcal{F}$, and $\mathcal{F}_i \neq \mathcal{F}$ for any $i \in X_{\mathcal{F}}$. A separating union closed family $\mathcal{F}$ is *irreducible (normalized)* if $|X_{\mathcal{F}}|$ is the minimum (maximum, resp.) with respect to the poset structure of $\mathcal{F}$. A set A in a given union closed family $\mathcal{F}$ is called a *single-parent-set* if A has only one parent in $\mathcal{F}$. We denote by $\mathrm{SPF}(\mathcal{F})$ the family of all single-parent-sets in $\mathcal{F}$.

Lemma 2.3.1. *[20] In a given union closed family $\mathcal{F}$ with a given element x, there is a unique maximal set in $\mathcal{F}$ not containing element x.*

Proof. Indeed, recall that $\mathcal{F}_{\bar{x}}$ is the family of all sets in $\mathcal{F}$ not containing x for any $x \in X_{\mathcal{F}}$. Then, $\mathcal{F}_i \neq \mathcal{F}$ for any $i \in X_{\mathcal{F}}$ implies that $\mathcal{F}_{\bar{x}}$ is not empty. Combining with $\mathcal{F}$ being union closed, we know that $\cup \mathcal{F}_{\bar{x}}$ serves as the maximal set in $\mathcal{F}$ not containing element x. □

Lemma 2.3.2. *In a given union closed family $\mathcal{F}$, if A has only one parent B in $\mathcal{F}$, then for any element $x \in B - A$, A is the maximal set in $\mathcal{F}$ not containing element x.*

Proof. Indeed, choose $x \in B - A$ and let A_x be the maximal set in $\mathcal{F}$ not containing element x, i.e., $A_x = \cup \mathcal{F}_{\bar{x}}$. We claim that $A = A_x$. Suppose, to the contrary, that $A \neq A_x$. Then, $x \notin A$ implies that $A \subsetneq A_x$. Combining this with $x \in B$ and B is a parent of A, we know that $B \nsim A_x$. Now, choose a set $C \in \mathcal{F}$ such that (i): $A \subset C \subseteq A_x$; and (ii): C is as minimal as possible with respect to (i). The minimality of C implies readily that C is a parent of A. Combining this with B being a parent of A and $B \neq C$, a contradiction to the fact that A has only one parent in $\mathcal{F}$ follows. □

2.3.1.1 bijection Before we get started, for all the union closed family $\mathcal{F}$ in the following, we always assume that $\mathcal{F}_i \neq \mathcal{F}$ for any $i \in X_{\mathcal{F}}$. In this subsection, a normalization and a reduction algorithm are given to settle Theorem 2.3.1 based on the following defined injective function.

For any separating family $\mathcal{F}$ and any element $x \in \mathcal{F}$, denote by $f_{\mathcal{F}}$ the function from $X_{\mathcal{F}}$ to $\mathcal{F} - \{X_{\mathcal{F}}\}$ such that $f_{\mathcal{F}}(x)$ is the maximal set in $\mathcal{F}$ not containing x. Lemma 2.3.1 gives us that $f_{\mathcal{F}}$ is well-defined. The following lemma characterizes separating family in terms of $f_{\mathcal{F}}$.

Lemma 2.3.3. *For any union closed family $\mathcal{F}, \mathcal{F}$ is separating if and only if $f_{\mathcal{F}}$ is injective.*

Proof. We first show the necessity of the above lemma. Indeed, for any $i \neq j \in X_{\mathcal{F}}$, we have $\mathcal{F}_i \neq \mathcal{F}_j$, i.e., there is a set $A \in \mathcal{F}$ such that $i \in A$ and $j \notin A$ or $j \in A$ and $i \notin A$. Hence, $A \nsubseteq f_{\mathcal{F}}(i)$ and $A \subseteq f_{\mathcal{F}}(j)$ or $A \nsubseteq f_{\mathcal{F}}(j)$ and $A \subseteq f_{\mathcal{F}}(i)$, which implies that $f_{\mathcal{F}}(i) \neq f_{\mathcal{F}}(j)$.

On the other hand, the sufficiency of the above lemma goes as follows. Since $f_{\mathcal{F}}$ is injective, $f_{\mathcal{F}}(x) \neq f_{\mathcal{F}}(y)$ for any pair of distinct elements x and y in $X_{\mathcal{F}}$. Noting that $f_{\mathcal{F}}(x) = \cup \mathcal{F}_{\bar{x}}$, we have $\mathcal{F}_{\bar{x}} \neq \mathcal{F}_{\bar{y}}$ for any given pair of distinct elements x and y in $X_{\mathcal{F}}$, i.e., $\mathcal{F}$ is a separating family. □

Lemma 2.3.4. *In any separating family $\mathcal{F}$, with a set $A \in \mathcal{F}$ such that $A \neq f_{\mathcal{F}}(x)$ for any $x \in X_{\mathcal{F}}$ and an element $y \notin X_{\mathcal{F}}$, define a function g from $\mathcal{F}$, such that $g(B) = B$ if $B \subseteq A$ and $g(B) = B \cup \{y\}$ otherwise. Then, $g(\mathcal{F})$ is separating union closed, sharing the same poset structure with $\mathcal{F}$.*

Proof. Note that for any two sets B and C in $\mathcal{F}$, we have $g(B) \cup g(C) = B \cup C = g(B \cup C)$ if $B \cup C \subseteq A$ and $g(B) \cup g(C) = B \cup C \cup \{y\} = g(B \cup C)$ otherwise. Hence, $g(\mathcal{F})$ is union closed. Since $y \notin A$ and the assumption that $\mathcal{F}_i \neq \mathcal{F}$ for any element $i \in X_{\mathcal{F}}$, lemma * implies readily that $f_{g(\mathcal{F})}$ is well-defined.

Noting the definition of $f_{\mathcal{F}}(x) = \cup \mathcal{F}_{\bar{x}}$ for any element $x \in X_{\mathcal{F}}$, we have readily that $f_{g(\mathcal{F})}(x) = f_{\mathcal{F}}(x)$ if $x \neq y$ and $f_{g(\mathcal{F})}(y) = A$ otherwise. Then, $f_{\mathcal{F}}$ being injective implies that $f_{g(\mathcal{F})}$ is injective, indicating that $g(\mathcal{F})$ is separating.

Then, it suffices to prove that g is injective from $\mathcal{F}$ to $g(\mathcal{F})$. Indeed, suppose not, then there are two distinct sets B and C in $\mathcal{F}$ such that $g(B) = g(C)$. Hence, B and C has only one element difference, i.e., either $B = C \cup \{y\}$ or $C = B \cup \{y\}$. In either case, $y \in X_{\mathcal{F}}$ and a contradiction thus follows. □

Combining this with the definition of normalized family, we have the following corollary.

Corollary 2.3.1. *[20] A separating family $\mathcal{F}$ is normalized if and only if $f_{\mathcal{F}}$ is bijective.*

Lemma 2.3.5. $|SPF(\mathcal{F})| \leq |X_{\mathcal{F}}| \leq |\mathcal{F}| - 1$ *for any separating family $\mathcal{F}$* [3].

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[3] *This lemma gives us the equivalence of our definitions and Wójcik's*

Proof. Lemma 2.3.2 gives us that $\mathrm{SPF}(\mathcal{F}) \subseteq f_{\mathcal{F}}(X_{\mathcal{F}})$. On the other hand, $f_{\mathcal{F}}$ being an injective function defined on $\mathcal{F} - \{X_{\mathcal{F}}\}$ implies that $|f_{\mathcal{F}}(X_{\mathcal{F}})| = |X_{\mathcal{F}}|$ and $|f_{\mathcal{F}}(X_{\mathcal{F}})| \leq |\mathcal{F}| - 1$. □

Then, based on Lemma 2.3.4 and Corollary 2.3.1, we have the following first normalization algorithm.

Algorithm 1:

- **for any** i with $1 \leq i \leq |X_{\mathcal{F}}|$, find the maximal set in $\mathcal{F}$ not containing i and label this set as A_i. Here, we assume that $X_{\mathcal{F}}$ is the unique maximal set in $\mathcal{F}$ not containing $|X_{\mathcal{F}}| + 1$ by our construction;
- $k \leftarrow |X_{\mathcal{F}}| + 2$;
- **while** (there is an unlabelled set in $\mathcal{F}$)
- choose an unlabelled set A and label it as A_k
- add an element k to each set in $\mathcal{F}$ which is not contained in A_k (we are adding elements to make A_k the maximal set in $\mathcal{F}$ not containing element k);
- update $X_{\mathcal{F}}$ to be $X_{\mathcal{F}} \cup \{k\}$;
- $k \leftarrow k + 1$;
- end **while**.

Remark 1: Recall that in [20], Wójcik defined a union closed family to be *normalized* if $|X_{\mathcal{F}}| = |\mathcal{F}| - 1$, $\emptyset \in \mathcal{F}$ and $\mathcal{F}_x \neq \mathcal{F}_y$ for any two distinct elements x and y in $X_{\mathcal{F}}$. We define a separating union closed family to be "normalized" if $|X_{\mathcal{F}}|$ is the maximum with respect to the lattice structure of $\mathcal{F}$. Indeed, his definition is equivalent to ours.

Remark 2: Recall that for any two distinct sets A and B in $\mathcal{F}$, we denote by $A \nsim B$ if A and B are incomparable. In a separating union closed family $\mathcal{F}$, for any two given elements i and j in $X_{\mathcal{F}}$, we have

1. $f_{\mathcal{F}}(i) \nsim f_{\mathcal{F}}(j)$ if and only if $i \in f_{\mathcal{F}}(j)$ and $j \in f_{\mathcal{F}}(i)$;

2. $f_{\mathcal{F}}(i) \subseteq f_{\mathcal{F}}(j)$ if and only if $j \notin f_{\mathcal{F}}(i)$.

Remark 3: In the following, for any separating family $\mathcal{F}$ and any element $x \in X_{\mathcal{F}}$, we denote the set $f_{\mathcal{F}}(x)$ by A_x, i.e., A_x is the maximal set in $\mathcal{F}$ not containing element x for any $x \in X_{\mathcal{F}}$.

Remark 4: Lemma 2.3.5 and the definition of irreducible family implies that $f_{\mathcal{F}}(X_{\mathcal{F}}) = \mathrm{SPF}(\mathcal{F})$. On the other hand, corollary 2.3.1 and the definition of normalized family imply that $f_{\mathcal{F}}(X_{\mathcal{F}}) = \mathcal{F} - \{X_{\mathcal{F}}\}$.

Then, the reduction part of Theorem 2 comes readily from the following lemma. Here, for a given set A and any element $x \in A$, we denote by $A - x$ the set obtained by deleting the element x from A.

Lemma 2.3.6. *Let $\mathcal{F}$ be a separating union closed family with an element $y \in X_{\mathcal{F}}$. If $f_{\mathcal{F}}(y) \notin SPF(\mathcal{F})$, then $\mathcal{F}^{y-} := \{A - y \mid A \in \mathcal{F}\}$ is a separating family sharing the same poset structure with $\mathcal{F}$.*

Proof. It suffices to prove that $|\mathcal{F}^{y-}| = |\mathcal{F}|$. Indeed, suppose, to the contrary, that $|\mathcal{F}^{y-}| \neq |\mathcal{F}|$. Then, there are two distinct sets A and B in $\mathcal{F}$ with $B = A \cup \{y\}$. Note that $y \notin A$. Then, we consider the following two cases. If $A = A_y$, i.e., if A is the maximal set in $\mathcal{F}$ not containing element y, then every set in $\mathcal{F}$ containing A must contain $A \cup \{y\}$. Thus, $A \cup \{y\}$ is the only parent if A_y in $\mathcal{F}$, contradicting to $A_y \notin \mathrm{SPF}(\mathcal{F})$. On the other hand, if $A \neq A_y$, then $B \cup A_y = A \cup \{y\} \cup A_y = A_y \cup \{y\} \in \mathcal{F}$, reducing to the previous case. □

2.3.1.2 Duals

In this subsection, we explore the normalization and reduction algorithm based on the duality of a given separating family.

