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On the Interaction between Transfer Restrictions and Crediting Strategies in Guaranteed Funds

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Abstract

Guaranteed funds with crediting rates for fixed periods determined by a Pension Provider or Insurance Company are common features of accumulation annuity contracts. Policyholders can transfer money back and forth between these accounts and Money Market accounts which give them features similar to demand deposits and yet they frequently credit a higher rate than the Money Market. Transfer restrictions are commonly employed to prevent arbitrage. In this paper, we model the interaction between company and policyholder as a multiperiod game in which the company maximizes risk-neutral expected present value of profits and the policyholder maximizes his expected discounted utility. We find that the optimal strategy on the part of the company is to credit a rate higher than money market rate in the first period to entice the policyholder to invest in the guaranteed fund. The company then credits the floor in the remaining periods as the policyholder transfers out the maximum amount. This does better for the policyholder in low interest rate environments and worse in high interest rate environments and acts as a type of “interest rate insurance” for the policyholder.

Keywords: Annuity Crediting Strategies, Optimal Policyholder Behavior

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1. Introduction

One of the major problems in modern financial planning is accumulating assets over a working lifetime to provide sufficient income in retirement. Defined Contribution Pension Plans have become increasingly common in recent years. Employees deposit money at regular intervals into a designated account. These contributions are frequently matched at some level by the employers. The employee can direct the funds to a number of different accounts. Subject to only a few restrictions, they can rebalance their portfolio whenever they want.

Most DC plans have stock funds, bond funds and mixed funds, all of which have the possibility of losing money in bad markets. In addition, many DC plans have a money market account which credits a short-term interest rate and cannot¹ lose money. A significant number of plans also contain a “Guaranteed Fund” which credits a rate guaranteed for a fixed period, often monthly or quarterly. These funds are backed by longer term assets and the rate quoted for the time period is usually dependent on the book return of these assets less a spread that covers expenses and insurer profits.

Unlike bond funds, which can lose money if interest rates rise and the bond market values fall, these funds are usually redeemable at book value and cannot lose money. In addition, there is a minimum crediting rate for these contracts. This rate is required by state non-forfeiture laws but the insurance company could set a higher rate for marketing reasons.

¹ It is theoretically possible for a money market fund to lose money. This had happened to only three funds in the 37 years prior to the recent financial crisis. Events of September 2008 prompted the US Treasury to guarantee Money Market funds.

Prevention of arbitrage between money market funds and Guaranteed Funds is a major issue for insurance companies. If there were no restrictions on transfers between these accounts, savvy policyholders would transfer their money into the highest earning account. The Money Market account would respond quickly to rises in interest rates, while the Guaranteed Funds would respond with a lag. Money would be transferred out of the Guaranteed Fund when rates are high, exactly the moment when the asset market value is lower than book value and assets would need to be sold at a loss. In practice, insurance companies try to mitigate this reaction by imposing transfer restrictions, whereby an individual can transfer out only a fixed percentage of his Guaranteed Fund in any given time period.

In this paper, we determine the optimal crediting strategy on Guaranteed Funds from the perspective of maximizing the risk-adjusted profit to the pension provider. We then compare this to crediting strategies observed in practice.

2. The Model

We will use a game-theoretical model to analyze the interplay between the actions of the Pension Provider (hereafter PP) and the Policyholder (hereafter PH). PP's goal is to maximize his present value of the expected future book profit stream under the Q measure. PH's goal is to maximize the expected discounted utility under the P measure. It could be argued that in the absence of frictions, PH should instead maximize the expected present value under the Q measure as well. There are, however, frictions in this case. The policyholder is unable to sell his pension to a third party and is typically unable

to inexpensively hedge his risk. In these situations, using expected utility under the P measure is arguably correct (see, for instance, Gao and Ulm (2012), Leung and Sircar (2009) or Shreve (2003) page 70).

At time t , the universe is in state (filtration) F_t . This includes the current interest rate environment, the insurers current assets and the policyholders' current allocation. Let $s_{t,i}$ represent the current zero-coupon rate for a duration of i years. Let $A_{t,ij}$ represent the dollar amount in a zero-coupon asset with a remaining duration of i years and $r_{t,ij}$ represent the book interest rate on that asset. j is an index that runs over all possible purchase dates for assets with a remaining duration of i years. For instance, a current bond with a two year duration could be a three-year bond purchased last year, a four-year bond purchased two years ago, and so on. These bonds would have different book rates since they were purchased at different times. Let ω_t represent the percentage of assets currently allocated to the money market account.

The "game" proceeds as follows. At time t :

1. PP picks r_c , the rate he will credit for the next time period.
2. PH picks his allocation, ω_{t+1} , which becomes a state variable for the next period.
3. PP buys assets, which become state variables for the next period.

2.1 Zero-Sum Analysis with no Transfer Restrictions and no Crediting Floor

To motivate the importance of a risk-averse policyholder who maximizes his expected utility under the P measure, we will here analyze the zero-sum case where the policyholder maximizes his expected value under the Q measure.

