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Homeomorphically Irreducible Spanning Trees, Halin Graphs, and Long Cycles in 3-connected Graphs with Bounded Maximum Degrees

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Under the Direction of Guantao Chen, PhD

ABSTRACT

A tree T with no vertex of degree 2 is called a *homeomorphically irreducible tree* (HIT) and if T is spanning in a graph, then T is called a *homeomorphically irreducible spanning tree* (HIST). Albertson, Berman, Hutchinson and Thomassen asked *if every triangulation of at least 4 vertices has a HIST* and *if every connected graph with each edge in at least two triangles contains a HIST*. These two questions were restated as two conjectures by Archdeacon in 2009. The first part of this dissertation gives a proof for each of the two conjectures. The second part focuses on some problems about *Halin graphs*, which is a class of graphs closely related to HITs and HISTs. A *Halin graph* is obtained from a plane embedding of a HIT of at least 4 vertices by connecting its leaves into a cycle following the cyclic order determined by the embedding. And a *generalized Halin graph* is obtained from a HIT of at least 4 vertices by connecting the leaves into a cycle. Let G be a sufficiently large n-vertex graph. Applying the Regularity Lemma and the Blow-up Lemma, it is shown that G contains a spanning Halin subgraph if it has minimum degree at least $(n+1)/2$ and G contains a spanning generalized Halin subgraph if it is 3-connected and has minimum degree at least $(2n+3)/5$. The minimum degree conditions are best possible. The last part estimates the length of longest cycles in 3-connected graphs with bounded maximum degrees. In 1993 Jackson and Wormald conjectured that for any positive integer $d \geq 4$, there exists a positive real number α depending only on d such that if G is a 3-connected n-vertex graph with maximum degree d, then G has a cycle of length at least $\alpha n^{\log_{d-1} 2}$. They showed that the exponent in the bound is best possible if the conjecture is true. The conjecture is confirmed for $d \geq 425$.

INDEX WORDS: Homeomorphically irreducible spanning tree, Halin graph, Genaralized Halin graph, 3-connected graphs, Tutte decomposition.

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DEDICATION

This dissertation is dedicated to my family and my advisor Professor Guantao Chen.

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LIST OF ABBREVIATIONS

- GSU - Georgia State University
- HIT - Homeomorphically irreducible tree
- HIST - Homeomorphically irreducible spanning tree
- SGHG - Spanning generalized Halin subgraph

PART 1

INTRODUCTION

Finding longest cycles, in particular a hamiltonian cycle in a graph, is one of a few fundamental yet very difficult problems in graph theory. In fact, to determine whether a graph is hamiltonian is a classic NP-complete problem. Moreover, Karger, Motwani, and Ramkumar [35] showed that, unless $\mathcal{P} = \mathcal{NP}$, it is impossible to find, in polynomial time, a path of length $n - n^{\epsilon}$ in an n-vertex hamiltonian graph for any $\epsilon < 1$. On the other hand, inspired by classic results obtained by Dirac [19] in 1954 and Tutte [51] in 1956, respectively, many sufficient conditions for hamiltonian graphs have been obtained. For examples, see [25]. Accompanying with each of these sufficient conditions, various stronger results such as being hamiltonian connected and pancyclic have also been established.

As an antithetical class to hamiltonian paths/cycles, *homeomorphically irreducible* graphs, graphs which have no vertex of degree 2, were introduced by graph theorists in 1970s. A *homeorphically irreducible tree* is called a *HIT*, and a *homeomorphically irreducible spanning tree* of a graph is called a *HIST* of the graph. As graphs of at most three vertices contain no HIST, we assume the graphs in consideration are of at least four vertices when we considering HISTs. In the first part of this dissertation, we show the existence of a HIST in surface triangulations and connected graphs with each edge contained in at least two triangles. This confirms the two conjectures raised by Albertson, Berman, Hutchinson, and Thomassen [1].

Another class of graphs which is closely related to HITs and HISTs is the class of *Halin graphs*. Let T be a HIT of at least 4 vertices. Then a *Halin graph* H is obtained from a plane embedding of T by connecting the leaves into a cycle C following the cyclic order determined by the plane embedding. In this notation, we may write the Halin graph as $H = T \cup C$. A wheel is an example of a Halin graph. Since a HIT of at least 4 vertices contains two leaves sharing the same parent, a Halin graph contains a triangle, and thus is not bipartite. Moreover, cubic Halin graphs are in one-to-one correspondence (via weak duality) with the plane triangulations of the disc. Halin constructed Halin graphs in [27] for the study of minimally 3-connected graphs. Lovász and Plummer named such graphs as Halin graphs in their study of planar bicritical graphs [40], which are planar graphs having a 1-factor after deleting any two vertices. It was conjectured by Lovász and Plummer [40] that *every 4-connected plane triangulation contains a spanning Halin subgraph* (disproved in [10]). Although the conjecture is not true, it inspires new questions and problems. We may ask, can we find any other class of graphs which contain a spanning Halin subgraph or a spanning generalized Halin subgraph? The second part of this dissertation considers the existence of spanning Halin subgraphs and spanning generalized Halin subgraphs in graphs with large minimum degree. Halin graphs possess very nice hamiltonicity properties. Hence finding the existence of a spanning Halin subgraph can be viewed as a generalization of finding hamiltonian paths/cycles in graphs.

Finally, in the last part, the problem of finding longest cycles in 3-connected graphs with bounded maximum degrees is investigated. In 1993 Jackson and Wormald conjectured that for any positive integer $d \geq 4$, there exists a positive real number α depending only on d such that if G is a 3-connected n-vertex graph with maximum degree d, then G has a cycle of length at least $\alpha n^{\log_{d-1} 2}$. They showed that the exponent in the bound is best possible if the conjecture is true. The conjecture is confirmed for $d \geq 425$.

Throughout this report, we limit our attention to simple and connected graphs, and further assume graphs to be finite unless we specify otherwise; and refer to Bondy and Murty [7] for notations and terminologies used but not defined. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S. Similarly, $G[F]$ is the subgraph induced on F if $F \subseteq E(G)$. The minimum degree and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Other specified notations are introduced in each chapters.

PART 2

THE EXISTENCE OF HISTS IN SURFACE TRIANGULATIONS AND CONNECTED GRAPHS WITH EACH EDGE IN AT LEAST TWO TRIANGLES

Recall that a tree is called *homeomorphically irreducible* if it does not contain vertices of degree 2 and a *homeomorphically irreducible spanning tree* of a graph G is called a *HIST* of G. Albertson, Berman, Hutchinson, and Thomassen [1] obtained various sufficient conditions for a graph to contain a HIST. They also showed that *it is NP-complete to decide whether a graph G contains a HIST*. Hill [28] conjectured that every *triangulation of the plane contains a HIST*. Malkevitch [41] conjectured that the same result hold for near-triangulations of the plane (2-connected plane graphs such that all, but at most one, faces are triangles). Albertson, Berman, Hutchinson, and Thomassen [1] confirmed the conjecture. Furthermore, they asked *whether every graph that triangulates some surface has a HIST*, and more generally *if every connected graph with each edge contained in two triangles contains a HIST*. To establish a strategy to tackle the problem, Ellingham [20] asked whether *every triangulation of a given surface with sufficiently large representativity contains a HIST*. Huneke observed that every triangulation of the projective plane contains a spanning plane subgraph such that every face is a triangle with one possible exception, so every triangulation of the projective plane contains a HIST. Davidow, Hutchinson, and Huneke [18] showed that every triangulation of the torus contains a HIST. In 2009, Achdeacon [4] (Chapter 15) restated the above two questions as two conjectures.

Conjecture 2.1. *Every surface triangulation contains a HIST.*

Conjecture 2.2. *Every connected graph with each edge in at least two triangles contains a HIST.*

We confirm the two conjectures in this Chapter. The proofs can also be found in [11] and [13].

2.1 Proof of Conjecture 2.1

A graph G is *locally connected* if for every vertex $v \in V(G)$, the subgraph induced by the neighborhood $N(v)$ is connected. Ringel [46] showed that every triangulation (includes orientable and nonorientable) is a connected and locally connected graph. In this section, we prove the following much more general result, which confirms the conjecture by Archdeacon and answers the first question asked by Albertson, Berman, Hutchinson, and Thomassen positively.

Theorem 2.1.1. *Every connected and locally connected graph with order at least four contains a HIST.*

Corollary 2.1.1. *Let* Π *be a surface (orientable or nonorientable). Then every triangulation of* Π *with at least four vertices contains a HIST.*

Let G be a graph. Write $v \in G$ if $v \in V(G)$ and similarly $e \in G$ if $e \in E(G)$.

2.1.1 Proof of Theorem 2.1.1

Let k be a positive integer. A graph G is called a *k-tree* if there is an ordering $v_1 \prec v_2 \prec \cdots \prec v_n$ of $V(G)$ such that (i) $G[\{v_1, v_2, \ldots, v_k\}]$ is a complete graph and (ii), for each $i > k$, $N(v_i) \cap \{v_1, v_2, \ldots, v_{i-1}\}$ induces a clique of order k. Clearly, 1-trees are the same as trees. Hwang, Richards, and Winter [31] proved that 2-trees are *maximal series-parallel graphs*. As shown in Lemma 2.2 and 2.3, we observe that *every 2-tree with more than three vertices contains a HIST*. However, not every connected and locally connected graph contains a 2-tree as a spanning subgraph. Let $W_n := K_1 + C_n$ be a wheel of order $n + 1$ and let G_n be obtained from W_n by adding n new vertices such that each is adjacent to a distinct pair of two consecutive vertices on the cycle C_n. It is not difficult to verify that G_n does not contain a spanning 2-tree. G_4 is depicted below.

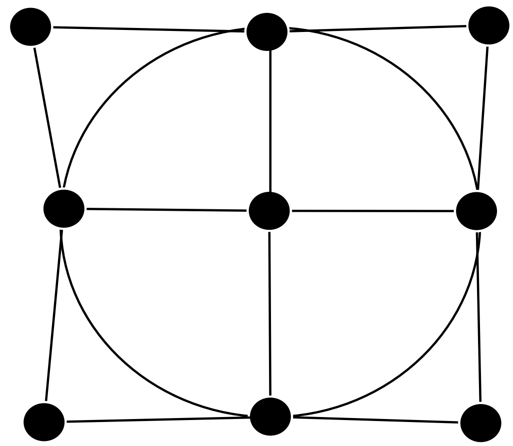


Figure (2.1) G_4, a locally connected graph without a spanning 2-tree

Let G be a graph and v be a vertex not in $V(G)$. We write $H = G \oplus v$ if there exist two distinct vertices $u_1, u_2 \in G$ such that $V(H) = V(G) \cup \{v\}$ and $E(H) = E(G) \cup \{u_1v, u_2v, u_1u_2\}$. Note that the edge u_1u_2 may already exist in G. We let $P(v) := \{u_1, u_2\}$ and call u_1 and u_2 the *parents* of v.

Definition 2.1.1. *A graph T of order $n \geq 3$ is called a weak 2-tree (W2-tree) if there is an ordering $\prec$: $v_1 \prec v_2 \prec \cdots \prec v_n$ of vertices of T and a sequence of graphs $G_3 \subset G_4 \subset \cdots \subset G_n = T$ such that the following properties hold.*

(1) $G_3 = T[\{v_1, v_2, v_3\}] \cong K_3$, and

(2) for each $i = 3, 4, \ldots, n-1$, $G_{i+1} \cong G_i \oplus v_{i+1}$.

In addition, we call the ordering $\prec$ a W2-tree ordering of T.

Clearly, every 2-tree is a W2-tree. However, the converse is not true, for example, the above graphs G_n are W2-trees but not 2-trees.

Given a W2-tree with a W2-tree ordering $\prec$, if we shift a degree 2 vertex to the end and keep the remaining ordering unchanged, we obtain another W2-tree ordering. So, the following result holds.

Lemma 2.1.1. *Let G be a W2-tree with $n \geq 4$ vertices. Let $w \in G$ be a degree 2 vertex and $N(w) = \{u, v\}$. Then either $G - w$ or $G - w - uv$ is a W2-tree.*

Lemma 2.1.2. *Let T be W2-tree with $n \geq 4$ vertices. Then, there exist two vertices u and v such that $T = (T' \oplus u) \oplus v$ and $N[u] \cap N(v) \neq \emptyset$, where T' is a W2-tree, K_3, or K_2. In this case, $\{u, v\}$ is called a* removable pair *of T.*

Proof. We prove Lemma 2.1.2 by applying induction on $n = |V(G)|$. Since K_4^- (K_4 minus an edge) is the unique W2-tree with 4 vertices, Lemma 2.1.2 holds for $n = 4$.

Suppose $n \geq 5$ and that Lemma 2.1.2 holds for all W2-trees with less than n vertices. Let T be a W2-tree with n vertices and w be the last vertex in a W2-ordering of T. Moreover, we assume that $T = T' \oplus w$, where T' is a W2-tree with $n-1$ vertices. Suppose that $\{u, v\}$ is a removable pair of T' and $T' = (T^* \oplus u) \oplus v$, where T^* is a W2-tree, K_3, or K_2. We complete the proof by considering the following five cases regarding $N(w) \cap \{u, v\}$.

- if $N(w) \cap \{u, v\} = \emptyset$, then $T = [(T^* \oplus w) \oplus u] \oplus v$, so $\{u, v\}$ is a removable pair of T;
- if $N(w) \cap \{u, v\} = \{v\}$, then $T = [(T^* \oplus u) \oplus v] \oplus w$, so $\{v, w\}$ is a removable pair of T;
- if $N(w) \cap \{u, v\} = \{u\}$ and $uv \notin E(T')$, then $T = [(T^* \oplus v) \oplus u] \oplus w$, so $\{u, w\}$ is a removable pair of T;
- if $N(w) \cap \{u, v\} = \{u\}$ and $uv \in E(T')$, then $T = [(T^* \oplus u) \oplus v] \oplus w$, so $\{v, w\}$ is a removable pair of T;
- if $N(w) = \{u, v\}$, then $T = [(T^* \oplus u) \oplus v] \oplus w$, so $\{v, w\}$ is a removable pair of T.

$\square$

Lemma 2.1.3. *A W2-tree with at least* 4 *vertices contains a HIST.*

Proof. Let G be a W2-tree with $n \geq 4$ vertices. We proceed by induction on n. If $n = 4$, then $G = K_4^-$, which contains a spanning star. If $n = 5$, by case analysis, we can show that G contains a spanning star, so a *HIST*.

Assume $n \geqslant 6$ and let G be a W2-tree with n vertices. By Lemma 2.1.2, let $\{u, v\}$ be a removable pair of G and assume $G = (G' \oplus u) \oplus v$, where G' is a W2-tree with $n-2$ vertices. By the induction hypothesis, G' contains a HIST, say, T'. Since $\{u, v\}$ is a removable pair of G, $N[u] \cap N(v) \neq \emptyset$. If $uv \notin E(G)$, then $N(u) \cap N(v) \neq \emptyset$; if $uv \in E(G)$, by the definition of $\oplus$, the other neighbor of v is adjacent to u. In either case, there exists a vertex $w \in N(u) \cap N(v)$. Then, $T := T' \cup \{wu, wv\}$ is a HIST of G. □

Lemma 2.1.4. *Every connected and locally connected graph with at least three vertices contains a spanning W2-tree.*

Proof. Let G be a connected and locally connected graph of order $n \geq 3$. Since every triangle is a W2-tree, G contains W2-trees as subgraphs. Let $T \subseteq G$ be a W2-tree such that $|V(T)|$ is maximum. We claim that $V(T) = V(G)$. Otherwise, $W := V(G) - V(T) \neq \emptyset$. Since G is connected, there is a vertex $v \in V(T)$ such that $N_W(v) \neq \emptyset$, where $N_W(v)$ is the set of neighbors of v in W. Since T is a W2-tree, $N(v) \cap V(T) \supseteq N_T(v) \neq \emptyset$. Since $G[N(v)]$ is connected, there is an edge $uw \in E(G)$ with $u \in N_T(v)$ and $w \in N_W(v)$. Then, $T \oplus w$ is a W2-tree containing more vertices than T, where $P(w) = \{u, v\}$. Since $uv, wv, uw \in E(G)$ and $T \subseteq G$, we have $T \oplus w \subseteq G$, which contradicts the maximality of $|V(T)|$. □

So, the proof of Theorem 2.1.1 is completed.

2.2 Proof of Conjecture 2.2

We now answer the second question raised by Albertson et al. positively as follows, whose proof will be given in the next section. We would like to mention that the main proof technique used in the proof is similar to that for Conjecture 2.1 in the first section. However, the induction proceeded on the spanning $\Theta-$patch graph H (we will give the definition very shortly) of G is not straightforward. In fact, when H has property Q_2(defined in subsection 2), we can not directly proceed the induction. The new approach in dealing with this case, looks easy and natural, yet really took efforts to come out.

Theorem 2.2.1. *Let G be a graph with every edge in at least two triangles. Then G contains a HIST.*

2.2.1 Proof of Theorem 2.2.1

The proof consists of three main components: (1) define a class of graphs called Θ-patch graphs(we will define this class of graphs very shortly), and show that every graph with each edge in at least two triangles contains a spanning Θ-patch graph; (2) prove a rearrangeability of Θ-patch graphs; and (3) show every Θ-patch graph contains a HIST. Throughout this section, a graph isomorphic to K_4^- (K_4 with exactly one edge removed) is called a Θ-*graph.*

Definition 2.2.1. *Given a graph H and a vertex $v \notin V(H)$, let $H\Delta v$ be a graph with $V(H\Delta v) = V(H) \cup \{v\}$ and $E(H\Delta v) = E(H) \cup \{u_1v, u_2v, u_1u_2\}$, where $u_1,\ u_2 \in V(H)$ are two distinct vertices. That is, $H\Delta v$ is obtained from H by adding a new vertex v and edges u_1v, u_2v, and u_1u_2 if $u_1u_2 \notin E(H)$. We name such an operation* Δ-operation *and denote by $A(v) := \{u_1, u_2\}$, the set of attachments of v on H. Moreover, we let $A[v] := A(v) \cup \{v\}$.*

Note that u_1u_2 may or may not be an edge of H.

Definition 2.2.2. *Given a graph H and a Θ-graph F with a specified degree 3 vertex, let $H\Theta F$ be the graph obtained by identifying the specified vertex of F with a vertex u in H. Let $A(F) = \{u\}$ be the set of the attachment of F on H. Such an operation is called a Θ-operation.*

We use $\oplus$ to denote either a Δ-operation or a $\Theta-$operation.

Definition 2.2.3. *A graph G is called a Θ-patch graph if there exists a subgraph sequence $G_1 \subset G_2 \subset \cdots \subset G_s = G$ with $s \geqslant 2$ such that*

(1) $G_1 \cong K_3$, and

(2) G_{i+1} is obtained from G_i by a $\oplus$-operation for each i ($1 \leqslant i \leqslant s-1$).

By the above definition, a Θ-patch graph has at least 4 vertices, and a Θ-patch graph with exactly 4 vertices is a Θ-graph.

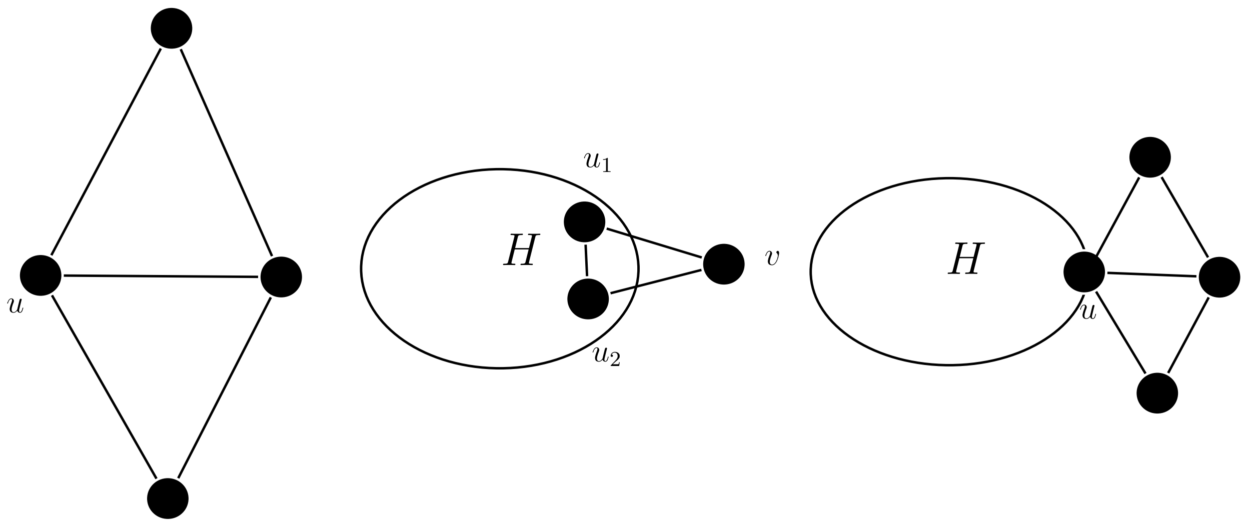


Figure (2.2) Θ-graph, Δ-operation, Θ-operation

Lemma 2.2.1. *A connected graph with every edge in at least two triangles contains a* Θ*-patch graph as a spanning subgraph.*

Proof. Let G be a graph such that every edge is in at least two triangles. Since two triangles sharing a common edge induce a Θ-graph, G contains a Θ-graph, which is also a Θ-patch graph by Definition 2.2.3. Let $H \subset G$ be a Θ-patch graph such that $|V(H)|$ is maximum. If $V(H) = V(G)$, the proof is completed. So assume the contrary: $W = V(G) - V(H) \neq \emptyset$. Since G is connected, there is an edge $uw \in E(G)$ such that $u \in V(H)$ and $w \in W$. Let v_1uwv_1 and v_2uwv_2 be two distinct triangles containing uw. If $v_i \in V(H)$ for some $i = 1, 2$, then $H\Delta w$ with $A(w) = \{u, v_i\}$ is a Θ-patch graph larger than H, contradicting the maximality of H. Hence, we have both $v_1, v_2 \in W$. Clearly, $G[\{u, v_1, v_2, w\}]$, the subgraph induced on $\{u, v_1, v_2, w\}$, contains a Θ-graph F. So $H\Theta F$ with $A(F) = \{u\}$ is a Θ-patch graph larger than H, contradicting the maximality of H. $\square$

It will be shown in the following lemma that the ordering of subgraph sequence in the definition of Θ-patch graphs can be rearranged to preserve a nice recursive property.

Lemma 2.2.2. *Let* G *be a* Θ*-patch graph of order* $n \geqslant 5$*. Then there exist a subgraph* H *which is either a* Θ*-patch graph or isomorphic to* K_3 *such that one of the following properties* *holds:*

$P:\ G = (H\Delta x_1)\Delta x_2$ *and* $A(x_2) \cap A[x_1] \neq \emptyset$*;*

$Q_k\,(0 \leq k \leq 3):$ *There exist vertices* $x_1, x_2, \cdots, x_k$ *such that* $G = (H\Theta F)\Delta x_1 \Delta x_2 \cdots \Delta x_k$ *(*$G = H\Theta F$ *when* $k = 0$*) with* $A[x_i] \cap A[x_j] = \emptyset$ *for all* $i \neq j$ *and* $A(x_i) \cap (V(F) - V(H)) \neq \emptyset$ *for* $i = 1, 2, \cdots, k$*.*

Proof. If $n = 5$, from the definition of Θ-patch graphs, there exist two vertices x_1 and x_2 such that $G = K_3 \Delta x_1 \Delta x_2$ and $A(x_2) \cap A[x_1] \neq \emptyset$, so P holds. We assume that $n \geqslant 6$ and Lemma 2.2.2 holds for graphs with order $< n$.

By the definition of Θ-patch graphs, $G = H^* \oplus F^*$, where H^* is a Θ-patch graph, and F^* is either a single vertex or a Θ-graph. If F^* is a Θ-graph, then Q_0 holds. So, we assume F^* is a single vertex graph, and say $V(F^*) = \{w\}$. By applying Lemma 2.2.2 to H^*, we divide the remaining proof into two cases below.

Case P**.** $H^* = (H\Delta x_1)\Delta x_2$ and $A(x_2) \cap A[x_1] \neq \emptyset$.

If $A(w) \cap \{x_1, x_2\} = \emptyset$, let $H' := H\Delta w$, which is a Θ-patch graph and a subgraph of G. Then $G = (H'\Delta x_1)\Delta x_2$, so P holds.

Suppose $A(w) \cap \{x_1, x_2\} \neq \emptyset$. If $x_1 \in A(x_2)$ or $x_2 \in A(w)$, $H' := H\Delta x_1 \subset G$ is a Θ-patch graph. Then, we have $G = (H'\Delta x_2)\Delta w$ and either $x_1 \in A(w) \cap A[x_2]$ or $x_2 \in A(w)$, so P holds. We may assume that $x_1 \notin A(x_2)$ and $x_2 \notin A(w)$. In this case, we have $x_1 \in A(w)$. Let $H' = H\Delta x_2$, which is a Θ-patch graph and a subgraph of G. Then $G = H'\Delta x_1 \Delta w$, so P holds.

Case Q_k**.** $H^* = (H\Theta F)\Delta x_1 \Delta x_2 \cdots \Delta x_k$, where F is a Θ-graph and x_i is a vertex in H^*.