In [12], Johnson et al introduced the *dual* of a given separating family which contains $\emptyset$. In the following, for the sake of completeness, we define the *dual* of a given separating family regardless of whether it contains $\emptyset$ or not from a different perspective. Recall that for any given subfamily $\mathcal{B}$ of $\mathcal{F}$, we denote by $< \mathcal{B} >$ the union closed family generated by $\mathcal{B}$, i.e., $< \mathcal{B} = \{\cup\mathcal{C} \mid \mathcal{C} \subseteq \mathcal{B}\} >$. For a separating family $\mathcal{F}$ with an element $x \in X_{\mathcal{F}} \cup \{n+1\}$, [4] we denote $f_{\mathcal{F}}(x)$ by A_x, i.e., A_x is the maximal set in $\mathcal{F}$ not containing element x for any $x \in X_{\mathcal{F}} \cup \{n+1\}$. Then, we define the subscript of the set A_x to be x. Now, for a subfamily $\mathcal{S}$ of $\mathcal{F}$, let $I_{\mathcal{S}}$ be the subscript set of $\mathcal{S}$, i.e., $I_{\mathcal{S}} = \{i \mid i \text{ is the subscript of some set in } \mathcal{S}\}$. Note that $|I_{\mathcal{F}}| = |X_{\mathcal{F}}| + 1$ for any given separating family $\mathcal{F}$. Recall that $\mathcal{F}_x$ is the family of all sets in $\mathcal{F}$ containing x for any $x \in X_{\mathcal{F}}$. For any separating union closed family $\mathcal{F}$, let $\mathcal{F}^* =< I_{\mathcal{F}_x}, x \in X_{\mathcal{F}\cup\{n+1\}} >$, i.e., $\mathcal{F}^*$ is the union closed family generated by $\{I_{\mathcal{F}_x}, x \in X_{\mathcal{F}} \cup \{n+1\}\}$, which is called the *dual family* of $\mathcal{F}$.

Recall that $\mathcal{C}(A) = \{B \mid B \in \text{SPF}(\mathcal{F}), B \supsetneq A$ and no single-parent-sets exist between B and $A\}$ is the *cover family* of A for any set A in a given separating union closed family $\mathcal{F}$. The following two lemmas express the relation between a set and its cover family.

Lemma 2.3.7. *For any set A in a given irreducible family $\mathcal{F}$, we have $A = \cap\mathcal{C}(A)$.*

Proof. If $A \in \text{SPF}(\mathcal{F})$ or $A = X_{\mathcal{F}}$, then the lemma follows readily. So, we always assume in the following that $A \notin \text{SPF}(\mathcal{F})$ and $A \neq X_{\mathcal{F}}$. Clearly, $A \subseteq \cap\mathcal{C}(A)$ holds. So, we only need to show $A \supseteq \cap\mathcal{C}(A)$. Suppose, to the contrary, this is not true. Then, there is an element $x \in \cap\mathcal{C}(A) - A$. Combining with $A \notin \text{SPF}(\mathcal{F})$, $A \subsetneq A_x$. $\mathcal{F}$ being irreducible implies that $A_x \in \text{SPF}(\mathcal{F})$. The definition of cover family implies that $A_x \not\subset B$ for any $B \in \mathcal{C}(A)$. Combining again with $x \in \cap\mathcal{C}(A)$, we have $A_x \nsim B$ for any $B \in \mathcal{C}(A)$, implying that $A_x \in \mathcal{C}(A)$. A contradiction thus follows. So, $\cap\mathcal{C}(A) \subseteq A$. Noting that $A \subseteq \cap\mathcal{C}(A)$, the proof is then finished. $\square$

Lemma 2.3.8. *For any set A in a given union closed family $\mathcal{F}$, $A = \wedge\mathcal{C}(A)$.*

Proof. Note that $A \subseteq \wedge\mathcal{C}(A)$. Thus, $\wedge\mathcal{C}(A)$ always exists. Let $Z = \wedge\mathcal{C}(A)$. Denote by $\mathcal{F}^I$ the irreducible family obtained from reducing $\mathcal{F}$, denote by A^I the set in $\mathcal{F}^I$ corresponding to A and denote by $\mathcal{C}_{\mathcal{F}^I}(A^I)$ the cover family of A^I in $\mathcal{F}^I$. The previous lemma implies that $A \cap I_{\text{SPF}(\mathcal{F})} = Z \cap I_{\text{SPF}(\mathcal{F})} = \cap\mathcal{C}_{\mathcal{F}^I}(A^I)$. Combining with the one to one correspondence between

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[4] Here, $n+1 \notin X_{\mathcal{F}}$, which indicates that $f_{\mathcal{F}}(n+1) = X_{\mathcal{F}}$ by our assumption.

the sets from $\mathcal{F}$ and the sets from $\mathcal{F}^I$, we know that $C \cap I_{\mathrm{SPF}(\mathcal{F})}$ completely determines the set C for any $C \in \mathcal{F}$. Hence, $A = Z$. $\square$

Next, the following lemma gives us an alternative characterization for normalized union closed family. Recall here that $G(\mathcal{F})$ is the family of all generators in $\mathcal{F}$.

Lemma 2.3.9. *Let $\mathcal{F}$ be a separating family which contains $\emptyset$. Then, the following statements are equivalent:*

1. *$\mathcal{F}$ is normalized with $\emptyset \in \mathcal{F}$.*
2. *$\{I_{\mathcal{F}_x}, x \in X_{\mathcal{F}}\}$ is union closed.*
3. *$\{I_{G(\mathcal{F})_x}, x \in X_{\mathcal{F}}\}$ is union closed.*

Proof. **1** $\Longrightarrow$ **2**: $\mathcal{F}$ being normalized implies that $f_{\mathcal{F}}$ is bijective. Since $\emptyset \in \mathcal{F}$, $\mathcal{F}$ with the set inclusion as a partial ordered relation is a lattice. For any two distinct elements i and j in $X_{\mathcal{F}}$, let $f_{\mathcal{F}}(k) = f_{\mathcal{F}}(i) \wedge f_{\mathcal{F}}(j)$. Then, it follows that $\mathcal{F}_{\subseteq f_{\mathcal{F}}(k)} = \mathcal{F}_{\subseteq f_{\mathcal{F}}(i)} \cap \mathcal{F}_{\subseteq f_{\mathcal{F}}(j)}$, i.e., $(I_{\mathcal{F}_k})^c = (I_{\mathcal{F}_i})^c \cap (I_{\mathcal{F}_j})^c$. Hence, $I_{\mathcal{F}_k} = I_{\mathcal{F}_i} \cup I_{\mathcal{F}_j}$.
2 $\Longleftrightarrow$ **3**: This follows from the fact that for any three distinct elements i, j and k in $X_{\mathcal{F}}$, $I_{\mathcal{F}_i} \cup I_{\mathcal{F}_j} = I_{\mathcal{F}_k}$ if and only if $I_{G(\mathcal{F})_i} \cup I_{G(\mathcal{F})_j} = I_{G(\mathcal{F})_k}$.
2 $\Longrightarrow$ **1**: It suffices to show that $f_{\mathcal{F}}$ is bijective. Indeed, for any given set A in $\mathcal{F} - \{X_{\mathcal{F}}\}$, lemma 2.3.7 and $\{I_{\mathcal{F}_x}, x \in X_{\mathcal{F}}\}$ being union closed imply that there is an element $z \in X_{\mathcal{F}}$ with $I_{\mathcal{F}_z} = \cup_{x \in f_{\mathcal{F}}^{-1}(\mathcal{C}(A))} I_{\mathcal{F}_x}$, i.e., any set in $\mathcal{F}$ contains z if and only if this set has a non-empty intersection with $f_{\mathcal{F}}^{-1}(\mathcal{C}(A))$. This is the same to say, any set in $\mathcal{F}$ does not contain z if and only if this set does not contain any element from $f_{\mathcal{F}}^{-1}(\mathcal{C}(A))$. Taking a maximal to both sides of the above equivalence relation and combining with $A = \wedge \mathcal{C}(A)$, A is the maximal set in $\mathcal{F}$ not containing element z. Hence, $f_{\mathcal{F}}$ is bijective. $\square$

Here, based on the previous lemma, we have the following an alternative normalization algorithm.

Algorithm 2:

- **while** (family $\{I_{G(\mathcal{F})_x}, x \in X_{\mathcal{F}}\}$ is not union closed)
- choose $I_{G(\mathcal{F})_a}$ and $I_{G(\mathcal{F})_b}$ from $\{I_{G(\mathcal{F})_x}, x \in X_{\mathcal{F}}\}$, such that $I_{G(\mathcal{F})_a} \cup I_{G(\mathcal{F})_b} \notin \{I_{G(\mathcal{F})_x}, x \in X_{\mathcal{F}}\}$
- add a new element c to all the sets in $G(\mathcal{F})$ whose indices are in $I_{G(\mathcal{F})_a} \cup I_{G(\mathcal{F})_b}$
- end **while**
- update the family $\mathcal{F}$ to be $< I_{G(\mathcal{F})_x}, x \in X_{\mathcal{F}} >+\{\emptyset\}$.

We define a family $\mathcal{S}$ to be almost union closed if $< \mathcal{S} >= \mathcal{S} + \{X_{<\mathcal{S}>}\}$. Then, we have the following another lemma.

Lemma 2.3.10. *Let $\mathcal{F}$ be a separating family which contains $\emptyset$. Then, the following statements are equivalent:*

1. *$\mathcal{F}$ is normalized with $\emptyset \in \mathcal{F}$*
2. *$\{I_{\mathcal{F}_x}, x \in X_{\mathcal{F}}\}$ is almost union closed.*
3. *$\{I_{G(\mathcal{F})_x}, x \in X_{\mathcal{F}}\}$ is almost union closed.*

Proof. **1** $\implies$ **2**: $\mathcal{F}$ being normalized implies that $f_{\mathcal{F}}$ is bijective. For any two distinct elements i and j in $X_{\mathcal{F}}$, if $f_{\mathcal{F}}(i) \wedge f_{\mathcal{F}}(j)$ exists, then the proof follows similarly as the previous lemma. Otherwise, if $\{_{\mathcal{F}}(i) \wedge f_{\mathcal{F}}(j)$ does not exist, then every set in $\mathcal{F}$ contains either element i or element j, i.e., $I_{\mathcal{F}_i} \cup I_{\mathcal{F}_j} = I_{\mathcal{F}} = \cup\{I_{\mathcal{F}_x}, x \in X_{\mathcal{F}}\}$.
2 $\iff$ **3**: This follows from the fact that for any three distinct elements i, j and k in $X_{\mathcal{F}}$, $I_{\mathcal{F}_i} \cup I_{\mathcal{F}_j} = I_{\mathcal{F}_k}$ if and only if $I_{G(\mathcal{F})_i} \cup I_{G(\mathcal{F})_j} = I_{G(\mathcal{F})_k}$.
2 $\implies$ **1**: Let x be a new element which is not contained in any set of $\mathcal{F}$. Now, consider $\mathcal{F}+x := \{A+x \mid A \in \mathcal{F}\}$, and then add $\emptyset$ to this new family. Denote the obtained family by $\mathcal{F}'$. Then $\{I_{\mathcal{F}_x}, x \in X_{\mathcal{F}}\}$ is union closed. By the previous lemma, $\mathcal{F}'$ is a normalized family which contains $\emptyset$. Noting the relation between $\mathcal{F}$ and $\mathcal{F}'$, the claim thus follows readily. □

Remark: From the above discussion, it follows readily that for every normalized family $\mathcal{F}$, $|\mathcal{F}^*| = |\mathcal{F}|$ if $\emptyset \in \mathcal{F}$ and $|\mathcal{F}^*| = |\mathcal{F}| - 1$ otherwise.

