Spring 5-11-2015

Perfect Matchings, Tilings and Hamilton Cycles in Hypergraphs

Jie Han

PERFECT MATCHINGS, TILINGS AND HAMILTON CYCLES IN HYPERGRAPHS

by

JIE HAN

Under the Direction of Yi Zhao, PhD

ABSTRACT

This thesis contains problems in finding spanning subgraphs in graphs, such as, perfect matchings, tilings and Hamilton cycles. First, we consider the tiling problems in graphs, which are natural generalizations of the matching problems. We give new proofs of the multipartite Hajnal-Szemerédi Theorem for the tripartite and quadripartite cases.

Second, we consider Hamilton cycles in hypergraphs. In particular, we determine the minimum codegree thresholds for Hamilton ℓ-cycles in large k-uniform hypergraphs for $\ell < k/2$. We also determine the minimum vertex degree threshold for loose Hamilton cycle in

large 3-uniform hypergraphs. These results generalize the well-known theorem of Dirac for graphs.

Third, we determine the minimum codegree threshold for near perfect matchings in large k-uniform hypergraphs, thereby confirming a conjecture of Rödl, Ruciński and Szemerédi. We also show that the decision problem on whether a k-uniform hypergraph with certain minimum codegree condition contains a perfect matching can be solved in polynomial time, which solves a problem of Karpiński, Ruciński and Szymańska completely.

At last, we determine the minimum vertex degree threshold for perfect tilings of $\mathcal{C}_4^3$ in large 3-uniform hypergraphs, where $\mathcal{C}_4^3$ is the unique 3-uniform hypergraph on four vertices with two edges.

INDEX WORDS: Absorbing method, Hypergraph, Perfect matching, Graph tiling, Graph packing, Hamilton cycle, Minimum degree condition.

PERFECT MATCHINGS, TILINGS AND HAMILTON CYCLES IN HYPERGRAPHS

by

JIE HAN

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

in the College of Arts and Sciences

Georgia State University

2015

PERFECT MATCHINGS, TILINGS AND HAMILTON CYCLES IN HYPERGRAPHS

by

JIE HAN

Committee Chair: Yi Zhao

Committee: Guantao Chen

Peter Keevash

Hein van der Holst

Electronic Version Approved:

Office of Graduate Studies

College of Arts and Sciences

Georgia State University

May 2015

DEDICATION

This dissertation is dedicated to my parents.

ACKNOWLEDGEMENTS

First I want to express my sincere gratitude to my thesis advisor, Yi Zhao. None of these works would be possible without his selfless help. He has been my advisor, mentor, collaborator and friend and helped me both mathematically and non-mathematically.

I appreciate Professors Guantao Chen, Hein van der Holst, and Peter Keevash who kindly agreed to serve on my dissertation committee.

I appreciate Professors Liming Xiong from Beijing Institute of Technology and my advisor, Yi Zhao, who brought me from China to the United States. I also thank Professors Guantao Chen and Zhongshan Li who have given me continuous help throughout my graduate study and other professors from our department for educating me in various courses.

Thanks also go to the combinatorics group in Georgia State; Nana Li, Suil O, Songling Shan, Ping Yang, Amy Yates and Chuanyun Zang. We have had many helpful discussion about math, some of which led to successful collaborations.

Also my sincere appreciation goes to the couple Hiệp Hàn and Andrea Jiménez. I have got many helpful discussions and great encouragements from them. I also thank Mihyun Kang from TU Graz – we had so many wonderful conversations throughout the month in Minneapolis! Special thanks go to the IMA (Institute for Mathematics and its Applications), who host my stay from September to December, 2014, when this thesis was written down.

At last, I want to thank everyone who has helped me throughout my graduate life, especially my dear girlfriend Wei Gao and my parents. I would not even stand here without their continuous support.

The research led to this dissertation was partially supported by NSF grant DMS 1400073.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS . . . . . **v**

LIST OF FIGURES . . . . . **ix**

PART 1 INTRODUCTION **1**

1.1 Perfect tiling in multi-partite graphs 2

1.2 Hamilton cycles in k-uniform hypergraphs 3

1.3 Perfect matchings in k-uniform hypergraphs 6

1.4 Perfect tiling in hypergraphs 8

1.5 Notations 9

PART 2 HYPERGRAPH REGULARITY METHOD AND ABSORBING METHOD **11**

2.1 Weak Hypergraph Regularity Lemma 11

2.2 Absorbing method 12

PART 3 ON MULTIPARTITE HAJNAL-SZEMERÉDI THEOREM **15**

3.1 Introduction 15

3.2 Proof of the Absorbing Lemma 20

PART 4 MINIMUM CODEGREE THRESHOLD FOR HAMILTON ℓ-CYCLES IN K-UNIFORM HYPERGRAPHS **37**

4.1 Introduction 37

4.2 Proof of Theorem 4.4 40

4.2.1 Auxiliary lemmas and Proof of Theorem 4.4 40

4.2.2 Proof of Lemma 4.8 43

4.2.3 Proof of Lemma 4.11 47

4.3 The Extremal Theorem 50
4.3.1 Classification of vertices 51
4.3.2 Classification of ℓ-sets in B' 52
4.3.3 Building a short path $\mathcal{Q}$ 56
4.3.4 Completing the Hamilton cycle 62

PART 5 MINIMUM VERTEX DEGREE THRESHOLD FOR LOOSE HAMILTON CYCLES IN 3-UNIFORM HYPERGRAPHS 70

5.1 Introduction 70
5.2 Proof of Theorem 5.6 73
5.2.1 Auxiliary lemmas and Proof of Theorem 5.6 73
5.2.2 Proof of Lemma 5.9 75
5.2.3 Proof of $\mathcal{C}$-tiling Lemma (Lemma 5.12) 79
5.3 The Extremal Theorem 87
5.3.1 Classifying vertices 88
5.3.2 Building a short path 90
5.3.3 Completing a Hamilton cycle 95

PART 6 NEAR PERFECT MATCHINGS IN K-UNIFORM HYPERGRAPHS 99

6.1 Introduction 99
6.2 Proof of Theorem 6.4 103
6.3 Proof of Theorem 6.5 105

PART 7 DECISION PROBLEM FOR PERFECT MATCHINGS IN DENSE K-UNIFORM HYPERGRAPHS 109

7.1 Introduction 109
7.1.1 Perfect matchings under minimum degree conditions 109
7.1.2 Lattice-based constructions 111

7.2 The Main structural theorem 113
7.3 The Non-extremal Case 118
7.3.1 Tools 118
7.3.2 Proof of Theorem 7.14 119
7.3.3 Proof of the Absorbing Lemma 122
7.3.4 Proof of Lemma 7.11 123
7.4 The Extremal Case 128
7.4.1 Preliminary and the proof of Theorem 7.15 129
7.4.2 Proof of Lemma 7.27 130
7.4.3 Proofs of Lemma 7.28 and Lemma 7.29 136
7.5 Algorithms and the proof of Theorem 7.3 137
7.5.1 A straightforward but slower algorithm 137
7.5.2 A faster algorithm 138

PART 8 MINIMUM VERTEX DEGREE THRESHOLD FOR $\mathcal{C}_4^3$-TILING **141**
8.1 Introduction 141
8.2 Proof of Lemma 8.6 145
8.3 Proof of Theorem 8.4 151

REFERENCES **159**

LIST OF FIGURES

Figure 1.1 A (3,1)-cycle 4

Figure 1.2 $\mathcal{C}_4^3$ 9

Figure 3.1 $\Theta_{3\times 3}$: dotted lines stand for non-edges 18

Figure 5.1 Constructions in Proposition 5.3 72

Figure 6.1 $\delta_{k-1}(H)$ and $\nu(H)$ 101

Figure 7.1 Space Barrier for $k = 3$ 110

Figure 7.2 Constructions 7.4 and 7.5 coincide when $k = 3$ 112

PART 1

INTRODUCTION

One of the most fundamental problems in every branch of mathematics is describing the structures of its objects. There are several way to understand the structure of graphs. An easy and obvious way is to find the substructures, the subgraphs. Given an n-vertex graph H and a g-vertex graph G, a natural question to ask is: when does a graph H contain a fixed graph G? The celebrated Mantel's Theorem [53] says that if H has more than $n^2/4$ edges, then H contains a K_3. This result has been generalized by Turán [70] for $G = K_g$. In general, the Turán number for a graph G is the maximum number of edges in an n-vertex graph H such that H does not contain a copy of G. Many work has been done on Turán type problems for graphs and hypergraphs, see surveys [14, 29]. However, for example, we still do not know the Turán number for complete bipartite graphs.

Given a graph H, it is also natural to consider the spanning subgraphs F in H. For instance, for the case that F is a perfect matching, the celebrated theorem from Tutte [71] characterizes the graphs G that contain a perfect matching and Edmonds's algorithm [11] finds a perfect matching in polynomial time if one exists. However, for most graphs F, we do not know such a nice characterization and the decision problem whether a graph H has a spanning subgraph F is NP-complete. Thus, it is natural to study the sufficient conditions that force the existence of such spanning subgraphs.

A hypergraph is a natural generalization of a graph. Finding certain spanning subhypergraphs in a hypergraph H is also a natural and desirable problem in hypergraph theory. However, we know much less on hypergraphs than the graph case, and even finding a perfect matching in a k-uniform hypergraph for $k \geq 3$ is NP-complete by Karp [26]. It is natural to find sufficient conditions to guarantee such spanning subhypergraphs. In this thesis, we will discuss spanning subhypergraphs such as perfect matchings, tilings and Hamilton cycles in

graphs and uniform hypergraphs.

1.1 Perfect tiling in multi-partite graphs

Graph/hypergraph *packing* (alternatively called *tiling*) is a natural extension of matching problems and has received much attention in the last two decades (see [44] for a survey). Given two (hyper)graphs G and H, a *perfect G-tiling*, or a *G-factor*, of H is a spanning subgraph of H that consists of vertex-disjoint copies of G. Define $t_d(n, G)$ to be the smallest integer t such that every k-graph H of order $n \in g\mathbb{N}$ with $\delta_d(H) \geq t$ contains a G-factor.

The celebrated theorem of Hajnal and Szemerédi [16] says that every n-vertex graph G with $\delta(G) \geq (1 - 1/k)n$ contains a K_k-factor, namely, $t_1(n, K_k) = (1 - 1/k)n$. (The case when $k = 3$ was proved earlier by Corrádi and Hajnal [5].) Komlós, Sárközy and Szemerédi [39] generalize this result to arbitrary graphs H by showing that there exists a constant $C = C(H)$ such that every graph G with $\delta(G) \geq (1 - 1/\chi(H))n + C$ contains an H-factor, where $\chi(H)$ is the chromatic number of H. This improves an earlier result of Alon and Yuster [3]. As observed in [3], there are graphs H for which the above constant C cannot be omitted completely. Kühn and Osthus [45] determined up to an additive constant the minimum degree threshold that forces an H-factor for arbitrary graph H. The threshold can be written as $(1-1/\chi^*(H))n+C$, where the value of $\chi^*(H)$ depends on the relative sizes of the color classes in the optimal colorings of H and satisfies $\chi(H) - 1 < \chi^*(H) \leq \chi(H)$.

Here we consider the analogues of the tiling results above in multipartite graphs. Let $\mathcal{G}_{n,k}$ denote the family of k-partite graphs with n vertices in each part. Given $G \in \mathcal{G}_{n,k}$, let $\delta^*(G)$ denote the minimum degree from a vertex in one part to any other part. Fischer [13] conjectured a multipartite Hajnal-Szemerédi theorem, which says that every $G \in \mathcal{G}_{n,k}$ with $\delta^*(G) \geq \frac{k-1}{k}n$ contains a K_k-factor. Magyar and Martin [52] noticed that Fischer's conjecture is false for odd k and proved that $\delta^*(G) \geq \frac{2}{3}n+1$ suffices for $k = 3$. Martin and Szemerédi [55] proved Fischer's conjecture for $k = 4$. Both of these proofs used the Szemerédi's Regularity Lemma. With my advisor, Yi Zhao, in Chapter 3, we give new proofs of these results in [52, 55] by the absorbing method (without the Regularity Lemma), thus extending the

results to more values of n. In fact, the absorbing lemma in [24] works for all $k \geq 3$. Lo and Markström [51] recently proved Fischer's conjecture asymptotically by using the absorbing method (but their absorbing lemma is weaker than ours). Very recently, Keevash and Mycroft [33] proved the (modified) Fischer's conjecture completely by using the hypergraph regularity method.

1.2 Hamilton cycles in k-uniform hypergraphs

A *Hamilton* (also called *Hamiltonian*) cycle in a graph is a cycle that covers all vertices of the graph. Hamilton cycles have been studied since 1857, when William Hamilton found a Hamilton cycle in the graph of dodecahedron. It is well-known that finding a Hamilton cycle in graphs is an NP-complete problem. Thus, finding sufficient conditions to guarantee the existence of such cycles is a desirable work. In fact, it has been one of the central problems in graph theory and received much attention for over one hundred years. The following celebrated theorem was proved by Dirac in 1952. It is easy to see that its degree condition is best possible.

Theorem 1.1. *[10] Every graph G on n vertices with $n \geq 3$ and minimum degree $\delta(G) \geq n/2$ contains a Hamilton cycle.*

Given $k \geq 2$, a k-uniform hypergraph (in short, *k-graph*) consists of a vertex set $V(H)$ and an edge set $E(H) \subseteq \binom{V(H)}{k}$, where every edge is a k-element subset of $V(H)$. A *matching* in H is a collection of vertex-disjoint edges of H. A *perfect matching* M in H is a matching that covers all vertices of H. Clearly a perfect matching in H exists only if k divides $|V(H)|$. When k does not divide $n = |V(H)|$, we call a matching M in H a *near perfect matching* if $|M| = \lfloor n/k \rfloor$. Given a k-graph H with a set S of d vertices (where $1 \leq d \leq k-1$) we define $\deg_H(S)$ to be the number of edges containing S (the subscript H is omitted if it is clear from the context). The *minimum d-degree* $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d-vertex sets S in H. We refer to $\delta_{k-1}(H)$ as the *minimum codegree* of H.

In recent years, researchers have worked on extending this classical result to k-uniform

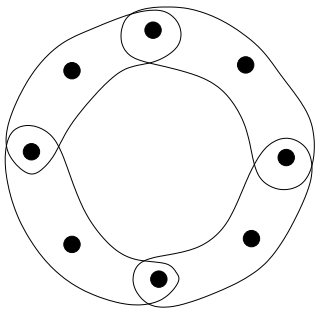

Figure 1.1. A (3,1)-cycle

hypergraphs. There are several notions of cycles in hypergraphs. Besides the so-called Berge cycle, another notion of cycle has become more and more popular in recent years.

Definition 1.2. *For $0 \leq \ell \leq k-1$, a (k,ℓ)-*cycle *is a k-graph whose vertices can be ordered cyclically in such a way that the edges are sets of consecutive k-vertices and every two consecutive edges share exactly ℓ vertices. A Hamilton ℓ-cycle in a k-graph H is then defined as a (k,ℓ)-cycle in H that contains all vertices of H.*

Since a (k,ℓ)-cycle on n vertices contains $n/(k-\ell)$ edges, it is necessary that $k-\ell$ divides n. Note that for $\ell = 0$ the ℓ-cycle reduces to a matching. We call the $(k,k-1)$-cycle a *tight cycle*. We refer a (k,ℓ)-cycle as a *loose cycle* when $\ell < k-1$. Suppose that $0 \leq d \leq k-1$ and $k-\ell$ divides n. Define $h_d^\ell(k,n)$ to be the smallest integer h such that every n-vertex k-graph H satisfying $\delta_d(H) \geq h$ contains a Hamilton ℓ-cycle.

The first result on this trend was obtained by Katona and Kierstead who proved in [28] that

$$\left\lfloor \frac{n-k+3}{2} \right\rfloor \leq h_{k-1}^{k-1}(k,n) \leq \left(1 - \frac{1}{2k}\right) n + O(1)$$

and conjectured the lower bound also suffices. Rödl, Ruciński and Szemerédi [60, 62] confirmed their conjecture asymptotically.

Theorem 1.3 ([62])**.** *Let $k \geq 3$, $\gamma > 0$, and let H be a k-graph on n vertices, where n is sufficiently large. If $\delta_{k-1}(H) \geq (1/2+\gamma)n$, then H contains a tight Hamilton cycle. In other words, $h_{k-1}^{k-1}(k,n) \approx \frac{n}{2}$.*

With long and involved arguments, the same authors [64] improved this to an exact result for $k = 3$, namely, $h_2^2(3, n) = \lfloor n/2 \rfloor$ for large n.

Other Hamilton cycles were first studied by Kühn and Osthus [41], who proved that $h_2^1(3, n) \approx n/4$ for large (even) n. By a series of work [18, 32, 40], $h_{k-1}^{\ell}(k, n)$ was determined asymptotically for all ℓ such that $\frac{k}{k-\ell} \notin \mathbb{Z}$. Note that for the divisible cases, the asymptotic values of $h_{k-1}^{\ell}(k, n)$ are answered by Theorem 1.3. In fact, note that the (k, ℓ)-cycle contains a perfect matching. So it is clear that $h_d^{\ell}(k, n) \geq m_d(k, n)$ for k, ℓ such that $\frac{k}{k-\ell} \in \mathbb{Z}$. The following conjecture was proposed in [58] for the case $\ell = k - 1$, but we think it is also true for other value of ℓ satisfying the divisibility condition.

Conjecture 1.4. *For all $1 \leq d, \ell \leq k - 1$, such that $\frac{k}{k-\ell} \in \mathbb{Z}$, $h_d^{\ell}(k, n) \approx m_d(k, n)$.*

Theorem 1.3 verifies this conjecture for $d = \ell = k - 1$.

Theorem 1.5 ([40])**.** *For $0 < \ell < k$ such that $\frac{k}{k-\ell} \notin \mathbb{Z}$,*

$$h_{k-1}^{\ell}(k, n) \approx \frac{n}{\lceil \frac{k}{k-\ell} \rceil (k - \ell)}.$$

The following construction of k-graphs $H = (V, E)$ shows that the degree condition in Theorem 1.5 is asymptotically best possible. Let V be partitioned into $A \cup B$ of vertices in V such that $|A| = \frac{n}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} - 1$. Let the edge set E be all the k-sets intersecting at least one vertex in A. Suppose H contains a Hamilton (k, ℓ)-cycle C. There are $\frac{n}{k-\ell}$ edges in C and every vertex in A is contained in at most $\lceil \frac{k}{k-\ell} \rceil$ edges in C. Thus, there are at least $\lceil \frac{k}{k-\ell} \rceil$ edges of C whose vertices are completely from B. But due to the construction, B is independent, so H contains no Hamilton (k, ℓ)-cycle.

Recently, Czygrinow and Molla [9] determined $h_2^1(3, n)$ exactly. Independent from their work, in Chapter 4, we show that $h_{k-1}^{\ell}(k, n) = \lceil \frac{n}{2(k-\ell)} \rceil$ for $\ell < k/2$, thus improving the results in [18, 32, 41].

As the first asymptotic result in the case when $d < k - 1$, Buß, Hàn, and Schacht [4] showed that $h_1^1(3, n) \approx \frac{7}{16}\binom{n}{2}$. In Chapter 5, we improve this to an exact result for large n

(Theorem 5.2). Theorem 5.2 is the first exact result on vertex degree conditions for Hamilton cycles in hypergraphs.

It seems that the lower bound constructions designed for $(k-1)$-degree cases still give the best lower bounds. Thus, we propose the following conjecture.

Conjecture 1.6. *For $1 \le d, \ell \le k$ such that $\frac{k}{k-\ell} \notin \mathbb{Z}$, let $a = \lceil \frac{k}{k-\ell} \rceil$, then*

$$h_d^\ell(k,n) \approx \binom{n-d}{k-d} - \binom{(1-\frac{1}{a(k-\ell)})n}{k-d}$$

for large $n \in (k-\ell)\mathbb{N}$.

Theorem 5.2 verifies the conjecture for $(k,d,\ell) = (3,1,1)$.

1.3 Perfect matchings in k-uniform hypergraphs

Over the last few years there has been a strong focus in establishing minimum d-degree thresholds that force a perfect matching in a k-graph [1, 8, 17, 35, 36, 42, 47, 54, 57, 58, 61, 63, 68, 69]. Let $m_d(k,n)$ be the smallest integer m such that any k-graph H with $\delta_d(H) \ge m$ contains a perfect matching. The story started from [60], in which Rödl, Ruciński and Szemerédi proved an asymptotically best possible minimum $(k-1)$-degree condition of a k-graph H that forces the existence of a tight Hamilton cycle in H. Since such a Hamilton cycle contains a perfect matching as a subhypergraph, as a corollary, this gives that $m_{k-1}(k,n) \le n/2 + o(n)$. The upper bound of $m_{k-1}(k,n)$ has been sharpened in a series of papers by different authors [42, 61, 59]. Finally, the exact value of $m_{k-1}(k,n)$ was determined by Rödl, Ruciński and Szemerédi [63], which is $\frac{n}{2} - k + C$, where $C \in \{3/2, 2, 5/2, 3\}$ depends on the values of n and k.

For other values of d, Pikhurko [57] proved that for $d \ge k/2$, $m_d(k,n) \approx \frac{1}{2}\binom{n-d}{k-d}$, which is asymptotically best possible. Treglown and Zhao [68, 69] determined the exact values of $m_d(k,n)$ when $d \ge k/2$ (independently Czygrinow and Kamat [8] determined the exact value of $m_2(4,n)$). Kühn, Osthus and Treglown [47], and independently Khan [36] determined the exact value of $m_1(3,n)$. Khan [35] also determined $m_1(4,n)$ exactly.

All known results and constructions (see Chapter 7) support the following conjecture.

Conjecture 1.7. *For $k \geq 3$ and $1 \leq d \leq k-1$,*

$$m_d(k,n) \approx \max\left\{\frac{1}{2},\ 1-\left(\frac{k-1}{k}\right)^{k-d}\right\}\binom{n-d}{k-d}.$$

Note that the case when $d \geq k/2$ has been verified in [57]. Alon, Frankl, Huang, Rödl, Ruciński and Sudakov [1] verified Conjecture 1.7 for the case $k-4 \leq d \leq k-1$, which gives (new) asymptotic values of $m_1(5,n), m_2(5,n), m_2(6,n)$ and $m_3(7,n)$.

For the general bound on $m_d(k,n)$ for $1 \leq d < k/2$, Hàn, Person and Schacht [17] showed that

$$m_d(k,n) \leq \left(\frac{k-d}{k}+o(1)\right)\binom{n-d}{k-d}.$$

Markström and Ruciński lowered the bound slightly as

$$m_d(k,n) \leq \left(\frac{k-d}{k}-\frac{1}{k^{k-d}}+o(1)\right)\binom{n-d}{k-d}.$$

Very recently, Kühn, Osthus and Townsend [46] further improved the upper bound to

$$m_d(k,n) \leq \left(\frac{k-d}{k}-\frac{k-d-1}{k^{k-d}}+o(1)\right)\binom{n-d}{k-d}.$$

by using fractional matchings. This is the current state of art by our knowledge, which is still away from the bound in Conjecture 1.7.

In contrast, the author of [63] also proved that the minimum codegree threshold that ensures a near perfect matching in a k-graph on $n \notin k\mathbb{N}$ vertices is between $\lfloor \frac{n}{k} \rfloor$ and $\frac{n}{k} + O(\log n)$. This is a quite surprising phenomenon from the Dirac threshold perspective, that a near perfect matching (almost perfect matching) appears much sooner than a perfect one. It is conjectured, in [63] and [58, Problem 3.3], that this threshold should be $\lfloor \frac{n}{k} \rfloor$. In Chapter 6, we prove this conjecture for large n.

It is also natural to ask for the relation between the minimum codegree and the *matching*

number of k-graphs, which is, the size of a maximum matching. Let $\nu(H)$ be the size of a maximum matching in H. The authors of [63] showed that for every k-uniform hypergraph H on n vertices, $\nu(H) \geq \delta_{k-1}(H)$ if $\delta_{k-1}(H) \leq \left\lfloor \frac{n}{k} \right\rfloor - k + 2$. Note that for $n \in k\mathbb{N}$ and $\frac{n}{k} \leq \delta_{k-1}(H) \leq \frac{n}{2} - k$, H may not contain a perfect matching, namely, a matching of size $\frac{n}{k}$ (see [63]). So the only open cases are when $\left\lfloor \frac{n}{k} \right\rfloor - k + 3 \leq \delta_{k-1}(H) < \frac{n}{k}$. In Chapter 6, we close this gap for large n.

For $k \geq 3$, by the result of Karp [26], it is NP-complete to determine whether a k-graph has a perfect matching. The result in [63] says that every k-graph with $\delta_{k-1}(H) \geq n/2$ has a perfect matching. A k-graph with $\delta_{k-1}(H) \in (0, n/2)$ may not contain a perfect matching. So it is natural to know if there is an efficient algorithm determining if such k-graphs have perfect matchings. For any $\gamma > 0$, Szymańska [67] proved that for the class of k-graphs with $\delta_{k-1}(H) < n/k - \gamma n$ the problem is NP-complete by reducing the problem to the perfect matching problem without degree restriction. Answering a question of Karpiński, Ruciński and Szymańska, we show that the decision problem is in P when $\delta_{k-1}(H) \geq n/k$ in Chapter 7. Previously Keevash, Knox and Mycroft [31] gave an asymptotic answer to this problem, that is, for any $\gamma > 0$, the decision problem is in P when $\delta_{k-1}(H) \geq n/k + \gamma n$. Moreover, they also constructed a polynomial-time algorithm to find a perfect matching provided one exists.

1.4 Perfect tiling in hypergraphs

For hypergraphs, only a few tiling results are known. Let K_4^3 be the complete 3-graph on four vertices, and let $K_4^3 - e$ be the (unique) 3-graph on four vertices with three edges. Recently, Lo and Markström [49] proved that $t_2(n, K_4^3) = (1+o(1))3n/4$, and independently Keevash and Mycroft [34] determined the exact value of $t_2(n, K_4^3)$ for sufficiently large n. In [50], Lo and Markström proved that $t_2(n, K_4^3 - e) = (1 + o(1))n/2$.

Let $\mathcal{C}_4^3$ be the unique 3-graph on four vertices with two edges (also denoted by $D, \mathcal{Y}$ or cherry in different papers). The perfect $\mathcal{C}_4^3$-tiling was first studied by Kühn and Osthus [41] who showed that $t_2(n, \mathcal{C}_4^3) = (1 + o(1))n/4$, and Czygrinow, DeBiasio and Nagle [7] recently

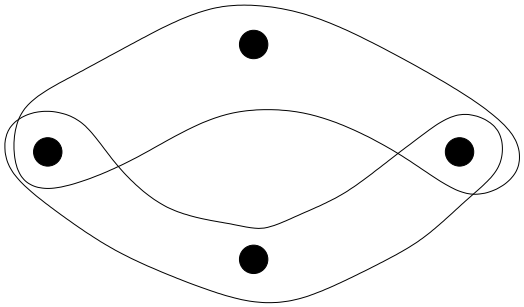

Figure 1.2. $\mathcal{C}_4^3$

determined $t_2(n, \mathcal{C}_4^3)$ exactly for large n. In Chapter 8, we determine the exact value of $t_1(n, \mathcal{C}_4^3)$ for large n. Our result is one of the first (exact) results on the vertex degree for hypergraph packing problems. Independently, Czygrinow [6] proved a similar result.

For general k-graphs, Mycroft [56] determined $t_{k-1}(n, K)$ asymptotically for many k-partite k-graphs using hypergraph blow-up lemma (instead of absorbing method).

The content of Chapters 3, 4, 5 and 8 is based on the joint work [24, 22, 23, 25] with my advisor, Yi Zhao. The content of Chapters 6 and 7 contains the work in [20, 19].

1.5 Notations

Given an integer $k \geq 0$, a k-set is a set with k elements. For a set X, we denote by $\binom{X}{k}$ the family of all k-subsets of X. We write $[r]$ to denote the set of integers from 1 to r. For two sets X and Y, we write $A \dot{\cup} B$ for $A \cup B$ when sets A, B are disjoint.

Given a k-graph H with a set S of at most $k-1$ vertices, the *link (hyper)graph* of S is the $(k - |S|)$-graph with vertex set $V(H) \setminus S$ and edge set $\{e \setminus S : e \in E(H), S \subseteq e\}$.

Given a k-graph $H = (V, E)$ and a set $\mathcal{E}$ of $(k-1)$-sets in $\binom{V}{k-1}$ (which can be viewed as a $(k-1)$-graph), let $\deg_H(v, \mathcal{E}) = |N_H(v) \cap \mathcal{E}|$. When $\mathcal{E} = \binom{X}{k-1}$ for some $X \subseteq V$, we write $\deg_H(v, \binom{X}{k-1})$ as $\deg_H(v, X)$ for short. Let $\overline{\deg}_H(v, \mathcal{E}) = |\mathcal{E} \cap \binom{V \setminus \{v\}}{k-1}| - \deg_H(v, \mathcal{E})$. When

$k = 3$, given not necessarily disjoint subsets X, Y, Z of V, define

$$E_H(XYZ) = \{xyz \in E(H) : x \in X, y \in Y, z \in Z\}$$
$$\overline{E}_H(XYZ) = \left\{xyz \in \binom{V}{3} \setminus E(H) : x \in X, y \in Y, z \in Z\right\},$$

and $e_H(XYZ) = |E_H(XYZ)|$, $\overline{e}_H(XYZ) = |\overline{E}_H(XYZ)|$. We often omit the subscript H if it is clear from the context.

We use bold font for vectors and normal fonts for their coordinates, *e.g.*, $\mathbf{v} = (v_1, v_2, \ldots, v_d)$. We write $x \ll y$ means that for any $y \geq 0$ there exists $x_0 \geq 0$ such that for any $x \leq x_0$ the following statement holds. Similar statements with more constants are defined similarly. Throughout this thesis we omit floor and ceiling symbols where they do not affect the argument.

PART 2

HYPERGRAPH REGULARITY METHOD AND ABSORBING METHOD

2.1 Weak Hypergraph Regularity Lemma

Szemerédi's Regularity Lemma [66] has been proved to be an incredibly powerful and useful tool in graph theory as well as in Ramsey theory, combinatorial number theory and other areas of mathematics and theoretical computer science. The lemma essentially says that large dense graphs can be approximated by a random-like graph. The lemma has many powerful variations and in particular, in this thesis, we use the so-called Weak Hypergraph Regularity Lemma, which is a straightforward extension of Szemerédi's regularity lemma for graphs.

Let $H = (V, E)$ be a k-graph and let $A_1, \ldots, A_k$ be mutually disjoint non-empty subsets of V. We define $e(A_1, \ldots, A_k)$ to be the number of edges with one vertex in each A_i, $i \in [k]$, and the density of H with respect to $(A_1, \ldots, A_k)$ as

$$d(A_1, \ldots, A_k) = \frac{e(A_1, \ldots, A_k)}{|A_1| \cdots |A_k|}.$$

We say a k-tuple $(V_1, \ldots, V_k)$ of mutually disjoint subsets $V_1, \ldots, V_k \subseteq V$ is (ϵ, d)-*regular*, for $\epsilon > 0$ and $d \geq 0$, if

$$|d(A_1, \ldots, A_k) - d| \leq \epsilon$$

for all k-tuples of subsets $A_i \subseteq V_i$, $i \in [k]$, satisfying $|A_i| \geq \epsilon |V_i|$. We say $(V_1, \ldots, V_k)$ is ϵ-*regular* if it is (ϵ, d)-regular for some $d \geq 0$. It is immediate from the definition that in an (ϵ, d)-regular k-tuple $(V_1, \ldots, V_k)$, if $V_i' \subset V_i$ has size $|V_i'| \geq c|V_i|$ for some $c \geq \epsilon$, then $(V_1', \ldots, V_k')$ is $(\max\{\epsilon/c, 2\epsilon\}, d)$-regular.

Theorem 2.1 (Weak Regularity Lemma)**.** *Given $t_0 \geq 0$ and $\epsilon > 0$, there exist $T_0 = T_0(t_0, \epsilon)$ and $n_0 = n_0(t_0, \epsilon)$ so that for every k-graph $H = (V, E)$ on $n > n_0$ vertices, there exists a*

partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ *such that*

(i) $t_0 \le t \le T_0$,

(ii) $|V_1| = |V_2| = \cdots = |V_t|$ *and* $|V_0| \le \epsilon n$,

(iii) for all but at most $\epsilon\binom{t}{k}$ k*-subsets* $\{i_1, \ldots, i_k\} \subset [t]$, *the* k*-tuple* $(V_{i_1}, \ldots, V_{i_k})$ *is* ϵ*-regular.*

The partition given in Theorem 2.1 is called an ϵ-regular partition of H. Given an ϵ-regular partition of H and $d \ge 0$, we refer to $V_i, i \in [t]$ as *clusters* and define the *cluster hypergraph* $\mathcal{K} = \mathcal{K}(\epsilon, d)$ with vertex set $[t]$ and $\{i_1, \ldots, i_k\} \subset [t]$ is an edge if and only if $(V_{i_1}, \ldots, V_{i_k})$ is ϵ-regular and $d(V_{i_1}, \ldots, V_{i_k}) \ge d$.

The following corollary shows that the cluster hypergraph inherits the minimum degree of the original hypergraph. Its proof is almost the same as in [18, Proposition 16] after we replace $\frac{1}{2(k-\ell)} + \gamma$ by c – we thus omit the proof.

Corollary 2.2. *[18] Given* $c, \epsilon, d > 0$ *and integers* $k \ge 3, t_0$ *such that* $0 < \epsilon < d^2/4$ *and* $t_0 \ge 2k/d$, *there exist* T_0 *and* n_0 *such that the following holds. Let* H *be a* k*-graph on* $n > n_0$ *vertices such that* $\delta_{k-1}(H) \ge cn$. *If* H *has an* ϵ*-regular partition* $V_0 \cup V_1 \cup \cdots \cup V_t$ *with* $t_0 \le t \le T_0$ *and* $\mathcal{K} = \mathcal{K}(\epsilon, d)$ *is the cluster hypergraph, then at most* $\sqrt{\epsilon}t^{k-1}$ $(k-1)$*-subsets* S *of* $[t]$ *violate* $\deg_{\mathcal{K}}(S) \ge (c - 2d)t$.

We will use the Weak Hypergraph Regularity Lemma in Chapters 4 and 5.

2.2 Absorbing method

The absorbing method, initiated by Rödl, Ruciński, and Szemerédi [60], has been shown to be effective handling extremal problems in graphs and hypergraphs. Roughly speaking, the absorbing method reduces the task of finding a spanning structure to finding an almost spanning structure. One example is the re-proof of Posa's conjecture by Levitt, Sárközy, and Szemerédi [48], while the original proof of Komlós, Sárközy, and Szemerédi [37] used the Regularity Lemma.

One crucial part of the absorbing method is the probabilistic arguments. We include the well-known Chernoff's bound and Markov's bound [2] here.

Proposition 2.3 (Chernoff's bound)**.** *Let $0 < p < 1$ and let $X_1, \dots, X_n$ be mutually independent indicator random variables with $\mathbb{P}[X_i = 1] = p$ for all i, and let $X = \sum X_i$. Then for all $a > 0$,*

$$\mathbb{P}[|X - \mathbb{E}[X]| > a] \leq 2e^{-a^2/2n}.$$

Proposition 2.4 (Markov's bound)**.** *If X is any nonnegative random variable and $a > 0$, then*

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$

To illustrate the absorbing method, we state and prove a typical component of the absorbing method, which will be used in Chapter 8. Recall that $\mathcal{C}_4^3$ is the unique 3-graph on four vertices with two edges. For positive integer $b \in 4\mathbb{N}$, we say that a b-set F *absorbs a 4-set* S if both $H[F]$ and $H[F \cup S]$ contain $\mathcal{C}_4^3$-factors.

Lemma 2.5. *Assume $1/n \ll \beta < 1/4$ and $b \in 4\mathbb{N}$. Suppose H is an n-vertex 3-graph such that every 4-vertex set S has at least βn^b b-sets F that absorb S. Then there is a vertex set $W' \in V(H)$ with $|W'| \in 4\mathbb{N}$ and $|W'| \leq b\beta^2 n$ such that for any vertex subset U with $U \cap W' = \emptyset$, $|U| \in 4\mathbb{N}$ and $|U| \leq \beta^4 n$ both $H[W']$ and $H[W' \cup U]$ contain $\mathcal{C}_4^3$-factors.*

Proof of Lemma 2.5. We choose a family $\mathcal{F} \subset \binom{V}{b}$ of b-sets by selecting each b-set randomly and independently with probability $p = \beta^2 n^{1-b}$. Then $|\mathcal{F}|$ follows the binomial distribution $B(\binom{n}{b}, p)$ with expectation $\mathbb{E}(|\mathcal{F}|) = p\binom{n}{b}$. Furthermore, for every 4-set S, let $f(S)$ denote the number of members of $\mathcal{F}$ that absorb S. Then $f(S)$ follows the binomial distribution $B(N, p)$ with $N \geq \beta n^b$ by the assumption of the lemma. Hence $\mathbb{E}(f(S)) \geq p\beta n^b$. Finally, since there are at most $\binom{n}{b} \cdot b \cdot \binom{n}{b-1} < \frac{1}{2} n^{2b-1}$ pairs of intersecting b-sets, the expected number of the intersecting pairs of b-sets in $\mathcal{F}$ is at most $p^2 \cdot \frac{1}{2} n^{2b-1} = \beta^4 n/2$.

Applying Chernoff's bound on the first two properties and Markov's bound on the last one, we know that, with positive probability, $\mathcal{F}$ satisfies the following properties:

- $|\mathcal{F}| \leq 2p\binom{n}{b} < \beta^2 n$,
- for any 4-set S, $f(S) \geq \frac{p}{2} \cdot \beta n^b = \beta^3 n/2$,
- the number of intersecting pairs of elements in $\mathcal{F}$ is at most $\beta^4 n$.

Thus, by deleting one member from each intersecting pair and the non-absorbing members from $\mathcal{F}$, we obtain a family $\mathcal{F}'$ consisting of at most $\beta^2 n$ b-sets and for each 4-set S, at least $\beta^3 n/2 - \beta^4 n > \beta^4 n$ members in $\mathcal{F}'$ absorb S. So we get the desired absorbing set $W' = V(\mathcal{F}')$ satisfying $|W'| \leq b\beta^2 n$. □

We finish this section by giving a quick introductory on the newly-developed lattice-based absorbing method. When the (hyper)graph is dense enough, the absorbing method provides a powerful, global (small) absorbing structure that can absorb any (smaller) set of leftover vertices. This reduces the job of finding a spanning structure into the one of finding an almost spanning structure. Interestingly, when the minimum degree condition falls below the critical threshold for which the absorbing structure exists, a partite structure appears in the (hyper)graph (see [34, 31]). Instead, we may partition the vertex set of the graph into a few parts and build the lattice-based absorbing structure on the partition. Our lattice-based absorbing structure works under the subcritical degree conditions and gives enough structural information in some applications (see Chapter 7).

PART 3

ON MULTIPARTITE HAJNAL-SZEMERÉDI THEOREM

3.1 Introduction

Let H be a graph on h vertices, and let G be a graph on n vertices. *Packing* (or *tiling*) problems in extremal graph theory are investigations of conditions under which G must contain many vertex disjoint copies of H (as subgraphs), where minimum degree conditions are studied the most. An H-matching of G is a subgraph of G which consists of vertex-disjoint copies of H. A *perfect H-matching*, or *H-factor*, of G is an H-matching consisting of $\lfloor n/h \rfloor$ copies of H. Let K_k denote the complete graph on k vertices. The celebrated theorem of Hajnal and Szemerédi [16] says that every n-vertex graph G with $\delta(G) \geq (k-1)n/k$ contains a K_k-factor (see [28] for another proof).

Using the Regularity Lemma of Szemerédi [66], researchers have generalized this theorem for packing arbitrary H [3, 39, 65, 45]. Results and methods for packing problems can be found in the survey of Kühn and Osthus [44].

In this chapter we consider multipartite packing, which restricts G to be a k-partite graph for $k \geq 2$. A k-partite graph is called *balanced* if its partition sets have the same size. Given a k-partite graph G, it is natural to consider the minimum partite degree $\delta^*(G)$, the minimum degree from a vertex in one partition set to any other partition set. When $k = 2$, $\delta^*(G)$ is simply $\delta(G)$. In most of the rest of this chapter, the minimum degree condition stands for the minimum partite degree for short.

Let $\mathcal{G}_k(n)$ denote the family of balanced k-partite graphs with n vertices in each of its partition sets. It is easy to see (e.g. using the König-Hall Theorem) that every bipartite graph $G \in \mathcal{G}_2(n)$ with $\delta^*(G) \geq n/2$ contains a 1-factor. Fischer [13] conjectured that if $G \in \mathcal{G}_k(n)$ satisfies

$$\delta^*(G) \geq \frac{k-1}{k}n, \tag{3.1}$$

then G contains a K_k-factor and proved the existence of an *almost* K_k-factor for $k = 3, 4$. Magyar and Martin [52] noticed that the condition (3.1) is not sufficient for odd k and instead proved the following theorem for $k = 3$. (They actually showed that when n is divisible by 3, there is only one graph in $\mathcal{G}_3(n)$, denoted by $\Gamma_3(n/3)$, that satisfies (3.1) but fails to contain a K_3-factor, and adding any new edge to $\Gamma_3(n/3)$ results in a K_3-factor.)

Theorem 3.1. *[52] There exists an integer n_0 such that If $n \geq n_0$ and $G \in \mathcal{G}_3(n)$ satisfies $\delta^*(G) \geq 2n/3 + 1$, then G contains a K_3-factor.*

On the other hand, Martin and Szemerédi [55] proved the original conjecture holds for $k = 4$.

Theorem 3.2. *[55] There exists an integer n_0 such that if $n \geq n_0$ and $G \in \mathcal{G}_4(n)$ satisfies $\delta^*(G) \geq 3n/4$, then G contains a K_4-factor.*

Recently Keevash and Mycroft [34] and independently Lo and Markström [51] proved that Fischer's conjecture is asymptotically true, namely, $\delta^*(G) \geq \frac{k-1}{k}n + o(n)$ guarantees a K_k-factor for all $k \geq 3$. Very recently, Keevash and Mycroft [33] improved this to an exact result.

In this chapter we give a new proof of Theorems 3.1 and 3.2 by the absorbing method. Our approach is similar to that of [51] (in contrast, a geometric approach was employed in [34]). However, in order to prove exact results by the absorbing lemma, one needs only assume $\delta^*(G) \geq (1 - 1/k)n$, instead of $\delta^*(G) \geq (1 - 1/k + \alpha)n$ for some $\alpha > 0$ as in [51]. In fact, our absorbing lemma uses an even weaker assumption $\delta^*(G) \geq (1 - 1/k - \alpha)n$ and has a more complicated absorbing structure.

The absorbing method, initiated by Rödl, Ruciński, and Szemerédi [60], has been shown to be effective handling extremal problems in graphs and hypergraphs. One example is the re-proof of Posa's conjecture by Levitt, Sárközy, and Szemerédi [48], while the original proof of Komlós, Sárközy, and Szemerédi [37] used the Regularity Lemma. Our proof is another example of replacing the regularity method with the absorbing method. Compared with the

threshold n_0 in Theorems 3.1 and 3.2 derived from the Regularity Lemma, the value of our n_0 is much smaller.

Before presenting our proof, let us first recall the approach used in [52, 55]. Given a k-partite graph $G \in \mathcal{G}_k(n)$ with parts $V_1, \ldots, V_k$, the authors said that G is Δ-extremal if each V_i contains a subset A_i of size $\lfloor n/k \rfloor$ such that the density $d(A_i, A_j) \leq \Delta$ for all $i \neq j$. Using standard but involved graph theoretic arguments, they solved the extremal case for $k = 3, 4$ [52, Theorem 3.1], [55, Theorem 2.1].

Theorem 3.3. *Let $k = 3, 4$. There exists Δ and n_0 such that the following holds. Let $n \geq n_0$ and $G \in \mathcal{G}_k(n)$ be a k-partite graph satisfying $\delta^*(G) \geq (2/3)n + 1$ when $k = 3$ and (3.1) when $k = 4$. If G is Δ-extremal, then G contains a K_k-factor.*

To handle the non-extremal case, they proved the following lemma ([52, Lemma 2.2] and [55, Lemma 2.2]).

Lemma 3.4 (Almost Covering Lemma)**.** *Let $k = 3, 4$. Given $\Delta > 0$, there exists $\alpha > 0$ such that for every graph $G \in \mathcal{G}_k(n)$ with $\delta^*(G) \geq (1 - 1/k)n - \alpha n$ either G contains an almost K_k-factor that leaves at most $C = C(k)$ vertices uncovered or G is Δ-extremal.*

To improve the almost K_k-factor obtained from Lemma 3.4, they used the Regularity Lemma and Blow-up Lemma [38]. Here is where we need our absorbing lemma whose proof is given in Section 3.2. Our lemma actually gives a more detailed structure than what is needed for the extremal case when G does not satisfy the absorbing property.

We need some definitions. Given positive integers k and r, let $\Theta_{k\times r}$ denote the graph with vertices a_{ij}, $i = 1, \ldots, k$, $j = 1, \ldots, r$, and a_{ij} is adjacent to $a_{i'j'}$ if and only if $i \neq i'$ and $j \neq j'$. In addition, given a positive integer t, the graph $\Theta_{k\times r}(t)$ denotes the blow-up of $\Theta_{k\times r}$, obtained by replacing vertices a_{ij} with sets A_{ij} of size t, and edges $a_{ij}a_{i'j'}$ with complete bipartite graphs between A_{ij} and $A_{i'j'}$. Given $\epsilon, \Delta > 0$ and $t \geq 1$ (not necessarily an integer), we say that a k-partite graph G is (ϵ, Δ)-approximate to $\Theta_{k\times r}(t)$ if each of its partition sets V_i can be partitioned into $\bigcup_{i=1}^{r} V_{ij}$ such that $||V_{ij}| - t| \leq \epsilon t$ for all i, j and $d(V_{ij}, V_{i'j}) \leq \Delta$ whenever $i \neq i'$.

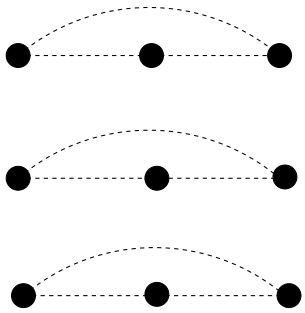

Figure 3.1. $\Theta_{3\times 3}$: dotted lines stand for non-edges

Lemma 3.5 (Absorbing Lemma)**.** *Given $k \geq 3$ and $\Delta > 0$, there exists $\alpha = \alpha(k, \Delta) > 0$ and an integer $n_1 > 0$ such that the following holds. Let $n \geq n_1$ and $G \in \mathcal{G}_k(n)$ be a k-partite graph on $V_1 \cup \cdots \cup V_k$ such that $\delta^*(G) \geq (1 - 1/k)n - \alpha n$. Then one of the following cases holds.*

1. *G contains a K_k-matching M of size $|M| \leq 2(k-1)\alpha^{4k-2}n$ in G such that for every $W \subset V \backslash V(M)$ with $|W \cap V_1| = \cdots = |W \cap V_k| \leq \alpha^{8k-6}n/4$, there exists a K_k-matching covering exactly the vertices in $V(M) \cup W$.*

2. *We may remove some edges from G so that the resulting graph G' satisfies $\delta^*(G') \geq (1 - 1/k)n - \alpha n$ and is $(\Delta/6, \Delta/2)$-approximate to $\Theta_{k\times k}(\frac{n}{k})$.*

The K_k-matching M in Lemma 3.5 has the so-called *absorbing* property: it can absorb *any* balanced set with a much smaller size.

Proof of Theorems 3.1 and 3.2. Let $k = 3, 4$. Let $\alpha \ll \Delta$, where Δ is given by Theorem 3.3 and α satisfies both Lemmas 3.4 and 3.5. Suppose that n is sufficiently large. Let $G \in \mathcal{G}_k(n)$ be a k-partite graph satisfying $\delta^*(G) \geq (2/3)n + 1$ when $k = 3$ and (3.1) when $k = 4$. By Lemma 3.5, either G contains a subgraph which is $(\Delta/6, \Delta/2)$-approximate to $\Theta_{k\times k}(\frac{n}{k})$ or G contains an absorbing K_k-matching M. In the former case, for $i = 1, \ldots, k$, we add or remove at most $\frac{\Delta n}{6k}$ vertices from V_{i1} to obtain a set $A_i \subset V_i$ of size $\lfloor n/k \rfloor$. For $i \neq i'$, we

have

$$\begin{aligned} e(A_i, A_{i'}) &\le e(V_{i1}, V_{i'1}) + \frac{\Delta n}{6k}(|A_i| + |A_{i'}|) \\ &\le \frac{\Delta}{2}|V_{i1}||V_{i'1}| + 2\frac{\Delta n}{6k}\left\lfloor \frac{n}{k} \right\rfloor \\ &\le \frac{\Delta}{2}\left(1 + \frac{\Delta}{6}\right)^2 \left(\frac{n}{k}\right)^2 + \frac{\Delta n}{3k}\left\lfloor \frac{n}{k} \right\rfloor \\ &\le \Delta \left\lfloor \frac{n}{k} \right\rfloor \left\lfloor \frac{n}{k} \right\rfloor, \end{aligned}$$

which implies that $d(A_i, A_{i'}) \le \Delta$. Thus G is Δ-extremal. By Theorem 3.3, G contains a K_k-factor. In the latter case, G contains a K_k-matching M is of size $|M| \le 2(k-1)\alpha^{4k-2}n$ such that for every $W \subset V \backslash V(M)$ with $|W \cap V_1| = \cdots = |W \cap V_k| \le \alpha^{8k-6}n/4$, there exists a K_k-matching on $V(M) \cup W$. Let $G' = G \setminus V(M)$ be the induced subgraph of G on $V(G) \setminus V(M)$, and $n' = |V(G')|$. Clearly G' is balanced. As $\alpha \ll 1$, we have

$$\delta^*(G') \ge \delta^*(G) - |M| \ge \left(1 - \frac{1}{k}\right) n - 2(k-1)\alpha^{4k-2}n \ge \left(1 - \frac{1}{k} - \alpha\right) n'.$$

By Lemma 3.4, G' contains a K_k-matching M' such that $|V(G') \setminus V(M')| \le C$. Let $W = V(G') \setminus V(M')$. Clearly $|W \cap V_1| = \cdots = |W \cap V_k|$. Since $C/k \le \alpha^{8k-6}n/4$ for sufficiently large n, by the absorbing property of M, there is a K_k-matching M'' on $V(M) \cup W$. This gives the desired K_k-factor $M' \cup M''$ of G. □

Remarks.

- Since our Lemma 3.5 works for all $k \ge 3$, it has the potential of proving a general multipartite Hajnal-Szemerédi theorem. To do it, one only needs to prove Theorem 3.3 and Lemma 3.4 for $k \ge 5$.

- Since our Lemma 3.5 gives a detailed structure of G when G does not have desired absorbing K_k-matching, it has the potential of simplifying the proof of the extremal case. Indeed, if one can refine Lemma 3.4 such that it concludes that G either contains

an almost K_k-factor or it is approximate to $\Theta_{k\times k}(\frac{n}{k})$ and other extremal graphs, then in Theorem 3.3 we may assume that G is actually approximate to these extremal graphs.

3.2 Proof of the Absorbing Lemma

In this section we prove the Absorbing Lemma (Lemma 3.5). We first introduce the concepts of reachability.

Definition 3.6. *In a graph G, a vertex x is reachable from another vertex y by a set $S \subseteq V(G)$ if both $G[x \cup S]$ and $G[y \cup S]$ contain K_k-factors. In this case, we say S connects x and y.*

The following lemma plays a key role in constructing absorbing structures. We postpone its proof to the end of the section.

Lemma 3.7 (Reachability Lemma)**.** *Given $k \geq 3$ and $\Delta > 0$, there exists $\alpha = \alpha(k, \Delta) > 0$ and an integer $n_2 > 0$ such that the following holds. Let $n \geq n_2$ and $G \in \mathcal{G}_k(n)$ be a k-partite graph on $V_1 \cup \cdots \cup V_k$ such that $\delta^*(G) \geq (1 - 1/k)n - \alpha n$. Then one of the following cases holds.*

1. *For any x and y in V_i, $i \in [k]$, x is reachable from y by either at least $\alpha^3 n^{k-1}$ $(k-1)$-sets or at least $\alpha^3 n^{2k-1}$ $(2k-1)$-sets in G.*

2. *We may remove some edges from G so that the resulting graph G' satisfies $\delta^*(G') \geq (1 - 1/k)n - \alpha n$ and is $(\Delta/6, \Delta/2)$-approximate to $\Theta_{k\times k}(\frac{n}{k})$.*

With the aid of Lemma 3.7, the proof of Lemma 3.5 becomes standard counting and probabilistic arguments, as shown in [17].

Proof of Lemma 3.5. We assume that G does not satisfy the second property stated in the lemma.

For a *crossing* k-tuple $T = (v_1, \cdots, v_k)$, with $v_i \in V_i$, for $i = 1, \cdots, k$, we call a set A an *absorbing set* for T if both $G[A]$ and $G[A \cup T]$ contain K_k-factors. Let $\mathcal{L}(T)$ denote the family of all $2k(k-1)$-sets that absorb T.

Claim 3.8. *For every crossing k-tuple T, we have $|\mathcal{L}(T)| > \alpha^{4k-3}n^{2k(k-1)}$.*

Proof. Fix a crossing k-tuple T. First we try to find a copy of K_k containing v_1 and avoiding $v_2, \ldots, v_k$. By the minimum degree condition, there are at least

$$\prod_{i=2}^{k}\left(n-1-(i-1)\left(\frac{1}{k}+\alpha\right)n\right) \geq \prod_{i=2}^{k}\left(n-(i-1)\frac{n}{k}-((k-1)\alpha n+1)\right)$$

such copies of K_k. When $n \geq 3k^2$ and $\frac{1}{\alpha} \geq 3k^2$, we have $(k-1)\alpha n + 1 \leq n/(3k)$ and thus the number above is at least

$$\prod_{i=2}^{k}\left(n-(i-1)\frac{n}{k}-\frac{n}{3k}\right) \geq \left(\frac{n}{k}\right)^{k-1}, \text{ when } k \geq 3.$$

Fix such a copy of K_k on $\{v_1, u_2, u_3, \cdots, u_k\}$. Consider u_2 and v_2. By Lemma 3.7 and the assumption that G does not satisfy the second property of the lemma, we can find at least $\alpha^3 n^{k-1}$ $(k-1)$-sets or $\alpha^3 n^{2k-1}$ $(2k-1)$-sets to connect u_2 and v_2. If S is a $(k-1)$-set that connects u_2 and v_2, then $S \cup K$ also connects u_2 and v_2 for any k-set K such that $G[K] \cong K_k$ and $K \cap S = \emptyset$. There are at least

$$(n-2)\prod_{i=2}^{k}\left(n-1-(i-1)\left(\frac{1}{k}+\alpha\right)n\right) \geq \frac{n}{2}\left(\frac{n}{k}\right)^{k-1}$$

copies of K_k in G avoiding u_2, v_2 and S. If there are at least $\alpha^3 n^{k-1}$ $(k-1)$-sets that connect u_2 and v_2, then at least

$$\alpha^3 n^{k-1} \cdot \frac{n}{2}\left(\frac{n}{k}\right)^{k-1}\frac{1}{\binom{2k-1}{k-1}} \geq 2\alpha^4 n^{2k-1}$$

$(2k-1)$-sets connect u_2 and v_2 because a $(2k-1)$-set can be counted at most $\binom{2k-1}{k-1}$ times. Since $2\alpha^4 < \alpha^3$, we can assume that there are always at least $2\alpha^4 n^{2k-1}$ $(2k-1)$-sets connecting u_2 and v_2. We inductively choose disjoint $(2k-1)$-sets that connects v_i and u_i for $i = 2, \ldots, k$. For each i, we must avoid T, $u_2, \ldots, u_k$, and $i-2$ previously selected $(2k-1)$-sets. Hence

there are at least $2\alpha^4 n^{2k-1} - (2k-1)(i-1)n^{2k-2} > \alpha^4 n^{2k-1}$ choices of such $(2k-1)$-sets for each $i \geq 2$. Putting all these together, and using the assumption that α is sufficiently small, we have

$$|\mathcal{L}(T)| \geq \binom{n}{k}^{k-1} \cdot (\alpha^4 n^{2k-1})^{k-1} > \alpha^{4k-3} n^{2k(k-1)}.$$

□

Every set $S \in \mathcal{L}(T)$ is *balanced* because $G[S]$ contains a K_k-factor and thus $|S \cap V_1| = \cdots = |S \cap V_k| = 2(k-1)$. Note that there are $\binom{n}{2(k-1)}^k$ balanced $2k(k-1)$-sets in G. Let $\mathcal{F}$ be the random family of $2k(k-1)$-sets obtained by selecting each balanced $2k(k-1)$-set from $V(G)$ independently with probability $p := \alpha^{4k-3} n^{1-2k(k-1)}$. Then by Chernoff's bound, since n is sufficiently large, with probability $1 - o(1)$, the family $\mathcal{F}$ satisfies the following properties:

$$|\mathcal{F}| \leq 2\mathbb{E}(|\mathcal{F}|) \leq 2p\binom{n}{2(k-1)}^k \leq \alpha^{4k-2} n, \tag{3.2}$$

$$|\mathcal{L}(T) \cap \mathcal{F}| \geq \frac{1}{2}\mathbb{E}(|\mathcal{L}(T) \cap \mathcal{F}|) \geq \frac{1}{2}p|\mathcal{L}(T)| \geq \frac{\alpha^{8k-6} n}{2} \text{ for every crossing } k\text{-tuple } T. \tag{3.3}$$

Let Y be the number of intersecting pairs of members of $\mathcal{F}$. Since each fixed balanced $2k(k-1)$-set intersects at most $2k(k-1)\binom{n-1}{2(k-1)-1}\binom{n}{2(k-1)}^{k-1}$ other balanced $2k(k-1)$-sets in G,

$$\mathbb{E}(Y) \leq p^2 \binom{n}{2(k-1)}^k 2k(k-1)\binom{n-1}{2k-3}\binom{n}{2(k-1)}^{k-1} \leq \frac{1}{8}\alpha^{8k-6} n.$$

By Markov's bound, with probability at least $\frac{1}{2}$, $Y \leq \alpha^{8k-6} n/4$. Therefore, we can find a family $\mathcal{F}$ satisfying (3.2), (3.3) and having at most $\alpha^{8k-6} n/4$ intersecting pairs. Remove one set from each of the intersecting pairs and the sets that have no K_k-factor from $\mathcal{F}$, we get a subfamily $\mathcal{F}'$ consisting of pairwise disjoint absorbing $2k(k-1)$-sets which satisfies $|\mathcal{F}'| \leq |\mathcal{F}| \leq \alpha^{4k-2} n$ and for all crossing T,

$$|\mathcal{L}(T) \cap \mathcal{F}'| \geq \frac{\alpha^{8k-6} n}{2} - \frac{\alpha^{8k-6} n}{4} \geq \frac{\alpha^{8k-6} n}{4}.$$

Since $\mathcal{F}'$ consists of disjoint absorbing sets and each absorbing set is covered by a K_k-matching, $V(\mathcal{F}')$ is covered by some K_k-matching M. Since $|\mathcal{F}'| \leq \alpha^{4k-2}n$, we have $|M| \leq 2k(k-1)\alpha^{4k-2}n/k = 2(k-1)\alpha^{4k-2}n$. Now consider a balanced set $W \subseteq V(G)\backslash V(\mathcal{F}')$ such that $|W \cap V_1| = \cdots = |W \cap V_k| \leq \alpha^{8k-6}n/4$. Arbitrarily partition W into at most $\alpha^{8k-6}n/4$ crossing k-tuples. We absorb each of the k-tuples with a different $2k(k-1)$-set from $\mathcal{L}(T) \cap \mathcal{F}'$. As a result, $V(\mathcal{F}') \cup W$ is covered by a K_k-matching, as desired. □

The rest of the chapter is devoted to proving Lemma 3.7. First we prove a useful lemma. A weaker version of it appears in [55, Proposition 1.4] with a brief proof sketch.