Proposition 2.1: PP's asset purchase strategy is independent of his crediting strategy and independent of PH's choices.

Proof: PP attempts to maximize

$$\sum_{t=1}^{\infty} E_Q \left[\frac{\Delta BVA - \Delta BVL}{1 + s_t} \right] = \sum_{t=1}^{\infty} E_Q \left[\frac{\Delta BVA}{1 + s_t} \right] - \sum_{t=1}^{\infty} E_Q \left[\frac{\Delta BVL}{1 + s_t} \right].$$

This can be done by choosing assets to maximize the first sum, and playing the game with PH in order to minimize the second sum. Therefore, the insurer acts to maximize the asset values and minimize the liability values.

The result of Proposition 2.1 might be counterintuitive for actuaries, since Guaranteed Funds frequently credit a rate that is tied, at least loosely, to the returns on the underlying asset portfolio and Proposition 2.1 says this is not optimal.

Proposition 2.1 holds even if the insurer is required to back money-market funds with short-term assets in a separate account. If he desires less short-term exposure than this, PP can adjust the overall asset portfolio by borrowing short-term to buy extra long-term assets in the General Account backing the Guaranteed Fund.

Proposition 2.2: PP is indifferent to his asset strategy.

Proof: This is a basic consequence of the above propositions and the Modigliani-Miller Theorems (1958, 1961) stating that companies are indifferent to capital structure and dividend policy.

The result of Proposition 2.2 might be counterintuitive to actuaries who are used to attempting to match the durations of assets to the durations of liabilities, but indifference to asset strategy is common in the financial literature as seen in the Modigliani-Miller theorem (1958). This indifference to asset strategy will not hold in the presence of frictions regarding borrowing costs, differential tax treatment, or bankruptcy costs. We assume here that this contract is a small enough piece of PP's overall portfolio that the firm can borrow internally and the contract has a negligible effect on PP's overall bankruptcy probability.

As a consequence of Proposition 2.2, we will allow the insurer to invest 100% in short-term assets for ease of analysis. In this case, the only state variable needed at a particular time is the short-term rate and a full yield curve model is unnecessary.

Proposition 2.3: If there are no transfer restrictions, PP will credit a rate $r_c < r_{t,1}$ and PH will allocate $\omega_{t+1} = 1$ or PP will credit $r_c = r_{t,1}$ and PH will allocate $0 \leq \omega_{t+1} \leq 1$.

Proof: Suppose the PP credits $r_c > r_{t,1}$. If the policyholder invests $\omega_{t+1} > 0$ he earns a rate less than r_c and can improve by investing $\omega_{t+1} = 0$ in the current period. This doesn't affect his choice set in the next period, so the policyholder gains by a strategy change. If

the policyholder invests $\omega_{t+1} = 0$, the PP could improve his result by lowering his crediting rate. Therefore, no Nash equilibrium exists with $r_c > r_{t,1}$.

If the PP credits $r_c = r_{t,1}$ and PH allocates $0 \leq \omega_{t+1} \leq 1$, the PH earns $r_{t,1}$ and cannot improve by a deviation. If the PP lowers his crediting rate, the PH transfers to $\omega_{t+1} = 0$ and there is no improvement. Therefore, this is an equilibrium.

If the PP credits $r_c < r_{t,1}$ and PH allocates $\omega_{t+1} < 1$, the PH earns less than $r_{t,1}$ and cannot improve by allocating $\omega_{t+1} = 1$. Therefore, this is not an equilibrium.

If the PP credits $r_c < r_{t,1}$ and PH allocates $\omega_{t+1} = 1$, the PH earns $r_{t,1}$ and neither benefits from a deviation. Therefore, this is an equilibrium.

Proposition 2.4: At any given time and state with $\omega_t = 1$, the expected present value of future book profits under Q is the market value of the assets less the book value of the assets. Specifically, the expectation at initiation of the contract is 0.

Proof: The proof is by induction. We assume from proposition 2.4 that liabilities always earn $r_{t-1,1}$ in period $t-1$. The expected present value of future book profits at time $t-1$ is equal to the book profits earned in the next period plus the discounted expected present value of book profits at time t . Assume that in all states j at time t , the expected present value of book profits from that moment forward is $MV_t(j) - BV_t(j)$. Assume there are n assets of book value A_j and book return r_j . The value of book profits is the change in

book value of assets, $\sum_{i=1}^n A_i(1+r_i)$, less the change in book value of liabilities,

$\sum_{i=1}^n A_i(1+r_{t-1,i})$ since the sum of the book asset values is the account value. Therefore, the

expected present value of future book profits at time $t-1$ is:

$$\frac{1}{(1+r_{t-1,i})} E_Q[MV_t(j)] - \frac{1}{(1+r_{t-1,i})} E_Q[BV_t(j)] + \sum_{i=1}^n \frac{A_i(r_i - r_{t-1,i})}{(1+r_{t-1,i})} \quad (1)$$