If $A(w) \cap ((V(F) - V(H)) \cup \{x_1, x_2, \cdots, x_k\}) = \emptyset$, then

$$G = ((H\Delta w)\Theta F)\,\Delta x_1 \Delta x_2 \cdots \Delta x_k,$$

so Q_k holds. If $A(w) \cap A[x_i] \neq \emptyset$, w.l.o.g., say $A(w) \cap A[x_k] \neq \emptyset$, then

$$G = (H\Theta F \Delta x_1 \Delta x_2 \cdots \Delta x_{k-1})\,\Delta x_k \Delta w,$$

so P holds. Hence, we assume $A(w) \cap (V(F) - V(H)) \neq \emptyset$ and $A(w) \cap A[x_i] = \emptyset$ for $i = 1, 2, \cdots, k$. Under this assumption together with the assumption that $A[x_i] \cap A[x_j] = \emptyset$ for $i \neq j$ and $A(x_i) \cap (V(F) - V(H)) \neq \emptyset$ for $i = 1, 2, \cdots, k$, we have $k \leq 2$. Then, we have

$$G = (H\Theta F)\Delta x_1 \Delta x_2 \cdots \Delta x_k \Delta w,$$

so Q_{k+1} holds. □

Lemma 2.2.3. *Every Θ-patch graph contains a HIST.*

Proof. We use induction on $n = |V(G)|$. When $n = 4$, $G \cong K_4^-$ is a Θ-graph. Clearly, G contains a HIST. Suppose $n \geqslant 5$, and assume that Lemma 2.2.3 holds for graphs of order $< n$. We divide the remaining proof into five cases according to the five properties given in Lemma 2.2.2.

If G has property Q_i for some $i = 0, 1, 2$ or 3, we follow the notations given in Lemma 2.2.2, and assume that $A(F) = \{u\}$ and $V(F) - V(H) = \{v_1, v_2, v_3\}$. If G has property P then u is a specially selected vertex in H. We let T be a HIST of H if H is a Θ-patch graph, and let $T \cong P_3$ with $d_T(u) = 2$ if $H \cong K_3$. The case that G satisfies property Q_2 is the most complicated one, and we can not straightforwardly play induction on it, so we defer this case to the end.

Property P holds. Suppose that $G = H\Delta x_1 \Delta x_2$ and $A(x_2) \cap A[x_1] \neq \emptyset$.

In this case, we first show that $N(x_1) \cap N(x_2) \cap V(H) \neq \emptyset$. This is clearly true if $A(x_1) \cap A(x_2) \neq \emptyset$, so we may assume $x_1 \in A(x_2)$. Let u be the other vertex in $A(x_2)$. Since $E(G) = E((H\Delta x_1)\Delta x_2) = E(H\Delta x_1) \cup \{x_2u, x_2x_1, ux_1\}$, we have $ux_1 \in E(G)$, that is, $u \in N(x_1) \cap N(x_2)$.

Let $u \in N(x_1) \cap N(x_2)$. Then, it is readily seen that $T \cup \{ux_1, ux_2\}$ is a HIST of G.

Property Q_0 holds. Let $G = H\Theta F$.

In this case, $T \cup \{uv_1, uv_2, uv_3\}$ is a HIST of G.

Property Q_1 holds. Let $G = (H\Theta F)\Delta x_1$, and assume, without loss of generality, $v_1 \in A(x_1) \cap (V(F) - V(H))$, and let w_1 be another vertex of $A(x_1)$.

In this case, $T \cup \{w_1v_1, w_1x_1, uv_2, uv_3\}$ is a HIST of G regardless of whether $w_1 \in V(F)$ or not.

Property Q_3 holds. Let $G = (H\Theta F)\Delta x_1 \Delta x_2 \Delta x_3$ and assume that $A(x_i) = \{v_i, w_i\}$ for each $i = 1, 2, 3$ with $w_1, w_2, w_3 \in V(H)$.

By the definition of Δ-operation, all three edges w_1v_1, w_2v_2, w_3v_3 are in $E(G)$. Then, $T \cup \{w_1x_1, w_1v_1, w_2x_2, w_2v_2, w_3x_3, w_3v_3\}$ is a HIST in G.

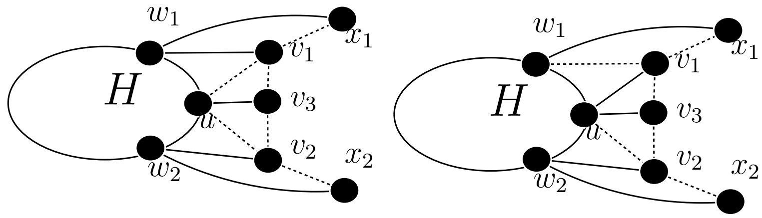
Property Q_2 holds. Let $G = (H\Theta F)\Delta x_1 \Delta x_2$ such that $A(x_i) \cap (V(F) - V(H)) \neq \emptyset$ for each $i = 1, 2$, and $A[x_2] \cap A[x_1] = \emptyset$. Assume that $A(x_i) = \{v_i, w_i\}$ for $i = 1, 2$.

We may assume $w_i \neq u$ for each $i = 1, 2$; otherwise, say $w_1 = u$, then $T \cup \{w_2v_2, w_2x_2, uv_1, ux_1, uv_3\}$ is a HIST of G. Since $A[x_2] \cap A[x_1] = \emptyset$, we may assume that $w_1 \in V(H) - \{u\}$. Moreover, under the assumption that $w_1 \in V(H) - \{u\}$, let notation be chosen so that v_1 is the degree 2 vertex in $F - u$ whenever it is possible, that is, if $w_2 \in V(H) - \{u\}$ and v_2 is the degree two vertex in $F - u$, we rename x_2, v_2 and w_2 as x_1, v_1 and w_1, and vice versa.

Let $z \notin V(G)$ be a vertex and $G' := H\Delta z$ with $A(z) = \{u, w_1\}$. Clearly, $uw_1 \in E(G')$ although uw_1 may not be in $E(G)$. Clearly, G' is a Θ-patch graph and $|V(G'| < n$, so it contains a HIST T'. Since $d_{G'}(z) = 2$, z is a degree 1 vertex of T'. So, we have either $w_1z \in E(T')$ or $uz \in E(T')$ but not both. Let $T_H := T' - z$.

Subcase 1. $uw_1 \notin E(T')$ or $uw_1 \in E(T') \cap E(G)$.

Note that $d_{T'}(z) = 1$. If $uz \in E(T')$, let $T^* := T_H \cup \{uv_3, w_1v_1, w_1x_1, w_2v_2, w_2x_2\}$, as depicted in Figure 2.3. It is routine to check that T^* is a spanning tree of G and the following



$uz \in E(T')$ $\qquad$ $w_1 z \in E(T')$

Figure (2.3) $uw_1 \notin E(T')$ or $uw_1 \in E(T') \cap E(G)$

equalities/inequalities hold.

$$\begin{aligned}
d_{T^*}(u) &= d_{T'}(u) - |\{uz\}| + |\{uv_3\}| = d_{T'}(u) \neq 2 \\
d_{T^*}(w_1) &= d_{T'}(w_1) + |\{w_1v_1, w_1x_2\}| = d_{T'}(w_1) + 2 \neq 2 \\
d_{T^*}(w_2) &= \begin{cases} d_{T'}(w_2) + |\{w_2v_2, w_2x_2\}| = d_{T'}(w_2) + 2 \neq 2, & \text{if } w_2 \in V(H); \\ |\{w_2v_2, w_2x_2, uv_3\}| = 3, & \text{if } w_2 = v_3. \end{cases} \\
d_{T^*}(x) &= d_{T'}(x) \neq 2 \quad \text{ for all other vertices } x \in V(H) \text{, and} \\
d_{T^*}(x) &\neq 2 \quad \text{ for each vertex } x \in \{v_1, v_2, v_3, x_1, x_2\}.
\end{aligned}$$

Consequently, T^* is a HIST of G.

If $w_1z \in E(T')$, let $T^* := T_H \cup \{w_1x_1, uv_1, uv_3, w_2v_2, w_2x_2\}$, as depicted in Figure 2.3. ($w_2 = v_3$ may occur.) As in the previous case, we can show that T^* is a HIST of G.

Subcase 2. $uw_1 \in E(T') - E(G)$.

In this case, $T_1 := T_H - uw_1$ has exactly two components. We construct a HIST of G from T_1 according to whether $uz \in E(T')$ or $w_1z \in E(T')$.

If $uz \in E(T')$, let $T^* = T_1 \cup \{uv_3, uv_1, v_1w_1, v_1x_1, w_2v_2, w_2x_2\}$, as depicted in Figure 2.4. It is routine to check that T^* is a spanning tree of G and the following equalities/inequalities hold.

$$
\begin{aligned}
d_{T^*}(u) &= d_{T'}(u) - |\{uw_1, uz\}| + |\{uv_1, uv_3\}| = d_{T'}(u) \neq 2 \\
d_{T^*}(w_1) &= d_{T'}(w_1) - |\{uw_1\}| + |\{v_1w_1\}| = d_{T'}(w_1) \neq 2 \\
d_{T^*}(w_2) &= \begin{cases} d_{T'}(w_2) + |\{w_2v_2, w_2x_2\}| = d_{T'}(w_2) + 2 \neq 2, & \text{if } w_2 \in V(H); \\ |\{w_2v_2, w_2x_2, uv_3\}| = 3, & \text{if } w_2 = v_3. \end{cases} \\
d_{T^*}(x) &= d_{T'}(x) \neq 2 \quad \text{for all other vertices } x \in V(H), \text{ and} \\
d_{T^*}(x) &\neq 2 \quad \text{for each vertex } x \in \{v_1, v_2, v_3, x_1, x_2\}.
\end{aligned}
$$

So, T^* is a HIST of G.

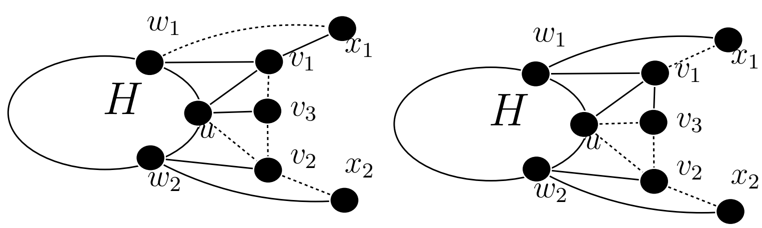
In the case $w_1z \in E(T')$, if $\underline{v_1v_3 \in E(G)}$, let

$$T^* = T_1 \cup \{w_1x_1, w_1v_1, v_1u, v_1v_3, w_2v_2, w_2x_2\},$$

as depicted in Figure 2.4. As in the previous case, we can show that T^* is a HIST of G. To complete the proof, we show that the vertex v_1 can be chosen such that $\underline{v_1v_3 \in E(G)}$. If $v_1v_3 \notin E(G)$, then both v_1 and v_3 are degree 1 vertices in $F - u \cong P_3$. So, v_2 is the degree 2 vertex in $F - u$. If $w_2 \in V(H)$, we would pick x_2 as x_1 and v_2 as our v_1 in the very beginning. So, $w_2 = v_3$. In this case, we can simply swap v_2 and v_3(also w_2) to ensure that $v_1v_3 \in E(G)$.

$\square$

Clearly, the combination of the above three Lemmas gives Theorem 2.2.1.



$uz \in E(T')$ $\qquad$ $w_1 z \in E(T')$

Figure (2.4) $uw_1 \in E(T') - E(G)$

PART 3

MINIMUM DEGREE CONDITION FOR SPANNING HALIN GRAPHS AND SPANNING GENERALIZED HALIN GRAPHS

3.1 Notations and definitions

We consider simple and finite graphs only. Let G be a graph. Denote by $e(G)$ the cardinality of $E(G)$. Let $v \in V(G)$ be a vertex and $S \subseteq V(G)$ a subset. The notation $\Gamma_G(v, S)$ denotes the set of neighbors of v in S, and $deg_G(v, S) = |\Gamma_G(v, S)|$. We let $\Gamma_{\overline{G}}(v, S) = S - \Gamma_G(v, S)$ and $deg_{\overline{G}}(v, S) = |\Gamma_{\overline{G}}(v, S)|$. Given another set $U \subseteq V(G)$, define $\Gamma_G(U, S) = \cap_{u \in U} \Gamma_G(u, S)$, $deg_G(U, S) = |\Gamma_G(U, S)|$, and $N_G(U, S) = \cup_{u \in U} \Gamma_G(u, S)$. When $U = \{u_1, u_2, \cdots, u_k\}$, we may write $\Gamma_G(U, S)$, $deg_G(U, S)$, and $N_G(U, S)$ as $\Gamma_G(u_1, u_2, \cdots, u_k, S)$, $deg_G(u_1, u_2, \cdots, u_k, S)$, and $N_G(u_1, u_2, \cdots, u_k, S)$, respectively, in specifying the vertices in U. When $S = V(G)$, we only write $\Gamma_G(U)$, $deg_G(U)$, and $N_G(U)$. Let $U_1, U_2 \subseteq V(G)$ be two disjoint subsets. Then $\delta_G(U_1, U_2) = \min\{deg_G(u_1, U_2) \,|\, u_1 \in U_1\}$ and $\Delta_G(U_1, U_2) = \max\{deg_G(u_1, U_2) \,|\, u_1 \in U_1\}$. Notice that the notations $\delta_G(U_1, U_2)$ and $\Delta_G(U_1, U_2)$ are not symmetric with respect to U_1 and U_2. We denote by $E_G(U_1, U_2)$ the set of edges with one end in U_1 and the other in U_2, the cardinality of $E_G(U_1, U_2)$ is denoted as $e_G(U_1, U_2)$. We may omit the index G if there is no risk of confusion. Let $u, v \in V(G)$ be two vertices. We write $u \sim v$ if u and v are adjacent. A path connecting u and v is called a (u, v)-path. If G is a bipartite graph with partite sets A and B, we denote G by $G(A, B)$ in emphasizing the two partite sets. A *matching* in G is a set of independent edges; a $\wedge$-*matching* is a set of vertex-disjoint copies of $K_{1,2}$; and a *claw-matching* is a set of vertex-disjoint copies of $K_{1,3}$. The set of degree 2 vertices in a $\wedge$-matching is called the center of the $\wedge$-*matching* ; and the set of degree 3 vertices in a claw-matching is called the center of the claw-matching. A cycle C in a graph G is *dominating* if $G - V(C)$ is an edgeless graph.

3.2 The Regularity Lemma and the Blow-up Lemma

The Regularity Lemma of Szemerédi [50] and Blow-up lemma of Komlós et al. [36] are main tools used in finding a spanning Halin subgraph or spanning generalized Halin subgraph. For any two disjoint non-empty vertex-sets A and B of a graph G, the *density* of A and B is the ratio $d(A, B) := \frac{e(A,B)}{|A||B|}$. Let ε and δ be two positive real numbers. The pair (A, B) is called ε-regular if for every $X \subseteq A$ and $Y \subseteq B$ with $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$, $|d(X, Y) - d(A, B)| < \varepsilon$ holds. In addition, if $deg(a, B) > \delta|B|$ for each $a \in A$ and $deg(b, A) > \delta|A|$ for each $b \in B$, we say (A, B) an (ε, δ)-super regular pair.

Lemma 3.2.1 (**Regularity lemma-Degree form [50]**). *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if G is any graph with n vertices and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set $V(G)$ into $l + 1$ clusters $V_0, V_1, \cdots, V_l$, and there is a spanning subgraph $G' \subseteq G$ with the following properties.*

- *$l \leq M$;*
- *$|V_0| \leq \varepsilon n$, all clusters $|V_i| = |V_j| \leq \lceil \varepsilon n \rceil$ for all $1 \leq i \neq j \leq l$;*
- *$deg_{G'}(v) > deg_G(v) - (d + \varepsilon)n$ for all $v \in V(G)$;*
- *$e(G'[V_i]) = 0$ for all $i \geq 1$;*
- *all pairs (V_i, V_j) $(1 \leq i < j \leq l)$ are ε-regular, each with a density either 0 or greater than d.*

Lemma 3.2.2 (**Blow-up lemma-weak version [36]**). *Given a graph R of order r and positive parameters δ, Δ, there exists a positive $\varepsilon = \varepsilon(\delta, \Delta, r)$ such that the following holds. Let $n_1, n_2, \cdots, n_r$ be arbitrary positive integers and let us replace the vertices $v_1, v_2, \cdots, v_r$ with pairwise disjoint sets $V_1, V_2, \cdots, V_r$ of sizes $n_1, n_2, \cdots, n_r$ (blowing up). We construct two graphs on the same vertex set $V = \bigcup V_i$. The first graph K is obtained by replacing each edge v_iv_j of R with the complete bipartite graph between the corresponding vertex sets V_i and V_j. A sparser graph G is constructed by replacing each edge v_iv_j arbitrarily with an*

(ε, δ)-super regular pair between V_i and V_i. If a graph H with $\Delta(H) \le \Delta$ is embeddable into K then it is already embeddable into G.

Lemma 3.2.3 (**Blow-up lemma-strengthened version [36]**). *Given $c > 0$, there are positive numbers $\varepsilon = \varepsilon(\delta, \Delta, r, c)$ and $\gamma = \gamma(\delta, \Delta, r, c)$ such that the Blow-up lemma in the equal size case (all $|V_i|$ are the same) remains true if for every i there are certain vertices x to be embedded into V_i whose images are a priori restricted to certain sets $C_x \subseteq V_i$ provided that*

(i) each C_x within a V_i is of size at least $c|V_i|$;

(ii) the number of such restrictions within a V_i is not more than $\gamma|V_i|$.

Besides the above two lemmas, we also need the two lemmas below regarding regular pairs.

Lemma 3.2.4. *If (A, B) is an ε-regular pair with density d, then for any $A' \subseteq A$ with $|A'| > \varepsilon|A|$, there are at most $\varepsilon|B|$ vertices $b \in B$ such that $deg(b, A') \le (d - \varepsilon)|A'|$.*

Lemma 3.2.5 (**Slicing lemma**). *Let (A, B) be an ε-regular pair with density d, and for some $\nu > \varepsilon$, let $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \ge \nu|A|$, $|B'| \ge \nu|B|$. Then (A', B') is an ε'-regular pair of density d', where $\varepsilon' = \max\{\varepsilon/\nu, 2\varepsilon\}$ and $d' > d - \varepsilon$.*

The following two results on hamiltonicity are used in finding hamiltonian cycles in the proofs.

Lemma 3.2.6 ([45]). *If G is a graph of order n satisfying $d(x) + d(y) \ge n + 1$ for every pair of nonadjacent vertices $x, y \in V(G)$, then G is hamiltonian-connected.*

Lemma 3.2.7 ([42]). *Let G be a balanced bipartite graph with $2n$ vertices. If $d(x) + d(v) \ge n + 1$ for any two non-adjacent vertices $x, y \in V(G)$, then G is hamiltonian.*

3.3 Dirac's sondition for spanning Halin graphs

3.3.1 Introduction

A classic theorem of Dirac [19] from 1952 asserts that every graph on n vertices with minimum degree at least $n/2$ is hamiltonian if $n \ge 3$. Following Dirac's result, numerous results on hamiltonicity properties on graphs with restricted degree conditions have been obtained (see, for instance, [26] and [25]). Traditionally, under similar conditions, results for a graph being hamiltonian, hamiltonian-connected, and pancyclic are obtained separately. We may ask, under certain conditions, if it is possible to uniformly show a graph possessing several hamiltonicity properties. The work on finding the square of a hamiltonian cycle in a graph can be seen as an attempt in this direction. However, it requires quite strong degree conditions for a graph to contain the square of a hamiltonian cycle, for examples, see [21], [22], [37], [9], and [49]. For bipartite graphs, finding the existence of a spanning ladder is a way of simultaneously showing the graph having many hamiltonicity properties (see [16] and [17]). In this paper, we introduce another approach of uniformly showing the possession of several hamiltonicity properties in a graph: we show the existence of a spanning *Halin graph* in a graph under given minimum degree condition.

A tree with no vertex of degree 2 is called a *homeomorphically irreducible tree* (HIT). A *Halin graph* H is obtained from a HIT T of at least 4 vertices embedded in the plane by connecting its leaves into a cycle C following the cyclic order determined by the embedding. According to the construction, the Halin graph H is denoted as $H = T \cup C$, and the HIT T is called the underlying tree of H. A wheel graph is an example of a Halin graph, where the underlying tree is a star. Halin constructed Halin graphs in [27] for the study of minimally 3-connected graphs. Lovász and Plummer named such graphs as Halin graphs in their study of planar bicritical graphs [40], which are planar graphs having a 1-factor after deleting any two vertices. Intensive researches have been done on Halin graphs. Bondy [5] in 1975 showed that a Halin graph is hamiltonian. In the same year, Lovász and Plummer [40] showed that not only a Halin graph itself is hamiltonian, but each of the subgraph obtained by deleting a vertex is hamiltonian. In 1987, Barefoot [2] proved that Halin graphs are hamiltonian-connected, i.e., there is a hamiltonian path connecting any two vertices of the graph. Furthermore, it was proved that each edge of a Halin graph is contained in a hamiltonian cycle and is avoided by another [48]. Bondy and Lovász [6], and Skowrońska [47], independently, in 1985, showed that a Halin graph is almost pancyclic and is pancyclic if the underlying tree has no vertex of degree 3, where an n-vertex graph is *almost pancyclic* if it contains cycles of length from 3 to n with the possible exception of a single even length, and is *pancyclic* if it contains cycles of length from 3 to n. Some problems that are NP-complete for general graphs have been shown to be polynomial time solvable for Halin graphs. For example, Cornuéjols, Naddef, and Pulleyblank [15] showed that in a Halin graph, a hamiltonian cycle can be found in polynomial time. It seems so promising to show the existence of a spanning Halin subgraph in a given graph in order to show the graph having many hamiltonicity properties. But, nothing comes for free, it is NP-complete to determine whether a graph contains a (spanning) Halin graph [30].

Despite all these nice properties of Halin graphs mentioned above, the problem of determining whether a graph contains a spanning Halin subgraph has not yet well studied except a conjecture proposed by Lovász and Plummer [40] in 1975. The conjecture states that *every 4-connected plane triangulation contains a spanning Halin subgraph* (disproved recently [10]). In this paper, we investigate the minimum degree condition for implying the existence of a spanning Halin subgraph in a graph, and thereby giving another approach for uniformly showing the possession of several hamiltonicity properties in a graph under a given minimum degree condition. We obtain the following result.

Theorem 3.3.1. *There exists $n_0 > 0$ such that for any graph G with $n \geq n_0$ vertices, if $\delta(G) \geq (n+1)/2$, then G contains a spanning Halin subgraph.*

Note that an n-vertex graph with minimum degree at least $(n+1)/2$ is 3-connected if $n \geq 4$. Hence, the minimum degree condition in Theorem 3.3.1 implies the 3-connectedness, which is a necessary condition for a graph to contain a spanning Halin subgraph, since every Halin graph is 3-connected. A Halin graph contains a triangle, and bipartite graphs are triangle-free. Hence, $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ contains no spanning Halin subgraph. Immediately, we see that the minimum degree condition in Theorem 3.3.1 is best possible.

3.3.2 Ladders and "ladder-like" Halin graphs

In constructing Halin graphs, we use ladder graphs and a class of "ladder-like" graphs as substructures. We give the description of these graphs below.

Definition 3.3.1. *An n-ladder $L_n = L_n(A, B)$ is a balanced bipartite graph with $A = \{a_1, a_2, \cdots, a_n\}$ and $B = \{b_1, b_2, \cdots, b_n\}$ such that $a_i \sim b_j$ iff $|i - j| \leq 1$. We call a_ib_i the i-th rung of L_n. If $2n(mod\ 4) \equiv 0$, we call each of the shortest (a_1, b_n)-path and (b_1, a_n)-path a side of L_n; otherwise we call each of the shortest (a_1, a_n)-path and (b_1, b_n)-path a side of L_n.*

Let L be a ladder with xy as one of its rungs. For an edge gh, we say xy and gh are *adjacent* if $x \sim g, y \sim h$ or $x \sim h, y \sim g$. Suppose L has its first rung as ab and its last rung as cd. We denote L by $ab - L - cd$ in specifying the two rungs, and we always assume that the distance between a and c is $|V(L)|/2$ (we make this assumption for being convenient in constructing other graphs based on ladders). Under this assumption, we denote L as $\overrightarrow{ab} - L - \overrightarrow{cd}$. Let A and B be two disjoint vertex sets. We say the rung xy of L is *contained* in $A \times B$ if either $x \in A, y \in B$ or $x \in B, y \in A$. Let L' be another ladder vertex-disjoint with L. If the last rung of L is adjacent to the first rung of L', we write LL' for the new ladder obtained by concatenating L and L'. In particular, if $L' = gh$ is an edge, we write LL' as Lgh.

We now define five types of "ladder-like" graphs, call them H_1, H_2, H_3, H_4 and H_5, respectively. Let L_n be a ladder with a_1b_1 and a_nb_n as the first and last rung, respectively, and x, y, z, w, u five new vertices. Then each of H_i is obtained from L_n by adding some specified vertices and edges as follows. Additionally, for each i with $1 \leq i \leq 5$, we define a graph T_i associated with H_i.

H_1: Adding two new vertices x, y and the edges xa_1, xb_1, ya_n, yb_n and xy.

Let $T_1 = H_1[\{x, y, a_1, b_1, a_n, b_n\}]$.

H_2: Adding three new vertices x, y, z and the edges $za_1, zb_1, xz, xb_1, ya_n, yb_n$ and xy.

Let $T_2 = H_2[\{x, y, z, a_1, b_1, a_n, b_n\}]$.

H_3: Adding three new vertices x, y, z and the edges xa_1, xb_1, ya_n, yb_n, either za_i or zb_i for some $2 \leq i \leq n-1$ and xz, yz.

Let $T_3 = H_3[\{x, y, z, a_1, b_1, a_n, b_n\}]$.

H_4: Adding four new vertices x, y, z, w and the edges $wa_1, wb_1, xw, xb_1, ya_n, yb_n$, either za_i or zb_i for some $2 \leq i \leq n-1$ and xz, yz.

Let $T_4 = H_4[\{x, y, z, w, a_1, b_1, a_n, b_n\}]$.

H_5: Adding five new vertices x, y, z, w, u.