PART 3

MAXIMUM DIRECTED CUT IN GIVEN CONNECTED DIGRAHS

Remark: This part of dissertation has already been published on Journal of Graph Theory with all copy rights reserved, see [7].

3.1 Introduction

In this part of dissertation, all graphs and digraphs are finite with no loops and no parallel edges. Let D be a digraph with vertex set $V(D)$ and edge set $E(D)$. For convenience, we let $m = |E(D)|$ throughout this part. We denote by xy the directed edge from x to y instead of the more cumbersome notation $\overrightarrow{xy}$. For each $v \in V$, we denote by $d(v)$, $d^+(v)$, and $d^-(v)$ the degree, outdegree, and indegree of v (that is, the number of edges incident with v, leaving from v, and heading to v), respectively. A set F of edges in a digraph D is called a *dicut* (directed cut) if there exists a nontrivial partition (X, Y) of $V(D)$ such that F consists of all directed edges from X to Y. Clearly, the edge connectivity $\lambda(D)$ is the minimum size of a dicut in D. However, for this this part of dissertation, we consider the maximum size of a dicut in D and denote it by $\Lambda(D)$.

It is well known that an undirected graph with m edges contains an edge-cut with more than $m/2$ edges. Yannakakis [15] showed that determining the maximum size of edge cuts for undirected graphs is an NP-hard problem, even with restriction to *triangle-free cubic* graphs. Bondy and Locke [3] provided a polynomial time algorithm to find an edge cut for any *triangle-free subcubic* undirected graph G with at least $\frac{4m}{5}$ edges. Xu and Yu [21] proved that there are precisely seven triangle-free subcubic undirected graphs whose maximum edge cuts are exactly $\frac{4m}{5}$, which was originally conjectured by Bondy and Locke [3]. Noting that, in a cubic digraph D, either $d^+(v) \le 1$ or $d^-(v) \le 1$ for each vertex $v \in V(D)$, Cropper et al. [8] introduced the following notion $\mathcal{D}(k, \ell)$. For each pair of nonnegative integers k and

ℓ, we denote by $\mathcal{D}(k,\ell)$ the set of digraphs D such that $d^+(v) \leq k$ or $d^-(v) \leq \ell$ for each vertex v in D. Clearly, every subcubic digraph belongs to $\mathcal{D}(1,1)$. Alon et al.[1] proved that $\Lambda(D) \geq \frac{m}{3}$ for any $D \in \mathcal{D}(1,1)$ and that $\Lambda(D) \geq \frac{2m}{5}$ for any acyclic $D \in \mathcal{D}(1,1)$. In a recent paper [22], Xu and Yu characterized the acyclic digraphs in $\mathcal{D}(1,1)$ with m edges such that $\Lambda(D) = \frac{2m}{5}$. In an earlier published this part of dissertation, Lehel et al. [14] proved that $\Lambda(D) \geq \frac{2m-t}{5}$ if $D \in \mathcal{D}(1,1)$ and contains at most t pairwise disjoint directed triangles. Moreover, without counting the number of disjoint triangles, they showed that $\Lambda(D) \geq \frac{7m}{20}$ for every *connected* digraph $D \in \mathcal{D}(1,1)$. In the same paper, they proposed a few open problems, including the following two.

Problem 3.1.1 (Lehel, Maffray, and Preissmann [14]). *For every $\varepsilon > 0$, there is a constant M such that $\Lambda(D) > (\frac{3}{8} - \varepsilon)m$ for every connected digraph $D \in \mathcal{D}(1,1)$ with $m > M$ edges.*

Problem 3.1.2 (Lehel, Maffray, and Preissmann [14]). *If a connected digraph $D \in \mathcal{D}(1,1)$ with m edges contains no directed triangles and has s vertices with zero indegree or outdegree, then $\Lambda(D) \geq \frac{2m+s}{5}$.*

We will provide a positive answer for Problem 3.1.1 and will show that Problem 3.1.2 is true for trees, i.e., when the underlying undirected graph of D is a tree.

Theorem 3.1.1. *If $D \in \mathcal{D}(1,1)$ is a connected digraph with m edges, then $\Lambda(D) \geq \frac{3m-1}{8}$.*

Theorem 3.1.2. *If the underlying undirected graph of $D \in \mathcal{D}(1,1)$ is a tree with m edges and D has s vertices with zero indegree or outdegree, then $\Lambda(D) \geq \lfloor \frac{2m+s}{5} \rfloor$.*

We introduce two types of graphs, H_1 (see Figure 1) and H_2 (see Figures 2 and 3), which will be used heavily in our proof. Throughout this this part of dissertation, when we mention that a graph is isomorphic to either H_1 or H_2 without labeling the vertices, we always assume that its vertices are labeled as in Figure 1, Figure 2 or Figure 3 unless otherwise specified. We say that D contains an *F-component* if D contains a component which is isomorphic to F.

For two disjoint vertex sets X and Y of a digraph D, let $E_D(X,Y)$ and $\overrightarrow{E_D}(X,Y)$ denote the set of edges between X and Y and directed from X to Y, respectively. Let $e_D(X,Y) = |E_D(X,Y)|$ and $\overrightarrow{e_D}(X,Y) = |\overrightarrow{E_D}(X,Y)|$. Clearly, $e_D(X,Y) = \overrightarrow{e_D}(X,Y) + \overrightarrow{e_D}(Y,X)$.

Let H be an induced subgraph of a digraph D and $v \in V(H)$. If $e_D(V(H), V(D) - V(H)) = 1$, then H is called an *edge-suspended subgraph of* D with suspended edge e, where $\{e\} = E_D(V(H), V(D) - V(H))$. If $e_D(V(H) - \{v\}, V(D) - V(H)) = 0$, then H is called a *vertex-suspended subgraph of* D with suspended vertex v. In particular, if $H \cong H_1$ and $\{e\} = E_D(V(H), V(D) - V(H))$, then H is called an *e-edge-suspended-H_1* (or an e-ES-H_1 for short). If $H \cong H_2$ and $e_D(V(H) - \{b_7\}, V(D) - V(H)) = 0$, then H is called a *b_7-vertex-suspended-H_2* (or a b_7-VS-H_2 for short). Note that if $D \cong H_2$, then D is a b_7-VS-H_2 itself. We denote by $I(H)$ the set of all vertices in H with both indegree and outdegree at least 1. Clearly, $I(H) = V(H)$ if $H \cong H_1$, and $I(H) = V(H) - \{v_7\}$ if $H \cong H_2$. In the following, for the clarity, we let $d_D(H) = e_D(I(H), V(D) - V(H))$ if either $H \cong H_1$ or $H \cong H_2$. Clearly, H is an edge-suspended-H_1 if and only if $d_D(H) = 1$, and H is a vertex-suspended-H_2 if and only if $d_D(H) = 0$.

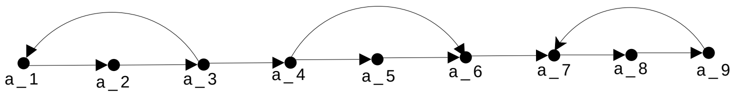

Figure 3.1 H_1

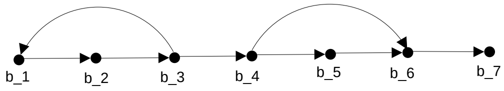

Figure 3.2 one version of H_2

Note that $\Lambda(H_1) = 4 = \frac{3\cdot|E(H_1)|-1}{8}$ and $\Lambda(H_2) = 3 = \frac{3\cdot|E(H_2)|}{8}$, so H_1 shows that Theorem 3 is best possible. Moreover, we construct infinitely many digraphs in $\mathcal{D}(1,1)$, showing that

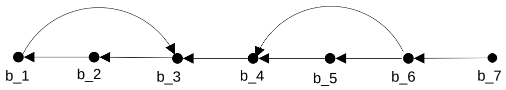

Figure 3.3 another version of H_2

the bound in Theorem 3 is tight: taking k copies of H_2 in Figure 2 or Figure 3 and one copy of H_1, we create a digraph D with $|E(D)| = 8k + 11$ and $\Lambda(D) = 3k + 4 = \frac{3m-1}{8}$ by identifying b_7 in each copy of H_2 with any vertex not in that copy of H_2, as long as it satisfies either $d^+(v) \leq 1$ or $d^-(v) \leq 1$ for any $v \in V(D)$ (see Figure 22).

For any $S \subseteq E(D)$, let $D - S$ denote the graph obtained from D by removing all the edges in S and *the resulting isolated vertices* (if any). We denote by $A \cong B$ if digraphs A and B are isomorphic. The *inverse digraph* of a digraph D is obtained by reversing the direction of each edge in D. We denote by $\overrightarrow{C_3}$ and $\overrightarrow{P_3}$ the *directed triangle* and the *directed path on three vertices*, respectively.

3.2 Preliminary Results

Starting this section with the following two theorems from [14], we present a few results which will be used in the proof of Theorem 3.

Theorem 3.2.1 (Lehel, Maffray, and Preissmann[14]). *If $D \in \mathcal{D}(1,1)$ has m edges and contains at most t pairwise disjoint directed triangles, then $\Lambda(D) \geq \frac{2m-t}{5}$.*

Theorem 3.2.2 (Lehel, Maffray, and Preissmann[14]). *If a connected digraph $D \in \mathcal{D}(1,1)$ has m edges, then $\Lambda(D) \geq \frac{7m}{20}$ unless $D \cong \overrightarrow{C_3}$.*

The following definition was given in [14]. A pair (A, B) of disjoint edge sets of a digraph D is called a *reducing pair* if any $\overrightarrow{P_3}$ with one edge in A has the other edge in B; equivalently, if A contains an edge xy then B contains all edges of D in the form of vx and yz. It is clear that a dicut contains no $\overrightarrow{P_3}$ by definition. The idea of introducing "reducing pair" is justified by the obvious fact that every $\overrightarrow{P_3}$-free edge set can be extended into a dicut of D. This observation is formulated into the following technical lemma.

Lemma 3.2.1. *Let* (A, B) *be a reducing pair of a digraph* D*, then* $\Lambda(D) \geq \Lambda(D - (A \cup B)) + |A|$*. Moreover, if* K *is a dicut of* $D - (A \cup B)$*, then there exists a dicut* K^* *in* D *with* $K^* \supseteq K \cup A$*.*

Let $F \subset E(D)$ and $D[F]$ be the subgraph induced by F. A vertex $x \in D[F]$ is called *F-saturated* if $d^+_{D[F]}(x)d^-_{D[F]}(x) \geq 1$ and *F-unsaturated* otherwise. An edge $xy \in F$ is called *F-saturated* if at least one of x and y is F-saturated and *unsaturated* if at least one of x and y is F-unsaturated. Clearly, xy is both saturated and unsaturated if and only if one of the x and y is F-saturated and the other one is F-unsaturated. We call F *saturated* if all edges in F are F-saturated. We denote by F^0 the set of all unsaturated edges in F. Clearly, if F is saturated, then F^0 is the set of edges which are both F-saturated and F-unsaturated.

Lemma 3.2.2. *Let* $D \in \mathcal{D}(1,1)$*,* $F \subseteq E(D)$ *be saturated, and* $H^{(1)}, H^{(2)}, H^{(3)}, \cdots, H^{(t)}$ *be* t *induced subgraphs of* $D - F$ *such that each of them is isomorphic to either* H_1 *or* H_2*. If* $I(H^{(i)}) \cap I(H^{(j)}) = \emptyset$ *for* $1 \leq i \neq j \leq t$*, then*

$$\sum_{i=1}^{t} d_D(H^{(i)}) - |F^0| \leq \sum_{i=1}^{t} d_{D-F}(H^{(i)}).$$

Proof. Since $D \in \mathcal{D}(1,1)$ and F is saturated, $F \cap E(H^{(i)}) = \emptyset$ for each $1 \leq i \leq t$. Moreover, the following two properties hold.