Now, the first term is just the definition of the market value of assets MV_{t-1} . The book

value of assets in the next period is independent of state and equal to $\sum_{i=1}^n A_i(1+r_i)$ so the

expected present value of future book profits at time $t-1$ is:

$$MV_{t-1} - \frac{\sum_{i=1}^n A_i(1+r_i)}{(1+r_{t-1,i})} + \sum_{i=1}^n \frac{A_i(r_i - r_{t-1,i})}{(1+r_{t-1,i})} = MV_{t-1} - \sum_{i=1}^n A_i = BV_{t-1} - MV_{t-1} \quad (2)$$

2.2 Zero-Sum Analysis with Transfer Restrictions

We will now extend the analysis to include the existence of transfer restrictions.

At the end of any period, the money in the money market account is free to be transferred in whole or in part to the guaranteed fund. On the other hand, only a percentage x can be transferred out of the guaranteed account into the money market account. A percentage $1-x$ must remain in the guaranteed account for the next period.

Proposition 2.5: In the presence of transfer restrictions, the only reasonable allocations in the period $t+1$ are $\omega_{t+1} = 0$ and $\omega_{t+1} = (1-x)\omega_t + x$ (or complete indifference to allocation). The decision of which allocation to choose is independent of the current allocation ω_t .

Proof: Imagine the PH has three independent accounts:

1. A guaranteed account of $(1-x)(1-\omega_t)$ which must remain in the guaranteed account and cannot be affected by the PH's current choice.
2. A guaranteed account of $x(1-\omega_t)$ currently allocated to the guaranteed account but fully allocatable in the next period.
3. A money market account of ω_t currently allocated to the money market account but fully allocatable in the next period.

This is identical to the situation in the presence of transfer restrictions. Since Funds 2 and 3 are identical going forward they should be allocated identically in the next period and should have the same present value to the PH. Consider Fund 3 first. In the next period, some of it will be allocated to Fund 1, some to Fund 2 and some to Fund 3. The total value is the weighted average of the amount allocated to Funds 1, 2 and 3. Fund 2 and 3 are equally valuable, so if Fund 1 going forward is more valuable than Fund 3, all of Fund 3 should be moved to Fund 1. Otherwise, it should all be retained in Fund 3. The same is true of Fund 2. Therefore, either all of Funds 2 and 3 should be moved to Fund 1 or all should be move to Fund 3. These situations correspond to $\omega_{t+1} = 0$ and $\omega_{t+1} = (1-x)\omega_t + x$ respectively. Indifference is obtained if Funds 1 and 3 are equally

valuable going forward. The decision is based entirely on the future values of Funds 1 and 3 and is therefore independent of current allocation ω_t .

The arguments in the above proofs are very useful because they show that the PP's strategy can be analyzed solely by the effect it produces on the actions of a policyholder invested in Fund 3, i.e. the Money Market Fund. This will be valuable in the proof of the main result in this section.

Proposition 2.6: In the first period, the policyholder is free to invest at any value of $0 \leq \omega_1 \leq 1$. If there are transfer restrictions, PP will credit a rate $r_c \leq r_{crit}$ where $r_{crit} \geq r_{1,1}$ and depends on time and state. PH will allocate $\omega_1 = 1$ if $r_c < r_{crit}$ and $0 \leq \omega_1 \leq 1$ if

$$r_c = r_{crit}.$$

Proof: If the PP credits $r_c \leq r_{1,1}$, there is no advantage to PH to investing $\omega_1 < 1$ since the profit in the first period would be less than (or equal to) the profits at $\omega_1 = 1$ and the options are limited in the next period. In fact, $\omega_1 = 1$ is a strict result even at $r_c = r_{1,1}$ since an allocation with $\omega_1 < 1$ allows the PP to credit "0" in subsequent periods and the policyholder takes the loss as he slowly transfers his Guaranteed Funds back to the money market² To compensate for the losses when "trapped", the PP will have to credit an amount greater than $r_{1,1}$ to induce PH to transfer any funds at all into the Guaranteed

² We have not yet shown that the two parties do not have superior strategies to this one, but the mere existence of this strategy is sufficient to prove the Proposition.

Account. There will be a rate $r_c = r_{crit}$ in which the profit in the first period exactly compensates for the expected present value of losses from the “trap”. If $r_c < r_{crit}$, the policyholder will invest $\omega_1 = 1$ since the first period gains are insufficient to cover the expected losses in future periods. If $r_c = r_{crit}$, PH is indifferent to choice of fund allocation and can choose any $0 \leq \omega_1 \leq 1$. PP will not credit $r_c > r_{crit}$ since it gives away money in the first period without changing PH behavior beyond that produced by $r_c = r_{crit}$.

Proposition 2.6 implies that the contract has a value of “0” at initiation, since crediting $r_c = r_{t,1}$ and allocating $\omega_t = 1$ is always a possible equilibrium and has a value of “0”. Crediting $r_c = r_{crit}$ makes PH indifferent to this outcome, and therefore must also have a value to PH of “0” and, by the zero-sum property of the game, to PP as well.