If $2(n-1)(mod\ 4) \equiv 2$, adding the edges $wa_1, wb_1, xw, xb_1, ua_n, ub_n, yu, yb_n$, either za_i or zb_i for some $2 \leq i \leq n-1$ and xz, yz ;

and if $2(n-1)(mod\ 4) \equiv 0$, adding the edges $wa_1, wb_1, xw, xb_1, ua_n, ub_n, yu, ya_n$, either za_i or zb_i for some $2 \leq i \leq n-1$ and xz, yz.

Let $T_5 = H_5[\{x, y, z, w, u, a_1, b_1, a_n, b_n\}]$.

Let $i = 1, 2, \cdots, 5$. Notice that each of H_i is a Halin graph and except H_1, each H_i has a unique underlying tree. Notice also that xy is an edge on the cycle along the leaves of any underlying tree of H_i. For each H_i, call x the *left end* and y the *right end*, and call a vertex of degree at least 3 in the underlying tree of H_i a *Halin constructible vertex*. By analyzing the structure of H_i, we see that each of the vertices on one side of the ladder $H_i - \{x, y, z, w, u\}$ is a Halin constructible vertex. Noting that any vertex in $V(H_1) - \{x, y\}$ can be a Halin constructible vertex. In Figure 3.1, we depict a ladder L_4, H_1, H_2, H_3, H_4, H_5 constructed from L_4, and the graph T_i associated with H_i. We call a_1b_1 the head link of T_i and a_nb_n the tail link of T_i, and for each of T_3, T_4, T_5, we call the vertex z not contained in any triangles the *pendent vertex* . The notations of H_i and T_i are fixed hereafter.

Let $T \in \{T_1, \cdots, T_5\}$ be a subgraph of a graph G. Suppose that T has head link ab, tail link cd, and possibly the pendent vertex z. It is clear that if $G - V(T)$ contains a spanning ladder L with first rung c_1d_1 and last rung c_nd_n such that c_1d_1 is adjacent to ab, c_nd_n is adjacent to cd, and z is adjacent some vertex z' on some internal rung of L if z exists, then $abLcd \cup T$ or $abLcd \cup T \cup \{zz'\}$ when z exists is a spanning Halin subgraph of G. This technique is frequently used later on in constructing a Halin graph. The following proposition gives another way of constructing a Halin graph based on H_1 and H_2.

Proposition 3.3.1. *For $i = 1, 2$, let $G_i \in \{H_1, H_2\}$ with left end x_i and right end y_i be defined as above, and let $u_i \in V(G_i)$ be a Halin constructible vertex, then $G_1 \cup G_2 - \{x_1y_1, x_2y_2\} \cup \{x_1x_2, y_1y_2, u_1u_2\}$ is a Halin graph spanning on $V(G_1) \cup V(G_2)$.*

Proof. For $i = 1, 2$, let G_i be embedded in the plane, and let T_{G_i} be a underlying plane tree of G_i. Then $T' := T_{G_1} \cup T_{G_2} \cup \{u_1u_2\}$ is a homeomorphically irreducible tree spanning on $V(G_1) \cup V(G_2)$. Moreover, we can draw the edge u_1u_2 such that $T_{G_1} \cup T_{G_2} \cup \{u_1u_2\}$ is a plane graph. Since $G_i[E(G_i - T_{G_i}) - \{x_iy_i\}]$ is an (x_i, y_i)-path spanning on the leaves of T_{G_i} obtained by connecting the leaves following the order determined by the embedding, we see $G_1[E(G_1 - T_{G_1}) - \{x_1y_1\}] \cup G_2[E(G_2 - T_{G_2}) - \{x_2y_2\}] \cup \{x_1x_2, y_1y_2\}$ is a cycle spanning on the leaves of T' obtained by connecting the leaves following the order determined by the embedding of T'. Thus $G_1 \cup G_2 - \{x_1y_1, x_2y_2\} \cup \{x_1x_2, y_1y_2, u_1u_2\}$ is a Halin graph. □

3.3.3 Proof of Theorem 3.3.1

In this section, we prove Theorem 3.3.1. Following the standard setup of proofs applying the Regularity Lemma, we divide the proof into non-extremal case and extremal cases. For this purpose, we define the two extremal cases in the following.

Let G be an n-vertex graph and V its vertex set. Given $0 \le \beta \le 1$, the two extremal cases are defined as below.

Extremal Case 1. G has a vertex-cut of size at most $5\beta n$.

Extremal Case 2. There exists a partition $V_1 \cup V_2$ of V such that $|V_1| \ge (1/2 - 7\beta)n$ and $\Delta(G[V_1]) \le \beta n$.

Non-extremal case. We say that an n-vertex graph with minimum degree at least $(n+1)/2$ is in *non-extremal case* if it is in neither of Extremal Case 1 and Extremal Case 2.

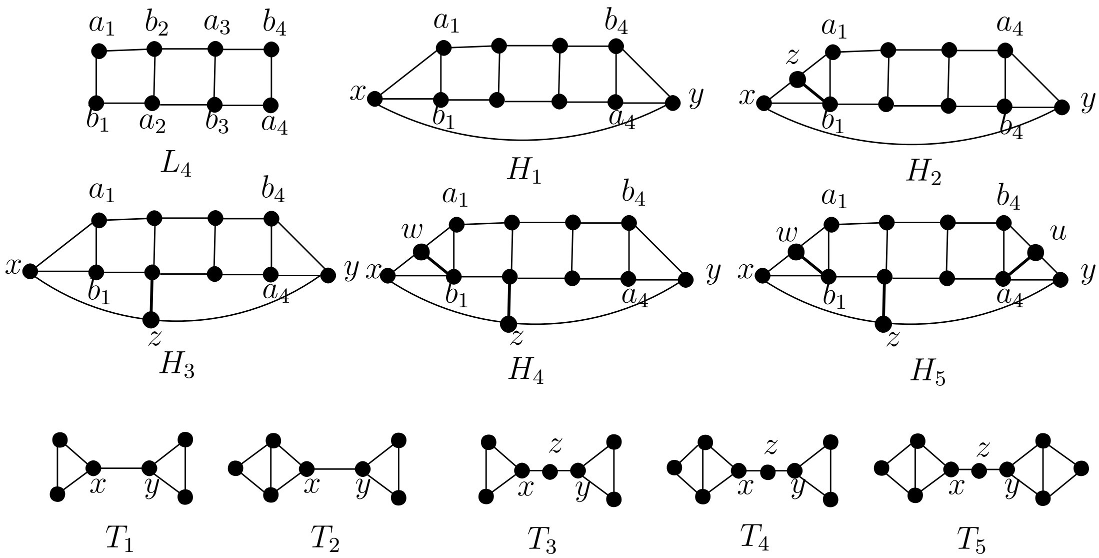


Figure (3.1) L_4, H_i constructed from L_4, and T_i associated with H_i for each $i = 1, 2, \cdots, 5$

The following three theorems deal with the non-extremal case and the two extremal cases, respectively, and thus give a proof of Theorem 3.3.1.

Theorem 3.3.2. *Suppose that $0 < \beta \ll 1/(20 \cdot 17^3)$ and n is a sufficiently large integer. Let G be a graph on n vertices with $\delta(G) \geq (n+1)/2$. If G is in Extremal Case 1, then G contains a spanning Halin subgraph.*

Theorem 3.3.3. *Suppose that $0 < \beta \ll 1/(20 \cdot 17^3)$ and n is a sufficiently large integer. Let G be a graph on n vertices with $\delta(G) \geq (n+1)/2$. If G is in Extremal Case 2, then G contains a spanning Halin subgraph.*

Theorem 3.3.4. *Let n be a sufficiently large integer and G an n-vertex graph with $\delta(G) \geq (n+1)/2$. If G is in the Non-extremal case, then G has a spanning Halin subgraph.*

We need the following "Absorbing Lemma" in each of the proofs of Theorems 3.3.2 - 3.4.3 in dealing with "garbage" vertices.

Lemma 3.3.1 (Absorbing Lemma)**.** *Let F be a graph such that $V(F)$ is partitioned as $S \cup R$. Suppose that (i) $\delta(R, S) \geq 3|R|$, (ii) for any two vertices $u, v \in N(R, S)$, $deg(u, v, S) \geq 6|R|$,* *and (iii) for any three vertices $u, v, w \in N(N(R,S),S)$, $deg(u,v,w,S) \geq 7|R|$. Then there is a ladder spanning on R and some other $7|R|-2$ vertices from S.*

Proof. Let $R = \{w_1, w_2, \cdots, w_r\}$. Consider first that $|r| = 1$. Choose $x_{11}, x_{12}, x_{13} \in \Gamma(w_1, S)$. By (ii), there are distinct vertices $y^1_{12} \in \Gamma(x_{11}, x_{12}, S)$ and $y^1_{23} \in \Gamma(x_{12}, x_{13}, S)$. Then the graph L on $\{w_1, x_{11}, x_{12}, x_{13}, y^1_{12}, y^1_{23}\}$ with edges in

$$\{w_1x_{11}, w_1x_{12}, w_1x_{13}, y^1_{12}x_{11}, y^1_{12}x_{12}, y^1_{23}x_{12}, y^1_{23}x_{13}\}$$

is a ladder covering R with $|V(L)| = 6$. Suppose now $r \geq 2$. For each i with $1 \leq i \leq r$, choose distinct (and unchosen) vertices $x_{i1}, x_{i2}, x_{i3} \in \Gamma(w_i, S)$. This is possible since $deg(x,S) \geq 3|R|$ for each $x \in R$. By (ii), we choose distinct vertices $y^1_{12}, y^1_{23}, \cdots, y^r_{12}, y^r_{23}$ different from the existing vertices already chosen such that $y^i_{12} \in \Gamma(x_{i1}, x_{i2}, S)$ and $y^i_{23} \in \Gamma(x_{i2}, x_{i3}, S)$ for each i, and at the same time, we chose distinct vertices $z_1, z_2, \cdots, z_{r-1}$ from the unchosen vertices in S such that $z_i \in \Gamma(x_{i3}, x_{(i+1),1}, S)$ for each $1 \leq i \leq r-1$. Finally, by (iii), choose distinct vertices $u_1, u_2, \cdots, u_{r-1}$ from the unchosen vertices in S such that $u_i \in \Gamma(x_{i3}, x_{i+1,1}, z_i, S)$. Let L be the graph with

$$V(L) = R \cup \{x_{i1}, x_{i2}, x_{i3}, y^i_{12}, y^i_{23}, z_i, u_i, x_{r1}, x_{r2}, x_{r3}, y^r_{12}, y^r_{23} \,|\, 1 \leq i \leq r-1\} \quad \text{and}$$

$E(L)$ consisting of the edges $w_rx_{r1}, w_rx_{r2}, w_rx_{r3}, y^r_{12}x_{r1}, y^r_{12}x_{r2}, y^r_{23}x_{r2}, y^r_{23}x_{r3}$ and the edges indicated below for each $1 \leq i \leq r-1$:

$$w_i \sim x_{i1}, x_{i2}, x_{i3};\ y^i_{12} \sim x_{i1}, x_{i2};\ y^i_{23} \sim x_{i2}, x_{i3};\ z_i \sim x_{i3}, x_{i+1,1};\ u_i \sim x_{i3}, x_{i+1,1}, z_i.$$

It is easy to check that L is a ladder covering R with $|V(L)| = 8r-2$. Figure 3.2 gives a depiction of L for $|R| = 2$. □

The following simple observation is heavily used in the proofs explicitly or implicitly.

Lemma 3.3.2. *Let $U = \{u_1, u_2 \cdots, u_k\}, S \subseteq V(G)$ be subsets. Then $deg(u_1, u_2, \cdots, u_k, S) \geq |S| - (deg_{\overline{G}}(u_1, S) + \cdots + deg_{\overline{G}}(u_k, S)) \geq |S| - k(|S| - \delta(U,S))$.*

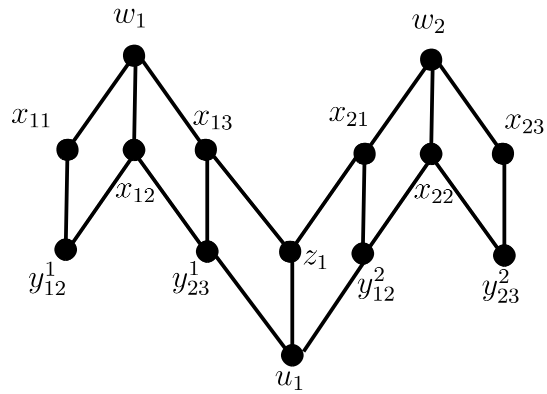


Figure (3.2) Ladder L of order 14

Extremal Case 1 is relatively easy among the three cases, therefore we prove Theorem 3.3.2 first below.

3.3.3.1 Proof of Theorem 3.3.2 We assume that G has a vertex-cut W such that $|W| \leq 5\beta n$. As $\delta(G) \geq (n+1)/2$, by simply counting degrees we see $G - W$ has exactly two components. Let V_1 and V_2 be the vertex set of the two components, respectively. Then $(1/2 - 5\beta)n \leq |V_i| \leq (1/2 + 5\beta)n$. We partition W into two subsets as follows:

$$W_1 = \{w \in W \mid deg(w, V_1) \geq (n+1)/4 - 2.5\beta n\} \quad \text{and} \quad W_2 = W - W_1.$$

As $\delta(G) \geq (n+1)/2$, we have $deg(w, V_2) \geq (n+1)/4 - 2.5\beta n$ for any $w \in W_2$. Since G is 3-connected and $(1/2 - 5\beta)n > 3$, there are three independent edges p_1p_2, q_1q_2, and r_1r_2 between $G[V_1 \cup W_1]$ and $G[V_2 \cup W_2]$ with $p_1, q_1, r_1 \in V_1 \cup W_1$ and $p_2, q_2, r_2 \in V_2 \cup W_2$.

For $i = 1, 2$, by the partition of W_i, we see that $\delta(W_i, V_i) \geq 3|W_i| + 3$. As $\delta(G) \geq (n+1)/2$, we have $\delta(G[V_i]) \geq (1/2 - 5\beta)n$. Then, as $|V_i| \leq (1/2 + 5\beta)n$, for any $u, v \in V_i$, $deg(u, v, V_i) \geq (1/2 - 25\beta)n \geq 6|W_i| + 2$, and for any $u, v, w \in V_i$, $deg(u, v, w, V_i) \geq (1/2 - 35\beta)n \geq 7|W_i| + 2$. By Lemma 3.4.2, we can find a ladder L_i spans $W_i - \{p_i, q_i\}$ and another $7|W_i - \{p_i, q_i\}| - 2$ vertices from $V_i - \{p_i, q_i\}$ if $W_i - \{p_i, q_i\} \neq \emptyset$. Denote a_ib_i and c_id_i the first and last rung of L_i (if L_i exists), respectively. Let

$$G_i = G[V_i - V(L_i)] \quad \text{and} \quad n_i = |V(G_i)|.$$

Then for $i = 1, 2$,

$$n_i \geq (n+1)/2 - 5\beta n - 7|W_i| \geq (n+1)/2 - 40\beta n \quad \text{and} \quad \delta(G_i) \geq \delta(G[V_i]) - 7|W_i| \geq (n+1)/2 - 40\beta n.$$

Let $i = 1, 2$. We now show that G_i contains a spanning subgraph isomorphic to either H_1 or H_2 as defined in the beginning of this section. Since $|n_i| \leq (1/2 + 5\beta)n$ and $\delta(G_i) \geq (n+1)/2 - 40\beta n$, any subgraph of G_i induced on at least $(1/4 - 40\beta)n$ vertices has minimum degree at least $(1/4 - 85\beta)n$, and thus has a matching of size at least 2. Hence, when n_i is even, we can choose independent edges $e_i = x_i y_i$ and $f_i = z_i w_i$ with

$$x_i, y_i \in \Gamma_{G_i}(p_i) - \{q_i\} \quad \text{and} \quad z_i, w_i \in \Gamma_{G_i}(q_i) - \{p_i\}.$$

(Notice that p_i or q_i may be contained in W_i, and in this case we have $deg_{G_i}(p_i), deg_{G_i}(q_i) \geq (1/4 - 40\beta)n$.) And if n_i is odd, we can choose independent edges $g_i y_i$ and $f_i = z_i w_i$ with

$$g_i, x_i, y_i \in \Gamma_{G_i}(p_i) - \{q_i\}, x_i \in \Gamma_{G_i}(g_i, y_i) - \{p_i, q_i\} \quad \text{and} \quad z_i, w_i \in \Gamma_{G_i}(q_i) - \{x_i, p_i\},$$

where the existence of the vertex x_i is possible since the subgraph of G_i induced on $\Gamma_{G_i}(p_i)$ has minimum degree at least $(1/2 - 40\beta)n - ((1/2 + 5\beta)n - |\Gamma_{G_i}(p_i)|) \geq |\Gamma_{G_i}(p_i)| - 45\beta n$, and hence contains a triangle. In this case, again, denote $e_i = x_i y_i$. Let

$$\begin{cases} G_i' = G_i - \{p_i, q_i\}, & \text{if } n_i \text{ is even;} \\ G_i' = G_i - \{p_i, q_i, g_i\}, & \text{if } n_i \text{ is odd.} \end{cases}$$

By the definition above, $|V(G_i')|$ is even.

The following claim is a modification of (1) of Lemma 2.2 in [17].

Claim 3.3.1. *For $i = 1, 2$, let $a'_i b'_i, c'_i d'_i \in E(G'_i)$ be two independent edges. Then G'_i contains two vertex disjoint ladders Q_{i1} and Q_{i2} spanning on $V(G'_i)$ such that Q_{i1} has $e_i = x_i y_i$ as its first rung, $a'_i b'_i$ as its last rung, and Q_{i2} has $c'_i d'_i$ as its first rung and $f_i = z_i w_i$ as its last rung, where e_i and f_i are defined prior to this claim.*

Proof. We only show the claim for $i = 1$ as the case for $i = 2$ is similar. Notice that by the definition of G'_1, $|V(G'_1)|$ is even. Since $|V(G'_1)| \leq (1/2 + 5\beta)n$ and $\delta(G'_1) \geq (n + 1)/2 - 40\beta n - 2 \geq |V(G'_1)|/2 + 8$, G'_1 has a perfect matching M containing $e_1, f_1, a'_1 b'_1, c'_1 d'_1$. We identify a'_1 and c'_1 into a vertex called s', and identify b'_1 and d'_1 into a vertex called t'. Denote G''_1 as the resulting graph and let $s't' \in E(G''_1)$ if the two vertices are not adjacent. Partition $V(G''_1)$ arbitrarily into U and V with $|U| = |V|$ such that $x_1, z_1, s' \in U$, $y_1, w_1, t' \in V$, and let $M' := M - \{a'_1 b'_1, c'_1 d'_1\} \cup \{s't'\} \subseteq E_{G'_1}(U, V)$. Define an auxiliary graph H' with vertex set M' and edge set defined as follows. If $xy, uv \in M'$ with $x, u \in U$ then $xy \sim_{H'} uv$ if and only if $x \sim_{G'_1} v$ and $y \sim_{G'_1} u$ (we do not include the case that $x \sim_{G'_1} u$ and $y \sim_{G'_1} v$ as we defined a bipartition here). Particularly, for any $pq \in M' - \{s't'\}$ with $p \in U$, $pq \sim_{H'} s't'$ if and only if $p \sim_{G'_1} b'_1, d'_1$ and $q \sim_{G'_1} a'_1, c'_1$. Notice that a ladder with rungs in M' is corresponding to a path in H' and vice versa. Since $(1/2 - 40\beta)n - 2 \leq |V(G'_1)| \leq (1/2 + 5\beta)n - 2$ and $\delta(G'_1) \geq (n + 1)/2 - 40\beta n - 2$, any two vertices in G'_1 has at least $(1/2 - 130\beta)n$ common neighbors. This together with the fact that $|U| = |V| \leq |V(G''_1)|/2 \leq (1/4 + 2.5\beta)n$ gives that $\delta(U, V), \delta(V, U) \geq (1/4 - 132.5\beta)n$. Hence

$$\delta(H') \geq (1/4-132.5\beta)n-((1/4 + 2.5\beta)n - (1/4 - 132.5\beta)n) = (1/4-267.5\beta)n \geq |V(H')|/2+1,$$

since $\beta < 1/2200$ and n is very large. Hence H' has a hamiltonian path starting with e_1, ending with f_1, and having $s't'$ as an internal vertex. The path with $s't'$ replaced by $a'_1 b'_1$ and $c'_1 d'_1$ is corresponding to the required ladders in G'_1. $\square$

We may assume n_1 is even and n_2 is odd and construct a spanning Halin subgraph of G (the construction for the other three cases follow a similar argument). Recall that $p_1 p_2, q_1 q_2, r_1 r_2$ are the three prescribed independent edges between $G[V_1 \cup W_1]$ and $G[V_2 \cup W_2]$, where $p_1, q_1, r_1 \in V_1 \cup W_1$ and $p_2, q_2, g_2, r_2 \in V_2 \cup W_2$. For a uniform discussion, we may assume that both of the ladders L_1 and L_2 exist. Let $i = 1, 2$. Recall that L_i has a_ib_i as its first rung and c_id_i as its last rung. Choose $a_i' \in \Gamma_{G_i'}(a_i)$, $b_i' \in \Gamma_{G_i'}(b_i)$ such that $a_i'b_i' \in E(G)$ and $c_i' \in \Gamma_{G_i'}(c_i)$, $d_i' \in \Gamma_{G_i'}(d_i)$ such that $c_i'd_i' \in E(G)$. This is possible as $\delta(G_i') \geq (n+1)/2 - 40\beta n - 2$. Let Q_{1i} and Q_{2i} be the ladders of G_i' given by Claim 3.3.1. Set $H_a = Q_{11}L_1Q_{12} \cup \{p_1x_1, p_1y_1, q_1z_1, q_1w_1\}$. Assume $Q_{21}L_2Q_{22}$ is a ladder can be denoted as $\overrightarrow{x_2y_2} - Q_{21}L_2Q_{22} - \overrightarrow{z_2w_2}$. To make r_2 a Halin constructible vertex, we let $H_b = Q_{21}L_2Q_{22} \cup \{g_2x_2, g_2y_2, p_2g_2, p_2y_2, q_2z_2, q_2w_2\}$ if r_2 is on the shortest (y_2, w_2)-path in $Q_{21}L_2Q_{22}$, and let $H_b = Q_{21}L_2Q_{22} \cup \{g_2x_2, g_2y_2, p_2g_2, p_2x_2, q_2z_2, q_2w_2\}$ if r_2 is on the shortest (x_2, z_2)-path (recall that $g_1, x_1, y_1 \in \Gamma_{G_1}(p_1)$). Let $H = H_a \cup H_b \cup \{p_1p_2, r_1r_2, q_1q_2\}$. Then H is a spanning Halin subgraph of G by Proposition 3.3.1 as $H_a \cup \{p_1q_1\} \cong H_1$ and $H_b \cup \{p_2q_2\} \cong H_2$. Figure 3.3 gives a construction of H for the above case when r_2 is on the shortest (y_2, w_2)-path in $Q_{21}L_2Q_{22}$.

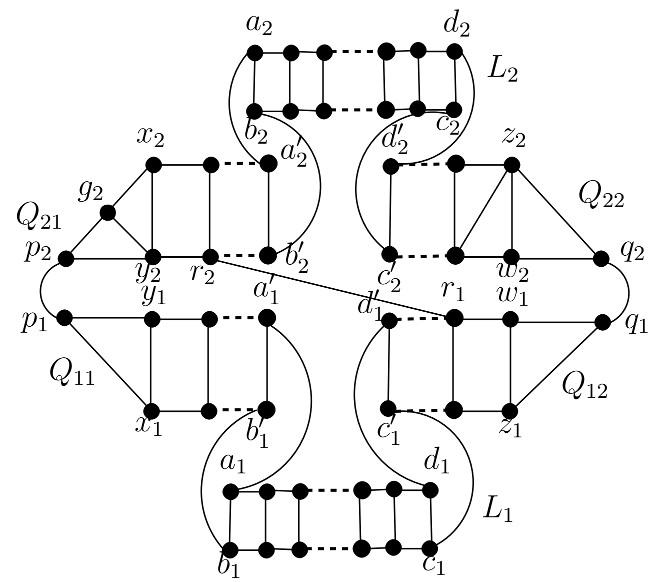


Figure (3.3) A Halin graph H

3.3.3.2 Proof of Theorem 3.3.3 Recall Extremal Case 2: There exists a partition $V_1 \cup V_2$ of V such that $|V_1| \geq (1/2 - 7\beta)n$ and $\Delta(G[V_1]) \leq \beta n$. Since $\delta(G) \geq (n+1)/2$, the assumptions imply that

$$(1/2 - 7\beta)n \leq |V_1| \leq (1/2 + \beta)n \quad \text{and} \quad (1/2 - \beta)n \leq |V_2| \leq (1/2 + 7\beta)n.$$

Let β and α be real numbers satisfying $\beta \leq \alpha/20$ and $\alpha \leq (1/17)^3$. Set $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = \alpha^{2/3}$ We first repartition $V(G)$ as follows.

$$\begin{aligned} V_2' &= \{v \in V_2 \,|\, deg(v, V_1) \geq (1 - \alpha_1)|V_1|\}, V_{01} = \{v \in V_2 - V_2' \,|\, deg(v, V_2') \geq (1 - \alpha_1)|V_2'|\}, \\ V_1' &= V_1 \cup V_{01}, \quad \text{and} \quad V_0 = V_2 - V_2' - V_{01}. \end{aligned}$$

Claim 3.3.2. $|V_{01}|, |V_0| \leq |V_2 - V_2'| \leq \alpha_2 |V_2|$.