- For each $1 \leq i \leq t$, $E(I(H^{(i)}), V(D) - V(H^{(i)})) \cap F \subseteq F^0$.
- For each pair $1 \leq i < j \leq t$, $E(I(H^{(i)}), I(H^{(j)})) \cap F = \emptyset$.

So $d_{D-F}(H^{(i)}) \geq d_D(H^{(i)}) - |F^0 \cap E(I(H^{(i)}, V(D) - \cup_{j=1}^{t} V(H^{(j)}))|$ for each $1 \leq i \leq t$, which in turn gives Lemma 3.2.2. □

Lemma 3.2.3. *Let* $D \in \mathcal{D}(1,1)$ *be a connected digraph and let* H^* *and* H^{**} *be two two distinct ES-*H_1*. If* $V(H^*) \cap V(H^{**}) \neq \emptyset$*, then* $D = H^* \cup H^{**}$*. Moreover,* D *is isomorphic to the digraph depicted in Figure 4, its inverse, or the digraph depicted in Figure 5. Consequently,* D *contains a directed triangle* T *with* $e_D(V(T), V(D) - V(T)) = 2$*.*

Proof. Denote by e^* and e^{**} the suspended edge of H^* and H^{**}, respectively. If $V(D) - V(H^* \cup H^{**}) \neq \emptyset$, we may assume e^* is the edge between $V(H^* \cup H^{**})$ and $V(D) - V(H^* \cup H^{**})$. Hence H^* is not a component of $D - e^*$, so a contradiction follows. Thus, $V(D) = V(H^* \cup H^{**})$, which in turn implies $D = H^* \cup H^{**}$ and the second result. □

Lemma 3.2.4. *Let $D \in \mathcal{D}(1,1)$ and let H^* and H^{**} be two two distinct VS-H_2. If $V(H^*) \cap V(H^{**}) \neq \emptyset$, then $D \cong H_1$ (see Figure 6), or is isomorphic to the digraph in Figure 3.7, or contains $V(H^*) \cup V(H^{**})$ as an induced subgraph with $V(H^*) \cap V(H^{**}) = \{b_7\}$ (see Figure 3.8 or Figure 3.9 where b_7 is the same as b_7^* or b_7^{**}). Moreover, if $I(H^*) \cap I(H^{**}) \neq \emptyset$, then $D \cong H_1$.*

Proof. We denote by b_7^* and b_7^{**} the suspended vertex of H^* and H^{**} , respectively. Since H^* and H^{**} are two distinct VS-H_2, $V(H^*) \cap V(H^{**}) \neq \emptyset$ and $D \in \mathcal{D}(1,1)$, we have $\{b_7^*, b_7^{**}\} \subset V(H^*) \cap V(H^{**})$ and $|V(H^*) \cap V(H^{**})| = 1, 2$ or 5. If $|V(H^*) \cap V(H^{**})| = 5$, then $D \cong H_1$ (see Figure 3.6). If $|V(H^*) \cap V(H^{**})| = 2$, then $V(H^*) \cap V(H^{**}) = \{b_7^*, b_7^{**}\}$ and that is the digraph depicted in Figure 3.7. If $|V(H^*) \cap V(H^{**})| = 1$, then $b_7^* = b_7^{**} = V(H^*) \cap V(H^{**})$, and that is exactly the digraph depicted in Figure 3.8 or Figure 3.9. □

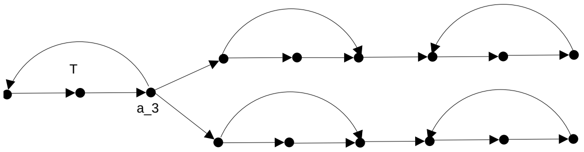

Figure 3.4

Lemma 3.2.5. *Let $D \in \mathcal{D}(1,1)$ and H be an e-ES-H_1 in D. If a subgraph H^* is a b_7-VS-H_2 in $D - E(H)$, then it is also a b_7-VS-H_2 in D.*

Proof. If $b_7 \notin V(H)$, then H^* is a b_7-VS-H_2 in D. Otherwise, b_6 and b_7 must be the endvertices of the edge e. So, $H^* = D - E(H)$, which in turn shows that H^* is also a b_7-VS-H_2 in D. □

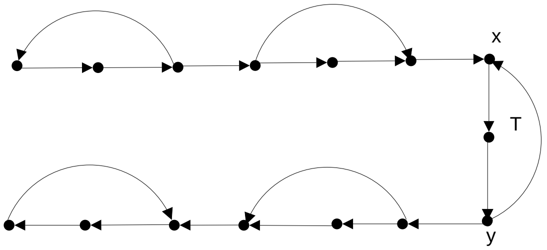

Figure 3.5

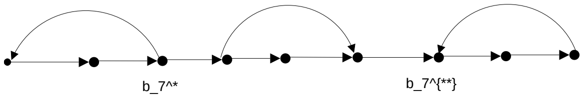

Figure 3.6 H_1

Lemma 3.2.6. *Let $D \in \mathcal{D}(1,1)$ and H be either an ES-H_1 or a b_7-VS-H_2 in D. If T is a directed triangle with $V(T) \nsubseteq I(H)$, then $V(T) \cap I(H) = \emptyset$.*

Proof. Since $V(T) \nsubseteq I(H)$, $|V(T) \cap I(H)| \leq 2$. If H is an ES-H_1, then $I(H) = V(H)$. Since in this case, $e(V(H), V(D) - V(H)) = 1$, we have $V(T) \cap I(H) = \emptyset$. If H is a b_7-VS-H_2, then $V(T) \cap V(H) \subseteq \{b_7\}$. Therefore, $V(T) \cap I(H) = \emptyset$. $\square$

Lemma 3.2.7. *Let $D \in \mathcal{D}(1,1)$. If there is a directed triangle T which is contained in every VS-H_2 of D, then D contains at most one VS-H_2.*

Proof. Suppose, to the contrary, D contains two distinct VS-H_2, say H^* and H^{**}. Since $I(H^*) \cap I(H^{**}) \supseteq V(T) \neq \emptyset$, by Lemma 10, $D \cong H_1$. But H_1 contains exactly two copies of H_2, which do not share a common directed triangle. A contradiction thus follows. $\square$

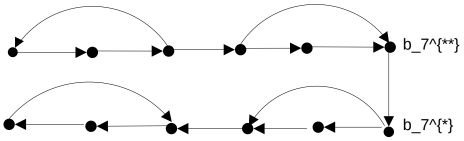

Figure 3.7

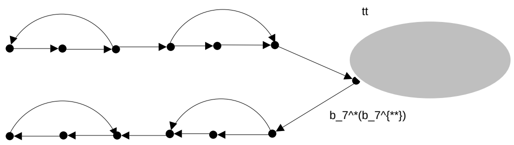

Figure 3.8 $V(H^*) \cup V(H^{**})$ induces a subgraph of the above digraph.

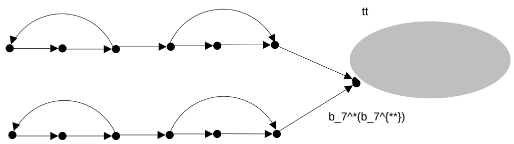

Figure 3.9 $V(H^*) \cup V(H^{**})$ induces a subgraph of the above digraph.

Lemma 3.2.8. *Let $D \in \mathcal{D}(1,1)$, $F \subset E(D)$ be saturated with $|F^0| \leq 3$ and $D' = D - F$. Suppose D' contains no H_1-components, $d_D(H) \geq 3$ for any induced H_1-subgraph H of D and $d_D(H) \geq 2$ for any induced H_2-subgraph H of D. Then D' contains at most one ES-H_1 or at most one VS-H_2, but not both.*

Proof. We may assume that D' contains an ES-H_1 or a VS-H_2 since the result follows immediately otherwise. Accordingly, the proof is divided into two cases.

Case 1: D' contains an ES-H_1.

Let H^* be an ES-H_1 in D'. We will show that H^* is the unique ES-H_1 and D does not contain any VS-H_2. Suppose that D' contains another ES-H_1 subgraph H^{**}. If $V(H^*) \cap V(H^{**}) = \emptyset$, by Lemma 8,

$$3 = 2 \times 3 - 3 \leq d_D(H^*) + d_D(H^{**}) - |F^0| \leq d_{D'}(H^*) + d_{D'}(H^{**}) \leq 2.$$

A contradiction thus follows. Hence, $V(H^*) \cap V(H^{**}) \neq \emptyset$. By Lemma 9, $D' = H^* \cup H^{**}$ is isomorphic to the digraph in Figure 4, its inverse, or Figure 5. Considering $D' = D - F$ and $|F^0| \leq 3$, we conclude that D contains either a VS-H_2, or an induced H_2-subgraph H with $d_D(H) = 1$ or an induced H_1-subgraph H with $d_D(H) = 2$, which in turn gives a contradiction.

We next claim that $D' - E(H^*)$ contains no VS-H_2. Indeed, suppose H^{**} is a VS-H_2 in $D' - E(H^*)$. Then $I(H^*) \cap I(H^{**}) = \emptyset$. Applying Lemma 8 and Lemma 11, we obtain

$$2 = 2 + 3 - 3 \leq d_D(H^*) + d_D(H^{**}) - |F^0| \leq d_{D'}(H^*) + d_{D'}(H^{**}) \leq 1,$$

a contradiction.

Case 2: D' does not contain any ES-H_1.

In this case, we claim that D' contains at most one VS-H_2. Indeed, assume H^* and H^{**} are two distinct VS-H_2 in D'. If $I(H^*) \cap I(H^{**}) = \emptyset$, Lemma 8 implies that

$$1 \leq d_D(H^*) + d_D(H^{**}) - |F^0| \leq d_{D'}(H^*) + d_{D'}(H^{**}) = 0,$$

a contradiction. Hence, $I(H^*) \cap I(H^{**}) \neq \emptyset$. Then, by Lemma 10, the only possible situation for D' is an H_1-component (see Figure 6), contradicting the assumption. $\square$

The following result was implicitly given in Lehel et al [14]. For the completeness, we give the outline of proof here.

Lemma 3.2.9. *If a digraph* $D \in \mathcal{D}(1,1)$ *is not a union of vertex-disjoint directed triangles, then* D *contains a reducing pair* (A, B) *with* $\frac{|A|}{|A|+|B|} \geq \frac{2}{5}$.

Proof. Let D^+ be the subgraph of D induced by $V^+ = \{v \in V(D) \mid d^+(v) \geq 2\}$; and let D^- be the subgraph of D induced by $V^- = \{v \in V(D) \mid d^-(v) \geq 2\}$. Let $V^0 = V(D) - (V^+ \cup V^-)$. Suppose D contains no reducing pair (A, B) with $\frac{|A|}{|A|+|B|} \geq \frac{2}{5}$. Then, we can show that the following claims stated in the proof of Theorem 1 in [14] hold by the same arguments there:

Claim 2: $V^+ \cup V^- \neq \emptyset$. (here, the condition that D is not a union of disjoint directed triangles is used.)

Claim 3: Each of D^+ and D^- is a disjoint union of directed cycles. Furthermore, every vertex in D^+ or D^- is incident with exactly one edge of $D - (D^+ \cup D^-)$.

Claim 4: All directed cycles in D^+ and D^- have odd length.

Claim 5: There is no edge between V^0 and $V^+ \cup V^-$.

Claim 6: Let M be the loopless bipartite multigraph obtained from the subgraph of D induced by $V^+ \cup V^-$ by contracting every directed cycle into one vertex. Then, M is a simple graph.

Applying the six claims listed above and following the same arguments in [14] , we get a desired reducing pair (A, B) with $\frac{|A|}{|A|+|B|} \geq \frac{2}{5}$. A contradiction thus follows. $\square$

3.3 Theorem 3.1.1

We first show that Theorem 3.1.1 is a consequence of the following result, whose proof will be given later.