Lemma 3.9. *Let $k \geq 2$ be an integer, $t \geq 1$ and $\epsilon \ll 1$. Let H be a k-partite graph on $V_1 \cup \cdots \cup V_k$ such that $|V_i| \geq (k-1)(1-\epsilon)t$ for all i and each vertex is nonadjacent to at most $(1+\epsilon)t$ vertices in each of the other color classes. Then either H contains at least $\epsilon^2 t^k$ copies of K_k, or H is $(16k^4\epsilon^{1/2^{k-2}}, 16k^4\epsilon^{1/2^{k-2}})$-approximate to $\Theta_{k\times(k-1)}(t)$.*

Proof. First we derive an upper bound for $|V_i|$, $i \in [k]$. Suppose for example, that $|V_k| \geq (k-1)(1+\epsilon)t + \epsilon t$. Then if we greedily construct copies of K_k while choosing the last vertex from V_k, by the minimum degree condition and $\epsilon \ll 1$, there are at least

$$\begin{aligned}
&|V_1| \cdot (|V_2| - (1+\epsilon)t) \cdots (|V_{k-1}| - (k-2)(1+\epsilon)t) \cdot (|V_k| - (k-1)(1+\epsilon)t) \\
\geq &(k-1)(1-\epsilon)t \cdot (k-2-k\epsilon)t \cdots (1-(2k-3)\epsilon)t \cdot \epsilon t \\
\geq &(k-1-\tfrac{1}{2})(k-2-\tfrac{1}{2}) \cdots (1-\tfrac{1}{2})\epsilon t^k \geq \tfrac{\epsilon}{2}t^k
\end{aligned}$$

copies of K_k in H, so we are done. We thus assume that for all i,

$$|V_i| \leq (k-1)(1+\epsilon)t + \epsilon t < (k-1)(1+2\epsilon)t. \tag{3.4}$$

Now we proceed by induction on k. The base case is $k = 2$. If H has at least $\epsilon^2 t^2$ edges, then we are done. Otherwise $e(H) < \epsilon^2 t^2$. Using the lower bound for $|V_i|$, we obtain that

$$d(V_1, V_2) < \frac{\epsilon^2 t^2}{|V_1| \cdot |V_2|} \leq \frac{\epsilon^2}{(1-\epsilon)^2} < \epsilon.$$

Hence H is $(2\epsilon, \epsilon)$-approximate to $\Theta_{2\times 1}(t)$. When $k = 2$, $16k^4\epsilon^{1/2^{k-2}} = 256\epsilon$, so we are done.

Now assume that $k \geq 3$ and the conclusion holds for $k-1$. Let H be a k-partite graph satisfying the assumptions and assume that H contains less than $\epsilon^2 t^k$ copies of K_k.

For simplicity, write $N_i(v) = N(v) \cap V_i$ for any vertex v. Let $V_1' \subset V_1$ be the vertices which are in at least ϵt^{k-1} copies of K_k in H, and let $\tilde{V}_1 = V_1 \setminus V_1'$. Note that $|V_1'| < \epsilon t$ otherwise we get at least $\epsilon^2 t^k$ copies of K_k in H. Fix $v_0 \in \tilde{V}_1$. For $2 \leq i \leq k$, by the minimum degree condition and $k \geq 3$,

$$|N_i(v_0)| \geq (k-1)(1-\epsilon)t - (1+\epsilon)t = (k-2)\left(1 - \frac{k}{k-2}\epsilon\right)t \geq (k-2)(1-3\epsilon)t.$$

On the other hand, following the same arguments as we used for (3.4), we derive that

$$|N_i(v_0)| \leq (k-2)(1+2\epsilon t). \tag{3.5}$$

The minimum degree condition implies that a vertex in $N(v_0)$ misses at most $(1+\epsilon)t$ vertices in each $N_i(v_0)$. We now apply induction with $k-1$, t and 3ϵ on $H[N(v_0)]$. Because of the definition of V_1', we conclude that $N(v_0)$ is (ϵ', ϵ')-approximate to $\Theta_{(k-1)\times(k-2)}(t)$, where

$$\epsilon' := 16(k-1)^4(3\epsilon)^{1/2^{k-3}}.$$

This means that we can partition $N_i(v_0)$ into $A_{i1} \cup \cdots A_{i(k-2)}$ for $2 \leq i \leq k$ such that

$$\forall\ 2 \leq i \leq k,\ 1 \leq j \leq k-2, \quad (1-\epsilon')t \leq |A_{ij}| \leq (1+\epsilon')t \qquad \text{and} \tag{3.6}$$

$$\forall\ 2 \leq i < i' \leq k,\ 1 \leq j \leq k-2, \quad d(A_{ij}, A_{i'j}) \leq \epsilon'. \tag{3.7}$$

Furthermore, let $A_{i(k-1)} := V_i \setminus N(v_0)$ for $i = 2, \cdots, k$. By (3.5) and the minimum degree condition, we get that

$$(1-(3k-5)\epsilon)t \leq |A_{i(k-1)}| \leq (1+\epsilon)t, \tag{3.8}$$

for $i = 2, \cdots, k$.

Let $A_{ij}^c = V_i \setminus A_{ij}$ denote the complement of A_{ij}. Let $\bar{e}(A,B) = |A||B| - e(A,B)$ denote the number of non-edges between two disjoint sets A and B, and $\bar{d}(A,B) = \bar{e}(A,B)/(|A||B|) = 1 - d(A,B)$. Given two disjoint sets A and B (with density close to one) and $\alpha > 0$, we call a vertex $a \in A$ is α-*typical* to B if $\deg_B(a) \geq (1-\alpha)|B|$.

Claim 3.10. *Let $2 \leq i \neq i' \leq k$, $1 \leq j \neq j' \leq k-1$.*

1. *$d(A_{ij}, A_{i'j'}) \geq 1 - 3\epsilon'$ and $d(A_{ij}, A_{i'j}^c) \geq 1 - 3\epsilon'$.*
2. *All but at most $\sqrt{3\epsilon'}$ vertices in A_{ij} are $\sqrt{3\epsilon'}$-typical to $A_{i'j'}$; at most $\sqrt{3\epsilon'}$ vertices in A_{ij} are $\sqrt{3\epsilon'}$-typical to $A_{i'j}^c$.*

Proof. (1). Since $A_{i'j}^c = \bigcup_{j' \neq j} A_{i'j'}$, the second assertion $d(A_{ij}, A_{i'j}^c) \geq 1 - 3\epsilon'$ immediately follows from the first assertion $d(A_{ij}, A_{i'j'}) \geq 1 - 3\epsilon'$. Thus it suffices to show that $d(A_{ij}, A_{i'j'}) \geq 1 - 3\epsilon'$, or equivalently that $\bar{d}(A_{ij}, A_{i'j'}) \leq 3\epsilon'$.

Assume $j \geq 2$. By (3.7), we have $e(A_{ij}, A_{i'j}) \leq \epsilon'|A_{ij}||A_{i'j}|$. So $\bar{e}(A_{ij}, A_{i'j}) \geq (1-\epsilon')|A_{ij}||A_{i'j}|$. By the minimum degree condition and (3.6),

$$\begin{aligned}
\bar{e}(A_{ij}, A_{i'j}^c) &\leq [(1+\epsilon)t - (1-\epsilon')|A_{i'j}|]|A_{ij}| \\
&\leq [(1+\epsilon)t - (1-\epsilon')(1-\epsilon')t]|A_{ij}| \\
&< (\epsilon + 2\epsilon')t|A_{ij}|,
\end{aligned}$$

which implies that $\bar{e}((A_{ij}, A_{i'j'}) \leq (\epsilon + 2\epsilon')t|A_{ij}|$ for any $j' \neq j$ and $1 \leq j' \leq k-1$. By (3.6) and (3.8), we have $|A_{i'j'}| \geq (1-\epsilon')t$. Hence

$$\bar{d}(A_{ij}, A_{i'j'}) \leq (\epsilon + 2\epsilon')\frac{t}{|A_{i'j'}|} \leq (\epsilon + 2\epsilon')\frac{t}{(1-\epsilon')t} \leq 3\epsilon',$$

where the last inequality holds because $\epsilon \ll \epsilon' \ll 1$.

(2) Given two disjoint sets A and B, if $\bar{d}(A,B) \leq \alpha$ for some $\alpha > 0$, then at most $\sqrt{\alpha}|A|$ vertices $a \in A$ satisfy $\deg_B(a) < (1-\sqrt{\alpha})|B|$. Hence Part (2) immediately follows from Part (1). $\square$

We need a lower bound for the number of copies of K_k in a dense k-partite graph.

Proposition 3.11. *Let G be a k-partite graph with vertex class $V_1, \cdots, V_k$. Suppose for every two vertex classes, the pairwise density $d(V_i, V_j) \geq 1 - \alpha$ for some $\alpha \leq (k+1)^{-4}$, then there are at least $\frac{1}{2}\prod_i |V_i|$ copies of K_k in G.*

Proof. Given two disjoint sets V_i and V_j, if $\bar{d}(V_i, V_j) \leq \alpha$ for some $\alpha > 0$, then at most $\sqrt{\alpha}|V_i|$ vertices $v \in V_i$ satisfy $\deg_{V_j}(v) < (1 - \sqrt{\alpha})|V_j|$. Thus, by choosing typical vertices greedily and the assumption $\alpha \leq (k+1)^{-4}$, there are at least

$$(1-\sqrt{\alpha})|V_1|(1-2\sqrt{\alpha})|V_2|\cdots(1-k\sqrt{\alpha})|V_k| > (1-(1+\cdots+k)\sqrt{\alpha})\prod_i |V_i| > \frac{1}{2}\prod_i |V_i|$$

copies of K_k in G. □

Let $\epsilon'' = 2k\sqrt{\epsilon'}$. Now we want to study the structure of $\tilde{V}_1$.

Claim 3.12. *Given $v \in \tilde{V}_1$ and $2 \leq i \leq k$, there exists $j \in [k-1]$, such that $|N_{A_{ij}}(v)| < \epsilon'' t$.*

Proof. Suppose instead, that there exist $v \in \tilde{V}_1$ and some $2 \leq i_0 \leq k$, such that $|N_{A_{i_0 j}}(v)| \geq \epsilon'' t$ for all $j \in [k-1]$. By the minimum degree condition, for each $2 \leq i \leq k$, there is at most one $j \in [k-1]$ such that $|N_{A_{ij}}(v)| < t/3$. Therefore we can greedily choose $k-2$ distinct j_i for $i \neq i_0$, such that $|N_{A_{ij_i}}(v)| \geq t/3$. Let j_{i_0} be the the (unique) unused index. Note that

$$\forall\ i \neq i_0, \quad \frac{|A_{ij_i}|}{|N_{A_{ij_i}}(v)|} \leq \frac{(1+\epsilon')t}{t/3} < 4, \quad \text{and} \quad \frac{|A_{i_0 j_{i_0}}|}{|N_{A_{i_0 j_{i_0}}}(v)|} \leq \frac{(1+\epsilon')t}{\epsilon'' t} < \frac{2}{\epsilon''}$$

So for any $i \neq i'$, by Claim 3.10 and the definition of ϵ'', we have

$$\bar{d}(N_{A_{ij_i}}(v), N_{A_{i'j_{i'}}}(v)) \leq \frac{3\epsilon'|A_{ij_i}||A_{i'j_{i'}}|}{|N_{A_{ij_i}}(v)||N_{A_{i'j_{i'}}}(v)|} \leq 3\epsilon' \cdot 4 \cdot \frac{2}{\epsilon''} = \frac{6}{k^2}\epsilon''. \tag{3.9}$$

Since $\epsilon \ll \epsilon'' \ll 1$, by Proposition 3.11, there are at least

$$\frac{1}{2}\prod_i N_{A_{ij_i}}(v) \geq \frac{1}{2}\cdot\epsilon'' t\left(\frac{t}{3}\right)^{k-2} = \frac{\epsilon''}{2\cdot 3^{k-2}}t^{k-1} > \epsilon t^{k-1}$$

copies of K_{k-1} in $N(v)$, contradicting the assumption $v \in \tilde{V}_1$. □

Note that if $\deg_{A_{ij}}(v) < \epsilon'' t$, at least $|A_{ij}| - \epsilon'' t$ vertices of A_{ij} are not in $N(v)$. By the minimum degree condition, (3.6) and (3.8), it follows that

$$|A_{ij}^c \setminus N(v)| \le (1+\epsilon)t - (|A_{ij}| - \epsilon'' t) \le (1+\epsilon)t - (1-\epsilon')t + \epsilon'' t \le 2\epsilon'' t. \tag{3.10}$$

Fix a vertex $v \in \tilde{V}_1$. Given $2 \le i \le k$, let ℓ_i denote the (unique) index such that $|N_{A_{i\ell_i}}(v)| < \epsilon'' t$ (the existence of ℓ_i follows from Claim 3.12).

Claim 3.13. *We have $\ell_2 = \ell_3 = \cdots = \ell_k$.*

Proof. Otherwise, say $\ell_2 \neq \ell_3$, then we set $j_2 = \ell_3$ and for $3 \le i \le k$, greedily choose distinct $j_k, j_{k-1}, \ldots, j_3 \in [k-1] \setminus \{\ell_3\}$ such that $j_i \neq \ell_i$ (this is possible as j_3 is chosen at last). Let us bound the number of copies of K_{k-1} in $\bigcup_{i=2}^k N_{A_{ij_i}}(v)$. By 3.10, we get $|N_{A_{ij_i}}(v)| \ge |A_{ij_i}| - 2\epsilon'' t \ge t/2$ for all i. As in (3.9), for any $i \neq i'$, we derive that $\bar{d}(N_{A_{ij_i}}(v), N_{A_{i'j_{i'}}}(v)) \le 3\epsilon'' \cdot 4 \cdot 4 = 48\epsilon''$. Since $\epsilon'' \ll 1$, by Proposition 3.11, we get at least $\frac{1}{2}\left(\frac{t}{2}\right)^{k-1} > \epsilon t^{k-1}$ copies of K_{k-1} in $N(v)$, a contradiction. □

We define $A_{1j} := \{v \in \tilde{V}_1 : |N_{A_{2j}}(v)| < \epsilon'' t\}$ for $j \in [k-1]$. By Claims 3.12 and 3.13, this yields a partition of $\tilde{V}_1 = \bigcup_{j=1}^{k-1} A_{1j}$ such that

$$d(A_{1j}, A_{ij}) < \frac{\epsilon'' t |A_{1j}|}{|A_{1j}||A_{ij}|} \le \frac{\epsilon'' t}{(1-\epsilon')t} < (1+2\epsilon')\epsilon'' \quad \text{for } i \ge 2 \text{ and } j \ge 1. \tag{3.11}$$

By (3.6), (3.8) and (3.10), as $(3k-5)\epsilon \le \epsilon'$, we have

$$\bar{d}(A_{1j}, A_{ij'}) < \frac{|A_{1j}| 2\epsilon'' t}{|A_{1j}||A_{ij'}|} \le \frac{2\epsilon'' t}{(1-\epsilon')t} < 3\epsilon'' \quad \text{for } i \ge 2 \text{ and } j \neq j'. \tag{3.12}$$

We claim $|A_{1j}| \le (1+\epsilon)t + (1+2\epsilon')\epsilon''|A_{1j}|$ for all j. Otherwise, by the minimum degree condition, we have $\deg_{A_{1j}}(v) > (1+2\epsilon')\epsilon''|A_{1j}|$ for all $v \in A_{ij}$, and consequently $d(A_{1j}, A_{ij}) >$

$(1+2\epsilon')\epsilon''$, contradicting (3.11). We thus conclude that

$$|A_{1j}| \leq \frac{1+\epsilon}{1-(1+2\epsilon')\epsilon''}t < (1+2\epsilon'')t. \tag{3.13}$$

Since $|V_1'| \leq \epsilon t$, we have $|\bigcup_{j=1}^{k-1} A_{1j}| = |V_1 \setminus V_1'| \geq |V_1| - \epsilon t$. Using (3.13), we now obtain a lower bound for $|A_{1j}|$, $j \in [k-1]$:

$$|A_{1j}| \geq (k-1)(1-\epsilon)t - (k-2)(1+2\epsilon'')t - \epsilon t \geq (1-2k\epsilon'')t. \tag{3.14}$$

It remains to show that for $2 \leq i \neq i' \leq k$, $d(A_{i(k-1)}, A_{i'(k-1)})$ is small. Write $N(v_1 \cdots v_m) = \bigcap_{1\leq i\leq m} N(v_i)$.

Claim 3.14. *$d(A_{i(k-1)}, A_{i'(k-1)}) \leq 6\epsilon''$ for $2 \leq i, i' \leq k$.*

Proof. Suppose to the contrary, that say $d(A_{(k-1)(k-1)}, A_{k(k-1)}) > 6\epsilon''$. Note that there are $(k-2)!$ choices of the sets $\{A_{ij_i}\}_{i\in[k-2]}$ such that $j_i \in [k-2]$ and every pair of the sets is dense. We construct copies of K_{k-2} in such sets, for example, $A_{11}, A_{22}, \cdots, A_{(k-2)(k-2)}$. Pick arbitrary $v_1 \in A_{11}$. For $2 \leq i \leq k-2$, we select $v_i \in N_{A_{ii}}(v_1 \cdots v_{i-1})$ such that v_i is $\sqrt{3\epsilon'}$-typical to $A_{(k-1)(k-1)}$, $A_{k(k-1)}$ and all A_{jj}, $i < j \leq k-2$. By Claim 3.10 and (3.10), there are at least $(1-(k-2)\sqrt{3\epsilon'})|A_{ii}| - 2\epsilon''t \geq t/2$ choices for each v_i. After selecting $v_1, \ldots, v_{k-2}$, we select adjacent vertices $v_{k-1} \in A_{(k-1)(k-1)}$ and $v_k \in A_{k(k-1)}$ such that $v_{k-1}, v_k \in N(v_1 \cdots v_{k-2})$. For $j \in \{k-1, k\}$, we know that $N(v_1)$ misses at most $2\epsilon''t$ vertices in $A_{j(k-1)}$, and at most $(k-3)\sqrt{3\epsilon'}|A_{j(k-1)}|$ vertices of $A_{j(k-1)}$ are not in $N(v_2 \cdots v_{k-2})$. Since $d(A_{(k-1)1}, A_{k1}) > 6\epsilon''$ and $\epsilon'' = 2k\sqrt{\epsilon'}$, there are at least

$$\begin{aligned}
&6\epsilon''|A_{(k-1)(k-1)}||A_{k(k-1)}| - 2\epsilon''t(|A_{(k-1)(k-1)}| + |A_{k(k-1)}|) - 2(k-3)\sqrt{3\epsilon'}|A_{(k-1)(k-1)}||A_{k(k-1)}| \\
&\geq (6\epsilon'' - 4\epsilon'' - 4(k-3)\sqrt{\epsilon'})|A_{(k-1)(k-1)}||A_{k(k-1)}| \\
&= 12\sqrt{\epsilon'}|A_{(k-1)(k-1)}||A_{k(k-1)}| \geq 6\sqrt{\epsilon'}t^2
\end{aligned}$$

such pairs v_{k-1}, v_k. In total, we obtain at least $(k-2)!(\frac{t}{2})^{k-2}\,6\sqrt{\epsilon'}t^2 > \epsilon t^k$ copies of K_k, a

contradiction. □

In summary, by (3.6), (3.8), (3.13) and (3.14), we have $(1-2k\epsilon'')t \le |A_{ij}| \le (1+2\epsilon'')t$ for all i and j. In order to make $\bigcup_{j=1}^{k-1} A_{1j}$ a partition of V_1, we move the vertices of V_1' to A_{11}. Since $|V_1'| < \epsilon t$, we still have $||A_{ij}| - t| \le 2k\epsilon'' t$ after moving these vertices. On the other hand, by (3.7), (3.11), and Claim 3.14, we have $d(A_{ij}, A_{i'j}) \le 6\epsilon'' \le 2k\epsilon''$ for $i \ne i'$ and all j (we now have $d(A_{11}, A_{i1}) \le 2\epsilon''$ for all $i \ge 2$ because $|A_{11}|$ becomes slightly larger). Therefore H is $(2k\epsilon'', 2k\epsilon'')$-approximate to $\Theta_{k\times(k-1)}(t)$. By the definitions of ϵ'' and ϵ',

$$2k\epsilon'' = 4k^2\sqrt{\epsilon'} = 4k^2\sqrt{16(k-1)^4(3\epsilon)^{1/2^{k-3}}} \le 16k^4\epsilon^{1/2^{k-2}},$$

where the last inequality is equivalent to $(\frac{k-1}{k})^2\, 3^{1/2^{k-2}} \le 1$ or $3^{1/2^{k-1}} \le \frac{k}{k-1}$, which holds because $3 \le 1 + \frac{2^{k-1}}{k-1} \le (1+\frac{1}{k-1})^{2^{k-1}}$ for $k \ge 2$.

This completes the proof of Lemma 3.9. □

We are ready to prove Lemma 3.7.

Proof of Lemma 3.7. First assume that $G \in \mathcal{G}_3(n)$ is minimal, namely, G satisfies the minimum partite degree condition but removing any edge of G will destroy this condition. Note that this assumption is only needed by Claim 3.20.

Given $0 < \Delta \le 1$, let

$$\alpha = \frac{1}{2k}\left(\frac{\Delta}{24k(k-1)\sqrt{2k}}\right)^{2^{k-1}}. \tag{3.15}$$

Without loss of generality, assume that $x, y \in V_1$ and y is *not* reachable by $\alpha^3 n^{k-1}$ $(k-1)$-sets or $\alpha^3 n^{2k-1}$ $(2k-1)$-sets from x.

For $2 \le i \le k$, define

$$A_{i1} = V_i \cap (N(x) \setminus N(y)), \quad A_{ik} = V_i \cap (N(y) \setminus N(x)),$$
$$B_i = V_i \cap (N(x) \cap N(y)), \quad A_{i0} = V_i \setminus (N(x) \cup N(y)).$$

Let $B = \bigcup_{i\geq 2} B_i$. If there are at least $\alpha^3 n^{k-1}$ copies of K_{k-1} in B, then x is reachable from y by at least $\alpha^3 n^{k-1}$ $(k-1)$-sets. We thus assume there are less than $\alpha^3 n^{k-1}$ copies of K_{k-1} in B.

Clearly, for $i \geq 2$, A_{i1}, A_{ik}, B_i and A_{i0} are pairwise disjoint. The following claim bounds the sizes of A_{ik}, B_i and A_{i0}.

Claim 3.15. *1.* $(1-k^2\alpha)\frac{n}{k} < |A_{i1}|, |A_{ik}| \leq (1+k\alpha)\frac{n}{k}$,

2. $(k-2-2k\alpha)\frac{n}{k} \leq |B_i| < (k-2+k(k-1)\alpha)\frac{n}{k}$,

3. $|A_{i0}| < (k+1)\alpha n$.

Proof. For $v \in V$, and $i \in [k]$, write $N_i(v) := N(v) \cap V_i$. By the minimum degree condition, we have $|A_{i1}|, |A_{ik}| \leq (1/k+\alpha)n$. Since $N_i(x) = A_{i1} \cup B_i$, it follows that

$$|B_i| \geq (\tfrac{k-1}{k} - \alpha)n - (\tfrac{1}{k} + \alpha)n. \tag{3.16}$$

If some B_i, say B_k, has at least $(\frac{k-2}{k} + (k-1)\alpha)n$ vertices, then there are at least $\prod_{i=2}^{k} |B_i| - (i-2)\left(\frac{1}{k}+\alpha\right)n$ copies of K_{k-1} in B. By (3.16) and $|B_k| \geq (\frac{k-2}{k} + (k-1)\alpha)n$, this is at least

$$\begin{aligned}
&\alpha n \cdot \prod_{i=2}^{k-1} \left(\frac{k-1}{k} - \alpha\right) n - (i-1)\left(\frac{1}{k}+\alpha\right)n \\
=&\alpha n \cdot \prod_{i=2}^{k-1} \left(\frac{k-i}{k} - i\alpha\right) n \\
\geq&\alpha n \cdot \prod_{i=2}^{k-1} \left(\frac{k-i-\frac{1}{2}}{k}\right) n \quad \text{because } 2k^2\alpha \leq 1, \\
\geq&\alpha n \cdot \frac{1}{2}\left(\frac{n}{k}\right)^{k-2} \\
\geq&\alpha^2 n^{k-1} \quad \text{because } 2k^{k-2}\alpha \leq 1.
\end{aligned}$$

This is a contradiction.

We may thus assume that $|B_i| < (\frac{k-2}{k} + (k-1)\alpha)n$ for $2 \leq i \leq k$, as required for Part (2). As $N_i(x) = A_{i1} \cup B_i$, it follows that

$$|A_{i1}| > (\tfrac{k-1}{k} - \alpha)n - (\tfrac{k-2}{k} + (k-1)\alpha)n = (\tfrac{1}{k} - k\alpha)n.$$

The same holds for $|A_{ik}|$ thus Part (1) follows. Finally

$$|A_{i0}| = |V_i| - |N_i(x)| - |A_{ik}| < n - (\tfrac{k-1}{k} - \alpha)n - (\tfrac{1}{k} - k\alpha)n = (k+1)\alpha n,$$

as required for Part (3). □

Let $t = n/k$ and $\epsilon = 2k\alpha$. By the minimum degree condition, every vertex $u \in B$ is nonadjacent to at most $(1 + k\alpha)n/k < (1 + \epsilon)t$ vertices in other color classes of B. By Claim 3.15, $|B_i| \geq (k - 2 - 2k\alpha)\frac{n}{k} = (k - 2 - \epsilon)t \geq (k-2)(1-\epsilon)t$. Thus $G[B]$ is a $(k-1)$-partite graph that satisfies the assumptions of Lemma 3.9. We assumed that B contains less than $\alpha^3 n^{k-1} < \epsilon^2 t^{k-1}$ copies of K_{k-1}, so by Lemma 3.9, B is (α', α')-approximate to $\Theta_{(k-1)\times(k-2)}(\frac{n}{k})$, where

$$\alpha' := 16(k-1)^4(2k\alpha)^{1/2^{k-3}}.$$

This means that we can partition B_i, $2 \leq i \leq k$, into $A_{i2} \cup \cdots \cup A_{i(k-1)}$ such that $(1-\alpha')\frac{n}{k} \leq |A_{ij}| \leq (1+\alpha')\frac{n}{k}$ for $2 \leq j \leq k-1$ and

$$\forall\ 2 \leq i < i' \leq k, 2 \leq j \leq k-1, \quad d(A_{ij}, A_{i'j}) \leq \alpha'. \tag{3.17}$$

Together with Claim 3.15 Part (1), we obtain that (using $k^2\alpha \leq \alpha'$)

$$\forall\ 2 \leq i \leq k, 1 \leq j \leq k, \quad (1-\alpha')\tfrac{n}{k} \leq |A_{ij}| \leq (1+\alpha')\tfrac{n}{k}. \tag{3.18}$$

Let $A_{ij}^c = V_i \setminus A_{ij}$ denote the complement of A_{ij}. The following claim is an analog of Claim 3.10, and its proof is almost the same – after we replace $(1+\epsilon)t$ with $(1+k\alpha)n/k$ and ϵ' with α' (and we use $\alpha \ll \alpha'$). We thus omit the proof.

Claim 3.16. *Let $2 \le i \ne i' \le k$, $1 \le j \ne j' \le k$, and $\{j, j'\} \ne \{1, k\}$.*

1. *$d(A_{ij}, A_{i'j'}) \ge 1 - 3\alpha'$ and $d(A_{ij}, A^c_{i'j}) \ge 1 - 3\alpha'$.*
2. *All but at most $\sqrt{3\alpha'}$ vertices in A_{ij} are $\sqrt{3\alpha'}$-typical to $A_{i'j'}$; at most $\sqrt{3\alpha'}$ vertices in A_{ij} are $\sqrt{3\alpha'}$-typical to $A^c_{i'j}$.* □

Now let us study the structure of V_1. Let $\alpha'' = 2k\sqrt{\alpha'}$. Recall that $N(xv) = N(x) \cap N(v)$. Let V_1' be the set of the vertices $v \in V_1$ such that there are at least αn^{k-1} copies of K_{k-1} in each of $N(xv)$ and $N(yx)$. We claim that $|V_1'| < 2\alpha n$. Otherwise, since a $(k-1)$-set intersects at most $(k-1)n^{k-2}$ other $(k-1)$-sets, there are at least

$$2\alpha n \cdot \alpha n^{k-1}(\alpha n^{k-1} - (k-1)n^{k-2}) > \alpha^3 n^{2k-1}$$

copies of $(2k-1)$-sets connecting x and y, a contradiction.

Let $\tilde{V}_1 := V_1 \setminus V_1'$. The following claim is an analog of Claim 3.12 for Lemma 3.9 and can be proved similarly. The only difference between their proofs is that here we find at least αn^{k-1} copies of K_{k-1} in each of $N(xv)$ and $N(yv)$, and thus obtain a contradiction with $v \in \tilde{V}_1$.

Claim 3.17. *Given $v \in \tilde{V}_1$ and $2 \le i \le k$, there exists $j \in [k]$ such that $|N_{A_{ij}}(v)| < \alpha'' t$.* □

Fix an vertex $v \in \tilde{V}_1$. Claim 3.17 implies that for each $2 \le i \le k$, there exists ℓ_i such that $|N_{A_{i\ell_i}}(v)| < \alpha'' t$. Our next claim is an analog of Claim 3.13 for Lemma 3.9 and can be proved similarly.

Claim 3.18. *We have $\ell_2 = \ell_3 = \cdots = \ell_k$.* □

We now define $A_{1j} := \{v \in \tilde{V}_1 : |N_{A_{2j}}(v)| < \alpha'' t\}$ for $j \in [k]$. By Claims 3.17 and 3.18, this yields a partition of $\tilde{V}_1 = \bigcup_{j=1}^k A_{1j}$ such that

$$d(A_{1j}, A_{ij}) < \frac{\alpha'' t |A_{1j}|}{|A_{1j}||A_{ij}|} \le \frac{\alpha'' t}{(1-\alpha')t} < (1 + 2\alpha')\alpha'' \text{ for } i \ge 2, j \ge 1. \tag{3.19}$$

For $v \in A_{1j}$, we have $|N_{A_{ij}}(v)| < \alpha'' t$ for $i \geq 2$. By the minimum degree condition and (3.18),

$$|A_{ij}^c \setminus N(v)| \leq (\tfrac{1}{k} + \alpha)n - (|A_{ij}| - \alpha'' t) < 2\alpha'' t. \tag{3.20}$$

By (3.18) and (3.20), we derive that

$$\bar{d}(A_{1j}, A_{ij'}) < \frac{|A_{1j}| \cdot 2\alpha'' t}{|A_{1j}||A_{ij'}|} \leq \frac{2\alpha'' t}{(1-\alpha')t} < 3\alpha'' \text{ for } i \geq 2, j \neq j'. \tag{3.21}$$

We claim $|A_{1j}| \leq (1+\alpha)t + (1+2\alpha')\alpha''|A_{1j}|$ for all j. Otherwise, by the minimum degree condition, we have $\deg_{A_{1j}}(v) > (1+2\alpha')\alpha''|A_{1j}|$ for all $v \in A_{ij}$, and consequently $d(A_{1j}, A_{ij}) > (1+2\alpha')\alpha''$, contradicting (3.19). We thus conclude that

$$|A_{1j}| \leq \frac{1+\alpha}{1-(1+2\alpha')\alpha''} t < (1+2\alpha'')\frac{n}{k}. \tag{3.22}$$

Since $|V_1'| \leq 2\alpha n$, we have $|\bigcup_{j=1}^{k} A_{1j}| = |V_1 \setminus V_1'| \geq |V_1| - 2\alpha n$. Using (3.22), we now obtain a lower bound for $|A_{1j}|$, $j \in [k]$.

$$|A_{1j}| \geq n - (k-1)(1+2\alpha'')\frac{n}{k} - 2\alpha n \geq (1 - 2k\alpha'')\frac{n}{k}. \tag{3.23}$$

It remains to show that $d(A_{i1}, A_{i'1})$ and $d(A_{ik}, A_{i'k})$, $2 \leq i, i' \leq k$, are small. First we show that if both densities are reasonably large then there are too many reachable $(2k-1)$-sets from x to y. The proof resembles the one of Claim 3.14.

Claim 3.19. *For $2 \leq i \neq i' \leq k$, either $d(A_{i1}, A_{i'1}) \leq 6\alpha''$ or $d(A_{ik}, A_{i'k}) \leq 6\alpha''$.*

Proof. Suppose instead, that say $d(A_{(k-1)1}, A_{k1}), d(A_{(k-1)k}, A_{kk}) > 6\alpha''$. Fix a vertex $v_1 \in A_{12}$. We construct two vertex disjoint copies of K_{k-1} in $N(xv_1)$ and $N(yv_1)$ as follows. Note that there are $(k-3)!$ choices of the sets $\{A_{ij_i}\}_{2 \leq i \leq k-2}$ such that $3 \leq j_i \leq k-1$ and every pair of the sets is dense. We construct copies of K_{k-3} in $N(xv_1)$ or $N(yv_1)$ from such sets, for example, $A_{23}, \ldots, A_{(k-2)(k-1)}$.

First, we construct a copy of K_{k-1} in $N(xv_1)$ from $A_{23}, \ldots, A_{(k-2)(k-1)}$. For $2 \leq i \leq$

$k-2$, we select $v_i \in N_{A_{i(i+1)}}(v_1 \cdots v_{i-1})$ that is $\sqrt{3\alpha'}$-typical to $A_{(k-1)1}$, A_{k1} and $A_{j(j+1)}$, $i < j \leq k-2$. By Claim 3.16 and (3.20), there are at least

$$(1-(k-2)\sqrt{3\alpha'})|A_{i(i+1)}| - (k\alpha+\alpha'+\alpha'')\frac{n}{k} \geq \frac{n}{2k}$$

such v_i. After selecting $v_2, \ldots, v_{k-2}$, we select two adjacent vertices $v_{k-1} \in A_{(k-1)1}$ and $v_k \in A_{k1}$ such that v_{k-1} and v_k are in $N(v_1 \cdots v_{k-2})$. For $j = k-1, k$, we know that $N(v_1)$ misses at most $(k\alpha+\alpha'+\alpha'')n/k$ vertices in A_{j1} and at most $(k-3)\sqrt{3\alpha'}|A_{j1}|$ vertices of A_{j1} are not in $N(v_2 \cdots v_{k-2})$. Since $d(A_{(k-1)1}, A_{k1}) > 6\alpha''$, there are at least

$$\begin{aligned} 6\alpha''|A_{(k-1)1}||A_{k1}| - (k\alpha+\alpha'+\alpha'')\frac{n}{k}(|A_{(k-1)1}|+|A_{k1}|) \\ -2(k-3)\sqrt{3\alpha'}|A_{(k-1)1}||A_{k1}| \geq 6\sqrt{\alpha'}\left(\frac{n}{k}\right)^2 \end{aligned}$$

such pairs v_{k-1}, v_k. Hence, $N(xv_1)$ contains at least

$$(k-3)!\left(\frac{n}{2k}\right)^{k-3} 6\sqrt{\alpha'}\left(\frac{n}{k}\right)^2 \geq \sqrt{\alpha'}\left(\frac{n}{k}\right)^{k-1} \geq \sqrt{\alpha}n^{k-1}$$

copies of K_{k-1}. Let C be such a copy of K_{k-1}. Then we follow the same procedure and construct a copy of K_{k-1} on $N(yv_1) \setminus C$. After fixing $k-3$ sets A_{ij} with $2 \leq i \leq k-2$ and $3 \leq j \leq k-1$ such that no two of them are on the same row or column, still there are at least $\frac{n}{2k}$ such v_i for $2 \leq i \leq k-2$. Then, as $d(A_{ik}, A_{i'k}) > 6\alpha''$, there are at least $6\sqrt{\alpha'}\left(\frac{n}{k}\right)^2$ choices of $v_{k-1} \in A_{(k-1)k}$ and $v_k \in A_{kk}$ such that v_{k-1} and v_k are in $N(v_1 \cdots v_{k-2})$. This gives at least $\sqrt{\alpha}n^{k-1}$ copies of K_{k-1} in $N(yv_1)$. Then, since there are at least $|V_1| - |A_{11}| - |A_{1k}| \geq \alpha n$ choices of v_1, totally there are at least $\alpha n(\sqrt{\alpha}n^{k-1})^2 = \alpha^2 n^{2k-1}$ reachable $(2k-1)$-sets from x to y, a contradiction. □

Next we show that if any of $d(A_{i1}, A_{i'1})$ or $d(A_{ik}, A_{i'k})$, $2 \leq i, i' \leq k$, is sufficiently large, then we can remove edges from G such that the resulting graph still satisfies the minimum degree condition, which contradicts the assumption that G is minimal.

Claim 3.20. *For $2 \le i \ne i' \le k$, $d(A_{i1}, A_{i'1}), d(A_{ik}, A_{i'k}) \le 6k\sqrt{\alpha''}$.*

Proof. Without loss of generality, assume that $d(A_{2k}, A_{3k}) > 6k\sqrt{\alpha''}$. By Claim 3.19, we have $d(A_{21}, A_{31}) < 6\alpha''$. Combining this with (3.17), we have $d(A_{2j}, A_{3j}) < 6\alpha''$ for all $j \in [k-1]$. Now fix $j \in [k-1]$. The number of non-edges between A_{2j} and A_{3j} satisfies $\bar{e}(A_{2j}, A_{3j}) > (1 - 6\alpha'')|A_{2j}||A_{3j}|$. By the minimum degree condition and (3.18),

$$\bar{e}(A_{2k}, A_{3j}) < (1 + \ k\alpha)\frac{n}{k}|A_{3j}| - (1 - 6\alpha'')|A_{2j}||A_{3j}| \le 7\alpha''\frac{n}{k}|A_{3j}|.$$

Using (3.18) again, we obtain that

$$d(A_{2k}, A_{3j}) \ge 1 - \frac{7\alpha''\frac{n}{k}|A_{3j}|}{|A_{2k}||A_{3j}|} \ge 1 - 8\alpha''.$$

Consequently $d(A_{2k}, A^c_{3k}) \ge 1 - 8\alpha''$, which implies $d(A_{ik}, A^c_{i'k}) \ge 1 - 8\alpha''$ for $\{i, i'\} = \{2, 3\}$ by symmetry. For $\{i, i'\} = \{2, 3\}$, define A^T_{ik} as the set of the vertices in A_{ik} that are $\sqrt{8\alpha''}$-typical to $A^c_{i'k}$. Note that $|A_{ik} \setminus A^T_{ik}| \le \sqrt{8\alpha''}|A_{ik}|$.

Let $A^{T_1}_{ik} = \{v \in A^T_{ik} : \deg_{A_{jk}}(v) \le \sqrt{8\alpha''}|A^c_{jk}|\}$ and $A^{T_2}_{ik} = A^T_{ik} \setminus A^{T_1}_{ik}$. For $u \in A^{T_2}_{2k}$, we have

$$\deg_{V_3}(u) = \deg_{A^c_{3k}}(u) + \deg_{A_{3k}}(u) > (1 - \sqrt{8\alpha''})|A^c_{3k}| + \sqrt{8\alpha''}|A^c_{3k}| = |A^c_{3k}|.$$

Since $|A^c_{3k}| \ge \deg_{V_3}(x)$ and $|A^c_{3k}|$ is an integer, we conclude that $\deg_{V_3}(u) \ge \deg_{V_3}(x) + 1$. Similarly we can derive that $\deg_{V_2}(v) \ge \deg_{V_2}(x) + 1$ for every $v \in A^{T_2}_{3k}$. If there is an edge uv joining some $u \in A^{T_2}_{2k}$ and some $v \in A^{T_2}_{3k}$, then we can delete this edge and the resulting graph still satisfies the minimum degree condition. Therefore we may assume that there is

no edge between $A_{2k}^{T_2}$ and $A_{3k}^{T_2}$. Then

$$\begin{aligned} e(A_{2k}, A_{3k}) &= e(A_{2k} \setminus A_{2k}^T, A_{3k}) + e(A_{2k}, A_{3k} \setminus A_{3k}^T) + e(A_{2k}^{T_1}, A_{3k}^T) + e(A_{2k}^T, A_{3k}^{T_1}) \\ &\leq 2\sqrt{8\alpha''}|A_{2k}||A_{3k}| + |A_{2k}^{T_1}|\sqrt{8\alpha''}|A_{3k}^c| + |A_{3k}^{T_1}|\sqrt{8\alpha''}|A_{2k}^c| \\ &\leq \sqrt{8\alpha''}\left(2|A_{2k}||A_{3k}| + |A_{2k}||A_{3k}^c| + |A_{3k}||A_{2k}^c|\right) \\ &= \sqrt{8\alpha''}\left(|A_{2k}||V_3| + |A_{3k}||V_2|\right) \\ &\leq 3\sqrt{\alpha''} \cdot 2k|A_{2k}||A_{3k}|. \end{aligned}$$

Therefore $d(A_{2k}, A_{3k}) \leq 6k\sqrt{\alpha''}$. □

In summary, by (3.18), (3.22) and (3.23), we have $(1 - 2k\alpha'')\frac{n}{k} \leq |A_{ij}| \leq (1 + 2\alpha'')\frac{n}{k}$ for all i and j. In order to make $\bigcup_{j=1}^{k} A_{ij}$ a partition of V_i, we move the vertices of V_1' to A_{11} and the vertices of A_{i0} to A_{i2} for $2 \leq i \leq k$. By $|V_1'| < 2\alpha n$ and (3.18), we still have $||A_{ij}| - \frac{n}{k}| \leq 2k\alpha''\frac{n}{k}$. On the other hand, by (3.17), (3.19), and Claim 3.20, we have $d(A_{ij}, A_{i'j}) \leq 6k\sqrt{\alpha''}$ for $i \neq i'$ and all j (at present $d(A_{11}, A_{i1}) \leq 2\alpha''$ for $i \geq 2$ because we added at most $2\alpha n$ vertices to A_{11}. Similarly $d(A_{i2}, A_{i'2}) \leq 2\alpha'$ for $i, i' \geq 2$). Therefore after deleting edges, G is $(2k\alpha'', 6k\sqrt{\alpha''})$-approximate to $\Theta_{k\times k}(n/k)$. By (3.15), and the definitions of α'' and α', G is $(\Delta/6, \Delta/2)$-approximate to $\Theta_{k\times k}(n/k)$. □

PART 4

MINIMUM CODEGREE THRESHOLD FOR HAMILTON ℓ-CYCLES IN K-UNIFORM HYPERGRAPHS

4.1 Introduction

A well-known result of Dirac [10] states that every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamilton cycle. In recent years, researchers have extended this result to hypergraphs in various ways (see [58] for a survey). In order to state these results, we need to define degrees and Hamilton cycles for hypergraphs.

Given $k \geq 2$, a k-uniform hypergraph (in short, *k-graph*) consists of a vertex set V and an edge set $E \subseteq \binom{V}{k}$, where every edge is a k-element subset of V. Given a k-graph H with a set S of d vertices (where $1 \leq d \leq k-1$) we define $\deg_H(S)$ to be the number of edges containing S (the subscript H is omitted if it is clear from the context). The *minimum d-degree* $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d-vertex sets S in H. We refer to $\delta_1(H)$ as the *minimum vertex degree* and $\delta_{k-1}(H)$ the *minimum codegree* of H. For $1 \leq \ell < k$, a k-graph is a called an *ℓ-cycle* if its vertices can be ordered cyclically such that each of its edges consists of k consecutive vertices and every two consecutive edges (in the natural order of the edges) share exactly ℓ vertices. In k-graphs, a $(k-1)$-cycle is often called a *tight* cycle while a 1-cycle is often called a *loose* cycle. We say that a k-graph contains a *Hamilton ℓ-cycle* if it contains an ℓ-cycle as a spanning subhypergraph. Note that a k-uniform ℓ-cycle on n vertices contains exactly $n/(k-\ell)$ edges, implying that $k-\ell$ divides n.

Confirming a conjecture of Katona and Kierstead [28], Rödl, Ruciński and Szemerédi [60, 62] showed that for any fixed k, every k-graph H on n vertices with $\delta_{k-1}(H) \geq n/2+o(n)$ contains a tight Hamilton cycle. When $k-\ell$ divides k, a $(k-1)$-cycle on V trivially contains an ℓ-cycle on V (provided $k-\ell$ divides $|V|$). Thus the result in [62] implies that for all $1 \leq \ell < k$ such that $k-\ell$ divides k, every k-graph H on $n \in (k-\ell)\mathbb{N}$ vertices with

$\delta_{k-1}(H) \geq n/2 + o(n)$ contains a Hamilton ℓ-cycle. It is not hard to see that these results are best possible up to the $o(n)$ term. With long and involved arguments, Rödl, Ruciński and Szemerédi [64] determined the minimum codegree threshold for tight Hamilton cycles in 3-graphs.

Loose Hamilton cycles were first studied by Kühn and Osthus [41], who proved that every 3-graph on n vertices with $\delta_2(H) \geq n/4 + o(n)$ contains a loose Hamilton cycle. It is easy to see that this is asymptotically best possible. It was generalized to arbitrary k by Keevash, Kühn, Mycroft, and Osthus [32] and to arbitrary k and arbitrary $\ell < k/2$ by Hàn and Schacht [18].

Theorem 4.1. *[18] Fix integers $k \geq 3$ and $1 \leq \ell < k/2$. Assume that $\gamma > 0$ and $n \in (k-\ell)\mathbb{N}$ is sufficiently large. If $H = (V, E)$ is a k-graph on n vertices such that $\delta_{k-1}(H) \geq (\frac{1}{2(k-\ell)} + \gamma)n$, then H contains a Hamilton ℓ-cycle.*

Later Kühn, Mycroft, and Osthus [40] proved that whenever $k - \ell$ does not divide k, every k-graph on n vertices with $\delta_{k-1}(H) \geq \frac{n}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} + o(n)$ contains a Hamilton l-cycle. Since $\lceil k/(k-\ell) \rceil = 2$ when $\ell < k/2$, this generalizes Theorem 4.1 and is best possible up to the $o(n)$ term.

Rödl and Ruciński [58, Problem 2.9] asked for the exact minimum codegree threshold for Hamilton ℓ-cycles in k-graphs. The $k = 3$ and $\ell = 1$ case was answered by Czygrinow and Molla [9] recently. In this chapter we determine this threshold for all $k \geq 3$ and $\ell < k/2$.

Theorem 4.2. *Fix integers $k \geq 3$ and $1 \leq \ell < k/2$. Assume that $n \in (k-\ell)\mathbb{N}$ is sufficiently large. If $H = (V, E)$ is a k-graph on n vertices such that*

$$\delta_{k-1}(H) \geq \frac{n}{2(k-\ell)}, \tag{4.1}$$

then H contains a Hamilton ℓ-cycle.

A simple well-known construction shows that Theorem 4.2 is best possible – in fact, it works for all $\ell < k$. Let $H_0 = (V, E)$ be an n-vertex k-graph in which V is partitioned into

sets A and B such that $|A| = \left\lceil \frac{n}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} \right\rceil - 1$. The edge set E consists of all k-sets that intersect A. It is easy to see (e.g. [40, Proposition 2.2]) that $\delta_{k-1}(H_0) = |A|$ and H_0 contains no Hamilton ℓ-cycle.

Using the typical approach of obtaining exact results, our proof of Theorem 4.2 consists of an *extremal case* and a *nonextremal case*.

Definition 4.3. *Let $\Delta > 0$, a k-graph H on n vertices is called Δ-extremal if there is a set $B \subset V(H)$, such that $|B| = \lfloor \frac{2(k-\ell)-1}{2(k-\ell)} n \rfloor$ and $e(B) \le \Delta n^k$.*

Theorem 4.4 (Nonextremal Case)**.** *For any integer $k \ge 3$, $1 \le \ell < k/2$ and $0 < \Delta < 1$ there exists $\gamma > 0$ such that the following holds. Suppose that H is a k-graph on n vertices such that $n \in (k-\ell)\mathbb{N}$ is sufficiently large. If H is not Δ-extremal and satisfies $\delta_{k-1}(H) \ge (\frac{1}{2(k-\ell)} - \gamma)n$, then H contains a Hamilton ℓ-cycle.*

Theorem 4.5 (Extremal Case)**.** *For any integer $k \ge 3$, $1 \le \ell < k/2$ there exists $\Delta > 0$ such that the following holds. Suppose H is a k-graph on n vertices such that $n \in (k-\ell)\mathbb{N}$ is sufficiently large. If H is Δ-extremal and satisfies* (4.1)*, then H contains a Hamilton ℓ-cycle.*

Theorem 4.2 follows from Theorem 4.4 and 4.5 immediately by choosing Δ from Theorem 4.5.

Let us compare our proof with those in aforementioned papers. There is no extremal case in [18, 32, 40, 41] because only asymptotic results were proved. Our Theorem 4.5 is new and more general than [9, Theorem 3.1]. Following previous work [60, 62, 64, 18, 40], we prove Theorem 4.4 by using the absorbing method. More precisely, we find the desired Hamilton ℓ-cycle by applying the Absorbing Lemma (Lemma 4.6), the Reservoir Lemma (Lemma 4.7), and the Path-cover Lemma (Lemma 4.8). In fact, when $\ell < k/2$, the Absorbing Lemma and the Reservoir Lemma are not very difficult and already proven in [18] (in contrast, when $\ell > k/2$, the Absorbing Lemma in [40] is more difficult to prove). Thus the main step is to prove the Path-cover Lemma. As shown in [18, 40], after the Regularity Lemma is applied, it

suffices to prove that the cluster k-graph $\mathcal{K}$ can be tiled almost perfectly by the k-graph $\mathcal{F}_{k,\ell}$, whose vertex set consists of disjoint sets $A_1, \ldots, A_{a-1}, B$ of size $k-1$, and whose edges are all the k-sets of the form $A_i \cup \{b\}$ for $i = 1, \ldots, a-1$ and all $b \in B$, where $a = \lceil \frac{k}{k-\ell} \rceil (k-\ell)$. In this chapter we reduce the problem to tile $\mathcal{K}$ with a much simpler k-graph $\mathcal{Y}_{k,2\ell}$, which consists of two edges sharing 2ℓ vertices. Because of the simple structure of $\mathcal{Y}_{k,2\ell}$, we can easily find an almost perfect $\mathcal{Y}_{k,2\ell}$-tiling unless $\mathcal{K}$ is in the extremal case (thus the original k-graph H is in the extremal case). Interestingly $\mathcal{Y}_{3,2}$-tiling was studied in the very first paper [41] on loose Hamilton cycles but as a separate problem. Our recent project indeed used $\mathcal{Y}_{3,2}$-tiling as a tool to prove the corresponding path-cover lemma (see Chapter 5). On the other hand, the authors of [9] used a different approach (without the Regularity Lemma) to prove the Path-tiling Lemma (though they did not state such lemma explicitly).

We prove Theorem 4.4 in Section 4.2 and Theorem 4.5 in Section 4.3.

Extra notations. Given a k-graph H with two vertex sets S, R such that $|S| < k$, we denote by $\deg_H(S, R)$ the number of $(k-|S|)$-sets $T \subseteq R$ such that $S \cup T$ is an edge of H (in this case T is called a *neighbor* of S). We define $\overline{\deg}_H(S, R) = \binom{|R \setminus S|}{k-|S|} - \deg(S, R)$ as the number of *non-edges* on $S \cup R$ that contain S. When $R = V(H)$ (and H is obvious), we simply write $\deg(S)$ and $\overline{\deg}(S)$. When $S = \{v\}$, we use $\deg(v, R)$ instead of $\deg(\{v\}, R)$.

A k-graph $\mathcal{P}$ is an *ℓ-path* if there is an ordering $(v_1, \ldots, v_t)$ of its vertices such that every edge consists of k consecutive vertices and two consecutive edges intersect in exactly ℓ vertices. Note that this implies that $k-\ell$ divides $t-\ell$. In this case we write $\mathcal{P} = v_1 \cdots v_t$ and call two ℓ-sets $\{v_1, \ldots, v_\ell\}$ and $\{v_{t-\ell+1}, \ldots, v_t\}$ *ends* of $\mathcal{P}$.

4.2 Proof of Theorem 4.4

In this section we prove Theorem 4.4 by following the same approach as in [18].

4.2.1 Auxiliary lemmas and Proof of Theorem 4.4

We need [18, Lemma 5] and [18, Lemma 6] of Hàn and Schacht, in which any linear codegree is sufficient.

Lemma 4.6 (Absorbing lemma,[18]). *For all integers $k \geq 3$ and $1 \leq \ell < k/2$ and every $\gamma_1 > 0$ there exist $\eta > 0$ and an integer n_0 such that the following holds. Let H be a k-graph on $n \geq n_0$ vertices with $\delta_{k-1}(H) \geq \gamma_1 n$. Then there is an ℓ-path $\mathcal{P}$ with $|V(\mathcal{P})| \leq \gamma_1^5 n$ such that for all subsets $U \subset V \setminus V(\mathcal{P})$ of size $|U| \leq \eta n$ and $|U| \in (k-\ell)\mathbb{N}$ there exists an ℓ-path $\mathcal{Q} \subset H$ with $V(\mathcal{Q}) = V(\mathcal{P}) \cup U$ such that $\mathcal{P}$ and $\mathcal{Q}$ have exactly the same ends (we say $\mathcal{P}$ absorbs U in this case).*

Lemma 4.7 (Reservoir lemma, [18]). *For all integers $k \geq 3$ and $1 \leq \ell < k/2$ and every $d, \gamma_2 > 0$ there exists an n_0 such that the following holds. Let H be a k-graph on $n > n_0$ vertices with $\delta_{k-1}(H) \geq dn$, then there is a set R of size at most $\gamma_2 n$ such that for all $(k-1)$-sets $S \in \binom{V}{k-1}$ we have $\deg(S, R) \geq d\gamma_2 n/2$.*

The main step in our proof of Theorem 4.4 is the following lemma, which is stronger than [18, Lemma 7].

Lemma 4.8 (Path-cover lemma). *For all integers $k \geq 3$, $1 \leq \ell < k/2$, and every $\gamma_3, \alpha > 0$ there exist integers p and n_0 such that the following holds. Let H be a k-graph on $n > n_0$ vertices with $\delta_{k-1}(H) \geq (\frac{1}{2(k-\ell)} - \gamma_3)n$, then there is a family of at most p vertex disjoint ℓ-paths that together cover all but at most αn vertices of H, or H is $14\gamma_3$-extremal.*

We can now prove Theorem 4.4 in a similar way as in [18].

Proof of Theorem 4.4. Given $k \geq 3$, $1 \leq \ell < k/2$ and $0 < \Delta < 1$, let $\gamma = \min\{\frac{\Delta}{43}, \frac{1}{4k^2}\}$ and $n \in (k-\ell)\mathbb{N}$ be sufficiently large. Suppose that $H = (V, E)$ is a k-graph on n vertices with $\delta_{k-1}(H) \geq (\frac{1}{2(k-\ell)} - \gamma)n$. Since $\frac{1}{2(k-\ell)} - \gamma > \gamma$, we can apply Lemma 4.6 with $\gamma_1 = \gamma$ and obtain $\eta > 0$ and an absorbing path $\mathcal{P}_0$ with ends S_0, T_0 such that $\mathcal{P}_0$ can absorb any u vertices outside $\mathcal{P}_0$ if $u \leq \eta n$ and $u \in (k-\ell)\mathbb{N}$.