Proposition 2.7: The value of r_{crit} is independent of the state variable ω_t .

Proof: This is a direct consequence of Proposition 2.5, i.e. that a strategy can be evaluated only by its effect on policyholders invested solely in Money Market Funds and that PP optimal strategies are independent of allocation ω_t .

Proposition 2.8: If $\omega_t > 0$, PP should set $r_c = 0$

Proof: We will first prove that the only rates that PP should credit are $r_c = 0$ and $r_c = r_{crit}$ and then show that $r_c = 0$ gives the more favorable result to PP. From Propositions 2.5-2.7, if PP credits $0 \leq r_c < r_{crit}$, the policyholder will wish to invest Funds 2 and 3 in Money Market accounts and therefore no value of r_c in this range will change behavior, or alter the profit on Funds 2 and 3. On the other hand, the lower the value of r_c , the greater the gain on Fund 1 to PP. Therefore $r_c = 0$ does better for PP than any other value of $r_c < r_{crit}$.

Similarly, if PP credits $r_c \geq r_{crit}$, the policyholder will wish to invest in Fund 1 and therefore no value of r_c in this range will change behavior. On the other hand, the lower the value of r_c , the lower the loss on Fund 1 to PP. Therefore $r_c = r_{crit}$ does better than any other value of $r_c > r_{crit}$. We therefore need only evaluate $r_c = 0$ or $r_c = r_{crit}$ from the perspective of PP.

Now, from Propositions 2.6 and 2.7, $r_c = r_{crit}$ is the rate that makes a person indifferent between Money Market and Guaranteed Accounts and crediting $r_c = r_{crit}$ is revenue-neutral relative to crediting $r_c = r_{t,1}$ in perpetuity³. Now, clearly, crediting $r_c = 0$ followed by crediting $r_c = r_{t,1}$ in perpetuity is better than this, and crediting $r_c = 0$ followed by optimal crediting in future periods is, by definition, at least as good as

³ Crediting $r_c = r_{t,1}$ in perpetuity is not optimal according to this argument, but it does not need to be for the argument to carry through. I need only show that $r_c = r_{crit}$ followed by subsequent optimal crediting is equivalent to crediting $r_c = r_{t,1}$ in perpetuity and that crediting $r_c = 0$ in the first period followed by optimal crediting does better than crediting $r_c = r_{t,1}$ in perpetuity.

crediting $r_c = r_{t,1}$ in future periods. In fact, it is strictly better since we've shown that $r_c = r_{t,1}$ is not optimal in the next period, only $r_c = r_{crit}$ or $r_c = 0$ could be.

Propositions 2.6-2.8 are interesting results, as they allows the PP to credit a rate on the Guaranteed Account that is higher than that on the Money Market Account, which is seen empirically. On the other hand, they imply that the insurer will credit "0" on Guaranteed Accounts after the first year, which disagrees with real PP practice. They also imply that PH will place no funds in the Guaranteed Account at initiation of the contract unless $r_c = r_{crit}$ exactly. It is possible that companies can overcome the implication that they must credit "0" through contractual precommitments. This could explain the prevalence of situations where PP credit a spread below their portfolio rates. It also explains situations where policyholders can exit the General Account with an annuity whose rate is related to the current market rates.

Proposition 2.9: If PP credits an interest rate larger than r_{crit} , and PH can borrow and lend at prevailing rates outside the pension plan, an arbitrage opportunity exists for PH.

Proof: Neither the PP or PH strategy is state dependent if PP credits $r_c > r_{crit}$ followed by $r_c = 0$ in subsequent periods and PH puts 100% of his money in the Guaranteed Account at time 0 and moves x percent deterministically to the money market every period

afterwards. The present value of this under the Q measure is $\frac{1+r_c}{1+r_{crit}} > 1$ since Proposition

2.7 implies that PH is indifferent between the Guaranteed Account and the Money Market Account worth \$1 if $r_c = r_{crit}$.

To set up the arbitrage, PH borrows \$1 to invest in the Pension Guaranteed Account. He borrows at prevailing rates in such a way as to repay $x(1-x)^t(1+r_{crit})$ at integer times $t > 0$. The present value of this stream is \$1 from Proposition 2.7. He repays these values by borrowing at money-market rates, and accumulates an outstanding debt at retirement equal to the value of these cash-flows accumulated at short-term rates.

Inside the account, PH receives cash flows of $x(1-x)^t(1+r_c)$ to invest in the Money Market. These funds accumulate at retirement to a value $\frac{1+r_c}{1+r_{crit}}$ times his accumulated external debt. When the assets and debts are netted at retirement, the amount is guaranteed to be positive.

This case, where PP credits “0” in subsequent periods is worst case for PH. If PP credits $r_c > 0$, the internal invested cash flows are even higher and the net amount available at retirement is an even larger positive number.