Theorem 3.3.1. *Let $D \in \mathcal{D}(1,1)$ with $D \ncong \overrightarrow{C_3}$ and $D \ncong H_1$. If D satisfies one of the following three properties, then $\Lambda(D) \geq \frac{3m}{8}$.*

(i) D contains a unique ES-H_1, say H, and $D - E(H)$ does not contain any VS-H_2.

(ii) D contains a unique VS-H_2 and does not contain any ES-H_1.

(iii) D contains neither an ES-H_1 nor a VS-H_2.

Theorem 3.3.1 implies Theorem 3.1.1:
Proof. We may assume that D is connected since we could consider each component of D otherwise. Theorem 3.1.1 is trivial for $m = 1, 2, 3$. For all the m satisfying $4 \leq m \leq 10$, Theorem 3.2.2 implies $\Lambda(D) \geq \lceil 7m/20 \rceil \geq 3m/8$. Hence, we assume $m > 10$ and Theorem 3.1.1 is true for digraphs with less than m edges.

If D contains an ES-H_1 subgraph H, then let $E_D(V(H), D - V(H)) = \{xy\}$. Assume, without loss of generality, $y \in V(H)$. Since $d^+_{H_1}(a_3) = 2$, $d^-_{H_1}(a_3) = 1$ and $D \in \mathcal{D}(1,1)$, $y \neq a_3$. Similarly, $y \neq a_4$. Let

$$A = \begin{cases} \{a_3a_1, a_4a_5, a_4a_6, a_7a_8\}, \text{ if } y = a_8, \\ \\ \{a_3a_1, a_4a_5, a_4a_6, a_8a_9\}, \text{ Otherwise,} \end{cases}$$

and $B = E(H) - A$. Clearly, (A, B) is a reducing pair. Let $D' = D - (A \cup B)$, that is, D' is obtained from D by deleting all vertices of H except y; let D^* be obtained from D' by attaching a directed triangle T to y. Applying the induction hypothesis to the connected graph D^*, $\Lambda(D^*) \geq \frac{3|E(D^*)|-1}{8}$. Since each dicut of D^* contains at most one edge of T, $\Lambda(D) \geq (\Lambda(D^*) - 1) + |A| \geq \frac{3m-1}{8}$.

If D contains a VS-H_2, say H, then one can find a reducing pair (A,B) with $A \cup B = E(H)$ and $|A| = 3$. Let $D' = D - (A \cup B)$. Applying the induction hypothesis to the connected graph D', $\Lambda(D') \geq \frac{3|E(D')|-1}{8}$. Therefore, $\Lambda(D) \geq \Lambda(D') + |A| \geq \frac{3m-1}{8}$.

If D contains neither an ES-H_1 nor a VS-H_2, then we have $\Lambda(D) \geq \frac{3m}{8} \geq \frac{3m-1}{8}$ by Theorem 3.3.1. □

We will use induction on the size of D to attack Theorem 3.3.1. In the proof, an appropriate reducing pair will be found and removed from D which might disconnect D. The possible situations will be handled by a case-by-case analysis corresponding to the above properties (i),(ii) and (iii).

Proof of Theorem 16:

By Theorem 3.2.2, $\Lambda(D) \geq \lceil \frac{7m}{20} \rceil \geq \frac{3m}{8}$ for $m = 1, 2, ..., 10$. So Theorem 16 holds for $m = 1, 2, \ldots, 10$. Suppose that $m > 10$ and it holds for all digraphs with less than m edges.

The proof will be divided into three cases according to the properties (i), (ii) and (iii).

Case 1: D satisfies property (iii), i.e., D contains neither an ES-H_1 nor a VS-H_2.

Case 1.1: There is an induced H_1-subgraph H with $d_D(H) = 2$.

Let $\{e_1, e_2\} = E_D(V(H), V(D) - V(H))$. We distinguish the following four subcases according to the endvertices of e_1 and e_2 in H.

(a) e_1 and e_2 are attached with the same directed triangle in H;

(b) e_1 and e_2 are attached with distinct directed triangles in H;

(c) exactly one of e_1 and e_2 is attached to a directed triangle in H;

(d) neither e_1 nor e_2 is attached with directed triangles in H.

For all the four subcases, it is not difficult to show that there is a reducing pair (A,B) of D with $A \cup B = E(H)$ and $|A| = 4$. For example, in the following four depicted situations, we may take $(A,B) = (\{a_3a_1, a_4a_5, a_4a_6, a_7a_8\},\{a_2a_3, a_1a_2, a_3a_4, a_5a_6, a_6a_7, a_8a_9, a_9a_7\})$ as the reducing pair for each situation (see Figures 3.10 - 3.13, where the edges in A are depicted in thicker lines).

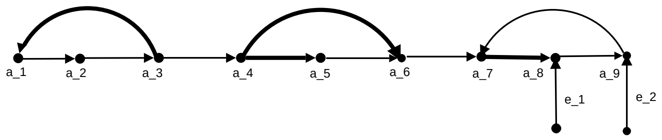

Figure 3.10 A situation for subcase (a)

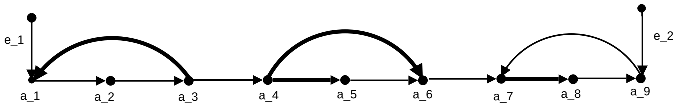

Figure 3.11 A situation for subcase (b)

Let D^* be obtained from $D - (A \cup B)$ by attaching a directed triangle T the ends of e_1 and e_2 in H according to the following rules: if the endvertices of e_1 and e_2 in H are distinct, we identify them to distinct vertices in T; otherwise, we identify the endvertices of e_1 and e_2 in H to a single vertex of T. Since $d_H^+(v)d_H^-(v) \geq 1$ for every vertex $v \in H_1$, $D^* \in \mathcal{D}(1,1)$.

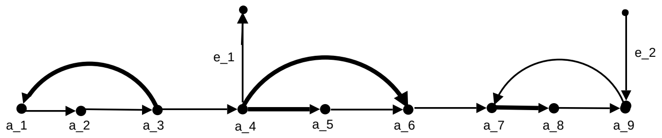

Figure 3.12 A situation for subcase (c)

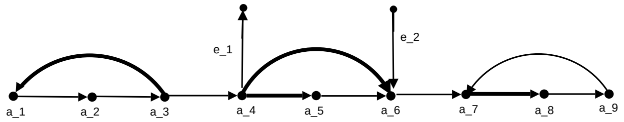

Figure 3.13 A situation for subcase (d)

Clearly, $D^* \ncong \overrightarrow{C_3}$. Since in H_1 each of two directed triangles is connected to the remaining graph through a single edge, $D^* \ncong H_1$.

We claim that D^* contains no VS-H_2. Otherwise, let H^* be a VS-H_2 in D^*. Since $e_{D^*}(V(T), V(D^*) - V(T)) = 2$, $V(T) \nsubseteq V(H^*)$. Thus, by Lemma 12, $V(T) \cap (I(H^*)) = \emptyset$. Because D can be obtained from D^* by replacing T with H, H^* is also a b_7-VS-H_2 in D, contradicting the assumption of Case 1.

We claim that D^* contains no ES-H_1. Otherwise, suppose H^* is an ES-H_1 in D^*. By Lemma 3.2.6, either $V(T) \subseteq V(H^*)$ or $V(T) \cap V(H^*) = \emptyset$. If $V(T) \subseteq V(H^*)$, then H^* is either an e_1-ES-H_1 or e_2-ES-H_1. Thus, $H^* - E(T)$ is a VS-H_2 in D. If $V(T) \cap V(H^*) = \emptyset$, then neither $e_1 \in H^*$ nor $e_2 \in H^*$, which implies H^* is an ES-H_1 in D. In either case, a contradiction to the assumption of Case 1 follows.

Therefore, D^* satisfies property (iii). Applying the induction hypothesis to the connected graph D^*, $\Lambda(D^*) \geq \frac{3|E(D^*)|}{8}$. Hence, $\Lambda(D) \geq \frac{3(m-8)}{8} + 3 = \frac{3m}{8}$ follows easily.

Case 1.2: There is an induced H_2-subgraph H with $d_D(H) = 1$.

Note that D contains neither an ES-H_1, nor a VS-H_2. Then, it can be reduced to Case 1.1 if there is an induced H_1-subgraph H in D with $d_D(H) = 2$. Thus, we may assume $d_D(H^*) \geq 3$ for any induced H_1-subgraph H^* and $d_D(H^*) \geq 1$ for any induced H_2-subgraph H^* in D.

Since D does not contain an ES-H_1, $D' = D - E(H)$ contains neither $\overrightarrow{C_3}$-components nor H_1-components. Since $H \cong H_2$ and $d_D(H) = 1$, there is a reducing pair (A,B) with $A \cup B = E(H)$ and $|A| = 3$. It is easy to see that $|(A \cup B)^0|$=1. Thus, from Lemma 8, $d_{D'}(H^*) \geq d_D(H^*) - 1 \geq 2$ for any induced H_1-subgraph H^* in D' and $d_{D'}(H^*) \geq d_D(H^*) - 1 \geq 0$ for any induced H_2-subgraph H^* in D', which implies that D' may contain a VS-H_2, but no ES-H_1. Moreover, if H^* and H^{**} are two distinct VS-H_2 in D', then Lemma 8 implies $I(H^*) \cap I(H^{**}) \neq \emptyset$. So, if D' contains more than one VS-H_2, then $D' \cong H_1$ by Lemma 10. Note that there is no vertices of degree 1 in H_1, contradicting the fact that $d_D(H) = 1$ and $D' = D - E(H)$. Hence, D' contains at most one VS-H_2.

Note that D' satisfies either property (ii) or property (iii). Applying the induction hypothesis to each component of D', $\Lambda(D') \geq \frac{3(m-8)}{8}$. Therefore, $\Lambda(D) \geq \Lambda(D') + |A| \geq \frac{3m}{8}$.

Case 1.3: There is an induced H_2-subgraph H with $d_D(H) \geq 2$.

In this case, we assume, without loss of generality, H is the graph shown in Figure 2, in particular, $d^+_H(b_4) = 2$.

If $d_D(H^*) = 2$ for an induced H_1-subgraph H^* or $d_D(H^*) = 1$ for an induced H_2-subgraph H^* in D, then it is reduced to either Case 1.1 or Case 1.2. Hence, we may assume $d_D(H^*) \geq 3$ for any induced H_1-subgraph H^* and $d_D(H^*) \geq 2$ for any induced H_2-subgraph H^* in D.

Starting with the vertex $v_0 := b_4$ in H, let $P = v_0v_1v_2, \cdots, v_l$ be a *maximal* directed path such that $d^+_D(v_j) \geq 2$ for $j = 0, 1, ..., l$. Since $D \in \mathcal{D}(1,1)$, $V(P) \cap \{b_1, b_2, b_3\} = \emptyset$. Let $x := v_l$ and $x' := x_{l-1}$ if $l \geq 1$ and $x' := b_3$ otherwise, and let xy and xz be two edges leaving x. Since P is maximal, $d^+_D(y) \leq 1$ and $d^+_D(z) \leq 1$. Denote by yy' (resp. zz') the possible edge leaving y (resp. z).

Case 1.3.1: $x \neq b_4$, i.e., $l > 0$.

Case 1.3.1.a: Either $yy' \in E(D)$ and $y' = z$ or $zz' \in E(D)$ and $z' = y$.

Suppose, without loss of generality, that $yy' \in E(D)$ and $y' = z$ (see Figure 3.14).

Let $A = \{xy, xz\}$,

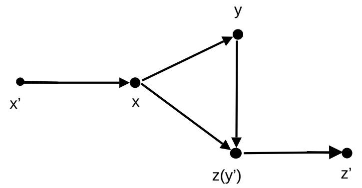

Figure 3.14 y' and z are the same vertex.

$$B = \begin{cases} \{x'x, yy', zz'\}, \text{ if } zz' \text{ exists,} \\ \\ \{x'x, yy'\}, \text{ otherwise,} \end{cases}$$

and $D' = D - (A \cup B)$.