Let $V_1 = (V \setminus V(\mathcal{P}_0)) \cup S_0 \cup T_0$ and $H_1 = H[V_1]$. Note that $|V(\mathcal{P}_0)| \leq \gamma^5 n$ implies that $\delta_{k-1}(H_1) \geq (\frac{1}{2(k-\ell)} - \gamma)n - \gamma^5 n \geq \frac{1}{2k} n$ since $\gamma < \frac{1}{4k^2}$ and $\ell \geq 1$. We next apply Lemma 4.7 with $d = \frac{1}{2k}$ and $\gamma_2 = \min\{\eta/2, \gamma\}$ to H_1 and get a reservoir $R \subset V_1$ such that for any

$(k-1)$-set $S \subset V_1$, we have

$$\deg(S, R) \geq d\gamma_2|V_1|/2 \geq d\gamma_2 n/4. \tag{4.2}$$

Let $V_2 = V \setminus (V(\mathcal{P}_0) \cup R)$, $n_2 = |V_2|$, and $H_2 = H[V_2]$. Note that $|V(\mathcal{P}_0) \cup R| \leq \gamma_1^5 n + \gamma_2 n \leq 2\gamma n$, so

$$\delta_{k-1}(H_2) \geq \left(\frac{1}{2(k-\ell)} - \gamma\right) n - 2\gamma n \geq \left(\frac{1}{2(k-\ell)} - 3\gamma\right) n_2.$$

Applying Lemma 4.8 to H_2 with $\gamma_3 = 3\gamma$ and $\alpha = \eta/2$, we obtain at most p vertex disjoint ℓ-paths that cover all but at most αn_2 vertices of H_2, unless H_2 is $14\gamma_3$-extremal. In the latter case, there exists $B' \subseteq V_2$ such that $|B'| = \lfloor \frac{2k-2\ell-1}{2(k-\ell)} n_2 \rfloor$ and $e(B') \leq 42\gamma n_2^k$. Then we add at most $n - n_2 \leq 2\gamma n$ vertices from $V \setminus B'$ to B' and obtain a vertex set $B \subseteq V(H)$ such that $|B| = \lfloor \frac{2k-2\ell-1}{2(k-\ell)} n \rfloor$ and

$$e(B) \leq 42\gamma n_2^k + 2\gamma n \cdot \binom{n-1}{k-1} \leq 42\gamma n^k + \gamma n^k \leq \Delta n^k,$$

which means that H is Δ-extremal, a contradiction. In the former case, denote these ℓ-paths by $\{\mathcal{P}_i\}_{i\in[p']}$ for some $p' \leq p$, and their ends by $\{S_i, T_i\}_{i\in[p']}$. Note that both S_i and T_i are ℓ-sets for $\ell < k/2$. We arbitrarily pick disjoint $(k-2\ell-1)$-sets $X_0, X_1, \ldots, X_{p'} \subset R \setminus (S_0 \cup T_0)$ (note that $k - 2\ell - 1 \geq 0$). Let $T_{p'+1} = T_0$. By (4.2), we get for $0 \leq i \leq p'$,

$$\deg\left(S_i \cup T_{i+1} \cup X_i, R \setminus \bigcup_{0\leq i\leq p'} (S_i \cup T_i \cup X_i)\right) \geq d\gamma_2 n/4 - (p'+1)(k-1) \geq p+1,$$

as n is large enough. So we can connect $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{p'}$ by using vertices from R and get an ℓ-cycle $\mathcal{C}$. Note that $|V(H) \setminus V(\mathcal{C})| \leq |R| + \alpha n_2 \leq \gamma_2 n + \alpha n \leq \eta n$ and since $n \in (k-\ell)\mathbb{N}$, $|V \setminus V(\mathcal{C})|$ is also a multiple of $k - \ell$. So we can use $\mathcal{P}_0$ to absorb all unused vertices in R and uncovered vertices in V_2 thus obtaining a Hamilton ℓ-cycle in H. □

The rest of this section is devoted to the proof of Lemma 4.8.

4.2.2 Proof of Lemma 4.8

Let H be a k-partite k-graph with partition classes $V_1, \dots, V_k$. Then we call an ℓ-path $\mathcal{P}$ of H with edges $\{E_1, \dots, E_t\}$ *canonical* with respect to $(V_1, \dots, V_k)$ if

$$E_i \cap E_{i+1} \subseteq \bigcup_{j \in [\ell]} V_j \quad \text{or} \quad E_i \cap E_{i+1} \subseteq \bigcup_{j \in [2\ell] \setminus [\ell]} V_j$$

for $i = 1, \dots, t-1$. Note that a canonical ℓ-path with an odd length t contains $\frac{t+1}{2}$ vertices of V_i for $i \in [2\ell]$ and t vertices of V_i for $i > 2\ell$.

We also need the following proposition from [18].

Proposition 4.9. *[18, Proposition 19] Suppose H is a k-partite, k-graph with partition classes $V_1, \dots, V_k$, $|V_i| = m$ for all $i \in [k]$, and $|E(H)| \geq dm^k$. Then there exists a canonical ℓ-path in H with $t > \frac{dm}{2(k-\ell)}$ edges.*

In [18] the authors used Proposition 4.9 to cover an (ϵ, d)-regular tuple $(V_1, \dots, V_k)$ of sizes $|V_1| = \cdots = |V_{k-1}| = (2k - 2\ell - 1)m$ and $|V_k| = (k-1)m$ with vertex disjoint ℓ-paths. Our next lemma shows that an (ϵ, d)-regular tuple $(V_1, \dots, V_k)$ of sizes $|V_1| = \cdots = |V_{2\ell}| = m$ and $|V_i| = 2m$ for $i > 2\ell$ can be covered with ℓ-paths.

Lemma 4.10. *Fix $k \geq 3$, $1 \leq \ell < k/2$ and $\epsilon, d > 0$ such that $d > 2\epsilon$. Let $m > \frac{2k^2}{\epsilon^2(d-\epsilon)}$. Suppose $\mathcal{V} = (V_1, V_2, \dots, V_k)$ is an (ϵ, d)-regular k-tuple with*

$$|V_1| = \cdots = |V_{2\ell}| = m \quad \text{and} \quad |V_{2\ell+1}| = \cdots = |V_k| = 2m. \tag{4.3}$$

Then there are at most $\frac{4(k-\ell)}{(d-\epsilon)\epsilon}$ vertex disjoint ℓ-paths that together cover all but at most $2k\epsilon m$ vertices of $\mathcal{V}$.

Proof. We greedily find disjoint canonical ℓ-paths of odd length by Proposition 4.9 in $\mathcal{V}$ until less than ϵm vertices are uncovered in V_1. Suppose that we have obtained odd ℓ-paths $\mathcal{P}_1, \dots, \mathcal{P}_p$ by Proposition 4.9 for some $p \geq 0$. Let $t = \sum_{j=1}^{p} e(\mathcal{P}_j)$. Since all $e(\mathcal{P}_j)$ are odd, $\bigcup_{j=1}^{p} \mathcal{P}_i$ contains $\frac{t+p}{2}$ vertices of V_i for $i \in [2\ell]$ and t vertices of V_i for $i > 2\ell$. For $i \in [k]$, let

U_i be the set of uncovered vertices of V_i and assume that $|U_1| \geq \epsilon m$. Using (4.3), we derive that $|U_1| = \cdots = |U_{2\ell}| \geq \epsilon m$ and

$$|U_{2\ell+1}| = \cdots = |U_k| = 2|U_1| + p. \tag{4.4}$$

We pick an arbitrary k-partite subhypergraph $\mathcal{V}'$ with $|U_1|$ vertices in each U_i for $i \in [k]$. By regularity, $\mathcal{V}'$ contains at least $(d-\epsilon)|U_1|^k$ edges so that we can apply Proposition 4.9 and find an ℓ-path of odd length at least $\frac{(d-\epsilon)\epsilon m}{2(k-\ell)} - 1$ (dismiss one edge if needed). We continue this process until $|U_1| < \epsilon m$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_p$ be the ℓ-paths obtained in $\mathcal{V}$ after the iteration stops. Since $|V_1 \cap V(\mathcal{P}_j)| \geq \frac{(d-\epsilon)\epsilon m}{4(k-\ell)}$ for every j, we have

$$p \leq \frac{m}{\frac{(d-\epsilon)\epsilon m}{4(k-\ell)}} = \frac{4(k-\ell)}{(d-\epsilon)\epsilon}.$$

Since $m > \frac{2k^2}{\epsilon^2(d-\epsilon)}$, we further have

$$p(k-2\ell) \leq \frac{4(k-\ell)(k-2\ell)}{(d-\epsilon)\epsilon} < \frac{4k^2}{(d-\epsilon)\epsilon} < 2\epsilon m.$$

By (4.4), the total number of uncovered vertices in $\mathcal{V}$ is

$$\begin{aligned}\sum_{i=1}^{k} |U_i| &= |U_1|2\ell + (2|U_1| + p)(k-2\ell) = 2(k-\ell)|U_1| + p(k-2\ell) \\ &< 2(k-1)\epsilon m + 2\epsilon m = 2k\epsilon m.\end{aligned}$$

□

Given $k \geq 3$ and $1 \leq b < k$, recall that $\mathcal{Y}_{k,b}$ is a k-graph with two edges that share exactly b vertices. The following lemma is the main step in our proof of Lemma 4.8 and we prove it in the next subsection. Note that it generalizes [7, Lemma 3.1] of Czygrinow, DeBiasio, and Nagle.

Lemma 4.11 ($\mathcal{Y}_{k,b}$-tiling Lemma)**.** *Given integers $k \geq 3$, $1 \leq b < k$ and constants $\gamma, \beta > 0$, there exist $0 < \epsilon' < \gamma\beta$ and an integer n_0 such that the following holds. Suppose H is a k-*

graph on $n > n_0$ vertices with $\deg(S) \geq (\frac{1}{2k-b} - \gamma)n$ *for all but at most* $\epsilon' n^{k-1}$ *sets* $S \in \binom{V}{k-1}$, *then there is a* $\mathcal{Y}_{k,b}$*-tiling that covers all but at most* βn *vertices of* H *unless* H *contains a vertex set* B *such that* $|B| = \lfloor \frac{2k-b-1}{2k-b} n \rfloor$ *and* $e(B) < 6\gamma n^k$.

Now we are ready to prove Lemma 4.8.

Proof of Lemma 4.8. Fix such integers k, ℓ, $0 < \gamma_3, \alpha < 1$. Let ϵ' be the constant returned from Lemma 4.11 with $b = 2\ell$, $\gamma = 2\gamma_3$, and $\beta = \alpha/2$. So $\epsilon' < \gamma\beta = \gamma_3\alpha$. Furthermore, let $p = \frac{4T_0}{(d-\epsilon)\epsilon}$, where T_0 is the constant returned from Corollary 2.2 with $c = \frac{1}{2(k-\ell)} - \gamma_3$, $\epsilon = (\epsilon')^2/16$, and $d = \gamma_3/2$.

Let n be a sufficiently large integer and let H be a k-graph on n vertices with $\delta_{k-1}(H) \geq (\frac{1}{2(k-\ell)} - \gamma_3)n$. By applying Corollary 2.2 with the constants chosen above we obtain an ϵ-regular partition and a cluster hypergraph $\mathcal{K} = \mathcal{K}(\epsilon, d)$ such that for all but at most $\sqrt{\epsilon}t^{k-1}$ $(k-1)$-sets $S \in \binom{[t]}{k-1}$,

$$\deg_{\mathcal{K}}(S) \geq \left(\frac{1}{2(k-\ell)} - \gamma_3 - 2d\right) t = \left(\frac{1}{2(k-\ell)} - 2\gamma_3\right) t,$$

because $d = \gamma_3/2$. Let m be the size of each cluster except V_0, then $(1-\epsilon)\frac{n}{t} \leq m \leq \frac{n}{t}$. Applying Lemma 4.11 with the constants chosen above, we derive that either there is a $\mathcal{Y}_{k,2\ell}$-tiling $\mathscr{Y}$ of $\mathcal{K}$ which covers all but at most βt vertices of $\mathcal{K}$ or there exists a set $B \subseteq V(\mathcal{K})$, such that $|B| = \lfloor \frac{2k-2\ell-1}{2(k-\ell)} t \rfloor$ and $e_{\mathcal{K}}(B) \leq 12\gamma_3 t^k$. In the latter case, let $B' \subseteq V(H)$ be the union of the clusters in B. By regularity,

$$e_H(B') \leq e_{\mathcal{K}}(B) \cdot m^k + \binom{t}{k} \cdot d \cdot m^k + \epsilon \cdot \binom{t}{k} \cdot m^k + \binom{m}{2}\binom{n}{k-2},$$

where the right-hand side bounds the number of edges from regular k-tuples with high density, edges from regular k-tuples with low density, edges from irregular k-tuples and edges that lie in at most $k-1$ clusters. Since $m \leq \frac{n}{t}$, $\epsilon < \gamma_3$, $d = \gamma_3/2$, and $t^{-2} < t_0^{-2} < \gamma_3$,

we obtain that

$$e_H(B') \le 12\gamma_3 t^k \cdot \binom{n}{t}^k + \binom{t}{k} d \binom{n}{t}^k + \epsilon \binom{t}{k} \binom{n}{t}^k + \binom{n/t}{2} \binom{n}{k-2} < 13\gamma_3 n^k.$$

Note that $|B'| = \lfloor \frac{2k-2\ell-1}{2(k-\ell)} t \rfloor m \le \frac{2k-2\ell-1}{2(k-\ell)} t \cdot \frac{n}{t} = \frac{2k-2\ell-1}{2(k-\ell)} n$, and consequently $|B'| \le \lfloor \frac{2k-2\ell-1}{2(k-\ell)} n \rfloor$. On the other hand,

$$\begin{aligned} |B'| &= \left\lfloor \frac{2k-2\ell-1}{2(k-\ell)} t \right\rfloor m \ge \left(\frac{2k-2\ell-1}{2(k-\ell)} t - 1 \right) (1-\epsilon) \frac{n}{t} \\ &\ge \left(\frac{2k-2\ell-1}{2(k-\ell)} t - \epsilon t \right) \frac{n}{t} = \frac{2k-2\ell-1}{2(k-\ell)} n - \epsilon n. \end{aligned}$$

By adding at most ϵn vertices from $V \setminus B'$ to B', we get a set $B'' \subseteq V(H)$ of size exactly $\lfloor \frac{2k-2\ell-1}{2(k-\ell)} n \rfloor$, with $e(B'') \le e(B') + \epsilon n \cdot n^{k-1} < 14\gamma_3 n^k$. Hence H is $14\gamma_3$-extremal.

In the former case, the union of the clusters covered by $\mathscr{Y}$ contains all but at most $\beta t m + |V_0| \le \alpha n/2 + \epsilon n$ vertices. We apply Lemma 4.10 to each member $\mathcal{Y}' \in \mathscr{Y}$. Suppose that $\mathcal{Y}'$ has the vertex set $[2k-2\ell]$ with edges $\{1, \dots, k\}$ and $\{k-2\ell+1, \dots, 2k-2\ell\}$. For $i \in [2k-2\ell]$, let W_i denote the corresponding cluster in H. We split each W_i, $i = k-2\ell+1, \dots, k$, into two disjoint sets W_i^1 and W_i^2 of equal size. Then the k-tuples $(W_{k-2\ell+1}^1, \dots, W_k^1, W_1, \dots, W_{k-2\ell})$ and $(W_{k-2\ell+1}^2, \dots, W_k^2, W_{k+1}, \dots, W_{2k-2\ell})$ are $(2\epsilon, d)$-regular and of sizes $\frac{m}{2}, \dots, \frac{m}{2}, m, \dots, m$. Applying Lemma 4.10 to these two k-tuples with $m' = \frac{m}{2}$, we find a family of disjoint loose paths in each k-tuple covering all but at most $2k\epsilon m' = k\epsilon m$ vertices.

Since $|\mathscr{Y}| \le \frac{t}{2k-2\ell}$, we obtain a path-tiling that consists of at most $2\frac{t}{2k-2\ell}\frac{4(k-\ell)}{(d-\epsilon)\epsilon} \le \frac{4T_0}{(d-\epsilon)\epsilon} = p$ paths and covers all but at most

$$2k\epsilon m \cdot \frac{t}{2k-2\ell} + \alpha n/2 + \epsilon n < 3\epsilon n + \alpha n/2 < \alpha n$$

vertices, where we use $2k - 2\ell > k$ and $\epsilon = (\epsilon')^2/16 < (\gamma_3\alpha)^2/16 < \alpha/6$. This completes the proof. □

4.2.3 Proof of Lemma 4.11

We first give an upper bound on the size of k-graphs containing no copy of $\mathcal{Y}_{k,b}$. Throughout the rest of the chapter, we frequently use the simple identity $\binom{m}{b}\binom{m-b}{k-b} = \binom{m}{k}\binom{k}{b}$, which holds for all integers $1 \le b \le k \le m$.

Fact 4.12. *Let $1 \le b < k \le m$. If H is a k-graph on m vertices containing no copy of $\mathcal{Y}_{k,b}$, then $e(H) < \binom{m}{k-1}$.*

Proof. Fix any b-set $S \subseteq V(H)$ and consider its link graph L_S. Since H contains no copy of $\mathcal{Y}_{k,b}$, any two edges of L_S intersect. By the Erdős–Ko–Rado Theorem [12], $|L_S| \le \binom{m-b-1}{k-b-1}$. Thus,

$$\begin{aligned} e(H) &\le \frac{1}{\binom{k}{b}}\binom{m}{b}\cdot\binom{m-b-1}{k-b-1} = \frac{1}{\binom{k}{b}}\binom{m}{b}\binom{m-b}{k-b}\frac{k-b}{m-b} = \binom{m}{k}\frac{k-b}{m-b} \\ &= \binom{m}{k-1}\frac{k-b}{k}\,\frac{m-k+1}{m-b} < \binom{m}{k-1}. \end{aligned}$$

□

Proof of Lemma 4.11. Given $\gamma, \beta > 0$, let $\epsilon' = \frac{\gamma\beta^{k-1}}{(k-1)!}$ and $n \in \mathbb{N}$ be sufficiently large. Let H be a k-graph on n vertices that satisfies $\deg(S) \ge (\frac{1}{2k-b} - \gamma)n$ for all but at most $\epsilon' n^{k-1}$ $(k-1)$-sets S. Fix a largest $\mathcal{Y}_{k,b}$-tiling $\mathscr{Y} = \{\mathcal{Y}_1, \dots, \mathcal{Y}_m\}$ and write $V_i = V(\mathcal{Y}_i)$ for $i \in [m]$. Let $V' = \bigcup_{i\in[m]} V_i$ and $U = V(H) \setminus V'$. Assume that $|U| > \beta n$ – otherwise we are done.

Let C be the set of vertices $v \in V'$ such that $\deg(v, U) \ge (2k-b)^2\binom{|U|}{k-2}$. We will show that $|C| \le \frac{n}{2k-b}$ and C covers almost all the edges of H, which implies that $H[V \setminus C]$ is sparse and H is in the extremal case. We first observe that every $\mathcal{Y}_i \in \mathscr{Y}$ contains at most one vertex in C. Suppose instead, two vertices $x, y \in V_i$ are both in C. Since $\deg(x, U) \ge (2k-b)^2\binom{|U|}{k-2} > \binom{|U|}{k-2}$, by Fact 4.12, there is a copy of $\mathcal{Y}_{k-1,b-1}$ in the link graph of x on U, which gives rise to $\mathcal{Y}'$, a copy of $\mathcal{Y}_{k,b}$ on $\{x\} \cup U$. Since the link graph of y on $U \setminus V(\mathcal{Y}')$ has at least

$$(2k-b)^2\binom{|U|}{k-2} - (2k-b-1)\binom{|U|}{k-2} > \binom{|U \setminus V(\mathcal{Y}')|}{k-2}$$

edges, we can find another copy of $\mathcal{Y}_{k,b}$ on $\{y\}\cup(U\setminus V(\mathcal{Y}'))$ by Fact 4.12. Replacing $\mathcal{Y}_i$ in $\mathscr{Y}$ with these two copies of $\mathcal{Y}_{k,b}$ creates a $\mathcal{Y}_{k,b}$-tiling larger than $\mathscr{Y}$, contradiction. Consequently,

$$\begin{aligned}\sum_{S\in\binom{U}{k-1}}\deg(S,V') &\le |C|\binom{|U|}{k-1}+|V'\setminus C|(2k-b)^2\binom{|U|}{k-2}\\ &< |C|\binom{|U|}{k-1}+(2k-b)^2n\binom{|U|}{k-2}\quad\text{because } |V'\setminus C|<n\\ &= \binom{|U|}{k-1}\left(|C|+\frac{(2k-b)^2n(k-1)}{|U|-k+2}\right).\end{aligned}\tag{4.5}$$

Second, by Fact 4.12, $e(U)\le\binom{|U|}{k-1}$ since $H[U]$ contains no copy of $\mathcal{Y}_{k,b}$, which implies

$$\sum_{S\in\binom{U}{k-1}}\deg(S,U)\le k\binom{|U|}{k-1}.\tag{4.6}$$

By the definition of ϵ', we have

$$\epsilon' n^{k-1}=\frac{\gamma\beta^{k-1}}{(k-1)!}n^{k-1}<\frac{\gamma|U|^{k-1}}{(k-1)!}<2\gamma\binom{|U|}{k-1},$$

since $|U|$ is large enough. At last, by the degree condition, we have

$$\sum_{S\in\binom{U}{k-1}}\deg(S)\ge\left(\binom{|U|}{k-1}-\epsilon' n^{k-1}\right)\left(\frac{1}{2k-b}-\gamma\right)n>(1-2\gamma)\binom{|U|}{k-1}\left(\frac{1}{2k-b}-\gamma\right)n,\tag{4.7}$$

Since $\deg(S)=\deg(S,U)+\deg(S,V')$, we combine (4.5), (4.6) and (4.7) and get

$$|C|>(1-2\gamma)\left(\frac{1}{2k-b}-\gamma\right)n-k-\frac{(2k-b)^2n(k-1)}{|U|-k+2}.$$

Since $|U|>16k^3/\gamma$, we get

$$\frac{(2k-b)^2n(k-1)}{|U|-k+2}<\frac{4k^3n}{|U|/2}<\gamma n/2,$$

As $2\gamma^2 n > k$ and $2k - b \geq 4$, it follows that $|C| > \left(\frac{1}{2k-b} - 2\gamma\right) n$.

Let I_C be the set of all $i \in [m]$ such that $V_i \cap C \neq \emptyset$. Since each V_i, $i \in I_C$, contains one vertex of C, we have

$$|I_C| = |C| \geq \left(\frac{1}{2k-b} - 2\gamma\right) n \geq m - 2\gamma n. \tag{4.8}$$

Let $A = (\bigcup_{i \in I_C} V_i \setminus C) \cup U$.

Claim 4.13. *$H[A]$ contains no copy of $\mathcal{Y}_{k,b}$, thus $e(A) \leq \binom{n}{k-1}$.*

Proof. The first half of the claim implies the second half by Fact 4.12. Suppose instead, $H[A]$ contains a copy of $\mathcal{Y}_{k,b}$, denoted by $\mathcal{Y}_0$. Note that $V(\mathcal{Y}_0) \not\subseteq U$ because $H[U]$ contains no copy of $\mathcal{Y}_{k,b}$. Without loss of generality, suppose that $V_1, \ldots, V_j$ contain the vertices of $\mathcal{Y}_0$ for some $j \leq 2k - b$. For $i \in [j]$, let c_i denote the unique vertex in $V_i \cap C$. We greedily construct vertex-disjoint copies of $\mathcal{Y}_{k,b}$ on $\{c_i\} \cup U$, $i \in [j]$ as follows. Suppose we have found $\mathcal{Y}'_1, \ldots, \mathcal{Y}'_i$ (copies of $\mathcal{Y}_{k,b}$) for some $i < j$. Let U_0 denote the set of the vertices of U covered by $\mathcal{Y}_0, \mathcal{Y}'_1, \ldots, \mathcal{Y}'_i$. Then $|U_0| \leq (i+1)(2k-b-1) \leq (2k-b)(2k-b-1)$. Since $\deg(c_{i+1}, U) \geq (2k-b)^2 \binom{|U|}{k-2}$, the link graph of c_{i+1} on $U \setminus U_0$ has at least

$$(2k-b)^2 \binom{|U|}{k-2} - |U_0| \binom{|U|}{k-2} > \binom{|U|}{k-2}$$

edges. By Fact 4.12, there is a copy of $\mathcal{Y}_{k,b}$ on $\{c_{i+1}\} \cup (U \setminus U_0)$. Let $\mathcal{Y}'_1, \ldots, \mathcal{Y}'_j$ denote the copies of $\mathcal{Y}_{k,b}$ constructed in this way. Replacing $\mathcal{Y}_1, \ldots, \mathcal{Y}_j$ in $\mathscr{Y}$ with $\mathcal{Y}_0, \mathcal{Y}'_1, \ldots, \mathcal{Y}'_j$ gives a $\mathcal{Y}_{k,b}$-tiling larger than $\mathscr{Y}$, contradiction. □

Note that the edges not incident to C are either contained in A or intersect some V_i, $i \notin I_C$. By (4.8) and Claim 4.13,

$$\begin{aligned} e(V \setminus C) &\leq e(A) + (2k-b) \cdot 2\gamma n \binom{n-1}{k-1} < \binom{n}{k-1} + (4k-2b)\gamma n \binom{n}{k-1} \\ &< 4k\gamma n \binom{n}{k-1} < \frac{4k}{(k-1)!} \gamma n^k \leq 6\gamma n^k, \end{aligned}$$

where the last inequality follows from $k \geq 3$. Since $|C| \leq \frac{n}{2k-b}$, we can pick a set $B \subseteq V \setminus C$ of order $\lfloor \frac{2k-b-1}{2k-b} n \rfloor$ such that $e(B) < 6\gamma n^k$. □

4.3 The Extremal Theorem

In this section we prove Theorem 4.5. Assume that $k \geq 3$, $1 \leq \ell < k/2$ and $0 < \Delta \ll 1$. Let $n \in (k-\ell)\mathbb{N}$ be sufficiently large. Let H be a k-graph on V of n vertices such that $\delta_{k-1}(H) \geq \frac{n}{2(k-\ell)}$. Furthermore, assume that H is Δ-extremal, namely, there is a set $B \subseteq V(H)$, such that $|B| = \lfloor \frac{(2k-2\ell-1)n}{2(k-\ell)} \rfloor$ and $e(B) \leq \Delta n^k$. Let $A = V \setminus B$. Then $|A| = \lceil \frac{n}{2(k-\ell)} \rceil$.

Let us give an outline of our proof first. We denote by A' and B' the sets of "typical" vertices of A and B, respectively. Let $V_0 = V \setminus (A' \cup B')$. It is not hard to see that $A' \approx A$, $B' \approx B$, and thus $V_0 \approx \emptyset$. In the ideal case when $V_0 = \emptyset$ and $|B'| = (2k-2\ell-1)|A'|$, we assign a cyclic order to the vertices of A', construct $|A'|$ copies of $\mathcal{Y}_{k,\ell}$ such that each copy contains one vertex of A' and $2k-\ell-1$ vertices of B', and any two consecutive copies of $\mathcal{Y}_{k,\ell}$ share exactly ℓ vertices of B'. This gives rise to the desired Hamilton ℓ-cycle of H. In the general case, we first construct an ℓ-path $\mathcal{Q}$ with ends L_0 and L_1 such that $V_0 \subseteq V(\mathcal{Q})$ and $|B_1| = (2k-2\ell-1)|A_1| + \ell$, where $A_1 = A' \setminus V(\mathcal{Q})$ and $B_1 = (B \setminus V(\mathcal{Q})) \cup L_0 \cup L_1$. Next we complete the Hamilton ℓ-cycle by constructing an ℓ-path on $A_1 \cup B_1$ with ends L_0 and L_1.

For the convenience of later calculations, we let $\epsilon_0 = 2k!e\Delta \ll 1$ and claim that $e(B) \leq \epsilon_0 \binom{|B|}{k}$. Indeed, since $2(k-\ell)-1 \geq k$, we have

$$\frac{1}{e} \leq \left(1 - \frac{1}{2(k-\ell)}\right)^{2(k-\ell)-1} \leq \left(1 - \frac{1}{2(k-\ell)}\right)^{k}.$$

Thus we get

$$e(B) \leq \frac{\epsilon_0}{2k!e} n^k \leq \epsilon_0 \left(1 - \frac{1}{2(k-\ell)}\right)^k \frac{n^k}{2k!} \leq \epsilon_0 \binom{|B|}{k}. \tag{4.9}$$

In general, given two disjoint vertex sets X and Y and two integers $i, j \geq 0$, a set $S \subset X \cup Y$ is called an X^iY^j-set if $|S \cap X| = i$ and $|S \cap Y| = j$. When X, Y are two disjoint subsets of $V(H)$ and $i + j = k$, we denote by $H(X^iY^j)$ the family of all edges

of H that are X^iY^j-sets, and let $e_H(X^iY^j) = |H(X^iY^j)|$ (the subscript may be omitted if it is clear from the context). We use $\overline{e}_H(X^iY^{k-i})$ to denote the number of non-edges among X^iY^{k-i}-sets. Given a set $L \subseteq X \cup Y$ with $|L \cap X| = l_1 \leq i$ and $|L \cap Y| = l_2 \leq k - i$, we define $\deg(L, X^iY^{k-i})$ as the number of edges in $H(X^iY^{k-i})$ that contain L, and $\overline{\deg}(L, X^iY^{k-i}) = \binom{|X|-l_1}{i-l_1}\binom{|Y|-l_2}{k-i-l_2} - \deg(L, X^iY^{k-i})$. Our earlier notation $\deg(S, R)$ may be viewed as $\deg(S, S^{|S|}(R \setminus S)^{k-|S|})$.

4.3.1 Classification of vertices

Let $\epsilon_1 = {\epsilon_0}^{1/3}$ and $\epsilon_2 = 2\epsilon_1^2$. Assume that the partition $V(H) = A \cup B$ satisfies that $|B| = \lfloor \frac{(2k-2\ell-1)n}{2(k-\ell)} \rfloor$ and (4.9). In addition, assume that $e(B)$ is the smallest among all such partitions. We now define

$$\begin{aligned} A' &:= \left\{ v \in V : \deg(v, B) \geq (1-\epsilon_1)\binom{|B|}{k-1} \right\}, \\ B' &:= \left\{ v \in V : \deg(v, B) \leq \epsilon_1\binom{|B|}{k-1} \right\}, \\ V_0 &:= V \setminus (A' \cup B'). \end{aligned}$$

Claim 4.14. *$A \cap B' \neq \emptyset$ implies that $B \subseteq B'$, and $B \cap A' \neq \emptyset$ implies that $A \subseteq A'$.*

Proof. First, assume that $A \cap B' \neq \emptyset$. Then there is some $u \in A$ such that $\deg(u, B) \leq \epsilon_1\binom{|B|}{k-1}$. If there exists some $v \in B \setminus B'$, namely, $\deg(v, B) > \epsilon_1\binom{|B|}{k-1}$, then we can switch u and v and form a new partition $A'' \cup B''$ such that $|B''| = |B|$ and $e(B'') < e(B)$, which contradicts the minimality of $e(B)$.

Second, assume that $B \cap A' \neq \emptyset$. Then some $u \in B$ satisfies that $\deg(u, B) \geq (1 - \epsilon_1)\binom{|B|}{k-1}$. Similarly, by the minimality of $e(B)$, we get that for any vertex $v \in A$, $\deg(v, B) \geq (1-\epsilon_1)\binom{|B|}{k-1}$, which implies that $A \subseteq A'$. □

Claim 4.15. *$\{|A \setminus A'|, |B \setminus B'|, |A' \setminus A|, |B' \setminus B|\} \leq \epsilon_2|B|$ and $|V_0| \leq 2\epsilon_2|B|$.*

Proof. First assume that $|B \setminus B'| > \epsilon_2|B|$. By the definition of B', we get that

$$e(B) > \frac{1}{k}\epsilon_1\binom{|B|}{k-1} \cdot \epsilon_2|B| > 2\epsilon_0\binom{|B|}{k},$$

which contradicts (4.9).

Second, assume that $|A \setminus A'| > \epsilon_2|B|$. Then by the definition of A', for any vertex $v \notin A'$, we have that $\overline{\deg}(v, B) > \epsilon_1\binom{|B|}{k-1}$. So we get

$$\overline{e}(AB^{k-1}) > \epsilon_2|B| \cdot \epsilon_1\binom{|B|}{k-1} = 2\epsilon_0|B|\binom{|B|}{k-1}.$$

Together with (4.9), this implies that

$$\begin{aligned}\sum_{S\in\binom{B}{k-1}} \overline{\deg}(S) &= k\overline{e}(B) + \overline{e}(AB^{k-1}) \\ &> k(1-\epsilon_0)\binom{|B|}{k} + 2\epsilon_0|B|\binom{|B|}{k-1} \\ &= ((1-\epsilon_0)(|B|-k+1) + 2\epsilon_0|B|)\binom{|B|}{k-1} > |B|\binom{|B|}{k-1}.\end{aligned}$$

where the last inequality holds because n is large enough. By the pigeonhole principle, there exists a set $S \in \binom{B}{k-1}$, such that $\overline{\deg}(S) > |B| = \lfloor\frac{(2k-2\ell-1)n}{2(k-\ell)}\rfloor$, contradicting (4.1).

Consequently,

$$\begin{aligned}&|A' \setminus A| = |A' \cap B| \leq |B \setminus B'| \leq \epsilon_2|B|, \\ &|B' \setminus B| = |A \cap B'| \leq |A \setminus A'| \leq \epsilon_2|B|, \\ &|V_0| = |A \setminus A'| + |B \setminus B'| \leq \epsilon_2|B| + \epsilon_2|B| = 2\epsilon_2|B|.\end{aligned}$$ □

4.3.2 Classification of ℓ-sets in B'

In order to construct our Hamilton ℓ-cycle, we need to connect two ℓ-paths. To make this possible, we want the ends of our ℓ-paths to be ℓ-sets in B' that have high degree in $H[A'B'^{k-1}]$. Formally, we call an ℓ-set $L \subset V$ *typical* if $\deg(L, B) \leq \epsilon_1\binom{|B|}{k-\ell}$, otherwise

atypical. We prove several properties related to typical ℓ-sets in this subsection.

Claim 4.16. *The number of atypical ℓ-sets in B is at most $\epsilon_2\binom{|B|}{\ell}$.*

Proof. Let m be the number of atypical ℓ-sets in B. By (4.9), we have

$$\frac{m\epsilon_1\binom{|B|}{k-\ell}}{\binom{k}{\ell}} \le e(B) \le \epsilon_0\binom{|B|}{k},$$

which gives that $m \le \frac{\epsilon_0\binom{k}{l}\binom{|B|}{k}}{\epsilon_1\binom{|B|}{k-\ell}} = \frac{\epsilon_2}{2}\binom{|B|-k+\ell}{\ell} < \epsilon_2\binom{|B|}{\ell}$. □

Claim 4.17. *Every typical ℓ-set $L \subset B'$ satisfies $\overline{\deg}(L, A'B'^{k-1}) \le 4k\epsilon_1\binom{|B'|-\ell}{k-\ell-1}|A'|$.*

Proof. Fix a typical ℓ-set $L \subset B'$, consider the following sum,

$$\sum_{L\subset D\subset B', |D|=k-1} \deg(D) = \sum_{L\subset D\subset B', |D|=k-1} (\deg(D, A') + \deg(D, B') + \deg(D, V_0)).$$

By (4.1), the left hand side is at least $\binom{|B'|-\ell}{k-\ell-1}|A|$. On the other hand,

$$\sum_{L\subset D\subset B', |D|=k-1} (\deg(D, B') + \deg(D, V_0)) \le (k-\ell)\deg(L, B') + \binom{|B'|-\ell}{k-\ell-1}|V_0|.$$

Since L is typical and $|B' \setminus B| \le \epsilon_2|B|$ (Claim 4.15), we have

$$\begin{aligned}\deg(L, B') &\le \deg(L, B) + |B' \setminus B|\binom{|B'|-1}{k-\ell-1} \\ &\le \epsilon_1\binom{|B|}{k-\ell} + \epsilon_2|B|\binom{|B'|-1}{k-\ell-1}.\end{aligned}$$

Since $\epsilon_2 \ll \epsilon_1$ and $||B| - |B'|| \le \epsilon_2|B|$, it follows that

$$(k-\ell)\deg(L, B') \le \epsilon_1|B|\binom{|B|-1}{k-\ell-1} + (k-\ell)\epsilon_2|B|\binom{|B'|-1}{k-\ell-1} \le 2\epsilon_1|B|\binom{|B'|-\ell}{k-\ell-1}.$$

Putting these together and using Claim 4.15, we obtain that

$$\begin{aligned}\sum_{L\subset D\subset B',|D|=k-1}\deg(D,A') &\geq \binom{|B'|-\ell}{k-\ell-1}(|A|-|V_0|)-2\epsilon_1|B|\binom{|B'|-\ell}{k-\ell-1}\\ &\geq \binom{|B'|-\ell}{k-\ell-1}(|A'|-3\epsilon_2|B|-2\epsilon_1|B|)\,.\end{aligned}$$

Note that $\deg(L,A'B'^{k-1})=\sum_{L\subset D\subset B',|D|=k-1}\deg(D,A')$. Since $|B|\leq(2k-2\ell-1)|A|\leq(2k-2\ell)|A'|$, we finally derive that

$$\deg(L,A'B'^{k-1})\geq\binom{|B'|-\ell}{k-\ell-1}(1-(2k-2\ell)(3\epsilon_2+2\epsilon_1))|A'|\geq(1-4k\epsilon_1)\binom{|B'|-\ell}{k-\ell-1}|A'|.$$

as desired. □

We next show that we can connect any two disjoint typical ℓ-sets of B' with an ℓ-path of length two while avoiding any given $\frac{n}{4(k-\ell)}$ vertices of V.

Claim 4.18. *Given two disjoint typical ℓ-sets L_1,L_2 in B' and a vertex set $U\subseteq V$ with $|U|\leq\frac{n}{4(k-\ell)}$, there exist a vertex $a\in A'\setminus U$ and a $(2k-3\ell-1)$-set $C\subset B'\setminus U$ such that $L_1\cup L_2\cup\{a\}\cup C$ spans an ℓ-path (of length two) ended at L_1,L_2.*

Proof. Fix two disjoint typical ℓ-sets L_1,L_2 in B'. Using Claim 4.15, we obtain that $|U|\leq\frac{n}{4(k-\ell)}\leq\frac{|A|}{2}<\frac{2}{3}|A'|$ and

$$\frac{n}{4(k-\ell)}\leq\frac{|B|+1}{2(2k-2\ell-1)}\leq\frac{(1+2\epsilon_2)|B'|}{2k}<\frac{|B'|}{k}.$$

Thus $|A'\setminus U|>\frac{|A'|}{3}$ and $|B'\setminus U|>\frac{k-1}{k}|B'|$. Consider a $(k-\ell)$-graph $\mathcal{G}$ on $(A'\cup B')\setminus U$ such that an $A'B'^{k-\ell-1}$-set T is an edge of $\mathcal{G}$ if and only if $T\cap U=\emptyset$ and T is a common neighbor of L_1 and L_2 in H. By Claim 4.17, we have

$$\begin{aligned}\overline{e}(\mathcal{G})&\leq 2\cdot 4k\epsilon_1\binom{|B'|-\ell}{k-\ell-1}|A'|<8k\epsilon_1\binom{\frac{k}{k-1}|B'\setminus U|}{k-\ell-1}\cdot 3|A'\setminus U|\\ &\leq 24k\epsilon_1\left(\frac{k}{k-1}\right)^{k-1}\binom{|B'\setminus U|}{k-\ell-1}|A'\setminus U|.\end{aligned}$$

Consequently $e(\mathcal{G}) > \frac{1}{2}\binom{|B'\setminus U|}{k-\ell-1}|A'\setminus U|$. Hence there exists a vertex $a \in A'\setminus U$ such that $\deg_{\mathcal{G}}(a) > \frac{1}{2}\binom{|B'\setminus U|}{k-\ell-1} > \binom{|B'\setminus U|}{k-\ell-2}$. By Fact 4.12, the link graph of a contains a copy of $\mathcal{Y}_{k-\ell-1,\ell-1}$ (two edges of the link graph sharing $\ell-1$ vertices). In other words, there exists a $(2k-3\ell-1)$-set $C \subset B'\setminus U$ such that $C\cup\{a\}$ contains two edges of $\mathcal{G}$ sharing ℓ vertices. Together with L_1, L_2, this gives rise to the desired ℓ-path (in H) of length two ended at L_1, L_2. □

The following claim shows that we can always extend a typical ℓ-set to an edge of H by adding one vertex from A' and $k-\ell-1$ vertices from B' such that every ℓ new vertices form a typical ℓ-set. This can be done even when at most $\frac{n}{4(k-\ell)}$ vertices of V are not available.

Claim 4.19. *Given a typical ℓ-set $L \subseteq B'$ and a set $U \subseteq V$ with $|U| \le \frac{n}{4(k-\ell)}$, there exists an $A'B'^{k-\ell-1}$-set $C \subset V\setminus U$ such that $L\cup C$ is an edge of H and every ℓ-subset of $C\cap B'$ is typical.*

Proof. First, since L is typical in B', by Claim 4.17, $\overline{\deg}(L, A'B'^{k-1}) \le 4k\epsilon_1\binom{|B'|-\ell}{k-\ell-1}|A'|$. Second, note that a vertex in A' is contained in $\binom{|B'|}{k-\ell-1}$ $A'B'^{k-\ell-1}$-sets, while a vertex in B' is contained in $|A'|\binom{|B'|-1}{k-\ell-2}$ $A'B'^{k-\ell-1}$-sets. It is easy to see that $|A'|\binom{|B'|-1}{k-\ell-2} < \binom{|B'|}{k-\ell-1}$ (as $|A'| \approx \frac{n}{2k-2\ell}$ and $|B'| \approx \frac{2k-2\ell-1}{2k-2\ell}n$). We thus derive that at most

$$|U|\binom{|B'|}{k-\ell-1} \le \frac{n}{4(k-\ell)}\binom{|B'|}{k-\ell-1}$$

$A'B'^{k-\ell-1}$-sets intersect U. Finally, by Claim 4.16, the number of atypical ℓ-sets in B is at most $\epsilon_2\binom{|B|}{\ell}$. Using Claim 4.15, we derive that the number of atypical ℓ-sets in B' is at most

$$\epsilon_2\binom{|B|}{\ell} + |B'\setminus B|\binom{|B'|-1}{\ell-1} \le 2\epsilon_2\binom{|B'|}{\ell} + \epsilon_2|B|\binom{|B'|-1}{\ell-1} < 3\ell\epsilon_2\binom{|B'|}{\ell}.$$

Hence at most $3\ell\epsilon_2\binom{|B'|}{\ell}|A'|\binom{|B'|-\ell}{k-2\ell-1}$ $A'B'^{k-\ell-1}$-sets contain an atypical ℓ-set. In summary, at most

$$4k\epsilon_1\binom{|B'|-\ell}{k-\ell-1}|A'| + \frac{n}{4(k-\ell)}\binom{|B'|}{k-\ell-1} + 3\ell\epsilon_2\binom{|B'|}{\ell}\binom{|B'|-\ell}{k-2\ell-1}|A'|$$

$A'B'^{k-\ell-1}$-sets fail some of the desired properties. Since $\epsilon_1, \epsilon_2 \ll 1$ and $|A'| \approx \frac{n}{2(k-\ell)}$, the desired $A'B'^{k-\ell-1}$-set always exists. □

4.3.3 Building a short path $\mathcal{Q}$

The following claim is the only place where we used the exact codegree condition (4.1).

Claim 4.20. *Suppose that $|A \cap B'| = q > 0$. Then there exists a family $\mathcal{P}_1$ of vertex-disjoint $2q$ edges in B', each of which contains two disjoint typical ℓ-sets.*

Proof. Let $|A \cap B'| = q > 0$. Since $A \cap B' \neq \emptyset$, by Claim 4.14, we have $B \subseteq B'$, and consequently $|B'| = \lfloor \frac{2k-2\ell-1}{2(k-\ell)} n \rfloor + q$. By Claim 4.15, we have $q \leq |A \setminus A'| \leq \epsilon_2 |B|$.

Let $\mathcal{B}$ denote the family of the edges in B' that contain two disjoint typical ℓ-sets. We derive a lower bound for $|\mathcal{B}|$ as follows. We first pick a $(k-1)$-subset of B (recall that $B \subseteq B'$) that contains no atypical ℓ-subset. Since $2\ell \leq k-1$, such a $(k-1)$-set contains two disjoint typical ℓ-sets. By Claim 4.16, there are at most $\epsilon_2 \binom{|B|}{\ell}$ atypical ℓ-sets in $B \cap B' = B$ and in turn, there are at most $\epsilon_2 \binom{|B|}{\ell} \binom{|B|-\ell}{k-\ell-1}$ $(k-1)$-subsets of B that contain an atypical ℓ-subset. Thus there are at least

$$\binom{|B|}{k-1} - \epsilon_2 \binom{|B|}{\ell} \binom{|B|-\ell}{k-\ell-1} = \left(1 - \binom{k-1}{\ell} \epsilon_2\right) \binom{|B|}{k-1}$$

$(k-1)$-subsets of B that contain no atypical ℓ-subset. After picking such a $(k-1)$-set $S \subset B$, we find a neighbor of S by the codegree condition. Since $|B'| = \lfloor \frac{2k-2\ell-1}{2(k-\ell)} n \rfloor + q$, by (4.1), we have $\deg(S, B') \geq q$. We thus derive that

$$|\mathcal{B}| \geq \left(1 - \binom{k-1}{\ell} \epsilon_2\right) \binom{|B|}{k-1} \frac{q}{k},$$

in which we divide by k because every edge of $\mathcal{B}$ is counted at most k times.

We claim that $\mathcal{B}$ contains $2q$ disjoint edges. Suppose instead, a maximum matching in

$\mathcal{B}$ has $i < 2q$ edges. By the definition of B, for any vertex $b \in B'$, we have

$$\begin{aligned}\deg(b, B') &\le \deg(b, B) + |B' \setminus B|\binom{|B'|-1}{k-2} \\ &\le \epsilon_1\binom{|B|}{k-1} + \epsilon_2|B|\binom{|B'|-1}{k-2} < 2\epsilon_1\binom{|B|}{k-1}. \end{aligned} \tag{4.10}$$

Thus at most $2qk \cdot 2\epsilon_1\binom{|B|}{k-1}$ edges of B' intersect the i edges in the matching. Hence, the number of edges of $\mathcal{B}$ that are disjoint from these i edges is at least

$$\frac{q}{k}\left(1 - \binom{k-1}{\ell}\epsilon_2\right)\binom{|B|}{k-1} - 4k\epsilon_1 q\binom{|B|}{k-1} \ge \left(\frac{1}{k} - (4k+1)\epsilon_1\right)q\binom{|B|}{k-1} > 0,$$

as $\epsilon_2 \ll \epsilon_1 \ll 1$. We may thus obtain a matching of size $i+1$, a contradiction. □

Claim 4.21. *There exists a non-empty ℓ-path $\mathcal{Q}$ in H with the following properties:*

- $V_0 \subseteq V(\mathcal{Q})$,
- $|V(\mathcal{Q})| \le 10k\epsilon_2|B|$,
- *two ends L_0, L_1 of $\mathcal{Q}$ are typical ℓ-sets in B',*
- $|B_1| = (2k - 2\ell - 1)|A_1| + \ell$, *where $A_1 = A' \setminus V(\mathcal{Q})$ and $B_1 = (B' \setminus V(\mathcal{Q})) \cup L_0 \cup L_1$.*

Proof. We split into two cases here.

Case 1. $A \cap B' \neq \emptyset$.

By Claim 4.14, $A \cap B' \neq \emptyset$ implies that $B \subseteq B'$. Let $q = |A \cap B'|$. We first apply Claim 4.20 and find a family $\mathcal{P}_1$ of vertex-disjoint $2q$ edges in B'. Next we associate each vertex of V_0 with $2k - \ell - 1$ vertices of B (so in B') forming an ℓ-path of length two such that these $|V_0|$ paths are pairwise vertex-disjoint, and also vertex-disjoint from the paths in $\mathcal{P}_1$, and all these paths have typical ends. To see it, let $V_0 = \{x_1, \ldots, x_{|V_0|}\}$. Suppose that we have found such ℓ-paths for $x_1, \ldots, x_{i-1}$ with $i \le |V_0|$. Since $B \subseteq B'$, it follows that $A \setminus A' = (A \cap B') \cup V_0$. Hence $|V_0| + q = |A \setminus A'| \le \epsilon_2|B|$ by Claim 4.15. Therefore

$$(2k - \ell - 1)(i-1) + |V(\mathcal{P}_1)| < 2k|V_0| + 2kq \le 2k\epsilon_2|B|$$

and consequently at most $2k\epsilon_2|B|\binom{|B|-1}{k-2} < 2k^2\epsilon_2\binom{|B|}{k-1}$ $(k-1)$-sets of B intersect the existing paths (including $\mathcal{P}_1$). By the definition of V_0, $\deg(x_i, B) > \epsilon_1\binom{|B|}{k-1}$. Let $\mathcal{G}_{x_i}$ be the $(k-1)$-graph on B such that $e \in \mathcal{G}_{x_i}$ if

- $\{x_i\} \cup e \in E(H)$,
- e does not contain any vertex from the existing paths,
- e does not contain any atypical ℓ-set.

By Claim 4.16, the number of $(k-1)$-sets in B containing at least one atypical ℓ-set is at most $\epsilon_2\binom{|B|}{\ell}\binom{|B|-\ell}{k-\ell-1} = \epsilon_2\binom{k-1}{\ell}\binom{|B|}{k-1}$. Thus, we have

$$e(\mathcal{G}_{x_i}) \geq \epsilon_1\binom{|B|}{k-1} - 2k^2\epsilon_2\binom{|B|}{k-1} - \epsilon_2\binom{k-1}{\ell}\binom{|B|}{k-1} > \frac{\epsilon_1}{2}\binom{|B|}{k-1} > \binom{|B|}{k-2},$$

because $\epsilon_2 \ll \epsilon_1$ and $|B|$ is sufficiently large. By Fact 4.12, $\mathcal{G}_{x_i}$ contains a copy of $\mathcal{Y}_{k-1,\ell-1}$, which gives the desired ℓ-path of length two containing x_i.

Denote by $\mathcal{P}_2$ the family of ℓ-paths we obtained so far. Now we need to connect paths of $\mathcal{P}_2$ together to a single ℓ-path. For this purpose, we apply Claim 4.18 repeatedly to connect the ends of two ℓ-paths while avoiding previously used vertices. This is possible because $|V(\mathcal{P}_2)| = (2k-\ell)|V_0| + 2kq$ and $(2k-3\ell)(|V_0| + 2q - 1)$ vertices are needed to connect all the paths in $\mathcal{P}_2$ – the set U (when we apply Claim 4.18) thus satisfies

$$|U| \leq (4k-4\ell)|V_0| + (6k-6\ell)q - 2k + 3\ell \leq 6(k-\ell)\epsilon_2|B| - 2k + 3\ell.$$

Let $\mathcal{P}$ denote the resulting ℓ-path. We have $|V(\mathcal{P}) \cap A'| = |V_0| + 2q - 1$ and

$$\begin{aligned} |V(\mathcal{P}) \cap B'| &= k \cdot 2q + (2k-\ell-1)|V_0| + (2k-3\ell-1)(|V_0| + 2q - 1) \\ &= 2(2k-2\ell-1)|V_0| + 2(3k-3\ell-1)q - (2k-3\ell-1). \end{aligned}$$

Let $s = (2k - 2\ell - 1)|A' \setminus V(\mathcal{P})| - |B' \setminus V(\mathcal{P})|$. We have

$$\begin{aligned} s &= (2k - 2\ell - 1)(|A'| - |V_0| - 2q + 1) - |B'| + 2(2k - 2\ell - 1)|V_0| + 2(3k - 3\ell - 1)q - (2k - 3\ell - 1) \\ &= (2k - 2\ell - 1)|A'| - |B'| + (2k - 2\ell - 1)|V_0| + (2k - 2\ell)q + \ell. \end{aligned}$$

Since $|A'| + |B'| + |V_0| = n$, we have

$$s = (2k - 2\ell)(|A'| + |V_0| + q) - n + \ell. \tag{4.11}$$

Note that $|A'| + |V_0| + q = |A|$ and

$$(2k - 2\ell)|A| - n = \begin{cases} 0, & \text{if } \frac{n}{k-\ell} \text{ is even} \\ k - \ell, & \text{if } \frac{n}{k-\ell} \text{ is odd.} \end{cases} \tag{4.12}$$

Thus $s = \ell$ or $s = k$. If $s = k$, then we extend $\mathcal{P}$ to an ℓ-path $\mathcal{Q}$ by applying Claim 4.19, otherwise let $\mathcal{Q} = \mathcal{P}$. Then

$$|V(\mathcal{Q})| \leq |V(\mathcal{P})| + (k - \ell) \leq 6k\epsilon_2|B|,$$

and $\mathcal{Q}$ has two typical ends $L_0, L_1 \subset B'$. We claim that

$$(2k - 2\ell - 1)|A' \setminus V(\mathcal{Q})| - |B' \setminus V(\mathcal{Q})| = \ell. \tag{4.13}$$

Indeed, when $s = \ell$, this is obvious; when $s = k$, $V(\mathcal{Q}) \setminus V(\mathcal{P})$ contains one vertex of A' and $k - \ell - 1$ vertices of B' and thus

$$(2k - 2\ell - 1)|A' \setminus V(\mathcal{Q})| - |B' \setminus V(\mathcal{Q})| = s - (2k - 2\ell - 1) + (k - \ell - 1) = \ell.$$

Let $A_1 = A' \setminus V(\mathcal{Q})$ and $B_1 = (B' \setminus V(\mathcal{Q})) \cup L_0 \cup L_1$. We derive that $|B_1| = (2k - 2\ell - 1)|A_1| + \ell$ from (4.13).

Case 2. $A \cap B' = \emptyset$.

Note that $A \cap B' = \emptyset$ means that $B' \subseteq B$. Then we have

$$|A'| + |V_0| = |V \setminus B'| = |A| + |B \setminus B'|. \tag{4.14}$$

If $V_0 \neq \emptyset$, we handle this case similarly as in Case 1 except that we do not need to construct $\mathcal{P}_1$. By Claim 4.15, $|B \setminus B'| \leq \epsilon_2 |B|$ and thus for any vertex $x \in V_0$,

$$\begin{aligned}\deg(x, B') &\geq \deg(x, B) - |B \setminus B'| \cdot \binom{|B|-1}{k-2} \\ &\geq \epsilon_1 \binom{|B|}{k-1} - (k-1)\epsilon_2 \binom{|B|}{k-1} > \frac{\epsilon_1}{2}\binom{|B'|}{k-1}.\end{aligned}$$

As in Case 1, we let $V_0 = \{x_1, \ldots, x_{|V_0|}\}$ and cover them with vertex-disjoint ℓ-paths of length two. Indeed, for each $i \leq |V_0|$, we construct $\mathcal{G}_x$ as before and show that $e(\mathcal{G}_{x_i}) \geq \frac{\epsilon_1}{4}\binom{|B'|}{k-1}$. We then apply Fact 4.12 to $\mathcal{G}_{x_i}$ obtaining a copy of $\mathcal{Y}_{k-1,\ell-1}$, which gives an ℓ-path of length two containing x_i. As in Case 1, we connect these paths to a single ℓ-path $\mathcal{P}$ by applying Claim 4.18 repeatedly. Then $|V(\mathcal{P})| = (2k-\ell)|V_0| + (2k-3\ell)(|V_0|-1)$. Define s as in Case 1. Thus (4.11) holds with $q = 0$. Applying (4.14) and (4.12), we derive that

$$s = 2(k-\ell)(|A| + |B \setminus B'|) - n + \ell = \begin{cases} \ell + 2(k-\ell)|B \setminus B'|, & \text{if } \frac{n}{k-\ell} \text{ is even} \\ k + 2(k-\ell)|B \setminus B'|, & \text{if } \frac{n}{k-\ell} \text{ is odd,} \end{cases} \tag{4.15}$$

which implies that $s \equiv \ell \mod (k-\ell)$. We extend $\mathcal{P}$ to an ℓ-path $\mathcal{Q}$ by applying Claim 4.19 $\frac{s-\ell}{k-\ell}$ times. Then

$$|V(\mathcal{Q})| = |V(\mathcal{P})| + s - \ell \leq (4k-4\ell)|V_0| - 2k + 3\ell + k - \ell + 2(k-\ell)|B \setminus B'| \leq 10k\epsilon_2|B|$$

by Claim 4.15. Note that $\mathcal{Q}$ has two typical ends $L_0, L_1 \subset B'$. Since $V(\mathcal{Q}) \setminus V(\mathcal{P})$ contains

$\frac{s-\ell}{k-\ell}$ vertices of A' and $\frac{s-\ell}{k-\ell}(k-\ell-1)$ vertices of B', we have

$$(2k-2\ell-1)|A' \setminus V(\mathcal{Q})| - |B' \setminus V(\mathcal{Q})| = s - \frac{s-\ell}{k-\ell}(2k-2\ell-1) + \frac{s-\ell}{k-\ell}(k-\ell-1) = \ell.$$

We define A_1 and B_1 in the same way and similarly we have $|B_1| = (2k-2\ell-1)|A_1| + \ell$.

When $V_0 = \emptyset$, we pick an arbitrary vertex $v \in A'$ and form an ℓ-path $\mathcal{P}$ of length two with typical ends such that v is in the intersection of the two edges. This is possible by the definition of A'. Define s as in Case 1. It is easy to see that (4.15) still holds. We then extend $\mathcal{P}$ to $\mathcal{Q}$ by applying Claim 4.19 $\frac{s-\ell}{k-\ell}$ times. Then $|V(\mathcal{Q})| = 2k - \ell + s - \ell \leq 2k\epsilon_2|B|$ because of (4.15). The rest is the same as in the previous case. □

Claim 4.22. *The A_1, B_1 and L_0, L_1 defined in Claim 4.21 satisfy the following properties:*

1. $|B_1| \geq (1-\epsilon_1)|B|$,
2. *for any vertex $v \in A_1$,* $\overline{\deg}(v, B_1) < 3\epsilon_1\binom{|B_1|}{k-1}$,
3. *for any vertex $v \in B_1$,* $\overline{\deg}(v, A_1B_1^{k-1}) \leq 3k\epsilon_1\binom{|B_1|}{k-1}$,
4. $\overline{\deg}(L_0, A_1B_1^{k-1}) \leq 5k\epsilon_1\binom{|B_1|}{k-\ell}$, $\overline{\deg}(L_1, A_1B_1^{k-1}) \leq 5k\epsilon_1\binom{|B_1|}{k-\ell}$.

Proof. Part (1): By Claim 4.15, we have $|B_1 \setminus B| \leq |B' \setminus B| \leq \epsilon_2|B|$. Furthermore,

$$|B_1| \geq |B'| - |V(\mathcal{Q})| \geq |B| - \epsilon_2|B| - 10k\epsilon_2|B| \geq (1-\epsilon_1)|B|.$$

Part (2): For a vertex $v \in A_1$, since $\overline{\deg}(v, B) \leq \epsilon_1\binom{|B|}{k-1}$, we have

$$\begin{aligned}
\overline{\deg}(v, B_1) &\leq \overline{\deg}(v, B) + |B_1 \setminus B|\binom{|B_1|-1}{k-2} \\
&\leq \epsilon_1\binom{|B|}{k-1} + \epsilon_2|B|\binom{|B_1|-1}{k-2} \\
&< \epsilon_1\binom{|B|}{k-1} + \epsilon_1\binom{|B_1|}{k-1} < 3\epsilon_1\binom{|B_1|}{k-1},
\end{aligned}$$

where the last inequality follows from Part (1).