Proof. Notice that $e(V_1, V_2) \geq (1/2 - 7\beta)n|V_2| \geq \frac{1/2 - 7\beta}{1/2 + \beta}|V_1||V_2| \geq (1 - \alpha)|V_1||V_2|$ as $\beta \leq \alpha/20$. Hence,

$$(1 - \alpha)|V_1||V_2| \leq \quad e(V_1, V_2) \quad \leq e(V_1, V_2') + e(V_1, V_2 - V_2') \leq |V_1||V_2'| + (1 - \alpha_1)|V_1||V_2 - V_2'|.$$

This gives that $|V_2 - V_2'| \leq \alpha_2 |V_2|$, and thus $|V_{01}|, |V_0| \leq |V_2 - V_2'| \leq \alpha_2 |V_2|$. $\square$

As a result of moving vertices from V_2 to V_1 and by Claim 3.3.2, we have the following.

$$\begin{aligned} \Delta(G[V_1']) &\leq \beta n + |V_{01}| \leq \beta n + \alpha_2 |V_2|, \\ \delta(V_1', V_2') &\geq (1/2 - \beta)n - |V_2 - V_2'| \geq (1/2 - \beta)n - \alpha_2 |V_2|, \\ \delta(V_2', V_1') &\geq (1 - \alpha_1)|V_1| \geq (1 - \alpha_1)(1/2 - 7\beta)n, \\ \delta(V_0, V_1') &\geq (n+1)/2 - (1 - \alpha_1)|V_2'| - |V_0| \geq 3\alpha_2 n + 8 \geq 3|V_0| + 10, \\ \delta(V_0, V_2') &\geq (n+1)/2 - (1 - \alpha_1)|V_1| - |V_0| \geq 3\alpha_2 n + 8 \geq 3|V_0| + 10, \end{aligned} \tag{3.1}$$

where the last two inequalities hold because we have $7\beta + 10/n \leq \alpha$, and $\alpha \leq (1/8)^3$.

Claim 3.3.3. *We may assume that $\Delta(G) < n-1$.*

Proof. Suppose on the contrary and let $w \in V(G)$ such that $deg(w) = n-1$. Then by $\delta(G) \geq (n+1)/2$ we have $\delta(G-w) \geq (n-1)/2$, and thus $G-w$ has a hamiltonian cycle. This implies that G has a spanning wheel subgraph, in particular, a spanning Halin subgraph of G. $\square$

Claim 3.3.4. *There exists a subgraph $T \subseteq G$ such that $|V(T)| \equiv n \,(mod\, 2)$, where T is isomorphic to some graph in $\{T_1, T_2, \cdots, T_5\}$. Assume that T has head link x_1x_2 and tail link y_1y_2. Let $m = n - |V(T)|$. Then $G - V(T)$ contains a balanced spanning bipartite graph G' with partite sets U_1 and U_2 and a subset W of $U_1 \cup U_2$ with at most $\alpha_2 n$ vertices such that the following holds:*

(i) $deg_{G'}(x, V(G') - W) \geq (1 - \alpha_1 - 2\alpha_2)m$ *for all* $x \notin W$;

(ii) There exists $x_1'x_2', y_1'y_2' \in E(G')$ *such that* $x_i', y_i' \in U_i - W$, $x_{3-i}' \sim x_i$, *and* $y_{3-i}' \sim y_i$, *for* $i = 1, 2$; *and if T has a pendent vertex, then the vertex is contained in* $V_1' \cup V_2' - W$.

(iii) There are $|W|$ vertex-disjoint 3-stars ($K_{1,3}$s) in $G'' - \{x_1', x_2', y_1', y_2'\}$ *with the vertices in W as their centers.*

Proof. By (3.1), for $i = 1, 2$, we notice that for any $u, v, w \in V_i'$,

$$deg(u, v, w, V_{3-i}') \geq |V_{3-i}'| - 3(|V_{3-i}'| - \delta(V_i', V_{3-i}')) \geq (1/2 - 28\beta - 3\alpha_1)n > n/4. \quad (3.2)$$

We now separate the proof into two cases according to the parity of n.

Case 1. n is even.

Suppose first that $\max\{|V_1'|, |V_2'|\} \leq n/2$. We arbitrarily partition V_0 into V_{10} and V_{20} such that $|V_1' \cup V_{10}| = |V_2' \cup V_{20}| = n/2$. Suppose $G[V_1']$ contains an edge x_1u_1 and there is a vertex $u_2 \in \Gamma(u_1, V_2')$ such that u_2 is adjacent to a vertex $y_2 \in V_2'$. By (3.2), there exist distinct vertices $x_2 \in \Gamma(x_1, u_1, V_2') - \{y_2, u_1\}, y_1 \in \Gamma(y_2, u_2, V_1') - \{x_1, u_1\}$. Then $G[\{x_1, u_1, x_2, y_1, u_1, y_2\}]$ contains a subgraph T isomorphic to T_1. So we assume $G[V_1']$

contains an edge x_1u_1 and no vertex in $\Gamma(u_1, V_2')$ is adjacent to any vertex in V_2'. As $\delta(G) \geq (n+1)/2$, $\delta(G[V_2' \cup V_{20}]) \geq 1$. Let $u_2 \in \Gamma(u_1, V_2')$ and $u_2y_2 \in E(G[V_2' \cup V_{20}])$. Since $deg(u_2, V_1') \geq (n+1)/2 - |V_0| > |V_1' \cup V_{10}| - |V_0|$ and $deg(y_2, V_1') \geq 3|V_0| + 10$, $\deg(u_2, y_2, V_1' \cup V_{10}) \geq 2|V_0| + 10$. Let $x_2 \in \Gamma(x_1, u_1, V_2') - \{y_2, u_2\}$, $y_1 \in \Gamma(y_2, u_2, V_1') - \{x_1, u_1\}$. Then $G[\{x_1, u_1, x_2, y_1, u_2, y_2\}]$ contains a subgraph T isomorphic to T_1. By symmetry, we can find $T \cong T_1$ if $G[V_2']$ contains an edge. Hence we assume that both V_1' and V_2' are independent sets. Again, as $\delta(G) \geq (n+1)/2$, $\delta(G[V_1' \cup V_{10}]), \delta(G[V_2' \cup V_{20}]) \geq 1$. Let $x_1u_1 \in E(G[V_1' \cup V_{10}])$ and $y_2u_2 \in E(G[V_2' \cup V_{20}])$ such that $x_1 \in V_1'$ and $u_2 \in \Gamma(u_1, V_2')$. Since $deg(x_1, V_2') \geq (n+1)/2 - |V_0| > |V_2' \cup V_{20}| - |V_0|$ and $deg(u_1, V_2') \geq 3|V_0| + 10$, we have $deg(x_1, u_1, V_2') \geq 2|V_0| + 10$. Hence, there exists $x_2 \in \Gamma(x_1, u_1, V_2') - \{y_2, u_2\}$. Similarly, there exists $y_1 \in \Gamma(y_2, u_2, V_1') - \{x_1, u_1\}$. Then $G[\{x_1, u_1, x_2, y_1, u_2, y_2\}]$ contains a subgraph T isomorphic to T_1. Let $m = (n-6)/2, U_1 = (V_1' - V(T)) \cup V_{10}$ and $U_2 = (V_2' - V(T)) \cup V_{20}$, and $W = V_0 - V(T)$. We then have $|U_1| = |U_2| = m$.

Let $G' = (V(G) - V(T), E_G(U_1, U_2))$ be the bipartite graph with partite sets U_1 and U_2. Notice that $|W| \leq |V_0| \leq \alpha_2|V_2| < \alpha_2 n$. By (3.1), we have $deg_{G'}(x, V(G') - W) \geq (1 - \alpha_1 - 2\alpha_2)m$ for all $x \notin W$. This shows (i). By the construction of T above, we have $x_1, y_1 \in V_1'$. Let $i = 1, 2$. By (3.1), we have $\delta(V_0, U_i - W) \geq 3|V_0| + 6$. Applying statement (i), we have $e_{G'}(\Gamma_{G'}(x_1, U_2 - W), \Gamma_{G'}(x_2, U_1 - W)), e_{G'}(\Gamma_{G'}(y_1, U_2 - W), \Gamma_{G'}(y_2, U_1 - W)) \geq (3|V_0| + 4)(1 - 2\alpha_1 - 4\alpha_2)m > 2m$. Hence, we can find independent edges $x_1'x_2'$ and $y_1'y_2'$ such that $x_i', y_i' \in U_i - W$, $x_{3-i}' \sim x_i$, and $y_{3-i}' \sim y_i$. This gives statement (ii). Finally, as $\delta(V_0, U_i - W) \geq 3|V_0| + 6$, we have $\delta(V_0, U_i - W - \{x_1', x_2', y_1', y_2'\}) \geq 3|V_0| + 2$. Hence, there are $|W|$ vertex-disjoint 3-stars with their centers in W.

Otherwise we have $\max\{|V_1'|, |V_2'|\} > n/2$. Assume, w.l.o.g., that $|V_1'| \geq n/2 + 1$. Then $\delta(G[V_1']) \geq 2$ and thus $G[V_1']$ contains two vertex-disjoint paths isomorphic to P_3 and P_2, respectively. Let $m = (n-8)/2$. We consider three cases here. Case (a): $|V_1'| - 5 \leq m$. Then let $x_1u_1w_1, y_1v_1 \subseteq G[V_1']$ be two vertex-disjoint paths, and let $x_2 \in \Gamma(x_1, u_1, w_1, V_2'), y_2 \in \Gamma(y_1, v_1, V_2')$ and $z \in \Gamma(w_1, v_1, V_2')$ be three distinct vertices. Then $G[\{x_1, u_1, w_1, x_2, z, y_1, v_1, y_2\}]$ contains a subgraph T isomorphic to T_4. Notice that

$|V_2' - V(T)| \leq m$. We arbitrarily partition V_0 into V_{10} and V_{20} such that $|V_1' \cup V_{10}| = |V_2' \cup V_{20}| = m$. Let $U_1 = (V_1' - V(T)) \cup V_{10}$, $U_2 = (V_2' - V(T)) \cup V_{20}$, and $W = V_0$. Hence we assume $|V_1'| - 5 = m + t_1$ for some $t_1 \geq 1$. This implies that $|V_1'| \geq n/2 + t_1 + 1$ and thus $\delta(G[V_1']) \geq t_1 + 2$. Let V_1^0 be the set of vertices $u \in V_1'$ such that $deg(u, V_1') \geq \alpha_1 m$. Case (b): $|V_1^0| \geq |V_1'| - 5 - m$. Then we form a set W with $|V_1'| - 5 - m$ vertices from V_1^0 and all the vertices of V_0. Then $|V_1' - W| = m + 5 + t_1 - (|V_1'| - 5 - m) = m + 5 = n/2 + 1$, and hence $\delta(G[V_1' - W]) \geq 2$. Similarly as in Case (a), we can find a subgraph T of G contained in $G[V_1' - W]$ isomorphic to T_4. Let $U_1 = V_1' - V(T) - W$, $U_2 = (V_2' - V(T)) \cup W$. Then $|U_1| = |U_2| = m$. Thus we have Case (c): $|V_1^0| < |V_1'| - 5 - m$. Suppose that $|V_1' - V_1^0| = m + 5 + t_1' = n/2 + t_1' + 1$ for some $t_1' \geq 1$. This implies that $\delta(G[V_1' - V_1^0]) \geq t_1' + 2$. We show that $G[V_1' - V_1^0]$ contains $t_1' + 2$ vertex-disjoint 3-stars. To see this, suppose $G[V_1' - V_1^0]$ contains a subgraph M of at most $s < t_1' + 2$ 3-stars. By counting the number of edges between $V(M)$ and $V_1' - V_1^0 - V(M)$ in two ways, we get that $t_1'|V_1' - V_1^0 - V(M)| \leq e_{G-V_1^0}(V(M), V_1' - V_1^0 - V(M)) \leq 4s\Delta(G[V_1' - V_1^0]) \leq 4s\alpha_1 m$. Since $|V_1' - V_1^0| = m + 5 + t_1' = n/2 + t_1' + 1$, $|V_1' - V_1^0 - V(M)| \geq m - 3t_1' \geq m - 6\alpha_2 m$, where the last inequality holds as $|V_1'| \leq (1/2 + \beta)n + \alpha_2|V_2'|$ implying that $t_1' \leq |V_1'| - m - 5 \leq 2\alpha_2 m$. This, together with the assumption that $\alpha \leq (1/8)^3$ gives that $s \geq t_1' + 2$, showing a contradiction. Hence we have $s \geq t_1' + 2$. Let $x_1 u_1 w_1$ and $y_1 v_1$ be two paths taken from two 3-stars in M. Then we can find a subgraph T of G isomorphic to T_4 the same way as in Case (a). We take exactly t_1' 3-stars from the remaining ones in M and denote the centers of these stars by W'. Let $U_1 = V_1' - V_1^0 - V(T) - W'$, $W = W' \cup V_1^0 \cup V_0$, and $U_2 = (V_2' - V(T)) \cup W$. Then $|U_1| = |U_2| = m$.

For the partition of U_1 and U_2 in all the cases discussed in the paragraph above, we let $G' = (V(G) - V(T), E_G(U_1, U_2))$ be the bipartite graph with partite sets U_1 and U_2. Notice that $|W| \leq |V_0| \leq \alpha_2 n$ if Case (a) occurs, $|W| \leq |V_0| + |V_1'| - m - 5 \leq (1/2 + \beta)n + |V_0| - n/2 \leq \alpha_2 n$ if Case (b) occurs, and $|W| = |W' \cup V_1^0 \cup V_0| = |V_1' - U_1 - V(T)| + |V_0| \leq (1/2 + \beta)n - (1/2 - 4)n + |V_0| \leq \alpha_2 n$ if Case (c) occurs. Since $\delta(V_2', V_1') \geq (1 - \alpha_1)|V_1|$ from (3.1) and $|V_1' - U_1| \leq 2\alpha_2 m$, we have $\delta(U_2 - W, U_1 - W) \geq (1 - \alpha_1 - 2\alpha_2)m$. On the other hand, from (3.1),

$\delta(V_1', V_2') \geq (1/2 - \beta)n - \alpha_2|V_2|$. This gives that $\delta(U_1 - W, U_2 - W) \geq (1 - \alpha_1 - 2\alpha_2)m$. Hence, we have $deg_{G'}(x, V(G') - W) \geq (1 - \alpha_1 - 2\alpha_2)m$ for all $x \notin W$. Applying statement (i), we have $e_{G'}(\Gamma_{G'}(x_1, U_2 - W), \Gamma_{G'}(x_2, U_1 - W)), e_{G'}(\Gamma_{G'}(y_1, U_2 - W), \Gamma_{G'}(y_2, U_1 - W)) \geq (3|V_0| + 4)(1 - 2\alpha_1 - 4\alpha_2)m > 2m$. Hence, we can find independent edges $x_1'x_2'$ and $y_1'y_2'$ such that $x_i', y_i' \in U_i - W$, $x_{3-i}' \sim x_i$, and $y_{3-i}' \sim y_i$. By the construction of T, T is isomorphic to T_4, and the pendent vertex $z \in V_2' \subseteq V_1 \cup V_2' - W$. This gives statement (ii). Finally, as $\delta(V_0, U_1 - W) \geq 3\alpha_2 n + 5 \geq 3|W| + 5$, we have $\delta(V_0, U_1 - W - \{x_1', x_2', y_1', y_2'\}) \geq 3|W| + 1$. By the definition of V_1^0, we have $\delta(V_1^0, V_1' - W - \{x_1', x_2', y_1', y_2'\}) \geq \alpha_1 m - \alpha_2 n - 4 \geq 3|W|$. For the vertices in W' in Case (c), we already know that there are vertex-disjoint 3-stars in G' with centers in W'. Hence, regardless of the construction of W, we can always find $|W|$ vertex-disjoint 3-stars with their centers in W.

Case 2. n **is odd.**

Suppose first that $\max\{|V_1'|, |V_2'|\} \leq (n+1)/2$ and let $m = (n-7)/2$. We arbitrarily partition V_0 into V_{10} and V_{20} such that, w.l.o.g., say $|V_1' \cup V_{10}| = (n+1)/2$ and $|V_2' \cup V_{20}| = (n-1)/2$. We show that $G[V_1' \cup V_{10}]$ either contains two independent edges or is isomorphic to $K_{1,(n-1)/2}$. As $\delta(G) \geq (n+1)/2$, we have $\delta(G[V_1' \cup V_{10}]) \geq 1$. Since n is sufficiently large, $(n+1)/2 > 3$. Then it is easy to see that if $G[V_1' \cup V_{10}] \ncong K_{1,(n-1)/2}$, then $G[V_1' \cup V_{10}]$ contains two independent edges. Furthermore, we can choose two independent edges x_1u_1 and y_1v_1 such that $u_1, v_1 \in V_1'$. This is obvious if $|V_{10}| \leq 1$. So we assume $|V_{10}| \geq 2$. As $\delta(V_0, V_1') \geq 3|V_0| + 10$, by choosing $x_1, y_1 \in V_{10}$, we can choose distinct vertices $u_1 \in \Gamma(x_1, V_1')$ and $v_1 \in \Gamma(y_1, V_1')$. Let $x_2 \in \Gamma(x_1, u_1, V_2'), y_2 \in \Gamma(y_1, v_1, V_2')$ and $z \in \Gamma(u_1, v_1, V_2')$. Then $G[\{x_1, u_1, x_2, y_1, v_1, y_2, z\}]$ contains a subgraph T isomorphic to T_3. We assume now that $G[V_1' \cup V_{10}]$ is isomorphic to $K_{1,(n-1)/2}$. Let u_1 be the center of the star $K_{1,(n-1)/2}$. Then each leave of the star has at least $(n-1)/2$ neighbors in $V_2' \cup V_{20}$. Since $|V_2' \cup V_{20}| = (n-1)/2$, we have $\Gamma(v, V_2' \cup V_{20}) = V_2' \cup V_{20}$ if $v \in V_1' \cup V_{10} - \{u_1\}$. By the definition of V_0, $\Delta(V_0, V_1') < (1 - \alpha_1)|V_1|$ and $\Delta(V_0, V_2') < (1 - \alpha_1)|V_2'|$, and so $u_1 \in V_1'$, $V_{10} = \emptyset$ and $V_{20} = \emptyset$. We claim that V_2' is not an independent set. Otherwise, by $\delta(G) \geq (n+1)/2$, for each $v \in V_2'$, $\Gamma(v, V_1') = V_1'$. This in turn shows that u_1 has degree

$n-1$, showing a contradiction to Claim 3.3.3. So let $y_2v_2 \in E(G[V_2'])$ be an edge. Let $w_1 \in \Gamma(v_2, V_1') - \{u_1\}$ and $w_1u_1x_1$ the path containing w_1. Choose $y_1 \in \Gamma(y_2, v_2, V_1') - \{w_1, u_1, x_1\}$ and $x_2 \in \Gamma(x_1, u_1, w_1, V_2') - \{y_1, v_1\}$. Then $G[\{x_1, u_1, x_2, w_1, v_2, y_2, y_1\}]$ contains a subgraph T isomorphic to T_2. Let $U_1 = (V_1' - V(T)) \cup V_{10}$ and $U_2 = (V_2' - V(T)) \cup V_{20}$ and $W = V_0 - V(T)$. We have $|U_1| = |U_2| = m$ and $|W| \leq |V_0| \leq \alpha_2 n$.

Otherwise we have $\max\{|V_1'|, |V_2'|\} \geq (n+1)/2 + 1$. Assume, w.l.o.g., that $|V_1'| \geq (n+1)/2 + 1$. Then $\delta(G[V_1']) \geq 2$ and thus $G[V_1']$ contains two independent edges. Let $m = (n-7)/2$ and V_1^0 be the set of vertices $u \in V_1'$ such that $deg(u, V_1') \geq \alpha_1 m$. We consider three cases here. Since $|V_1'| \geq (n+1)/2 + 1 > m + 4$, we assume $|V_1'| = m + 4 + t_1$ for some $t_1 \geq 1$. Case (a): $|V_1^0| \geq |V_1'| - m - 4$. Then we form a set W with $|V_1'| - 4 - m$ vertices from V_1^0 and all the vertices of V_0. Then $|V_1' - W| = m + 4 + t_1 - (|V_1'| - 4 - m) = m + 4 = (n+1)/2 + 1$. Then we have $\delta(G[V_1' - W]) \geq 2$. Hence $G[V_1' - W]$ contains two independent edges. Let $x_1u_1, y_1v_1 \subseteq E(G[V_1' - W])$ be two independent edges, and let $x_2 \in \Gamma(x_1, u_1, V_2'), y_2 \in \Gamma(y_1, v_1, V_2')$ and $z \in \Gamma(w_1, v_1, V_2')$ be three distinct vertices. Then $G[\{x_1, u_1, x_2, z, y_1, v_1, y_2\}]$ contains a subgraph T isomorphic to T_3. Let $U_1 = V_1' - V(T) - W$, $U_2 = (V_2' - V(T)) \cup W$. Then $|U_1| = |U_2| = m$ and $|W| \leq |V_0| + |V_1' - U_1| \leq |V_2 - V_2'| + \beta n + 4\alpha_2 n$. Thus we have $|V_1^0| < |V_1'| - 4 - m$. Suppose that $|V_1' - V_1^0| = m + 4 + t_1' = (n+1)/2 + t_1'$ for some $t_1' \geq 1$. This implies that $\delta(G[V_1' - V_1^0]) \geq t_1' + 1$. Case (b): $t_1' \geq 2$. We show that $G[V_1' - V_1^0]$ contains $t_1' + 2$ vertex-disjoint 3-stars. To see this, suppose $G[V_1' - V_1^0]$ contains a subgraph M of at most s vertex disjoint 3-stars. We may assume that $s < t_1' + 2$. Then we have $(t_1 - 1)|V_1' - V_1^0 - V(M)| \leq e_{G - V_1^0}(V(M), V_1' - V_1^0 - V(M)) \leq 4s\Delta(G[V_1' - V_1^0])$. Since $|V_1' - V_1^0| = m + 5 + t_1' = (n+1)/2 + t_1'$, $|V_1' - V_1^0 - V(M)| \geq m - 3t_1' \geq m - 6\alpha_2 m$, where the last inequality holds as $|V_1'| \leq (1/2 + \beta)n + \alpha_2|V_2'|$ implying that $t_1' \leq |V_1'| - m - 5 \leq 2\alpha_2 m$. This, together with the assumption that $\alpha \leq (1/8)^3$ gives that $s \geq t_1' + 2$, showing a contradiction. Hence we have $s \geq t_1' + 2$. Let x_1u_1 and y_1v_1 be two paths taken from two 3-stars in M, and we can find a subgraph T of G isomorphic to T_3 the same way as in Case (a). We take exactly t_1' 3-stars from the remaining ones in M and denote the centers of these stars by W'. Let $U_1 = V_1' - V_1^0 - V(T) - W'$, $W = W' \cup V_1^0 \cup V_0$, and $U_2 = (V_2' - V(T)) \cup W$. Then

$|U_1| = |U_2| = m$. Case (c): $t'_1 = 1$. In this case, we let $m = (n-9)/2$. If $G[V'_1 - V^0_1]$ contains a vertex adjacent to all other vertices in $V'_1 - V^0_1$, we take this vertex to V'_2. This gets back to Case (a). Hence, we assume that $G[V'_1 - V^0_1]$ has no vertex adjacent to all other vertices in $V'_1 - V^0_1$. Then by the assumptions that $\delta(G) \geq (n+1)/2$ and $|V'_1 - V^0_1| = (n+1)/2 + 1$, we can find two copies of vertex disjoint P_3s in $G[V'_1 - V^0_1]$. Let $x_1u_1w_1$ and $y_1v_1z_1$ be two P_3s in $G[V'_1]$. There exist distinct vertices $x_2 \in \Gamma(x_1, u_1, w_1, V'_2), y_2 \in \Gamma(y_1, v_1, z_1, V'_2)$ and $z \in \Gamma(w_1, z_1, V'_2)$. Then $G[\{x_1, u_1, w_1, x_2, y_1, v_1, z_1, y_2, z\}]$ contains a subgraph T isomorphic to T_5. Let $U_1 = V'_1 - V^0_1 - V(T)$, $W = V^0_1 \cup V_0$, and $U_2 = V'_2 - V(T)$. Then $|U_1| = |U_2| = m$.

For the partition of U_1 and U_2 in all the cases discussed in Case 2, we let $G' = (V(G) - V(T), E_G(U_1, U_2))$ be the bipartite graph with partite sets U_1 and U_2. Similarly as in Case 1, we can show that all the statements (i)-(iii) hold.

□

Let $W_1 = U_1 \cap W$ and $W_2 = U_2 \cap W$. For $i = 1, 2$, by the definition of W, we see that $\delta(W_i, U_i - \{x'_1, y'_1, x'_2, y'_2\}) \geq 3|W_i|$. And for any $u, v \in U_i$, $\Gamma(u, v, U_{3-i}) \geq 6|W_i|$, and for any $u, v, w \in U_i$, $\Gamma(u, v, w, U_{3-i}) \geq 7|W_i|$. By Lemma 3.4.2, we can find ladder L_i spanning on W_i and another $7|W_i| - 2$ vertices from $U_i - \{x'_1, x'_2, y'_1, y'_2\}$ if $W_i \neq \emptyset$. Denote $a_{1i}a_{2i}$ and $b_{1i}b_{2i}$ the first and last rungs of L_i (if L_i exists), respectively, where $a_{1i}, b_{1i} \in U_1$. Let

$$U'_i = U_i - V(L_i), \qquad m' = |U'_1| = |U'_2|, \quad \text{and} \quad G'' = G''(U'_1 \cup U'_2, E_G(U'_1, U'_2)).$$

Since $|W| \leq \alpha_2 n$, $m \geq (n-9)/2$, and n is sufficiently large, we have $1/n + 7|W| \leq 15\alpha_2 m$. As $\delta(G' - W) \geq (1 - \alpha_1 - 2\alpha_2)m$ and $\alpha \leq (1/17)^3$, we obtain the following:

$$\delta(G'') \geq 7m'/8 + 1.$$

Let $a'_{2i} \in \Gamma(a_{1i}, U'_2)$, $a'_{1i} \in \Gamma(a_{2i}, U'_1)$ such that $a'_{1i}a'_{2i} \in E(G)$; and $b'_{2i} \in \Gamma(b_{1i}, U'_2)$, $b'_{1i} \in \Gamma(b_{2i}, U'_1)$ such that $b'_{1i}b'_{2i} \in E(G)$. We have the claim below.