It is readily seen that $A \cup B$ is saturated, and either $(A \cup B)^0 = \{x'x, zz'\}$ if zz' exists or $(A \cup B)^0 = \{x'x, yz, xz\}$ if zz' does not exist. By Lemma 14, D' contains at most one ES-H_1 or one VS-H_2, but not both of them, which implies that D' satisfies one of the three properties (i), (ii) or (iii).

We first consider the case that D' does not contain an H_1-component. By the maximality of P, D' contains at most one $\overrightarrow{C_3}$-component which is attached with $A \cup B$ by only the vertex z'. If this is the case, let the directed triangle be T and $V(T) - z' = \{u_1, u_2\}$. Assume, without loss of generality, $E(T) = \{z'u_1, u_1u_2, u_2z'\}$. Let $A' = \{xy, xz, u_1u_2\}$ and

$$B' = \begin{cases} \{x'x, yy', zz', u_2z', z'u_1\} \text{ if } zz' \text{ exists.} \\ \\ \{x'x, yy', u_2z, zu_1\} \text{ otherwise.} \end{cases}$$

Easy to see that (A', B') is a reducing pair. Obviously, the resulting graph $D^* = D - (A' \cup B')$ contains no $\overrightarrow{C_3}$-components. Recall that $d_D(H) \geq 3$ for any induced H_1-subgraph H in D^*, $d_D(H) \geq 2$ for any induced H_2-subgraph H in D^*, and $|(A' \cup B')^0| = 1$. It is readily seen that $A' \cup B'$ is saturated. By Lemma 8, D^* contains neither an ES-H_1 nor a VS-H_2. Applying the induction hypothesis to each component of D^*, $\Lambda(D^*) \geq \frac{3(m-8)}{8}$. Therefore, $\Lambda(D) \geq \Lambda(D^*) + |A'| = \frac{3m}{8}$. So, we may assume there are no $\overrightarrow{C_3}$-components in

D'. Then, applying induction hypothesis to each component of D', $\Lambda(D') \geq \frac{3|E(D')|}{8} = \frac{3(m-5)}{8}$. Therefore, $\Lambda(D) \geq \Lambda(D') + |A| > \frac{3m}{8}$.

We now assume that D' contains an H_1-component H^*. Since every vertex in H^* is saturated, $V(H^*) \cap V(D[A \cup B])$ is a subset of the unsaturated vertex set in $A \cup B$. Since $d_D(H^*) \geq 3$, edge zz' does not exist and $V(H^*) \cap V(D[A \cup B]) = \{x', z = y'\}$. Recall that $D \in \mathcal{D}(1,1)$. Then, all edges between $D[A \cup B] \cup H^*$ and the remaining vertices of D are incident to either x and y. We assume, without loss of generality, H^* is the one shown in Figure 1. Since $d^+_D(x) \geq 2$ and $d^+_D(y) = 1$, there is a reducing pair (A', B') such that A' is the union of $\{xy, xz, a_4a_5, a_4a_6\}$ and two edges from each directed triangle of H^*, respectively, while $B' = (A \cup B) \cup E(H^*) - A'$. Clearly, $|A'|/(|A'| + |B'|) = 6/15 > 3/8$. Similarly to the previous case, we can show that $\Lambda(D) \geq \frac{3m}{8}$.

Case 1.3.1.b: $yy' \in E(D)$, $zz' \in E(D)$, and y, y', z and z' are 4 distinct vertices (see Figure 3.15).

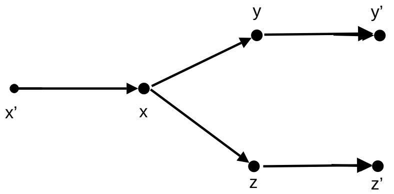

Figure 3.15 y' and z are different vertices.

Let $D' = D - \{A \cup B\}$ with $A = \{xy, xz\}$ and $B = \{x'x, yy', zz'\}$. Then, D' contains at most two $\overrightarrow{C_3}$-components which are attached to y' and z', respectively.

If D' has an H_1-component H^*, then $V(H^*) \cap \{x, y, z, x', y', z'\} = \{x', y', z'\}$. There are several cases regarding the position of $\{x', y', z'\}$ in the H_1-component. One case is shown in Figure 3.16. Regardless the positions of x', y' and z' in H^*, it is readily seen that (A^*, B^*)

is a reducing pair, where

$$\begin{aligned} A^* &= \{xy, xz, a_2a_3, a_4a_5, a_4a_6, a_7a_8\}, \\ B^* &= \{x'x, yy', zz', a_1a_2, a_3a_1, a_3a_4, a_5a_6, a_6a_7, a_8a_9, a_9a_7\}. \end{aligned}$$

Let $D^* = D - (A^* \cup B^*)$. Clearly, $(A^* \cup B^*)^0 = \emptyset$. Consequently, $A^* \cup B^*$ is saturated and D^* contains no $\overrightarrow{C_3}$-components. By Lemma 8, D^* contains neither an ES-H_1 nor a VS-H_2. Applying the induction hypothesis to each component of D^*, $\Lambda(D^*) \geq \frac{3|E(D^*)|}{8} = \frac{3(m-16)}{8}$. Therefore, $\Lambda(D) \geq \Lambda(D^*) + |A'| = \frac{3m}{8}$.

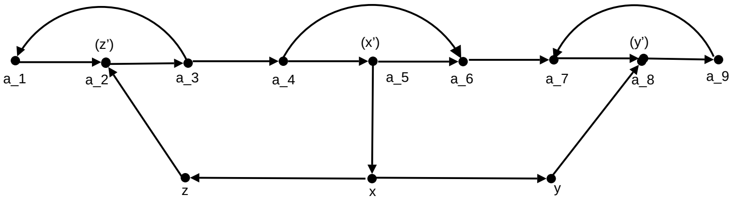

Figure 3.16 An isolated H_1-component in D'

Thus, in the following, we may assume D' contains no H_1-components.

Case 1.3.1.b.1: D' contains no $\overrightarrow{C_3}$-components.

In this case, D' contains neither $\overrightarrow{C_3}$-components nor H_1-components. Lemma 14 implies that D' satisfies one of the three properties (i), (ii) and (iii). Applying the induction hypothesis to each component of D', $\Lambda(D) \geq \Lambda(D') + |A| = \frac{3(m-5)}{8} + 2 > \frac{3m}{8}$.

Case 1.3.1.b.2: D' contains a unique $\overrightarrow{C_3}$-component T.

In this case, T is attached with either y' or z'. Suppose, without loss of generality, T is attached with y'. Similar to the Case 1.3.1.a, by adding one edge of T into A and the other two edges into B, we get a new reducing pair (A', B'). Let $D^* = D - (A' \cup B')$. Since $D \in \mathcal{D}(1,1)$ and T is the unique $\overrightarrow{C_3}$-component in D', D^* contains no $\overrightarrow{C_3}$-components. Moreover, D^* contains at most one ES-H_1 or one VS-H_2, but not both of them by Lemma 14. So, D^* satisfies one of the three properties (i), (ii) and (iii). Applying the induction hypothesis to each component of D^*, $\Lambda(D) \geq \Lambda(D^*) + |A'| \geq \frac{3m}{8}$.

Case 1.3.1.b.3: D' contains two $\overrightarrow{C_3}$-components.

In this case, one triangle, say T_1, is attached with vertex y' and the other one, say T_2, attached with z'. Let $E(T_1) = \{v_1v_2, v_2v_3, v_3v_1\}$ and $E(T_2) = \{v_4v_5, v_5v_6, v_6v_4\}$, with $v_1 = y'$ and $v_4 = z'$. The following two situations are considered.

If $d^-_{D'}(y)d^-_{D'}(z) = 0$ (assume, without loss of generality, $d^-_{D'}(y) = 0$), then $(A', B') = (\{v_2v_3, yy'\}, \{xy, v_1v_2, v_3v_1\})$ is a reducing pair. Let $D^* = D - (A' \cup B')$. Clearly, $A' \cup B'$ is a saturated pair and $(A' \cup B')^0 = xy$. Since $d^+_D(z) = 1$, D^* does not have a $\overrightarrow{C_3}$-component. We also claim that D^* does not have a H_1-component. Otherwise, that H_1-component in D^* is an xy-ES-H_1 in D, contradicting the assumption of Case 1. Since $|(A' \cup B')^0| = 1$, by Lemma 8, D^* satisfies property (iii). Applying the induction hypothesis to each component of D^*, $\Lambda(D) \geq \Lambda(D^*) + |A'| \geq \frac{3(m-5)}{8} + 2 > \frac{3m}{8}$.

So, we may assume $d^-_{D'}(y)d^-_{D'}(z) \neq 0$. Then, let $A' = \{xy, xz, v_2v_3, v_5v_6\}$, and $B' = \{x'x, yy', zz', v_1v_2, v_3v_1, v_4v_5, v_6v_4\}$. Remove A' and B' from D, and attach a directed triangle T to y to get a new graph D^*.

We claim that D^* contains no H_1-components (no ES-H_1). Indeed, let H^* be an H_1-component (an ES-H_1) in D^*, i.e., $d_{D^*}(H^*) \leq 1$. By Lemma 12, we have either $V(T) \cap V(H^*) = \emptyset$ or $V(T) \subseteq V(H^*)$. Recall that $d_D(H) \geq 3$ for any induced H_1-subgraph H of D, $d_D(H) \geq 2$ for any induced H_2-subgraph H of D and $|(A' \cup B')^0| = |x'x| = 1$. If $V(T) \cap V(H^*) = \emptyset$, Lemma 8 gives $d_D(H^*) \leq 2$. A contradiction thus follows. If $V(T) \subseteq V(H^*)$, then $H^* - E(T)$ is a y-VS-H_2 in D (induced H_2-subgraph in D with $d_D(H^* - E(T)) = 1$), contradicting to Case 1 (reducing to Case 1.2). After that, we claim there is at most one VS-H_2 in D^*. Indeed, if T is contained in every VS-H_2 of D^*, then D^* contains at most one VS-H_2 by Lemma 13. On the other hand, if there is one VS-H_2 in D^*, say H^{**}, such that $T \subsetneq H^{**}$. Then, by Lemma 8, $d_D(H^{**}) \leq 1$. A contradiction thus follows.

Based on the above discussion, D^* contains neither $\overrightarrow{C_3}$-components nor H_1-components, and satisfies either property (ii) or property (iii). By induction hypothesis, $\Lambda(D^*) \geq$

$\frac{3|E(D^*)|}{8} = \frac{3(m-8)}{8}$, which implies $\Lambda(D^* - E(T)) \geq \frac{3(m-8)}{8} - 1$. Therefore, $\Lambda(D) \geq \Lambda(D^* - E(T)) + |A'| = \frac{3m}{8}$.

Case 1.3.1.c: There are no edges leaving y or z.

Let $A = \{xy, xz\}$, $B = \{x'x\}$ and $D' = D - (A \cup B)$. Since $d^+_D(y) = 0$, $d^+_D(z) = 0$, and x' belongs to the chosen vertex sequence, D' contains no $\overrightarrow{C_3}$-components. Because each H_1-component in D' is an $x'x$-ES-H_1 in D and D does not contain any ES-H_1, D' contains no H_1-components.

Note that $d_D(H) \geq 3$ for any induced H_1-subgraph H in D', $d_D(H) \geq 2$ for any induced H_2-subgraph H in D', and only x' could possibly be in the induced H_1-subgraph or induced H_2-subgraph of D'. Hence, by Lemma 8, D' does not contain any ES-H_1 or VS-H_2. Applying induction hypothesis to each component of D', $\Lambda(D) \geq \Lambda(D') + |A| > \frac{3m}{8}$.