Part (3): Consider the sum $\sum \deg(S\cup\{v\})$ taken over all $S \in \binom{B'\setminus\{v\}}{k-2}$. Since $\delta_{k-1}(H) \geq |A|$, we have $\sum \deg(S\cup\{v\}) \geq \binom{|B'|-1}{k-2}|A|$. On the other hand,

$$\sum \deg(S\cup\{v\}) = \deg(v, A'B'^{k-1}) + \deg(v, V_0B'^{k-1}) + (k-1)\deg(v, B').$$

By (4.10), $\deg(v, B') \leq 2\epsilon_1\binom{|B|}{k-1}$. We thus derive that

$$\begin{aligned}
\deg(v, A'B'^{k-1}) &\geq \binom{|B'|-1}{k-2}|A| - \deg(v, V_0B'^{k-1}) - (k-1)\deg(v, B') \\
&\geq \binom{|B'|-1}{k-2}(|A'| - \epsilon_2|B|) - 2\epsilon_2|B|\binom{|B'|-1}{k-2} - 2(k-1)\epsilon_1\binom{|B|}{k-1} \\
&\geq \binom{|B'|-1}{k-2}|A'| - 2k\epsilon_1\binom{|B|}{k-1}.
\end{aligned}$$

Thus, by Part (1), we have

$$\overline{\deg}(v, A_1B_1^{k-1}) \leq \overline{\deg}(v, A'B'^{k-1}) \leq 2k\epsilon_1\binom{|B|}{k-1} \leq 3k\epsilon_1\binom{|B_1|}{k-1}.$$

Part (4): By Claim 4.17, for any typical $L \subseteq B'$, we have $\overline{\deg}(L, A'B'^{k-1}) \leq 4k\epsilon_1\binom{|B'|-\ell}{k-\ell-1}|A'|$. Thus,

$$\overline{\deg}(L_0, A_1B_1^{k-1}) \leq \overline{\deg}(L_0, A'B'^{k-1}) \leq 4k\epsilon_1\binom{|B'|-\ell}{k-\ell-1}|A'| \leq 5k\epsilon_1\binom{|B_1|}{k-\ell},$$

where the last inequality holds because $|B'| \leq |B_1| + |V(\mathcal{Q})| \leq (1+\epsilon_1)|B_1|$. The same holds for L_1. □

4.3.4 Completing the Hamilton cycle

We finally complete the proof of Theorem 4.5 by applying the following lemma with $X = A_1$, $Y = B_1$, $\rho = 5k\epsilon_1$, and L_0, L_1.

Lemma 4.23. *Fix $1 \leq \ell < k/2$. Let $0 < \rho \ll 1$ and n be sufficiently large. Suppose that H is a k-graph with a partition $V(H) = X \cup Y$ and the following properties:*

- $|Y| = (2k - 2\ell - 1)|X| + \ell$,
- *for every vertex* $v \in X$, $\overline{\deg}(v, Y) \leq \rho\binom{|Y|}{k-1}$ *and for every vertex* $v \in Y$, $\overline{\deg}(v, XY^{k-1}) \leq \rho\binom{|Y|}{k-1}$,
- *there are two disjoint* ℓ*-sets* $L_0, L_1 \subset Y$ *such that*

$$\overline{\deg}(L_0, XY^{k-1}), \overline{\deg}(L_1, XY^{k-1}) \leq \rho\binom{|Y|}{k-\ell}. \tag{4.16}$$

Then H *contains a Hamilton* ℓ*-path with* L_0 *and* L_1 *as ends.*

In order to prove Lemma 4.23, we apply two results of Glebov, Person, and Weps [15]. Given $1 \leq l \leq k-1$ and $0 \leq \rho \leq 1$, an ordered set $(x_1, \ldots, x_l)$ is *ρ-typical* in a k-graph $\mathcal{G}$ if for every $i \in [l]$

$$\overline{\deg}_{\mathcal{G}}(\{x_1, \ldots, x_i\}) \leq \rho^{k-i}\binom{|V(\mathcal{G})| - i}{k-i}.$$

It was shown in [15] that every k-graph $\mathcal{G}$ with very large minimum vertex degree contains a tight Hamilton cycle. The proof of [15, Theorem 2] actually shows that we can obtain a tight Hamilton cycle by extending any fixed tight path of constant length with two typical ends. This implies the following theorem that we will use.

Theorem 4.24. *[15] Given* $1 \leq l \leq k$ *and* $0 < \alpha \ll 1$*, there exists an* m_0 *such that the following holds. Suppose that* $\mathcal{G}$ *is a* k*-graph on* V *with* $|V| = m \geq m_0$ *and* $\delta_1(\mathcal{G}) \geq (1-\alpha)\binom{m-1}{k-1}$*. Then given any two* $(22\alpha)^{\frac{1}{k-1}}$*-typical ordered* l*-sets* $(x_1, \ldots, x_l)$ *and* $(y_1, \ldots, y_l)$*, there exists a tight Hamilton path* $\mathcal{P} = x_l x_{l-1} \cdots x_1 \cdots\cdots y_1 y_2 \cdots y_l$ *in* $\mathcal{G}$*.*

We also use [15, Lemma 3], in which V^{2k-2} denotes the set of all $(2k-2)$-tuples $(v_1, \ldots, v_{2k-2})$ such that $v_i \in V$ (v_i's are not necessarily distinct).

Lemma 4.25. *[15] Let* $\mathcal{G}$ *be the* k*-graph given in Lemma 4.24. Suppose that* $(x_1, \ldots, x_{2k-2})$ *is selected uniformly at random from* V^{2k-2}*. Then the probability that all* x_i*'s are pairwise distinct and* $(x_1, \ldots, x_{k-1}), (x_k, \ldots, x_{2k-2})$ *are* $(22\alpha)^{\frac{1}{k-1}}$*-typical is at least* $\frac{8}{11}$*.*

Proof of Lemma 4.23. In this proof we often write the union $A \cup B \cup \{x\}$ as ABx, where A, B are sets and x is an element.

Let $t = |X|$. Our goal is to write X as $\{x_1, \ldots, x_t\}$ and partition Y as $\{L_i, R_i, S_i, R_i' : i \in [t]\}$ with $|L_i| = \ell$, $|R_i| = |R_i'| = k - 2\ell$, and $|S_i| = \ell - 1$ such that

$$L_i R_i S_i x_i,\ S_i x_i R_i' L_{i+1} \in E(H) \tag{4.17}$$

for all $i \in [t]$, where $L_{t+1} = L_0$. Consequently

$$L_1\, R_1\, S_1\, x_1\, R_1'\, L_2\, R_2\, S_2\, x_2\, R_2'\, \cdots\, L_t\, R_t\, S_t\, x_t\, R_t'\, L_{t+1}$$

is the desired Hamilton ℓ-path of H.

Let $\mathcal{G}$ be the $(k-1)$-graph on Y whose edges are all $(k-1)$-sets $S \subseteq Y$ such that $\deg_H(S, X) > (1 - \sqrt{\rho})t$. The following is an outline of our proof. We first find a small subset $Y_0 \subset Y$ with a partition $\{L_i, R_i, S_i, R_i' : i \in [t_0]\}$ such that for every $x \in X$, we have $L_i R_i S_i x, S_i x R_i' L_{i+1} \in E(H)$ for many $i \in [t_0]$. Next we apply Theorem 4.24 to $\mathcal{G}[Y \setminus Y_0]$ and obtain a tight Hamilton path, which, in particular, partitions $Y \setminus Y_0$ into $\{L_i, R_i, S_i, R_i' : t_0 < i \le t\}$ such that $L_i R_i S_i,\ S_i R_i' L_{i+1} \in E(\mathcal{G})$ for $t_0 < i \le t$. Finally we apply the Marriage Theorem to find a perfect matching between X and $[t]$ such that (4.17) holds for all matched x_i and i.

We now give details of the proof. First we claim that

$$\delta_1(\mathcal{G}) \ge (1 - 2\sqrt{\rho})\binom{|Y| - 1}{k - 2}, \tag{4.18}$$

and consequently,

$$\overline{e}(\mathcal{G}) \le 2\sqrt{\rho}\binom{|Y|}{k-1}. \tag{4.19}$$

Suppose instead, some vertex $v \in Y$ satisfies $\overline{\deg}_{\mathcal{G}}(v) > 2\sqrt{\rho}\binom{|Y|-1}{k-2}$. Since every non-neighbor S' of v in $\mathcal{G}$ satisfies $\overline{\deg}_H(S'v, X) \ge \sqrt{\rho}t$, we have $\overline{\deg}_H(v, XY^{k-1}) > 2\sqrt{\rho}\binom{|Y|-1}{k-2}\sqrt{\rho}t$. Since

$|Y| = (2k-2\ell-1)t+\ell$, we have

$$\overline{\deg}_H(v, XY^{k-1}) > 2\rho\frac{|Y|-\ell}{2k-2\ell-1}\binom{|Y|-1}{k-2} > \rho\frac{|Y|}{k-1}\binom{|Y|-1}{k-2} = \rho\binom{|Y|}{k-1},$$

contradicting our assumption (the second inequality holds because $|Y|$ is sufficiently large).

Let Q be a $(2k-\ell-1)$-subset of Y. We call Q *good* (otherwise *bad*) if every $(k-1)$-subset of Q is an edge of $\mathcal{G}$ and every ℓ-set $L \subset Q$ satisfies

$$\overline{\deg}_{\mathcal{G}}(L) \le \rho^{1/4}\binom{|Y|-\ell}{k-\ell-1}. \tag{4.20}$$

Furthermore, we say Q is *suitable* for a vertex $x \in X$ if $x \cup T \in E(H)$ for every $(k-1)$-set $T \subset Q$. Note that if a $(2k-\ell-1)$-set is good, by the definition of $\mathcal{G}$, it is suitable for at least $(1-\binom{2k-\ell-1}{k-1}\sqrt{\rho})t$ vertices of X. Let $Y' = Y \setminus (L_0 \cup L_1)$.

Claim 4.26. *For any $x \in X$, at least $(1-\rho^{1/5})\binom{|Y|}{2k-\ell-1}$ $(2k-\ell-1)$-subsets of Y' are good and suitable for x.*

Proof. Since $\rho + \rho^{1/2} + 3\binom{2k-\ell-1}{\ell}\rho^{1/4} \le \rho^{1/5}$, the claim follows from the following three assertions:

- At most $2\ell\binom{|Y|-1}{2k-\ell-2} \le \rho\binom{|Y|}{2k-\ell-1}$ $(2k-\ell-1)$-subsets of Y are not subsets of Y'.
- Given $x \in X$, at most $\rho^{1/2}\binom{|Y|}{2k-\ell-1}$ $(2k-\ell-1)$-sets in Y are not suitable for x.
- At most $3\binom{2k-\ell-1}{\ell}\rho^{1/4}\binom{|Y|}{2k-\ell-1}$ $(2k-\ell-1)$-sets in Y are bad.

The first assertion holds because $|Y \setminus Y'| = 2\ell$. The second assertion follows from the degree condition of H, namely, for any $x \in X$, the number of $(2k-\ell-1)$-sets in Y that are not suitable for x is at most $\rho\binom{|Y|}{k-1}\binom{|Y|-k+1}{k-\ell} \le \sqrt{\rho}\binom{|Y|}{2k-\ell-1}$.

To see the third one, let m be the number of ℓ-sets $L \subseteq Y$ that fail (4.20). By (4.19),

$$m\frac{\rho^{1/4}\binom{|Y|-\ell}{k-\ell-1}}{\binom{k-1}{\ell}} \le \overline{e}(\mathcal{G}) \le 2\sqrt{\rho}\binom{|Y|}{k-1},$$

which implies that $m \le 2\rho^{1/4}\binom{|Y|}{\ell}$. Thus at most

$$2\rho^{1/4}\binom{|Y|}{\ell}\cdot\binom{|Y|-\ell}{2k-2\ell-1}$$

$(2k-\ell-1)$-subsets of Y contain an ℓ-set L that fails (4.20). On the other hand, by (4.19), at most

$$\overline{e}(\mathcal{G})\binom{|Y|-k+1}{k-\ell} \le 2\sqrt{\rho}\binom{|Y|}{k-1}\binom{|Y|-k+1}{k-\ell}$$

$(2k-\ell-1)$-subsets of Y contain a non-edge of $\mathcal{G}$. Putting these together, the number of bad $(2k-\ell-1)$-sets in Y is at most

$$2\rho^{1/4}\binom{|Y|}{\ell}\binom{|Y|-\ell}{2k-2\ell-1}+2\sqrt{\rho}\binom{|Y|}{k-1}\binom{|Y|-k+1}{k-\ell} \le 3\binom{2k-\ell-1}{\ell}\rho^{1/4}\binom{|Y|}{2k-\ell-1},$$

as $\rho \ll 1$. □

We will pick a family of disjoint good $(2k-\ell-1)$-sets in Y' such that for any $x \in X$, many members of this family are suitable for x. To achieve this, we pick a family $\mathcal{F}$ by selecting each good $(2k-\ell-1)$-subsets of Y' randomly and independently with probability $p = 6\sqrt{\rho}|Y|/\binom{|Y|}{2k-\ell-1}$. Since there are at most $\binom{|Y|}{2k-\ell-1}\cdot(2k-\ell-1)\cdot\binom{|Y|-1}{2k-\ell-2}$ pairs of intersecting $(2k-\ell-1)$-sets in Y, the expected number of intersecting pairs of $(2k-\ell-1)$-sets in $\mathcal{F}$ is at most

$$p^2\binom{|Y|}{2k-\ell-1}\cdot(2k-\ell-1)\cdot\binom{|Y|-1}{2k-\ell-2} = 36(2k-\ell-1)^2\rho|Y|.$$

By applying Chernoff's bound on the first two properties and Markov's bound on the last one below, we can find, with positive probability, a family $\mathcal{F}$ of good $(2k-\ell-1)$-subsets of Y' that satisfies

- $|\mathcal{F}| \le 2p\binom{|Y'|}{2k-\ell-1} \le 12\sqrt{\rho}|Y|$,

- for any vertex $x \in X$, because of Claim 4.26, at least

$$\frac{p}{2}(1-\rho^{1/5})\binom{|Y|}{2k-\ell-1} \geq 2\sqrt{\rho}|Y|$$

  members of $\mathcal{F}$ are suitable for x.

- the number of intersecting pairs of $(2k-\ell-1)$-sets in $\mathcal{F}$ is at most $72(2k-\ell-1)^2\rho|Y|$.

After deleting one $(2k-\ell-1)$-set from each of the intersecting pairs from $\mathcal{F}$, we obtain a family $\mathcal{F}' \subseteq \mathcal{F}$ consisting of at most $12\sqrt{\rho}|Y|$ disjoint good $(2k-\ell-1)$-subsets of Y' and for each $x \in X$, at least

$$2\sqrt{\rho}|Y| - 72(2k-\ell-1)^2\rho|Y| \geq \frac{3}{2}\sqrt{\rho}|Y| \tag{4.21}$$

members of $\mathcal{F}'$ are suitable for x.

Denote $\mathcal{F}'$ by $\{Q_2, Q_4, \ldots, Q_{2q}\}$ for some $q \leq 12\sqrt{\rho}|Y|$. We arbitrarily partition each Q_{2i} into $L_{2i} \cup P_{2i} \cup L_{2i+1}$ such that $|L_{2i}| = |L_{2i+1}| = \ell$ and $|P_{2i}| = 2k-3\ell-1$. Since Q_{2i} is good, both L_{2i} and L_{2i+1} satisfy (4.20). We claim that L_0 and L_1 satisfy (4.20) as well. Let us show this for L_0. By the definition of $\mathcal{G}$, the number of $XY^{k-\ell-1}$-sets T such that $T \cup L_0 \notin E(H)$ is at least $\overline{\deg}_{\mathcal{G}}(L_0)\sqrt{\rho}t$. Using (4.16), we derive that $\overline{\deg}_{\mathcal{G}}(L_0)\sqrt{\rho}t \leq \rho\binom{|Y|}{k-\ell}$. Since $|Y| \leq (2k-2\ell)t$, it follows that $\overline{\deg}_{\mathcal{G}}(L_0) \leq 2\sqrt{\rho}\binom{|Y|-1}{k-\ell-1} \leq \rho^{1/4}\binom{|Y|-\ell}{k-\ell-1}$.

Next we find disjoint $(2k-3\ell-1)$-sets $P_1, P_3, \ldots, P_{2q-1}$ from $Y' \setminus \bigcup_{i=1}^{q} Q_{2i}$ such that for $i \in [q]$, every $(k-\ell-1)$-subset of P_{2i-1} is a common neighbor of L_{2i-1} and L_{2i} in $\mathcal{G}$. Since $L_1, L_2, \ldots, L_{2q}$ all satisfy (4.20), at most

$$2 \cdot \rho^{1/4}\binom{|Y|-\ell}{k-\ell-1}\binom{|Y|-k+\ell+1}{k-2\ell}$$

$(2k-3\ell-1)$-subsets of Y contain a non-neighbor of L_{2i-1} or L_{2i}. Since $q \leq 12\sqrt{\rho}|Y|$ and $\rho \ll 1$, we can greedily find desired $P_1, P_3 \ldots, P_{2q-1}$.

Let $Y_1 = Y' \setminus \bigcup_{i=1}^{q}(P_{2i-1} \cup Q_{2i})$ and $\mathcal{G}' = \mathcal{G}[Y_1]$. Then $|Y_1| = |Y'| - (2k-2\ell-1)2q$.

Since $\overline{\deg}_{\mathcal{G}'}(v) \le \overline{\deg}_{\mathcal{G}}(v)$ for every $v \in Y_1$, we have, by (4.18),

$$\delta_1(\mathcal{G}') \ge \binom{|Y_1|-1}{k-2} - 2\sqrt{\rho}\binom{|Y|-1}{k-2} \ge (1-3\sqrt{\rho})\binom{|Y_1|-1}{k-2}.$$

Let $\alpha = 3\sqrt{\rho}$ and $\rho_0 = (22\alpha)^{\frac{1}{k-1}}$. We want to find two disjoint ρ_0-typical ordered subsets $(x_1, \dots, x_{k-\ell-1})$ and $(y_1, \dots, y_{k-\ell-1})$ of Y_1 such that

$$L_{2q+1} \cup \{x_1, \dots, x_{k-\ell-1}\},\ L_0 \cup \{y_1, \dots, y_{k-\ell-1}\} \in E(\mathcal{G}). \tag{4.22}$$

To achieve this, we choose $(x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1})$ from ${Y_1}^{2k-2}$ uniformly at random. By Lemma 4.25, with probability at least $\frac{8}{11}$, $(x_1, \dots, x_{k-\ell-1})$ and $(y_1, \dots, y_{k-\ell-1})$ are two disjoint ordered ρ_0-typical $(k-\ell-1)$-sets. Since L_0 satisfies (4.20), at most $(k-\ell-1)!\rho^{1/4}\binom{|Y|-\ell}{k-\ell-1}$ ordered $(k-\ell-1)$-subsets of Y are not neighbors of L_0 (the same holds for L_{2q+1}). Thus (4.22) fails with probability at most $2(k-\ell-1)!\rho^{1/4}$, provided that $x_1, \dots, x_{k-\ell-1}$, $y_1, \dots, y_{k-\ell-1}$ are all distinct. Therefore the desired $(x_1, \dots, x_{k-\ell-1})$ and $(y_1, \dots, y_{k-\ell-1})$ exist.

Next we apply Theorem 4.24 to $\mathcal{G}'$ and obtain a tight Hamilton path

$$\mathcal{P} = x_{k-\ell-1}x_{k-\ell-2}\cdots x_1 \cdots\cdots y_1y_2\cdots y_{k-\ell-1}.$$

Following the order of $\mathcal{P}$, we partition Y_1 into

$$R_{2q+1}, S_{2q+1}, R'_{2q+1}, L_{2q+2}, \dots, L_t, R_t, S_t, R'_t$$

such that $|L_i| = \ell$, $|R_i| = |R'_i| = k-2\ell$, and $|S_i| = \ell - 1$. Since $\mathcal{P}$ is a tight path in $\mathcal{G}$, we have

$$L_iR_iS_i,\ S_iR'_iL_{i+1} \in E(\mathcal{G}) \tag{4.23}$$

for $2q+2 \le i \le t-1$. Letting $L_{t+1} = L_0$, by (4.22), we also have (4.23) for $i = 2q+1$ and $i = t$.

We now arbitrarily partition P_i, $1 \le i \le 2q$ into $R_i \cup S_i \cup R'_i$ such that $|R_i| = |R'_i| = k-2\ell$,

and $|S_i| = \ell - 1$. By the choice of P_i, (4.23) holds for $1 \le i \le 2q$.

Consider the bipartite graph Γ between X and $Z := \{z_1, z_2, \dots, z_t\}$ such that $x \in X$ and $z_i \in Z$ are adjacent if and only if $L_i R_i S_i x, x S_i R_i' L_{i+1} \in E(H)$. For every $i \in [t]$, since (4.23) holds, we have $\deg_\Gamma(z_i) \ge (1 - 2\sqrt{\rho})t$ by the definition of $\mathcal{G}$. Let $Z' = \{z_{2q+1}, \dots, z_t\}$ and X_0 be the set of $x \in X$ such that $\deg_\Gamma(x, Z') \le |Z'|/2$. Then

$$|X_0|\frac{|Z'|}{2} \le \sum_{x \in X} \overline{\deg}_\Gamma(x, Z') \le 2\sqrt{\rho}t \cdot |Z'|,$$

which implies that $|X_0| \le 4\sqrt{\rho}t = 4\sqrt{\rho}\frac{|Y|-\ell}{2k-2\ell-1} \le \frac{4}{3}\sqrt{\rho}|Y|$ (note that $2k - 2\ell - 1 \ge k \ge 3$).

We now find a perfect matching between X and Z as follows.

Step 1: Each $x \in X_0$ is matched to some z_{2i}, $i \in [q]$ such that the corresponding $Q_{2i} \in \mathcal{F}'$ is suitable for x (thus x and z_{2i} are adjacent in Γ) – this is possible because of (4.21) and $|X_0| \le \frac{4}{3}\sqrt{\rho}|Y|$.

Step 2: Each of the unused z_i, $i \in [2q]$ is matched to a vertex in $X \setminus X_0$ – this is possible because $\deg_\Gamma(z_i) \ge (1 - 2\sqrt{\rho})t \ge |X_0| + 2q$.

Step 3: Let X' be the set of the remaining vertices in X. Then $|X'| = t - 2q = |Z'|$. Now consider the induced subgraph Γ' of Γ on $X' \cup Z'$. Since $\delta(\Gamma') \ge |X'|/2$, the Marriage Theorem provides a perfect matching in Γ'.

The perfect matching between X and Z gives rise to the desired Hamilton path of H. $\square$

PART 5

MINIMUM VERTEX DEGREE THRESHOLD FOR LOOSE HAMILTON CYCLES IN 3-UNIFORM HYPERGRAPHS

5.1 Introduction

The study of Hamilton cycles is an important topic in graph theory. In recent years, researchers have worked on extending the classical theorem of Dirac on Hamilton cycles to hypergraphs (see [58] for a survey and also Chapter 4).

Recently, Buß, Hàn, and Schacht [4] studied the minimum vertex degree that guarantees a loose Hamilton cycle in 3-graphs and obtained the following result.

Theorem 5.1. *[4, Theorem 3] For all $\gamma > 0$ there exists an integer n_0 such that the following holds. Suppose H is a 3-graph on $n > n_0$ with $n \in 2\mathbb{N}$ and*

$$\delta_1(H) > \left(\frac{7}{16} + \gamma\right)\binom{n}{2}.$$

Then H contains a loose Hamilton cycle.

In this chapter we improve Theorem 5.1 as follows.

Theorem 5.2. *There exists an $n_{5.2} \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-graph on $n > n_{5.2}$ with $n \in 2\mathbb{N}$ and*

$$\delta_1(H) \geq \binom{n-1}{2} - \binom{\lfloor \frac{3}{4}n \rfloor}{2} + c, \tag{5.1}$$

where $c = 2$ if $n \in 4\mathbb{N}$ and $c = 1$ otherwise. Then H contains a loose Hamilton cycle.

The following construction shows that Theorem 5.2 is best possible. It is slightly stronger than [4, Fact 4].

Proposition 5.3. *For every $n \in 2\mathbb{N}$ there exists a 3-graph on n vertices with minimum vertex degree $\binom{n-1}{2} - \binom{\lfloor \frac{3}{4}n \rfloor}{2} + c - 1$, where c is defined as in Theorem 5.2, and which contains no loose Hamilton cycle.*

Proof. Let $H_1 = (V_1, E_1)$ be the 3-graph on $n \in 2\mathbb{N} \setminus 4\mathbb{N}$ vertices such that $V_1 = A \dot{\cup} B$ with $|A| = \lceil \frac{n}{4} \rceil - 1$ and $|B| = \lfloor \frac{3n}{4} \rfloor + 1$, and E_1 consists of all triples intersecting A. Note that $\delta_1(H_1) = \binom{n-1}{2} - \binom{\lfloor \frac{3n}{4} \rfloor}{2}$. Suppose that H_1 contains a loose Hamilton cycle C. There are $n/2$ edges in C and every vertex in A is contained in at most two edges in C. Since $2|A| = \frac{n-2}{2}$, there is at least one edge of C whose vertices are completely from B. This is a contradiction because B is independent. So H_1 contains no loose Hamilton cycle.

Let $H_2 = (V_2, E_2)$ be a 3-graph on $n \in 4\mathbb{N}$ vertices such that $V_2 = A \dot{\cup} B$ with $|A| = \frac{n}{4} - 1$ and $|B| = \frac{3}{4}n + 1$, and E_2 consists of all triples intersecting A and those containing both b_1 and b_2, where b_1, b_2 are two fixed vertices in B. Then $\delta_1(H_2) = \binom{n-1}{2} - \binom{\frac{3}{4}n}{2} + 1$. Suppose that H_2 contains a loose Hamilton cycle C. There are $n/2$ edges in C and every vertex in A is contained in at most two edges in C. Thus, there are at least two edges of C whose vertices are completely from B. But due to the construction, every two edges in B share two vertices so they can not both appear in one loose cycle. This contradiction shows that H_2 contains no loose Hamilton cycle. □

As a typical approach of obtaining exact results, we distinguish the *extremal* case from the *nonextremal* case and solve them separately.

Definition 5.4. *Given $\Delta > 0$, a 3-graph H on n vertices is called Δ-extremal if there is a set $B \subseteq V(H)$, such that $|B| = \lfloor 3n/4 \rfloor$ and $e(B) \leq \Delta n^3$.*

Theorem 5.5 (Extremal Case)**.** *There exist $\Delta > 0$ and $n_{5.5} \in \mathbb{N}$ such that the following holds. Let $n > n_{5.5}$ be an even integer. Suppose that H is a 3-graph on n vertices satisfying* (5.1)*. If H is Δ-extremal, then H contains a loose Hamilton cycle.*

Theorem 5.6 (Nonextremal Case)**.** *For any $\Delta > 0$, there exist $\gamma > 0$ and $n_{5.6} \in \mathbb{N}$ such that the following holds. Let $n > n_{5.5}$ be an even integer. Suppose that H is a 3-graph on*

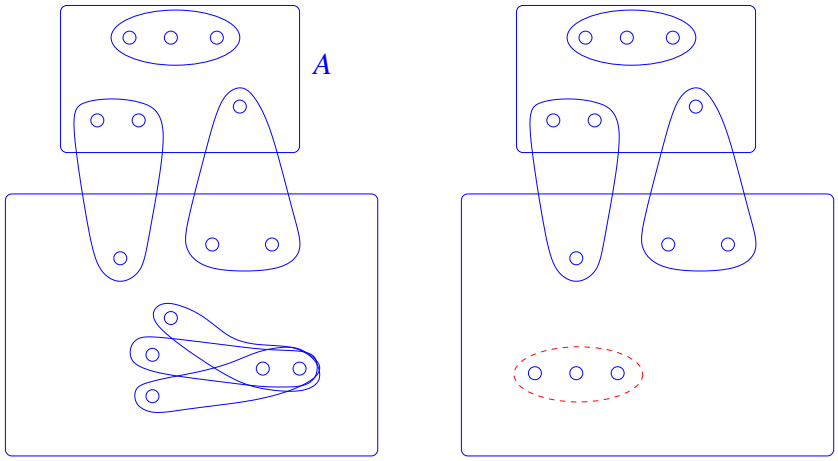

Figure 5.1. Constructions in Proposition 5.3

n vertices satisfying $\delta_1(H) \geq \left(\frac{7}{16} - \gamma\right)\binom{n}{2}$. If H is not Δ-extremal, then H contains a loose Hamilton cycle.

Theorem 5.2 follows from Theorems 5.5 and 5.6 immediately by choosing Δ from Theorem 5.5 and letting $n_{5.2} = \max\{n_{5.5}, n_{5.6}\}$.

Let us briefly discuss our proof ideas here. Since the proof of Theorem 5.5 is somewhat routine, the main task is to prove Theoreom 5.6. Following previous work [60, 63, 64, 18, 40, 4], we use the absorbing method. More precisely, we find the desired loose Hamilton cycle by applying the Absorbing Lemma (Lemma 5.7), the Reservoir Lemma (Lemma 5.8), and the Path-tiling Lemma (Lemma 5.9). In fact, the Absorbing Lemma and the Reservoir Lemma are not very difficult and already proven in [4]. Thus the main step is to prove the Path-tiling Lemma, under the assumption $\delta_1(H) \geq \left(\frac{7}{16} - \gamma\right)\binom{n}{2}$ and that H is not Δ-extremal (in contrast, $\delta_1(H) \geq (\frac{7}{16} + \gamma)\binom{n}{2}$ is assumed in [4]). As shown in [18, 4], after applying the (weak) Regularity Lemma, it suffices to prove that the cluster 3-graph $\mathcal{K}$ can be tiled almost perfectly by some particular 3-graph. For example, the 3-graph $\mathcal{M}$ given in [4] has the vertex set $[8] = \{1, 2, \ldots, 8\}$ and edges $123, 345, 456, 678$ (throughout this chapter, we often represent a set $\{v_1, v_2, \ldots, v_k\}$ as $v_1v_2\cdots v_k$). Since it is hard to find an $\mathcal{M}$-tiling directly, the authors of [4] found a fractional $\mathcal{M}$-tiling instead and converted it to an (integer) $\mathcal{M}$-tiling

by applying the Regularity Lemma again. We consider $\mathcal{C}_4^3$, a much simpler 3-graph, and obtain an almost perfect $\mathcal{C}_4^3$-tiling in $\mathcal{K}$ directly. Interestingly, $\mathcal{C}_4^3$-tiling was studied (via the codegree condition) in the very first paper on loose Hamilton cycles [41].

As far as we know, Theorem 5.2 is the second exact result on Hamilton cycles in hypergraphs (the one in [64] was the first). Comparing with [64], our proof is much shorter because their Absorbing and Reservoir Lemmas are much harder to prove.

We will prove Theorem 5.6 in Section 5.2 and Theorem 5.5 in Section 5.3.

5.2 Proof of Theorem 5.6

In this section we prove Theorem 5.6 by following the same approach as in [4].

5.2.1 Auxiliary lemmas and Proof of Theorem 5.6

A *loose* path $\mathcal{P} = v_1v_2 \cdots v_{2k+1}$ is a 3-graph on $\{v_1, v_2, \ldots, v_{2k+1}\}$ with edges $v_{2i-1}v_{2i}v_{2i+1}$ for all $i \in [k]$. The vertices v_1 and v_{2k+1} are called the *ends* of $\mathcal{P}$. For convenience, we rephrase the Absorbing Lemma [4, Lemma 7] as follows.[1]

Lemma 5.7 (Absorbing Lemma)**.** *For any $0 < \gamma_1 \le 10^{-14}$ there exists an integer $n_{5.7}$ such that the following holds. Let H be a 3-graph on $n > n_{5.7}$ vertices with $\delta_1(H) \ge \frac{13}{32}\binom{n}{2}$. Then there is a loose path $\mathcal{P}$ with $|V(\mathcal{P})| \le \gamma_1 n$ such that for every subset $U \subseteq V \setminus V(\mathcal{P})$ with $|U| \le \gamma_1^3 n$ and $|U| \in 2\mathbb{N}$ there exists a loose path $\mathcal{Q}$ with $V(\mathcal{Q}) = V(\mathcal{P}) \cup U$ such that $\mathcal{P}$ and $\mathcal{Q}$ have the same ends.*

We also need the Reservoir Lemma [4, Lemma 6].

Lemma 5.8 (Reservoir Lemma)**.** *For any $0 < \gamma_2 < 1/4$ there exists an integer $n_{5.8}$ such that for every 3-graph H on $n > n_{5.8}$ vertices satisfying*

$$\delta_1(H) \ge (1/4 + \gamma_2)\binom{n}{2},$$

[1] Lemma 7 in [4] assumes that $\delta_1(H) \ge (\frac{5}{8} + \gamma)^2\binom{n}{2}$ and returns $|V(\mathcal{P})| \le \gamma^7 n$ with $|U| \le \frac{\gamma^{14}}{14336}n$. We simply take their γ^7 as our γ_1 and thus $\gamma_1 \le \left(\sqrt{\frac{13}{32}} - \frac{5}{8}\right)^7 \approx 10^{-14}$.

there is a set R of size at most $\gamma_2 n$ with the following property: for every $k \le \gamma_2^3 n/12$ mutually disjoint pairs $\{a_i, b_i\}_{i\in[k]}$ of vertices from $V(H)$ there are $3k$ vertices u_i, v_i, w_i, $i \in [k]$ from R such that $a_i u_i v_i, v_i w_i b_i \in H$ for all $i \in [k]$.

The main step in our proof of Theorem 5.6 is the following lemma, which is stronger than Lemma 10 in [4].

Lemma 5.9 (Path-tiling lemma)**.** *For any $0 < \gamma_3, \alpha < 1$ there exist integers p and $n_{5.9}$ such that the following holds for $n > n_{5.9}$. Suppose H is a 3-graph on n vertices with minimum vertex degree*

$$\delta_1(H) \ge \left(\frac{7}{16} - \gamma_3\right)\binom{n}{2},$$

then there are at most p vertex disjoint loose paths in H that together cover all but at most αn vertices of H unless H is $2050\gamma_3$-extremal.

Proof of Theorem 5.6. Given $\Delta > 0$, let $\gamma = \min\{\frac{\Delta}{4101}, 10^{-14}\}$. We choose $n_{5.6} = \max\{n_{5.7}, 2n_{5.8}, 2n_{5.9}, 192(p+1)/(\gamma/3)^9\}$, where p is the constant returned from Lemma 5.9 with $\gamma_3 = 2\gamma$ and $\alpha = (\gamma/3)^3/2$. Let $n > n_{5.6}$ be an even integer.

Suppose that $H = (V, E)$ is a 3-graph on n vertices with $\delta_1(H) \ge \left(\frac{7}{16} - \gamma\right)\binom{n}{2}$. Since $\frac{7}{16} - \gamma > \frac{13}{32}$, we can apply Lemma 5.7 with $\gamma_1 = \gamma/3$ and obtain an absorbing path $\mathcal{P}_0$ with ends a_0, b_0. We next apply Lemma 5.8 with $\gamma_2 = (\gamma/3)^3/2$ to $H[(V \setminus V(\mathcal{P}_0)) \cup \{a_0, b_0\}]$ and obtain a reservoir R. Let $V' = V \setminus (V(\mathcal{P}_0) \cup R)$ and $n' = |V'|$. Note that $n - n' \le \gamma_1 n + \gamma_2 n < \gamma n/2$. The induced subhypergraph $H' = H[V']$ satisfies

$$\delta_1(H') \ge \left(\frac{7}{16} - \gamma\right)\binom{n}{2} - \frac{\gamma}{2} n \cdot (n-2) > \left(\frac{7}{16} - 2\gamma\right)\binom{n'}{2}.$$

Applying Lemma 5.9 to H' with $\gamma_3 = 2\gamma$ and $\alpha = (\gamma/3)^3/2$, we obtain at most p vertex disjoint loose paths that cover all but at most $\alpha n'$ vertices of H', unless H' is $2050\gamma_3$-extremal. In the latter case, there exists $B' \subseteq V'$ such that $|B'| = \lfloor \frac{3}{4} n' \rfloor$ and $e(B') \le 4100\gamma(n')^3$. Then we add $\lfloor \frac{3}{4} n \rfloor - \lfloor \frac{3}{4} n' \rfloor < \gamma n/2$ arbitrary vertices from $V \setminus B'$ to B' to get a vertex set B such

that $|B| = \lfloor \frac{3}{4}n \rfloor$ and

$$e(B) \leq 4100\gamma(n')^3 + \frac{\gamma n}{2}\binom{n-1}{2} < 4101\gamma n^3 \leq \Delta n^3,$$

which means that H is Δ-extremal, a contradiction. In the former case, denote these loose paths by $\{\mathcal{P}_i\}_{i\in[p']}$ for some $p' \leq p$, and their ends by $\{a_i, b_i\}_{i\in[p']}$. The choice of $n_{5.6}$ guarantees that $p'+1 \leq p+1 \leq \gamma_2^3 n/24$. We can thus connect $\{a_i, b_{i+1}\}_{0\leq i\leq p'-1} \cup \{a_{p'}, b_0\}$ by using vertices from R obtaining a loose cycle C. Since $|V\setminus C| \leq |R|+\alpha n' \leq \gamma_2 n+\gamma_2 n' \leq \gamma_1^3 n$, we can use $\mathcal{P}_0$ to absorb all unused vertices in R and uncovered vertices in V'. □

The rest of this section is devoted to the proof of Lemma 5.9.

5.2.2 Proof of Lemma 5.9

Following the approach in [4], we will use the weak regularity lemma and the cluster hypergraph introduced in Chapter 2. The following corollary of the weak regularity lemma (Theorem 2.1) shows that the cluster hypergraph inherits the minimum vertex degree of the original hypergraph. Its proof is the same as that of [4, Proposition 15] after we replace $7/16+\gamma$ by c (we thus omit the proof).

Corollary 5.10. *For $c > d > \epsilon > 0$ and $t_0 \geq 0$ there exist T_0 and n_0 such that the following holds. Suppose H is a 3-graph on $n > n_0$ vertices which has minimum vertex degree $\delta_1(H) \geq c\binom{n}{2}$. Then there exists an (ϵ, t)-regular partition $\mathcal{Q}$ with $t_0 < t < T_0$ such that the cluster hypergraph $\mathcal{K} = \mathcal{K}(\epsilon, d, \mathcal{Q})$ has minimum vertex degree $\delta_1(\mathcal{K}) \geq (c-\epsilon-d)\binom{t}{2}$.*

In 3-graphs, a loose path is 3-partite with partition sizes *about* $m, m, 2m$ for some integer m. Proposition 5.11 below shows that every regular triple with partition sizes $m, m, 2m$ contains an almost spanning loose path as a subhypergraph. In contrast, [4, Proposition 25] (more generally [18, Lemma 20]) shows that every regular triple with partition sizes $3m, 3m, 2m$ contains finitely many vertex disjoint loose paths. The proof of Proposition 5.11 uses the standard approach handling regularity.

For a vertex v and disjoint vertex sets S, T in a 3-graph, we denote by $\deg(v, S)$ the number of edges that contain v and two vertices from S, and denote by $\deg(v, ST)$ the number of edges that contain v, one vertex from S and one vertex from T.

Proposition 5.11. *Fix any $\epsilon > 0$, $d > 2\epsilon$, and an integer $m \geq \frac{d}{\epsilon(d-2\epsilon)}$. Suppose that $V(H) = V_1 \cup V_2 \cup V_3$ and (V_1, V_2, V_3) is (ϵ, d)-regular with $|V_i| = m$ for $i = 1, 3$ and $|V_2| = 2m$. Then there is a loose path P omitting at most $8\epsilon m/d + 3$ vertices of H.*

Proof. We will greedily construct the loose path $P = v_1 v_2 \cdots v_{2k+1}$ such that $v_{2i} \in V_2$, $v_{4i+1} \in V_1$ and $v_{4i+3} \in V_3$ until $|V_i \setminus V(P)| < \frac{2\epsilon}{d}|V_i|$ for some $i \in [3]$. For $j \in [3]$, let $U_j^0 = V_j$ and $U_j^i = V_j \setminus \{v_1, \ldots, v_{2i-1}\}$ for $i \in [k]$. In addition, we require that for $i = 0, \ldots, k$,

$$\deg(v_{2i+1}, U_2^i U_r^i) \geq (d - \epsilon)|U_2^i||U_r^i|, \tag{5.2}$$

where $r \equiv 2i - 1 \mod 4$. We proceed by induction on i. First we pick a vertex $v_1 \in V_1$ such that $\deg(v_1, V_2 V_3) \geq (d-\epsilon)|V_2||V_3|$ (thus (5.2) holds for $i = 0$). By regularity, all but at most $\epsilon|V_1|$ vertices can be chosen as v_1. Suppose that we have selected $v_1, \ldots, v_{2i-1}$. Without loss of generality, assume that $v_{2i-1} \in V_1$. Our goal is to choose $v_{2i} \in U_2^i, v_{2i+1} \in U_3^i$ such that

(i) $v_{2i-1} v_{2i} v_{2i+1} \in E(H)$,

(ii) $\deg(v_{2i+1}, U_1^i U_2^i) \geq (d - \epsilon)|U_1^i||U_2^i|$.

In fact, the induction hypothesis implies that $\deg(v_{2i-1}, U_2^{i-1} U_3^{i-1}) \geq (d-\epsilon)|U_2^{i-1}||U_3^{i-1}|$. Since $U_2^i = U_2^{i-1} \setminus \{v_{2i-2}\}$ and $U_3^i = U_3^{i-1}$, we have

$$\deg(v_{2i-1}, U_2^i U_3^i) \geq (d - \epsilon)|U_2^{i-1}||U_3^{i-1}| - |U_3^{i-1}| = ((d - \epsilon)|U_2^{i-1}| - 1)|U_3^{i-1}|.$$

By regularity, at most $\epsilon|V_3|$ vertices in V_3 does not satisfy (ii). So, at least

$$\deg(v_{2i-1}, U_2^i U_3^i) - \epsilon|V_3| \cdot |U_2^{i-1}| \geq ((d - \epsilon)|U_2^{i-1}| - 1)|U_3^{i-1}| - \epsilon|V_3| \cdot |U_2^{i-1}| \tag{5.3}$$

pairs of vertices can be chosen as v_{2i}, v_{2i+1}. Since $|U_3^{i-1}| \geq \frac{2\epsilon}{d}|V_3|$ and $|U_2^{i-1}| \geq \frac{2\epsilon}{d}|V_2| \geq \frac{4}{d-2\epsilon}$ (using $m \geq \frac{d}{\epsilon(d-2\epsilon)}$), the right side of (5.3) is at least

$$((d-\epsilon)|U_2^{i-1}|-1)\frac{2\epsilon}{d}|V_3| - \epsilon|V_3|\cdot|U_2^{i-1}| = \left((d-2\epsilon)|U_2^{i-1}|-2\right)\frac{\epsilon}{d}|V_3| \geq \frac{2\epsilon}{d}|V_3|,$$

thus the selection of v_{2i}, v_{2i+1} satisfying (i) and (ii) is guaranteed.

To calculate the number of the vertices omitted by $P = v_1v_2\cdots v_{2k+1}$, note that $|V_1 \cap V(P)| = \lceil \frac{k+1}{2} \rceil$, $|V_2 \cap V(P)| = k$, and $|V_3 \cap V(P)| = \lfloor \frac{k+1}{2} \rfloor$. Our greedy construction of P stops as soon as $|V_i \setminus V(P)| < \frac{2\epsilon}{d}|V_i|$ for some $i \in [3]$. As $|V_1| = |V_3| = m = |V_2|/2$, one of the following three inequalities holds:

$$m - \left\lceil \frac{k+1}{2} \right\rceil < \frac{2\epsilon}{d}m, \quad 2m - k < \frac{2\epsilon}{d}2m, \quad m - \left\lfloor \frac{k+1}{2} \right\rfloor < \frac{2\epsilon}{d}m.$$

Thus we always have $m - \lceil \frac{k+1}{2} \rceil < \frac{2\epsilon}{d}m$, which implies that $\frac{k+2}{2} > \left(1 - \frac{2\epsilon}{d}\right)m$ or $k > 2\left(1-\frac{2\epsilon}{d}\right)m - 2$. Consequently,

$$|V(H)\setminus V(P)| = 4m - (2k+1) < 4m - \left(4\left(1-\frac{2\epsilon}{d}\right)m - 4 + 1\right) = \frac{8\epsilon}{d}m + 3. \qquad \square$$

Recall that $\mathcal{C}_4^3$ is the unique 3-graph with four vertices and two edges. Throughout this chapter, we call it as $\mathcal{C}$ for short. The following lemma is the main step in our proof of Lemma 5.9. In general, given two (hyper)graphs $\mathcal{F}$ and $\mathcal{G}$, an *$\mathcal{F}$-tiling* is a sub(hyper)graph of $\mathcal{G}$ that consists of vertex disjoint copies of $\mathcal{F}$. The $\mathcal{F}$-tiling is *perfect* if it is a spanning sub(hyper)graph of $\mathcal{G}$.

Lemma 5.12 ($\mathcal{C}$-tiling Lemma)**.** *For any $\gamma > 0$, there exists an integer $n_{5.12}$ such that the following holds. Suppose H is a 3-graph on $n > n_{5.12}$ vertices with*

$$\delta_1(H) \geq \left(\frac{7}{16} - \gamma\right)\binom{n}{2},$$

then there is a $\mathcal{C}$-tiling covering all but at most $2^{19}/\gamma$ vertices of H unless H is $2^{10}\gamma$-extremal.

Now we are ready to prove Lemma 5.9 using the same approach as in [4].

Proof of Lemma 5.9. Given $0 < \gamma_3, \alpha < 1$, let $n_{5.9} = \max\{n_0, 4T_0/\epsilon\}$ and $p = T_0/2$, where T_0 and n_0 are the constants returned from Corollary 5.10 with $c = \frac{7}{16} - \gamma_3$, $d = \gamma_3/2$, $\epsilon = \frac{\alpha d}{8+\alpha}$, and $t_0 = \max\{n_{5.12}, \frac{2^{20}}{\gamma_3 \alpha}\}$.

Suppose that H is a 3-graph on $n > n_{5.9}$ vertices with $\delta_1(H) \geq (\frac{7}{16} - \gamma_3)\binom{n}{2}$. By applying Corollary 5.10 with the constants chosen above, we obtain an (ϵ, t)-regular partition $\mathcal{Q}$. The cluster hypergraph $\mathcal{K} = \mathcal{K}(\epsilon, d, \mathcal{Q})$ satisfies $\delta_1(\mathcal{K}) \geq (\frac{7}{16} - 2\gamma_3)\binom{t}{2}$. Let m be the size of each cluster except V_0, then $(1-\epsilon)\frac{n}{t} \leq m \leq \frac{n}{t}$. By Lemma 5.12, either $\mathcal{K}$ is $2^{10}(2\gamma_3)$-extremal, or there is a $\mathcal{C}$-tiling $\mathscr{C}$ of $\mathcal{K}$ that covers all but at most $2^{19}/(2\gamma_3)$ vertices of $\mathcal{K}$. In the first case, there exists a set $B \subseteq V(\mathcal{K})$ such that $|B| = \lfloor \frac{3t}{4} \rfloor$ and $e(B) \leq 2^{11}\gamma_3 t^3$. Let $B' \subseteq V(H)$ be the union of the clusters in B. By regularity,

$$e(B') \leq e(B) \cdot m^3 + \binom{t}{3} \cdot d \cdot m^3 + \epsilon \cdot \binom{t}{3} \cdot m^3 + \binom{m}{2} n,$$

where the right-hand side bounds the number of edges from regular triples with high density, edges from regular triples with low density, edges from irregular triples and edges that are from at most two clusters. Since $m \leq \frac{n}{t}$, $\epsilon < d < \gamma_3$, and $t^{-2} < t_0^{-2} < \gamma_3$, we get

$$e(B') \leq 2^{11}\gamma_3 t^3 \left(\frac{n}{t}\right)^3 + d\binom{t}{3}\left(\frac{n}{t}\right)^3 + \epsilon\binom{t}{3}\left(\frac{n}{t}\right)^3 + \binom{n/t}{2} n < 2049\gamma_3 n^3.$$

Note that $|B'| = \left\lfloor \frac{3t}{4} \right\rfloor m \leq \frac{3t}{4} \cdot \frac{n}{t} = \frac{3n}{4}$ implies that $|B'| \leq \lfloor \frac{3n}{4} \rfloor$. On the other hand,

$$|B'| = \left\lfloor \frac{3t}{4} \right\rfloor m \geq \left(\frac{3t}{4} - 1\right)(1-\epsilon)\frac{n}{t} \geq \left(\frac{3t}{4} - \epsilon t\right)\frac{n}{t} = \frac{3n}{4} - \epsilon n,$$

by adding at most ϵn vertices from $V \setminus B'$ to B', we get a set $B'' \subseteq V(H)$ of size exactly $\lfloor 3n/4 \rfloor$, with $e(B'') \leq e(B') + \epsilon n \cdot n^2 < 2050\gamma_3 n^3$. Hence H is $2050\gamma_3$-extremal.

In the second case, the union of the clusters covered by $\mathscr{C}$ contains all but at most $\frac{2^{19}}{2\gamma_3} m + |V_0| \leq \alpha n/4 + \epsilon n < 3\alpha n/8$ vertices (here we use $t \geq \frac{2^{20}}{\gamma_3 \alpha}$). We will apply Proposition 5.11 to each member $\mathcal{C}' \in \mathscr{C}$. Suppose that $\mathcal{C}'$ has the vertex set $[4]$ with edges $123, 234$.

For $i \in [4]$, let V_i denote the corresponding cluster in H. We split V_i, $i = 2, 3$, into two disjoint sets V_i^1 and V_i^2 of equal sizes. Then the triples (V_1, V_2^1, V_3^1) and (V_4, V_2^2, V_3^2) are $(2\epsilon, d-\epsilon)$-regular and of sizes $m, \frac{m}{2}, \frac{m}{2}$. Applying Proposition 5.11 to these two triples with $m' = \frac{m}{2}$, we find a loose path in each triple covering all but at most $\frac{8(2\epsilon)}{d-\epsilon}m' + 3 = \alpha m + 3$ vertices (here we need $\epsilon = \frac{\alpha d}{8+\alpha}$).

Since $|\mathscr{C}| \leq t/4$, we obtain a path tiling that consists of at most $2t/4 \leq T_0/2 = p$ paths and covers all but at most

$$2(\alpha m + 3)\frac{t}{4} + \frac{3\alpha}{8}n \leq \frac{\alpha}{2}n + \frac{3t}{2} + \frac{3\alpha}{8}n < \alpha n$$

vertices. This completes the proof. □

5.2.3 Proof of $\mathcal{C}$-tiling Lemma (Lemma 5.12)

Fact 5.13. *Let H be a 3-graph on m vertices which contains no copy of $\mathcal{C}$, then $e(H) \leq \frac{1}{3}\binom{m}{2}$.*

Proof. Since there is no copy of $\mathcal{C}$, then given any $u, v \in V(H)$, we have that $\deg(uv) \leq 1$, which implies $e(H) \leq \frac{1}{3}\binom{m}{2} \cdot 1 = \frac{1}{3}\binom{m}{2}$. □

Proof of Lemma 5.12. Fix $\gamma > 0$ and let $n \in \mathbb{N}$ be sufficiently large. Let H be a 3-graph on n vertices that satisfies $\delta_1(H) \geq (\frac{7}{16} - \gamma)\binom{n}{2}$. Fix a largest $\mathcal{C}$-tiling $\mathscr{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$ and let $V_i = V(\mathcal{C}_i)$ for $i \in [m]$. Let $V' = \bigcup_{i\in[m]} V_i$ and $U = V(H) \setminus V'$. Assume that $|U| > 2^{19}/\gamma$ – otherwise we are done.

Our goal is to find a set C of vertices in V' of size at most $n/4$ that covers almost all the edges, which implies that H is extremal.

Let $\mathcal{A}_i$ be the set of all edges with exactly i vertices in V', for $i = 0, 1, 2, 3$. Note that $|\mathcal{A}_0| \leq \frac{1}{3}\binom{|U|}{2}$ by Fact 5.13. We may assume that $|U| < \frac{3}{4}n$ and consequently

$$m > \frac{n}{16}. \tag{5.4}$$

Indeed, if $|U| \geq \frac{3}{4}n$, then taking $U' \subseteq U$ of size $\lfloor \frac{3}{4}n \rfloor$, we get that $e(U') \leq e(U) \leq \frac{1}{3}\binom{|U|}{2} \leq \frac{1}{6}n^2 < \gamma n^3$. Thus H is γ-extremal and we are done.

Claim 5.14. $|\mathcal{A}_1| \leq m\binom{|U|}{2} + 12m|U|$.

Proof. Let D be the set of vertices $v \in V'$ such that $\deg(v, U) \geq 4|U|$. First observe that every $\mathcal{C}_i \in \mathscr{C}$ contains at most one vertex in D. Suppose instead, two vertices $x, y \in V_i$ are both in D. Since $\deg(x, U) \geq 4|U| > |U|/2$, the link graph of x on U contains a path $u_1u_2u_3$ of length two. The link graph of y on $U \setminus \{u_1, u_2, u_3\}$ has size at least $4|U| - 3|U| > |U|/2$, so it also contains a path of length two, with vertices denoted by u_4, u_5, u_6. Note that $\{x, u_1, u_2, u_3\}$ and $\{y, u_4, u_5, u_6\}$ span two vertex disjoint copies of $\mathcal{C}$. Replacing $\mathcal{C}_i$ in $\mathscr{C}$ with them creates a larger $\mathcal{C}$-tiling, contradicting the maximality of $\mathscr{C}$. So we conclude that $|D| \leq m$. Consequently,

$$|\mathcal{A}_1| \leq |D| \cdot \binom{|U|}{2} + |V' \setminus D| \cdot 4|U| = m\binom{|U|}{2} + 3m \cdot 4|U| \leq m\binom{|U|}{2} + 12m|U|.$$

□

Fix $u \in U$, $i \neq j \in [m]$, denote $L_{i,j}(u)$ as the link graph of u on $[V_i, V_j]$, namely the bipartite link graph of u between V_i and V_j. Let $\mathcal{T}_{\leq 6}$ be the set of all triples uij, $u \in U$, $i, j \in [m]$ such that $e(L_{i,j}(u)) \leq 6$. Let $\mathcal{T}_7^1$ be the set of all triples uij, $u \in U$, $i, j \in [m]$ such that $L_{i,j}(u)$ contains exactly seven edges, and a vertex cover of two vertices with one from V_i and the other from V_j. Let $\mathcal{T}_{\geq 7}^2$ be the set of all triples uij, $u \in U$, $i, j \in [m]$ such that $L_{i,j}(u)$ contains at least seven edges, and a vertex cover of two vertices both from V_i or V_j. Let $\mathcal{T}_{\geq 7}^3$ be the set of all triples uij, $u \in U$, $i, j \in [m]$ such that $L_{i,j}(u)$ contains at least seven edges, and a matching of size three. Since a bipartite graph either contains a matching of size three or a vertex cover of size two (by the König–Egervary theorem), $\mathcal{T}_{\leq 6}$, $\mathcal{T}_7^1$, $\mathcal{T}_{\geq 7}^2$ and $\mathcal{T}_{\geq 7}^3$ form a partition of $U \times \binom{[m]}{2}$.

Fact 5.15. *1. H does not contain $i \neq j \in [m]$ and six vertices $u_1, \ldots, u_6 \in U$ such that $u_1, \ldots, u_6$ have the same (labeled) link graph on $[V_i, V_j]$ and $u_1ij \in \mathcal{T}_{\geq 7}^3$.*

2. *H does not contain distinct $i, j, k \in [m]$ and eight vertices $u_1, \dots, u_8 \in U$ such that the following holds. First, $u_1, \dots, u_4$ share the same link graph on $[V_i, V_j]$, and $u_5, \dots, u_8$ share the same link graph on $[V_i, V_k]$. Second, $u_1 ij \in \mathcal{T}^2_{\geq 7}$ with the vertex cover in V_j and $u_5 ik \in \mathcal{T}^2_{\geq 7}$ with the vertex cover in V_k.*

Proof. To see Part (1), since there is a matching of size three in the (same) link graph of $u_1, \dots, u_6$, say, a_1b_1, a_2b_2, a_3b_3, then $u_1u_2a_1b_1$, $u_3u_4a_2b_2$ and $u_5u_6a_3b_3$ span three copies of $\mathcal{C}$. Replacing $\mathcal{C}_i, \mathcal{C}_j$ by them gives a $\mathcal{C}$-tiling larger than $\mathscr{C}$, a contradiction.

To see Part (2), assume that $V_i = \{a, b, c, d\}$. Suppose that the vertex cover of $L_{i,j}(u_1)$ is $\{x_1, y_1\} \subseteq V_j$ and the vertex cover of $L_{i,k}(u_5)$ is $\{x_2, y_2\} \subseteq V_k$. Since $u_1 ij \in \mathcal{T}^2_{\geq 7}$, at most one pair from $\{x_1, y_1\} \times \{a, b\}$ is not in $L_{i,j}(u_1)$. Analogously at most one pair from $\{x_2, y_2\} \times \{c, d\}$ is not in $L_{i,k}(u_5)$. Thus, without loss of generality, we may assume that $x_1a, y_1b \in L_{i,j}(u_1)$ and $x_2c, y_2d \in L_{i,k}(u_5)$. Since $u_1, \dots, u_4$ share the same link graph on $[V_i, V_j]$, $u_1u_2x_1a$, $u_3u_4y_1b$ span two copies of $\mathcal{C}$. Similarly, $u_5u_6x_2c$ and $u_7u_8y_2d$ span two copies of $\mathcal{C}$. Replacing $\mathcal{C}_i, \mathcal{C}_j, \mathcal{C}_k$ by these four copies of $\mathcal{C}$ gives a $\mathcal{C}$-tiling larger than $\mathscr{C}$, a contradiction. □

We next show that all but at most $\gamma n^2|U|$ triples uij, $u \in U$, $i, j \in [m]$ are in $\mathcal{T}^1_7$.

Claim 5.16. $|\mathcal{T}^1_7| \geq \binom{m}{2}|U| - \gamma n^2|U|$.

Proof. First, we claim that

$$|\mathcal{T}^3_{\geq 7}| \leq \binom{m}{2} \cdot 2^{16} \cdot 5, \quad |\mathcal{T}^2_{\geq 7}| \leq 756\binom{m}{2} + m \cdot |U|. \tag{5.5}$$

To see the first inequality, by Part (1) of Fact 5.15, given $i, j \in [m]$ and a bipartite graph on $[V_i, V_j]$ containing a matching of size three, at most five vertices in U can share this link graph on $[V_i, V_j]$. Since there are 2^{16} (labeled) bipartite graphs on $[V_i, V_j]$, we get that $|\mathcal{T}^3_{\geq 7}| \leq \binom{m}{2} \cdot 2^{16} \cdot 5$.