Claim 3.3.5. *The balanced bipartite graph G'' contains three vertex-disjoint ladders Q_1, Q_2, and Q_3 spanning on $V(G'')$ such that the first rung of Q_1 is $x'_1x'_2$ and the last rung of Q_1 is*

$a'_{11}a'_{21}$, the first rung of Q_2 is $b'_{11}b'_{21}$ and the last rung of Q_2 is $a'_{12}a'_{22}$, the first rung of Q_3 is $b'_{12}b'_{22}$ and the last rung of Q_3 is $y'_1y'_2$.

Proof. Since $\delta(G'') \geq 7m'/8+5$, G'' has a perfect matching M containing the following edges: $x'_1x'_2, a'_{11}a'_{21}, b'_{11}b'_{21}, a'_{12}a'_{22}, b'_{12}b'_{22}, y'_1y'_2$. We identify a'_{11} and b'_{11}, a'_{21} and b'_{21}, a'_{12} and b'_{12}, and a'_{22} and b'_{22} as vertices called c'_{11}, c'_{21}, c'_{12}, and c'_{22}, respectively. Denote $G^* = G^*(U^*_1, U^*_2)$ as the resulting graph and let $c'_{11}c'_{21}, c'_{12}c'_{22} \in E(G^*)$ if they do not exist in $E(G^*)$. Denote $M' := M - \{a'_{11}a'_{21}, b'_{11}b'_{21}, a'_{12}a'_{22}, b'_{12}b'_{22}\} \cup \{c'_{11}c'_{21}, c'_{12}c'_{22}\}$. Define an auxiliary graph H' on M' as follows. If $xy, uv \in M'$ with $x, u \in U'_1$ then $xy \sim_{H'} uv$ if and only if $x \sim_{G'} v$ and $y \sim_{G'} u$. Particularly, for any $pq \in M' - \{c'_{11}c'_{21}, c'_{12}c'_{22}\}$ with $p \in U'_2$, $pq \sim_{H'} c'_{11}c'_{21}$ (resp. $pq \sim_{H'} c'_{12}c'_{22}$) if and only if $p \sim_{G'} a'_{11}, b'_{11}$ and $q \sim_{G'} a'_{21}, b'_{21}$ (resp. $p \sim_{G'} a'_{12}, b'_{12}$ and $q \sim_{G'} a'_{22}, b'_{22}$). Notice that there is a natural one-to-one correspondence between ladders with rungs in M' and paths in H'. Since $\delta_{G^*}(U^*_1, U^*_2), \delta_{G^*}(U^*_2, U^*_1) \geq 3m'/4 + 1$, we get $\delta(H') \geq m'/2 + 1$. Hence H' has a hamiltonian path starting with $x'_1x'_2$, ending with $y'_1y'_2$, and having $c'_{11}c'_{21}$ and $c'_{12}c'_{22}$ as two internal vertices. The path with the vertex $c'_{11}c'_{21}$ replaced by $a'_{11}a'_{21}$ and $b'_{11}b'_{21}$, and with the vertex $c'_{12}c'_{22}$ replaced by $a'_{12}a'_{22}$ and $b'_{12}b'_{22}$ is corresponding to the required ladders in G''. □

If $T \in \{T_1, T_2\}$, then

$$H = x_1x_2Q_1L_1Q_2L_2Q_3y_1y_2 \cup T.$$

is a spanning Halin subgraph of G. Suppose now that $T \in \{T_3, T_4, T_5\}$ and z is the pendent vertex. Then $z \in V'_1 \cup V'_2 - W$ by Claim 3.3.4. By (3.1) and the definition of U'_1 and U'_2, we get $deg_G(z, U'_1), deg_G(z, U'_2) \geq (1 - \alpha_1 - 9\alpha_2)m > m'/2$. So z has a neighbor on each side of the ladder $Q_1L_1Q_2L_2Q_3$. Let H' be obtained from $x_1x_2Q_1L_1Q_2L_2Q_3y_1y_2 \cup T$ by suppressing the degree 2 vertex z. Then H' is a Halin graph such that each vertex on one side of $Q_1L_1Q_2L_2Q_3$ is a degree 3 vertex on its underlying tree. Let z' be a neighbor of z such that z' has degree 3 in the underlying tree of H'. Then

$$H = x_1x_2Q_1L_1Q_2L_2Q_3y_1y_2 \cup T \cup \{zz'\},$$

is a spanning Halin subgraph of G.

3.3.3.3 Proof of Theorem 3.4.3 We first show that G contains a subgraph T isomorphic to T_1 if n is even and to T_2 if n is odd. Then by showing that $G - V(T)$ contains a spanning ladder L with its first rung adjacent to the head link of T and its last rung adjacent to the tail link of T, we get a spanning Halin subgraph H of G formed by $L \cup T$.

Finding a subgraph T

Claim 3.3.6. *Let n be a sufficient large integer and G an n-vertex graph with $\delta(G) \geq (n+1)/2$. If G is not in Extremal Case 2, then G contains a subgraph T isomorphic to T_1 if n is even and to T_2 if n is odd.*

Proof. Suppose first that n is even. Let $xy \in E(G)$ be an edge. We show that $G[N(x) - \{y\}]$ contains an edge x_1x_2 and $G[N(y) - \{x\}]$ contains an edge y_1y_2 such that the two edges are independent. Since G is not in Extremal Case 2, it has no independent set of size at least $(1/2 - 7\beta)n$. Hence, we can find the two desired edges, and $G[\{x, y, x_1, x_2, y_1, y_2\}]$ contains a subgraph T isomorphic to T_1. Then assume that n is odd. We show in the first step that G contains a subgraph isomorphic to K_4^- (K_4 with one edge removed). Let $yz \in E(G)$. As $\delta(G) \geq (n+1)/2$, there exists $y_1 \in \Gamma(y, z)$. If there exists $y_2 \in \Gamma(y, z) - \{y_1\}$, we are done. Otherwise, $(\Gamma(y) - \{y_1, z\}) \cap (\Gamma(z) - \{y_1, y\}) = \emptyset$. As $\delta(G) \geq (n+1)/2$, y_1 is adjacent to a vertex $y_2 \in \Gamma(y) \cup \Gamma(z) - \{y_1, y, z\}$. Assume $y_2 \in \Gamma(z) - \{y_1, y\}$. Then $G[\{y, y_1, z, y_2\}]$ contains a copy of K_4^-. Choose $x \in \Gamma(y) - \{z, y_1, y_2\}$ and choose an edge $x_1x_2 \in G[\Gamma(x) - \{y, y_1, y_2, z\}]$. Then $G[\{y, y_1, z, y_2, x, x_1, x_2\}]$ contains a subgraph T isomorphic to T_2. □

Let T be a subgraph of G as given by Claim 3.3.6. Suppose the head link of T is x_1x_2 and the tail link of T is y_1y_2. Let $G' = G - V(T)$. We show in next section that G' contains a spanning ladder with first rung adjacent to x_1x_2 and its last rung adjacent to y_1y_2. Let $n' = |V(G')|$. Then we have $\delta(G') \geq (n+1)/2 - 7 \geq n'/2 - 4 \geq (1/2 - \beta)n'$.

Finding a spanning ladder of G' with prescribed end rungs

Theorem 3.3.5. *Let n' be a sufficiently large even integer and G' the subgraph of G obtained by removing vertices in T. Suppose that $\delta(G') \geq (1/2-\beta)n'$ and $G = G[V(G') \cup V(T)]$ is in Non-extremal case, then G' contains a spanning ladder with first rung adjacent to x_1x_2 and its last rung adjacent to y_1y_2.*

Proof. We fix the following sequence of parameters

$$0 < \varepsilon \ll d \ll \beta \ll 1$$

and specify their dependence as the proof proceeds.

Let β be the parameter defined in the two extremal cases. Then we choose $d \ll \beta$ and choose

$$\varepsilon = \frac{1}{4}\epsilon(d/2, 3, 2, d/4)$$

following the definition of ϵ in the Blow-up Lemma.

Applying the Regularity Lemma to G' with parameters ε and d, we obtain a partition of $V(G')$ into $\ell+1$ clusters $V_0, V_1, \cdots, V_\ell$ for some $\ell \leq M \leq M(\varepsilon)$, and a spanning subgraph G'' of G' with all described properties in the Regularity Lemma. In particular, for all $v \in V(G')$,

$$deg_{G''}(v) > deg_{G'}(v) - (d+\varepsilon)n' \geq (1/2 - \beta - \varepsilon - d)n' \geq (1/2 - 2\beta)n' \tag{3.3}$$

provided that $\varepsilon + d \leq \beta$. On the other hand,

$$e(G'') \geq e(G') - \frac{(d+\varepsilon)}{2}(n')^2 > e(G') - d(n')^2$$

by $\varepsilon < d$.

We further assume that $\ell = 2k$ is even; otherwise, we eliminate the last cluster V_ℓ by removing all the vertices in this cluster to V_0. As a result, $|V_0| \leq 2\varepsilon n'$, and

$$(1-2\varepsilon)n' \leq \ell N = 2kN \leq n', \tag{3.4}$$

where $N = |V_i|$ for $1 \le i \le \ell$.

For each pair i and j with $1 \le i \ne j \le \ell$, we write $V_i \sim V_j$ if $d(V_i, V_j) \ge d$. As in other applications of the Regularity Lemma, we consider the *reduced graph* G_r, whose vertex set is $\{1, 2, \cdots, r\}$ and two vertices i and j are adjacent if and only if $V_i \sim V_j$. From $\delta(G'') > (1/2 - 2\beta)n'$, we claim that $\delta(G_r) \ge (1/2 - 2\beta)\ell$. Suppose not, and let $i_0 \in V(G_r)$ be a vertex with $deg_{Gr}(i_0) < (1/2 - 2\beta)\ell$. Let V_{i_0} be the cluster in G corresponding to i_0. Then we have

$$(1/2-\beta)n'|V_{i_0}| \le |E_{G'}(V_{i_0}, V - V_{i_0})| < (1/2-2\beta)\ell N|V_{i_0}| + 2\varepsilon n'|V_{i_0}| < (1/2-\beta)n'|V_{i_0}|.$$

This gives a contradiction by $\ell N \le n'$ from inequality (3.4).

Let $x \in V(G')$ be a vertex and A a cluster. We say x is *typical* to A if $deg(x, A) \ge (d - \varepsilon)|A|$, and in this case, we write $x \sim A$.

Claim 3.3.7. *Each vertex in $\{x_1, x_2, y_1, y_2\}$ is typical to at least $(1/2 - 2\beta)l$ clusters in $\{V_1, \cdots, V_l\}$.*

Proof. Suppose on the contrary that there exists $x \in \{x_1, x_2, y_2, y_2\}$ such that x is typical to less than $(1/2 - 2\beta)l$ clusters in $\{V_1, \cdots, V_l\}$. Then we have $deg_{G'}(x) < (1/2 - 2\beta)lN + (d+\varepsilon)n' \le (1/2-\beta)n'$ by $lN \le n'$ and $d + \varepsilon \le \beta$. $\square$

Let $x \in V(G')$ be a vertex. Denote by $\mathcal{V}_x$ the set of clusters to which x typical.

Claim 3.3.8. *There exist $V_{x_1} \in \mathcal{V}_{x_1}$ and $V_{x_2} \in \mathcal{V}_{x_2}$ such that $d(V_{x_1}, V_{x_2}) \ge d$.*

Proof. We show the claim by considering two cases based on the size of $|\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}|$.

Case 1. $|\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}| \le 2\beta l$.

Then we have $|\mathcal{V}_{x_1} - \mathcal{V}_{x_2}| \ge (1/2 - 4\beta)l$ and $|\mathcal{V}_{x_2} \cap \mathcal{V}_{x_1}| \ge (1/2 - 4\beta)l$. We conclude that there is an edge between $\mathcal{V}_{x_1} - \mathcal{V}_{x_2}$ and $\mathcal{V}_{x_2} - \mathcal{V}_{x_1}$ in G_r. For otherwise, let $\mathcal{U}$ be the union of clusters in $\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}$. Then $|V_0 \cup \mathcal{U} \cup V(T)| \le 5\beta n$ is a vertex-cut of G, implying that G is in Extremal Case 1.

Case 2. $|\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}| > 2\beta l$.

We may assume that $\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}$ is an independent set in G_r. For otherwise, we are done by finding an edge within $\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}$. Also we may assume that $E_{G_r}(\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}, \mathcal{V}_{x_1} - \mathcal{V}_{x_2}) = \emptyset$ and $E_{G_r}(\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}, \mathcal{V}_{x_2} - \mathcal{V}_{x_1}) = \emptyset$. Since $\delta(G_r) \geq (1/2 - 2\beta)l$ and $\delta_{G_r}(\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}, \mathcal{V}_{x_1} \cup \mathcal{V}_{x_2}) = 0$, we know that $l - |\mathcal{V}_{x_1} \cup \mathcal{V}_{x_2}| \geq (1/2 - 2\beta)l$. Hence, $|\mathcal{V}_{x_1} \cup \mathcal{V}_{x_2}| = |\mathcal{V}_{x_1}| + |\mathcal{V}_{x_2}| - |\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}| \leq (1/2 + 2\beta)l$. This gives that $|\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}| \geq |\mathcal{V}_{x_1}| + |\mathcal{V}_{x_2}| - (1/2 + 2\beta)l \geq (1/2 - 2\beta)l + (1/2 - 2\beta)l - (1/2 + 2\beta)l \geq (1/2 - 6\beta)l$. Let $\mathcal{U}$ be the union of clusters in $\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}$. Then $|\mathcal{U}| \geq (1/2 - 7\beta)n$ and $\Delta(G[\mathcal{U}]) \leq (d + \varepsilon)n' \leq \beta n$. This shows that G is in Extremal Case 2. $\square$

Similarly, we have the following claim.

Claim 3.3.9. *There exist $V_{y_1} \in \mathcal{V}_{y_1} - \{V_{x_1}, V_{x_2}\}$ and $V_{y_2} \in \mathcal{V}_{y_2} - \{V_{x_1}, V_{x_2}\}$ such that $d(V_{y_1}, V_{y_2}) \geq d$.*

Claim 3.3.10. *The reduced graph G_r has a hamiltonian path $X_1Y_1 \cdots X_kY_k$ such that $\{X_1, Y_1\} = \{V_{x_1}, V_{x_2}\}$ and $\{X_k, Y_k\} = \{V_{y_1}, V_{y_2}\}$.*

Proof. We contract the edges $V_{x_1}V_{x_2}$ and $V_{y_1}V_{y_2}$ in G_r. Denote the two new vertices as V_x' and V_y' respectively, and denote the resulting graph as G_r'. Then we show that G_r' contains a hamiltonian (V_x', V_y')-path. This path is corresponding to a required hamiltonian path in G_r.

To show G_r' has a hamiltonian (V_x', V_y')-path, we need the following generalized version of a result due to Nash-Williams [44] : Let Q be a 2-connected graph of order m. If $\delta(Q) \geq \max\{(m+2)/3 + 1, \alpha(Q) + 1\}$, then Q is hamiltonian connected, where $\alpha(Q)$ is the size of a largest independent set of Q.

We claim that G_r' is $2\beta l$-connected. For otherwise, let S be a vertex-cut of G_r' with $|S| < 2\beta l$ and $\mathcal{S}$ the vertex set corresponding to S in G . Then $|\mathcal{S} \cup V_0 \cup V(T)| \leq 2\beta n' + 2\varepsilon n' < 5\beta n$, showing that G is in Extremal Case 1. Since $n' = Nl + |V_0| \leq (l+2)\varepsilon n'$, we have $l \geq 1/\varepsilon - 2 \geq 1/\beta$. Hence, G_r' is 2-connected. As G is not in Extremal Case 2, $\alpha(G_r') \leq (1/2 - 7\beta)l$. By $\delta(G_r) \geq (1/2 - 2\beta)l$, we have $\delta(G_r') \geq (1/2 - 2\beta)l - 2 \geq \max\{(l+2)/3 + 1, (1/2 - 7\beta)l + 1\}$. Thus, by the result on hamiltonian connectedness given above, we know that G_r' contains a hamiltonian (V_x', V_y')-path. $\square$

Following the order of the clusters on the hamiltonian path given in Claim 3.3.10, for $i = 1, 2, \cdots, k$, we call X_i, Y_i partners of each other and write $P(X_i) = Y_i$ and $P(Y_i) = X_i$.

Claim 3.3.11. *For each $1 \le i \le k$, there exist $X'_i \subseteq X_i$ and $Y'_i \subseteq Y_i$ such that (X'_i, Y'_i) is $(2\varepsilon, d-3\varepsilon)$-super-regular, $|Y'_1| = |X'_1|+1$, $|Y'_k| = |X'_k|+1$, and $|X'_i| = |Y'_i|$ for $2 \le i \le k-1$. Additionally, each pair (Y'_i, X'_{i+1}) is 2ε-regular with density at least $d-\varepsilon$ for $i = 1, 2, \cdots, k$, where $X'_{k+1} = X'_1$.*

Proof. For each $1 \le i \le k$, let

$$\begin{aligned} X''_i &= \{x \in X_i \,|\, deg(x, Y_i) \ge (d-\varepsilon)N\}, \text{ and} \\ Y''_i &= \{y \in Y_i \,|\, deg(y, X_i) \ge (d-\varepsilon)N\}. \end{aligned}$$

If necessary, we either take a subset X'_i of X''_i or take a subset Y'_i of Y''_i such that $|Y'_1| = |X'_1|+1$, $|Y'_k| = |X'_k|+1$, and $|X'_i| = |Y'_i|$ for $2 \le i \le k-1$. Since (X_i, Y_i) is ε-regular, we have $|X''_i|, |Y''_i| \ge (1-\varepsilon)N$. This gives that $|X_1|', |X'_k| \ge (1-\varepsilon)N - 1$ and $|X'_i| = |Y'_i| \ge (1-\varepsilon)N$ for $2 \le i \le k-1$. As a result, we have $\deg(x, Y'_i) \ge (d-2\varepsilon)N$ for each $x \in X'_i$ and $deg(y, X'_i) \ge (d-2\varepsilon)N - 1 \ge (d-3\varepsilon)N$ for each $y \in Y'_i$. By Slicing lemma (Lemma 3.2.5), (X'_i, Y'_i) is 2ε-regular. Hence (X'_i, Y'_i) is $(2\varepsilon, d-3\varepsilon)$-super-regular for each $1 \le i \le k$. By Slicing lemma again, we know that (X'_i, Y'_{i+1}) is 2ε-regular with density at least $d-\varepsilon$. $\square$

For $1 \le i \le k$, we call (X'_i, Y'_i) a super-regularized cluster (sr-cluster). Denote $R = V_0 \cup (\bigcup_{i=1}^{k}((X_i \cup Y_i) - (X'_i \cup Y'_i)))$. Since $|(X_i \cup Y_i) - (X'_i \cup Y'_i)| \le 2\varepsilon N$ for $2 \le i \le k-1$ and $|(X_1 \cup Y_1) - (X'_1 \cup Y'_1)|, |(X_k \cup Y_k) - (X'_k \cup Y'_k)| \le 2\varepsilon N + 1$, we have $|R| \le 2\varepsilon n + 2k\varepsilon N + 2 \le 3\varepsilon n'$. As n' is even and $|X'_1| + |Y'_1| + \cdots + |X'_k| + |Y'_k|$ is even, we know $|R|$ is even. We arbitrarily group vertices in R into $|R|/2$ pairs. Given two vertices $u, v \in R$, we define a (u, v)-chain of length $2t$ as distinct clusters $A_1, B_1, \cdots, A_t, B_t$ such that $u \sim A_1 \sim B_1 \sim \cdots \sim A_t \sim B_t \sim v$ and each A_j and B_j are partners, in other words, $\{A_j, B_j\} = \{X_{i_j}, Y_{i_j}\}$ for some $i_j \in \{1, \cdots, k\}$. We call such a chain of length $2t$ a $2t$-chain.

Claim 3.3.12. *For each pair (u, v) in R, we can find a (u, v)-chain of length at most 4 such that every sr-cluster is used in at most $d^2N/5$ chains.*

Proof. Suppose we have found chains for the first $m < 2\varepsilon n'$ pairs of vertices in R such that no sr-cluster is contained in more than $d^2N/5$ chains. Let Ω be the set of all sr-clusters that are used exactly by $d^2N/5$ chains. Then

$$\frac{d^2N}{5}|\Omega| \quad \leq \quad 4m < 8\varepsilon n' \leq 8\varepsilon\frac{2kN}{1-2\varepsilon},$$

where the last inequality follows from (3.4). Therefore,

$$|\Omega| \quad \leq \quad \frac{80k\varepsilon}{d^2(1-2\varepsilon)} \leq \frac{80l\varepsilon}{d^2} \leq \beta l/2,$$

provided that $1 - 2\varepsilon \geq 1/2$ and $80\varepsilon \leq d^2\beta/2$.

Consider now a pair (w, z) of vertices in R which does not have a chain found so far, we want to find a (w, z)-chain using sr-clusters not in Ω. Let $\mathcal{U}$ be the set of all sr-clusters adjacent to w but not in Ω, and let $\mathcal{V}$ be the set of all sr-clusters adjacent to z but not in Ω. We claim that $|\mathcal{U}|, |\mathcal{V}| \geq (1/2 - 2\beta)l$. To see this, we first observe that any vertex $x \in R$ is adjacent to at least $(1/2 - 3\beta/2)l$ sr-clusters. For instead,

$$\begin{aligned}
(1/2-\beta)n' \quad &\leq \quad deg_{G'}(x) < (1/2-3\beta/2)lN + (d-2\varepsilon)lN + 3\varepsilon n', \\
&\leq \quad (1/2 - 3\beta/2 + d + 2\varepsilon)n' \\
&< \quad (1/2-\beta)n' \text{ (provided that } d + 2\varepsilon < \beta/2 \text{)},
\end{aligned}$$

showing a contradiction. Since $|\Omega| \leq \beta l/2$, we have $|\mathcal{U}|, |\mathcal{V}| \geq (1/2 - 2\beta)l$. Let $P(\mathcal{U})$ and $P(\mathcal{V})$ be the set of the partners of clusters in $\mathcal{U}$ and $\mathcal{V}$, respectively. By the definition of the chains, a cluster $A \in \Omega$ if and only its partner $P(A) \in \Omega$. Hence, $(P(\mathcal{U}) \cup P(\mathcal{V})) \cap \Omega = \emptyset$. Notice also that each cluster has a unique partner, and so we have $|P(\mathcal{U})| = |\mathcal{U}| \geq (1/2-2\beta)l$ and $|P(\mathcal{V})| = |\mathcal{V}| \geq (1/2 - 2\beta)l$.

If $E_{G_r}(P(\mathcal{U}), P(\mathcal{V})) \neq \emptyset$, then there exist two adjacent clusters $B_1 \in P(\mathcal{U})$, $A_2 \in P(\mathcal{V})$. If B_1 and A_2 are partners of each other, then $w \sim A_2 \sim B_1 \sim z$ gives a (w, z)-chain of length 2. Otherwise, assume $A_1 = P(B_1)$ and $B_2 = P(A_2)$, then $w \sim A_1 \sim B_1 \sim A_2 \sim B_2 \sim z$

gives a (w,z)-chain of length 4. Hence we assume that $E_{G_r}(P(\mathcal{U}),P(\mathcal{V}))=\emptyset$. We may assume that $P(\mathcal{U})\cap P(\mathcal{V})\neq\emptyset$. Otherwise, let $\mathcal{S}$ be the union of clusters contained in $V(G_r)-(P(\mathcal{U})\cup P(\mathcal{V}))$. Then $\mathcal{S}\cup R\cup V(T)$ with $|\mathcal{S}\cup R\cup V(T)|\le 4\beta n'+3\varepsilon n'+7\le 5\beta n$ (provided that $3\varepsilon+7/n'<\beta$) is a vertex-cut of G, implying that G is in Extremal Case 1. As $E_{G_r}(P(\mathcal{U}),P(\mathcal{V}))=\emptyset$, any cluster in $P(\mathcal{U})\cap P(\mathcal{V})$ is adjacent to at least $(1/2-2\beta)l$ clusters in $V(G_r)-(P(\mathcal{U})\cup P(\mathcal{V}))$ by $\delta(G_r)\ge(1/2-2\beta)l$. This implies that $|P(\mathcal{U})\cup P(\mathcal{V})|\le(1/2+2\beta)l$, and thus $|P(\mathcal{U})\cap P(\mathcal{V})|\ge|P(\mathcal{U})|+|P(\mathcal{V})|-|P(\mathcal{U})\cup P(\mathcal{V})|\ge(1/2-6\beta)l$. Then $P(\mathcal{U})\cap P(\mathcal{V})$ is corresponding to a subset V_1 of $V(G)$ such that $|V_1|\ge(1/2-6\beta)lN\ge(1/2-7\beta)n$ and $\Delta(G[V_1])\le(d+\varepsilon)n'\le\beta n$. This implies that G is in Extremal Case 2, showing a contradiction. $\square$

For each cluster $Z\in\{X_1',Y_1',\cdots,X_k',Y_k'\}$, let $R_2(Z)$ denote the set of vertices in R using Z in the 2-chains and $R_4(Z)$ denote the set of vertices in R using Z in the 4-chains given by Claim 3.3.12. By the definition of 2-chains and 4-chains, we have the following holds.