3. Analysis Assuming Utility Maximizing Policyholders

We now consider the non-zero sum case where the PP can hedge and therefore attempts to maximize the expected value of future profits under the Q measure whereas the PH attempts to maximize expected value of the utility of his ending fund under the P measure.

Proposition 3.1: The results of Propositions 2.3-2.4 hold even when PH attempts to maximize expected utility under the P measure.

Proof: The equilibrium arguments for PP deviations in the proof of Proposition 2.4 are still valid. Also, as long as utility is increasing in money amount, the PH deviation arguments in the proof of Proposition 2.3 remain valid. Therefore, the assumption that liabilities always earn $r_{t-1,1}$ in period $t-1$ remains valid and the argument in Proposition 2.4 carries through unchanged.

Now, an individual who is even risk-averse in fund outcomes will often prefer a crediting strategy of r_{crit} in period “1” followed by “0” in subsequent periods to a strategy where PP credits the money market rate at all periods. You might expect from the definition of r_{crit} , these two strategies would have equal mean outcomes, but the outcomes will typically be lower on average for the “credit r_{crit} ” strategy due to the effects of time-dependent discount rates. On the other hand, the first strategy produces better (worse) ending fund values in low (high) interest rate scenarios because the cost of crediting “0” is less (more) in these scenarios. Therefore, the first strategy might easily be preferred by a risk-averse investor.

Also, if the P measure has larger probabilities for low interest scenarios relative to the Q measure which is typical given the bias toward rising yield curves, an individual who maximizes expected values under the P measure could easily prefer a crediting strategy of r_{crit} in period “1” followed by “0” in subsequent periods to a strategy where

PP credits the money market rate at all periods. The P measure overweights low-interest rate scenarios where the first strategy produces larger values than the second strategy and underweights high-interest rate scenarios where the first strategy produces smaller values than the second strategy. This increases the mean outcome of the first strategy, and therefore raises its desirability to a risk-neutral investor who values under the P measure.

These results suggest that PPs in perfect competition will credit r_{crit} in period “1” followed by “0” in subsequent periods if frictions are such that *any* of their policyholders are influenced by the expectation of the utility of the fund under the real world probability measure.

Of course, if the PP credits the full value of r_{crit} in the first period, he has a zero profit and the entire surplus goes to the consumer. He could lower his first-period crediting rate to r_{crit}^P , the rate that would make a risk averse policyholder who values under the P measure infinitesimally prefer the Guaranteed Account. In this case all the surplus is captured by the producer. In reality, some value $r_{crit}^P \leq r_c \leq r_{crit}$ would be credited depending on the bargaining power of the two agents.

4. Analysis including the Effect of Minimum Guarantees

Now assume there is a minimum credited rate r_{min} which is either set by law or contractually guaranteed. The results of Proposition 2.6 follow through unchanged. Proposition 2.9 could be restated as “If $\omega_t > 0$, PP should set $r_c = r_{min}$ ”, but the proof is

similar. The arguments used in Section 3 regarding risk-averse policyholders under the P measure are still reasonable.

It is possible now, however, for r_{\min} to exceed r_{crit} at some times in some states of the world. In this case, PH will move all funds to the Guaranteed Accounts. Since PP credits more than r_{crit} , the expected profits to PP under the Q measure are negative. This contract, therefore, has a negative expectation at issue. This would seem to imply that the PP would not issue such a contract. On the other hand, his bargaining power may allow him to lower the first period crediting rate far enough to create an expectation of a positive profit and allow the contract to be issued.

While this situation does exist in practice, it is also similar to one where withdrawals are allowed by way of a “transfer payout annuity” with a fixed term and rate. The “minimum rate” in this case is usually time-dependent and tied to the market in some fashion. If this rate is contractually tied to a reference rate, this is a way PP can pre-commit to crediting more than “0” and reduce the value of r_{crit} necessary to entice policyholders to choose the Guaranteed Account.

5. Numerical Examples

We now turn our attention to some numerical calculations of the critical rate. The behavior of PP and PH is fully deterministic and not interest sensitive when $r_{\min} = 0$, so r_{crit} is completely determined by today’s yield curve and is not dependent on an interest

rate model. It does, however, depend on the transfer restriction x . When $r_{\min} > 0$, PH strategy does depend on the state of the world and we will need a full interest rate model.

In addition, r_{crit}^P does depend on the interest rate model used because it depends on the full distribution of final outcomes which is model sensitive. r_{crit}^P also depends on the precise form of the PH utility function and the Radon-Nikodym Derivative of the Q measure relative to the P measure.

5.1 Determination of r_{crit} when $r_{\min} = 0$

The case of a level yield curve with rate r can be straightforwardly evaluated and demonstrates the method that will be used for non-level yield curves. The value of any money in the money market account at $t = 0$ is \$1. Putting \$1 into the Guaranteed Account produces $\$(1 + r_{crit})$ in one year. The Guaranteed Account then no longer grows in future years. A fraction x is transferred out every year and the present value of these transfers must equal \$1 for PH to be indifferent between the funds. That is,

$$1 = (1 + r_{crit}) \sum_{t=1}^{\infty} \frac{x(1-x)^{t-1}}{(1+r)^t} \quad (3)$$

Which solves nicely for:

$$r_{crit} = \frac{r}{x} \quad (4)$$

When $x = 1$, $r_{crit} = r$ which agrees with Proposition 2.4.