To see the second inequality in (5.5), let $\mathcal{D}$ denote the digraph on $[m]$ such that $(i, j) \in E(\mathcal{D})$ if and only if at least eight vertices $u_1, \dots, u_8 \in U$ share the same link graph on $[V_i, V_j]$

such that $u_1ij \in \mathcal{T}^2_{\geq 7}$, and the vertex cover is in V_i. We claim that $d^-_{\mathcal{D}}(i) \leq 1$ for every $i \in [m]$ and consequently $e(\mathcal{D}) \leq m$. Suppose instead, there are $i, j, k \in [m]$ such that $(j, i), (k, i) \in E(\mathcal{D})$, then eight vertices of U share the same link graph on $[V_i, V_j]$, and (not necessarily different) eight vertices of U share the same link graph on $[V_i, V_k]$. Thus we can pick four distinct vertices for each of $[V_i, V_j]$ and $[V_i, V_k]$ and obtain a structure forbidden by Part (2) of Fact 5.15, a contradiction. Note that there are $2 \cdot \binom{4}{2} \cdot 8 + 2 \cdot \binom{4}{2} = 108$ (labeled) bipartite graphs on $[V_i, V_j]$ with at least seven edges and a vertex cover of two vertices both from V_i or V_j. Furthermore, fixing one of these bipartite graphs, if $(i, j), (j, i) \notin \mathcal{D}$, then at most seven vertices in U share this link graph by the definition of $\mathcal{D}$. So we get that

$$|\mathcal{T}^2_{\geq 7}| \leq \binom{m}{2} \cdot 108 \cdot 7 + m|U| = 756\binom{m}{2} + m|U|.$$

Recall that $\mathcal{A}_2$ is the set of all edges of H with exactly two vertices in V'. Then

$$|\mathcal{A}_2| \leq 6|\mathcal{T}_{\leq 6}| + 7|\mathcal{T}^1_7| + 8|\mathcal{T}^2_{\geq 7}| + 16|\mathcal{T}^3_{\geq 7}| + \binom{4}{2} m|U|.$$

Together with $|\mathcal{T}_{\leq 6}| + |\mathcal{T}^1_7| + |\mathcal{T}^2_{\geq 7}| + |\mathcal{T}^3_{\geq 7}| = \binom{m}{2}|U|$, we get,

$$\begin{aligned}
|\mathcal{A}_2| &\leq 7\binom{m}{2}|U| - |\mathcal{T}_{\leq 6}| + |\mathcal{T}^2_{\geq 7}| + 9|\mathcal{T}^3_{\geq 7}| + 6m|U| \\
&\leq 7\binom{m}{2}|U| - |\mathcal{T}_{\leq 6}| + \binom{m}{2} \cdot (2^{16} \cdot 45 + 756) + 7m|U| \quad \text{by (5.5)} \\
&< 7\binom{m}{2}|U| - |\mathcal{T}_{\leq 6}| + 2^{22}\binom{m}{2} + 7m|U|. \qquad (5.6)
\end{aligned}$$

We know that $\sum_{u \in U} \deg(u) = 3|\mathcal{A}_0| + 2|\mathcal{A}_1| + |\mathcal{A}_2|$. Thus, by $|\mathcal{A}_0| \leq \frac{1}{3}\binom{|U|}{2}$, Claim 5.14

and (5.6), we have

$$\begin{aligned}\sum_{u\in U}\deg(u) &\le \binom{|U|}{2} + 2m\binom{|U|}{2} + 24m|U| + 7\binom{m}{2}|U| - |\mathcal{T}_{\le 6}| + 2^{22}\binom{m}{2} + 7m|U| \\ &= \binom{|U|}{2} + m|U|^2 + 30m|U| + 7\binom{m}{2}|U| - |\mathcal{T}_{\le 6}| + 2^{22}\binom{m}{2} \\ &< \frac{7}{16}\binom{|U|}{2}|U| + \frac{7}{4}m|U|^2 + \frac{7}{16}\binom{4m}{2}|U| - |\mathcal{T}_{\le 6}| + 2^{22}\binom{m}{2} \quad \text{as } |U| > 40 \\ &= \frac{7}{16}\binom{n}{2}|U| - |\mathcal{T}_{\le 6}| + 2^{22}\binom{m}{2}, \end{aligned} \tag{5.7}$$

where the last inequality is due to $\binom{|U|}{2} + 4m|U| + \binom{4m}{2} = \binom{|U|+4m}{2} = \binom{n}{2}$.

On the other hand, $\delta_1(H) \ge (\frac{7}{16} - \gamma)\binom{n}{2}$ implies that $\sum_{u\in U}\deg(u) \ge (\frac{7}{16} - \gamma)\binom{n}{2}|U|$. Together with (5.7), this gives

$$|\mathcal{T}_{\le 6}| \le \gamma\binom{n}{2}|U| + 2^{22}\binom{m}{2}. \tag{5.8}$$

Note that (5.4) implies that $|U| < \frac{3}{4}n < \frac{3}{4}16m = 12m$. By (5.5) and (5.8), we have

$$\begin{aligned}|\mathcal{T}_7^1| &\ge \binom{m}{2}|U| - \left(\binom{m}{2}\cdot(2^{16}\cdot 5 + 756) + m|U|\right) - \gamma\binom{n}{2}|U| - 2^{22}\binom{m}{2} \\ &\ge \binom{m}{2}|U| - \gamma\binom{n}{2}|U| - 2^{23}\binom{m}{2} \quad \text{as } |U| < 12m \\ &\ge \binom{m}{2}|U| - \gamma\binom{n}{2}|U| - 2^{19}\binom{n}{2} \quad \text{as } m < \frac{n}{4} \\ &> \binom{m}{2}|U| - \gamma n^2|U| \quad \text{as } |U| > 2^{19}/\gamma. \end{aligned}$$

□

For a triple $uij \in \mathcal{T}_7^1$, we call $v_1 \in V_i$ and $v_2 \in V_j$ a pair of centers (in short, *centers*) for u if $\{v_1, v_2\}$ is the vertex cover of $L_{i,j}(u)$. Define G as the graph on the vertex set V' such that two vertices $v_1, v_2 \in V'$ are adjacent if and only if there are at least 16 vertices $u \in U$ such that v_1, v_2 are centers for u. Let C be the set of vertices $v \in V'$ such that $\deg_G(v) \ge 7$ and $\deg_G(v') \ge 2$ for some $v' \in N_G(v)$, where $N_G(v)$ denotes the neighborhood of v in G.

Fact 5.17. *For every $i \in [m]$, at most one vertex $v \in V_i$ satisfies* $\deg_G(v) > 0$.

Proof. Suppose to the contrary, some $V_i = \{a, b, c, d\}$ satisfies $\deg_G(a), \deg_G(b) > 0$. Let $a' \in N_G(a), b' \in N_G(b)$. First, assume that both a' and b' are in V_j for some $j \in [m] \setminus i$. Furthermore, assume $a' \neq b'$ and say $V_j = \{a', b', c', d'\}$. Then by the definition of G, we can find $u_1, \ldots, u_4, u'_1, \ldots, u'_4 \in U$ such that a, a' are centers for u_l and b, b' are centers for u'_l for $l = 1, \ldots, 4$. This gives four copies of $\mathcal{C}$ on $ac'u_1u_2, a'cu_3u_4, bd'u'_1u'_2, b'du'_3u'_4$. Replacing $\mathcal{C}_i, \mathcal{C}_j$ by them in $\mathscr{C}$ gives a larger $\mathcal{C}$-tiling, a contradiction. Otherwise, assume that $a' = b'$ and say $V_j = \{a', x, c', d'\}$. Then by the definition of G, we can find $u_1, \ldots, u_4, u'_1, u'_2 \in U$ such that a, a' are centers for u_l, $l = 1, \ldots, 4$ and b, a' are centers for u'_1 and u'_2. This gives three copies of $\mathcal{C}$ on $ac'u_1u_2, a'cu_3u_4, bd'u'_1u'_2$. Replacing $\mathcal{C}_i, \mathcal{C}_j$ by them in $\mathscr{C}$ gives a larger $\mathcal{C}$-tiling, a contradiction.

Second, assume that $a' \in V_j$ and $b' \in V_k$ for distinct $j, k \in [m] \setminus i$. Let $c' \in V_j \setminus a'$ and $d' \in V_k \setminus b'$. Then by the definition of G, we can find $u_1, \ldots, u_4, u'_1, \ldots, u'_4 \in U$ such that a, a' are centers for u_l and b, b' are centers for u'_l, for $l = 1, \ldots, 4$. This gives four copies of $\mathcal{C}$ on $ac'u_1u_2, a'cu_3u_4, bd'u'_1u'_2$, $b'du'_3u'_4$. Replacing $\mathcal{C}_i, \mathcal{C}_j, \mathcal{C}_k$ by them in $\mathscr{C}$ gives a larger $\mathcal{C}$-tiling, a contradiction. □

Claim 5.18. $(1 - 2^{11}\gamma)m \leq |C| \leq m$.

Proof. The upper bound follows from Fact 5.17 immediately.

To see the lower bound, we first show that

$$e(G) \geq (1 - 2^{10}\gamma)\binom{m}{2}. \tag{5.9}$$

To see this, let M be the set of pairs $i, j \in \binom{[m]}{2}$ such that there are at most 240 vertices $u \in U$ satisfying that $uij \in \mathcal{T}_7^1$. By Claim 5.16, the number of triples $uij \notin \mathcal{T}_7^1$ ($u \in U$, $i \neq j \in [m]$) is at most $\gamma n^2 |U|$. Thus

$$|M| \leq \frac{\gamma n^2 |U|}{|U| - 240} \leq \frac{\gamma n^2 |U|}{\frac{2}{3}|U|} = \frac{3\gamma n^2}{2} < \frac{3\gamma(16m)^2}{2} < 2^{10}\gamma\binom{m}{2}.$$

where the second last inequality follows from (5.4). Fix a pair $i, j \in \binom{[m]}{2} \setminus M$. There are

at least $241 = 16 \cdot 15 + 1$ vertices $u \in U$ satisfying that $uij \in \mathcal{T}_7^1$. Since $V_i \times V_j$ contains 16 pairs of vertices, by the pigeonhole principle, some pair of vertices $v_1 \in V_i, v_2 \in V_j$ are centers for at least 16 vertices $u \in U$, namely, $v_1v_2 \in G$. Thus (5.9) follows.

By Fact 5.17, there are at most m vertices with positive degree in G. For convenience, define $V'' \subset V'$ as an arbitrary set of m vertices that contains all the vertices with positive degree in G. Furthermore, for any integer $t < m$, let $D_t \subseteq V''$ denote the set of vertices v such that $\deg_G(v) \leq t$. Let $D_2' \subseteq (V'' \setminus D_1)$ denote the set of vertices v such that $N_G(v) \subseteq D_1$. We have

$$2e(G) \leq t|D_t| + (m-1)(m - |D_t|) = m(m-1) - (m-t-1)|D_t|.$$

Together with (5.9), it gives $|D_t| \leq 2^{10}\gamma\frac{m(m-1)}{m-t-1}$. By definition, each vertex $v \in D_2'$ satisfies $\deg_G(v) \geq 2$, and its neighborhood is contained in D_1 (thus the vertices in D_2' have disjoint neighborhoods). This implies that $|D_2'| \leq |D_1|/2$. Recall that $C = V'' \setminus (D_6 \cup D_2')$. Since D_6 and D_2' are not necessarily disjoint,

$$|C| \geq m - |D_6| - |D_2'| \geq m - 2^{10}\gamma\frac{m(m-1)}{m-7} - 2^{10}\gamma\frac{m(m-1)}{2(m-2)} \geq (1 - 2^{11}\gamma)m.$$

as claimed. □

Let I_C be the set of all $i \in [m]$ such that $V_i \cap C \neq \emptyset$. Fact 5.17 and Claim 5.18 together imply that $|I_C| = |C| \geq (1 - 2^{11}\gamma)m$. Let $A = (\bigcup_{i \in I_C} V_i \setminus C) \cup U$.

Claim 5.19. *$H[A]$ contains no copy of $\mathcal{C}$, thus $e(A) \leq \frac{1}{3}\binom{n}{2}$.*

Proof. The first half of the claim implies the second half by Fact 5.13. Suppose instead, $H[A]$ contains a copy of $\mathcal{C}$, denoted by $\mathcal{C}_0$, on V_0. Since $H[U]$ contains no copy of $\mathcal{C}$, V_0 must intersect some V_i with $i \in I_C$. Without loss of generality, suppose that $V_1, \ldots, V_j$ contain the vertices of $V_0 \setminus U$ for some $1 \leq j \leq 4$. Here we separate two cases.

Case 1. For any $i \in [j]$, $|V_i \cap V_0| \leq 2$.

For $i \in [j]$, let $c_i = V_i \cap C$, and suppose that $d_i \in V_i \setminus (V_0 \cup c_i)$. For each $i \in [j]$, since $\deg_G(c_i) \geq 7$, we can pick distinct $v_i \in N_G(c_i) \setminus (V_1 \cup \cdots \cup V_j)$. By Fact 5.17, $v_1, \ldots, v_j$ are contained in different members of $\mathscr{C}$ (also different from $\mathcal{C}_1, \ldots, \mathcal{C}_j$). Let $v'_1, \ldots, v'_j$ be arbitrary vertices in these members of $\mathscr{C}$, respectively, which are different from $v_1, \ldots, v_j$. For every $i \in [j]$, since c_i, v_i are centers for at least 16 vertices of U, we find a different set of four vertices $u_i^1, \ldots, u_i^4 \in U \setminus V_0$ such that c_i, v_i are centers for them. This is possible because $|V_0 \cap U| \leq 4 - j$ and the number of available vertices in U is thus at least $16 - (4 - j) = 12 + j \geq 4j$.

Note that for $i \in [j]$, $c_i v'_i u_i^1 u_i^2, d_i v_i u_i^3 u_i^4$ span two copies of $\mathcal{C}$. Together with $\mathcal{C}_0$, this gives $2j + 1$ copies of $\mathcal{C}$ while using vertices from $2j$ members of $\mathscr{C}$, contradicting the maximality of $\mathscr{C}$.

Case 2. There exists $i_0 \in [j]$, such that $|V_{i_0} \cap V_0| = 3$.

Note that $j = 1$ or 2 in this case. Without loss of generality, assume that $|V_1 \cap V_0| = 3$. First assume that $j = 1$ (then $|V_0 \cap U| = 1$). Let $c_1 = V_1 \cap C$. By the definition of C, there exists $c_2 \in N_G(c_1)$ such that $\deg_G(c_2) \geq 2$. Let $c_3 \neq c_1$ be a neighbor of c_2 in G. Assume that $\mathcal{C}_{i_2}, \mathcal{C}_{i_3} \in \mathscr{C}$ contains c_2, c_3, respectively. By the definition of G, we can find $u_1, \ldots, u_6 \in U \setminus V_0$ such that c_1, c_2 are centers for u_1, u_2, and c_2, c_3 are centers for u_3, u_4, u_5, u_6. Thus, $c_1 w_1 u_1 u_2$, $c_2 w_3 u_3 u_4$, $c_3 w_2 u_5 u_6$ span three copies of $\mathcal{C}$, where w_1, w_2 are two vertices in $V_{i_2} \setminus \{c_2\}$ and $w_3 \in V_{i_3} \setminus \{c_3\}$. Together with $\mathcal{C}_0$, it gives four copies of $\mathcal{C}$ while using vertices from three members of $\mathscr{C}$, contradicting the maximality of $\mathscr{C}$.

Now assume that $j = 2$, that is, $|V_0 \cap V_2| = 1$. We pick $c_2, c_3, u_1, \ldots, u_6$ in the same way as in the $j = 1$ case. If $c_2 \in V_2$, then this gives four copies of $\mathcal{C}$ by using vertices from three members of $\mathscr{C}$, a contradiction. Otherwise, let $c_4 = V_2 \cap C$ and pick $c_5 \in N_G(c_4) \setminus \{c_1, c_2, c_3\}$ (this is possible because $\deg_G(c_4) \geq 7$). Suppose that $\mathcal{C}_{i_5}$ contains c_5. We pick four new vertices $u_7, \ldots, u_{10} \in U$ for whom c_4, c_5 are centers. Thus, we can form two copies of $\mathcal{C}$ by using vertices from $\mathcal{C}_2, \mathcal{C}_{i_5}$ and $u_7, \ldots, u_{10}$. Together with the four copies of $\mathcal{C}$ given in the previous case, we obtain six copies of $\mathcal{C}$ while using vertices from five members of $\mathscr{C}$, a contradiction. $\square$

Note that the edges not incident to C are either contained in A or incident to some V_i, $i \notin I_C$. By Claim 5.19, C is incident to all but at most

$$\begin{aligned} e(A) + 4 \cdot 2^{11}\gamma m\binom{n-1}{2} &< \frac{1}{3}\binom{n}{2} + 2^{10}\gamma(4m)n^2 \\ &< 2^{10}\gamma n^2\left(\frac{1}{2^{10}\gamma} + 4m\right) < 2^{10}\gamma n^3, \end{aligned}$$

edges, where the last inequality holds because $|U| > \frac{1}{2^{10}\gamma}$. Since $|C| \leq m \leq n/4$, we can pick a set $B \subseteq V \setminus C$ of order $\lfloor \frac{3}{4}n \rfloor$. Then $e(B) < 2^{10}\gamma n^3$, which implies that H is $2^{10}\gamma$-extremal. □

In Claim 5.19 we proved that $H[A]$ contains no copy of $\mathcal{C}$, where, by Claim 5.18,

$$|A| = n - m - 3(m - |C|) \geq n - \frac{n}{4} - 3 \cdot 2^{11}\gamma m \geq (1 - 2^{11}\gamma)\frac{3}{4}n.$$

We summarize this in the following lemma. It is easy to see this lemma is equivalent to Lemma 8.7.

Lemma 5.20. *For any $\gamma > 0$, there exists an integer n_0 such that the following holds. Suppose H is a 3-graph on $n > n_0$ vertices with*

$$\delta_1(H) \geq \left(\frac{7}{16} - \gamma\right)\binom{n}{2},$$

then there is a $\mathcal{C}$-tiling covering all but at most $2^{19}/\gamma$ vertices of H unless H contains a set of order at least $(1 - 2^{11}\gamma)\frac{3}{4}n$ that contains no copy of $\mathcal{C}$.

5.3 The Extremal Theorem

In this section we prove Theorem 5.5. Let n be sufficiently large and H be a 3-graph on n vertices satisfying (5.1). Assume that H is Δ-extremal, namely, there is a set $B \subseteq V(H)$, such that $|B| = \lfloor \frac{3}{4}n \rfloor$ and $e(B) \leq \Delta n^3$. For the convenience of later calculations, we let

$\epsilon_0 = 18\Delta$ and derive that

$$e(B) < \epsilon_0 \binom{|B|}{3}. \tag{5.10}$$

Let $A = V(H) \setminus B$. We write $E_H(XYZ)$ as XYZ for short.

5.3.1 Classifying vertices

Let $\epsilon_1 = 8\sqrt{\epsilon_0}$. Assume that the partition A and B satisfies that $|B| = \lfloor \frac{3}{4}n \rfloor$ and (5.10). In addition, assume that $e(B)$ is the smallest among all the partitions satisfying the first two conditions. We now define

$$\begin{aligned} A' &:= \left\{ v \in V \mid \deg(v, B) \geq (1-\epsilon_1)\binom{|B|}{2} \right\}, \\ B' &:= \left\{ v \in V \mid \deg(v, B) \leq \epsilon_1 \binom{|B|}{2} \right\}, \\ V_0 &= V \setminus (A' \cup B'). \end{aligned}$$

Claim 5.21. *$A \cap B' \neq \emptyset$ implies that $B \subseteq B'$, and $B \cap A' \neq \emptyset$ implies that $A \subseteq A'$.*

Proof. First, assume that $A \cap B' \neq \emptyset$. Then there is some $u \in A$ satisfies that $\deg(u, B) \leq \epsilon_1 \binom{|B|}{2}$. If there exists some $v \in B \setminus B'$, namely, $\deg(v, B) > \epsilon_1 \binom{|B|}{2}$, then we can switch u and v and form a new partition $A'' \cup B''$ such that $|B''| = |B|$ and $e(B'') < e(B)$, which contradicts the minimality of $e(B)$.

Second, assume that $B \cap A' \neq \emptyset$. Then some $u \in B$ satisfies that $\deg(u, B) \geq (1-\epsilon_1)\binom{|B|}{2}$. Similarly, by the minimality of $e(B)$, we get that for any vertex $v \in A$, $\deg(v, B) \geq (1-\epsilon_1)\binom{|B|}{2}$, which implies that $A \subseteq A'$. □

Claim 5.22. *$\{|A \setminus A'|, |B \setminus B'|, |A' \setminus A|, |B' \setminus B|\} \leq \frac{\epsilon_1}{64}|B|$ and $|V_0| \leq \frac{\epsilon_1}{32}|B|$.*

Proof. First assume that $|B \setminus B'| > \frac{\epsilon_1}{64}|B|$. By the definition of B' and the assumption $\epsilon_1 = 8\sqrt{\epsilon_0}$, we get that

$$e(B) > \frac{1}{3}\epsilon_1 \binom{|B|}{2} \cdot \frac{\epsilon_1}{64}|B| > \frac{\epsilon_1^2}{64}\binom{|B|}{3} = \epsilon_0 \binom{|B|}{3},$$

which contradicts (5.10).

Second, assume that $|A \setminus A'| > \frac{\epsilon_1}{64}|B|$. Then by the definition of A', for any vertex $v \notin A'$, we have that $\overline{\deg}(v, B) > \epsilon_1\binom{|B|}{2}$. So we get

$$\overline{e}(ABB) > \frac{\epsilon_1}{64}|B| \cdot \epsilon_1\binom{|B|}{2} = \epsilon_0|B|\binom{|B|}{2} > 3\epsilon_0\binom{|B|}{3}.$$

Together with (5.10), this implies that

$$\sum_{b\in B}\overline{\deg}(b) \geq 3\overline{e}(B) + 2\overline{e}(ABB) > 3(1-\epsilon_0)\binom{|B|}{3} + 6\epsilon_0\binom{|B|}{3} = 3(1+\epsilon_0)\binom{|B|}{3}.$$

By the pigeonhole principle, there exists $b \in B$, such that

$$\overline{\deg}(b) > (1+\epsilon_0)\binom{|B|}{2} = (1+\epsilon_0)\binom{\lfloor\frac{3n}{4}\rfloor - 1}{2} > \binom{\lfloor\frac{3n}{4}\rfloor}{2},$$

where the last inequality follows from the assumption that n is large enough. This contradicts (5.1).

Consequently,

$$\begin{aligned}
&|A' \setminus A| = |A' \cap B| \leq |B \setminus B'| \leq \frac{\epsilon_1}{64}|B|,\\
&|B' \setminus B| = |A \cap B'| \leq |A \setminus A'| \leq \frac{\epsilon_1}{64}|B|,\\
&|V_0| = |A \setminus A'| + |B \setminus B'| \leq \frac{\epsilon_1}{64}|B| + \frac{\epsilon_1}{64}|B| = \frac{\epsilon_1}{32}|B|.
\end{aligned}$$ □

We next show that we can connect any two vertices of B' with a loose path of length two without using any fixed $\frac{n}{8}$ vertices of V.

Claim 5.23. *For every pair of vertices $u, v \in B'$ and every vertex set $S \subseteq V$ with $|S| \leq n/8$, there exist $a \in A' \setminus S$ and $b_1, b_2 \in B' \setminus S$ such that $ub_1a, ab_2v \in E(H)$.*

Proof. For any $x \in B'$, by (5.1), we have that $\overline{\deg}(x) \leq \binom{\lfloor\frac{3}{4}n\rfloor}{2} = \binom{|B|}{2}$. So by the definition

of B',

$$\overline{\deg}(x, AB) \leq \overline{\deg}(x) - \overline{\deg}(x, B) \leq \binom{|B|}{2} - (1-\epsilon_1)\binom{|B|}{2} = \epsilon_1\binom{|B|}{2}.$$

By Claim 5.22, we get that

$$\begin{aligned}\overline{\deg}(x, A'B') &\leq \overline{\deg}(x, AB) + |A' \setminus A| \cdot |B'| + |B' \setminus B| \cdot |A'| \\ &\leq \epsilon_1\binom{|B|}{2} + \frac{\epsilon_1}{64}|B|n \leq 2\epsilon_1\binom{|B|}{2}. \end{aligned} \tag{5.11}$$

Consider a bipartite graph G on $A \setminus S$ and $B \setminus S$ with pairs $ab \in E(G)$ if and only if $uab, vab \in E(H)$. Since $|S| \leq \frac{n}{8}$, we have $|A \setminus S| \geq \frac{|A|}{2} \geq \frac{|B|}{6}$ and $|B \setminus S| > \frac{|B|}{2}$, so $|A \setminus S| \cdot |B \setminus S| > \frac{1}{6}\binom{|B|}{2} > 8\epsilon_1\binom{|B|}{2}$. Consequently,

$$e(G) \geq |A \setminus S| \cdot |B \setminus S| - 4\epsilon_1\binom{|B|}{2} \geq \frac{1}{2}|A \setminus S| \cdot |B \setminus S| > |A \setminus S|.$$

Hence there exists a vertex $a \in A \setminus S$ such that $\deg_G(a) \geq 2$. By picking $b_1, b_2 \in N_G(a)$ we finish the proof. □

5.3.2 Building a short path

Claim 5.24. *Suppose that $|A \cap B'| = q > 0$. Then there exists a family $\mathcal{P}_1$ of vertex disjoint loose paths in B', where*

$$\mathcal{P}_1 \text{ consists of } \begin{cases} \text{one edge} & \text{if } q = 1 \text{ and } n \notin 4\mathbb{N} \\ \text{two edges } e_1, e_2 \text{ with } |e_1 \cap e_2| \leq 1 & \text{if } q = 1 \text{ and } n \in 4\mathbb{N} \\ 2q \text{ disjoint edges} & \text{if } q \geq 2 \end{cases}$$

Proof. Let $|A \cap B'| = q > 0$. Since $A \cap B' \neq \emptyset$, by Claim 5.21, we get $B \subseteq B'$, which implies $|B'| = \lfloor \frac{3}{4}n \rfloor + q$.

By Claim 5.22, we get that $q = |A \cap B'| \leq |A \setminus A'| \leq \frac{\epsilon_1}{64}|B|$. Hence for any vertex b in

B',

$$\begin{aligned}\deg(b, B') &\le \deg(b, B) + |B' \setminus B|(|B'| - 1) \\ &\le \epsilon_1 \binom{|B|}{2} + q(|B'| - 1) < 2\epsilon_1 \binom{|B|}{2}. \end{aligned} \tag{5.12}$$

Now we assume that $q = 1$, so $|B'| - 1 = \lfloor \frac{3}{4}n \rfloor$. By (5.1), for any $b \in B'$,

$$\deg(b, B') \ge \binom{n-1}{2} - \binom{\lfloor \frac{3}{4}n \rfloor}{2} + c - \left[\binom{n-1}{2} - \binom{|B'| - 1}{2} \right] = c,$$

where $c = 1$ if $n \notin 4\mathbb{N}$ and $c = 2$ otherwise. The $n \notin 4\mathbb{N}$ case is trivial since B' actually contains at least $|B'|/3 > 1$ edges. If $n \in 4\mathbb{N}$, then we have $\deg(b, B') \ge 2$. Assume that B' does not contain the desired structure. Then any two distinct edges of B' share exactly two vertices. Fix an edge $e_0 = v_1v_2v_3$ of B' and two vertices $u, u' \in B' \setminus e_0$. Then every edge of B' containing u must have its two other vertices in e_0. Since $\deg(u, B') \ge 2$, the link graph of u contains at least two pairs of vertices of e_0. So does the link graph of u'. We thus find a loose path of length two from u to u' because two distinct pairs on e_0 share exactly one vertex.

Second, assume that $q > 1$. In this case we construct $2q$ disjoint edges greedily. By (5.1) and $|B'| = \lfloor \frac{3}{4}n \rfloor + q$, for any $b \in B'$,

$$\begin{aligned}\deg(b, B') &\ge \binom{n-1}{2} - \binom{\lfloor \frac{3}{4}n \rfloor}{2} + c - \left[\binom{n-1}{2} - \binom{|B'| - 1}{2} \right] \\ &> \binom{|B'| - 1}{2} - \binom{\lfloor \frac{3}{4}n \rfloor}{2} \\ &\ge (q-1) \left\lfloor \frac{3}{4}n \right\rfloor, \end{aligned}$$

which implies that $e(B') > \frac{1}{3}|B'|(q-1)\lfloor \frac{3}{4}n \rfloor$. Suppose we have found $i < 2q$ disjoint edges of B'. By (5.12), there are at most $3(2q-1) \cdot 2\epsilon_1 \binom{|B|}{2}$ edges of B' intersecting these i edges.

Hence, there are at least

$$\begin{aligned} e(B') - 3(2q-1)\cdot 2\epsilon_1\binom{|B|}{2} &\geq \frac{1}{3}|B'|(q-1)\left\lfloor\frac{3}{4}n\right\rfloor - 6(2q-1)\epsilon_1\binom{|B|}{2} \\ &\geq \frac{2(q-1)}{3}\binom{|B|}{2} - 6(2q-1)\epsilon_1\binom{|B|}{2} \\ &= \frac{2}{3}\left[(q-1) - 9(2q-1)\epsilon_1\right]\binom{|B|}{2} \end{aligned}$$

edges not intersecting the existing i edges. This quantity is positive provided that $\epsilon_1 < \frac{q-1}{9(2q-1)}$. Thus, $\epsilon_1 < \frac{1}{27}$ suffices since the minimum of $\frac{q-1}{9(2q-1)}$, $q > 1$ is $\frac{1}{27}$ attained by $q = 2$. □

Remark 5.25. *Claim 5.24 is the only place where the constant c from* (5.1) *is used.*

The goal of this subsection is to prove the following claim.

Claim 5.26. *There exists a loose path P in H with the following properties:*

- $V_0 \subseteq V(P)$,
- $|V(P)| \leq \frac{\epsilon_1}{4}|B|$,
- $|B' \setminus V(P)| \leq 3|A' \setminus V(P)| - 1$,
- *two ends of P are in B'.*

Proof. We split into two cases here.

Case 1. $A \cap B' \neq \emptyset$.

By Claim 5.21, $A \cap B' \neq \emptyset$ implies that $B \subseteq B'$, which implies that $V_0 \subseteq A$. Let $q = |A \cap B'|$. We first apply Claim 5.24 and find a family $\mathcal{P}_1$ of vertex disjoint loose paths on at most $6q$ vertices of B'. Next we put each vertex of V_0 into a loose path of length two with four vertices from B (so in B') such that these paths are pairwise vertex disjoint and also vertex disjoint from the paths in $\mathcal{P}_1$. Let $V_0 = \{x_1, \ldots, x_{|V_0|}\}$. Suppose that we have found loose paths for $x_1, \ldots, x_i$ with $i < |V_0|$. Since $A \setminus A' = V_0 \dot\cup (A \cap B')$, by Claim 5.22, we have

$$q + |V_0| = |A \setminus A'| \leq \frac{\epsilon_1}{64}|B|. \tag{5.13}$$

Thus,

$$4i + 6q < 4|V_0| + 6q \leq 6(|V_0| + q) \leq \frac{3\epsilon_1}{32}|B|$$

and consequently at most $\frac{3\epsilon_1}{32}|B|(|B|-1) = \frac{3\epsilon_1}{16}\binom{|B|}{2}$ pairs of B intersect the existing paths. By the definition of V_0, $\deg(x_{i+1}, B) > \epsilon_1\binom{|B|}{2}$. Since every graph on $n \geq 4$ vertices and $m \geq n$ edges contains two vertex disjoint edges, we can find two vertex disjoint pairs in the link graph of x_{i+1} in B.

Denote by $\mathcal{P}_2$ the family of the loose paths that we obtained so far. Now we want to glue paths of $\mathcal{P}_2$ together to a single loose path. For this purpose, we apply Claim 5.23 repeatedly to connect the ends of two loose paths while avoiding previously used vertices. This is possible because $|V(\mathcal{P}_2)| \leq 5|V_0| + 6q$ and at most $3(|V_0| + 2q - 1)$ vertices will be used to connect the paths in $\mathcal{P}_2$. By (5.13), the resulting loose path P satisfies

$$|V(P)| \leq 8|V_0| + 12q - 3 < 12 \cdot \frac{\epsilon_1}{64}|B| < \frac{\epsilon_1}{4}|B|.$$

We next show that $|B' \setminus V(P)| \leq 3|A' \setminus V(P)| - 1$. To prove this, we split into three cases according to the structure of $\mathcal{P}_1$. Note that $|B'| = \lfloor \frac{3n}{4} \rfloor + q$ and $|A'| = \lceil \frac{n}{4} \rceil - |V_0| - q$.

First, assume that $q > 1$. Our construction shows that $\mathcal{P}_1$ consists of $2q$ disjoint edges in B'. So $|V(P) \cap A'| = |V_0| + 2q - 1$ and $|V(P) \cap B'| = 4|V_0| + 3 \cdot 2q + 2(|V_0| + 2q - 1) = 6|V_0| + 10q - 2$. Thus,

$$\begin{aligned} |B' \setminus V(P)| &= \left\lfloor \frac{3n}{4} \right\rfloor + q - (6|V_0| + 10q - 2) \\ &\leq 3\left(\left\lceil \frac{n}{4} \right\rceil - 2|V_0| - 3q + 1 \right) - 1 = 3|A' \setminus V(P)| - 1. \end{aligned}$$

Second, assume that $q = 1$ and $n \in 4\mathbb{N}$. Then $\mathcal{P}_1$ consists of a loose path of length two or two disjoint edges. For the first case, we have that $|V(P) \cap A'| = |V_0|$ and $|V(P) \cap B'| = 4|V_0| + 2|V_0| + 5 = 6|V_0| + 5$. Thus,

$$|B' \setminus V(P)| = \frac{3n}{4} + 1 - (6|V_0| + 5) = 3\left(\frac{n}{4} - 2|V_0| - 1 \right) - 1 = 3|A' \setminus V(P)| - 1.$$

In the second case, we have that $|V(P)\cap A'| = |V_0|+1$ and $|V(P)\cap B'| = 4|V_0|+2(|V_0|+1)+6 = 6|V_0|+8$. Thus,

$$|B' \setminus V(P)| = \frac{3n}{4} + 1 - (6|V_0|+8) = 3\left(\frac{n}{4} - 2|V_0| - 2\right) - 1 = 3|A' \setminus V(P)| - 1.$$

Third, assume that $q = 1$ and $n \notin 4\mathbb{N}$, so $\mathcal{P}_1$ contains only one edge. We have $|V(P) \cap A'| = |V_0|$ and $|V(P) \cap B'| = 4|V_0| + 2|V_0| + 3 = 6|V_0| + 3$. Let $n = 4k + 2$ with some $k \in \mathbb{Z}$, so $|A| = k + 1$, $|B| = 3k + 1$, $|B'| = 3k + 2$ and $|A'| = k - |V_0|$. Thus,

$$|B' \setminus V(P)| = 3k + 2 - (6|V_0| + 3) = 3(k - 2|V_0|) - 1 = 3|A' \setminus V(P)| - 1.$$

Case 2. $A \cap B' = \emptyset$.

Note that $A \cap B' = \emptyset$ means that $B' \subseteq B$. The difference from the first case is that we do not need to construct $\mathcal{P}_1$.

First we will put every vertex in V_0 into a loose path of length two together with four vertices from B'. By Claim 5.22, $|B \setminus B'| \leq \frac{\epsilon_1}{64}|B|$ and thus for any vertex $x \in V_0$,

$$\deg(x, B') \geq \deg(x, B) - |B \setminus B'| \cdot (|B| - 1) \geq \epsilon_1 \binom{|B|}{2} - \frac{\epsilon_1}{32}\binom{|B|}{2}. \tag{5.14}$$

Similar as in Case 1, let $V_0 = \{x_1, \ldots, x_{|V_0|}\}$ and suppose that we have found loose paths for $x_1, \ldots, x_i$ with $i < |V_0|$. By Claim 5.22, $|V_0| \leq \frac{\epsilon_1}{32}|B|$. Thus, we have $4i < 4|V_0| \leq \frac{\epsilon_1}{8}|B|$ and consequently at most $\frac{\epsilon_1}{8}|B|(|B'|-1) \leq \frac{\epsilon_1}{4}\binom{|B|}{2}$ pairs of B' intersect the existing i loose paths. Then by (5.14), we may find two vertex disjoint pairs in the link graph of x_{i+1} in B'.

As in Case 1, we connect the paths that we obtained to a single loose path by applying Claim 5.23 repeatedly. The resulting loose path P satisfies that

$$|V(P)| = 5|V_0| + 3(|V_0| - 1) < 8 \cdot \frac{\epsilon_1}{32}|B| = \frac{\epsilon_1}{4}|B|.$$

We next show that $|B' \setminus V(P)| \leq 3|A' \setminus V(P)| - 1$. Note that $|V(P) \cap A'| = |V_0| - 1$ and

$|V(P)\cap B'| = 4|V_0|+2(|V_0|-1) = 6|V_0|-2$. Since $B' \subseteq B$, we have $|A'| \geq |A'\cap A| = \lceil \frac{n}{4} \rceil - |V_0|$. Thus,

$$\begin{aligned} |B' \setminus V(P)| &= |B'| - (6|V_0| - 2) \leq 3\left\lceil \frac{n}{4} \right\rceil - 6|V_0| + 2 \\ &\leq 3\left(|A'| + |V_0| - 2|V_0| + 1\right) - 1 \\ &= 3(|A'| - |V(P) \cap A'|) - 1 = 3|A' \setminus V(P)| - 1. \end{aligned}$$ □

5.3.3 Completing a Hamilton cycle

Let P be the loose path given by Claim 5.26. Suppose that $|B'\setminus V(P)| = 3|A'\setminus V(P)| - l$ for some integer $l \geq 1$. Since P is a loose path, $|V(P)|$ is odd. Since $V = A' \cup B' \cup V_0$ and $V_0 \subset V(P)$, we have

$$|V(P)| + |B' \setminus V(P)| + |A' \setminus V(P)| = n. \tag{5.15}$$

Since n is even, it follows that $|B' \setminus V(P)| + |A' \setminus V(P)|$ is odd, which implies that $l = 3|A' \setminus V(P)| - |B' \setminus V(P)|$ is odd.

If $l > 1$, then we extend P as follows. Starting from an end u of P (note that $u \in B'$), we add an edge by using one vertex from A' and one from B'. This is guaranteed by Claim 5.23, which actually provides a loose path starting from u. We repeat this $\frac{l-1}{2}$ times. The resulting loose path P' satisfies $|B' \setminus V(P')| = 3|A' \setminus V(P')| - 1$. We claim that $|V(P')| \leq \frac{3\epsilon_1}{4}|B|$ (thus Claim 5.23 can be applied repeatedly). Indeed, by (5.15) and $|V(P)| \leq \frac{\epsilon_1}{4}|B|$,

$$\begin{aligned} l &= 3|A' \setminus V(P)| - |B' \setminus V(P)| = 4|A' \setminus V(P)| - (n - |V(P)|) \\ &\leq 4|A'| - n + \frac{\epsilon_1}{4}|B|. \end{aligned}$$

Since $|A'| \leq |A| + |B \setminus B'| \leq \lceil \frac{n}{4} \rceil + \frac{\epsilon_1}{64}|B|$ from Claim 5.22, we have $l \leq \frac{\epsilon_1}{2}|B|$. Since $|V(P')| = |V(P)| + l - 1$, we derive that $|V(P')| \leq \frac{3\epsilon_1}{4}|B|$.

Finally, since both ends of P' are vertices in B', we extend P' by one more ABB edge from each end, respectively. Denote the ends of the resulting path Q be $x_0, x_1 \in A'$. Let

$A_1 = (A' \setminus V(Q)) \cup \{x_0, x_1\}$ and $B_1 = B' \setminus V(Q)$. Note that we have $|B_1| = 3(|A_1| - 1)$. By Claim 5.22, we have $|B_1 \setminus B| \le |B' \setminus B| \le \frac{\epsilon_1}{64}|B|$. Furthermore,

$$|B_1| \ge |B'| - \frac{3\epsilon_1}{4}|B| \ge |B| - \frac{\epsilon_1}{64}|B| - \frac{3\epsilon_1}{4}|B| - 2 > (1-\epsilon_1)|B|. \tag{5.16}$$

For a vertex $v \in A_1$, since $\overline{\deg}(v, B) \le \epsilon_1 \binom{|B|}{2}$, we have

$$\begin{aligned}
\overline{\deg}\,(v, B_1) &\le \overline{\deg}(v, B) + |B_1 \setminus B| \cdot (|B_1| - 1) \\
&\le \epsilon_1 \binom{|B|}{2} + \frac{\epsilon_1}{64}|B| \left(1 + \frac{\epsilon_1}{64}\right)|B| \\
&< 2\epsilon_1 \binom{|B|}{2} < 3\epsilon_1 \binom{|B_1|}{2},
\end{aligned}$$

where the last inequality follows from (5.16). In addition, (5.11) and (5.16) imply that for any vertex $v \in B_1$,

$$\overline{\deg}(v, A_1B_1) \le \overline{\deg}(v, A'B') \le 2\epsilon_1 \binom{|B|}{2} < \epsilon_1 |B|^2 < 4\epsilon_1 |A_1||B_1|.$$

We finally complete the proof of Theorem 5.5 by applying the following lemma with $X = A_1$, $Z = B_1$, and $\rho = 4\epsilon_1$.

Lemma 5.27. *Suppose that $0 < \rho < 10^{-8}$ and n is sufficiently large. Let H be a 3-graph on n vertices with $V(H) = X \dot\cup Z$ such that $|Z| = 3(|X| - 1)$. Further, assume that for every vertex $v \in X$, $\overline{\deg}(v, Z) \le \rho \binom{|Z|}{2}$ and for every vertex $v \in Z$, $\overline{\deg}(v, XZ) \le \rho |X||Z|$. Then given any two vertices $x_0, x_1 \in X$, there is a loose Hamilton path from x_0 to x_1.*

Let us introduce some terminology. A bipartite graph $G = (A, B, E)$ with $|A| = |B| = n$ is called (d, ϵ)-regular if for any two subsets $A' \subseteq A$, $B' \subseteq B$ with $|A'|, |B'| \ge \epsilon n$,

$$(1-\epsilon)d \le \frac{e(A', B')}{|A'||B'|} \le (1+\epsilon)d,$$

and G is called (d, ϵ)-super-regular if in addition for every $v \in A \cup B$, $(1-\epsilon)dn \le \deg(v) \le$

$(1+\epsilon)dn$.

We use the following result of Kühn and Osthus in [43] in the proof of Lemma 5.27.

Lemma 5.28. *[43] For all positive constants $d, v_0, \eta \leq 1$ there is a positive $\epsilon = \epsilon(d, v_0, \eta)$ and an integer $N_0 = N_0(d, v_0, \eta)$ such that the following holds for all $n \geq N_0$ and all $v \geq v_0$. Let $G = (A, B, E)$ be a (d, ϵ)-super-regular bipartite graph whose vertex classes both have size n and let F be a subgraph of G with $|F| = v|E|$. Choose a perfect matching M uniformly at random in G. Then with probability at least $1 - e^{-\epsilon n}$ we have*

$$(1-\eta)vn \leq |M \cap E(F)| \leq (1+\eta)vn.$$

Proof of Lemma 5.27. Let G be the graph of all pairs uv in Z such that $\deg(uv, X) \geq (1-\sqrt{\rho})|X|$. We claim that for any vertex $v \in Z$,

$$\overline{\deg}_G(v) \leq \sqrt{\rho}|Z|. \tag{5.17}$$

Otherwise, some vertex $v \in Z$ satisfies $\overline{\deg}_G(v) > \sqrt{\rho}|Z|$. As each $u \notin N_G(v)$ satisfies $\overline{\deg}_H(uv, X) > \sqrt{\rho}|X|$, we have

$$\overline{\deg}_H(v, XZ) > \sqrt{\rho}|Z| \cdot \sqrt{\rho}|X| = \rho|Z||X|,$$

contradicting our assumption.

Arbitrarily partition Z into three sets Z_1, Z_2, Z_3, each of order $|X|-1$. By (5.17), we have $\overline{\deg}_G(v) \leq \sqrt{\rho}|Z| = 3\sqrt{\rho}(|X|-1)$ and $\delta(G[Z_1, Z_2]), \delta(G[Z_2, Z_3]) \geq (1-3\sqrt{\rho})(|X|-1)$. Thus, $G[Z_1, Z_2]$ and $G[Z_2, Z_3]$ are both $(1, 3\sqrt[4]{\rho})$-super-regular. For any $x \in X$, let $F_x^1 := \{zz' \in E(G[Z_1, Z_2]) : \{x, z, z'\} \in E(H)\}$ and let $F_x^2 := \{zz' \in E(G[Z_2, Z_3]) : \{x, z, z'\} \in E(H)\}$. Since $\overline{\deg}(x, Z) \leq \rho\binom{|Z|}{2} \leq 5\rho|X|^2$, we have $|F_x^1|, |F_x^2| \geq (1-3\sqrt{\rho})(|X|-1)^2 - 5\rho|X|^2 \geq (1-4\sqrt{\rho})(|X|-1)^2$. By applying Lemma 5.28 with $v = 1-4\sqrt{\rho}$ and $\eta = \rho$, then for any

$x \in X$, with probability at least $1 - e^{-\epsilon|X|}$ we have

$$|M_1 \cap E(F_x^1)|, |M_2 \cap E(F_x^2)| \geq (1-\eta)v(|X|-1) \geq (1-5\sqrt{\rho})|X|.$$

Thus, there is a matching M_1 in $G[Z_1, Z_2]$ and a matching M_2 in $G[Z_2, Z_3]$ such that $|M_1 \setminus F_x^1| \leq 5\sqrt{\rho}|X|$ and $|M_2 \setminus F_x^2| \leq 5\sqrt{\rho}|X|$ for every vertex $x \in X$. Label the vertices of Z so that $Z_1 = \{a_1, \ldots, a_{|X|-1}\}$, $Z_2 = \{b_1, \ldots, b_{|X|-1}\}$ and $Z_3 = \{c_1, \ldots, c_{|X|-1}\}$ such that $M_1 = \{a_1b_1, \ldots, a_{|X|-1}b_{|X|-1}\}$ and $M_2 = \{b_1c_1, \ldots, b_{|X|-1}c_{|X|-1}\}$. Let Γ be a bipartite graph with one part X and the other part $\{a_1b_1c_1, \ldots, a_{|X|-1}b_{|X|-1}c_{|X|-1}\}$ such that $\{x, a_ib_ic_i\} \in E(\Gamma)$ if and only if $xa_ib_i, xb_ic_i \in E(H)$. For every $1 \leq i \leq |X|-1$, since $a_ib_i, b_ic_i \in E(G)$, so $\deg_\Gamma(a_ib_ic_i) \geq (1-2\sqrt{\rho})|X|$ in Γ. On the other hand, by assumptions, we have $\deg_\Gamma(x) \geq (1-10\sqrt{\rho})|X|$ for any $x \in X$. Thus it is easy to see that there is a Hamilton path in Γ with ends x_0, x_1. Since for each $1 \leq i \leq |X|-1$, $\{x_i, a_ib_ic_i\}, \{x_{i+1}, a_ib_ic_i\} \in E(\Gamma)$ implies that $x_ia_ib_i, b_ic_ix_{i+1} \in E(H)$ (let $x_{|X|} = x_0$), we get a loose Hamilton path of H as

$$x_1\, a_1\, b_1\, c_1\, x_2\, a_2\, b_2\, c_2 \cdots x_{|X|-1}\, a_{|X|-1}\, b_{|X|-1}\, c_{|X|-1} x_{|X|} (= x_0) \qquad \square$$

PART 6

NEAR PERFECT MATCHINGS IN K-UNIFORM HYPERGRAPHS

6.1 Introduction

Given $k \geq 2$, a k-uniform hypergraph (in short, *k-graph*) consists of a vertex set $V(H)$ and an edge set $E(H) \subseteq \binom{V(H)}{k}$, where every edge is a k-element subset of $V(H)$. A *matching* in H is a collection of vertex-disjoint edges of H. A *perfect matching* M in H is a matching that covers all vertices of H. Clearly a perfect matching in H exists only if k divides $|V(H)|$. When k does not divide $n = |V(H)|$, we call a matching M in H a *near perfect matching* if $|M| = \lfloor n/k \rfloor$.

Given a k-graph H with a set S of d vertices (where $1 \leq d \leq k-1$) we define $\deg_H(S)$ to be the number of edges containing S (the subscript H is omitted if it is clear from the context). The *minimum d-degree* $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d-vertex sets S in H. We refer to $\delta_{k-1}(H)$ as the *minimum codegree* of H.

Over the last few years there has been a strong focus in establishing minimum d-degree thresholds that force a perfect matching in a k-graph [1, 8, 17, 35, 36, 42, 47, 54, 57, 58, 61, 63, 69]. In particular, Rödl, Ruciński and Szemerédi [63] determined the minimum codegree threshold that ensures a perfect matching in a k-graph on n vertices for all $k \geq 3$ and sufficiently large $n \in k\mathbb{N}$. The threshold is $\frac{n}{2} - k + C$, where $C \in \{3/2, 2, 5/2, 3\}$ depends on the values of n and k. In contrast, they proved that the minimum codegree threshold that ensures a near perfect matching in a k-graph on $n \notin k\mathbb{N}$ vertices is between $\lfloor \frac{n}{k} \rfloor$ and $\frac{n}{k} + O(\log n)$. It is conjectured, in [63] and [58, Problem 3.3], that this threshold should be $\lfloor \frac{n}{k} \rfloor$. In this chapter we verify this conjecture.

Theorem 6.1. *For any integer $k \geq 3$, let n be a sufficiently large integer which is not divisible by k. Suppose H is a k-uniform hypergraph on n vertices with $\delta_{k-1}(H) \geq \lfloor \frac{n}{k} \rfloor$. Then H contains a matching of size $\lfloor \frac{n}{k} \rfloor$.*

It is also natural to ask for the minimum codegree threshold for the *matching number* of k-graphs, namely, the size of a maximum matching. The following theorem [63, Fact 2.1] is obtained by a greedy algorithm. Let $\nu(H)$ be the size of a maximum matching in H.

Theorem 6.2. *[63] Let $n \geq k \geq 2$. For every k-uniform hypergraph H on n vertices,*

$$\nu(H) \geq \delta_{k-1}(H) \text{ if } \delta_{k-1}(H) \leq \left\lfloor \frac{n}{k} \right\rfloor - k + 2.$$

Note that for $n \in k\mathbb{N}$ and $\frac{n}{k} \leq \delta_{k-1}(H) \leq \frac{n}{2} - k$, H may not contain a perfect matching, namely, a matching of size $\frac{n}{k}$ (see [63]). So the only open cases are when $\lfloor \frac{n}{k} \rfloor - k + 3 \leq \delta_{k-1}(H) < \frac{n}{k}$. In this note, we close this gap for large n.

Corollary 6.3. *For any integer $k \geq 3$, let n be a sufficiently large integer. For every k-uniform hypergraph H on n vertices,*

$$\nu(H) \geq \delta_{k-1}(H) \text{ if } \delta_{k-1}(H) < \frac{n}{k}.$$

Proof. Let $\delta_{k-1}(H) = \lfloor \frac{n}{k} \rfloor - c$. We only prove Corollary 6.3 in the cases when $1 \leq c \leq k-3$, since Theorem 6.2 covers the cases when $c \geq k-2$ and Theorem 6.1 covers the case when $\delta_{k-1}(H) = \lfloor \frac{n}{k} \rfloor < \frac{n}{k}$. Let $r \equiv n \mod k$ such that $0 \leq r \leq k-1$. Note that $\lfloor \frac{n}{k} \rfloor = \lfloor \frac{n+c}{k} \rfloor$ if $r + c < k$ and $\lfloor \frac{n}{k} \rfloor + 1 = \lfloor \frac{n+c+1}{k} \rfloor$ otherwise. For the first case, we add c vertices to H and get H' such that H' contains all edges of H and all k-sets containing any of these new vertices. Note that H' has $n+c$ vertices and $\delta_{k-1}(H') = \lfloor \frac{n+c}{k} \rfloor$. Moreover, k does not divide $n+c$ since $1 \leq r+c < k$. We apply Theorem 6.1 on H' and get a near perfect matching M of H'. Deleting up to c edges from M that contain the new vertices, we get a matching in H of size $\lfloor \frac{n}{k} \rfloor - c$.

In the second case, we add $c+1$ vertices to H and get H' such that H' contains all edges of H and all k-sets containing any of these new vertices. Note that H' has $n+c+1$ vertices and $\delta_{k-1}(H') = \lfloor \frac{n}{k} \rfloor + 1 = \lfloor \frac{n+c+1}{k} \rfloor$. Moreover, k does not divide $n+c+1$ since $k+1 \leq r+c+1 \leq 2k-3$. Similarly we apply Theorem 6.1 on H' and get a near perfect

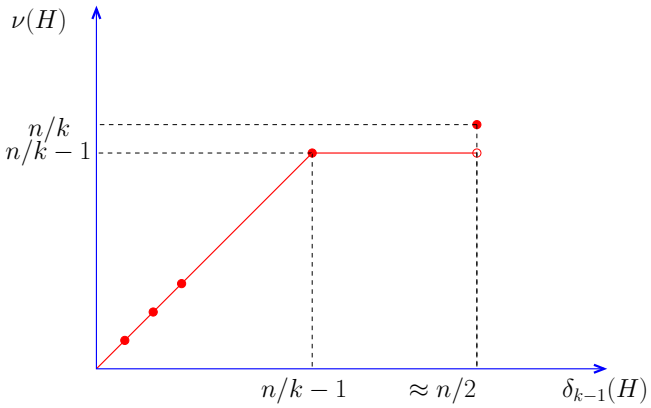

Figure 6.1. $\delta_{k-1}(H)$ and $\nu(H)$

matching M of H'. Deleting up to $c+1$ edges from M that contain the new vertices, we get a matching in H of size $\lfloor \frac{n}{k} \rfloor + 1 - (c+1) = \lfloor \frac{n}{k} \rfloor - c$. □

It is easy to see that Theorem 6.1 and Corollary 6.3 are best possible. For an integer $0 \le d < \frac{n}{k}$, let H be a k-graph with a partition $A \cup B$ of the vertex set $V(H)$ such that $|A| = d$ and $E(H)$ consists of all k-tuples that intersect A. Since every edge intersects A, we have $\nu(H) = \delta_{k-1}(H) = |A| = d$.

Let us describe this interesting phenomenon by the following dynamic process (see Figure 6.1). Consider a k-graph H on n vertices with $E(H) = \emptyset$ at the beginning and add edges to $E(H)$ gradually. Corollary 6.3 says $\nu(H) \ge \delta_{k-1}(H)$ when $\delta_{k-1}(H) < \frac{n}{k}$. In order to guarantee a perfect matching, $\delta_{k-1}(H)$ needs to be about $n/2$ [63].

As a typical approach to obtain exact results, our proof of Theorem 6.1 consists of an *extremal case* and a *nonextremal case*. We say that H is γ-*extremal* if $V(H)$ contains an independent subset B of order at least $(1-\gamma)\frac{k-1}{k}n$.

Theorem 6.4 (Nonextremal case)**.** *For any integer $k \ge 3$ and constant $\gamma > 0$, there is an integer n_0 such that the following holds. Let $n \ge n_0$ be an integer not divisible by k and let H be an n-vertex k-graph with $\delta_{k-1}(H) \ge \frac{n}{k} - \gamma n$. If H is not $5k\gamma$-extremal, then H contains a near perfect matching.*

Theorem 6.5 (Extremal case)**.** *For any integer $k \geq 3$, there exist an $\epsilon > 0$ and an integer n_1 such that the following holds. Let $n \geq n_1$ be an integer not divisible by k and let H be an n-vertex k-graph with $\delta_{k-1}(H) \geq \lfloor \frac{n}{k} \rfloor$. If H is ϵ-extremal, then H contains a near perfect matching.*

Theorem 6.1 follows from Theorem 6.4 and Theorem 6.5 immediately.

We prove Theorem 6.4 by the absorbing method, initiated by Rödl, Ruciński and Szemerédi [60]. Given a set S of $k+1$ vertices, we call an edge $e \in E(H)$ disjoint from S *S-absorbing* if there are two disjoint edges e_1 and e_2 in $E(H)$ such that $|e_1 \cap S| = k-1$, $|e_1 \cap e| = 1$, $|e_2 \cap S| = 2$, and $|e_2 \cap e| = k-2$. Note that this is not the absorbing in the usual sense because $e_1 \cup e_2$ misses one vertex of $S \cup e$. Let us explain how such absorbing works. Let S be a $(k+1)$-set and M be a matching, where $V(M) \cap S = \emptyset$, which contains an S-absorbing edge e. Then M can "absorb" S by replacing e in M by e_1 and e_2 (one vertex of e becomes uncovered). The following absorbing lemma was proved in [63, Fact 2.3] with the conclusion that *the number of S-absorbing edges in M is at least $k-2$*. However, its proof shows that $k-2$ can be replaced by any constant. Note that we do not require that k does not divide n in Lemma 6.6 and Lemma 6.7.

Lemma 6.6. *[63, Absorbing lemma] For all $c, \gamma > 0$ there exist $C > 0$ and n_2 such that if H is a k-graph with $n \geq n_2$ vertices and $\delta_{k-1}(H) \geq cn$, then there exists a matching M' in H of size $|M'| \leq C \log n$ and such that for every $(k+1)$-tuple S of vertices of H, the number of S-absorbing edges in M' is at least k/γ.*

We also need the following lemma, which provides a matching that covers all but a constant number of vertices when H is not extremal.

Lemma 6.7 (Almost perfect matching)**.** *For any integer $k \geq 3$ and constant $\gamma > 0$ the following holds. Let H be an n-vertex k-graph such that n is sufficiently large and $\delta_{k-1}(H) \geq \frac{n}{k} - \gamma n$. If H is not $2k\gamma$-extremal, then H contains a matching that covers all but at most k^2/γ vertices.*

Now let us compare our proof with the proof in [63], which showed that $\delta_{k-1}(H) \geq \frac{n}{k} + O(\log n)$ guarantees a near perfect matching. In [63], the authors first build an absorbing matching of size $C \log n$ and then apply Theorem 6.2 in the remaining k-graph. Finally, they absorb the leftover vertices and get the near perfect matching. In our proof, instead of Theorem 6.2, we apply Lemma 6.7 after building the absorbing matching. Lemma 6.7 only requires a weaker degree condition $\delta_{k-1}(H) \geq \frac{n}{k} - \gamma n$ and the condition that H is not extremal. We then handle the extremal case separately.