Claim 3.3.13. *For each $i=1,2,\cdots,k$, if $R_2(X_i')\neq\emptyset$, then $|R_2(X_i')|=|R_2(Y_i')|$; and if $R_4(X_i')\neq\emptyset$, then $|R_4(X_i')|=|R_4(Y_{i+1}')|$.*

Claim 3.3.14. *For each $i=1,2,\cdots,k$, if $R_2(X_i')\neq\emptyset$, then there exist vertex-disjoint ladders L_{2x}^i and L_{2y}^i covering all vertices in $R_2(X_i')\cup R_2(Y_i')$ such that $|X_i'\cap V(L_{2x}^i\cup L_{2y}^i)|=|Y_i'\cap V(L_{2x}^i\cup L_{2y}^i)|$; and if $R_4(X_i')\neq\emptyset$, then there exist three vertex disjoint ladders $L_{4x}^i, L_{4xy}^i, L_{4y}^{i+1}$ covering all vertices in $R_4(X_i')\cup R_4(Y_{i+1}')$ such that $V(L_{4x}^i)\subseteq X_i'\cup Y_i'$, $V(L_{4xy}^i)\subseteq Y_i'\cup X_{i+1}'$, and $V(L_{4y}^{i+1})\subseteq X_{i+1}'\cup Y_{i+1}'$, and that $|X_i'\cap V(L_{4x}^i\cup L_{4xy}^i)|=|Y_i'\cap V(L_{4x}^i\cup L_{4xy}^i\cup L_{4y}^{i+1})|=|X_{i+1}'\cap V(L_{4x}^i\cup L_{4xy}^i\cup L_{4y}^{i+1})|=|Y_{i+1}'\cap V(L_{4xy}^i\cup L_{4y}^{i+1})|$.*

Proof. Notice that by Claim 3.3.11, (X_i',Y_i') is $(2\varepsilon,d-3\varepsilon)$-super-regular and (Y_i,X_{i+1}) is 2ε-regular. Assume $R_2(X_i')\neq\emptyset$. By Claim 3.3.12 and Claim 3.3.13, we have $|R_2(X_i')|=|R_2(Y_i')|\le d^2N/5$. Let $R_2(X_i')=\{x_1,\cdots,x_r\}$. For each $j=1,\cdots,r$, since $|\Gamma(x_j,X_i')|\ge(d-2\varepsilon)|X_i'|>2\varepsilon|X_i'|$, by Lemma 3.2.4, there exists a vertex set $B_j\subseteq Y_i'$ with $|B_j|\ge(1-2\varepsilon)|Y_i'|$ such that B_j is typical to $\Gamma(x_j,X_i')$. If $r\ge 2$, for

$j = 1, \cdots, r-1$, there also exists a vertex set $B_{j,j+1} \subseteq Y_i'$ with $|B_{j,j+1}| \geq (1-4\varepsilon)|Y_i'|$ such that $B_{j,j+1}$ is typical to both $\Gamma(x_j, X_i')$ and $\Gamma(x_{j+1}, X_i')$. That is, for each vertex $b_1 \in B_j$, we have $deg(b_1, \Gamma(x_j, X_i')) \geq (d-5\varepsilon)|\Gamma(x_j, X_i')| > 4|R|$, and for each vertex $b_2 \in B_{j,j+1}$, we have $deg(b_2, \Gamma(x_j, X_i')), deg(b_2, \Gamma(x_{j+1}, X_i')) \geq (d-5\varepsilon)|\Gamma(x_j, X_i')| > 4|R|$. When $r \geq 2$, since $|B_j|, |B_{j,j+1}|, |B_{j+1}| \geq (d-4\varepsilon)|Y_i'| > 2\varepsilon|Y_i'|$, there is a set $A \subseteq X_i'$ with $|A| \geq (1-6\varepsilon)|X_i'| \geq |R|$ such that A is typical to each of B_j, B_{j+1} and B_{j+1}. Notice that $(d-5\varepsilon)|B_j|, (d-5\varepsilon)|B_{j,j+1}|, (d-5\varepsilon)|B_{j+1}| \geq (d-5\varepsilon)(1-4\varepsilon)|Y_i'| > 3|R|$. Hence we can choose distinct vertices $u_1, u_2, \cdots, u_{r-1} \in A$ such that $deg(u_j, B_j), deg(u_j, B_{j,j+1}), deg(u_j, B_{j+1}) \geq 3|R|$. Then we can choose distinct vertices $y_{23}^j \in \Gamma(u_j, B_j), z_j \in \Gamma(u_j, B_{j,j+1})$ and $y_{12}^{j+1} \in \Gamma(u_j, B_{j+1})$ for each j, and choose distinct and unchosen vertices $y_{12}^1 \in B_1$ and $y_{23}^r \in B_r$. Finally, as for each vertex $b_1 \in B_j$, we have $deg(b_1, \Gamma(x_j, X_i')) > 4|R|$ and for each vertex $b_2 \in B_{j,j+1}$, we have $deg(b_2, \Gamma(x_j, X_i')), deg(b_2, \Gamma(x_{j+1}, X_i')) > 4|R|$, we can choose $x_{j1}, x_{j2}, x_{j3} \in \Gamma(x_j, X_i') - \{u_1, \cdots, u_{r-1}\}$ such that $y_{12}^j \in \Gamma(x_{j1}, x_{j2}, Y_i')$, $y_{23}^j \in \Gamma(x_{j2}, x_{j3}, Y_i')$, and $z_j \in \Gamma(x_{i3}, x_{i+1,1}, Y_i')$. Let L_{2x}^i be the graph with

$$V(L_{2x}^i) = R_2(X_i') \cup \{x_{i1}, x_{i2}, x_{i3}, y_{12}^i, y_{23}^i, z_i, u_i, x_{r1}, x_{r2}, x_{r3}, y_{12}^r, y_{23}^r \,|\, 1 \leq i \leq r-1\} \quad \text{and}$$

$E(L_{2x}^i)$ consisting of the edges $x_r x_{r1}, x_r x_{r2}, x_r x_{r3}, y_{12}^r x_{r1}, y_{12}^r x_{r2}, y_{23}^r x_{r2}, y_{23}^r x_{r3}$ and the edges indicated below for each $1 \leq i \leq r-1$:

$$x_i \sim x_{i1}, x_{i2}, x_{i3};\; y_{12}^i \sim x_{i1}, x_{i2};\; y_{23}^i \sim x_{i2}, x_{i3};\; z_i \sim x_{i3}, x_{i+1,1};\; u_i \sim x_{i3}, x_{i+1,1}, z_i.$$

It is easy to check that L_{2x}^i is a ladder spanning on $R_2(X_i')$, $4|R_2(X_i')|-1$ vertices from X_i' and $3|R_2(X_i')|-1$ vertices from Y_i'. Similarly, we can find a ladder L_{2y}^i spanning on $R_2(Y_i')$, $4|R_2(Y_i')|-1$ vertices from X_i' and $3|R_2(X_i')|-1$ vertices from X_i'. Clearly, we have $|X_i' \cap V(L_{2x}^i \cup L_{2y}^i)| = |Y_i' \cap V(L_{2x}^i \cup L_{2y}^i)|$.

Assume now that $R_4(X_i') \neq \emptyset$. Then by Claim 3.3.12, we have $|R_4(X_i')| = |R_4(Y_{i+1}')|$. By the similar argument as above, we can find ladder L_{4x}^i, L_{4y}^{i+1} such that $R_4(X_i') \subseteq$

$V(L^i_{4x}), R_4(Y'_{i+1}) \subseteq V(L^{i+1}_{4y})$. Furthermore, we have

$$\begin{aligned} |X'_i \cap V(L^i_{4x})| &= 4|R_4(X'_i)| - 1, \quad |Y'_i \cap V(L^i_{4x})| \;=\; 3|R_4(X'_i)| - 1; \\ |Y'_{i+1} \cap V(L^{i+1}_{4y})| &= 4|R_4(Y'_{i+1})| - 1, \quad |X'_{i+1} \cap V(L^{i+1}_{4y})| \;=\; 3|R_4(Y'_{i+1})| - 1. \end{aligned}$$

Finally, we claim that we can find a ladder L^i_{4xy} between (Y'_i, X'_{i+1}) such that $|Y'_i \cap V(L^i_{4xy})| = |X'_{i+1} \cap V(L^i_{4xy})| = |R_4(Y'_{i+1})|$ and is vertex-disjoint from $L^i_{4x} \cup L^{i+1}_{4y}$. Since $3|R_4(Y'_{i+1})| \leq 3d^2N/5$ and (Y'_i, X'_{i+1}) is 2ε-regular with density at least $d - \varepsilon$ by Claim 3.3.11, a similar argument as in the proof of Lemma 3.3.11, we can find $Y''_i \subseteq Y'_i - V(L^i_{4x})$ and $X''_{i+1} \subseteq X'_{i+1} - V(L^{i+1}_{4y})$ such that (Y''_i, X''_{i+1}) is $(4\varepsilon, d - 5\varepsilon)$-super-regular and $|Y''_i| = |X''_{i+1}|$, and thus is $(4\varepsilon, d/2)$-super-regular (provided that $\varepsilon \leq d/10$). Notice that there are at least $(d - 9\varepsilon)|Y''_i| \geq d|Y''_i|/4$ vertices typical to X''_{i+1}, and there are at least $(d - 9\varepsilon)|X''_{i+1}| \geq d^2|X''_{i+1}|/4$ vertices typical to Y''_i. Applying the Below-up Lemma (Lemma 3.2.2), we can find a ladder L^i_{4xy} within (Y''_i, X''_{i+1}) such that $|Y'_i \cap V(L^i_{4xy})| = |X'_{i+1} \cap V(L^i_{4xy})| = |R_4(Y'_{i+1})|$. It is routine to check that $L^i_{4x}, L^{i+1}_{4y}, L^i_{4xy}$ are the desired ladders. $\square$

For each $i = 1, 2, \cdots, k$, let $X^{**}_i = X'_i - V(L^i_{2x} \cup L^i_{2y} \cup L^i_{4x} \cup L^i_{4xy} \cup L^i_{4y})$ and $Y^{**}_i = Y'_i - V(L^i_{2x} \cup L^i_{2y} \cup L^i_{4x} \cup L^i_{4xy} \cup L^i_{4y})$. Using Lemma 3.2.4, for $i \in \{1, \cdots, k-1\}$, choose $y^*_i \in Y^{**}_i$ such that $|A_{i+1}| \geq dN/4$, where $A_{i+1} := X^{**}_{i+1} \cap \Gamma(y^*_i)$. This is possible, as (Y^{**}_i, X^{**}_{i+1}) is 4ε-regular (applying Slicing lemma based on (Y'_i, X'_{i+1})). Similarly, choose $x^*_{i+1} \in A_{i+1}$ such that $|B_i| \geq dN/4$, where $B_i := Y^{**}_i \cap \Gamma(x^*_{i+1})$. Let $S = \{y^*_i, x^*_{i+1} \,|\, 1 \leq i \leq k-1\}$, and let $X^*_i = X^{**}_i - S$ and $Y^*_i = Y^{**}_i - S$. We have the following holds.

Claim 3.3.15. *For each $i = 1, 2, \cdots, k$, (X^*_i, Y^*_i) is $(4\varepsilon, d/2)$-super-regular such that $|Y^*_1| = |X^*_1| + 1$, $|Y^*_k| = |X^*_k| + 1$, and $|X^*_i| = |Y^*_i|$ for $2 \leq i \leq k-1$.*

Proof. Since $|R_2(X'_i)|, |R_4(Y'_{i+1})| \leq d^2N/5$ for each i, we have $|X^*_i|, |Y^*_i| \geq (1 - \varepsilon - d^2)N - 1$. As $\varepsilon, d \ll 1$, we can assume that $1 - \varepsilon - d^2 - 1/N < 1/2$. Thus, by Slicing lemma based on the 2ε-regular pair (X'_i, Y'_i), we know that (X^*_i, Y^*_i) is 4ε-regular. Recall from Claim 3.3.11 that (X'_i, Y'_i) is $(2\varepsilon, d - 3\varepsilon)$-super-regular, as $4|R_2(X'_i)|, 4|R_4(Y'_{i+1})| < d^2|Y^*_i|$, we know that for each $x \in X^*_i$, $deg(x, Y^*_i) \geq (d - 3\varepsilon - d^2)|Y^*_i| > d|Y^*_i|/2$. Similarly, we have for each $y \in Y_i^*$, $deg(y, X_i^*) \geq d|X_i^*|/2$. Thus (X_i^*, Y_i^*) is $(4\varepsilon, d/2)$-super-regular. Finally, Combining Claims 3.3.11 and 3.3.14, we have $|Y_1^*| = |X_1^*| + 1$, $|Y_k^*| = |X_k^*| + 1$, and $|X_i^*| = |Y_i^*|$ for $2 \leq i \leq k-1$. $\square$

For each $i = 1, 2, \cdots, k-1$, now set $B_{i+1} := Y_i^* \cap \Gamma(x_{i+1}^*)$ and $C_i := X_i^* \cap \Gamma(y_i^*)$. Since (X_i^*, Y_i^*) is $(4\varepsilon, d/2)$-super-regular, we have $|B_{i+1}|, |C_i| \geq d|X_i^*|/2 > d|X_i^*|/4$. Recall from Claim 3.3.10 that $\{X_1, Y_1\} = \{V_{x_1}, V_{x_2}\}$ and $\{X_k, Y_k\} = \{V_{y_1}, V_{y_2}\}$. We assume, w.l.o.g., that $X_1 = V_{x_1}$ and $X_k = V_{y_1}$. Let $A_1 = X_1^* \cap \Gamma(x_1)$, $B_1 = Y_1^* \cap \Gamma(x_2)$, $C_k = X_k^* \cap \Gamma(y_1)$, and $D_k = Y_k^* \cap \Gamma(y_2)$. Since $deg(x_1, X_1) \geq (d-\varepsilon)N$, we have $deg(x_1, X_1^*) \geq (d - \varepsilon - 2\varepsilon - d^2)N \geq d|X_1^*|/4$, and thus $|A_1| \geq d|X_1^*|/4$. Similarly, we have $|B_1|, |C_k|, |D_k| \geq d|X_1^*|/4$. For each $1 \leq i \leq k$, we assume that $L_{2x}^i = a_1^i b_1^i - L_{2x}^i - c_1^i d_1^i$, $L_{2y}^i = a_2^i b_2^i - L_{2y}^i - c_2^i d_2^i$, $L_{4x}^i = a_3^i b_3^i - L_{4x}^i - c_3^i d_3^i$, $L_{4xy}^i = a_4^i b_4^i - L_{4xy}^i - c_4^i d_4^i$, and $L_{4y}^i = a_5^i b_5^i - L_{4y}^i - c_5^i d_5^i$, where $a_j^i, c_j^i \in Y_i' \subseteq Y_i$ for $j = 1, 2, \cdots, 5$. For $j = 1, 2, \cdots, 5$, let $A_j^i = X_i^* \cap \Gamma(a_j^i)$, $C_j^i = X_i^* \cap \Gamma(c_j^i)$, $B_j^i = Y_i^* \cap \Gamma(b_j^i)$, and $D_j^i = Y_i^* \cap \Gamma(d_j^i)$. Since (X_i', Y_i') is $(2\varepsilon, d - 3\varepsilon)$-super-regular, for $j = 1, 2, 3, 5$, we have $|\Gamma(a_j^i, X_i')|, |\Gamma(c_j^i, X_i')| \geq (d - 3\varepsilon)|X_i'|$ and $|\Gamma(b_j^i, Y_i')|, |\Gamma(d_j^i, Y_i')| \geq (d-3\varepsilon)|Y_i'|$. From the proof of Claim 3.3.14, the pair (Y_i'', X_{i+1}'') is $(4\varepsilon, d-5\varepsilon)$-super-regular. Hence, $|\Gamma(a_4^i, X_{i+1}')|, |\Gamma(c_4^i, X_{i+1}')| \geq (d-4\varepsilon)|X_{i+1}'|$ and $|\Gamma(b_4^i, Y_i')|, |\Gamma(d_4^i, Y_i')| \geq (d-4\varepsilon)|Y_i'|$. Thus, we have $|A_j^i|, |B_j^i|, |C_j^i|, |D_j^i| \geq (d-4\varepsilon)|X_i'| - d^2 N \geq d|X_i^*|/4 = d|Y_i^*|/4$.

We now apply the Blow-up lemma on (X_i^*, Y_i^*) to find a spanning ladder L^i with its first and last rungs being contained in $A_i \times B_i$ and $C_i \times D_i$, respectively, and for $j = 1, 2, \cdots, 5$, its $(2j)$-th and $(2j+1)$-th rungs being contained in $A_j^i \times B_j^i$ and $C_j^i \times D_j^i$, respectively. We can then insert L_{2x}^i between the 2nd and 3rd rungs of L^i, L_{2y}^i between the 4th and 5th rungs of L^i, L_{4x}^i between the 6th and 7th rungs of L^i, L_{4xy}^i between the 8th and 9th rungs of L^i, and L_{4y}^i between the 10th and 11th rungs of L^i to obtained a ladder $\mathcal{L}^i$ spanning on $X_i \cup Y_i - S$. Finally, $\mathcal{L}^1 y_1^* x_2^* \mathcal{L}^2 \cdots y_{k-1}^* x_k^* \mathcal{L}^k$ is a spanning ladder of G' with its first rung adjacent to $x_1 x_2$ and its last rung adjacent to $y_1 y_2$.

The proof is then complete.

3.4 Minimum degree condition for spanning generalized Halin graphs

3.4.1 Introduction

A tree with no vertex of degree 2 is called a *homeomorphically irreducible tree (HIT)*, and a spanning tree with no vertex of degree 2 is a *homeomorphically irreducible spanning tree (HIST)*. A *Halin graph*, constructed by Halin in 1971 [27], is a graph formed from a plane embedding of a HIT T of at least 4 vertices by connecting its leaves into a cycle following the cyclic order determined by the embedding. Relaxing the planarity requirement, a generalized Halin graph is obtained from a HIT T of at least 4 vertices by connecting its leaves into a cycle. We call the HIT T the *underlying tree* or *underlying HIST* of the resulting (generalized) Halin graph.

Halin graphs possess many hamiltonicity properties. For examples, Halin graphs are hamiltonian [5], hamiltonian-connected [2] (there is a hamiltonian path between any two distinct vertices), and almost pancyclic [6] (contains all possible cycle lengths with one possible exception of a single even length). Compared to Halin graphs, generalized Halin graphs are less studied. Kaiser et al. in [34] showed that a generalized Halin graph is *prism hamiltonian*; that is, the Cartesian product of a generalized Halin graph and K_2 is hamiltonian. Since a tree with no degree 2 vertices has more leaves than the non-leaves, a generalized Halin graph contains a cycle of length at least half of its order. Also, one can notice that by contracting the non-leaves of the underlying tree of a generalized Halin graph into a singe vertex, a wheel graph is resulted with the contracted vertex as the hub, where a minor of a graph is obtained from the graph by deleting edges/contracting edges, or deleting vertices. Therefore, a generalized Halin graph contains a wheel-minor of order at least half of its order. Although a generalized Halin graph may not be hamiltonian, we conjecture that the lengths of a longest cycle in a generalized Halin graph is large.

Conjecture 3.1. *Let G be an n-vertex generalized Halin graph. Then the length of a longest cycle of G is at least $4n/5$.*

It was shown by Horton, Parker, and Borie [30] that *it is NP-complete to determine* *whether a graph contains a (spanning) Halin graph.* For generalized Halin graphs we obtain the following.

Theorem 3.4.1. *It is NP-hard to determine whether a graph contains a spanning generalized Halin graph.*

A classic theorem of Dirac [19] from 1952 asserts that every graph on n vertices with minimum degree at least $n/2$ is hamiltonian if $n \geq 3$. As a continuous "generalization" of Dirac's Theorem as well as an approach of showing many hamiltonicity properties simultaneously in a graph, the existence of a spanning Halin graph in graphs with large minimum degree was investigated in the previous section, and it was shown that any sufficiently large n-vertex graph with minimum degree at least $(n+1)/2$ contains a spanning Halin graph. We here determine the minimum degree threshold for a graph to contain a spanning generalized Halin graph.

Theorem 3.4.2. *There exists a positive integer n_0 such that every 3-connected graph with $n \geq n_0$ vertices and minimum degree at least $(2n+3)/5$ contains a spanning generalized Halin graph. The result is best possible in the sense of the connectivity and minimum degree constraints.*

Since a generalized Halin graph of order n contains a wheel-minor of order at least $n/2$, we get the following corollary.

Corollary 3.4.1. *There exists a positive integer n_0 such that every 3-connected graph with $n \geq n_0$ vertices and minimum degree at least $(2n+3)/5$ contains a wheel-minor of order at least $n/2$.*

For notational convenience, for a graph T, we denote by $L(T)$ the set of degree 1 vertices of T and $S(T) = V(T) - L(T)$. Also we abbreviate *spanning generalized Halin graph* as *SGHG* in what follows, and denote a generalized Halin graph as $H = T \cup C$, where T is the underlying HIST of H and C is the cycle spanning on $L(T)$.

3.4.2 Proof of Theorem 3.4.1 and the sharpness of Theorem 3.4.2

Proof of Theorem 3.4.1. To show the problem is NP-hard we assume the existence of a polynomial algorithm to test for an SGHG and use it to create a polynomial algorithm to test for a hamiltonian path between two vertices in an arbitrary graph. The decision problem for such hamiltonian paths is a classic NP-complete problem [24].

Let G be a graph and $x, y \in V(G)$. We want to determine whether there exists a hamiltonian path connecting x and y. We first construct a new graph G' and show that G contains a hamiltonian path between x and y if and only if G' contains a HIST (the proof of this part is the same as the proof of Albertson et al. in [1]). Then based on G', we construct a graph G'' and show that G' contains a HIST if and only if G'' contains an SGHG.

Let $\{z_1, z_2, \cdots, z_t\} = V(G) - \{x, y\}$. Then G' is formed by adding new vertices $\{z_1', z_2', \cdots, z_t'\}$ and new edges $\{z_i z_i' : 1 \le i \le t\}$. It is clear that if P is a hamiltonian path between x and y, then $P \cup \{z_i z_i' : 1 \le i \le t\}$ is a HIST of G'. Conversely, let T be a HIST of G'. Since $1 \le d_T(z_i') \le d_{G'}(z_i') = 1$, we get $d_T(z_i') = 1$ for each i. Since $N_{G'}(z_i') = \{z_i\}$ and T is a HIST, we have $d_T(z_i) \ge 3$. Hence $T - \{z_1', z_2', \cdots, z_t'\}$ is a tree with leaves possibly in $\{x, y\}$. Since each tree has at least 2 leaves and a tree with exactly two leaves is a path, we conclude that $T - \{z_1', z_2', \cdots, z_t'\}$ is a path between x and y.

Then based on G', we construct a graph G''. First, for each i with $1 \le i \le t$, we add new vertices $z_{i1}', z_{i2}', z_{i3}'$ and new edges $z_i' z_{i1}', z_i' z_{i2}', z_i' z_{i3}', z_{i1}' z_{i2}', z_{i2}' z_{i3}'$. Then we connect all vertices in $\{x, y\} \cup \{z_{i1}', z_{i2}', z_{i3}' : 1 \le i \le t\}$ into a cycle C'' such that $\{z_{i1}' z_{i2}', z_{i2}' z_{i3}' : 1 \le i \le t\} \subseteq E(C'')$. If T' is a HIST of G', then $T'' := T' \cup \{z_i' z_{i1}', z_i' z_{i2}', z_i' z_{i3}' : 1 \le i \le t\}$ is a HIST of G'' and $T'' \cup C''$ is an SGHG of G''. Conversely, suppose $H = T \cup C$ is an SGHG of G''. We claim that $C = C''$. This in turn gives that $T = T''$ and therefore $T'' - \{z_{i1}', z_{i2}', z_{i3}' : 1 \le i \le t\}$ is a HIST of G'. To show that $C = C''$, we first show that $z_{i2}' \in L(T)$ for each i. Suppose on the contrary and assume, without loss of generality, that $z_{12}' \in S(T)$. Then as $N_{G''}(z_{12}') = \{z_1', z_{11}', z_{13}'\}$, we get $\{z_{12}' z_1', z_{12}' z_{11}', z_{12}' z_{13}'\} \subseteq E(T)$. Since T is acyclic, $z_{11}' z_1', z_{13}' z_1' \notin E(T)$. This in turn shows that $\{z_1', z_{11}', z_{13}'\} \subseteq L(T)$. However, $\{z_{12}' z_1', z_{12}' z_{11}', z_{12}' z_{13}'\}$ forms a component of T, showing a contradiction. Then we show that

$z'_{i1}, z'_{i3} \in L(T)$ for each i. Suppose on the contrary and assume, without loss of generality, that $z'_{11} \in S(T)$. By the previous argument, we have $z'_{12} \in L(T)$. Then $z'_1, z'_{13} \in L(T)$ as z'_{12} is on C and z'_1 and z'_{13} are the only two neighbors of z'_{12} which can be on the cycle C. As $d_{G''}(z'_{11}) = 3$ and $\{z'_{12}, z'_1\} \subseteq N_{G''}(z'_{11})$, $z'_{11}z'_{12}, z'_{11}z'_1 \in E(T)$. Since $z'_{12} \in L(T)$ and $z'_1, z'_{13} \in L(T)$, we get $z'_{12}z'_{13}, z'_{12}z'_1, z'_1z'_{13} \notin E(T)$. Since $d_{G''}(z'_{12}) = d_{G''}(z'_{13}) = 3$, we have $z'_{12}z'_{13}, z'_{12}z'_1, z'_1z'_{13} \in E(C)$. However, $z'_{12}z'_{13}, z'_{12}z'_1, z'_1z'_{13}$ forms a triangle but $|V(C)| \geq 4$, showing a contradiction. So we have shown that $\{z'_{i1}, z'_{i2}, z'_{i3} : 1 \leq i \leq t\} \subseteq L(T)$. This indicates that in the tree $T - \{z'_{i1}, z'_{i2}, z'_{i3} : 1 \leq i \leq t\}$, each vertex z'_i has degree 1 and no vertices of degree 2. Hence $T - \{z'_{i1}, z'_{i2}, z'_{i3} : 1 \leq i \leq t\}$ is a HIST of G'.