Now, in cases where the yield curve is not flat, the denominator in Equation (3) is easily adjusted by replacing $(1+r)^t$ by $(1+s_t)^t$ where s_t represents the t year spot rate at the initiation of the contract. If the one-year forward rates after the first year are level at f_1 and the one-year spot rate is s_0 , Equation (3) becomes:

$$1 = (1+r_{crit}) \sum_{t=1}^{\infty} \frac{x(1-x)^{t-1}}{(1+s_0)(1+f_1)^{t-1}} \quad (5)$$

which again solves nicely for

$$r_{crit} * x = (1-x)f_1 \left(\frac{1+s_0}{1+f_1} \right) + xs_0 \quad (6)$$

This moves linearly from $f_1 \left(\frac{1+s_0}{1+f_1} \right)$ when transfers are completely disallowed to s_0 , in agreement with Proposition 2.4, when there are no transfer restrictions. This general pattern of movement from long-term to short term rates when transfer restrictions are removed is a general feature of the model for arbitrary yield curves.

Figure 1 shows the behavior of the critical rate from 1/1990 to 5/2014 when $x = 25\%$. We also examined the correlations between r_{crit} (for $x=1\%$, 5% , 10% , 25% and 50%) and treasury rates (at 1 year, 5 year, 10 year and 30 year durations). We find large correlations in general. The largest correlations for $x=1\%$, $x=5\%$ and $x=10\%$ are with the 10 year rate, $x=25\%$ with the 5 year rate and $x=50\%$ with the 1 year rate.

All of these maximum correlations are above 0.989. These correlations are shown in Table 1.

5.2 Determination of r_{crit} when $r_{min} > 0$

We first examine the case where $r_{crit} > r_{min}$ at all times and in all future states of world (the static case). In this situation, PP should credit r_{min} in future periods and the policyholder should withdraw the maximum amount possible. In the case of a level yield curve, Equation (3) becomes:

$$1 = (1 + r_{crit}) \sum_{t=1}^{\infty} \frac{x(1-x)^{t-1}(1+r_{min})^{t-1}}{(1+r)^t} \quad (7)$$

which solves for:

$$r_{crit} = r_{min} + \frac{r - r_{min}}{x} \quad (8)$$

which again equals the short-term rate when $x = 1$, consistent with Proposition 2.4.

The equivalent of Equation (5) now solves for:

$$r_{crit} * x = (1-x)(f_1 - r_{min}) \left(\frac{1+s_0}{1+f_1} \right) + xs_0 \quad (9)$$

Which can be either positive or negative depending on whether the long-term rate is larger or smaller than the minimum crediting rate. If the credited rate at time 0 is larger than this static amount, an arbitrage opportunity analogous to the one in Proposition 2.10 exists.

In reality, this “static” case ignores a number of important options possessed by PH. For instance, PH can empty his Guaranteed Account as described above and still retain the option to transfer back to the Guaranteed Account if r_{\min} exceeds r_{crit} at some point in the future, which increases the value of Guaranteed Account funds. In addition, Money Market funds are worth more than \$1 as the PH has the option to move money to the Guaranteed Account if r_{\min} ever exceeds r_{crit} . This implies that the true, dynamic r_{crit} must equal or exceed the short-term rate, otherwise the money market would be preferable as the option value on money market funds exceeds that on Guaranteed Funds.

To see the effect of these options on r_{crit} we calibrate a Black-Derman-Toy (BDT) model with volatility 14% to the treasury curves. This volatility is consistent with values in Coleman, Fisher and Ibbotson (1991), Radhakrishnan (1998) and Damberg and Gullnäs (2012). The results are not particularly sensitive to the choice of volatility parameter.

Figure 2 shows the dynamic and static values of r_{crit} for a minimum crediting rate of 3%. The values are nearly indistinguishable except in those cases where the dynamic rate is essentially equal to the one-year treasury rate. Although not visible in the graph, the dynamic value is about 3-5 bp below the static value reflecting the option value of re-entering the Guaranteed Account. If PP credits above the static value, an arbitrage

opportunity exists. If PP credits above the dynamic value but below the static one, the BDT model suggests that PH should put funds in the Guaranteed Account. This conclusion is dependent on the accuracy of the model, however, and does not necessarily represent an arbitrage opportunity for PH.

5.3 Determination of r_{crit}^P .

As in section 5.2, we will use the calibrated Black-Derman-Toy Model. We assume CRRA policyholders. That is, we assume they have a utility function

$U(w) = \frac{w^{1-\gamma}}{1-\gamma}$. Figure 3 shows the values of r_{crit}^P for $\gamma = 0$ and $\gamma = 3$. As expected, the

values for risk-neutral individuals ($\gamma = 0$) are above the critical rates in figure 1 and the values for reasonably risk-averse individuals ($\gamma = 3$) are below both the risk-neutral and Figure 1 values.