6.2 Proof of Theorem 6.4

In this section we prove Theorem 6.4 with the help of Lemma 6.6 and Lemma 6.7.

Proof of Lemma 6.7. Let $M = \{e_1, e_2, \ldots, e_m\}$ be a maximum matching of size m in H. Let V' be the set of vertices covered by M and let U be the set of vertices which are not covered by M. We assume that H is not $2k\gamma$-extremal and $|U| > k^2/\gamma$. Note that U is an independent set by the maximality of M. We arbitrarily partition all but at most $k-2$ vertices of U as disjoint $(k-1)$-sets $A_1, \ldots, A_t$ where $t = \lfloor \frac{|U|}{k-1} \rfloor > \frac{k}{\gamma}$.

Let D be the set of vertices $v \in V'$ such that $\{v\} \cup A_i \in E(H)$ for at least k sets A_i, $i \in [t]$. We claim that $|e_i \cap D| \leq 1$ for any $i \in [m]$. Otherwise, assume that $x, y \in e_i \cap D$. By the definition of D, we can pick A_i, A_j for some distinct $i, j \in [t]$ such that $\{x\} \cup A_i \in E(H)$ and $\{y\} \cup A_j \in E(H)$. We obtain a matching of size $m+1$ by replacing e_i in M by $\{x\} \cup A_i$ and $\{y\} \cup A_j$, contradicting the maximality of M.

Next we show that $|D| \geq (\frac{1}{k} - 2\gamma)n$. By the minimum degree condition, we have

$$t\left(\frac{1}{k} - \gamma\right) n \leq \sum_{i=1}^{t} \deg(A_i) \leq |D|t + n \cdot k,$$

where we use the fact that U is an independent set. So we get

$$|D| \geq \left(\frac{1}{k} - \gamma\right) n - \frac{nk}{t} > \left(\frac{1}{k} - 2\gamma\right) n,$$

where we use $t > k/\gamma$.

Let $V_D := \bigcup\{e_i, e_i \cap D \neq \emptyset\}$. Note that $|V_D \setminus D| = (k-1)|D| \geq (k-1)(\frac{1}{k} - 2\gamma)n$. Since H is not $2k\gamma$-extremal, $H[V_D \setminus D]$ contains at least one edge, denoted by e_0. We assume that e_0 intersects $e_{i_1}, \ldots, e_{i_l}$ in M for some $2 \leq l \leq k$. Suppose $\{v_{i_j}\} = e_{i_j} \cap D$ for all $j \in [l]$. By the definition of D, we can greedily pick $A_{i_1}, \ldots, A_{i_l}$ such that $\{v_{i_j}\} \cup A_{i_j} \in E(H)$ for all $j \in [l]$. Let M'' be the matching obtained from replacing the edges $e_{i_1}, \ldots, e_{i_l}$ by e_0 and $\{v_{i_j}\} \cup A_{i_j}$ for $j \in [l]$. Thus, M'' has $m+1$ edges, contradicting the maximality of M. □

Now we prove Theorem 6.4.

Proof of Theorem 6.4. Suppose H is a k-graph on $n \notin k\mathbb{N}$ vertices with $\delta_{k-1}(H) \geq n/k - \gamma n$ and H is not $5k\gamma$-extremal. In particular, $\gamma < \frac{1}{5k}$. Since $\delta_{k-1}(H) \geq \frac{n}{2k}$, we first apply Lemma 6.6 on H with $c = \frac{1}{2k}$ and find the absorbing matching M' of size at most $C \log n$ such that for every set S of $k+1$ vertices of H, the number of S-absorbing edges in M' is at least k/γ.

Let $H' = H[V(H) \setminus V(M')]$ and $n' = |V(H')|$. Note that $\delta_{k-1}(H') \geq \delta_{k-1}(H) - kC \log n > (\frac{1}{k} - 2\gamma)n'$. If H' is $4k\gamma$-extremal, namely, $V(H')$ contains an independent set B of order at least $(1 - 4k\gamma)\frac{k-1}{k}n'$, then since

$$(1 - 4k\gamma)\frac{k-1}{k}n' \geq (1 - 5k\gamma)\frac{k-1}{k}n,$$

we get that H is $5k\gamma$-extremal, a contradiction. Thus, H' is not $4k\gamma$-extremal and we can apply Lemma 6.7 on H' with parameter 2γ and get a matching M'' in H' that covers all but at most $k^2/(2\gamma)$ vertices. Since for every $(k+1)$-tuple S in $V(H)$, the number of S-absorbing edges in M' is at least k/γ, we can repeatedly absorb the leftover vertices (at most $k/(2\gamma)$ times, each time the number of leftover vertices is reduced by k) until the number of leftover vertices is at most k (strictly less than k by the assumption). Let $\tilde{M}$ denote the absorbing matching after the absorption. Then $\tilde{M} \cup M''$ is the desired near perfect matching in H. □

6.3 Proof of Theorem 6.5

We prove Theorem 6.5 in this section. We use the following result of Pikhurko [57], stated here in a less general form.

Theorem 6.8. *[57, Theorem 3] Let H be a k-partite k-graph with k-partition $V(H) = V_1 \cup V_2 \cup \cdots \cup V_k$ such that $|V_i| = m$ for all $i \in [k]$. Let $\delta_{\{1\}}(H) = \min\{|N(v_1)| : v_1 \in V_1\}$ and*

$$\delta_{[k]\setminus\{1\}}(H) = \min\{|N(v_2, \ldots, v_k)| : v_i \in V_i \text{ for every } 2 \le i \le k\}.$$

For sufficiently large integer m, if

$$\delta_{\{1\}}(H)m + \delta_{[k]\setminus\{1\}}(H)m^{k-1} \ge \frac{3}{2}m^k,$$

then H contains a perfect matching.

Proof of Theorem 6.5. Fix a sufficiently small $\epsilon > 0$. Suppose n is sufficiently large and not divisible by k. Let H be a k-graph on n vertices satisfying $\delta_{k-1}(H) \ge \lfloor \frac{n}{k} \rfloor$. Assume that H is ϵ-extremal, namely, there is an independent set $S \subseteq V(H)$ with $|S| \ge (1-\epsilon)\frac{k-1}{k}n$.

We partition $V(H)$ as follows. Let $\alpha = \epsilon^{1/2}$. Let C be a maximum independent set of $V(H)$. Define

$$A = \left\{ x \in V \setminus C : \deg(x, C) \ge (1-\alpha)\binom{|C|}{k-1} \right\}, \tag{6.1}$$

and $B = V \setminus (A \cup C)$. We first observe the following bounds of $|A|, |B|, |C|$.

Proposition 6.9. *$|A| \ge \lfloor \frac{n}{k} \rfloor - \alpha n$, $|B| \le \alpha n$, and $(1-\epsilon)\frac{(k-1)n}{k} \le |C| \le \lceil \frac{(k-1)n}{k} \rceil$.*

Proof. The lower bound for $|C|$ follows from our hypothesis immediately. For any $S \subseteq C$ of order $k-1$, we have $N(S) \subseteq A \cup B$. By the minimum degree condition, we have

$$\left\lfloor \frac{n}{k} \right\rfloor \le |N(S)| \le |A| + |B| = n - |C| \le \frac{n}{k} + \epsilon\frac{(k-1)n}{k}, \tag{6.2}$$

which gives the upper bound for $|C|$. By the definitions of A and B, we have

$$\left\lfloor \frac{n}{k} \right\rfloor \binom{|C|}{k-1} \leq e((A \cup B)C^{k-1}) \leq (1-\alpha)\binom{|C|}{k-1}|B| + \binom{|C|}{k-1}|A|,$$

where $e((A \cup B)C^{k-1})$ denotes the number of edges that contains $k-1$ vertices in C and one vertex in $A \cup B$. Thus, we get $\left\lfloor \frac{n}{k} \right\rfloor \leq |A| + |B| - \alpha|B|$, which gives that $\alpha|B| \leq |A| + |B| - \left\lfloor \frac{n}{k} \right\rfloor \leq \epsilon n$ by (6.2). So $|B| \leq \alpha n$ and $|A| \geq \left\lfloor \frac{n}{k} \right\rfloor - |B| \geq \left\lfloor \frac{n}{k} \right\rfloor - \alpha n$. □

We will build four disjoint matchings M_1, M_2, M_3, and M_4 in H, whose union gives the desired near perfect matching in H. Let $r \equiv n \mod k$ and $1 \leq r \leq k-1$. Note that $\left\lfloor \frac{n}{k} \right\rfloor = \frac{n-r}{k}$. For $i \in [3]$, let $A_i = A \setminus V(\cup_{j\in[i]} M_j)$ and $C_i = C \setminus V(\cup_{j\in[i]} M_j)$ be the sets of uncovered vertices of A and C, respectively. Let $n_i = |V(H) \setminus V(\cup_{j\in[i]} M_j)|$ and note that $n_i \equiv r \mod k$.

Step 1. Small matchings M_1 and M_2 covering B.

We build the first matching M_1 on vertices of $B \cup C$ of size t only if $t := \left\lfloor \frac{n}{k} \right\rfloor - |A| > 0$. Note that it is possible that $t \leq 0$ – in this case $M_1 = \emptyset$. By Proposition 6.9, we know that $t = \left\lfloor \frac{n}{k} \right\rfloor - |A| \leq \alpha n$. Since $\delta_{k-1}(H) \geq \left\lfloor \frac{n}{k} \right\rfloor$ and by the definition of t, we have $\delta_{k-1}(H[B \cup C]) \geq t$. Since $|C| \leq \left\lceil \frac{(k-1)n}{k} \right\rceil$, we have $|B| = n - |C| - |A| \geq \left\lfloor \frac{n}{k} \right\rfloor - |A| = t$. We pick arbitrary t disjoint $(k-1)$-sets from C. Since C is an independent set, each of the $(k-1)$-sets has at least t neighbors in B, so we can choose a matching M_1 of size t.

Next we build the second matching M_2 that covers all the vertices in $B \setminus V(M_1)$. For each $v \in B \setminus V(M_1)$, we pick $k-2$ arbitrary vertices from C not covered by the existing matching, and an uncovered vertex $v \in V$ to complete an edge and add it to M_2. Since $\delta_{k-1}(H) \geq \left\lfloor \frac{n}{k} \right\rfloor$ and the number of vertices covered by the existing matching is at most $k|B| \leq k\alpha n < \left\lfloor \frac{n}{k} \right\rfloor$, such an edge always exists.

Our construction guarantees that each edge in $M_1 \cup M_2$ contains at least one vertex from B and thus $|M_1 \cup M_2| \leq |B|$. We claim that $|A_1| \geq \frac{n_1-r}{k}$ and $|A_2| \geq \frac{n_2-r}{k}$. To see the

bound for $|A_1|$, we separate two cases depending on t. When $t > 0$, since $|M_1| = t$, we have

$$|A_1| = \frac{n-r}{k} - t = \frac{n-r-k|M_1|}{k} = \frac{n_1-r}{k}.$$

Otherwise $t \leq 0$, we have $n_1 = n$ and $|A_1| = |A| \geq \frac{n-r}{k} = \frac{n_1-r}{k}$. For the bound for $|A_2|$, since each edge of M_2 contains at most one vertex of A, we have

$$|A_2| \geq |A_1| - |M_2| \geq \frac{n_1-r}{k} - |M_2| = \frac{n_2-r}{k}.$$

Let $s := |A_2| - \frac{n_2-r}{k} \geq 0$. Since $n_2 = n - k|M_1 \cup M_2| \geq n - k|B| \geq n - k\alpha n$ and $|C| \geq (1-\epsilon)\frac{(k-1)n}{k}$ (Proposition 6.9), we get

$$s \leq n - |C| - \frac{n-k\alpha n-r}{k} \leq \epsilon\frac{(k-1)n}{k} + \alpha n + 1 \leq 2\alpha n.$$

Step 2. A small matching M_3.

Starting with $M_3 = \emptyset$, we will greedily add at most $2\alpha n$ edges to M_3 from $A_2 \cup C_2$ until we have $|A_3| - \frac{n_3-r}{k} \in \{0, 1\}$. Indeed, throughout the process, denote by n' the number of uncovered vertices of H and denote by A', C' the set of uncovered vertices in A, C, respectively. Let $c = |A'| - \frac{n'-r}{k}$. If $c \geq k-1$, then we arbitrarily pick $k-1$ vertices from A' and a vertex from $A' \cup C'$ to form an edge. As a result, $|A'| - \frac{n'-r}{k}$ decreases by $k-1$ or $k-2$. If $c < k-1$, then we pick c vertices from A', $k-c-1$ vertices from C', and form an edge with some vertex from $A' \cup C'$. In this case, $|A'| - \frac{n'-r}{k}$ decreases by c or $c-1$. The iteration stops when $|A'| - \frac{n'-r}{k}$ becomes 0 or 1 after at most $\lceil \frac{s}{k-2} \rceil \leq s \leq 2\alpha n$ steps. Note that we can always form an edge in each step because the number of covered vertices is at most $k|B| + k \cdot 2\alpha n \leq 3k\alpha n < \delta_{k-1}(H)$. So we get a matching M_3 of at most $2\alpha n$ edges.

Step 3. The last matching M_4.

Now we have two cases, $|A_3| - \frac{n_3-r}{k} = 0$ or 1. In the first case, we will find a matching M_4 of size $|A_3|$ which leaves r vertices in C_3. In the second case, we will find a matching M_4

of size $|A_3| - 1$ which leaves one vertex in A_3 and $r-1$ vertices in C_3. Note that in either case we are done since $M = M_1 \cup M_2 \cup M_3 \cup M_4$ is a matching that covers all but r vertices of $V(H)$.

We define A_3' and C_3' as follows. If $|A_3| - \frac{n_3 - r}{k} = 0$, we let $A_3' = A_3$ and obtain C_3' by deleting arbitrary r vertices from C_3. Otherwise, we obtain A_3' by deleting one arbitrary vertex from A_3 and obtain C_3' by deleting $r-1$ arbitrary vertices from C_3. Note that in both cases, we have $|A_3'| - \frac{|A_3'| + |C_3'|}{k} = 0$, which implies $|C_3'| = (k-1)|A_3'|$. Furthermore, we have

$$|A_3'| \geq |A| - |M_1 \cup M_2| - |M_3| - 1 \geq \left\lfloor \frac{n}{k} \right\rfloor - \alpha n - \alpha n - 2\alpha n - 1 \geq \left\lfloor \frac{n}{k} \right\rfloor - 5\alpha n,$$

because $|M_1 \cup M_2| \leq |B| \leq \alpha n$ and $|M_3| \leq 2\alpha n$.

Let $m := |A_3'|$. Next, we partition C_3' arbitrarily into $k-1$ parts $C^1, C^2, \ldots, C^{k-1}$ of the same size m. We want to apply Theorem 6.8 on the k-partite k-graph $H' := H[A_3', C^1, \ldots, C^{k-1}]$. Let us verify the assumptions. First, since C_3' is independent, for any set of $k-1$ vertices $v_1, \ldots, v_{k-1}$ such that $v_i \in C^i$ for $i \in [k-1]$, the number of its non-neighbors in $A \cup B$ is at most

$$|A| + |B| - \left\lfloor \frac{n}{k} \right\rfloor \leq \frac{n}{k} + \epsilon \frac{(k-1)n}{k} - \left\lfloor \frac{n}{k} \right\rfloor \leq \epsilon n \leq 2k\epsilon m,$$

where we use (6.2) and the last inequality follows from $m = |A_3'| \geq \lfloor \frac{n}{k} \rfloor - 5\alpha n > \frac{k-1}{k^2} n$. So we have $\delta_{[k]\setminus\{1\}}(H') \geq m - 2k\epsilon m = (1 - 2k\epsilon)m$. Next, by (6.1), for any $v \in A_3'$, we have

$$\overline{\deg}_H(v, C) \leq \alpha \binom{|C|}{k-1} \leq \alpha \frac{|C|^{k-1}}{(k-1)!} \leq \alpha \frac{\left(\frac{k-1}{k} n\right)^{k-1}}{(k-1)!} \leq \alpha \frac{(km)^{k-1}}{(k-1)!} = \alpha c_k m^{k-1},$$

where $c_k = \frac{k^{k-1}}{(k-1)!}$. This implies that $\delta_{\{1\}}(H') \geq (1 - \alpha c_k) m^{k-1}$. Thus, we have

$$\delta_{\{1\}}(H')m + \delta_{[k]\setminus\{1\}}(H')m^{k-1} \geq (1 - \alpha c_k)m^{k-1}m + (1 - 2k\epsilon)mm^{k-1} > \frac{3}{2}m^k.$$

By Theorem 6.8, we find a perfect matching M_4 on $V(H') = A_3' \cup C_3'$. □

PART 7

DECISION PROBLEM FOR PERFECT MATCHINGS IN DENSE K-UNIFORM HYPERGRAPHS

7.1 Introduction

The question of whether a given k-graph H contains a perfect matching is one of the most fundamental questions of combinatorics. In the graph case $k = 2$, Tutte's Theorem [71] gives necessary and sufficient conditions for H to contain a perfect matching, and Edmonds' Algorithm [11] finds such a matching in polynomial time. However, for $k \geq 3$ this problem was one of Karp's celebrated 21 NP-complete problems [26]. Since the general problem is intractable provided P $\neq$ NP, it is natural to ask conditions on H which make the problem tractable or even guarantee that a perfect matching exists. One well-studied class of such conditions are minimum degree conditions.

7.1.1 Perfect matchings under minimum degree conditions

Given a k-graph H with a set S of d vertices (where $1 \leq d \leq k-1$) we define $\deg_H(S)$ to be the number of edges containing S (the subscript H is omitted if it is clear from the context). The *minimum d-degree* $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d-vertex sets S in H. We refer to $\delta_{k-1}(H)$ as the *minimum codegree* of H.

Over the last few years there has been a strong focus in establishing minimum d-degree thresholds that force a perfect matching in a k-graph. In particular, Rödl, Ruciński and Szemerédi [63] determined the minimum codegree threshold that ensures a perfect matching in a k-graph on n vertices for large n and all $k \geq 3$. The threshold is $n/2 - k + C$, where $C \in \{3/2, 2, 5/2, 3\}$ depends on the values of n and k. In contrast, they proved that a k-graph H on n vertices satisfying $\delta_{k-1}(H) \geq n/k + O(\log n)$ contains a matching of size $n/k - 1$ (one edge away from a perfect matching). We improved this result in Chapter 6 by

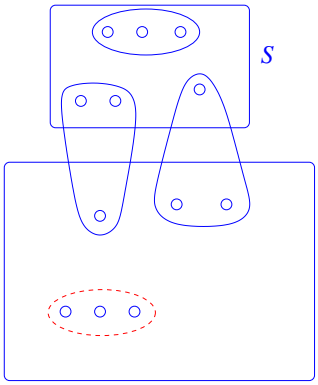

Figure 7.1. Space Barrier for $k = 3$

showing that $\delta_{k-1}(H) \geq n/k - 1$ suffices. The following construction, usually called *space barrier*, shows that this is best possible.

Construction 7.1 (Space Barrier)**.** *Let V be a set of size n and fix $S \subseteq V$ with $|S| < n/k$. Let H be the k-graph whose edges are all k-sets that intersect S.*

Note that the minimum codegree of H is $|S|$ and any matching in Construction 7.1 has at most $|S|$ edges.

Let $\mathbf{PM}(k, \delta)$ be the decision problem of determining whether a k-graph H with $\delta_{k-1}(H) \geq \delta n$ contains a perfect matching. Given the result of [63], a natural question to ask is the following: For which values of δ can $\mathbf{PM}(k, \delta)$ be decided in polynomial time? This holds for $\mathbf{PM}(k, 1/2)$ by the main result of [63]. On the other hand, $\mathbf{PM}(k, 0)$ includes no degree restriction on H at all, so is NP-complete by the result of Karp [26]. Szymańska [67] proved that for $\delta < 1/k$ the problem $\mathbf{PM}(k, \delta)$ admits a polynomial-time reduction to $\mathbf{PM}(k, 0)$ and hence $\mathbf{PM}(k, \delta)$ is also NP-complete. Karpiński, Ruciński and Szymańska showed that there exists $\epsilon > 0$ such that $\mathbf{PM}(k, 1/2 - \epsilon)$ is in P and asked the complexity of $\mathbf{PM}(k, \delta)$ for $\delta \in [1/k, 1/2)$.

Problem 7.2. *[27] What is the computational complexity of $\boldsymbol{PM}(k, \delta)$ for $\delta \in [1/k, 1/2)$?*

Recently, Keevash, Knox and Mycroft [31] gave a long and involved proof that shows

$\mathbf{PM}(k,\delta)$ is in P for any $\delta > 1/k$ that leaves only $\mathbf{PM}(k,1/k)$ unknown. Moreover, they also constructed a polynomial-time algorithm to find a perfect matching provided one exists. They [30] also expected that it would be difficult to solve the decision problem for $\delta = 1/k$, as n/k is the minimum codegree threshold at which a perfect fractional matching is guaranteed, so there is a clear behavioral change at this point. In this chapter, we give a short proof that shows $\mathbf{PM}(k,\delta)$ is in P for all $\delta \geq 1/k$ and thus solve Problem 7.2 completely.

Theorem 7.3. *Fix $k \geq 3$. Let H be an n-vertex k-graph with $\delta_{k-1}(H) \geq n/k$. Then there is an algorithm with running time $O(n^{3k^2-5k})$, which determines whether H contains a perfect matching.*

The proof of Theorem 7.3 follows the approach of [31], from which we use several definitions and results. The heart of the algorithm in that paper was a structural theorem [31, Theorem 1.10], which was proved by partitioning the k-graph H into a number of k-partite k-graphs, before finding a perfect matching in each of these k-partite k-graphs by using a theorem of Keevash and Mycroft [34]. Our main improvement is to replace this by a new structural theorem (Theorem 7.12) which significantly simplifies the argument in [31], and which applies in the exact case $\delta_{k-1}(H) \geq n/k$ (the structural theorem of [31] only applied for $\delta_{k-1}(H) \geq n/k + o(n)$). This already provides a polynomial-time algorithm deciding the existence of perfect matchings, and a faster algorithm as claimed in Theorem 7.3 is obtained by combining Theorem 7.12 with ideas from [31]. Our proof of Theorem 7.12 uses a lattice-based absorbing method which does not need the hypergraph regularity lemma or the main result of [34]. This novel approach, which combines the powerful absorbing technique with the 'divisibility barrier' structures considered in [34], may well be useful for other matching problems in hypergraphs.

7.1.2 Lattice-based constructions

It is shown in [34] that a k-graph H has a perfect matching or is close to a family of lattice-based constructions termed "divisibility barriers". The following examples of divisibility barriers were given in [63].

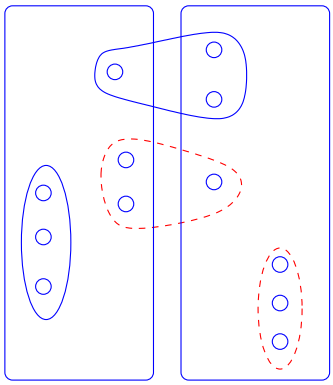

Figure 7.2. Constructions 7.4 and 7.5 coincide when $k = 3$

Construction 7.4. *Let X and Y be disjoint sets such that $|X \cup Y| = n$ and $|X|$ is odd, and let H be the k-graph on $X \cup Y$ whose edges are all k-sets which intersect X in an even number of vertices.*

Construction 7.5. *Let X and Y be disjoint sets such that $|X \cup Y| = n$ and $|X| - n/k$ is odd, and let H be the k-graph on $X \cup Y$ whose edges are all k-sets which intersect X in an odd number of vertices.*

To see why there is no perfect matching in Construction 7.5, note that a perfect matching has n/k edges, intersecting X in $n/k \pmod 2$ number of vertices. Since $|X| \not\equiv n/k \pmod 2$, a perfect matching does not exist. To describe divisibility barriers in general, we make the following definition. In this chapter, every partition has an implicit order on its parts.

Definition 7.6. *Let H be a k-graph and let $\mathcal{P}$ be a partition of $V(H)$ into d parts. Then the* index vector $\mathbf{i}_{\mathcal{P}}(S) \in \mathbb{Z}^d$ *of a subset $S \subseteq V(H)$ with respect to $\mathcal{P}$ is the vector whose coordinates are the sizes of the intersections of S with each part of $\mathcal{P}$, namely, $\mathbf{i}_{\mathcal{P}}(S)_X = |S \cap X|$ for $X \in \mathcal{P}$. Furthermore,*

(i) $I_{\mathcal{P}}(H)$ denotes the set of index vectors $\mathbf{i}_{\mathcal{P}}(e)$ of edges $e \in H$, and

(ii) $L_{\mathcal{P}}(H)$ denotes the lattice (i.e. additive subgroup) in $\mathbb{Z}^d$ generated by $I_{\mathcal{P}}(H)$.

A *divisibility barrier* is a k-graph H which admits a partition $\mathcal{P}$ of its vertex set V such that $\mathbf{i}_{\mathcal{P}}(V) \notin L_{\mathcal{P}}(H)$; To see that such an H contains no perfect matching, let M be a matching in H. Then $\mathbf{i}_{\mathcal{P}}(V(M)) = \sum_{e\in M} \mathbf{i}_{\mathcal{P}}(e) \in L_{\mathcal{P}}(H)$. But $\mathbf{i}_{\mathcal{P}}(V) \notin L_{\mathcal{P}}(H)$, so $V(M) \neq V$, namely, M is not perfect. For example, to see that this generates Construction 7.4, let $\mathcal{P}$ be the partition into parts X and Y; then $L_{\mathcal{P}}(H)$ is the lattice of vectors (x, y) in $\mathbb{Z}^2$ for which x is even and k divides $x + y$, and $|X|$ being odd implies that $\mathbf{i}_{\mathcal{P}}(V) \notin L_{\mathcal{P}}(H)$.

7.2 The Main structural theorem

We need the following definitions from [31] before giving the statement of our structural theorem.

Definition 7.7. *[31] Suppose L is a lattice in $\mathbb{Z}^d$.*

(*i*) *We say that $\mathbf{i} \in \mathbb{Z}^d$ is an r-*vector *if it has non-negative coordinates that sum to r. We write $\mathbf{u}_j$ for the 'unit' 1-vector that is 1 in coordinate j and 0 in all other coordinates.*

(*ii*) *We say that L is an* edge-lattice *if it is generated by a set of k-vectors.*

(*iii*) *We write $L^d_{\max}$ for the lattice generated by all k-vectors. So $L^d_{\max} = \{x \in \mathbb{Z}^d : k \text{ divides } \sum_{i\in[d]} x_i\}$.*

(*iv*) *We say that L is* complete *if $L = L^d_{\max}$, otherwise it is* incomplete.

(*v*) *A* transferral *is a non-zero difference $\mathbf{u}_i - \mathbf{u}_j$ of 1-vectors.*

(*vi*) *We say that L is* transferral-free *if it does not contain any transferral.*

(*vii*) *We say that a set I of k-vectors is* full *if for every $(k-1)$-vector $\mathbf{v}$ there is some $i \in [d]$ such that $\mathbf{v} + \mathbf{u}_i \in I$.*

(*viii*) *We say that L is* full *if it contains a full set of k-vectors and is transferral-free.*

We recall the following construction [31, Construction 1.6] in the case when $k = 4$.

Construction 7.8. *[31] Let $\mathcal{P} = \{V_1, V_2, V_3\}$ be a partition of vertex set $|V| = n$, with $|V_1| = n/3 - 2$, $|V_2| = n/3$ and $|V_3| = n/3 + 2$. Fix some vertex $x \in V_2$, and let H be the 4-graph such that $E(H)$ consists of all k-sets e with $\mathbf{i}_{\mathcal{P}}(e) = (3,0,1), (0,3,1), (0,0,4), (2,2,0)$ or $(1,1,2)$ and all k-sets e containing x with $\mathbf{i}_{\mathcal{P}}(e) = (0,1,3)$.*

Note that $\delta_3(H) = n/3 - 4$. It is not hard to see that $\mathbf{i}_{\mathcal{P}}(V) \in L_{\mathcal{P}}(H)$ but H does not contain a perfect matching. Indeed, if a matching M in H does not contain any edge e with index vector $(0,1,3)$, then $|V(M) \cap V_2| - |V(M) \cap V_1| \equiv 0 \pmod 3$. Otherwise M contains an edge with index vector $(0,1,3)$, thus we have $|V(M) \cap V_2| - |V(M) \cap V_1| \equiv 1 \pmod 3$. In either case, M is not perfect since $|V_2| - |V_1| = 2$. In fact, as shown in [31], $\mathbf{i}_{\mathcal{P}}(V) \in L_{\mathcal{P}}(H)$ holds for any $\mathcal{P}$ of $V(H)$. Thus, having a divisibility barrier is not a necessary condition for H not containing a perfect matching.

Note that when we determine if $\mathbf{i}_{\mathcal{P}}(V) \in L_{\mathcal{P}}(H)$, we are free to use any multiple of any vectors $\mathbf{i} \in I_{\mathcal{P}}(H)$. But in Construction 7.8, all edges e with $\mathbf{i}_{\mathcal{P}}(e) = (0,1,3)$ contain x, thus a matching in H can only contain one edge with index vector $(0,1,3)$. So although $\mathbf{i}_{\mathcal{P}}(V) \in L_{\mathcal{P}}(H)$, there is no perfect matching. Thus, it is natural to consider the following robust edge-lattice such that for every k-vector $\mathbf{i} \in I^{\mu}_{\mathcal{P}}(H)$, there are many edges e such that $\mathbf{i}_{\mathcal{P}}(e) = \mathbf{i}$.

Definition 7.9 (Robust edge-lattices)**.** *Let H be a k-graph and $\mathcal{P}$ be a partition of $V(H)$ into d parts. Then for any $\mu > 0$,*

(i) $I^{\mu}_{\mathcal{P}}(H)$ denotes the set of all $\mathbf{i} \in \mathbb{Z}^d$ such that at least $\mu|V(H)|^k$ edges $e \in H$ have $\mathbf{i}_{\mathcal{P}}(e) = \mathbf{i}$.

(ii) $L^{\mu}_{\mathcal{P}}(H)$ denotes the lattice in $\mathbb{Z}^d$ generated by $I^{\mu}_{\mathcal{P}}(H)$.

We will show that there exists a partition $\mathcal{P}$ of $V(H)$ and $\mu > 0$, such that if $\mathbf{i}_{\mathcal{P}}(V) \in L^{\mu}_{\mathcal{P}}(H)$, then H contains a perfect matching. Indeed, even a weaker condition suffices. If we can find a small matching M such that $\mathbf{i}_{\mathcal{P}}(V \setminus V(M)) \in L^{\mu}_{\mathcal{P}}(H[V \setminus V(M)]) = L^{\mu}_{\mathcal{P}}(H)$, then we can apply our proof above to show that $H[V \setminus V(M)]$ contains a perfect matching M'. Thus

$M \cup M'$ is a perfect matching of H. Note that we can guarantee $L^{\mu}_{\mathcal{P}}(H[V \setminus V(M)]) = L^{\mu}_{\mathcal{P}}(H)$ by selecting μ 'wisely' and requiring that M is small. The following definitions are essentially from [31]. The only difference is that a full pair defined in [31] has at most $k-1$ parts.

Definition 7.10. *[31] Let H be a k-graph.*

(*i*) *A* full pair $(\mathcal{P}, L)$ *for H consists of a partition $\mathcal{P}$ of $V(H)$ into $d \le k$ parts and a full edge-lattice $L \subset \mathbb{Z}_d$.*

(*ii*) *A (possibly empty) matching M of size at most $|\mathcal{P}|-1$ is a* solution *for $(\mathcal{P}, L)$ (in H) if $\mathbf{i}_{\mathcal{P}}(V(H) \setminus V(M)) \in L$; we say that $(\mathcal{P}, L)$ is* soluble *if it has a solution, otherwise* insoluble.

The following lemma provides a partition $\mathcal{P}_0$ such that we can develop the absorbing lemma on the pair $(\mathcal{P}_0, L^{\mu}_{\mathcal{P}_0}(H))$ for some $\mu > 0$. For a small enough $\mu > 0$, $I^{\mu}_{\mathcal{P}_0}(H)$ is full. However, the pair $(\mathcal{P}_0, L^{\mu}_{\mathcal{P}_0}(H))$ may not be full because it may contain transferrals. Then we will obtain a full pair $(\mathcal{P}'_0, L^{\mu}_{\mathcal{P}'_0}(H))$ from the pair $(\mathcal{P}_0, L^{\mu}_{\mathcal{P}_0}(H))$ by iteratively merging parts that contain transferrals.

We call that a vertex u is (β, i)-*reachable* to a vertex v if there are at least βn^{ik-1} $(ik-1)$-sets S such that both $H[S \cup u]$ and $H[S \cup v]$ have perfect matchings. We say a vertex set U is (β, i)-*closed* if any two vertices $u, v \in U$ are (β, i)-reachable to each other. For two partitions $\mathcal{P}, \mathcal{P}'$ of a set V, we say $\mathcal{P}$ *refines* $\mathcal{P}'$ if every vertex class of $\mathcal{P}$ is a subset of some vertex class of $\mathcal{P}'$.

Lemma 7.11. *Given an integer $k \ge 3$, for any $0 < \gamma \ll 1/k$, suppose that $1/n \ll \{\beta, \mu\} \ll \gamma$. Then for each k-graph H on n vertices with $\delta_{k-1}(H) \ge n/k - \gamma n$, we find partitions $\mathcal{P}_0 = \{V_1, \dots, V_d\}$ and $\mathcal{P}'_0 = \{V'_1, \dots, V'_{d'}\}$ of $V(H)$ in time $O(n^{2^{k-1}k+1})$ satisfying the following properties:*

(*i*) *$\mathcal{P}_0$ refines $\mathcal{P}'_0$ and $(\mathcal{P}'_0, L^{\mu}_{\mathcal{P}'_0}(H))$ is a full pair,*

(*ii*) *each partition set of $\mathcal{P}_0$ or $\mathcal{P}'_0$ has size at least $n/k - 2\gamma n$,*

(iii) for each $D \subseteq V(H)$ *such that* $\mathbf{i}_{\mathcal{P}_0'}(D) \in L^{\mu}_{\mathcal{P}_0'}(H)$*, we have* $\mathbf{i}_{\mathcal{P}_0}(D) \in L^{\mu}_{\mathcal{P}_0}(H)$*,*

(iv) for each $i \in [d]$*,* V_i *is* $(\beta, 2^{k-1})$*-closed.*

Given integers $n \geq k \geq 3$, let $\mathcal{H}_{n,k}$ be the collection of k-graphs H such that there is a partition of $V(H) = X \cup Y$ with $n/k - |X|$ is odd and all edges of H intersect X at an odd number of vertices. Note that the members of $\mathcal{H}_{n,k}$ are subhypergraphs of the k-graphs in Construction 7.5 and thus none of them has a perfect matching.

Now we are ready to state our main structural theorem.

Theorem 7.12. *Fix an integer* $k \geq 3$*. Suppose*

$$1/n_0 \ll \{\beta, \mu\} \ll \gamma \ll 1/k.$$

Let H *be a* k*-graph on* $n \geq n_0$ *vertices such that* $\delta_{k-1}(H) \geq n/k$ *with* $\mathcal{P}_0$ *and* $\mathcal{P}_0'$ *found by Lemma 7.11. Then* H *contains a perfect matching if and only if the full pair* $(\mathcal{P}_0', L^{\mu}_{\mathcal{P}_0'}(H))$ *is soluble and* $H \notin \mathcal{H}_{n,k}$*.*

We first prove the forward implication. The following lemma from [31] says that we can omit the condition on the size of M when considering solubility. Although the definition of full pairs is slightly different in [31], the same proof works in our case.

Lemma 7.13. *[31, Lemma 6.9] Let* $(\mathcal{P}, L)$ *be a full pair for a* k*-graph* H*, where* $k \geq 3$*. Then* $(\mathcal{P}, L)$ *is soluble if and only if there exists a matching* M *in* H *such that* $\mathbf{i}_{\mathcal{P}}(V(H) \setminus V(M)) \in L$*.*

Proof of the forward implication of Theorem 7.12. If H contains a perfect matching M, then $\mathbf{i}_{\mathcal{P}_0'}(V(H) \setminus V(M)) = \mathbf{0} \in L^{\mu}_{\mathcal{P}_0'}(H)$. Since $(\mathcal{P}_0', L^{\mu}_{\mathcal{P}_0'}(H))$ is a full pair, by Lemma 7.13, it is soluble. Furthermore, $H \notin \mathcal{H}_{n,k}$ because no member of $\mathcal{H}_{n,k}$ contains a perfect matching. □

The proof of the backward implication is more involved. For this purpose, we develop a lattice-based absorbing method. In order to use the absorbing method, we need to reserve $O(\log n)$ vertices for our absorbing matching and then look for an almost perfect matching

in the remaining k-graph H'. But an almost perfect matching may not exist if H' is close to the space barrier (Construction 7.1). This means that our absorbing technique works only if H is not extremal (not close to the space barrier). So we separate the proof into an *extremal case* and a *non-extremal case* and then handle the extremal case separately. More precisely, we say that H is γ-*extremal* if $V(H)$ contains an independent subset of order at least $(1-\gamma)\frac{k-1}{k}n$. By picking constants $0 < \gamma, \epsilon \ll 1/k$ such that $\epsilon = 11k\gamma$, the backward implication follows from the following two theorems immediately.

Theorem 7.14. *For any $0 < \gamma \ll 1/k$, suppose that $1/n \ll \{\beta, \mu\} \ll \gamma$. Let H be a k-graph on n vertices such that $\delta_{k-1}(H) \geq n/k - \gamma n$ with $\mathcal{P}_0$ and $\mathcal{P}_0'$ found by Lemma 7.11. Moreover, assume H is not $11k\gamma$-extremal and $(\mathcal{P}_0', L^{\mu}_{\mathcal{P}_0'}(H))$ is soluble, then H contains a perfect matching.*

Theorem 7.15. *For any $0 < \epsilon \ll 1/k$ and sufficiently large integer n the following holds. Suppose H is a k-graph on n vertices such that $\delta_{k-1}(H) \geq n/k$ and H is ϵ-extremal. If $H \notin \mathcal{H}_{n,k}$, then H contains a perfect matching.*

Note that we only need that $(\mathcal{P}_0', L^{\mu}_{\mathcal{P}_0'}(H))$ is soluble in the non-extremal case and $H \notin \mathcal{H}_{n,k}$ in the extremal case.

Let us compare our method and the traditional absorbing method and outline our proof of Theorem 7.14. The absorbing method, initialed by Rödl, Ruciński and Szemerédi [60], has been shown efficient in finding spanning structures in graphs and hypergraphs. For example, in order to get a perfect matching in a k-graph H, it is first shown that any k-set has many absorbing sets in H. Then we apply the probabilistic method to find a small matching that can absorb any (much smaller) collection of k-vertex sets.

However, with potential divisibility barriers, we cannot guarantee that every k-vertex set can be absorbed in general unless the minimum codegree is at least $(1/2+\gamma)n$. In this chapter, we develop a lattice-based absorbing method to overcome this difficulty. More precisely, we first find a partition $\mathcal{P}_0 = \{V_1, \ldots, V_d\}$ of $V(H)$ such that any two vertices from the same V_i are reachable to each other (property (iv) of Lemma 7.11). Then we build our

absorbing matching that can absorb any k-set S with index vector $\mathbf{i}_{\mathcal{P}_0}(S) \in I^{\mu}_{\mathcal{P}_0}(H)$. After applying the almost perfect matching theorem (Theorem 7.16), we will have only k vertices left unmatched. Then the solubility condition guarantees that we can release some edges from the partial matching such that the set of unmatched vertices can be partitioned into k-sets $S_1, \dots, S_{d''}$ for some constant d'' such that $\mathbf{i}_{\mathcal{P}_0}(S_i) \in I^{\mu}_{\mathcal{P}_0}(H)$, so we can absorb them by the absorbing matching and get a perfect matching of H.

The rest of the chapter is organized as follows. We prove Theorem 7.14 in Section 7.3 and prove Theorem 7.15 in Section 7.4, respectively. We show the algorithms and prove Theorem 7.3 in Section 7.5.

7.3 The Non-extremal Case

In this section we prove Theorem 7.14.

7.3.1 Tools

Theorem 6.4 easily implies the following theorem.

Theorem 7.16. *Suppose that $1/n \ll \gamma \ll 1/k$ and $n \in k\mathbb{N}$. Let H be a k-graph on n vertices with $\delta_{k-1}(H) \geq n/k - \gamma n$. If H is not $5k\gamma$-extremal, then H contains a matching that leaves k vertices uncovered.*

Although we are one step away from a perfect matching after applying Theorem 7.16, it is not easy to finish the last edge (in many cases impossible). Let us introduce the following definition and result in [31].

Definition 7.17. *Suppose L is an edge-lattice in $\mathbb{Z}^{|\mathcal{P}|}$, where $\mathcal{P}$ is a partition of a set V.*

(*i*) *The* coset group *of $(\mathcal{P}, L)$ is $G = G(\mathcal{P}, L) = L^{|\mathcal{P}|}_{\max}/L$.*

(*ii*) *For any $\mathbf{i} \in L^{|\mathcal{P}|}_{\max}$, the* residue *of $\mathbf{i}$ in G is $R_G(\mathbf{i}) = \mathbf{i} + L$. For any $A \subseteq V$ of size divisible by k, the* residue *of A in G is $R_G(A) = R_G(\mathbf{i}_{\mathcal{P}}(A))$.*

Lemma 7.18. *[31, Lemma 6.4] If $k \geq 3$ and L is a full lattice, then $|G(\mathcal{P}, L)| = |\mathcal{P}|$.*

Suppose I is a set of k-vectors of $\mathbb{Z}^d$ and $\mathbf{i}$ is an l-vector with $k \le l \le k^2$ such that $\mathbf{i}$ can be written as a linear combination of vectors in I, namely,

$$\mathbf{i} = \sum_{\mathbf{v} \in I} a_{\mathbf{v}} \mathbf{v}. \tag{7.1}$$

We denote by $C = C(d, k, I)$ as the maximum of $|a_{\mathbf{v}}|, \mathbf{v} \in I$ over all possible $\mathbf{i}$.

Fix an integer $i > 0$. For a k-vertex set S, we say a set T is an *absorbing i-set for S* if $|T| = i$ and both $H[T]$ and $H[T \cup S]$ contain perfect matchings. Now we may state our absorbing lemma.

Lemma 7.19 (Absorbing Lemma)**.** *Suppose*

$$1/n \ll 1/c \ll \{\beta, \mu\} \ll 1/k, 1/t,$$

and define C as above. Suppose that $\mathcal{P}_0 = \{V_1, \dots, V_d\}$ is a partition of $V(H)$ such that for $i \in [d]$, V_i is (β, t)-closed. Then there is a family $\mathcal{F}_{abs}$ of disjoint tk^2-sets with size at most $c \log n$ such that $H[V(\mathcal{F})]$ contains a perfect matching and every k-vertex set S with $\mathbf{i}_{\mathcal{P}_0}(S) \in I^{\mu}_{\mathcal{P}_0}(H)$ has at least $2k^k C$ absorbing tk^2-sets in $\mathcal{F}_{abs}$.

We postpone the proof of the absorbing lemma to the end of this section and prove the main goal of this section, Theorem 7.14 first.

7.3.2 Proof of Theorem 7.14

Proof of Theorem 7.14. Fix $0 < \gamma \ll 1/k$. Suppose

$$1/n \ll 1/c \ll \{\beta, \mu\} \ll \gamma.$$

Let H be a k-graph on n vertices such that $\delta_{k-1}(H) \ge n/k - \gamma n$ with $\mathcal{P}_0$ and $\mathcal{P}_0'$ found by Lemma 7.11 satisfying properties (i)-(iv). Moreover, assume that H is not $11k\gamma$-extremal and $(\mathcal{P}_0', L^{\mu}_{\mathcal{P}_0'}(H))$ is soluble. Let $\mathcal{P}_0 = \{V_1, \dots, V_d\}$ and $\mathcal{P}_0' = \{V_1', \dots, V_{d'}'\}$ and note that $d' \le d \le k$ by (ii). We first apply Lemma 7.19 on H with $t = 2^{k-1}$ and get a family $\mathcal{F}_{abs}$ of

$2^{k-1}k^2$-sets with size at most $c\log n$ such that every k-set S of vertices with $\mathbf{i}_{\mathcal{P}_0}(S) \in I^{\mu}_{\mathcal{P}_0}(H)$ has at least $2k^kC$ absorbing $2^{k-1}k^2$-sets in $\mathcal{F}_{abs}$.

Since $(\mathcal{P}_0', L^{\mu}_{\mathcal{P}_0'}(H))$ is soluble, there exists a matching M_1 of size at most $d'-1$ such that $\mathbf{i}_{\mathcal{P}_0'}(V(H)\setminus V(M_1)) \in L^{\mu}_{\mathcal{P}_0'}(H)$. Note that $V(M_1)$ may intersect $V(\mathcal{F}_{abs})$, but M_1 can only intersect at most $k(k-1)$ absorbing sets of $\mathcal{F}_{abs}$. Let $\mathcal{F}_0$ be the subfamily of $\mathcal{F}_{abs}$ obtained from removing the $2^{k-1}k^2$-sets that intersect $V(M_1)$. Let M_0 be the perfect matching on $V(\mathcal{F}_0)$ that consists of perfect matchings on each member of $\mathcal{F}_0$. Note that every k-set S of vertices with $\mathbf{i}_{\mathcal{P}_0}(S) \in I^{\mu}_{\mathcal{P}_0}(H)$ has at least $2k^kC - k(k-1)$ absorbing sets in $\mathcal{F}_0$.

Now we switch to $\mathcal{P}_0$. We want to 'store' some edges for each k-vector in $I^{\mu}_{\mathcal{P}_0}(H)$ for future use. More precisely, we find a matching M_2 in $V(H)\setminus V(M_0\cup M_1)$ which contains C edges e with $\mathbf{i}_{\mathcal{P}_0}(e) = \mathbf{i}$ for every $\mathbf{i} \in I^{\mu}_{\mathcal{P}_0}(H)$. So $|M_2| \le \binom{k+d-1}{k}C$ and the process is possible because H contains at least μn^k edges for each k-vector $\mathbf{i} \in I^{\mu}_{\mathcal{P}_0}(H)$ and $|V(M_0\cup M_1\cup M_2)| \le 2^{k-1}k^2c\log n + k(k-1) + \binom{k+d-1}{k}C < \mu n$.

Let $H' := H[V(H)\setminus V(M_0\cup M_1\cup M_2)]$. Note that $|V(H')| \ge n - \mu n$. So

$$\delta_{k-1}(H') \ge \delta_{k-1}(H) - \mu n \ge n/k - 2\gamma n \ge (1/k - 2\gamma)|V(H')|.$$

Moreover, if H' is $10k\gamma$-extremal, namely, $V(H')$ contains an independent subset of order at least

$$(1-10k\gamma)\frac{k-1}{k}|V(H')| \ge (1-10k\gamma)\frac{k-1}{k}(n-\mu n) \ge (1-11k\gamma)\frac{k-1}{k}n,$$

then H is $11k\gamma$-extremal, a contradiction. Now we can apply Theorem 7.16 on H' with parameter 2γ in place of γ and find a matching M_3 covering all but a set S_0 of k vertices of $V(H')$. Note that we can absorb S_0 by $\mathcal{F}_0$ and get a perfect matching of H immediately if $\mathbf{i}_{\mathcal{P}_0}(S_0) \in I^{\mu}_{\mathcal{P}_0}(H)$ (however, this may not be the case).

Now we step back to the full pair $(\mathcal{P}_0', L^{\mu}_{\mathcal{P}_0'}(H))$. Instead of index vectors, we consider the residues of S_0 and all edges in the matching $M_0\cup M_3$ with respect to $\mathcal{P}_0'$. Recall

that $\mathbf{i}_{\mathcal{P}_0'}(V(H) \setminus V(M_1)) \in L^{\mu}_{\mathcal{P}_0'}(H)$. Note that, since $\mathcal{P}_0$ refines $\mathcal{P}_0'$, the index vectors of all edges in M_2 are in $I^{\mu}_{\mathcal{P}_0'}(H)$. So we have $\mathbf{i}_{\mathcal{P}_0'}(V(H) \setminus V(M_1 \cup M_2)) \in L^{\mu}_{\mathcal{P}_0'}(H)$, namely, $R_G(V(H) \setminus V(M_1 \cup M_2)) = \mathbf{0} + L^{\mu}_{\mathcal{P}_0'}(H)$. Thus,

$$\sum_{e \in M_0 \cup M_3} R_G(e) + R_G(S_0) = \mathbf{0} + L^{\mu}_{\mathcal{P}_0'}(H).$$

Suppose $R_G(S_0) = \mathbf{v}_0 + L^{\mu}_{\mathcal{P}_0'}(H)$ for some $\mathbf{v}_0 \in L^{d'}_{\max}$ and we get

$$\sum_{e \in M_0 \cup M_3} R_G(e) = -\mathbf{v}_0 + L^{\mu}_{\mathcal{P}_0'}(H).$$

Claim 7.20. *There exist edges $e_1, \dots, e_{d''} \in M_0 \cup M_3$ for some $d'' \leq d' - 1$ such that*

$$\sum_{i \in [d'']} R_G(e_i) = -\mathbf{v}_0 + L^{\mu}_{\mathcal{P}_0'}(H).$$

Proof. We follow the proof of [31, Lemma 6.10]. Fix any set of edges $e_1, \dots, e_l \in M_0 \cup M_3$ for $l \geq d'$, consider $l+1$ partial sums $\sum_{i \in [j]} R_G(e_i)$ for $j = 0, 1, \dots, l$, where the sum equals $\mathbf{0} + L^{\mu}_{\mathcal{P}_0'}(H)$ when $j = 0$. Since $G = G(\mathcal{P}, L)$ is a group, the sums are still in G. By Lemma 7.18, $|G| = |\mathcal{P}_0'| = d'$, then by the pigeonhole principle two of the partial sums must be equal, that is, there exist $0 \leq l_1 < l_2 \leq l$ such that $\sum_{l_1 < i \leq l_2} R_G(e_i) = \mathbf{0} + L^{\mu}_{\mathcal{P}_0'}(H)$. So we can delete them from the equation. We can repeat this process until there are at most $d' - 1$ edges. □

So we have $\sum_{i \in [d'']} \mathbf{i}_{\mathcal{P}_0'}(e_i) + \mathbf{i}_{\mathcal{P}_0'}(S_0) \in L^{\mu}_{\mathcal{P}_0'}(H)$. Let $D := \bigcup_{i \in [d'']} e_i \cup S_0$ satisfying that $|D| = kd'' + k \leq k(d' - 1) + k \leq k^2$. At last, we switch to $(\mathcal{P}_0, L^{\mu}_{\mathcal{P}_0}(H))$ again and finish the perfect matching by absorption. Since $\mathbf{i}_{\mathcal{P}_0'}(D) \in L^{\mu}_{\mathcal{P}_0'}(H)$, by Lemma 7.11 (iii), we have $\mathbf{i}_{\mathcal{P}_0}(D) \in L^{\mu}_{\mathcal{P}_0}(H)$. Thus, we have the following equation

$$\mathbf{i}_{\mathcal{P}_0}(D) = \sum_{\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)} a_{\mathbf{v}} \mathbf{v},$$

where $a_{\mathbf{v}} \in \mathbb{Z}$ for all $\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)$. Since the equation above is a special case of equation (7.1),

we have $|a_{\mathbf{v}}| \leq C$ for all $\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)$. Noticing that $a_{\mathbf{v}}$ may be negative, we can assume $a_{\mathbf{v}} = b_{\mathbf{v}} - c_{\mathbf{v}}$ such that one of $b_{\mathbf{v}}, c_{\mathbf{v}}$ is a nonnegative integer and the other is zero for all $\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)$. So, we have

$$\sum_{\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)} c_{\mathbf{v}}\mathbf{v} + \mathbf{i}_{\mathcal{P}_0}(D) = \sum_{\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)} b_{\mathbf{v}}\mathbf{v}.$$

This equation means that given any family $\mathcal{F}$ consisting of disjoint $\sum_{\mathbf{v}} c_{\mathbf{v}}$ k-sets $e^{\mathbf{v}}_1, \ldots, e^{\mathbf{v}}_{c_{\mathbf{v}}} \subseteq V(H) \setminus D$ for $\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)$ such that $\mathbf{i}_{\mathcal{P}_0}(e^{\mathbf{v}}_i) = \mathbf{v}$ for all $i \in [c_{\mathbf{v}}]$, we can regard $V(\mathcal{F}) \cup D$ as the union of $b_{\mathbf{v}}$ k-sets $S^{\mathbf{v}}_1, \ldots, S^{\mathbf{v}}_{b_{\mathbf{v}}}$ such that $\mathbf{i}_{\mathcal{P}_0}(S^{\mathbf{v}}_j) = \mathbf{v}$, $j \in [b_{\mathbf{v}}]$ for all $\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)$. Since $c_{\mathbf{v}} \leq C$ for all $\mathbf{v}$ and $V(M_2) \cap D = \emptyset$, we can pick the family $\mathcal{F}$ as a subset of M_2. Thus, we regard $V(\mathcal{F}) \cup D$ as at most $\binom{k+d'-1}{d'}C + k \leq k^k C$ k-sets S with $\mathbf{i}_{\mathcal{P}_0}(S) \in I^{\mu}_{\mathcal{P}_0}(H)$. Note that by definition, D may intersect at most $k-1$ absorbing sets in $\mathcal{F}_0$, which cannot be used to absorb those sets we obtained above. Since each k-set S has at least $2k^kC - k(k-1) > k^kC + k - 1$ absorbing sets in $\mathcal{F}_0$, we can absorb them by $\mathcal{F}_0$ greedily and get a perfect matching of H. □

7.3.3 Proof of the Absorbing Lemma

Claim 7.21. *Suppose V_i is (β, t)-closed for all $i \in [d]$. Then any k-set S with $\mathbf{i}_{\mathcal{P}_0}(S) \in I^{\mu}_{\mathcal{P}_0}(H)$ has at least $\frac{\mu\beta^k}{2^{k+1}}n^{tk^2}$ absorbing tk^2-sets.*

Proof. For a k-set $S = \{y_1, \ldots, y_k\}$ with $\mathbf{i}_{\mathcal{P}_0}(S) \in I^{\mu}_{\mathcal{P}_0}(H)$, we construct absorbing tk^2-sets for S as follows. We first fix an edge $e = \{x_1, \ldots, x_k\}$ in H such that $\mathbf{i}_{\mathcal{P}_0}(e) = \mathbf{i}_{\mathcal{P}_0}(S) \in I^{\mu}_{\mathcal{P}_0}(H)$ and $e \cap S = \emptyset$. Note that we have at least $\mu n^k - kn^{k-1} > \frac{\mu}{2}n^k$ choices for such e. Without loss of generality, we may assume that for all $i \in [k]$, x_i, y_i are in the same partition set of $\mathcal{P}_0$. Since x_i is (β, t)-reachable to y_i, there are at least βn^{tk-1} $(tk-1)$-sets T_i such that both $H[T_i \cup x_i]$ and $H[T_i \cup y_i]$ have perfect matchings. We pick disjoint reachable $(tk-1)$-sets for each x_i, y_i, $i \in [k]$ greedily, while avoiding the existing vertices. Since the number of existing vertices is at most $tk^2 + k$, we have at least $\frac{\beta}{2}n^{tk-1}$ choices for such $(tk-1)$-sets in each step. Note that each of $e \cup T_1 \cup \cdots \cup T_k$ is an absorbing set for S. First, it contains a perfect

matching because each $T_i \cup x_i$ for $i \in [k]$ spans t disjoint edges. Second, $H[e \cup T_1 \cup \cdots \cup T_k \cup S]$ also contains a perfect matching because e is an edge and each $T_i \cup y_i$ for $i \in [k]$ spans t disjoint edges. So we find at least $\frac{\mu\beta^k}{2^{k+1}} n^{tk^2}$ absorbing tk^2-sets for S. □

Proof of Lemma 7.19. We pick a family $\mathcal{F}$ of tk^2-sets by including every tk^2-set with probability $p = cn^{-tk^2} \log n$ independently, uniformly at random. Then the expected number of elements in $\mathcal{F}$ is $p\binom{n}{tk^2} \le \frac{c}{tk^2} \log n$ and the expected number of intersecting pairs of tk^2-sets is at most

$$p^2 \binom{n}{tk^2} \cdot tk^2 \cdot \binom{n}{tk_2 - 1} \le \frac{c^2 (\log n)^2}{n} = o(1).$$

Then by Markov's inequality, with probability $1 - 1/(tk^2) - o(1)$, $\mathcal{F}$ contains at most $c \log n$ sets and they are pairwise vertex-disjoint.

For every k-set S with $\mathbf{i}_{\mathcal{P}_0}(S) \in I^{\mu}_{\mathcal{P}_0}(H)$, let X_S be the number of absorbing sets for S in $\mathcal{F}$. Then by Claim 7.21,

$$\mathbb{E}(X_S) \ge p \frac{\mu\beta^k}{2^{k+1}} n^{tk^2} = \frac{\mu\beta^k c \log n}{2^{k+1}}.$$

By Chernoff's bound,

$$\mathbb{P}\left(X_S \le \frac{1}{2}\mathbb{E}(X_S) \right) \le \exp\left\{ -\frac{1}{8}\mathbb{E}(X_S) \right\} \le \exp\left\{ -\frac{\mu\beta^k c \log n}{2^{k+4}} \right\} = o(n^{-k})$$

because $1/c \ll \{\beta, \mu\}$. Thus, with probability $1 - o(1)$, for each k-set S with $\mathbf{i}_{\mathcal{P}_0}(S) \in I^{\mu}_{\mathcal{P}_0}(H)$, there are at least

$$\frac{1}{2}\mathbb{E}(X_S) \ge \frac{\mu\beta^k c \log n}{2^{k+2}} \gg 2k^k C$$

absorbing sets for S in $\mathcal{F}$. We obtain $\mathcal{F}_{abs}$ by deleting the elements of $\mathcal{F}$ that are not absorbing sets for any k-set S and thus $|\mathcal{F}_{abs}| \le |\mathcal{F}| \le c \log n$. □

7.3.4 Proof of Lemma 7.11

In this subsection we prove Lemma 7.11. Our main goal is to build a partition $\mathcal{P} = \{V_1, \ldots, V_d\}$ of $V(H)$ for some $d \le k$ such that every V_i is $(\beta, 2^{k-1})$-closed for some $\beta > 0$.

For any $v \in V(H)$, let $\tilde{N}_{\beta,i}(v)$ be the set of vertices in $V(H)$ that are (β, i)-reachable to v.