Case 1.3.2: $x = b_4$, i.e., b_4 is the final vertex in the sequence which has outdegree more than one. It is readily seen that in this case (A, B) is a reducing pair, where $A = \{b_4b_5, b_4b_6\}$ and $B = \{b_3b_4, b_5b_6, b_6b_7\}$. Let $D' = D - (A \cup B)$. Clearly $A \cup B$ is saturated and $|(A \cup B)^0| = 2$. Consequently, D' contains at most two $\overrightarrow{C_3}$-components.

Case 1.3.2.a: D' contains two $\overrightarrow{C_3}$-components.

Let the two $\overrightarrow{C_3}$-components of D' be T_1 and T_2 with $V(T_1) = \{b_1, b_2, b_3\}$ and $V(T_2) = \{b_7, b_8, b_9\}$. Then, D contains an induced H_1-subgraph, say H, whose vertex set is $\{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9\}$. Since $D \in \mathcal{D}(1,1)$ and $d^+_D(x) \leq 1$ for every $b_4x \in E(D)$, $d^-_{D'}(b_4) = d^+_{D'}(b_5) = d^+_{D'}(b_6) = 0$, and $d_D(H) \geq 3$. Let $D'' = D - E(H)$. Then, either $d^+_{D''}(b_4) \geq 1$, or $d^-_{D''}(b_5) \geq 1$, or $d^-_{D''}(b_6) \geq 1$. Suppose, without loss of generality, $d^+_{D''}(b_4) \geq 1$ and attach a directed triangle T to b_4 in D''. Denote the resulting graph by D^*. Clearly, D^* contains no $\overrightarrow{C_3}$-components.

We claim that D^* contains no H_1-components (ES-H_1). Indeed, let H^* be an H_1-component (ES-H_1) in D^*. If $V(T) \nsubseteq V(H^*)$, then $V(T) \cap V(H^*) = \emptyset$, which implies H^* is an H_1-component (an ES-H_1) in D''. Note that $|H^0| = 0$ and $D'' = D - E(H)$. Thus, by Lemma 8, H^* is also an H_1-component (ES-H_1) in D, which is a contradiction to the assumption of Case 1. On the other hand, if $V(T) \subseteq V(H^*)$, then, $H^* - E(T)$ is a b_4-VS-H_2

in D (induced H_2-subgraph with $d_D(H^* - E(T)) = 1$), contradicting the assumption of Case 1.3 (reducing to Case 1.2). Next, we claim that D^* contains at most one VS-H_2. Indeed, suppose H^* is a VS-H_2 in D^*, then $T \subseteq H^*$. Otherwise, $V(T) \cap I(H^*) = \emptyset$ by Lemma 12, which implies H^* is a VS-H_2 in D''. Note that $|H^0| = 0$. Thus, H^* is a VS-H_2 in D, which is a contradiction to the assumption of Case 1.3. Hence, by Lemma 13, H^* is the unique VS-H_2 in D^*, i.e., D^* contains at most one VS-H_2.

Hence, D^* satisfies either property (ii) or property (iii). By induction hypothesis, $\Lambda(D^*) \geq \frac{3|E(D^*)|}{8} = \frac{3m}{8} - 3$. Therefore, $\Lambda(D) \geq \frac{3m}{8}$.

Case 1.3.2.b: D' does not contain two $\overrightarrow{C_3}$-components.

In this situation, similar to Case 1.3.1, we can show $\Lambda(D) \geq \frac{3m}{8}$.

Case 1.4: D contains no induced H_2-subgraphs.

Note that in this case any subgraph of D satisfies property (iii). We may assume that D contains an induced $\overrightarrow{C_3}$, otherwise by Theorem 5, $\Lambda(D) \geq \frac{2m}{5} > \frac{3m}{8}$ follows.

Case 1.4.1: D contains directed triangles, and for any directed triangle T, either $V(T) \subseteq V^+$ or $V(T) \subseteq V^-$, where $V^+ = \{v \in V(D) | d^+(v) \geq 2\}$ and $V^- = \{v \in V(D) | d^-(v) \geq 2\}$.

In this case, Lemma 3.2.9 gives us a reducing pair (A, B) with $\frac{|A|}{|A|+|B|} \geq \frac{2}{5}$ Let $D' = D - (A \cup B)$. Since D contains no induced H_2-subgraphs, D' contains no H_1-components. Suppose there are totally n $\overrightarrow{C_3}$-components in D'. Note that for a reducing pair (A, B), any $\overrightarrow{P_3}$ with one edge in A has the other edge in B. Hence, all the edges connected with the n $\overrightarrow{C_3}$-components in D' are in B. Note that $V(T) \subseteq V^+$ or $V(T) \subseteq V^-$ for any directed triangle T, so each triangle is incident to at least three edges in B. Hence, $|B| \geq 3n$. For each of the n triangles, we add one edge of the triangle into A, and the other two edges into B. In this way, the updated reducing pair (A', B') is obtained. Clearly, $|A'| = |A| + n$ and $|B'| = |B| + 2n$. Since $\frac{|A|}{|A|+|B|} \geq \frac{2}{5}$ and $|B| \geq 3n$, $\frac{|A'|}{|A'|+|B'|} = \frac{|A|+n}{|A|+|B|+3n} \geq \frac{3}{8}$.

Since there is neither $\overrightarrow{C_3}$-components nor H_1-components in the resulting graph $D^* = D - (A' \cup B')$, and each component in D^* satisfies property (iii), $\Lambda(D^*) \geq \frac{3|E(D^*)|}{8}$. Therefore, $\Lambda(D) \geq \Lambda(D^*) + |A'| \geq \frac{3|E(D^*)|}{8} + |A'| \geq \frac{3(|E(D^*)|+|A'|+|B'|)}{8} = \frac{3m}{8}$.

Case 1.4.2: There is a directed triangle T such that $V(T) \nsubseteq V^+$ and $V(T) \nsubseteq V^-$ (recall that $V^+ = \{v \in V(D) | d^+(v) \geq 2\}$ and $V^- = \{v \in V(D) | d^-(v) \geq 2\}$). Let $E(T) = \{v_1v_2, v_2v_3, v_3v_1\}$. Suppose, without loss of generality, $v_3 \in D^+$ and $v_3x \in E(D)$ for some $x \in V(D) - V(T)$.

Case 1.4.2.a: $x \notin V^+$.

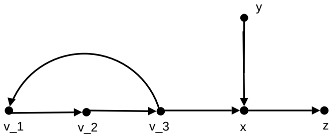

Figure 3.17 $x \notin D^+$

In this case, there may be edges coming into vertex x (see Figure 3.17). If $v_1 \notin D^+$, let $(A, B) = (\{v_3v_1, v_3x\},\{v_1v_2, v_2v_3, xz\})$; If $v_2 \notin D^+$, let $(A, B) = (\{v_1v_2, v_3x\}, \{v_2v_3, v_3v_1, xz\})$. Clearly (A, B) is a reducing pair and $\frac{|A|}{|A|+|B|} = \frac{2}{5}$. Let $D' = D - (A \cup B)$, then D' contains at most one $\overrightarrow{C_3}$-component which is attached with z in D. If this is the case, we update the reducing pair (A,B) by adding one edge of the triangle to A and the other two edges to B to get a new reducing pair (A', B') with $\frac{|A'|}{|A'|+|B'|} \geq \frac{3}{8}$. So, we may assume each component in $D' = D - (A \cup B)$ is neither a $\overrightarrow{C_3}$-component nor an H_1-component. Then, applying induction hypothesis to each component of D', $\Lambda(D') \geq \frac{3|E(D')|}{8}$. Therefore, $\Lambda(D) \geq \frac{3m}{8}$.

Case 1.4.2.b: $x \in V^+$.

Similar to Case 1.3, let $P := x_0(= x)x_1 \cdots x_l$ be a maximal path starting with x such that $d^+_D(x_i) \geq 2$ for each $0 \leq i \leq l$. If $l \geq 1$, we can show $\Lambda(D) \geq \frac{3m}{8}$ by following similar arguments of Case 1.3.1. So we may assume the path P only has one vertex x, that is, $d^+_D(y) \leq 1$ for any possible edge $xy \in E(D)$ (see Figure 3.18). In particular, we only need to consider the situation as in Figure 18, since the other situations can be handled similarly as Case 1.3.1.a and Case 1.3.1.c. Let $A = \{xy, xz\}$, $B = \{v_3x, yy', zz'\}$, and $D' = D - (A \cup B)$. If D' contains at most two $\overrightarrow{C_3}$-components, then similar to Case 1.3.2, $\Lambda(D) \geq \frac{3m}{8}$ follows easily.

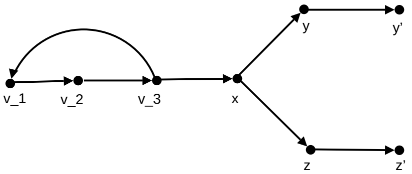

Figure 3.18 x is the final vertex of the sequence

If D' contains three $\overrightarrow{C_3}$-components, then except triangle $\{v_1v_2, v_2v_3, v_3v_1\}$, the other two $\overrightarrow{C_3}$-components T_1 and T_2 are attached with vertices y' and z' in D, respectively. There are edges coming into both y and z (otherwise $\Lambda(D) \geq \frac{3m}{8}$ follows easily as Case 1.3.1.b.3). For each $\overrightarrow{C_3}$-component, update the reducing pair by adding one edge from each triangle to A and the other two edges from that triangle to B. The new reducing pair (A',B') satisfies $A' \cup B' = \{v_1v_2, v_2v_3, v_3v_1, v_3x, xy, xz, yy', zz'\} \cup E(T_1) \cup E(T_2)$, $|A'| = 5$, and $|B'| = 9$.

Let $D^* = D - (A' \cup B')$. Since $(A' \cup B')^0 = \emptyset$, there are no H_1-components in D^*. We attach two directed triangles T_1' and T_2' to vertices y and z in D^* respectively, to get a new graph D^{**}. We claim that D^{**} contains no induced H_1-subgraphs. Indeed, note that D contains no induced H_2-subgraphs. Hence, if D^{**} contains an induced H_1-subgraph, say H^*, then both T_1' and T_2' are the two directed triangles in H^*. But, since both y and z are attached with edges coming into them in D^{**}, T_1' and T_2' can not be the two directed triangles of H^*.

We claim that $\Lambda(D^{**}) \geq \frac{3|E(D^{**})|}{8}$. Indeed, note that D^{**} may contain at most two induced H_2-subgraphs. If at most one of two induced H_2-subgraph is vertex-suspended, then D^{**} satisfies either property (ii) or (iii). By induction hypothesis, $\Lambda(D^{**}) \geq \frac{3|E(D^{**})|}{8}$. If both of the two induced H_2-subgraphs in D^{**} are vertex-suspended, a reducing pair (A'', B''), where $A'' \cup B''$ forms an induced H_2-subgraph and $|A''| \geq \frac{3(|A''|+|B''|)}{8}$, is obtained. Applying the induction hypothesis, $\Lambda(D^{**} - (A'' \cup B'')) \geq \frac{3|E(D^{**}-(A''\cup B'')|}{8}$. Thus, $\Lambda(D^{**}) \geq \Lambda(D^{**} - (A'' \cup B'')) + |A''| = \frac{3|E(D^{**})|}{8}$.

Hence, $\Lambda(D) \geq \Lambda(D^{**}) - 2 + |A'| = \Lambda(D^{**}) + 3 \geq \frac{3(|E(D^{**})|+8)}{8} = \frac{3|E(D)|}{8}$. This concludes the proof of Case 1. The rest consists in proving Case 2 and Case 3.

Case 2: D satisfies property (i), i.e., D contains a unique ES-H_1, say H, and $D - E(H)$ does not contain any VS-H_2.