6. Crediting in Practice

We now turn our attention to an empirical analysis of the typical crediting strategies of pension providers. Our analysis spans the period from 1990-2011 inclusive. We look at two questions. First, were there any companies and time periods where the

arbitrage relationship in Proposition 2.9 was present? Second, what aspects of a company's assets and the interest rate environment predict crediting rates?

We determine a company's credited rate from the information provided in publicly available NAIC statements. We take interest credited to be the tabular interest in the Group Annuities column of the "Analysis of Increase in Reserves and Deposit Funds During the Year". Prior to 2000 this was divided into "Reserves" and "Deposit Funds" but in 2000 and later the two amounts were combined. Because of this, it is unclear in some cases whether a particular rate is one that is credited on policyholder controlled funds. Relevant transfer restrictions are also unavailable. The crediting rate was estimated by dividing the tabular interest into the average of the beginning and ending reserves for the year. This will be the dependent variable in the later regression analysis.

Figure 4 shows the median credited rate, as well as the 90th and 10th percentiles, for those companies with positive (non-zero) group annuity reserves. It also shows some of the critical rates from Figures 1-3 as well as a short term rate. Companies typically, but not always, credit more than money market rates. For example, many policyholders between 1993-2000 as well as 2005-2007 would do well to transfer as much money as allowed into the money-markets. It also seems likely that at least some arbitrage possibilities existed between 2001-2004 and, more recently, 2011-2012.

We performed a least-squares regression on the data to find the determinants of company crediting strategies. The dependent variable was the amount credited and the independent variables were: "Assets", "NII on line", "Proportionally Allocated Company NII", "Short Term Interest Crediting", "5 Year Interest Crediting" and "10 Year Interest

Crediting”. The Assets were calculated as the average of the starting and ending reserves used in the denominator of the credited rate calculation. NII on line was obtained from the Analysis of Operations by Line of Business page of the NAIC statement for the Group Annuities column. “Proportionally Allocated Company NII” calculated what the NII would have been on the line of business had they had the same NII rate as the company as whole. The “Short Term Interest Crediting”, “5 Year Interest Crediting” and “10 Year Interest Crediting” variables were the amounts that would have been earned by the line if the assets had earned exactly the “Short Term”, “5 Year” and “10 Year” treasury rates respectively. The results of the regression are shown in Table 2. The R-squared of the regression is quite high, 0.9813.

All coefficients are quite statistically significant. The results are reasonable and can be interpreted straightforwardly. The negative intercept implies that a typical company builds in about \$3,500,000 of profit (after the effects of the various NII and interest rates) regardless of size. The average company has about \$1,100,000,000 in group annuity assets so this about 0.32% of assets for a typical company. The coefficient on “Assets” is 0.0061, implying that a typical company credits about 0.61% independent of external rates or its own investment performance. The credited rate averaged over all years and companies is about 6.23% so only about 1/10th of the interest credited is constant independent of company or economic circumstances. The typical spread profit per company per year (NII on line less Interest Credited) is about \$33,800,000 per year or about 3.08% of assets, mostly because interest rates have been declining through most of the period and Credited Rates have fallen faster than NII rates.

It appears that external rates matter more than internal investment performance, as suggested by Proposition 2.1. For example, if the company wide NII rate rises by 1% (in a way which causes the Line NII rate to also rise 1%), the Credited Rate will only rise by 0.12%. On the other hand, if the external yield curve rises by 1% (all three rates in parallel), the Credited Rate will rise by 0.86%. If all rates, internal and external, rise by 1%, the Credited Rate will rise by 0.986% so almost, but not quite all, of the extra return is passed through to policyholders.

The pattern of coefficients on the NII variables suggests that increasing the return on either line specific or company-wide non-line specific assets results in higher crediting rates although the effect of line specific assets is larger. The pattern of coefficients on Treasury Rate variables suggests that the 10-year rate is a strong determinant of Crediting Rates. In addition, the slope between the 5-year and 10-year rate is also quite important, suggesting that expectations of increasing returns in the future might produce higher Crediting Rates today (or that the Crediting Rates might depend on an even longer rate, say the 30 year, which is not always available).

7. Conclusion

In this paper, we examine the interaction between crediting strategies on guaranteed funds and transfer restrictions. We show that the optimal strategy for a pension provider is to credit a critical rate during the first year and credit the lowest

possible legal or contractually allowed rate thereafter. The policyholder's optimal strategy is to enter the guaranteed fund at initiation of the contract and then transfer the maximum possible amount into the money market until the guaranteed fund is emptied.