Proposition 7.22. *Suppose H is a k-graph on n vertices satisfying $\delta_{k-1}(H) \geq (1/k - \gamma)n$. For $\alpha > 0$ and any $v \in V(H)$, $|\tilde{N}_{\alpha,1}(v)| \geq (1/k - \gamma - 2k!\alpha)n$.*

Proof. First note that $\delta_{k-1}(H) \geq (1/k - \gamma)n$ implies that $\delta_1(H) \geq (1/k - \gamma)\binom{n-1}{k-1}$. Fix a vertex $v \in V(H)$, note that for any vertex u, $u \in \tilde{N}_{\alpha,1}(v)$ if and only if $|N_H(u) \cap N_H(v)| \geq \alpha n^{k-1}$. By double counting, we have

$$|N_H(v)|\delta_{k-1}(H) \leq \sum_{S \in N_H(v)} \deg_H(S) < |\tilde{N}_{\alpha,1}(v)| \cdot |N_H(v)| + n \cdot \alpha n^{k-1}.$$

Thus, $|\tilde{N}_{\alpha,1}(v)| > \delta_{k-1}(H) - \frac{\alpha n^k}{|N_H(v)|} \geq (1/k - \gamma - 2k!\alpha)n$ as $|N_H(v)| \geq \delta_1(H) \geq (1/k - \gamma)\binom{n-1}{k-1}$. □

The following lemma provides the partition $\mathcal{P}_0$ in Lemma 7.11. Note that it does not require the minimum codegree condition.

Lemma 7.23. *Given $0 < \alpha \ll \delta, \delta'$, there exists constant $\beta > 0$ satisfying the following. Assume an n-vertex k-graph H satisfies that $|\tilde{N}_{\alpha,1}(v)| \geq \delta' n$ for any $v \in V(H)$ and $\delta_1(H) \geq \delta\binom{n-1}{k-1}$. Then we can find a partition $\mathcal{P}_0$ of $V(H)$ into $V_1, \ldots, V_d$ with $d \leq \min\{\lfloor 1/\delta \rfloor, \lfloor 1/\delta' \rfloor\}$ such that for any $i \in [d]$, $|V_i| \geq (\delta' - \alpha)n$ and V_i is $(\beta, 2^{\lfloor 1/\delta \rfloor - 1})$-closed in H, in time $O(n^{2^{c-1}k+1})$.*

We will use the following simple result from [49] to prove Lemma 7.23.

Proposition 7.24. *[49, Proposition 2.1] For $\epsilon, \beta > 0$ and integer $i \geq 1$, there exists $\beta_0 > 0$ and an integer n_0 satisfying the following. Suppose H is a k-graph of order $n \geq n_0$ and there exists a vertex $x \in V(H)$ with $|\tilde{N}_{\beta,i}(x)| \geq \epsilon n$. Then for all $0 < \beta' \leq \beta_0$, $\tilde{N}_{\beta,i}(x) \subseteq \tilde{N}_{\beta',i+1}(x)$.*

Proof of Lemma 7.23. Let $c = \lfloor 1/\delta \rfloor$ (then $(c+1)\delta - 1 > 0$) and $\epsilon = \alpha/c$. We choose constants satisfying the following hierarchy

$$1/n \ll \beta = \beta_{c-1} \ll \beta_{c-2} \ll \cdots \ll \beta_1 \ll \beta_0 \ll \epsilon, (c+1)\delta - 1.$$

Throughout this proof, given $v \in V(H)$ and $i \in [c-1]$, we write $\tilde{N}_{\beta_i,2^i}(v)$ as $\tilde{N}_i(v)$ for short. Note that for any $v \in V(H)$, $|\tilde{N}_0(v)| = |\tilde{N}_{\beta_0,1}(v)| \geq |\tilde{N}_{\alpha,1}(v)| \geq \delta' n$ because $\beta_0 < \alpha$. We also say 2^i-reachable (or 2^i-closed) for $(\beta_i, 2^i)$-reachable (or $(\beta_i, 2^i)$-closed). By Proposition 7.24 and the choice of β_i's, we may assume that $\tilde{N}_i(v) \subseteq \tilde{N}_{i+1}(v)$ for all $0 \leq i < c-1$ and all $v \in V(H)$. Hence, if $W \subseteq V(H)$ is 2^i-closed in H for some $i \leq c-1$, then W is 2^{c-1}-closed.

Recall that two vertices u and v are 1-reachable to each other if $|N_H(u) \cap N_H(v)| \geq \beta_0 n^{k-1}$. We first note that any set of $c+1$ vertices in $V(H)$ contains two vertices that are 1-reachable to each other because $\delta_1(H) \geq \delta\binom{n-1}{k-1}$ and $(c+1)\delta - 1 \geq 2k!\beta_0$. Also we can assume that there are two vertices that are not 2^{c-1}-reachable to each other, as otherwise $V(H)$ is 2^{c-1}-closed and we get a trivial partition $\mathcal{P}_0 = \{V(H)\}$.

Let d be the largest integer such that there exist $v_1, \ldots, v_d \in V(H)$ such that no pair of them are 2^{c+1-d}-reachable to each other. Note that d exists by our assumption and $2 \leq d \leq c = \lfloor 1/\delta \rfloor$ by our observation. Fix such $v_1, \ldots, v_d \in V(H)$, by Proposition 7.24, we can assume that any two of them are not 2^{c-d}-reachable to each other. Consider $\tilde{N}_{c-d}(v_i)$ for all $i \in [d]$. Then we have the following facts.

(i) Any $v \in V(H) \setminus \{v_1, \ldots, v_d\}$ must be in $\tilde{N}_{c-d}(v_i)$ for some $i \in [d]$, as otherwise $v, v_1, \ldots, v_d$ contradicts the definition of d.

(ii) $|\tilde{N}_{c-d}(v_i) \cap \tilde{N}_{c-d}(v_j)| < \epsilon n$ because v_i, v_j are not 2^{c+1-d}-reachable to each other. Indeed, otherwise we get at least

$$\frac{1}{(2^{c+1-d}k-1)!}\epsilon n(\beta_{c-d}n^{2^{c-d}k-1} - n^{2^{c-d}k-2})(\beta_{c-d}n^{2^{c-d}k-1} - 2^{c-d}kn^{2^{c-d}k-2}) \geq \beta_{c+1-d}n^{2^{c+1-d}k-1}$$

reachable $(2^{c+1-d}k-1)$-sets for v_i, v_j, which means that they are 2^{c+1-d}-reachable, a contradiction.

Note that (ii) and $|\tilde{N}_{c-d}(v_i)| \geq |\tilde{N}_0(v_i)| \geq \delta' n$ for $i \in [d]$ imply $d\delta' n - \binom{d}{2}\epsilon n \leq n$. So we have $d \leq (1 + d^2\epsilon)/\delta'$. Since $\epsilon \leq \alpha \ll \delta'$, we have $d \leq \lfloor 1/\delta' \rfloor$ and thus, $d \leq \min\{\lfloor 1/\delta \rfloor, \lfloor 1/\delta' \rfloor\}$.

For $i \in [d]$, let $U_i = (\tilde{N}_{c-d}(v_i) \cup \{v_i\}) \setminus \bigcup_{j\in[d]\setminus\{i\}} \tilde{N}_{c-d}(v_j)$. Note that for $i \in [d]$, U_i is 2^{c-d}-closed. Indeed, if there exist $u_1, u_2 \in U_i$ that are not 2^{c-d}-reachable to each other, then $\{u_1, u_2\} \cup (\{v_1, \dots, v_d\} \setminus \{v_i\})$ contradicts the definition of d.

Let $U_0 = V(H) \setminus (U_1 \cup \cdots \cup U_d)$. By (i) and (ii), we have $|U_0| \le \binom{d}{2}\epsilon n$. We will move vertices of U_0 greedily to U_i for some $i \in [d]$. For any $v \in U_0$, since $|\tilde{N}_0(v) \setminus U_0| \ge \delta' n - |U_0| \ge d\epsilon n$, there exists $i \in [d]$ such that v is 1-reachable to at least ϵn vertices in U_i. In this case we add v to U_i (we add v to an arbitrary U_i if there are more than one such i). Let the resulting partition of $V(H)$ be $V_1, \dots, V_d$. Note that we have $|V_i| \ge |U_i| \ge |\tilde{N}_{c-d}(v_i)| - d\epsilon n \ge |\tilde{N}_0(v_i)| - c\epsilon n \ge (\delta' - \alpha)n$. Observe that in each V_i, the 'farthest' possible pairs are those two vertices both from U_0, which are $(2^{c-d}+2)$-reachable to each other. Thus, each V_i is $(2^{c-d}+2)$-closed, so 2^{c-1}-closed because $d \ge 2$.

We estimate the running time as follows. First, for every two vertices $u, v \in V(H)$, we determine if they are 2^i-reachable for $0 \le i \le c-1$. This can be done by testing if any $(2^i k - 1)$-set $S \in \binom{V(H)\setminus\{u,v\}}{2^i k - 1}$ is a reachable set for u and v, namely, if both $H[S \cup \{u\}]$ and $H[S \cup \{v\}]$ have perfect matchings or not, which can be checked by listing every set of 2^i edges on them, in constant time. If there are at least $\beta_i n^{2^i k-1}$ reachable $(2^i k - 1)$-sets for v_i and v_j, then they are 2^i-reachable. Since we need time $O(n^{2^{c-1}k-1})$ to list all $2^{c-1}k - 1$ sets for all pairs u, v of vertices, this can be done in time $O(n^{2^{c-1}k+1})$. Second, we search the set of vertices $v_1, \dots, v_d$ such that no pair of them are 2^{c+1-d}-reachable to each other for all $2 \le d \le c$. With the reachable information at hand, this can be done in time $O(n^c)$. We then fix the largest d as in the proof. If such d does not exist, then we get $\mathcal{P}_0 = \{V(H)\}$ and output $\mathcal{P}_0$. Otherwise, we fix any d-set $v_1, \dots, v_d$ such that no pair of them are 2^{c+1-d}-reachable to each other. We find the partition $\{U_0, U_1, \dots, U_d\}$ by identifying $\tilde{N}_{c-d}(v_i)$ for $i \in [d]$, in time $O(n)$. Finally we move vertices in U_0 to $U_1, \dots, U_d$, depending on $|\tilde{N}_0(v) \cap U_i|$ for $v \in U_0$ and $i \in [d]$, which can be done in time $O(n^2)$. Thus, the running time for finding a desired partition is $O(n^{2^{c-1}k+1})$. □

Proof of Lemma 7.11. Fix $0 < \gamma \ll 1/k$. We apply Lemma 7.23 with $\alpha \ll \gamma$, $\delta = 1/k - \gamma$,

and $\delta' = 1/k - \gamma - 2k!\alpha$ and get $\beta > 0$. Suppose

$$1/n \ll \{\beta, \mu_0\} \ll \gamma, 2^{-k}.$$

Let H be a k-graph on n vertices satisfying $\delta_{k-1}(H) \geq (1/k - \gamma)n$. By Proposition 7.22, for any $v \in V(H)$, $\tilde{N}_{\alpha,1}(v) \geq (1/k - \gamma - 2k!\alpha)n = \delta' n$. Since we also have $\delta_1(H) \geq \delta\binom{n-1}{k-1}$, we apply Lemma 7.23 on H and get a partition $\mathcal{P}_0 = \{V_1, \ldots, V_d\}$ of $V(H)$ in time $O(n^{2^{k-1}k+1})$. Note that $|V_i| \geq (\delta' - \alpha)n \geq (1/k - 2\gamma)n$ for all $i \in [d]$ because $\alpha \ll \gamma$. Also we know that $d \leq \lfloor 1/\delta \rfloor = k$ and each V_i is $(\beta, 2^{k-1})$-closed.

Let $K = (k+1)^{d-1}$. We pick a constant μ such that $K^{-\binom{k+d-1}{k}}\mu_0 \leq \mu \leq \mu_0$ and

$$L^{\mu}_{\mathcal{P}_0}(H) = L^{\mu/K}_{\mathcal{P}_0}(H). \tag{7.2}$$

Indeed, it suffices to pick such a μ so that $I^{\mu}_{\mathcal{P}_0}(H) = I^{\mu/K}_{\mathcal{P}_0}(H)$. This means that we will not 'witness' more vectors even if we loosen our selection parameter μ by a factor K. Note that $|L^d_{\max}| = \binom{k+d-1}{k}$. So if $I^{\mu_0}_{\mathcal{P}_0}(H) \neq I^{\mu_0/K}_{\mathcal{P}_0}(H)$, we pick μ_0/K as the new candidate, check it and repeat until we get the desired μ. Note that in each intermediate step for some μ', we witness at least one new vector in $I^{\mu'/K}_{\mathcal{P}_0}(H)$. So the process will terminate in at most $\binom{k+d-1}{k}$ steps and the resulting value μ satisfying $\mu \geq K^{-\binom{k+d-1}{k}}\mu_0$. Note that we find μ in constant time and we have the same hierarchy of constants after replacing μ_0 by μ.

It is possible that $(\mathcal{P}_0, L^{\mu}_{\mathcal{P}_0}(H))$ contains transferrals. We merge V_i and V_j into one vertex set if the transferral $\mathbf{u}_i - \mathbf{u}_j$ appears in $L^{\mu}_{\mathcal{P}_0}(H)$ and repeat until there is no transferral in the resulting pair, denoted by $(\mathcal{P}'_0, L^{\mu}_{\mathcal{P}'_0}(H))$, where $\mathcal{P}'_0 = \{V'_1, \ldots, V'_{d'}\}$ for some $d' \leq d \leq k$. Note that we get $\mathcal{P}'_0$ from $\mathcal{P}_0$ in time $O(n^k)$. Indeed, we merge parts at most $d-1$ times and in each step, we identify the set of robust edge vectors by visiting all edges of H and then determine if any transferral appears in the lattice in constant time. Thus, overall, we find the pair $(\mathcal{P}'_0, L^{\mu}_{\mathcal{P}'_0}(H))$ in time $O(n^k)$.

Claim 7.25. *Fix $\mu > 0$. Given a partition $\mathcal{P}_1 = \{V_1, \ldots, V_{|\mathcal{P}_1|}\}$ such that $\mathbf{u}_1 - \mathbf{u}_2 \in L^{\mu}_{\mathcal{P}_1}(H)$*

and let $\mathcal{P}_1'$ be the partition obtained from merging V_1, V_2 of $\mathcal{P}_1$. Then for any $D \subseteq V(H)$ such that $\mathbf{i}_{\mathcal{P}_1'}(D) \in L^{\mu}_{\mathcal{P}_1'}(H)$, we have $\mathbf{i}_{\mathcal{P}_1}(D) \in L^{\mu/(k+1)}_{\mathcal{P}_1}(H)$.

Proof. For any vector $\mathbf{v}$ with respect to $\mathcal{P}_1$, let $\mathbf{v}|_{\mathcal{P}_1'}$ be the projection of $\mathbf{v}$ on $\mathcal{P}_1'$, which is a vector with respect to $\mathcal{P}_1'$. Let $D \subseteq V(H)$ be any vertex set such that $\mathbf{i}_{\mathcal{P}_1'}(D) \in L^{\mu}_{\mathcal{P}_1'}(H)$. So we have the equation $\mathbf{i}_{\mathcal{P}_1'}(D) = \sum_{\mathbf{v}' \in I^{\mu}_{\mathcal{P}_1'}(H)} a_{\mathbf{v}'}\mathbf{v}'$, where $a_{\mathbf{v}'} \in \mathbb{Z}$ for all $\mathbf{v}' \in I^{\mu}_{\mathcal{P}_1'}(H)$. Note that for each $\mathbf{v}' \in I^{\mu}_{\mathcal{P}_1'}(H)$, there exist at most $k+1$ vectors $\mathbf{v}_i \in L^{|\mathcal{P}_1|}_{\max}$ such that $\mathbf{v}_i|_{\mathcal{P}_1'} = \mathbf{v}'$. Thus, by the pigeonhole principle, there exists $\mathbf{v} \in I^{\mu/(k+1)}_{\mathcal{P}_1}(H)$ such that $\mathbf{v}|_{\mathcal{P}_1'} = \mathbf{v}'$. Let $\mathbf{i}_0 = \sum_{\mathbf{v}' \in I^{\mu}_{\mathcal{P}_1'}(H)} a_{\mathbf{v}'}\mathbf{v}$, which is a $|D|$-vector in $L^{\mu/(k+1)}_{\mathcal{P}_1}(H)$. Note that $\mathbf{i}_{\mathcal{P}_1}(D)|_{\mathcal{P}_1'} = \mathbf{i}_{\mathcal{P}_1'}(D) = \mathbf{i}_0|_{\mathcal{P}_1'}$. This implies that $\mathbf{i}_{\mathcal{P}_1}(D) = \mathbf{i}_0$ or $\mathbf{i}_{\mathcal{P}_1}(D) - \mathbf{i}_0$ equals a multiple of $\mathbf{u}_1 - \mathbf{u}_2$. Since $\mathbf{u}_1 - \mathbf{u}_2 \in L^{\mu}_{\mathcal{P}_1}(H)$, we have $\mathbf{i}_{\mathcal{P}_1}(D) - \mathbf{i}_0 \in L^{\mu}_{\mathcal{P}_1}(H)$ and thus $\mathbf{i}_{\mathcal{P}_1}(D) = \mathbf{i}_{\mathcal{P}_1}(D) - \mathbf{i}_0 + \mathbf{i}_0 \in L^{\mu/(k+1)}_{\mathcal{P}_1}(H)$. □

Now let us show Lemma 7.11 (iii). Fix any $D \subseteq V(H)$ such that $\mathbf{i}_{\mathcal{P}_0'}(D) \in L^{\mu}_{\mathcal{P}_0'}(H)$. We apply Claim 7.25 $d - d'$ times and get that $\mathbf{i}_{\mathcal{P}_0}(D) \in L^{\mu/(k+1)^{d-d'}}_{\mathcal{P}_0}(H)$. Since $\mu/K \le \mu/(k+1)^{d-d'} \le \mu$, by (7.2), we get $\mathbf{i}_{\mathcal{P}_0}(D) \in L^{\mu/(k+1)^{d-d'}}_{\mathcal{P}_0}(H) = L^{\mu/K}_{\mathcal{P}_0}(H) = L^{\mu}_{\mathcal{P}_0}(H)$.

It remains to show that $(\mathcal{P}_0', L^{\mu}_{\mathcal{P}_0'}(H))$ is a full pair for H. Indeed, since $(\mathcal{P}_0', L^{\mu}_{\mathcal{P}_0'}(H))$ is transferral-free, it remains to show that $I^{\mu}_{\mathcal{P}_0'}(H)$ is full. Assume to the contrary, that there exists a $(k-1)$-vector $\mathbf{v}$ such that $\mathbf{v} + \mathbf{u}_i \notin I^{\mu}_{\mathcal{P}_0'}(H)$ for all $i \in [d']$. Note that since $\mathbf{v} + \mathbf{u}_i \notin I^{\mu}_{\mathcal{P}_0'}(H)$, there are less than μn^k edges e in H with $\mathbf{i}_{\mathcal{P}_0'}(e) = \mathbf{v} + \mathbf{u}_i$. So there are less than $d'\mu n^k$ edges that contain some $(k-1)$-set with index vector $\mathbf{v}$. But since there are at least $\binom{\min_{j\in[d']}|V_j'|}{k-1}$ $(k-1)$-sets with index vector $\mathbf{v}$ and $\delta_{k-1}(H) \ge n/k - \gamma n$, the number of such edges is at least $\frac{1}{k}\left(\frac{n}{k} - \gamma n\right)\binom{\min_{j\in[d']}|V_j'|}{k-1} \ge \frac{1}{k}\left(\frac{n}{k} - \gamma n\right)\binom{n/k - 2\gamma n}{k-1} > d'\mu n^k$, a contradiction. □

7.4 The Extremal Case

Our goal of this section is to prove Theorem 7.15. We remark that the k-graphs in Construction 7.4 do not appear in our proof because they achieve smaller minimum codegrees than those k-graphs in Construction 7.5 if k is even and Construction 7.4 and Construction

7.5 are the same if k is odd. A main ingredient of our proof is Theorem 6.8, a result of Pikhurko (see Chapter 6).

7.4.1 Preliminary and the proof of Theorem 7.15

Fix a sufficiently small $\epsilon > 0$. Let n be a sufficiently large integer. Suppose H is a k-graph on n vertices such that $\delta_{k-1}(H) \geq n/k$ and $H \notin \mathcal{H}_{n,k}$. Assume that H is ϵ-extremal, namely, there is an independent subset $S \subseteq V(H)$ with $|S| \geq (1-\epsilon)\frac{k-1}{k}n$. Let $\alpha = \epsilon^{1/3}$. We partition $V(H)$ as follows. Let C be a maximum independent subset of $V(H)$. Define

$$A = \left\{ x \in V \setminus C : \deg(x, C) \geq (1-\alpha)\binom{|C|}{k-1} \right\}, \tag{7.3}$$

and $B = V(H) \setminus (A \cup C)$. We first observe the following bounds of $|A|, |B|, |C|$.

Claim 7.26. *$|A| \geq n/k - \alpha^2 n$, $|B| \leq \alpha^2 n$, and $(1-\epsilon)\frac{(k-1)n}{k} \leq |C| \leq \frac{(k-1)n}{k}$.*

Proof. The lower bound for $|C|$ follows from our hypothesis immediately. For any $S \subseteq C$ of order $k-1$, we have $N(S) \subseteq A \cup B$. By the minimum degree condition, we have

$$\frac{n}{k} \leq |N(S)| \leq |A| + |B| = n - |C| \leq \frac{n}{k} + \epsilon\frac{(k-1)n}{k}, \tag{7.4}$$

which gives the upper bound for $|C|$. By the definitions of A and B, we have

$$\frac{n}{k}\binom{|C|}{k-1} \leq e((A \cup B)C^{k-1}) \leq (1-\alpha)\binom{|C|}{k-1}|B| + \binom{|C|}{k-1}|A|,$$

where $e((A \cup B)C^{k-1})$ denotes the number of edges that contain $k-1$ vertices in C and one vertex in $A \cup B$. Thus, we get $n/k \leq |A| + |B| - \alpha|B|$, which gives that $\alpha|B| \leq |A| + |B| - n/k \leq \epsilon n$ by (7.4). So $|B| \leq \alpha^2 n$ and by (7.4) again, $|A| \geq n/k - |B| \geq n/k - \alpha^2 n$. □

The partition which we will work on in this section is $\mathcal{P} = (A \cup B, C)$. For $0 \leq i \leq k$, we say an edges e is *an i-edge* if $|e \cap (A \cup B)| = i$. We remark that as mentioned before, since H is close to the space barrier, it is rather 'fragile' – even the bad choice of one edge may

lead the remaining k-graph into the space barrier, so we cannot use the robust edge-lattice and apply the discussions in Section 7.3.

Let us list our auxiliary lemmas.

Lemma 7.27. *Fix any even $2 \le i \le k$. Assume that $|A \cup B| \ge n/k + i - 1$ and H contains no j-edge for all even $0 \le j \le i-2$. If H contains an i-edge, then H contains a perfect matching.*

Lemma 7.28. *Fix any even $0 \le i \le k$. If $|A \cup B| = n/k + i$ and H contains no j-edge for all even $0 \le j \le i$, then H contains a perfect matching.*

Lemma 7.29. *If H contains no j-edge for all even $0 \le j \le k$ and $H \notin \mathcal{H}_{n,k}$, then H contains a perfect matching.*

We postpone the proofs of these lemmas to the following subsections and prove Theorem 7.15 first.

Proof of Theorem 7.15. The proof of Theorem 7.15 runs in an algorithmic way as follows. The case when $|A \cup B| = n/k$ is covered by Lemma 7.28 with $i = 0$. Next by Lemma 7.27, if $|A \cup B| \ge n/k+1$ and there is a 2-edge in H, then H contains a perfect matching. So we may assume that H contains no 2-edge. Consider any $(k-1)$-set S with $|S \cap (A \cup B)| = 2$, since there is no 2-edge, we get $N(S) \subseteq A \cup B$ and thus $|A \cup B| \ge n/k+2$. By Lemma 7.28 again, if $|A \cup B| = n/k + 2$ and H contains no 2-edge, then H contains a perfect matching. So we can assume that $|A \cup B| \ge n/k + 3$ and H contains no 2-edge. If H contains one 4-edge, then by Lemma 7.27, H has a perfect matching. After $\lfloor k/2 \rfloor$ iterations, we can assume that H contains no j-edge for all even $0 \le j \le k$. In this case, by Lemma 7.29, we find a perfect matching provided that $H \notin \mathcal{H}_{n,k}$. □

7.4.2 Proof of Lemma 7.27

Fix any even $2 \le i \le k$. Assume that $|A \cup B| \ge n/k + i - 1$ and H contains no j-edge for all even $0 \le j \le i-2$. Assume that H contains an i-edge.

Let us first outline our proof. Our main goal is to remove a small matching M that covers every vertex in B such that the sets of remaining vertices $A \setminus V(M)$ and $C \setminus V(M)$ satisfy $|C \setminus V(M)| = (k-1)|A \setminus V(M)|$. Then we partition $C \setminus V(M)$ into $k-1$ parts and apply Theorem 6.8 and get a perfect matching on $V(H) \setminus V(M)$. So we get a perfect matching of H.

Roughly speaking, since $|\mathcal{P}| = 2$, the 'divisibility' is reduced to 'parity', which means that if we need to 'repair' the divisibility, one edge is enough. An i-edge e_0 will be such edge for repairing – we will add e_0 to our matching at the very beginning of our proof. But the divisibility barrier may not appear, in which case, choosing e_0 makes the parity bad. However, we cannot foresee this at the beginning. So at some intermediate step, if we find out that we made the wrong decision, we just free e_0 from our partial matching and the parity will be good again (in this case, the parity was good at the beginning).

Now we start our proof. We separate two cases.

Case 1. $i = 2$ and there is a 2-edge e_0 such that $|e_0 \cap A| = |e_0 \cap B| = 1$.

Let $x = e_0 \cap B$. Since C is a maximum independent set, there exists a $(k-1)$-set $S_x \subseteq C$ such that $e_x := x \cup S_x \in E(H)$. Note that $S_x \setminus e_0$ may intersect $e_0 \cap C$. We reserve S_x for future use, which means, we will not use its vertices later until the very last step.

We will build four disjoint matchings M_1, M_2, M_3, and M_4 in H, whose union gives the desired perfect matching in H. For $i \in [3]$, let $A_i = A \setminus V(\cup_{j\in[i]} M_j)$ and $C_i = C \setminus V(\cup_{j\in[i]} M_j)$ be the sets of uncovered vertices of A and C, respectively. Let $n_i = |V(H) \setminus V(\cup_{j\in[i]} M_j)|$.

Step 1. Small matchings M_1 and M_2 covering B.

Let $t := n/k - |A|$. We let $M_1 = \{e_0\}$ if $t \le 0$. Otherwise, we build the first matching M_1 of size $t+1$ as follows. By Claim 7.26, we know that $t = n/k - |A| \le \alpha^2 n$. By $\delta_{k-1}(H) \ge n/k$ and the definition of t, we have $\delta_{k-1}(H[B \cup C]) \ge t$. Since $|C| \le \frac{(k-1)n}{k} - 1$, we have $|B| = n - |C| - |A| \ge n/k - |A| + 1 = t + 1$.

We claim that we can find a matching of t 1-edges in $(B \cup C) \setminus (e_0 \cup S_x)$. Let M_1 be the union of these edges and e_0. Indeed, we pick t arbitrary disjoint $(k-1)$-sets $S_1, \dots, S_t$ from

$C \setminus (e_0 \cup S_x)$. Since C is an independent set, each of S_i has at least $t-1$ neighbors in $B \setminus x$ for $i \in [t]$. Consider the bipartite graph between $B \setminus x$ and $\{S_1, \ldots, S_t\}$, in which we put an edge if $v \cup S_i \in E(H)$ for $v \in B \setminus x$ and $i \in [t]$. By the König-Egervary Theorem, either we have a matching of size t (then we are done), or there is a vertex cover of order $t-1$. Since the degree of any $S_1, \ldots, S_t$ is at least $t-1$ in the auxiliary bipartite graph, the vertex cover must be in $B \setminus x$, denoted by B' (of order $t-1$), and every vertex in B' is adjacent to all S_i for $i \in [t]$. Now consider $(k-1)$-sets in $C \setminus (\bigcup_{i\in[t]} S_i \cup e_0 \cup S_x)$. If our claim does not hold, namely, there is no t disjoint 1-edges, then all these $(k-1)$-sets are adjacent to all vertices in B'. Note that $|C \setminus (\bigcup_{i\in[t]} S_i \cup e_0 \cup S_x)| \geq |C| - (k-1)t - 2k \geq (1-2k\alpha^2)|C|$, because $t \leq \alpha^2 n \leq 2\alpha^2|C|$. So for any $v \in B'$, we have

$$\deg(v, C) \geq \binom{(1-2k\alpha^2)|C|}{k-1} \geq ((1-2k\alpha^2)^{k-1} - o(1))\binom{|C|}{k-1} > (1-\alpha)\binom{|C|}{k-1},$$

as α is small enough. This contradicts the fact that $v \notin A$. So the claim holds.

Next we build the second matching M_2 that covers all vertices in $B \setminus V(M_1)$. For each $v \in B \setminus V(M_1)$, we pick $k-2$ arbitrary vertices from $C \setminus S_x$ not covered by the existing matching, and an uncovered vertex in V to complete an edge and add it to M_2. Since $\delta_{k-1}(H) \geq n/k$ and the number of vertices covered by the existing matching is at most $k|B| \leq k\alpha^2 n < \delta_{k-1}(H)$, such edge always exists.

Our construction guarantees that each edge in $M_1 \cup M_2$ contains at least one vertex from B and thus $|M_1 \cup M_2| \leq |B|$. We claim that $|A_1| \geq n_1/k$ and $|A_2| \geq n_2/k$. To see the bound for $|A_1|$, we separate two cases depending on t. When $t > 0$, by the definition of M_1, we have

$$|A_1| = \frac{n}{k} - t - 1 = \frac{n - k|M_1|}{k} = \frac{n_1}{k}.$$

Otherwise $t \leq 0$, we have $n_1 = n - k$ and $|A_1| = |A| - 1 \geq n/k - 1 = n_1/k$. For the bound for $|A_2|$, since each edge of M_2 contains at most one vertex of A, we have

$$|A_2| \geq |A_1| - |M_2| \geq \frac{n_1}{k} - |M_2| = \frac{n_2}{k}.$$

Let $s := |A_2| - n_2/k \geq 0$. Since $n_2 = n - k|M_1 \cup M_2| \geq n - k|B| \geq n - k\alpha^2 n$ and $|C| \geq (1-\epsilon)\frac{(k-1)n}{k}$ (Claim 7.26), we get

$$s \leq n - |C| - \frac{n - k\alpha^2 n}{k} \leq \epsilon\frac{(k-1)n}{k} + \alpha^2 n \leq 2\alpha^2 n.$$

Step 2. A small matching M_3.

We will construct a matching M_3 of size at most $2\alpha^2 n$ on $A_2 \cup (C_2 \setminus S_x)$ such that $|A_3| - n_3/k \in \{0, -1\}$. To see that this is possible, at some intermediate step, denote by n' as the number of uncovered vertices of H and denote by A', C' as the sets of uncovered vertices in $A, C \setminus S_x$, respectively. Let $c = |A'| - n'/k$. If $c > 0$, then we arbitrarily pick two vertices from A', $k-3$ vertices from C' and one vertex from $A' \cup C'$ to form an edge. Note that we pick a 2-edge or a 3-edge in each step. As a result, c decreases by 1 or 2. The iteration stops when c becomes 0 or -1 after at most $s \leq 2\alpha^2 n$ steps. Note that we can always form an edge in each step because the number of covered vertices is at most $k|B| + k \cdot 2\alpha^2 n \leq 3k\alpha^2 n < \delta_{k-1}(H)$. So we get a matching M_3 of at most $2\alpha^2 n$ edges.

Step 3. The last matching M_4.

Now we have two cases, $|A_3| - n_3/k = -1$ or 0. In the former case, we delete the edge e_0 from M_1 and add e_x to M_1. Note that this is possible because $S_x \subseteq C_3$. Let the resulting sets of uncovered vertices be A_3', C_3', respectively. Also let $n_3' := |A_3'| + |C_3'| = n_3$. So $|A_3'| = |A_3| + 1$ and we have $|A_3'| - n_3'/k = 0$, that is, $|C_3'| = (k-1)|A_3'|$. In the latter case we let $A_3' = A_3$ and $C_3' = C_3$. We have $|C_3'| = (k-1)|A_3'|$ immediately. By definition, we have

$$|A_3'| \geq |A| - |M_1 \cup M_2| - 3|M_3| \geq n/k - \alpha^2 n - \alpha^2 n - 6\alpha^2 n \geq n/k - 8\alpha^2 n,$$

as $|M_1 \cup M_2| \leq |B| \leq \alpha^2 n$ and $|M_3| \leq 2\alpha^2 n$.

Let $m := |A_3'|$. Next, we partition C_3' arbitrarily into $k-1$ parts $C^1, C^2, \ldots, C^{k-1}$ of the same size m. We want to apply Theorem 6.8 on the k-partite k-graph $H' :=$

$H[A_3', C^1, \ldots, C^{k-1}]$. Let us verify the assumptions. First, since C_3' is independent, for any set of $k-1$ vertices $v_1, \ldots, v_{k-1}$ such that $v_i \in C^i$ for $i \in [k-1]$, the number of its non-neighbors in $A \cup B$ is at most

$$|A| + |B| - n/k \leq n/k + \epsilon\frac{(k-1)n}{k} - n/k \leq k\epsilon m,$$

where we use (7.4) in the first inequality and the last inequality follows from $m = |A_3'| \geq n/k - 8\alpha^2 n > \frac{k-1}{k^2}n$. So we have $\delta_{[k]\setminus\{1\}}(H') \geq m - k\epsilon m = (1-k\epsilon)m$. Next, by (7.3), for any $v \in A_3'$, we have

$$\overline{\deg}_H(v, C) \leq \alpha\binom{|C|}{k-1} \leq \alpha\frac{|C|^{k-1}}{(k-1)!} \leq \alpha\frac{\left(\frac{k-1}{k}n\right)^{k-1}}{(k-1)!} \leq \alpha\frac{(km)^{k-1}}{(k-1)!} = \alpha c_k m^{k-1},$$

where $c_k = \frac{k^{k-1}}{(k-1)!}$. This implies that $\delta_{\{1\}}(H') \geq (1-\alpha c_k)m^{k-1}$. Thus, we have

$$\delta_{\{1\}}(H')m + \delta_{[k]\setminus\{1\}}(H')m^{k-1} \geq (1-\alpha c_k)m^{k-1}m + (1-k\epsilon)mm^{k-1} > \frac{3}{2}m^k,$$

as ϵ is small enough. By Theorem 6.8, we find a perfect matching in H', which gives the perfect matching M_4 on $A_3' \cup C_3'$. So $M_1 \cup M_2 \cup M_3 \cup M_4$ gives a perfect matching of H.

Case 2. $i = 2$, there is a 2-edge e_0 and there is no 2-edge e such that $|e \cap A| = |e \cap B| = 1$; or i is even with $4 \leq i \leq k$ and there is an i-edge e_0.

We first observe the following fact.

Fact 7.30. *Assume that H contains no 2-edge e such that $|e \cap A| = |e \cap B| = 1$, then for any $(k-1)$-tuple S with $|S \cap B| = 1$ and $|S \cap C| = k-2$, we have $\deg(S, C) \geq n/k - \alpha^2 n$.*

Proof. Since there is no such 2-edge, $N(S) \subseteq B \cup C$. By the minimum degree condition and $|B| \leq \alpha^2 n$ by Claim 7.26, we have $\deg(S, C) \geq n/k - \alpha^2 n$. □

Note that Fact 7.30 works under either assumption in this case. This simplifies Step 1 – we only need to build one matching. But for uniformness, we set $M_2 = \emptyset$ in this case.

Step 1. A small matching M_1 covering B.

We build M_1 as follows. First we add the i-edge e_0 to M_1. By Fact 7.30 and $|B| \leq \alpha^2 n$, we greedily pick a matching M' of $|B|$ 1-edges from $B \cup (C \setminus e_0)$. Assume that $|e_0 \cap B| = j \leq i$. If $j > 0$, denote the vertices by $x_1, \ldots, x_j \in e_0 \cap B$ and let $S_{x_1}, \ldots, S_{x_j}$ be the $(k-1)$-sets in C that form edges $e_{x_1}, \ldots, e_{x_j}$ with $x_1, \ldots, x_j$ in the matching M', respectively. As in Case 1, we will reserve $S_{x_1} \setminus e_0, \ldots, S_{x_j} \setminus e_0$ for future use. If $j = 0$, we add all edges of M' to M_1. Otherwise, we add the $|B| - j$ edges of M' that do not contain $x_1, \ldots, x_j$ to M_1. So we have $|M_1| = |B| + 1 - j$.

We claim that $|A_1| \geq n_1/k$. Recall that

$$|A \cup B| \geq n/k + i - 1 = n_1/k + |M_1| + i - 1 = n_1/k + |B| + i - j.$$

Since $|e_0 \cap A| = i - j$, we have,

$$|A_1| = |A| - (i - j) = |A \cup B| - |B| - (i - j) \geq n_1/k.$$

Since $M_2 = \emptyset$, we have $|A_2| \geq n_2/k$.

So $s := |A_2| - n_2/k \geq 0$ and as in the previous case, $s \leq 2\alpha^2 n$.

Step 2. A small matching M_3.

We will construct a matching M_3 of 2-edges and 3-edges with size at most $2\alpha^2 n$ on $A_2 \cup (C_2 \setminus (S_{x_1} \cup \cdots \cup S_{x_j}))$ such that $|A_3| - n_3/k \in \{0, 1 - i\}$. Similar as in Case 1, if we add a 2-edge (or a 3-edge) to M_3, then the value of c decreases by 1 (or 2), respectively. So if there is one 2-edge, we can construct M_3 of size at most s such that $|A_3| - n_3/k = 0$ (we can choose to include or exclude this 2-edge in M_3). So if we cannot have $|A_3| - n_3/k = 0$, then there is no 2-edge in $H[A_2 \cup (C_2 \setminus (S_{x_1} \cup \cdots \cup S_{x_j}))]$ and s is odd. In this case we add $(s + i - 1)/2$ disjoint 3-edges to M_3 and therefore $|A_3| - n_3/k = 1 - i$. Note that we always can form 2-edges or 3-edges similarly as in Case 1. So we get a matching M_3 of at most $s \leq 2\alpha^2 n$ edges.

Step 3. The last matching M_4.

Now we have two cases, $|A_3| - n_3/k = 1 - i$ or 0. In the former case, we delete the i-edge e_0 from M_1 and add the edges $e_{x_1}, \ldots, e_{x_j}$ to M_1 (if $j > 0$). Let the resulting sets of uncovered vertices be A'_3, C'_3, respectively. Also let $n'_3 := |A'_3| + |C'_3| = n_3 + k - jk$. So $|A'_3| = |A_3| + i - j$ and we have $|A'_3| - n'_3/k = 0$, namely, $|C'_3| = (k-1)|A'_3|$. In the latter case we let $A'_3 = A_3$ and $C'_3 = C_3$. We have $|C'_3| = (k-1)|A'_3|$ immediately. By definition, we have

$$|A'_3| \geq |A| - |M_1| - 3|M_3| \geq n/k - \alpha^2 n - (\alpha^2 n + 1) - 6\alpha^2 n \geq n/k - 9\alpha^2 n,$$

as $|M_1| \leq |B| + 1 \leq \alpha^2 n + 1$ and $|M_3| \leq 2\alpha^2 n$.

Let $m := |A'_3|$. We partition C'_3 arbitrarily into $k-1$ parts $C^1, C^2, \ldots, C^{k-1}$ of the same size m. We apply Theorem 6.8 on the k-partite k-graph $H' := H[A'_3, C^1, \ldots, C^{k-1}]$ and get a perfect matching in H', which gives the perfect matching M_4 on $A'_3 \cup C'_3$. So $M_1 \cup M_2 \cup M_3 \cup M_4$ gives a perfect matching of H. We omit the similar calculations.

7.4.3 Proofs of Lemma 7.28 and Lemma 7.29

Proof of Lemma 7.28. Fix any even $0 \leq i \leq k$. Assume that $|A \cup B| = n/k + i$ and H contains no j-edge for all even $0 \leq j \leq i$. If $i = 0$, then we have $|A \cup B| = n/k$ and $|C| = \frac{k-1}{k}n$. By the minimum degree condition, every k-set containing exactly $k-1$ vertices in C is an edge of H. Thus, we partition $V(H)$ into n/k such k-sets and get a perfect matching of H. So we may assume $i \geq 2$.

Since there is no i-edge, we can take an $(i+1)$-edge e_0 such that $|e_0 \cap A| = i+1$. Indeed, we take i vertices from A and $k-i-1$ vertices from C and another vertex to form an edge. Since H contains no i-edge and $|B| \leq \alpha^2 n < n/k$, we can pick the last vertex from A and get the desired $(i+1)$-edge e_0.

Next by Fact 7.30, we find a matching of $|B|$ 1-edges that covers all vertices of B. Let A' and C' be the set of uncovered vertices of A and C, respectively. Note that we have

$|A'| = n/k + i - |B| - (i+1) = n/k - |B| - 1$ and

$$|C'| = \frac{k-1}{k}n - i - (k-i-1) - (k-1)|B| = (k-1)|A'|.$$

So as in the previous proofs, we partition C' arbitrarily into $k-1$ parts, apply Theorem 6.8 and get a perfect matching on $A' \cup C'$. Thus, we get a perfect matching of H. □

Proof of Lemma 7.29. Assume that H contains no j-edge for all even $0 \le j \le k$ and $H \notin \mathcal{H}_{n,k}$. Since there is no 2-edge, by Fact 7.30, we find a matching M_1 of $|B|$ 1-edges that covers all vertices of B. Let C' be the set of uncovered vertices of C. Let $n' = |A| + |C'|$ and note that $n'/k = n/k - |B|$. Let

$$s := |A| - n'/k = |A| + |B| - n/k = |A \cup B| - n/k.$$

So $0 \le s \le \epsilon n$ by (7.4). Moreover, we claim that s is even. Indeed, since all edges of H intersect $A \cup B$ in an odd number of vertices, if s is odd, then $H \in \mathcal{H}_{n,k}$, a contradiction. We greedily pick a matching M_2 of $s/2$ disjoint 3-edges, which is possible because $s \le \epsilon n$ and $\delta_{k-1}(H) \ge n/k$. Let A_2 and C_2 be the set of vertices not covered by $M_1 \cup M_2$. As in the previous proofs, we have $|C_2| = (k-1)|A_2|$. We partition C_2 arbitrarily into $k-1$ parts, apply Theorem 6.8 and get a perfect matching M_3 on $A_2 \cup C_2$. So we get a perfect matching $M_1 \cup M_2 \cup M_3$ of H. □

7.5 Algorithms and the proof of Theorem 7.3

7.5.1 A straightforward but slower algorithm

Let L_{odd} be the lattice generated by all two dimensional k-vectors with first coordinate odd, that is, $(1,k-1),(3,k-3),\ldots,(k-1,1)$ if k is even, and $(1,k-1),(3,k-3),\ldots,(k,0)$ if k is odd. It is easy to see that L_{odd} is full. To check if a k-graph $H \in \mathcal{H}_{n,k}$, we find the bipartitions $\mathcal{P}$ of $V(H)$ such that $\mathbf{i}_{\mathcal{P}}(e) \in L_{odd}$ for every $e \in H$. We use the algorithm Procedure ListPartitions in [31]. The following lemma [31, Lemma 2.2] estimates the computation

complexity of Procedure ListPartitions (although [31, Lemma 2.2] was proved under the codegree condition $\delta_{k-1}(H) \geq n/k + \gamma n$, we can weaken the codegree condition as explained in [31, Remark 2.3]).

Lemma 7.31. *[31] Suppose H is an n-vertex k-graph with $\delta_{k-1}(H) \geq n/k - 2k(k-2)$. For any $d \in [k]$ and full edge-lattice $L \subseteq \mathbb{Z}^d$, there are at most d^{2k-1} partitions $\mathcal{P}$ of $V(H)$ such that $\mathbf{i}_{\mathcal{P}}(e) \in L$ for every $e \in H$, and Procedure ListPartitions lists them in time $O(n^{k+1})$.*

By Theorem 7.12, the straightforward way to determine the existence of a perfect matching is to check if $(\mathcal{P}_0', L^{\mu}_{\mathcal{P}_0'}(H))$ is soluble and if $H \notin \mathcal{H}_{n,k}$.

Theorem 7.32. *Fix $k \geq 3$. Let H be an n-vertex k-graph with $\delta_{k-1}(H) \geq n/k$. Then there is an algorithm with running time $O(n^{2^{k-1}k+1})$, which determines whether H contains a perfect matching.*

Proof. Let H be an n-vertex k-graph with $\delta_{k-1}(H) \geq n/k$. Note that it is trivial to determine the existence of a perfect matching if $n < n_0$ given by Theorem 7.12. Our algorithm contains two parts when $n \geq n_0$. First we find the partition $\mathcal{P}_0$ and $\mathcal{P}_0'$ and check if $(\mathcal{P}_0', L^{\mu}_{\mathcal{P}_0'}(H))$ is soluble. Second, we check if $H \notin \mathcal{H}_{n,k}$. If both answers are 'true', then H contains a perfect matching by Theorem 7.12.

By Lemma 7.11, we find $\mathcal{P}_0$ and $\mathcal{P}_0'$ in time $O(n^{2^{k-1}k+1})$. To check the solubility, we check if $\mathbf{i}_{\mathcal{P}_0'}(V(H) \setminus V(M)) \in L^{\mu}_{\mathcal{P}_0'}(H))$ for each matching M of size at most $k-1$, which can be done in time $O(n^{k(k-1)})$. To check if $H \in \mathcal{H}_{n,k}$, by Lemma 7.31 with $d = 2$ and $L = L_{odd}$, we find the bipartitions for L_{odd} in time $O(n^{k+1})$. Then for each bipartition $\mathcal{P} = \{V_1, V_2\}$, we check if $n/k - |V_1|$ is odd in constant time. Thus, the overall running time is $O(n^{2^{k-1}k+1})$. □

7.5.2 A faster algorithm

An *s-certificate* for H is an insoluble full pair $(\mathcal{P}, L)$ for which some set of s vertices intersects every edge $e \in H$ with $\mathbf{i}_{\mathcal{P}}(e) \notin L$. Note that if a full pair $(\mathcal{P}, L)$ is soluble, then it is not an s-certificate for any s. Recall that we allow the partition of a full pair to have k

parts and in contrast, the partition of a full pair in [31] has at most $k-1$ parts. Modifying the proof of [31, Lemma 8.14], we can get the following lemma.

Lemma 7.33. *[31] Suppose that $k \geq 3$ and H is a k-graph such that there is no $2k(k-2)$-certificate for H. Then every full pair for H is soluble.*

Now we give the following structural theorem.

Theorem 7.34. *Suppose $1/n_0 \ll \{\beta, \mu\} \ll \gamma \ll 1/k$. Let H be a k-graph on $n \geq n_0$ vertices such that $\delta_{k-1}(H) \geq n/k$ with $\mathcal{P}_0$ and $\mathcal{P}_0'$ found by Lemma 7.11. Then the following properties are equivalent.*

(i) H contains a perfect matching.

(ii) There is no $2k(k-2)$-certificate for H.

(iii) The full pair $(\mathcal{P}_0', L^{\mu}_{\mathcal{P}_0'}(H))$ is soluble and $H \notin \mathcal{H}_{n,k}$.

Proof. We will show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. Note that the proof of $(i) \Rightarrow (ii)$ is the same as the forward implication of proof of Theorem 7.12 and $(iii) \Rightarrow (i)$ by Theorem 7.12. It remains to show $(ii) \Rightarrow (iii)$. Assume that there is no $2k(k-2)$-certificate for H, then by Lemma 7.33, every full pair for H is soluble.

Since $(\mathcal{P}_0', L^{\mu}_{\mathcal{P}_0'}(H))$ is a full pair, it is soluble. Second, assume to the contrary, that $H \in \mathcal{H}_{n,k}$. Then there is a partition $\mathcal{P}_1 = \{X, Y\}$ of $V(H)$ such that $L_{\mathcal{P}_1}(H) \subseteq L_{odd}$ and $|X| - n/k$ is odd. Consider any $(k-1)$-set S with $|S \cap X| = a$ for some even $0 \leq a \leq k$, since H contains no even edge and $\delta_{k-1}(H) > 0$, we have $(a+1, k-a-1) \in I_{\mathcal{P}_1}(H)$ and thus $L_{\mathcal{P}_1}(H) = L_{odd}$. Also, $L_{\mathcal{P}_1}(H) = L_{odd}$ is transferral-free and thus $(\mathcal{P}_1, L_{\mathcal{P}_1}(H))$ is a full pair. Note that by definition, the first coordinate of each $\mathbf{i} \in I_{\mathcal{P}_1}(H)$ is odd and thus for any $(x, y) \in L_{\mathcal{P}_1}(H)$, we have $k \mid (x+y)$ and $x \equiv (x+y)/k \pmod 2$. So $\mathbf{i}_{\mathcal{P}_1}(V) = (|X|, |Y|) \notin L_{\mathcal{P}_1}(H)$ because $|X| - n/k$ is odd. Moreover, fix any edge e of H with $\mathbf{i}_{\mathcal{P}_1}(e) = (a, k-a)$ for some odd $a \in [k]$, then $\mathbf{i}_{\mathcal{P}_1}(V \setminus e) = (|X| - a, |Y| - k + a) \notin L_{\mathcal{P}_1}(H)$ because $|X| - a - (n-k)/k = |X| - n/k - a + 1$ is odd. So for any matching M of size at most 1, $\mathbf{i}_{\mathcal{P}_1}(V(H) \setminus V(M)) \notin L_{\mathcal{P}_1}(H)$. Thus, $(\mathcal{P}_1, L_{\mathcal{P}_1}(H))$ is an insoluble full pair, a contradiction. □

Proof of Theorem 7.3. Let H be an n-vertex k-graph with $\delta_{k-1}(H) \geq n/k$. Note that it is trivial to determine the existence of a perfect matching if $n < n_0$ given by Theorem 7.34. If $n \geq n_0$, by Theorem 7.34, to determine if H contains a perfect matching, we only need to search the existence of a $2k(k-2)$-certificate for H. This can be done by Procedure DeterminePM constructed in [31]. We estimate the running time as follows. There are at most $n^{2k(k-2)}$ choices of sets S, and these can be generated in time $O(n^{2k(k-2)})$. Also, there are only a constant number of choices for d and L, and these can be generated in constant time. For each choice of S, d and L, we apply Procedure ListPartitions on $H[V \setminus S]$ and then add the vertices of S arbitrarily to the partition we obtained. This generates the list of partitions $\mathcal{P}$ in time $O(n^{k+1})$ by Lemma 7.31. Furthermore, the number of choices for $\mathcal{P}$ is constant, and for each one it takes time $O(n^{k(k-1)})$ to check the existence of the matching M of size at most $d-1$ such that $\mathbf{i}_{\mathcal{P}}(V(H) \setminus V(M)) \in L_{\mathcal{P}}(H)$. Note that $k(k-1) > k+1$ for all $k \geq 3$ and the total running time is $O(n^{2k(k-2)+k(k-1)}) = O(n^{3k^2-5k})$. $\square$

PART 8

MINIMUM VERTEX DEGREE THRESHOLD FOR $\mathcal{C}_4^3$-TILING

8.1 Introduction

As a natural extension of the matching problem, tiling has been an active area in the past two decades (see surveys [44, 58]). Much work has been done on the problem for graphs ($k = 2$), see *e.g.*, [16, 3, 39, 45]. In particular, Kühn and Osthus [45] determined $t_1(n, G)$, for any graph G, up to an additive constant. Tiling problems become much harder for hypergraphs. For example, despite much recent progress [1, 8, 35, 36, 47, 63, 69], we still do not know the 1-degree threshold for a perfect matching in k-graphs for arbitrary k.

Other than the matching problem, only a few tiling thresholds are known. Let K_4^3 be the complete 3-graph on four vertices, and let $K_4^3 - e$ be the (unique) 3-graph on four vertices with three edges. Recently Lo and Markström [49] proved that $t_2(n, K_4^3) = (1 + o(1))3n/4$, and independently Keevash and Mycroft [34] determined the exact value of $t_2(n, K_4^3)$ for sufficiently large n. In [50], Lo and Markström proved that $t_2(n, K_4^3 - e) = (1 + o(1))n/2$. Let $\mathcal{C}_4^3$ be the unique 3-graph on four vertices with two edges. This 3-graph was denoted by $K_4^3 - 2e$ in [7], and by $\mathcal{Y}$ in [23]. Here we follow the notation in [41] and view it as a *cycle* on four vertices. Kühn and Osthus [41] showed that $t_2(n, \mathcal{C}_4^3) = (1 + o(1))n/4$, and Czygrinow, DeBiasio and Nagle [7] recently determined $t_2(n, \mathcal{C}_4^3)$ exactly for large n. In this chapter we determine $t_1(n, \mathcal{C}_4^3)$ for sufficiently large n. From now on, we simply write $\mathcal{C}_4^3$ as $\mathcal{C}$.

Previously we only knew $t_1(n, K_3^3)$ [36, 47] and $t_1(n, K_4^4)$ [35] exactly, and $t_1(n, K_5^5)$ [1], $t_1(n, K_3^3(m))$ and $t_1(n, K_4^4(m))$ [49] asymptotically, where K_k^k denotes a single k-edge, and $K_k^k(m)$ denotes the complete k-partite k-graph with m vertices in each part. So Theorem 8.1 below is one of the first (exact) results on vertex degree conditions for hypergraph tiling.

Theorem 8.1. *Suppose H is a 3-graph on n vertices such that $n \in 4\mathbb{N}$ is sufficiently large*

and

$$\delta_1(H) \geq \binom{n-1}{2} - \binom{\frac{3}{4}n}{2} + \frac{3}{8}n + c(n), \tag{8.1}$$

where $c(n) = 1$ *if* $n \in 8\mathbb{N}$ *and* $c(n) = -1/2$ *otherwise. Then* H *contains a perfect* $\mathcal{C}$*-tiling.*

Proposition 8.2 below shows that Theorem 8.1 is best possible. Theorem 8.1 and Proposition 8.2 together imply that $t_1(n, \mathcal{C}) = \binom{n-1}{2} - \binom{\frac{3}{4}n}{2} + \frac{3}{8}n + c(n)$.

Proposition 8.2. *For every* $n \in 4\mathbb{N}$ *there exists a 3-graph of order* n *with minimum vertex degree* $\binom{n-1}{2} - \binom{\frac{3}{4}n}{2} + \frac{3}{8}n + c(n) - 1$*, which does not contain a perfect* $\mathcal{C}$*-tiling.*

Proof. We give two constructions similar to those in [7]. Let $V = A \dot\cup B$ with $|A| = \frac{n}{4} - 1$ and $|B| = \frac{3n}{4} + 1$. A Steiner system $S(2, 3, m)$ is a 3-graph $\mathcal{S}$ on n vertices such that every pair of vertices has degree one – so $S(2, 3, m)$ contains no copy of $\mathcal{C}$. It is well-known that an $S(2, 3, m)$ exists if and only if $m \equiv 1, 3 \mod 6$.

Let $H_0 = (V, E_0)$ be the 3-graph on $n \in 8\mathbb{N}$ vertices as follows. Let E_0 be the set of all triples intersecting A plus a Steiner system $S(2, 3, \frac{3}{4}n+1)$ in B. Since for the Steiner system $S(2, 3, \frac{3}{4}n+1)$, each vertex is in exactly $\frac{3}{4}n/2 = \frac{3}{8}n$ edges, we have $\delta_1(H_0) = \binom{n-1}{2} - \binom{\frac{3n}{4}}{2} + \frac{3}{8}n$. Furthermore, since B contains no copy of $\mathcal{C}$, the size of the largest $\mathcal{C}$-tiling in H_0 is $|A| = \frac{n}{4} - 1$. So H_0 does not contain a perfect $\mathcal{C}$-tiling.

On the other hand, let $H_1 = (V, E_1)$ be the 3-graph on $n \in 4\mathbb{N} \setminus 8\mathbb{N}$ vertices as follows. Let G be a Steiner system of order $\frac{3}{4}n + 4$. This is possible since $\frac{3}{4}n + 4 \equiv 1 \mod 6$. Then pick an edge abc in G and let G' be the induced subgraph of G on $V(G) \setminus \{a, b, c\}$. Finally let E_1 be the set of all triples intersecting A plus G' induced on B. Since G is a regular graph with vertex degree $\frac{1}{2}(\frac{3}{4}n + 4 - 1) = \frac{3}{8}n + \frac{3}{2}$, we have that $\delta_1(G') = \frac{3}{8}n + \frac{3}{2} - 3 = \frac{3}{8}n - \frac{3}{2}$. Thus, $\delta_1(H_1) = \binom{n-1}{2} - \binom{\frac{3n}{4}}{2} + \frac{3}{8}n - \frac{3}{2}$. As in the previous case, H_1 does not contain a perfect $\mathcal{C}$-tiling. □

As a typical approach of obtaining exact results, we distinguish the *extremal* case from the *nonextremal* case and solve them separately. Given a 3-graph H of order n, we say that H is $\mathcal{C}$-free if H contains no copy of $\mathcal{C}$. In this case, clearly, every pair of vertices has degree

at most one. Every vertex has degree at most $\frac{n-1}{2}$ because its link graph contains no vertex of degree two.

Definition 8.3. *Given $\epsilon > 0$, a 3-graph H on n vertices is called ϵ-extremal if there is a set $S \subseteq V(H)$, such that $|S| \geq (1-\epsilon)\frac{3n}{4}$ and $H[S]$ is $\mathcal{C}$-free.*

Theorem 8.4 (Extremal Case)**.** *There exists $\epsilon > 0$ such that for every 3-graph H on n vertices, where $n \in 4\mathbb{N}$ is sufficiently large, if H is ϵ-extremal and satisfies* (8.1)*, then H contains a perfect $\mathcal{C}$-tiling.*

Theorem 8.5 (Nonextremal Case)**.** *For any $\epsilon > 0$, there exists $\gamma > 0$ such that the following holds. Let H be a 3-graph on n vertices, where $n \in 4\mathbb{N}$ is sufficiently large. If H is not ϵ-extremal and satisfies $\delta_1(H) \geq \left(\frac{7}{16} - \gamma\right)\binom{n}{2}$, then H contains a perfect $\mathcal{C}$-tiling.*

Theorem 8.1 follows Theorems 8.4 and 8.5 immediately by choosing ϵ from Theorem 8.4. The proof of Theorem 8.4 is somewhat routine and will be presented in Section 8.3.