Let $E_D(V(H), D-V(H)) = \{e\} = \{xy\}$. Assume, without loss of generality, $y \in V(H)$. Similar to the proof of Theorem 3, one can find a reducing pair (A, B) with $A \cup B = E(H)$ and $|A| = 4$. Let $D' = D - E(H)$ and attach a directed triangle T to y to get a new graph D^* (see Figure 3.19).

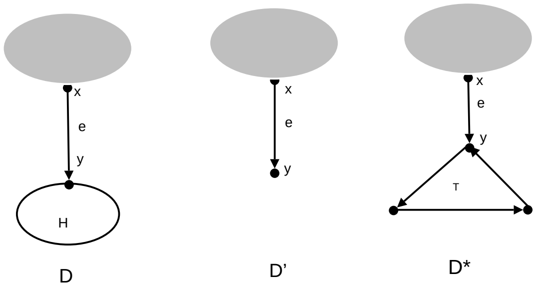

Figure 3.19 The three graphs D, D' and D^*

D^* is connected and $D^* \ncong \overrightarrow{C_3}$. We claim that $D^* \ncong H_1$. Otherwise, $D' \cong H_2$, so, $D - E(H)$ contains a VS-H_2, a contradiction.

Case 2.1: There exists an ES-H_1, say H^*, in D^*. We will show that D^* satisfies property (i).

We claim that the attached triangle $T \subset H^*$ if H^* is an ES-H_1 in D^*. Otherwise, by Lemma 3.2.5, $V(H^*) \cap V(T) = \emptyset$, which in turn shows that H^* is also an ES-H_1 in D, contradicting the uniqueness of ES-H_1 in D.

We claim H^* is the unique ES-H_1 in D^*. Otherwise, let H^{**} be another ES-H_1 in D^*. Then, $E(T) \subseteq E(H^*) \cap E(H^{**})$. From Lemma 9, $D^* = H^* \cup H^{**}$ and $e_{D^*}(V(T), D^* - V(T)) = 2$ for T in D^*. Since D^* is obtained from D by replacing H with T, we have $d_D(H) = 2$, contradicting the fact that H is an ES-H_1 in D.

We claim that $D^* - E(H^*)$ contains no VS-H_2. Otherwise, let H^{**} be a b_7-VS-H_2 in $D^* - E(H^*)$. By Lemma 11, H^{**} is also a b_7-VS-H_2 in D^*. So, $V(H^{**}) \cap V(T) = \emptyset$. Thus, H^{**} is a b_7-VS-H_2 in $D^* - E(T)$=$D - E(H)$, contradicting the assumption of Case 1.

Case 2.2: D^* contains no ES-H_1. We will show that D^* satisfies either property (ii) or property (iii).

We claim that there is a unique VS-H_2 in D^*. Indeed, if there is no VS-H_2 in D^*, then D^* satisfies property (iii). Suppose there is a VS-H_2, say H^*, in D^*. We claim that $T \subseteq H^*$. Otherwise, by Lemma 3.2.6, $V(T) \cap I(H^*) = \emptyset$, which implies $V(T) \cap (V(H^*) - y) = \emptyset$. Consequently, H^* is a VS-H_2 in $D - E(H)$, contradicting there being no VS-H_2 in $D - E(H)$. Hence, by Lemma 13, H^* is the unique VS-H_2 in D^*, which implies that D^* contains a unique VS-H_2.

Therefore, D^* satisfies one of the properties (i), (ii) or (iii). Since D^* is connected and $|E(D^*)| < |E(D)|$, $\Lambda(D^*) \geq \frac{3(m-11+3)}{8} = \frac{3m}{8} - 3$. Thus, by Lemma 7, $\Lambda(D) \geq \Lambda(D^*) - 1 + |A| \geq \frac{3m}{8}$.

Case 3: D satisfies property (ii), i.e., D contains a unique VS-H_2, say H, but does not contain any ES-H_1. Let $D' = D - E(H)$. Since $D \ncong H_1$, $D' \ncong \overrightarrow{C_3}$ and D does not contain any ES-H_1, we have $D' \ncong H_1$.

Case 3.1: There exists an ES-H_1, say H^*, in D'.

We claim that H^* is the unique ES-H_1 in D'. Otherwise, suppose there is another ES-H_1, say H^{**}, in D'. Since D does not contain any ES-H_1, we have $b_7 \in V(H^*) \cap V(H^{**})$. Thus, by Lemma 9, D' contains two VS-H_2 and there is a directed triangle T such that $b_7 \in V(T) = V(H^*) \cap V(H^{**})$. Since $D' = D - E(H)$ and H is attached with the single vertex b_7 in D, D contains at least two VS-H_2, contradicting the assumption of Case 3. Next, we claim that $D' - E(H^*)$ contains no VS-H_2. Otherwise, let H^{**} be another VS-H_2 in $D' - E(H^*)$ with suspended vertex b_7^*. Thus, $d_{D'-E(H^*)}(H^{**}) = 0$, i.e., $d_{D-(E(H)\cup E(H^*))}(H^{**}) = 0$. Since $b_7 \in V(H^*)$, $(E(H) \cup E(H^*))^0 = \emptyset$. By Lemma 8, $d_D(H^{**}) - 0 \leq d_{D-(E(H)\cup E(H^*))}(H^{**}) = 0$. So, $d_D(H^{**}) = 0$. Therefore, H^{**} is a VS-H_2 in D, contradicting the uniqueness of VS-H_2 in D.

Case 3.2: D' contains no ES-H_1.

We claim that D' contains at most one VS-H_2. Otherwise, let H^* and H^{**} be two distinct VS-H_2 in D' with suspended vertices b_7^* and b_7^{**}, respectively. If $I(H^*) \cap I(H^{**}) = \emptyset$, by Lemma 8, $d_D(H^*) + d_D(H^{**}) \leq 1$, which implies either H^* or H^{**} is a VS-H_2 in D. Therefore, a contradiction to the uniqueness of VS-H_2 in D follows. On the other hand, if $I(H^*) \cap I(H^{**}) \neq \emptyset$, then by Lemma 10, D' is isomorphic to H_1. Note that $D' = D - E(H)$ and $|E(H)^0| = 1$. Hence, D' is an ES-H_1 in D, contradicting the assumption of Case 3.

Therefore, in case 3, D' satisfies one of the properties (i), (ii) or (iii). Since $|E(D')| < m$ and D' is connected, $\Lambda(D') \geq \frac{3(m-8)}{8} = \frac{3}{8}m - 3$. Thus, by Lemma 7, $\Lambda(D) \geq \Lambda(D') + 3 = \frac{3}{8}m$.

□

3.4 Problem 2

In this section, we will give an infinite class of graphs showing that the Problem 2 in [3] is not true. In addition, we show that it is true if the underlying undirected graph of the digraph D considered is a tree. Indeed, the construction goes as follows. Let Ω be the graph obtained from a directed path $P_5 = v_1v_2v_3v_4v_5$ by adding the edge v_2v_4. Then, $C = \{v_2v_4, v_3v_4\}$ is a dicut, and it is easy to verify that $\Lambda(\Omega) = 2 = \frac{2|E(\Omega)|}{5}$. Let G be obtained from s copies of vertex disjoint Ω by identifying v_1 in the i-th copy of Ω to one vertex, say, v_1, v_2 or v_3, in any previous copy of Ω for $i = 2, 3, \cdots\cdots, s$. Then, $\Lambda(G) = \frac{2|E(G)|}{5}$, but there are s vertices whose outdegree or indegree is 0.

Proof of Theorem 4

Suppose Theorem 4 is not true, and let D be a counterexample with the minimum number of edges.

Claim: There are no 2 leaves sharing one common neighbor in D.

Suppose, to the contrary, that there is a vertex x which is adjacent to at least two leaves. If there are two leaves v_1 and v_2, such that $v_1x, xv_2 \in E(D)$, we have $d^+_{D-\{v_1,v_2\}}(x)d^-_{D-\{v_1,v_2\}}(x) = 0$. Hence, $\Lambda(D - \{v_1, v_2\}) \geq \lfloor\frac{(2(m-2)+s-2+1)}{5}\rfloor = \lfloor\frac{(2m+s)}{5} - 1\rfloor$. Then, $\Lambda(D) \geq \lfloor(2m + s)/5\rfloor$, contradicting D being a counterexample. So, we may assume that $E(\{x\}, Y) = \overrightarrow{E}(\{x\}, Y)$, where Y is the set of all leaves which are adjacent to x in D. Let $|Y| = t$. Then, $\Lambda(D - Y - \{x\}) \geq \lfloor\frac{2(m-(t+1))+s-t}{5}\rfloor$. Hence, $\Lambda(D) \geq \Lambda(D - Y - \{x\}) + t \geq \lfloor\frac{2m+s}{5}\rfloor$, contradicting D being a counterexample.

By the above Claim, the second vertex of the longest path in the underlying graph of D has degree two. Then, there is a vertex-suspended P_3 (it is not necessarily a directed path) in the graph D. Thus, $\Lambda(D - E(P_3)) \geq \lfloor \frac{2(m-2)+s-1}{5} \rfloor = \lfloor \frac{2m+s}{5} \rfloor - 1$. So, we have $\Lambda(D) \geq \lfloor \frac{2m+s}{5} \rfloor$. □

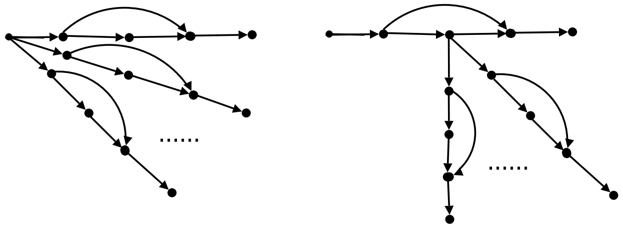

Figure 3.20

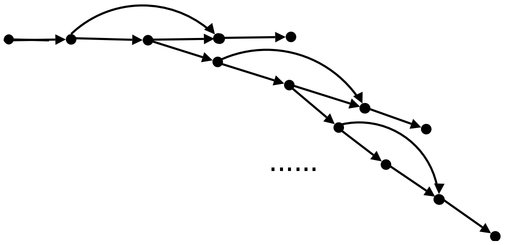

Figure 3.21

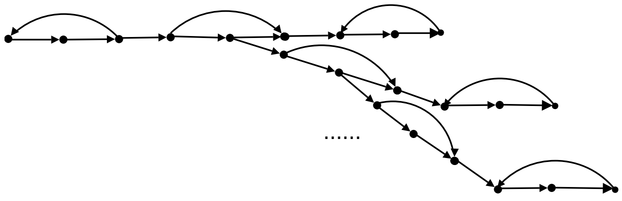

Figure 3.22

Remark: it is easy to obtain that $\Lambda(T) \geq \frac{|E(T)|}{2}$ if the underlying undirected graph of T is a tree. So, the above proposition can be generalized to $\Lambda(T) \geq \max\{\frac{|E(T)|}{2}, \frac{2|E(T)|+s}{5}\}$

for every digraph $T \in \mathcal{D}(1,1)$ whose underlying undirected graph is a tree. Note that if $s > \frac{|E(T)|}{2}$, then $\frac{2|E(T)|+s}{5}$ is a better bound than $\frac{|E(T)|}{2}$. There are infinitely many examples to illustrate that the theorem is not true if the underlying undirected graph of $D \in \mathcal{D}(1,1)$ is not a tree (see Figure 3.20 and 3.21).

PART 4

CONCLUSIONS

In this dissertation, we study the following two topics: the union closed set conjecture and the maximum edge cut in connected digraphs.

For the union-closed-set-conjecture-topic, we surveyed necessary and important results based on different techniques developed as the time goes by. More importantly, we present our results which could potentially lead to a full proof for the union closed set conjecture.

On the other hand, for the topic related to the maximum edge cuts in connected digraphs, we give a detailed exploration of its historical development and also present our proof techniques to solve the two problems posed by other authors.

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