If the pension provider credits more than the critical rate during the first year, an arbitrage opportunity exists for the policyholder. This has likely happened during some years for some companies since 1990. The effect of the arbitrage is mitigated somewhat since it is not scalable (policyholders have a maximum amount they can deposit in tax-deferred accounts) and policyholders cannot, in practice, borrow at the money-market rate.

We also examine how Credited Rates are determined in practice for U.S. insurance companies. We find that the effect of external treasury rates is far larger than the effect of internal investment returns, consistent with theoretical expectations.

8. Acknowledgment

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Figure 1. r_{crit} vs. time for $r_{min} = 0$



Figure 2. Static and Dynamic Critical Rates for $r_{\min} = 3\%$

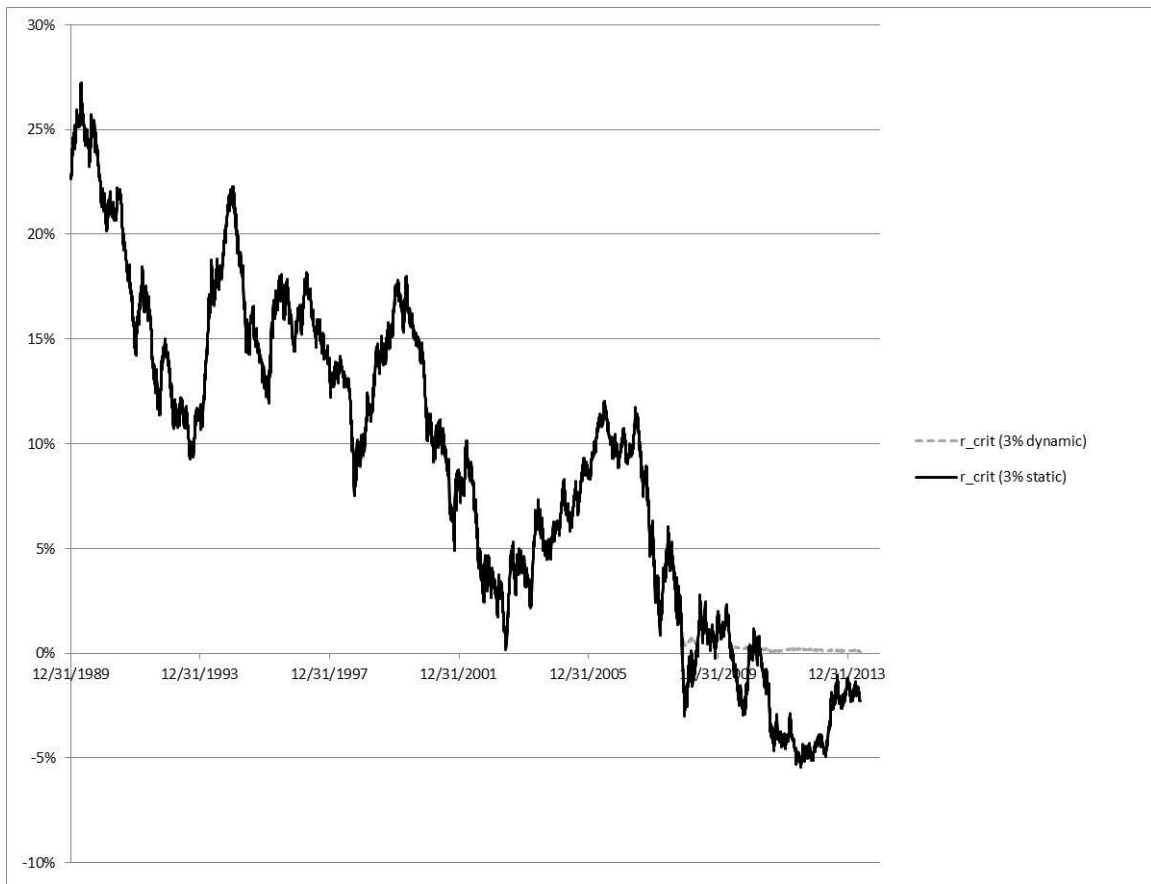


Figure 3. r_{crit}^P vs. Time for Risk-Averse Policyholders.



Figure 4. Actual and Critical Credited Rates.



Table 1. Correlation Between Critical Rates and Treasury Rates for Varying Durations and Transfer Restrictions.

Treasury Duration	Transfer Restriction				
	1%	5%	10%	25%	50%
1	0.862	0.892	0.919	0.962	0.989
5	0.971	0.984	0.994	0.999	0.985
10	0.996	0.999	0.998	0.980	0.946
30	0.996	0.987	0.975	0.937	0.885

Table 2. Regression Analysis

	<i>Coefficients</i>	<i>Standard Error</i>	<i>P-value</i>
Intercept	-\$3,547,190	\$446,845	2.38E-15
Assets	0.006	0.001	5.57E-17
NII on Line	0.076	0.003	4.7E-101
NII Proportional	0.049	0.009	1.59E-07
Short Term	0.288	0.019	1.38E-49
5 Year	-1.634	0.073	3.9E-108
10 Year	2.208	0.065	2.5E-234