The proof of Theorem 8.5, as the one of [7, Theorem 1.5], uses the *absorbing method*. More precisely, we find the perfect $\mathcal{C}$-tiling by applying the Absorbing Lemma below and the $\mathcal{C}$-tiling Lemma [23, Lemma 2.15] together.

Lemma 8.6 (Absorbing Lemma)**.** *For any $\gamma > 0$, there exist $\beta' > 0$ and an integer $n_0 > 0$ such that the following holds. Suppose H is a 3-graph on $n \geq n_0$ vertices and $\delta_1(H) \geq (1/3+\gamma)\binom{n}{2}$. Then there exists a vertex set $W \subseteq V(H)$ with $|W| \leq \gamma n/2$ such that for any vertex set $U \subset V \setminus W$ with $|U| \leq \beta' n$ and $|U| \in 4\mathbb{N}$, both $H[W]$ and $H[U \cup W]$ have perfect $\mathcal{C}$-tilings.*

Lemma 8.7 ($\mathcal{C}$-tiling Lemma)**.** *For any $0 < \gamma < 1$, there exists an integer $n_{8.7}$ such that the following holds. Suppose H is a 3-graph on $n > n_{8.7}$ vertices with*

$$\delta_1(H) \geq \left(\frac{7}{16} - \gamma\right)\binom{n}{2},$$

then H contains a $\mathcal{C}$-tiling covering all but at most $2^{19}/\gamma$ vertices or H is $2^{11}\gamma$-extremal.

We postpone the proof of lemmas later and prove Theorem 8.5 first.

Proof of Theorem 8.5. Without loss of generality, assume $0 < \epsilon < 1$. Let $\gamma = 2^{-13}\epsilon$. We find β' by applying Lemma 8.6. Choose $n \in 4\mathbb{N}$ which is large enough. Let $H = (V, E)$ be a 3-graph on n vertices. Suppose that H is not ϵ-extremal and $\delta_1(H) \geq \left(\frac{7}{16} - \gamma\right)\binom{n}{2}$. First we apply Lemma 8.6 to H and find the absorbing set W with $|W| \leq \gamma n/2$. Let $H' = H[V \setminus W]$ and $n' = n - |W|$. Note that,

$$\delta_1(H') \geq \delta_1(H) - |W|(n-1) \geq \left(\frac{7}{16} - 2\gamma\right)\binom{n}{2} \geq \left(\frac{7}{16} - 2\gamma\right)\binom{n'}{2}.$$

Second we apply Lemma 8.7 to H' with parameter 2γ in place of γ and derive that either H' is $2^{12}\gamma$-extremal or H' contains a $\mathcal{C}$-tiling covering all but at most $2^{18}/\gamma$ vertices. In the former case, since

$$(1 - 2^{12}\gamma)\frac{3n'}{4} > (1 - 2^{12}\gamma)\frac{3}{4}\left(n - \frac{\gamma n}{2}\right) > (1 - 2^{13}\gamma)\frac{3n}{4} = (1 - \epsilon)\frac{3n}{4},$$

H is ϵ-extremal, a contradiction. In the latter case, let U be the set of uncovered vertices in H'. Then we have $|U| \in 4\mathbb{N}$ and $|U| \leq 2^{18}/\gamma \leq \beta' n$ as n is large enough. By Lemma 8.6, $H[W \cup U]$ contains a perfect $\mathcal{C}$-tiling. Together with the $\mathcal{C}$-tiling provided by Lemma 8.7, this gives a perfect $\mathcal{C}$-tiling of H. □

The Absorbing Lemma and $\mathcal{C}$-tiling Lemma in [7] are not very difficult to prove because of the co-degree condition. In contrast, our corresponding lemmas are harder. We have proved Lemma 8.7 in Chapter 5 (as a key step for finding a loose Hamilton cycle in 3-graphs). In order to prove Lemma 8.6, we will use a baby version of the lattice-based absorbing method (for a full strength of this argument, see [21]).

We prove Lemma 8.6 in Section 8.2, and prove Theorem 8.4 in Section 8.3.

8.2 Proof of Lemma 8.6

We remark that the absorbing lemma for $\mathcal{C}$-tiling can be proved under a weaker condition $\delta_1(H) \geq (1/4+\gamma)\binom{n}{2}$, which is best possible (see [22, 21]). Since the main purpose of the proof here is to give an expository of our method, we prove it under the stronger condition which shortens the argument and the case analysis.

For $\beta > 0$, integer $i \geq 1$ and two vertices $u, v \in V(H)$, we call that u is (β, i)-*reachable* to v if and only if there are at least βn^{4i-1} $(4i-1)$-sets W such that both $H[u \cup W]$ and $H[v \cup W]$ contain $\mathcal{C}$-factors. In this case, we call W a reachable set for u and v. A vertex set A is (β, i)-*closed* if every pair of vertices in A are (β, i)-reachable. Similar definitions for the absorbing method can be found in [49, 50].

Proposition 8.8. *Suppose $x, y \in V$ such that $|N_H(x) \cap N_H(y)| \geq \gamma\binom{n}{2}$, then x and y are $(\gamma^2/9, 1)$-reachable to each other.*

Proof. Let $G = N_H(x) \cap N_H(y)$ be a graph on V, then since $e(G) \geq \gamma\binom{n}{2}$, the number of paths of length 3 in G is at least

$$\sum_{v \in V} \binom{\deg_G(v)}{2} \geq \frac{1}{n}\binom{\sum \deg_G(v)}{2} \geq \frac{1}{n}\binom{2\gamma\binom{n}{2}}{2} \geq \gamma^2 n^3/3.$$

Since each path of length 3 in G is a reachable 3-set for x and y, then the number of reachable 3-sets for x and y is at least $\gamma^2 n^3/9$, which implies that x and y are $(\gamma^2/9, 1)$-reachable to each other. □

Lemma 8.9. *Suppose H is an n-vertex 3-graph. Let V_1 and V_2 be disjoint vertex subsets of $V(H)$ such that both V_1 and V_2 are (β, c)-closed. Suppose there exist at least ηn^4 copies F of $\mathcal{C}$ such that $|V(F) \cap V_1| = |V(F) \cap V_2| = 2$ and there exist at least ηn^4 copies F' of $\mathcal{C}$ such that $|V(F') \cap V_1| = 3$ and $|V(F') \cap V_2| = 1$. Then $V_1 \cup V_2$ is $(\eta^2\beta^6, 5c+1)$-closed in H.*

Proof. Fix vertex-disjoint F, F' in H which are copies of K_4^- satisfying the assumptions in the lemma. Pick any vertex $x \in V(F') \cap V_1$ and $y \in V(F) \cap V_2$, and note that $V(F) \setminus \{y\}$ and

$V(F') \setminus \{x\}$ both have two vertices in V_1 and one vertex in V_2. Label them as $V(F) \setminus \{y\} = \{v_1, v_2, v_3\}$ and $V(F) \setminus \{y\} = \{v_1', v_2', v_3'\}$ such that $v_1, v_2, v_1', v_2' \in V_1$ and $v_3, v_3' \in V_2$. Since each V_i is (β, c)-closed, there are at least βn^{4c-1} reachable $(4c-1)$-sets for (v_i, v_i') and $i \in [3]$. We pick vertex-disjoint reachable sets S_i for (v_i, v_i') and $i \in [3]$ such that they are also vertex-disjoint with $V(F)$ and $V(F')$. Let $S = \{v_1, v_2, v_3, v_1', v_2', v_3'\} \cup S_1 \cup S_2 \cup S_3$ and note that each S is a reachable set for x and y. Indeed, $H[x \cup S]$ has a $\mathcal{C}$-factor as the union of $xv_1'v_2'v_3'$ and $v_i \cup S_i$ for $i \in [3]$, and $H[y \cup S]$ has a $\mathcal{C}$-factor as the union of $yv_1v_2v_3$ and $v_i' \cup S_i$ for $i \in [3]$.

Now fix any two vertices $x' \in V_1$ and $y' \in V_2$. Since x' and x are (β, c)-reachable, they have at least βn^{4c-1}-reachable $(4c-1)$-sets and the same holds for y' and y. We pick reachable sets X and Y for them such that X, Y, S are pairwise disjoint. Observe that $Z = X \cup Y \cup S \cup \{x, y\}$ is a reachable set for x' and y'. Indeed, $H[Z \cup x']$ has a $\mathcal{C}$-factor as the union of $X \cup \{x'\}$, $S \cup \{x\}$ and $Y \cup \{y\}$, and $H[Z \cup y']$ has a $\mathcal{C}$-factor as the union of $Y \cup \{y'\}$, $S \cup \{y\}$ and $X \cup \{x\}$.

Note that we have at least

$$\frac{1}{(4(3c+1)+1)!}\eta n^4 \cdot \frac{\eta n^4}{2} \cdot \left(\frac{\beta n^{4c-1}}{2}\right)^5 \geq \eta^2 \beta^6 n^{4(5c+1)-1}$$

reachable $(4(5c+1)-1)$-sets for x' and y', as β is small enough. Thus, $V_1 \cup V_2$ is $(\eta^2\beta^6, 5c+1)$-closed in H. □

Let H be a 3-graph on n vertices with $\delta_1(H) \geq (1/3 + \gamma)\binom{n}{2}$. A pair of vertices (x, y) is called *α-good* if the number of pairs $p \in N(x) \cap N(y)$ with $\deg(p) \geq \alpha n$ is at least $\alpha\binom{n}{2}$. Fix $\epsilon > 0$. If an edge $e \in H$ contains a pair $p \in \binom{V(H)}{2}$ with $\deg(p) \leq \epsilon^2 n$, it is called *weak*, otherwise called *strong*. Note that the number of weak edges in H is at most $\binom{n}{2}\epsilon^2 n$. Let

$$V_\epsilon = \left\{ v \in V(H) : v \text{ is contained in at least } \epsilon\binom{n}{2} \text{ weak edges} \right\}.$$

We observe that $|V_\epsilon| \leq 3\epsilon n$, as otherwise there are more than $3\epsilon n \epsilon \binom{n}{2}/3 = \binom{n}{2}\epsilon^2 n$ weak

edges in H, a contradiction. For any $u \in V \setminus V_\epsilon$, it is adjacent to at most $3\epsilon n(n-2)$ edges containing $v \in V_\epsilon$, so $\delta_1(H[V \setminus V_\epsilon]) \geq \delta_1(H) - 6\epsilon\binom{n}{2}$. Every $u \in V \setminus V_\epsilon$ is contained in at most $\epsilon\binom{n}{2}$ weak edges. Define subhypergraph H_ϵ on $V \setminus V_\epsilon$ with $E(H_\epsilon)$ consisting of only strong edges. Then

$$\delta_1(H_\epsilon) \geq \delta_1(H) - 7\epsilon\binom{n}{2} \geq (1/3 + \gamma/2)\binom{n}{2}.$$

For any $v \in V(H_\epsilon)$, let $\tilde{N}_{\beta,i}(v)$ be the set of vertices in $V(H_\epsilon)$ that are (β, i)-reachable to v. We have the following proposition.

Proposition 8.10. *For $\alpha > 0$ and any $x \in V(H_\epsilon)$, $|\tilde{N}_{\epsilon^3,1}(x)| \geq \frac{3}{4}\epsilon^2 n$.*

Proof. Let t be the number of pairs (p, y) where $p \in N_{H_\epsilon}(x)$ and $y \in N_{H_\epsilon}(p)$, and obviously $t \geq \deg_{H_\epsilon}(x) \cdot \epsilon^2 n$. Suppose there are m vertices who have at least $\frac{1}{12}\epsilon^2\binom{n}{2}$ common neighbors with x. Then by double counting, we get

$$\deg_{H_\epsilon}(x) \cdot \epsilon^2 n \leq t \leq (n-m) \cdot \frac{1}{12}\epsilon^2\binom{n}{2} + m \cdot \deg_{H_\epsilon}(x).$$

Together with $\delta_1(H_\epsilon) \geq \frac{1}{3}\binom{n}{2}$, we have $m \geq \frac{3}{4}\epsilon^2 n$. Therefore, there are at least $\frac{3}{4}\epsilon^2 n$ vertices y with $|N_{H_\epsilon}(x) \cap N_{H_\epsilon}(y)| \geq \frac{1}{12}\epsilon^2\binom{n}{2}$. Then for each such pair x and y, by Proposition 8.8, they are $(\epsilon^3, 1)$-reachable to each other as ϵ is small enough. □

Lemma 8.11. *There exists $\beta \gg 1/n$ and a vertex set V_0 of size at most $\epsilon^3 n$ such that $V(H_\epsilon) \setminus V_0$ is $(\beta, 6)$-closed.*

Proof. By $\delta_1(H_\epsilon) \geq (1/3 + \gamma/2)\binom{n}{2}$, any three vertices x, y, z contain two vertices, say, x and y, such that $|N_H(x) \cap N_H(y)| \geq \gamma\binom{n}{2}$, which implies that x and y are $(\gamma^2/9, 1)$-reachable to each other by Proposition 8.8.

Suppose $0 < \beta \ll \beta_2 \ll \beta_1 \ll \epsilon \ll \gamma$. By Proposition 7.24 and $\beta \ll \beta_2 \ll \beta_1$, we may assume that $\tilde{N}_{\beta_1,1}(v) \subseteq \tilde{N}_{\beta_2,2}(v) \subseteq \tilde{N}_{\beta,6}(v)$ for every $v \in V(H_\epsilon)$. This implies that if X is (β_i, i)-closed for $i = 1$ or 2, then X is $(\beta, 6)$-closed.

First if all pairs of vertices are $(\beta_2, 2)$-reachable, then we are done as $V(H)$ is $(\beta_2, 2)$-closed. So we may assume that there are $x, y \in V$ which are not $(\beta_2, 2)$-reachable to each other. Let A and B be the set of vertices which are $(\beta_1, 1)$-reachable to x and y, respectively. By Proposition 7.24, we may assume that $y \notin \tilde{N}_{\beta_1,1}(x)$, that is, x and y are not $(\beta_1, 1)$-reachable to each other. Thus, for any vertex $z \in V(H_\epsilon) \setminus \{x, y\}$, we have $z \in A$ or $z \in B$, by our observation at the beginning of the proof.

For any pair $u, v \in A \setminus B$, since neither u, v are $(\beta_1, 1)$-reachable to y, u and v must be $(\beta, 1)$-reachable to each other. The same holds for any pair of vertices in $B \setminus A$. Furthermore, we have $|A \cap B| \leq \epsilon^3 n$, as otherwise there are $\epsilon^3 n$ vertices that are $(\beta_1, 1)$-reachable to both x and y, which implies that the number of reachable 7-sets for x and y is at least

$$\frac{1}{7!}\epsilon^3 n \cdot (\beta_1 n^3/2)^2 \geq \beta_2 n^7,$$

that is, x and y are $(\beta_2, 2)$-reachable, a contradiction.

Let $V_1 = \{x\} \cup (A \setminus B)$, $V_2 = \{y\} \cup (B \setminus A)$ and $V_0 = V \setminus (V_1 \cup V_2) = A \cap B$. We have showed that $|V_0| \leq \epsilon^3 n$. By Proposition 8.10, we know that $|V_1|, |V_2| \geq |\tilde{N}_{\beta_1,1}(x)| - |V_0| \geq |\tilde{N}_{\epsilon^3,1}(x)| - \epsilon^3 n \geq \frac{1}{2}\epsilon^2 n$. Observe that both V_1 and V_2 are $(\beta_1, 1)$-closed. Next we will show that in fact $V_1 \cup V_2$ is $(\beta, 6)$-closed.

Note that since $\delta_1(H_\epsilon) \geq (1/3 + \gamma/2)\binom{n}{2}$, there must be at least $2\epsilon^3 n^3$ edges e of H that are crossing, namely, $e \cap V_1 \neq \emptyset$ and $e \cap V_2 \neq \emptyset$. Indeed, otherwise, note that the smaller set of V_1 and V_2 has at most $n/2$ vertices and by averaging, it contains a vertex v that is in at most $\frac{2\epsilon^3 n^3}{\min\{|V_1|,|V_2|\}} \leq \frac{2\epsilon^3 n^3}{\epsilon^2 n/2} = 4\epsilon n^2$ crossing edges. So we have that

$$\deg_{H_\epsilon}(v) \leq \binom{n/2}{2} + 4\epsilon n^2 + |V_0| n \leq \frac{1}{4}\binom{n}{2} + 5\epsilon n^2 < \delta_1(H_\epsilon),$$

a contradiction, where we used $|V_0| \leq \epsilon^3 n$. Without loss of generality, we can assume that $e_{H_\epsilon}(V_1 V_1 V_2) \geq \epsilon^3 n^3$.

We want to conclude the proof by applying Lemma 8.9. So let us show that we indeed have such copies of $\mathcal{C}$. First, note that four vertices $x_1, x_2 \in V_1$ and $y_1, y_2 \in V_2$ form a copy

of $\mathcal{C}$ if $y_1, y_2 \in N_{H_\epsilon}(x_1x_2)$. So the number of copies of $\mathcal{C}$ with exactly two vertices in V_1 is at least

$$\sum_{x_1,x_2\in V_1} \binom{\deg_{H_\epsilon}(x_1x_2, V_2)}{2} \geq \frac{1}{\binom{|V_1|}{2}} \binom{\sum \deg_{H_\epsilon}(x_1x_2, V_2)}{2}$$
$$= \frac{1}{\binom{|V_1|}{2}} \binom{e_{H_\epsilon}(V_1V_1V_2)}{2} \geq \epsilon^6 n^4,$$

where we used the convexity in the first inequality. Second, note that four vertices $x, x_1, x_2 \in V_1$ and $y \in V_2$ form a copy of $\mathcal{C}$ if $x_1, x_2 \in N_{H_\epsilon}(xy)$. Similarly, the number of copies of $\mathcal{C}$ with exactly three vertices in V_1 is at least

$$\sum_{x\in V_1, y\in V_2} \binom{\deg_{H_\epsilon}(xy, V_1)}{2} \geq \frac{1}{|V_1||V_2|} \binom{\sum \deg_{H_\epsilon}(xy, V_1)}{2}$$
$$= \frac{1}{|V_1||V_2|} \binom{2e_{H_\epsilon}(V_1V_1V_2)}{2} \geq \epsilon^6 n^4.$$

Thus, by Lemma 8.9, $V_1 \cup V_2 = V(H_\epsilon) \setminus V_0$ is $(\beta, 6)$-closed as $(\epsilon^6)^2\beta_1^6 > \beta$. □

Now we are ready to prove Lemma 8.6.

Proof of Lemma 8.6. Suppose we have the constants $1/n \ll \beta' \ll \beta \ll \epsilon \ll \gamma$. Let H be a 3-graph on n vertices with $\delta_1(H) \geq (1/3+\gamma)n$. We first apply the arguments at the beginning of this section and find V_ϵ. Then we apply Lemma 8.11 and get V_0. Let $V' = V_0 \cup V_\epsilon$ and thus $|V'| \leq 4\epsilon n$. There are two steps in our proof. In the first step, we build an absorbing family $\mathcal{F}'$ such that for any small portion of vertices in $V(H) \setminus V'$, we can absorb them using members of $\mathcal{F}'$. In the second step, we put the vertices in V' not covered by any member of $\mathcal{F}'$ into a set $\mathcal{A}$ of copies of $\mathcal{C}$. Thus, the union of $\mathcal{F}'$ and $\mathcal{A}$ gives the desired absorbing set.

We say that a set A absorbs another set B if $A \cap B = \emptyset$ and both $H[A]$ and $H[A \cup B]$ contains $\mathcal{C}$-factors. Fix any 4-set $S = \{v_1, v_2, v_3, v_4\} \in V \setminus V'$, we will show that there are many 72-sets absorbing S. First, we find vertices u_2, u_3, u_4 such that $v_1u_2u_3u_4$ spans a copy of $\mathcal{C}$. Indeed, consider the link graph H_{v_1} of v_1 on $V \setminus V'$, which contains at least $(\frac{1}{3}+\gamma)\binom{n}{2} - |V'|n \geq (\frac{1}{3}+\gamma/2)\binom{n}{2}$ edges. By convexity, the number of paths of length two

in H_{v_1} is

$$\sum_{x\in V\setminus(V'\cup\{v_1\})}\binom{\deg_{H_{v_1}}(x)}{2}\geq(n-4\epsilon n-1)\binom{\frac{1}{n-4\epsilon n-1}\sum\deg_{H_{v_1}}(x)}{2}$$
$$\geq(n-4\epsilon n-1)\binom{(\frac{1}{3}+\gamma/2)n}{2}>\frac{1}{18}n^3,$$

where the last inequality holds because $\epsilon\ll\gamma$. Since $v_1u_2u_3u_4$ spans a copy of $\mathcal{C}$ if $u_2u_3u_4$ is a path of length two in H_{v_1}, then there are at least $\frac{1}{18}n^3$ choices for such $u_2u_3u_4$.

Second, we find reachable 23-sets C_i for u_i and v_i, for $i=2,3,4$, which is possible because u_i is $(\beta,6)$-reachable to v_i, for $i=2,3,4$. Since in each step we need to avoid at most 59 previously selected vertices, there are at least $\frac{\beta}{2}n^{23}$ choices for each C_i. In total, we get $\frac{1}{18}n^3\cdot(\frac{\beta}{2}n^{23})^3>\beta^4n^{72}$ 72-sets $F=C_1\cup C_2\cup C_3\cup\{u_2,u_3,u_4\}$ (because β is small enough). It is easy to see that F absorbs S. Indeed, $H[F]$ has a $\mathcal{C}$-factor since $C_i\cup\{u_i\}$ spans six copies of $\mathcal{C}$ for $i=2,3,4$. In addition, $H[F\cup S]$ has a $\mathcal{C}$-factor since $v_1u_2u_3u_4$ spans a copy of $\mathcal{C}$ and $C_i\cup\{v_i\}$ six two copies of $\mathcal{C}$ for $i=2,3,4$.

Let $n'=n-|V'|$. We apply Lemma 2.5 with $b=72$ and β^4 on $H[V\setminus V']$ and get a set W' with $|W'|\in4\mathbb{N}$ and $|W'|\leq72\beta^8n'$ such that for any vertex subset U with $U\cap W'=\emptyset$, $|U|\in4\mathbb{N}$ and $|U|\leq\beta^{16}n'$ both $H[W']$ and $H[W'\cup U]$ contain $\mathcal{C}$-factors.

At last, we will greedily build $\mathcal{A}$, a collection of copies of $\mathcal{C}$ to cover the vertices in V' only using vertices in $V\setminus W'$. Indeed, assume that we have built $a<|V'|\leq4\epsilon n$ copies of $\mathcal{C}$. Together with the vertices in W', there are at most $4a+72\beta^8n'<\gamma n/2$ vertices already selected. Then at most $\gamma n^2/2$ pairs of vertices intersect these vertices. So for any remaining vertex $v\in V'$, there are at least

$$\deg(v)-\gamma n^2/2\geq\left(\frac{1}{4}+\gamma\right)\binom{n}{2}-\gamma n^2/2>n/2$$

edges containing v and not intersecting the existing vertices. So there is a path of length two in the link graph of v not intersecting the existing vertices, which gives a copy of $\mathcal{C}$ containing v.

We get the desired absorbing set $W = V(\mathcal{A}) \cup W'$ satisfying $|W| \leq 4 \cdot 4\epsilon n + 72\beta^8 n' < \gamma n/2$ which can be used to absorb a vertex set U with $|U| \in 4\mathbb{N}$ and $|U| \leq \beta' n$ as $\beta' \ll \beta$. □

8.3 Proof of Theorem 8.4

In this section we prove Theorem 8.4. Our proof is similar to the one of [7, Theorem 1.4]. The following fact is the only place where we need the *exact* degree condition (8.1).

Fact 8.12. *Let H be a 3-graph on n vertices with $n \in 4\mathbb{N}$ satisfying* (8.1)*. If $S \subseteq V(H)$ spans no copy of $\mathcal{C}$, then $|S| \leq \frac{3}{4}n$.*

Proof. Assume to the contrary, that $S \subseteq V(H)$ spans no copy of $\mathcal{C}$ and is of size at least $\frac{3}{4}n+1$. Take $S_0 \subseteq S$ with size exactly $\frac{3}{4}n+1$. Then for any $v \in S_0$, $\deg(v, S_0) \leq \frac{|S_0|-1}{2} = \frac{3}{8}n$. We split into two cases.

Case 8.1. $n \in 8\mathbb{N}$.

In this case, for any $v \in S_0$, since $\deg(v, S_0) \leq \frac{3}{8}n$, we have that

$$\deg(v) = \deg(v, S_0) + \deg\left(v, \binom{V}{2} \setminus \binom{S_0}{2}\right) \leq \frac{3}{8}n + \binom{n-1}{2} - \binom{\frac{3}{4}n}{2} < \delta_1(H),$$

contradicting (8.1).

Case 8.2. $n \in 4\mathbb{N} \setminus 8\mathbb{N}$.

In this case, for any $v \in S_0$, $\deg(v, S_0) \leq \frac{3}{8}n$ implies that $\deg(v, S_0) \leq \frac{3}{8}n - \frac{1}{2}$ because $n \in 4\mathbb{N} \setminus 8\mathbb{N}$. So we have

$$3e(S_0) = \sum_{v \in S_0} \deg(v, S_0) \leq \left(\frac{3}{8}n - \frac{1}{2}\right)\left(\frac{3}{4}n + 1\right) = \frac{3n-4}{8} \cdot \frac{3n+4}{4}.$$

However, neither $\frac{3n-4}{8}$ or $\frac{3n+4}{4}$ is a multiple of 3. Thus $\sum_{v \in S_0} \deg(v, S_0) < \frac{3n-4}{8} \cdot \frac{3n+4}{4}$, which implies that there exists $v_0 \in S_0$ such that $\deg(v_0, S_0) < \frac{3}{8}n - \frac{1}{2}$. Consequently,

$$\deg(v_0) = \deg(v_0, S_0) + \deg\left(v_0, \binom{V}{2} \setminus \binom{S_0}{2}\right) < \frac{3}{8}n - \frac{1}{2} + \binom{n-1}{2} - \binom{\frac{3}{4}n}{2} \leq \delta_1(H),$$

contradicting (8.1). □

Proof of Theorem 8.4. Take $\epsilon = 10^{-18}$ and let n be sufficiently large. We write $\alpha = \epsilon^{1/3} = 10^{-6}$. Let $H = (V, E)$ be a 3-graph of order n satisfying (8.1) which is ϵ-extremal, namely, there exists a set $S \subseteq V(H)$ such that $|S| \geq (1-\epsilon)\frac{3n}{4}$ and $H[S]$ is $\mathcal{C}$-free.

Let $C \subseteq V$ be a maximum set for which $H[C]$ is $\mathcal{C}$-free. Define

$$A = \left\{ x \in V \setminus C : \deg(x, C) \geq (1-\alpha)\binom{|C|}{2} \right\}, \tag{8.2}$$

and $B = V \setminus (A \cup C)$. We first claim the following bounds of $|A|, |B|, |C|$.

Claim 8.13. *$|A| > \frac{n}{4}(1-4\alpha^2), |B| < \alpha^2 n$ and $\frac{3n}{4}(1-\epsilon) \leq |C| \leq \frac{3n}{4}$.*

Proof. The estimate on $|C|$ follows from our hypothesis and Fact 8.12. We now estimate $|B|$. For any $v \in C$, we have $\deg(v, C) \leq \frac{|C|-1}{2}$, which gives $\overline{\deg}(v, C) \geq \binom{|C|-1}{2} - \frac{|C|-1}{2}$. By (8.1), $\overline{\deg}(v) \leq \binom{\frac{3}{4}n}{2} - \frac{3}{8}n + \frac{1}{2}$. Thus

$$\begin{aligned}
\overline{\deg}\left(v, \binom{V}{2} \setminus \binom{C}{2}\right) &\leq \binom{\frac{3}{4}n}{2} - \frac{3}{8}n + \frac{1}{2} - \binom{|C|-1}{2} + \frac{|C|-1}{2} \\
&\leq \binom{\frac{3}{4}n}{2} - \binom{|C|-1}{2} \quad \text{because } |C| \leq \frac{3}{4}n \\
&= \left(\frac{3}{4}n - |C| + 1\right) \cdot \frac{1}{2}\left(\frac{3}{4}n + |C| - 2\right).
\end{aligned}$$

The estimate on $|C|$ gives $\frac{3}{4}n \leq \frac{|C|}{1-\epsilon} < (1+2\epsilon)(|C|-1)$. Hence

$$\begin{aligned}
\overline{\deg}\left(v, \binom{V}{2} \setminus \binom{C}{2}\right) &< \left(\frac{3}{4}n - |C| + 1\right) \cdot \frac{1}{2}\Big((1+2\epsilon)(|C|-1) + |C| - 1\Big) \\
&= \left(\frac{3}{4}n - |C| + 1\right) \cdot (1+\epsilon)(|C|-1) \qquad (8.3) \\
&\leq \left(\frac{3}{4}\epsilon n + 1\right) \cdot (1+\epsilon)(|C|-1) < \epsilon n \cdot (|C|-1). \qquad (8.4)
\end{aligned}$$

Consequently $\overline{e}(CC(A \cup B)) < \frac{1}{2}|C| \cdot \epsilon n \cdot (|C|-1) = \epsilon n \cdot \binom{|C|}{2}$. Together with the definition

of A and B, we have

$$(|A\cup B|-\epsilon n)\binom{|C|}{2} < e(CC(A\cup B)) \le (1-\alpha)\binom{|C|}{2}|B| + \binom{|C|}{2}|A|,$$

so that $|A\cup B|-\epsilon n < |A|+|B|-\alpha|B|$. Since A and B are disjoint, we get that $|B|<\alpha^2 n$. Finally, $|A| = n-|B|-|C| > n-\alpha^2 n-\frac{3}{4}n = \frac{n}{4}(1-4\alpha^2)$. □

In the rest of the section, we will build four vertex-disjoint $\mathcal{C}$-tilings $\mathcal{Q},\mathcal{R},\mathcal{S},\mathcal{T}$ whose union is a perfect $\mathcal{C}$-tiling of H. In particular, when $|A|=n/4$, $B=\emptyset$ and $|C|=3n/4$, we have $\mathcal{Q}=\mathcal{R}=\mathcal{S}=\emptyset$ and the perfect $\mathcal{C}$-tiling $\mathcal{T}$ of H will be provided by Lemma 3.9. The purpose of $\mathcal{C}$-tilings $\mathcal{Q},\mathcal{R},\mathcal{S}$ is covering the vertices of B and adjusting the sizes of A and C such that we can apply Lemma 3.9 after $\mathcal{Q},\mathcal{R},\mathcal{S}$ are removed.

The $\mathcal{C}$-tiling $\mathcal{Q}$. Let $\mathcal{Q}$ be a largest $\mathcal{C}$-tiling in H on $B\cup C$ and $q=|\mathcal{Q}|$. We claim that $|B|/4 \le q \le |B|$. Since C contains no copy of $\mathcal{C}$, every element of $\mathcal{Q}$ contains at least one vertex of B and consequently $q\le|B|$. On the other hand, suppose that $q<|B|/4$, then $(B\cup C)\setminus V(\mathcal{Q})$ spans no copy of $\mathcal{C}$ and has order

$$|B|+|C|-4q > |B|+|C|-|B| = |C|.$$

which contradicts the assumption that C is a maximum $\mathcal{C}$-free subset of $V(H)$.

Claim 8.14. $q+|A|\ge\frac{n}{4}$.

Proof. Let $l=\frac{n}{4}-|A|$. There is nothing to show if $l\le 0$. If $l=1$, we have $|B\cup C|=\frac{3}{4}n+1$, and thus Fact 8.12 implies that $H[B\cup C]$ contains a copy of $\mathcal{C}$. Thus $q\ge 1=l$ and we are done. We thus assume $l\ge 2$ and $l>q\ge|B|/4$, which implies that $|B|\le 4(l-1)$. In this case $|B|\ge 2$ because $|C|\le\frac{3}{4}n$.

For any $v\in C$, by (8.3), we have $\overline{\deg}(v,BC) < \left(\frac{3}{4}n-|C|+1\right)\cdot(1+\epsilon)(|C|-1)$. By definition, $\frac{3}{4}n-|C| = |A|+|B|-\frac{n}{4} = |B|-l$. So we get

$$\overline{e}(BCC) < \frac{1}{2}|C|\left(\frac{3}{4}n-|C|+1\right)\cdot(1+\epsilon)(|C|-1) = (1+\epsilon)(|B|-l+1)\binom{|C|}{2}.$$

Together with $|B| \leq 4(l-1)$, this implies

$$\begin{aligned} e(BCC) &> (|B| - (1+\epsilon)(|B| - l + 1))\binom{|C|}{2} \\ &= ((1+\epsilon)(l-1) - \epsilon|B|)\binom{|C|}{2} \\ &\geq ((1+\epsilon)(l-1) - 4\epsilon(l-1))\binom{|C|}{2} = (1-3\epsilon)(l-1)\binom{|C|}{2}. \end{aligned} \tag{8.5}$$

On the other hand, we want to bound $e(BCC)$ from above and then derive a contradiction. Assume that $\mathcal{Q}'$ is the maximum $\mathcal{C}$-tiling of size q' such that each element of $\mathcal{Q}'$ contains exactly one vertex in B and three vertices in C. Note that $q' \geq 1$ because C is a maximum $\mathcal{C}$-free set and $B \neq \emptyset$. Write $B_{\mathcal{Q}'}$ for the set of vertices of B covered by $\mathcal{Q}'$ and $C_{\mathcal{Q}'}$ for the set of vertices of C covered by $\mathcal{Q}'$. Clearly, $|B_{\mathcal{Q}'}| = q'$, $|C_{\mathcal{Q}'}| = 3q'$ and $q' \leq q \leq l-1$. For any vertex $v \in B \setminus B_{\mathcal{Q}'}$, $\deg(v, C) \leq 3q'(|C|-1) + \frac{1}{2}|C| < 4q'|C|$. Together with the definition of B and Claim 8.13, we get

$$\begin{aligned} e(BCC) &= e(B_{\mathcal{Q}'}CC) + e((B \setminus B_{\mathcal{Q}'})CC) \\ &\leq q'(1-\alpha)\binom{|C|}{2} + |B| \cdot 4q'|C| < q'(1-\alpha)\binom{|C|}{2} + 4\alpha^2 nq'|C|. \end{aligned} \tag{8.6}$$

Putting (8.5) and (8.6) together and using $q' \leq l-1$ and $|C| > n/2$, we get

$$1 - 3\alpha^3 = 1 - 3\epsilon < 1 - \alpha + \frac{8\alpha^2 n}{|C|-1} < 1 - \alpha + 16\alpha^2 < 1 - \frac{\alpha}{2},$$

which is a contradiction since $\alpha = 10^{-6}$. □

Let B_1 and C_1 be the vertices in B and C not covered by $\mathcal{Q}$, respectively. By Claim 8.13,

$$|C_1| \geq |C| - 3q \geq |C| - 3|B| > \frac{3}{4}n(1-\epsilon) - 3\alpha^2 n > \frac{3}{4}n - 4\alpha^2 n + 1. \tag{8.7}$$

The $\mathcal{C}$-tiling $\mathcal{R}$. Next we will build our $\mathcal{C}$-tiling $\mathcal{R}$ which covers B_1 such that every element in $\mathcal{R}$ contains one vertex from A, one vertex from B_1 and two vertices from C_1. Since $\mathcal{Q}$

is a maximum $\mathcal{C}$-tiling on $B \cup C$, for every vertex $v \in B_1$, we have that $\deg(v, C_1) \leq \frac{|C_1|}{2}$. Together with (8.7), this implies that

$$\overline{\deg}(v, C_1) \geq \binom{|C_1|}{2} - \frac{|C_1|}{2} = \frac{|C_1|(|C_1|-2)}{2} > \frac{(\frac{3}{4}n - 4\alpha^2 n)^2 - 1}{2}.$$

Together with (8.1), we get that for every $v \in B_1$,

$$\begin{aligned}\overline{\deg}(v, AC_1) &< \binom{\frac{3}{4}n}{2} - \frac{3}{8}n + \frac{1}{2} - \frac{(\frac{3}{4}n - 4\alpha^2 n)^2 - 1}{2} \\ &= \frac{1}{2}\left(\frac{3}{2}n - 4\alpha^2 n\right) 4\alpha^2 n - \frac{3}{4}n + 1 < 3\alpha^2 n^2.\end{aligned}$$

By Claim 8.13 and (8.7), we have that $|A||C_1| > (1-4\alpha^2)\frac{n}{4} \cdot (\frac{3}{4} - 4\alpha^2)n > \frac{3}{17}n^2$. Thus, $\overline{\deg}(v, AC_1) < 3\alpha^2 n^2 < 17\alpha^2 |A||C_1|$, equivalently, $\deg(v, AC_1) > (1-17\alpha^2)|A||C_1|$. For every $v \in B_1$, we greedily pick a copy of $\mathcal{C}$ containing v by picking a path of length two with center in A and two ends in C_1 from the link graph of v. Suppose we have found $i < |B_1|$ copies of $\mathcal{C}$, then for any remaining vertex $v \in B_1$, by Claim 8.13, the number of pairs not intersecting the existing vertices is at least

$$\deg(v, AC_1) - 3i \cdot (|A| + |C_1|) > (1-17\alpha^2)|A||C_1| - 3|B_1| \cdot 2|C_1| > |A|,$$

which guarantees a path of length two centered at A, so a copy of $\mathcal{C}$ containing v.

Now all vertices of B are covered by $\mathcal{Q}$ or $\mathcal{R}$. Let A_2 denote the set of vertices of A not covered by $\mathcal{Q}$ or $\mathcal{R}$ and define C_2 similarly. By the definition of $\mathcal{Q}$ and $\mathcal{R}$, we have $|A_2| = |A| - |B_1|$ and $|C_2| = |B| + |C| - 4q - 3|B_1|$. Define $s = \frac{1}{4}(3|A_2| - |C_2|)$. Then

$$s = \frac{1}{4}(3|A| - 3|B_1| - |B| - |C| + 4q + 3|B_1|) = \frac{1}{4}(4|A| - n + 4q) = q + |A| - \frac{n}{4}.$$

Thus $s \in \mathbb{Z}$, and $s \geq 0$ by Claim 8.14. Since $q \leq |B|$, by Claim 8.13,

$$s = q + |A| - \frac{n}{4} \leq |B| + |A| - \frac{n}{4} = \frac{3}{4}n - |C| \leq \frac{3}{4}\epsilon n. \tag{8.8}$$

The definition of $\mathcal{Q}$ and $\mathcal{R}$ also implies that $|C \setminus C_2| \le 3|B|$ and

$$|C_2| \ge |C| - 3|B| > |C| - 3 \cdot 2\alpha^2|C| = (1 - 6\alpha^2)|C|, \tag{8.9}$$

where the second inequality follows from $|B| < \alpha^2 n < 2\alpha^2|C|$.

The $\mathcal{C}$-tiling $\mathcal{S}$. Next we will build our $\mathcal{C}$-tiling $\mathcal{S}$ of size s such that every element of $\mathcal{S}$ contains two vertices in A_2 and two vertices in C_2. Note that for any vertex $v \in A_2$, by (8.2) and (8.9),

$$\overline{\deg}(v, C_2) \le \alpha \binom{|C|}{2} \le \alpha \binom{\frac{1}{1-6\alpha^2}|C_2|}{2} < 2\alpha \binom{|C_2|}{2}.$$

Suppose that we have found $i < s$ copies of $\mathcal{C}$ of the desired type. We next select two vertices a_1, a_2 in A_2 and note that they have at least $(1 - 4\alpha)\binom{|C_2|}{2}$ common neighbors in C_2. By (8.8) and(8.9),

$$(1 - 4\alpha)\binom{|C_2|}{2} - 2s|C_2| \ge (1 - 4\alpha)\binom{|C_2|}{2} - \frac{3}{2}\epsilon n|C_2| \ge (1 - 5\alpha)\binom{|C_2|}{2} > 0.$$

So we can pick a common neighbor c_1c_2 of a_1 and a_2 from unused vertices of C_2 such that $\{a_1, a_2, c_1, c_2\}$ spans a copy of $\mathcal{C}$.

Let A_3 be the set of vertices of A not covered by $\mathcal{Q}, \mathcal{R}, \mathcal{S}$ and define C_3 similarly. Then $|A_3| = |A_2| - 2s = \frac{1}{2}(|C_2| - |A_2|)$ and $|C_3| = |C_2| - 2s = \frac{3}{2}(|C_2| - |A_2|)$, so $|C_3| = 3|A_3|$. Furthermore, by (8.8) and (8.9), we have

$$|C_3| = |C_2| - 2s \ge (1 - 6\alpha^2)|C| - \frac{3}{2}\epsilon n > (1 - 6\alpha^2)|C| - 3\epsilon|C| > (1 - 7\alpha^2)|C|.$$

Hence, for every vertex $v \in A_3$,

$$\overline{\deg}(v, C_3) \le \alpha \binom{|C|}{2} \le \alpha \binom{\frac{1}{1-7\alpha^2}|C_3|}{2} < 2\alpha \binom{|C_3|}{2}.$$

Since $|C_3| \geq (1-7\alpha^2)|C| \geq (1-7\alpha^2)(1-\epsilon)\frac{3}{4}n$, by (8.4), we know that for any vertex $v \in C_3$,

$$\overline{\deg}(v, A_3C_3) < \epsilon n \cdot (|C| - 1) < 2\epsilon|C_3|^2 = 6\epsilon|A_3||C_3|.$$

The $\mathcal{C}$-tiling $\mathcal{T}$. Finally we use the following lemma to find a $\mathcal{C}$-tiling $\mathcal{T}$ covering A_3 and C_3 such that every element of $\mathcal{T}$ contains one vertex in A_3 and three vertices in C_3. Note that in [7], this was done by applying a general theorem of Pikhurko [57, Theorem 3] (but impossible here because we do not have the co-degree condition).

Lemma 8.15. *Suppose that $0 < \rho \leq 2 \cdot 10^{-6}$ and n is sufficiently large. Let H be a 3-graph on n vertices with $V(H) = X \dot\cup Z$ such that $|Z| = 3|X|$. Further, assume that for every vertex $v \in X$, $\overline{\deg}(v, Z) \leq \rho\binom{|Z|}{2}$ and for every vertex $v \in Z$, $\overline{\deg}(v, XZ) \leq \rho|X||Z|$. Then H contains a perfect $\mathcal{C}$-tiling.*

Applying Lemma 8.15 with $X = A_3$, $Z = C_3$, $\rho = 2\alpha$ finishes the proof of Theorem 8.4. □

Proof of Lemma 8.15. Let G be the graph of all pairs uv in Z such that $\deg(uv, X) \geq (1 - \sqrt{\rho})|X|$. We claim that for any vertex $v \in Z$,

$$\overline{\deg}_G(v) \leq \sqrt{\rho}|Z|. \tag{8.10}$$

Otherwise, some vertex $v \in Z$ satisfies $\overline{\deg}_G(v) > \sqrt{\rho}|Z|$. As each $u \notin N_G(v)$ satisfies $\overline{\deg}_H(uv, X) > \sqrt{\rho}|X|$, we have

$$\overline{\deg}_H(v, XZ) > \sqrt{\rho}|Z| \cdot \sqrt{\rho}|X| = \rho|Z||X|,$$

contradicting our assumption.

Arbitrarily partition Z into three sets Z_1, Z_2, Z_3, each of order $|X|$. By (8.10), we have $\overline{\deg}_G(v) \leq \sqrt{\rho}|Z| = 3\sqrt{\rho}|X|$ and $\delta(G[Z_1, Z_2]), \delta(G[Z_2, Z_3]) \geq (1 - 3\sqrt{\rho})|X|$. Thus,

$G[Z_1, Z_2]$ and $G[Z_2, Z_3]$ are both $(1, 3\sqrt[4]{\rho})$-super-regular. For any $x \in X$, let $F_x^1 := \{zz' \in E(G[Z_1, Z_2]) : \{x, z, z'\} \in E(H)\}$ and let $F_x^2 := \{zz' \in E(G[Z_2, Z_3]) : \{x, z, z'\} \in E(H)\}$. Since $\overline{\deg}(x, Z) \leq \rho\binom{|Z|}{2} \leq 5\rho|X|^2$, we have $|F_x^1|, |F_x^2| \geq (1 - 3\sqrt{\rho})|X|^2 - 5\rho|X|^2 \geq (1 - 4\sqrt{\rho})|X|^2$. By applying Lemma 5.28 with $v = 1 - 4\sqrt{\rho}$ and $\eta = \rho$, then for any $x \in X$, with probability at least $1 - e^{-\epsilon|X|}$ we have

$$|M_1 \cap E(F_x^1)|, |M_2 \cap E(F_x^2)| \geq (1 - \eta)v|X| \geq (1 - 5\sqrt{\rho})|X|.$$

Thus, there is a matching M_1 in $G[Z_1, Z_2]$ and a matching M_2 in $G[Z_2, Z_3]$ such that $|M_1 \setminus F_x^1| \leq 5\sqrt{\rho}|X|$ and $|M_2 \setminus F_x^2| \leq 5\sqrt{\rho}|X|$ for every vertex $x \in X$. Label the vertices of Z so that $Z_1 = \{a_1, \ldots, a_{|X|}\}$, $Z_2 = \{b_1, \ldots, b_{|X|}\}$ and $Z_3 = \{c_1, \ldots, c_{|X|}\}$ such that $M_1 = \{a_1b_1, \ldots, a_{|X|}b_{|X|}\}$ and $M_2 = \{b_1c_1, \ldots, b_{|X|}c_{|X|}\}$. Let Γ be a bipartite graph with one part X and the other part $\{a_1b_1c_1, \ldots, a_{|X|}b_{|X|}c_{|X|}\}$ such that $\{x, a_ib_ic_i\} \in E(\Gamma)$ if and only if $xa_ib_i, xb_ic_i \in E(H)$. For every $1 \leq i \leq |X|$, since $a_ib_i, b_ic_i \in E(G)$, so $\deg_\Gamma(a_ib_ic_i) \geq (1 - 2\sqrt{\rho})|X|$ in Γ. On the other hand, by assumptions, we have $\deg_\Gamma(x) \geq (1 - 10\sqrt{\rho})|X|$ for any $x \in X$. Thus we can find a perfect matching in Γ, which gives a perfect $\mathcal{C}$-tiling in H. □

REFERENCES

[1] N. Alon, P. Frankl, H. Huang, V. Rödl, A. Ruciński, and B. Sudakov. Large matchings in uniform hypergraphs and the conjecture of Erdős and Samuels. *J. Combin. Theory Ser. A*, 119(6):1200–1215, 2012.

[2] N. Alon and J. Spencer. *The probabilistic method*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2008. With an appendix on the life and work of Paul Erdős.

[3] N. Alon and R. Yuster. H-factors in dense graphs. *J. Combin. Theory Ser. B*, 66(2):269–282, 1996.

[4] E. Buß, H. Hàn, and M. Schacht. Minimum vertex degree conditions for loose Hamilton cycles in 3-uniform hypergraphs. *J. Combin. Theory Ser. B*, 103(6):658–678, 2013.

[5] K. Corradi and A. Hajnal. On the maximal number of independent circuits in a graph. *Acta Math. Acad. Sci. Hungar.*, 14:423–439, 1963.

[6] A. Czygrinow. Minimum degree condition for C_4-tiling in 3-uniform hypergraphs. *submitted*.

[7] A. Czygrinow, L. DeBiasio, and B. Nagle. Tiling 3-uniform hypergraphs with $K_4^3 - 2e$. *Journal of Graph Theory*, 75(2):124–136, 2014.

[8] A. Czygrinow and V. Kamat. Tight co-degree condition for perfect matchings in 4-graphs. *Electron. J. Combin.*, 19(2):Paper 20, 16, 2012.

[9] A. Czygrinow and T. Molla. Tight codegree condition for the existence of loose Hamilton cycles in 3-graphs. *SIAM J. Discrete Math.*, 28(1):67–76, 2014.

[10] G. A. Dirac. Some theorems on abstract graphs. *Proc. London Math. Soc. (3)*, 2:69–81, 1952.

[11] J. Edmonds. Paths, trees, and flowers. *Canad. J. Math.*, 17:449–467, 1965.

[12] P. Erdős, C. Ko, and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 12:313–320, 1961.

[13] E. Fischer. Variants of the Hajnal-Szemerédi theorem. *J. Graph Theory*, 31(4):275–282, 1999.

[14] Z. Füredi. Turán type problems. In *Surveys in combinatorics, 1991 (Guildford, 1991)*, volume 166 of *London Math. Soc. Lecture Note Ser.*, pages 253–300. Cambridge Univ. Press, Cambridge, 1991.

[15] R. Glebov, Y. Person, and W. Weps. On extremal hypergraphs for Hamiltonian cycles. *European J. Combin.*, 33(4):544–555, 2012.

[16] A. Hajnal and E. Szemerédi. Proof of a conjecture of P. Erdős. In *Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969)*, pages 601–623. North-Holland, Amsterdam, 1970.

[17] H. Hàn, Y. Person, and M. Schacht. On perfect matchings in uniform hypergraphs with large minimum vertex degree. *SIAM J. Discrete Math*, 23:732–748, 2009.

[18] H. Hàn and M. Schacht. Dirac-type results for loose Hamilton cycles in uniform hypergraphs. *Journal of Combinatorial Theory. Series B*, 100:332–346, 2010.

[19] J. Han. Decision problem for perfect matchings in dense uniform hypergraphs. *submitted.*

[20] J. Han. Near perfect matchings in k-uniform hypergraphs. *Combin. Probab. Comput., in press.*

[21] J. Han, C. Zang, and Y. Zhao. Tiling 3-partite 3-uniform hypergraphs. *in preparation.*

[22] J. Han and Y. Zhao. Minimum vertex degree threshold for C_4^3-tiling. *Journal of Graph Theory, in press.*

[23] J. Han and Y. Zhao. Minimum vertex degree threshold for loose Hamilton cycles in 3-graphs. *Journal of Combinatorial Theory, Series B, accepted.*

[24] J. Han and Y. Zhao. On multipartite Hajnal-Szemerédi theorems. *Discrete Math.*, 313(10):1119–1129, 2013.

[25] J. Han and Y. Zhao. Minimum codegree threshold for Hamilton ℓ-cycles in k-uniform hypergraphs. *Journal of Combinatorial Theory, Series A*, 132(0):194 – 223, 2015.

[26] R. M. Karp. Reducibility among combinatorial problems. In *Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972)*, pages 85–103. Plenum, New York, 1972.

[27] M. Karpiński, A. Ruciński, and E. Szymańska. Computational complexity of the perfect matching problem in hypergraphs with subcritical density. *Internat. J. Found. Comput. Sci.*, 21(6):905–924, 2010.

[28] G. Katona and H. Kierstead. Hamiltonian chains in hypergraphs. *Journal of Graph Theory*, 30(2):205–212, 1999.

[29] P. Keevash. Hypergraph Turán problems. In *Surveys in combinatorics 2011*, volume 392 of *London Math. Soc. Lecture Note Ser.*, pages 83–139. Cambridge Univ. Press, Cambridge, 2011.

[30] P. Keevash, F. Knox, and R. Mycroft. Polynomial-time perfect matchings in dense hypergraphs. *Proceedings of the 45th STOC (2013)*, 2013.

[31] P. Keevash, F. Knox, and R. Mycroft. Polynomial-time perfect matchings in dense hypergraphs. *Advances in Mathematics*, 269(0):265 – 334, 2015.

[32] P. Keevash, D. Kühn, R. Mycroft, and D. Osthus. Loose Hamilton cycles in hypergraphs. *Discrete Mathematics*, 311(7):544–559, 2011.

[33] P. Keevash and R. Mycroft. A multipartite Hajnal-Szemerédi theorem. *preprint.*

[34] P. Keevash and R. Mycroft. A geometric theory for hypergraph matching. *Memoirs of the American Mathematical Society*, 233(Monograph 1908), 2014.

[35] I. Khan. Perfect matchings in 4-uniform hypergraphs. *arXiv:1101.5675*.

[36] I. Khan. Perfect matchings in 3-uniform hypergraphs with large vertex degree. *SIAM J. Discrete Math.*, 27(2):1021–1039, 2013.

[37] J. Komlós, G. Sárközy, and E. Szemerédi. On the square of a Hamiltonian cycle in dense graphs. In *Proceedings of the Seventh International Conference on Random Structures and Algorithms (Atlanta, GA, 1995)*, volume 9, pages 193–211, 1996.

[38] J. Komlós, G. Sárközy, and E. Szemerédi. Blow-up lemma. *Combinatorica*, 17(1):109–123, 1997.

[39] J. Komlós, G. Sárközy, and E. Szemerédi. Proof of the Alon-Yuster conjecture. *Discrete Math.*, 235(1-3):255–269, 2001. Combinatorics (Prague, 1998).

[40] D. Kühn, R. Mycroft, and D. Osthus. Hamilton ℓ-cycles in uniform hypergraphs. *Journal of Combinatorial Theory. Series A*, 117(7):910–927, 2010.

[41] D. Kühn and D. Osthus. Loose Hamilton cycles in 3-uniform hypergraphs of high minimum degree. *Journal of Combinatorial Theory. Series B*, 96(6):767–821, 2006.

[42] D. Kühn and D. Osthus. Matchings in hypergraphs of large minimum degree. *J. Graph Theory*, 51(4):269–280, 2006.

[43] D. Kühn and D. Osthus. Multicolored Hamilton cycles and perfect matchings in pseudorandom graphs. *SIAM J. Discrete Math.*, 20(2):273–286 (electronic), 2006.

[44] D. Kühn and D. Osthus. Embedding large subgraphs into dense graphs. In *Surveys in combinatorics 2009*, volume 365 of *London Math. Soc. Lecture Note Ser.*, pages 137–167. Cambridge Univ. Press, Cambridge, 2009.

[45] D. Kühn and D. Osthus. The minimum degree threshold for perfect graph packings. *Combinatorica*, 29(1):65–107, 2009.

[46] D. Kühn, D. Osthus, and T. Townsend. Fractional and integer matchings in uniform hypergraphs. *European J. Combin.*, 38:83–96, 2014.

[47] D. Kühn, D. Osthus, and A. Treglown. Matchings in 3-uniform hypergraphs. *J. Combin. Theory Ser. B*, 103(2):291–305, 2013.

[48] I. Levitt, G. Sárközy, and E. Szemerédi. How to avoid using the regularity lemma: Pósa's conjecture revisited. *Discrete Math.*, 310(3):630–641, 2010.

[49] A. Lo and K. Markström. F-factors in hypergraphs via absorption. *Graphs and Combinatorics, to appear.*

[50] A. Lo and K. Markström. Minimum codegree threshold for $(K_4^3 - e)$-factors. *J. Combin. Theory Ser. A*, 120(3):708–721, 2013.

[51] A. Lo and K. Markström. A multipartite version of the Hajnal-Szemerédi theorem for graphs and hypergraphs. *Combin. Probab. Comput.*, 22(1):97–111, 2013.

[52] C. Magyar and R. Martin. Tripartite version of the Corrádi-Hajnal theorem. *Discrete Math.*, 254(1-3):289–308, 2002.

[53] W. Mantel. Problem 28. *Wiskundige Opgaven*, 10:60–61, 1907.

[54] K. Markström and A. Ruciński. Perfect Matchings (and Hamilton Cycles) in Hypergraphs with Large Degrees. *Eur. J. Comb.*, 32(5):677–687, July 2011.

[55] R. Martin and E. Szemerédi. Quadripartite version of the Hajnal-Szemerédi theorem. *Discrete Math.*, 308(19):4337–4360, 2008.

[56] R. Mycroft. Packing k-partite k-uniform hypergraphs. *Electronic Notes in Discrete Mathematics*, 38(0):663 – 668, 2011. The Sixth European Conference on Combinatorics, Graph Theory and Applications, EuroComb 2011.

[57] O. Pikhurko. Perfect matchings and K_4^3-tilings in hypergraphs of large codegree. *Graphs Combin.*, 24(4):391–404, 2008.

[58] V. Rödl and A. Ruciński. Dirac-type questions for hypergraphs a survey (or more problems for endre to solve). *An Irregular Mind*, Bolyai Soc. Math. Studies 21:561–590, 2010.

[59] V. Rödl, A. Ruciński, M. Schacht, and E. Szemerédi. A note on perfect matchings in uniform hypergraphs with large minimum collective degree. *Comment. Math. Univ. Carolin.*, 49(4):633–636, 2008.

[60] V. Rödl, A. Ruciński, and E. Szemerédi. A Dirac-type theorem for 3-uniform hypergraphs. *Combinatorics, Probability and Computing*, 15(1-2):229–251, 2006.

[61] V. Rödl, A. Ruciński, and E. Szemerédi. Perfect matchings in uniform hypergraphs with large minimum degree. *European J. Combin.*, 27(8):1333–1349, 2006.

[62] V. Rödl, A. Ruciński, and E. Szemerédi. An approximate Dirac-type theorem for k-uniform hypergraphs. *Combinatorica*, 28(2):229–260, 2008.

[63] V. Rödl, A. Ruciński, and E. Szemerédi. Perfect matchings in large uniform hypergraphs with large minimum collective degree. *J. Combin. Theory Ser. A*, 116(3):613–636, 2009.

[64] V. Rödl, A. Ruciński, and E. Szemerédi. Dirac-type conditions for Hamiltonian paths and cycles in 3-uniform hypergraphs. *Advances in Mathematics*, 227(3):1225–1299, 2011.

[65] A. Shokoufandeh and Y. Zhao. Proof of a tiling conjecture of Komlós. *Random Structures Algorithms*, 23(2):180–205, 2003.

[66] Endre Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages 399–401. CNRS, Paris, 1978.

[67] E. Szymańska. The complexity of almost perfect matchings and other packing problems in uniform hypergraphs with high codegree. *European J. Combin.*, 34(3):632–646, 2013.

[68] A. Treglown and Y. Zhao. Exact minimum degree thresholds for perfect matchings in uniform hypergraphs. *J. Combin. Theory Ser. A*, 119(7):1500–1522, 2012.

[69] A. Treglown and Y. Zhao. Exact minimum degree thresholds for perfect matchings in uniform hypergraphs II. *J. Combin. Theory Ser. A*, 120(7):1463–1482, 2013.

[70] P. Turán. On an extremal problem in graph theory, (Hungarian). *Mat. Fiz. Lapok*, 48:436–452, 1941.

[71] W. T. Tutte. The factorization of linear graphs. *J. London Math. Soc.*, 22:107–111, 1947.