Combining the arguments in the two paragraphs above, we see that G has a hamiltonian path between x and y if and only if G'' has an SGHG. Hence a polynomial SGHG-tester becomes a polynomial path-tester. $\square$

Since a generalized Halin graph is 3-connected, the connectivity requirement in Theorem 3.4.2 is necessary. To show that the minimum degree requirement is best possible, we show the following proposition.

Proposition 3.4.1. *Let $G(A, B) = K_{a,b}$ be a complete bipartite graph with $|A| = a$ and $|B| = b$. Then $G(A, B)$ has no HIST T with $|L(T) \cap A| = |L(T) \cap B|$ if $b > \frac{3(a-1)}{2}$.*

If a bipartite graph $G(A, B)$ contains an SGHG $H = T \cup C$, then $|L(T) \cap A| = |L(T) \cap B|$. Thus, by Proposition 3.4.1, it is easy to see that the complete bipartite graphs $K_{a,b}$ with $b = \frac{3a-1}{2}$ when a is odd and $b = \frac{3a-2}{2}$ when a is even does not have an SGHG. Let $n = a + b$. By direct computation, we get $\delta(K_{a,b}) = \frac{2n+1}{5}$ when $b = \frac{3a-1}{2}$ and $\delta(K_{a,b}) = \frac{2n+2}{5}$ when $b = \frac{3a-2}{2}$. We now prove Proposition 3.4.1.

Proof of Proposition 3.4.1. Suppose on the contrary that $G(A, B)$ contains a HIST T such that $|L(T) \cap A| = |L(T) \cap B|$. Then

$$\begin{aligned} |S(T) \cap B| - |S(T) \cap A| &= |B| - |L(T) \cap B| - (|A| - |L(T) \cap A)| \\ &= |B| - |A| > \frac{3(a-1)}{2} - a = \frac{a-3}{2}. \end{aligned}$$

Since $G(A,B)$ is bipartite and T is a HIST of $G(A,B)$, we have $|S(T)\cap A|\geq 1$. Thus, from the inequalities above, we obtain $|S(T)\cap B|>(a-1)/2$. Since T is a HIST, we have $d_T(y)\geq 3$ for each $y\in S(T)\cap B$. Let $E_B=\{e\in E(T)\ :\ e \text{ is incident to a vertex in } S(T)\cap B\}$. Denote by T' the subgraph of T induced on E_B. Notice that T' is a forest of at least $3|S(T)\cap B|$ edges. Hence T' has at least $3|S(T)\cap B|+1$ vertices. As T' is a bipartite graph with one partite set as $S(T)\cap B$, and another as a subset of A, we conclude that $|V(T)\cap A|=|V(T)|-|S(T)\cap B|\geq 2|S(T)\cap B|+1$. Since $|S(T)\cap B|>(a-1)/2$, we then have $|V(T)\cap A|>a$. This gives a contradiction to the assumption $|A|=a$. $\square$

3.4.3 Proof of Theorem 3.4.2

Given $0\leq\beta\ll\alpha\ll 1$, we define the two extremal cases with parameters α and β as follows.

Extremal Case 1. There exists a partition of $V(G)$ into V_1 and V_2 such that $|V_i|\geq (2/5-4\beta)n$ and $d(V_1,V_2)<\alpha$. Furthermore, $deg(v_1,V_2)\leq 2\beta n$ for each $v_1\in V_1$.

Extremal Case 2. There exists a partition of $V(G)$ into V_1 and V_2 such that $|V_1|> (3/5-\alpha)n$ and $d(V_1,V_2)\geq 1-3\alpha$. Furthermore, $deg(v_1,V_2)\geq (2n+3)/5-2\beta n$ for each $v_1\in V_1$.

Then Theorem 3.4.2 is shown through the following three theorems.

Theorem 3.4.3 (**Non-extremal Case**). *For every $\alpha>0$, there exists $\beta>0$ and a positive integer n_0 such that if G is a 3-connected graph with $n\geq n_0$ vertices and $\delta(G)\geq(2n+3)/5-\beta n$, then G contains an SGHG or G is in one of the two extremal cases.*

Theorem 3.4.4 (**Extremal Case 1**). *Suppose that $0<\beta\ll\alpha\ll 1$ and n is a sufficiently large integer. Let G be a 3-connected graph on n vertices with $\delta(G)\geq(2n+3)/5$. If G is in Extremal Case 1, then G contains an SGHG.*

Theorem 3.4.5 (**Extremal Case 2**). *Suppose that $0<\beta\ll\alpha\ll 1$ and n is a sufficiently large integer. Let G be a 3-connected graph on n vertices with $\delta(G)\geq(2n+3)/5$. If G is in Extremal Case 2, then G contains an SGHG.*

We show Theorems 3.4.3-3.4.5 separately in the following three subsections.

3.4.3.1 Proof of Theorem 3.4.3 We fix the following sequence of parameters,

$$0 < \varepsilon \ll d \ll \beta \ll \alpha < 1, \tag{3.5}$$

and specify their dependence as the proof proceeds. We let $\beta \ll \alpha$ be the same α and β as defined in the two extremal cases. Then we choose $d \ll \beta$. Finally we choose

$$\varepsilon = \min\left\{\frac{1}{4}\varepsilon\left(\frac{d}{2}, \left\lceil\frac{2}{d^3}\right\rceil, 2, \frac{d}{2}\right), \frac{1}{9}\varepsilon\left(\frac{d}{2}, \left\lceil\frac{3}{d^3}\right\rceil, 3\right), \frac{1}{4}\varepsilon\left(\frac{d}{2}, 2, 2, \frac{d}{2}\right)\right\}, \tag{3.6}$$

where $\varepsilon\left(\frac{d}{2}, \left\lceil\frac{3}{d^3}\right\rceil, 3\right)$ follows from the definition of the ε in the weak version of the Blow-up lemma and $\varepsilon\left(\frac{d}{2}, \left\lceil\frac{2}{d^3}\right\rceil, 2, \frac{d}{2}\right)$ and $\varepsilon\left(\frac{d}{2}, 2, 2, \frac{d}{2}\right)$ follow from the definition of the ε in the strengthened version of the Blow-up lemma. Choose n to be sufficiently large. In the proof, we omit non-necessary ceiling and floor functions.

Let G be a graph of order n such that $\delta(G) \geq (2n+3)/5 - \beta n$ and suppose that G is not in any of the two extremal cases. Applying the regularity lemma to G with parameters ε and d, we obtain a partition of $V(G)$ into $l+1$ clusters $V_0, V_1, \cdots, V_l$ for some $l \leq M = M(\varepsilon)$, and a spanning subgraph G' of G with all described properties in Lemma 3.2.1 (the Regularity lemma). In particular, for all $v \in V$,

$$\begin{aligned} deg_{G'}(v) &> deg_G(v) - (d+\varepsilon)n \geq (2/5 - \beta - d - \varepsilon)n \\ &\geq (2/5 - 2\beta)n \text{ (provided that } \varepsilon + d \leq \beta), \end{aligned} \tag{3.7}$$

and

$$e(G') \geq e(G) - \frac{(d+\varepsilon)}{2}n^2 \geq e(G) - dn^2,$$

by using $\varepsilon < d$.

We further assume that $l = 2k$ is even; otherwise, we eliminate the last cluster V_l by removing all the vertices in this cluster to V_0. As a result, $|V_0| \le 2\varepsilon n$ and

$$(1-2\varepsilon)n \le lN = 2kN \le n, \tag{3.8}$$

here we assume that $|V_i| = N$ for $i \ge 1$.

For each pair i and j with $1 \le i < j \le l$, we write $V_i \sim V_j$ if $d(V_i, V_j) \ge d$. We now consider the reduced graph G_r, whose vertex set is $\{1, 2, \cdots, l\}$, and two vertices i and j are adjacent if and only if $V_i \sim V_j$. We claim that $\delta(G_r) \ge (2/5 - 2\beta)l$. Suppose not, and let $i_0 \in V(G_r)$ such that $deg(i_0, V(G_r)) < (2/5 - 2\beta)l$. Then, for the corresponding cluster V_{i_0} we have $e_{G'}(V_{i_0}, V(G') - V_{i_0}) < |V_{i_0}|(2/5 - 2\beta)lN$. On the other hand, by using (3.7), we have $e_{G'}(V_{i_0}, V(G') - V_{i_0}) \ge |V_{i_0}|(2/5 - 2\beta)n$. As $lN \le n$ from (3.8), we obtain a contradiction. The rest of the proof consists of the following steps.

Step 1. Show that G_r contains a dominating cycle C and there is a $\wedge$-matching in G_r with all vertices in $V(G_r) - V(C)$ as its center. We distinguish two cases in Step 1, and each of the other steps will be separated into two cases correspondingly.

Case A. $C = X_1Y_1X_2Y_2 \cdots X_tY_t$ is an even cycle for some $t \le k$.

Case B. $C = X_0X_1Y_1X_2Y_2 \cdots X_tY_t$ is an odd cycle for some $t < k$.

Notice that in Case B there is at least one vertex in $V(G_r) - V(C)$ by the assumption that $|V(G_r)| = l$ is even. In what follows, if we denote a vertex of G_r by a capital letter, it means either a vertex of G_r or the corresponding cluster in G, but the exact meaning will be clear from the context. For $1 \le i \le t$, we call X_i and Y_i the partners of each other, and write as $P(X_i) = Y_i$ and $P(Y_i) = X_i$.

Since C is not necessarily hamiltonian in G_r, we need to take care of the clusters of G which are not represented on C. For each vertex $F \in V(G_r) - V(C)$, we partition the corresponding cluster F into two small clusters F_1 and F_2 such that $-1 \le |F_1| - |F_2| \le 1$. We call each F_1 and F_2 a *half-cluster*. Then we group all the original clusters and the partitioned clusters into pairs (A, B) and triples (C, D, F) with F as a half-cluster such that each pair (A, B) and (C, D) is still ε-regular with density d and the pair (D, F) is 2.1ε-regular with density $d - \varepsilon$. Having the cluster groups like this, in the end, we will find "small" HITs within each pair (A, B) or among each triple (C, D, F).

Step 2. For each $1 \leq i \leq t-1$, initiate two independent edges connecting Y_i and X_{i+1}. In Case A, also initiate two independent edges connecting X_1 and Y_t; and in Case B, initiate two independent edges connecting the clusters in each pair of X_0 and X_1, and X_0 and Y_t.

Step 3. Make each regular pair in the new grouped pairs and triples given in Step 1 super-regular.

Step 4. Construct HITs covering all vertices in V_0 using vertices from the super-regular pairs obtained from Step 3, and obtain new super-regular pairs.

Step 5. Apply the Blow-up lemma to find a HIT between a super-regular pair resulted from Step 4 or among a triple (A, B, F), where both (A, F) and (A, B) are super-regular pairs resulted from Step 4, and F is a half cluster. In addition, in the construction, for each triple (A, B, F), we require the HIT to use as many vertices as possible from F as non-leaves.

Step 6. Apply the Blow-up Lemma again on the regular-pairs induced on the leaves of each HIT obtained in Step 5 to find two disjoint paths covering all the leaves. Then connect all the HITs into a HIST of G using edges guaranteed by the regularity and connect the disjoint paths into a cycle using the edges initiated in Step 2. The union of the HIST and the cycle gives an SGHG of G.

We now give details of each step. The assumption that G is not in any of the two extremal cases leads to the following claim, which will be used in Step 1.

Claim 3.4.1. *Each of the following holds for G_r.*

(a) *G_r contains no cut-vertex set of size at most βl;*

(b) *G_r contains no independent set of size more than $(3/5 - \alpha/2)l$.*

Proof. (a) Suppose instead that G_r contains a vertex-cut W of size at most βl. As $\delta(G_r) \geq (2/5 - 2\beta)l$, then each component of $G_r - W$ has at least $(2/5 - 3\beta)l$ vertices. Let U be the vertex set of one of the components of $G_r - W$, $A = \bigcup_{i \in U} V_i$, and $B = V(G) - A$.

We see that $|A|, |B| \geq (2/5 - 3\beta)lN \geq (2/5 - 4\beta)n$, and since $e(G) \leq e(G') + dn^2$, we have

$$\begin{aligned}
e_G(A,B) &\leq e_{G'}(A,B) + dn^2 \leq |W||A| + dn^2 \\
&\leq \beta lN(3/5+3\beta)lN + dn^2 \leq (3\beta/5 + 3\beta^2 + d)n^2 \quad (\text{as}\,|A| \leq (3/5+3\beta)lN \,\text{and}\, ln \leq n) \\
&\leq \frac{25}{3}(3\beta/5 + 3\beta^2 + d)|A||B| \quad (\text{since}\,|A||B| \geq 3n^2/25) \\
&< \alpha|A||B| \quad (\text{provided that}\,\frac{25}{3}(3\beta/5 + 3\beta^2 + d) < \alpha).
\end{aligned}$$

This shows that $d(A,B) < \alpha$. Since $deg_{G_r}(u, V(G_r) - U) = deg_{G_r}(u, W) \leq \beta l$ for each $u \in U$, we see that $deg_G(a, B) \leq \beta lN + (d+\varepsilon)n \leq 2\beta n$ for each $a \in A$ provided that $d + \varepsilon \leq \beta$. However, the above argument shows that G is in Extremal Case 1, showing a contradiction.

(b) Suppose instead that G_r contains an independent set U of size larger than $(3/5 - \alpha/2)l$. Let $U' = V(G_r) - U$, $A = \bigcup_{i \in U} V_i$, and $B = V(G) - A$. Then $|A| \geq (3/5 - \alpha/2)lN \geq (3/5 - \alpha)n$. For each vertex $v \in A$, since $deg_G(v, A) \leq deg_{G'}(v, A) + (d+\varepsilon)n \leq \beta n$, we have $deg_G(v, B) \geq (2n+3)/5 - \beta n - \beta n \geq (2n+3)/5 - 2\beta n$. This gives that

$$d(A,B) \geq \frac{(2/5 - 2\beta)n}{|B|} \geq \frac{(2/5-2\beta)n}{(2/5+\alpha)n} \geq 1 - 3\alpha,$$

provided that $\beta \leq \alpha/10 + 3\alpha^2/2$. We see that G is in Extremal Case 2. □

Step 1. Show that G_r contains a dominating cycle C, and there is a $\wedge$-matching in G_r with all vertices in $V(G_r) - V(C)$ as its center.

We need some results on longest cycles and paths as follows.

Lemma 3.4.1 ([44])**.** *Let G be a 2-connected graph on n vertices with $\delta(G) \geq (n+2)/3$. Then every longest cycle in G is a dominating cycle.*

Lemma 3.4.2 ([3])**.** *Let G be a 2-connected graph on n vertices with $\delta(G) \geq (n+2)/3$. Then G contains a cycle of length at least* $\min\{n, n + \delta(G) - \alpha(G)\}$*, where $\alpha(G)$ is the size of a largest independent set in G.*

Lemma 3.4.3 ([38])**.** *If G is a 3-connected graph of order n such that the degree sum of any* *four independent vertices is at least 3n/2+1, then the number of vertices on a longest path and that on a longest cycle differs at most by 1.*

By (a) of Claim 3.4.1, G_r is βl-connected. Since $n = Nl + |V_0| \leq (l+2)\varepsilon n$, we get $l \geq 1/\varepsilon - 2$. Since $1/\varepsilon - 2 \geq 3/\beta$ (provided that $\beta \geq 3\varepsilon/(1-2\varepsilon)$), we then have $\beta l \geq 3$. So G_r is 3-connected. By Claim 3.4.1 (b), G_r has no independent set of size more than $(3/5 - \alpha/2)l$. Notice that $\delta(G_r) \geq (2/5 - 2\beta)l > (l+2)/3$. Applying Lemma 3.4.1 and Lemma 3.4.2 on G_r, we see that there is a cycle C in G_r which is longest, dominating, and has length at least $(4/5 + \alpha/2 - 2\beta)l$. Let $\mathcal{W} = V(G_r) - V(C)$. In Case B, we order and label the vertices of C such that X_0 is adjacent to a vertex, say $Y_0 \in \mathcal{W}$ (recall that $\mathcal{W} \neq \emptyset$ in this case). We fix (X_0, Y_0) as a pair at the first place ($X_0Y_0 \in E(G_r)$, as cluster in G, (X_0, Y_0) is an ε-regular pair with density d). Let

$$\mathcal{W}' = \begin{cases} \mathcal{W}, & \text{if in Case A;} \\ \mathcal{W} - \{Y_0\}, & \text{if in Case B.} \end{cases}$$

We have $|\mathcal{W}'| \leq (1/5 - \alpha/2 + 2\beta)l$ if in Case A and $|\mathcal{W}'| \leq (1/5 - \alpha/2 + 2\beta)l - 1$ if in Case B. So $2|\mathcal{W}'| \leq (2/5 - \alpha + 4\beta)l < (2/5 - 2\beta)l$ (provided that $\beta < \alpha/6$) if in Case A and $2|\mathcal{W}'| \leq (2/5 - \alpha + 4\beta)l - 2 < (2/5 - 2\beta)l - 1$ (provided that $\beta < \alpha/6$) if in Case B. Thus there is a $\wedge$-matching centered in all vertices in $\mathcal{W}'$; furthermore, if in Case B, we can choose the matching such that X_0 is not covered by it. Let $M_\wedge$ be such a matching. For a vertex $X \in \mathcal{W}'$, denote by $M_\wedge(X)$ the two vertices from $V(C)$ to which X is adjacent in $M_\wedge$. Then we have two facts about vertices in $M_\wedge(X)$.

Fact 3.4.1. *Let $X \in \mathcal{W}'$. Then the two vertices in $M_\wedge(X)$ are non-consecutive on C. (By the assumption that C is longest.)*

Fact 3.4.2. *Let $X \neq Y \in \mathcal{W}'$. Then no two vertices from $M_\wedge(X) \cup M_\wedge(Y)$ are adjacent on C. (By applying Lemma 3.4.3.)*

For a complete bipartite graph, if it contains an SGHG, then the ratio of the cardinalities of the two partite sets should be greater than 2/3 as shown in Proposition 3.4.1. Since a longest dominating cycle in G_r is not necessarily hamiltonian, we need to take care of the clusters of G which are not represented by the vertices on C. One possible consideration is that for each $F \in V(G_r) - V(C)$, suppose F is adjacent to $A \in V(C)$, recall $P(A)$ is the partner of A. Then as clusters, we consider the bipartite graph of G with partite sets A and $P(A) \cup F$. However, $|A|/|P(A) \cup F|$ is about $1/2$, which is less than $2/3$. For this reason, we partition $F \in V(G_r) - V(C)$ into two parts to attain the right ratio in the corresponding bipartite graphs. Suppose $M_\wedge(F) = \{D_1, D_2\} \subseteq V(C)$. As a cluster of G, we partition F into F_1 and F_2 arbitrarily such that

$$|F_1| = \left\lfloor \frac{|F|}{2} \right\rfloor = \left\lfloor \frac{N}{2} \right\rfloor \quad \text{and} \quad |F_2| = \left\lceil \frac{|F|}{2} \right\rceil = \left\lceil \frac{N}{2} \right\rceil .$$

We call each F_i a half-cluster of G. Then we create two pairs (D_i, F_i), and call D_i the dominator of F_i, and F_i the follower of D_i, and (D_i, F_i) a DF-pair, for $i = 1, 2$. We have the following fact about a DF-pair.

Fact 3.4.3. *Each DF-pair (D, F) is 2.1ε-regular with density at least $d - \varepsilon$. (By Slicing lemma.)*

Also, by Fact 3.4.1 and Fact 3.4.2, if $D \in V(C)$ is a dominator, then $P(D)$, the partner of D, is not a dominator for any followers. As $X_0 \notin V(\mathcal{W}')$, we know that X_0 is not a dominator for any half-clusters. We group the clusters and half-clusters of G into *H-pairs* and *H-triples* in a way below. For each pair (X_i, Y_i) on C, if $\{X_i, Y_i\} \cap V(M_\wedge) = \emptyset$, we take (X_i, Y_i) as an H-pair. Otherwise, $|\{X_i, Y_i\} \cap V(M_\wedge)| = 1$ by Fact 3.4.1 and Fact 3.4.2. Since there is no difference for the proof for the case that $X_i \in V(M_\wedge)$ or the case that $Y_i \in V(M_\wedge)$, throughout the remaining proof, we always assume that $Y_i \in V(M_\wedge)$ if $\{X_i, Y_i\} \cap V(M_\wedge) \neq \emptyset$. In this case, there is a unique half-cluster F with Y_i as its dominator. Then we take (X_i, Y_i, F) as an H-triple. We assign (X_0, Y_0) as an H-pair.

Step 2. Initiating connecting edges.

Given an ε-regular pair (A, B) of density d and a subset $B' \subseteq B$, we say a vertex $a \in A$ *typical* to B' if $deg(a, B') \geq (d - \varepsilon)|B'|$. Then by the regularity of (A, B), the fact below holds.

Fact 3.4.4. *If (A, B) is an ε-regular pair, then at most $\varepsilon|A|$ vertices of A are not typical to $B' \subseteq B$ whenever $|B'| > \varepsilon|B|$.*

For each $1 \le i \le t-1$, choose $y_i^* \in Y_i$ typical to both X_i and X_{i+1}, and $y_i^{**} \in Y_i$ typical to each of X_i, X_{i+1}, and $\Gamma(y_i^*, X_i)$. Correspondingly, choose $x_{i+1}^* \in \Gamma(y_i^*, X_{i+1})$ typical to Y_{i+1}, and $x_{i+1}^{**} \in \Gamma(y_i^{**}, X_{i+1})$ typical to both Y_{i+1} and $\Gamma(x_{i+1}^*, Y_{i+1})$. For $i = t$, we choose y_t^* and y_t^{**} the same way as for $i < t$, but if in Case A, choose $x_1^* \in \Gamma(y_t^{**}, X_1)$ typical to Y_1, and $x_1^{**} \in \Gamma(y_t^*, X_1)$ typical to both Y_1 and $\Gamma(x_1^*, Y_1)$; and if in Case B, choose $x_0^* \in \Gamma(y_t^{**}, X_0)$ typical to X_1, and $x_0^{**} \in \Gamma(y_t^*, X_0)$ typical to both X_1 and $\Gamma(x_0^*, X_1)$. Furthermore, in Case B, we choose $y_{t+1}^* \in X_0$ typical to both Y_0 and X_1, and $y_{t+1}^{**} \in X_0$ typical to each of Y_0, X_1, and $\Gamma(y_{t+1}^*, Y_0)$. Correspondingly, choose $x_1^* \in \Gamma(y_{t+1}^*, X_1)$ typical to Y_1 and $x_1^{**} \in \Gamma(y_{t+1}^{**}, X_1)$ typical to both Y_1 and $\Gamma(x_1^*, Y_1)$. Additionally, we choose $y_0^* \in \Gamma(y_{t+1}^*, Y_0)$ such that y_0^* is typical to X_0, and choose $y_0^{**} \in \Gamma(y_{t+1}^{**}, Y_0)$ such that y_0^{**} is typical to X_0. Notice that by the choice of these vertices above, we have the following.

$$\begin{cases} y_i^* x_{i+1}^*, y_i^{**} x_{i+1}^{**} \in E(G), & \text{for } 1 \le i \le t-1; \\ x_1^* y_t^{**}, x_1^{**} y_t^* \in E(G), & \text{in Case A;} \\ x_0^* y_t^{**}, x_0^{**} y_t^*, x_1^* y_{t+1}^*, x_1^{**} y_{t+1}^{**}, y_0^* y_{t+1}^*, y_0^{**} y_{t+1}^{**} \in E(G), & \text{in Case B.} \end{cases}$$

By Fact 3.4.4, for each $0 \le i \le t$, we have $|\Gamma(x_i^*, Y_i) \cap \Gamma(x_i^{**}, Y_i)|, |\Gamma(y_i^*, X_i) \cap \Gamma(y_i^{**}, X_i)| \ge (d-\varepsilon)^2 N$, and $|\Gamma(y_{t+1}^*, Y_0) \cap \Gamma(y_{t+1}^{**}, Y_0)| \ge (d-\varepsilon)^2 N$.

Step 3. Super-regularizing the regular pairs in each H-pair and H-triple given in Step 1.

For each $0 \le i \le t$, if (X_i, Y_i) is an H-pair, let

$$X_i' = \{x \in X_i : deg(x, Y_i) \ge (d-\varepsilon)N\} \quad \text{and} \quad Y_i' = \{y \in Y_i : deg(y, X_i) \ge (d-\varepsilon)N\}.$$

By Fact 3.4.4, we have $|X_i'|, |Y_i'| \ge (1-\varepsilon)N$. Recall that $x_i^*, x_i^{**} \in X_i$ and $y_i^*, y_i^{**} \in Y_i$ are the initiated vertices in Step 2. For $1 \le i \le t$, if $|X_i' - \{x_i^*, x_i^{**}\}| \ne |Y_i' - \{y_i^*, y_i^{**}\}|$, say $|X_i' - \{x_i^*, x_i^{**}\}| > |Y_i' - \{y_i^*, y_i^{**}\}|$, we then remove $|X_i' - \{x_i^*, x_i^{**}\}| - |Y_i' - \{y_i^*, y_i^{**}\}|$ vertices out from $X_i' - \{x_i^*, x_i^{**}\}$, and denote the remaining set still as X_i'. Denote $Y_i' - \{y_i^*, y_i^{**}\}$ still as Y_i'. We see that $|X_i'| = |Y_i'|$. As $|Y_i'| \geq (1-\varepsilon)N$ (to be precise, the lower bound should be $(1-\varepsilon)N - 2$, however, the constant 2 can be made vanished by adjusting the ε factor, we ignore the slight different of the ε-factor here), we have that $|X_i \cup Y_i - (X_i' \cup Y_i')| \leq 2\varepsilon N$. For $i = 0$, if $|X_i' - \{x_i^*, x_i^{**}, y_{t+1}^*, y_{t+1}^{**}\}| \neq |Y_i' - \{y_i^*, y_i^{**}\}|$, say $|X_i' - \{x_i^*, x_i^{**}, y_{t+1}^*, y_{t+1}^{**}\}| > |Y_i' - \{y_i^*, y_i^{**}\}|$, then we remove $|X_i' - \{x_i^*, x_i^{**}, y_{t+1}^*, y_{t+1}^{**}\}| - |Y_i' - \{y_i^*, y_i^{**}\}|$ vertices out from $X_i' - \{x_i^*, x_i^{**}, y_{t+1}^*, y_{t+1}^{**}\}$ and denote the remaining set still as X_i'. Denote $Y_i' - \{y_i^*, y_i^{**}\}$ still as Y_i'. We see that $|X_i'| = |Y_i'|$. We call the resulting H-pairs *supper-regularized H-pairs*. By Slicing lemma (Lemma 3.2.5) and the definitions of X_i', Y_i', we see that

Fact 3.4.5. *Each supper-regularized H-pair (X_i', Y_i') is a $(2\varepsilon, d - 2\varepsilon)$-super-regular pair.*

For each H-triple (X_i, Y_i, F), by Fact 3.4.3, (Y_i, F) is 2.1ε-regular with density at least $d - \varepsilon$. Let

$$\begin{aligned}
X_i' &= \{x \in X_i : deg(x, Y_i) \geq (d-\varepsilon)N\}, \\
Y_i' &= \{y \in Y_i : deg(y, X_i) \geq (d-\varepsilon)N, deg(y, F) \geq (d - 3.1\varepsilon)|F|\}, \text{and} \\
F' &= \{f \in F : deg(f, Y_i) \geq (d - 3.1\varepsilon)N\}.
\end{aligned}$$

Recall that $x_i^*, x_i^{**} \in X_i$ and $y_i^*, y_i^{**} \in Y_i$ are the initiated vertices in Step 2. We remove x_i^*, x_i^{**} out from X_i', and remove y_i^*, y_i^{**} out from Y_i'. Still denote the resulted clusters as X_i' and Y_i', respectively. Remove $\lceil d^3 N \rceil$ vertices out from F, which consists of all vertices in $F - F'$ and any $\lceil d^3 N \rceil - |F - F'|$ vertices from F' (we need to increase the ratio $|Y_i'|/|X_i' \cup F'|$ a little as later on we may use vertices in Y_i' in constructing HITs covering vertices in V_0). Denote the resulting set still by F'. Then we see that $|X_i'| \geq (1-\varepsilon)N$, $|Y_i'| \geq (1 - 3.1\varepsilon)N$, and $|F'| \geq (1 - 2.1\varepsilon)|F| - d^3 N \geq (1 - 2.1\varepsilon - 2d^3)|F|$. We call the resulted H-triples *supper-regularized H-triples*. By the Slicing Lemma and the definitions above, the following is true.

Fact 3.4.6. *For each super-regularized H-triple (X_i', Y_i', F'), (X_i', Y_i') is $(2\varepsilon, d - 3.1\varepsilon)$- super-regular, and (Y_i', F') is $(4.2\varepsilon, d - 3.1\varepsilon - 2d^3)$-super-regular.*

Let V_0^1 be the union of the set of vertices from each $(X_i \cup Y_i - (X_i' \cup Y_i')) - \{x_i^*, x_i^{**}, y_i^*, y_i^{**}\} - \{y_{t+1}^*, y_{t+1}^{**}\}$ ($\{y_{t+1}^*, y_{t+1}^{**}\}$ exists only if in Case B), where (X_i, Y_i) is an H-pair, and let V_0^2 be the union of the set of vertices from each $(X_i \cup Y_i \cup F - (X_i' \cup Y_i' \cup F')) - \{x_i^*, x_i^{**}, y_i^*, y_i^{**}\}$, where (X_i, Y_i, F) is an H-triple. Notice that for each H-pair (X_i, Y_i), we have $|X_i \cup Y_i - (X_i' \cup Y_i')| \leq 2\varepsilon N$; and for each H-triple (X_i, Y_i, F), we have $|X_i - X_i'| \leq \varepsilon N$, $|Y_i - Y_i'| \leq (\varepsilon + 2.1\varepsilon)N$, and $|F - F'| \leq d^3 N$. Hence by using the facts that $|\mathcal{W}'| \leq (1/5 - \alpha/2 + 2\beta)l$, $t = l/2$, and $Nl \leq n$ from inequality (3.8), we get

$$|V_0^1| + |V_0^2| \leq 2\varepsilon Nl/2 + 2(1/5 - \alpha + 2\beta)l(d^3N + 2.1\varepsilon N) \leq 2d^3Nl/5 + 2\varepsilon Nl \leq 2d^3n/5 + 2\varepsilon n.$$

Let $V_0' = V_0 \cup V_0^1 \cup V_0^2$. Then

$$|V_0'| \leq 2\varepsilon n + 2d^3n/5 + 2\varepsilon n \leq d^3n/2 \quad \text{(provided that } \varepsilon \leq d^3/40). \tag{3.9}$$

Step 4. Construct small HITs covering all vertices in V_0'.

Consider a vertex $x \in V_0'$ and a cluster or a half-cluster A, we say that x is *adjacent to* A, denoted by $x \sim A$, if $deg(x, A) \geq (d - \varepsilon)|A|$. We call A the *partner* of x.

Claim 3.4.2. *For each vertex $x \in V_0'$, there is a cluster or a half-cluster A such that $x \sim A$, where A is not a dominator, and we can assign all vertices in V_0' to their partners which are not dominators such that each of the cluster or half-cluster is used by at most $\frac{d^2N}{20}$ vertices from V_0'.*

Proof. Suppose we have found partners for the first $m < d^3n/2$ (recall that $|V_0'| \leq d^3n/2$) vertices of V_0' such that no cluster or half-cluster is used by at most $\frac{d^2N}{20}$ vertices. Let Ω be the set of all clusters and half-clusters that are used exactly by $\frac{d^2N}{20}$ vertices. Then

$$\begin{aligned} \frac{d^2N}{20}|\Omega| &\leq m < d^3n/2 \leq d^3(2kN + 2\varepsilon n)/2 \\ &\leq d^3kN + d^3\frac{2kN}{1 - 2\varepsilon}, \end{aligned}$$

by inequality (3.8). Therefore,

$$
\begin{aligned}
|\Omega| &\leq \frac{20d^3k}{d^2} + \frac{20d^3l}{d^2(1-2\varepsilon)} \\
&\leq 10dl + 40dl \text{ (provided that } 1-2\varepsilon \geq 1/2 \text{)} \\
&\leq \beta l \text{ (provided that } 50d \leq \beta \text{).}
\end{aligned}
$$

Consider now a vertex $v \in V_0'$ not having a partner found so far. Let $\mathcal{U}$ be the set of all non-dominator clusters and half-clusters adjacent to v not contained in Ω. We claim that $|\mathcal{U}| \geq (\alpha - 7\beta)l$. To see this, we first observe that any vertex $v \in V_0'$ is adjacent to at least $(\alpha - 6\beta)l$ non-dominator clusters and half-clusters. For instead, as v may adjacent to $2|\mathcal{W}'|$ dominators, vertices in V_0', or clusters A with $deg(v, A) < (d-\varepsilon)|A|$, we have

$$
\begin{aligned}
(2/5 - \beta)n &\leq deg_G(v) < (\alpha - 6\beta)lN + (2/5 + 4\beta - \alpha)lN + d^3n/2 + (d-\varepsilon)lN \\
&\leq (2/5 - 2\beta + d^3/2 + d - \varepsilon)n \\
&< (2/5 - 3\beta/2)n \text{ (provided that } d - \varepsilon + d^3/2 < \beta/2 \text{),}
\end{aligned}
$$

showing a contradiction. Since $|\Omega| \leq \beta l$, we have $|\mathcal{U}| \geq (2\alpha - 7\beta)l$. □

Now for each non-dominator cluster A (A is either a cluster X_i', Y_i', or a half cluster F'), let $I(A)$ be the set of vertices from V_0' such that each of them has A as its partner. By Claim 3.4.2, we have $|I(A)| \leq \frac{d^2N}{20}$.

We need three operations below for constructing small HITs covering vertices in V_0'.

Operation I Let (A, B) be an (ε', δ)-super-regular pair, and I a set of vertices disjoint from $A \cup B$. Suppose that (i) $deg(x, B) \geq d'|B| > \varepsilon'|B|$ and $deg(x, B) \geq d'|B| \geq 3|I|$ for any $x \in I$; (ii) $(\delta - \varepsilon')d'|B| \geq 3|I|$; (iii) $(\delta - \varepsilon')|A| > |I|$; and (iv) $\delta|A| > 4|I|$. Then we can do the following operations on (A, B) and I.

Let $I = \{x_1, x_2, \cdots, x_{|I|}\}$. We first assume that $|I| \geq 2$.

Since (A, B) is (ε', δ)-super-regular, for each $v \in \Gamma(x_i, B)$, $|\Gamma(v, A)| \geq \delta|A|$. By condition (i), we have $|\Gamma(x_i, B)| > \varepsilon'|B|$ for each i. Applying Fact 3.4.4, we then know that there are at least $(\delta - \varepsilon')|A| > |I|$ vertices from $\Gamma(v, A)$ typical to $\Gamma(x_{i+1}, B)$ for each $1 \leq i \leq |I| - 1$. That is, there exists $A_1 \subseteq \Gamma(v, A)$ with $|A_1| \geq (\delta - \varepsilon')|A| > |I|$ such that for each $a_1 \in A_1$, $|\Gamma(a_1, \Gamma(x_{i+1}, B))| \geq (\delta - \varepsilon')d'|B| \geq 3|I|$. As $deg(x, B) \geq d'|B| \geq 3|I|$ for any $x \in I$ and $(\delta - \varepsilon')d'|B| \geq 3|I|$, combining the above argument, we know there is a claw-matching M_I from I to B centered in I such that one vertex from $\Gamma(x_i, V(M_I))$ and one vertex from $\Gamma(x_{i+1}, V(M_I))$ have at least $(\delta - \varepsilon')|A| > |I|$ common neighbors in A. Let x_{i1}, x_{i2}, x_{i3} be the three neighbors of x_i in M_I (in fact in B) and suppose that $|\Gamma(x_{i3}, A) \cap \Gamma(x_{i+1,1}, A)| \geq |I|$. For $1 \leq i \leq |I| - 1$, we then choose distinct vertices $y_i \in \Gamma(x_{i3}, A) \cap \Gamma(x_{i+1,1}, A)$. By condition (iv), there is a $\wedge$-matching M_2 between the vertex set $\{x_{i3} : 1 \leq i \leq |I| - 1\}$ and the vertex set $A - \{y_i : 1 \leq i \leq |I| - 1\}$ centered in the first set, a matching M_3 between $\{x_{i+1,1} : 1 \leq i \leq |I| - 1\}$ and $A - \{y_i : 1 \leq i \leq |I| - 1\} - V(M_2)$ covering the first set, and a matching M_4 between the vertex set $\{y_i : 1 \leq i \leq |I| - 1\}$ and $B - V(M_I)$ covering the first set. Finally, by using (iv) again, we can find three distinct vertices $y_{31}, y_{32}, y_{33} \in \Gamma(x_{13}, A) - \{y_i : 1 \leq i \leq |I| - 1\} - V(M_2) - V(M_3)$. Let T_B be the graph with

$$V(T_B) = V(M_I) \cup \{y_i : 1 \leq i \leq |I| - 1\} \cup V(M_2) \cap V(M_3) \cup V(M_4) \cup \{y_{31}, y_{32}, y_{33}\}$$

and

$$E(T_B) = M_I \cup \{y_i x_{i3}, y_i x_{i+1,1} : 1 \leq i \leq |I| - 1\} \cup M_2 \cup M_3 \cup M_4 \cup \{x_{13}y_{31}, x_{13}y_{32}, x_{13}y_{33}\}.$$

If $|I| = 1$, we choose $x_{11}, x_{12}, x_{13} \in \Gamma(x_1, B)$ and $y_{31}, y_{32}, y_{33} \in \Gamma(x_{13}, A)$. Then let T_B be the graph with

$$V(T_B) = \{x_1, x_{11}, x_{12}, x_{13}, y_{31}, y_{32}, y_{33}\}$$

and

$$E(T_B) = \{x_1x_{11}, x_1x_{12}, x_1x_{13}, x_{13}y_{31}, x_{13}y_{32}, x_{13}y_{33}\}.$$

In any case, we see that T_B is a HIT satisfying

$$\begin{aligned} |V(T_B) \cap B| &= |V(T_B) \cap A| = 4|I| - 1, \\ |L(T_B) \cap B| &= \min\{2|I| + 1, 3|I| - 1\},\ |L(T_B) \cap A| = 3|I|. \end{aligned} \tag{3.10}$$

We call T_B the insertion HIT associated with B. Figure 3.4 gives a depiction of T_B for $|I| = 1, 3$, respectively.

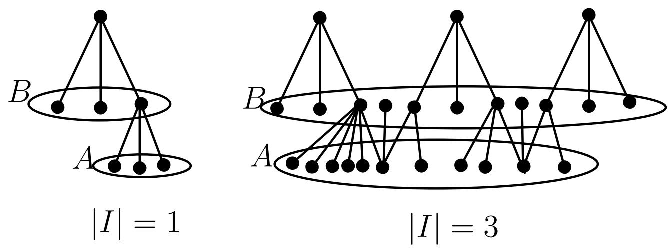


Figure (3.4) The HIT T_B

Operation II Let (A, B) be an (ε', δ)-super-regular pair, and I a set of vertices disjoint from $A \cup B$. Suppose that (i) $deg(x, A) \geq d'|A| > \varepsilon'|A|$ and $deg(x, A) \geq d'|A| \geq 3|I|$ for any $x \in I$; (ii) $(\delta - \varepsilon')d'|A| \geq 3|I|$; (iii) $(\delta - 2\varepsilon')|B| > |I|$; and (iv) $\delta|B| > 3|I|$. Then we can do the following operations on (A, B) and I.

Let $I = \{x_1, x_2, \cdots, x_{|I|}\}$. We first assume that $|I| \geq 3$.

Since (A, B) is (ε', δ)-super-regular, for each $v \in \Gamma(x_i, A)$, $|\Gamma(v, B)| \geq \delta|B|$. By condition (i), we have $|\Gamma(x_i, A)| > \varepsilon'|A|$ for each i. Applying Fact 3.4.4, we then know that there are at least $(\delta - 2\varepsilon')|B| > |I|$ vertices from $\Gamma(v, B)$ typical to both $\Gamma(x_{i+1}, A)$ and $\Gamma(x_{i+2}, A)$ for each $1 \leq i \leq |I| - 2$. That is, there exists $B_1 \subseteq \Gamma(v, B)$ with $|B_1| \geq (\delta - 2\varepsilon')|B| > |I|$ such that for each $b_1 \in B_1$, $|\Gamma(b_1, \Gamma(x_{i+1}, A))|, |\Gamma(b_1, \Gamma(x_{i+2}, A))| \geq (\delta - \varepsilon')d'|A| \geq 3|I|$. As $deg(x, A) \geq d'|A| \geq 3|I|$ for any $x \in I$ and $(\delta - \varepsilon')d'|A| \geq 3|I|$, combining the above argument, we know there is a claw-matching M_I from I to A centered in I such that any one vertex from $\Gamma(x_i, V(M_I))$, any one vertex from $\Gamma(x_{i+1}, V(M_I))$, and any one vertex from $\Gamma(x_{i+2}, V(M_I))$ have at least $|I|$ common neighbors in B. Let x_{i1}, x_{i2}, x_{i3} be the three neighbors of x_i in M_I (in fact in A). For $i = 1$, choose $y_0 \in \Gamma(x_{13}, A) \cap \Gamma(x_{23}, A) \cap \Gamma(x_{33}, A)$. Let $h = \lceil (|I| - 3)/2 \rceil$. For $1 \le k \le h$, we then choose distinct vertices $y_k \in \Gamma(x_{1+2k,2}, A) \cap \Gamma(x_{2+2k,3}, A) \cap \Gamma(x_{3+2k,3}, A)$ (if $|I| = 2 + 2k$, let $\Gamma(x_{3+2k,3}, A) = A$). By condition (iv), there is a matching M between the vertex set $\{x_{i3}, x_{1+2k,2} : 1 \le i \le |I|, 1 \le k \le h\}$ and the vertex set $B - \{y_0, y_k : 1 \le k \le h\}$ covering the first set. If $|I|$ is even, choose $y_{31}, y_{32} \in \Gamma(x_{13}, B)$ such that they have not been chosen before; if $|I|$ is odd, choose $y_{31}, y_{32}, y_{33} \in \Gamma(x_{13}, B)$ such that they have not been chosen before. Let T_A be the graph with

$$V(T_A) = \begin{cases} V(M_I) \cup V(M) \cup \{y_0, y_k : 1 \le k \le h\} \cup \{y_{31}, y_{32}\}, & \text{if } |I| \text{ is even;} \\ V(M_I) \cup V(M) \cup \{y_0, y_k : 1 \le k \le h\} \cup \{y_{31}, y_{32}, y_{33}\}, & \text{if } |I| \text{ is odd;} \end{cases}$$

and $E(T_A)$ containing all edges in $M_I \cup M \cup \{y_0x_{13}, y_0x_{23}, y_0x_{33}\}$ and all edges in

$$\begin{cases} \{x_{1+2k,2}y_k, x_{2+2k,2}y_k, x_{3+2k,2}y_k, x_{1+2h,2}y_h, x_{2+2h,2}y_h : 1 \le k \le h-1\} \cup \{y_{31}, y_{32}\}, & \text{if } |I| \text{ is even;} \\ \{x_{1+2k,2}y_k, x_{2+2k,2}y_k, x_{3+2k,2}y_k : 1 \le k \le h\} \cup \{y_{31}, y_{32}, y_{33}\}, & \text{if } |I| \text{ is odd.} \end{cases}$$

If $|I| = 1$, we choose $x_{11}, x_{12}, x_{13} \in \Gamma(x_1, A)$ and $y_{31}, y_{32} \in \Gamma(x_{13}, B)$, and then let T_A be the graph with

$$V(T_B) = \{x_1, x_{11}, x_{12}, x_{13}, y_{31}, y_{32}\} \text{ and } E(T_B) = \{x_1x_{11}, x_1x_{12}, x_1x_{13}, x_{13}y_{31}, x_{13}y_{32}\}.$$

If $|I| = 2$, we choose $x_{11}, x_{12}, x_{13} \in \Gamma(x_1, A)$, $x_{11}, x_{12}, x_{13} \in \Gamma(x_2, A)$, $y \in \Gamma(x_{13}, B) \cap \Gamma(x_{21}, B)$, $y_{11}, y_{12} \in \Gamma(x_{13}, B)$, and $y_{21}, y_{22} \in \Gamma(x_{21}, B)$ such that they are all distinct, then let T_A be the graph with

$$V(T_B) = \{x_i, x_{i1}, x_{i2}, x_{i3}, y, y_{i1}, y_{i2} : i = 1, 2\} \quad \text{and}$$

$$E(T_B) = \{x_ix_{i1}, x_ix_{i2}, x_ix_{i3}, x_{13}y, x_{21}y, x_{13}y_{11}, x_{13}y_{12}, x_{21}y_{21}, x_{21}y_{22}\}.$$

We see that T_A is a tree which has a degree 2 vertex y only if $|I| = 2$ and a degree 2 vertex

y_h only if $|I| > 2$ and $|I|$ is even. In addition, T_A satisfies the following.

$$|V(T_A) \cap A| = 3|I| \quad \text{and} \quad |L(T_A) \cap A| = \begin{cases} 2|I|, & \text{if } |I| = 1, 2; \\ 2|I| - \left\lceil \frac{|I|-3}{2} \right\rceil, & \text{if } |I| \geq 3; \quad \text{and} \end{cases}$$

$$|V(T_A) \cap B| = \begin{cases} 2, & \text{if } |I| = 1; \\ 2|I| + 1, & \text{if } |I| \geq 2; \quad \text{and} \end{cases}$$

$$|L(T_A) \cap B| = \begin{cases} 2|I|, & \text{if } |I| = 1, 2; \\ 2|I| - \left\lceil \frac{|I|-3}{2} \right\rceil, & \text{if } |I| \geq 3. \end{cases} \tag{3.11}$$

In this case, we call T_A the insertion tree associated with A. Notice that $|L(T_A) \cap A| = |L(T_A) \cap B|$ always holds. Figure 3.4 gives a depiction of T_A for $|I| = 1, 2, 5, 6$, respectively.

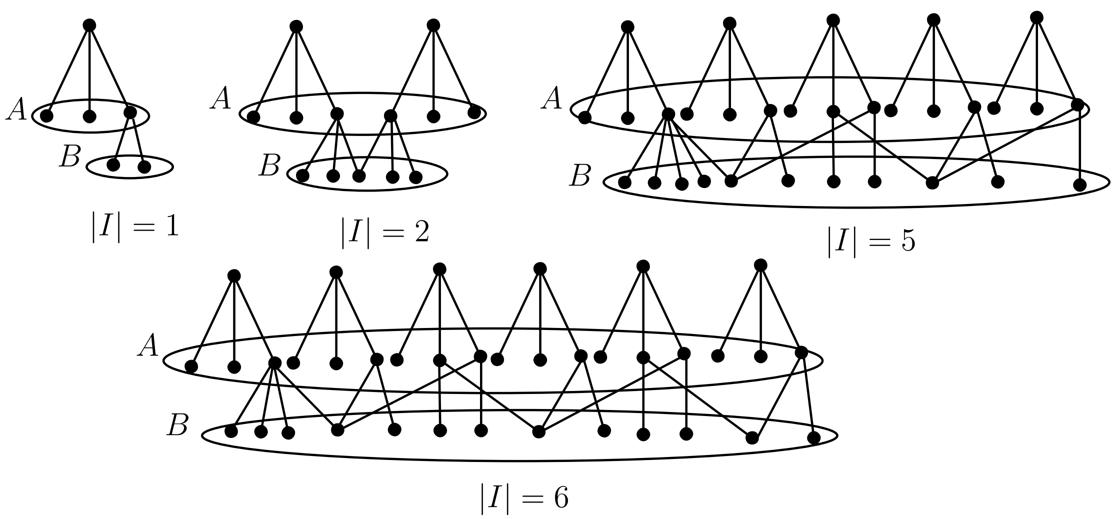


Figure (3.5) The tree T_A

Operation III Let (B, F) be an (ε', δ)-super-regular pair, and I a set of vertices disjoint from $B \cup F$. Suppose that $deg(x, F) \geq d'|F| \geq 3|I|$ for any $x \in I$ and $\delta|B| \geq 6|I|$. Then we can do the following operations on (A, B) and I.

Let $I = \{x_1, x_2, \cdots, x_{|I|}\}$. Since $deg(x, B) \geq d'|B| \geq 3|I|$ for any $x \in I$, there is a claw-matching M_I from I to F centered in I. Then as $\delta|B| \geq 6|I|$, there is a $\wedge$-matching

$M_\wedge$ from $V(M_I) \cap F$ to B centered in $V(M_I) \cap F$. Let T_F be the graph with

$$V(T_B) = V(M_I) \cup V(M_\wedge) \quad \text{and} \quad E(T_B) = M_I \cup M_\wedge.$$

We see that T_F is a forest with no vertex of degree 2 satisfying

$$|V(T_F) \cap F| = |S(T_F) \cap F| \;\; = \;\; 3|I| \text{ and } |V(T_F) \cap B| = |L(T_F) \cap B| = 6|I|. \quad (3.12)$$

We call T_F the insertion forest associated with F.

Now for each H-pair (X_i', Y_i'), we may assume that $I(X_i') \neq \emptyset$ and $I(Y_i') \neq \emptyset$ for a uniform discussion, as the consequent argument is independent of the assumptions. Recall that (X_i', Y_i') is $(2\varepsilon, d-2\varepsilon)$-super-regular by Fact 3.4.5. Notice that $deg(x, X_i') \geq (d-\varepsilon)|X_i'|$ for each $x \in I(X_i')$, $|I(X_i')| \leq \frac{d^2N}{20}$, and $|X_i'|, |Y_i'| \geq (1-\varepsilon)N$. By simple calculations, we see that (i) $deg(x, X_i') \geq (d-\varepsilon)|X_i'| > 2\varepsilon|X_i'|$ and $(d-\varepsilon)|X_i'| \geq 3d^2N/20$ for each $x \in I(X_i')$; (ii) $(d - 2\varepsilon - 2\varepsilon)(d-\varepsilon)|X_i'| > 3d^2N/20$; (iii) $(d-4\varepsilon)|Y_i'| > d^2N/20$; and (iv) $(d-2\varepsilon)|Y_i'| > d^2N/5 \geq 4I(X_i')$. Thus all the conditions in Operation I are satisfied. So we can find a HIT $T_{X_i'}$ associated with X_i'. As $|V(T_{X_i'}) \cap X_i'| = |V(T_{X_i'}) \cap Y_i'| \leq 4|I(X_i')| \leq \frac{d^2N}{5}$, we know that $(X_i' - V(T_{X_i'}), Y_i' - V(T_{X_i'}))$ is $(4\varepsilon, d - 2\varepsilon - d^2N/5)$-super regular. Since $deg(y, Y_i') \geq (d-\varepsilon)|Y_i'|$ for each $y \in I(Y_i')$, we get $deg(y, Y_i' - V(T_{X_i'})) \geq (d - \varepsilon - d^2/5)|Y_i'|$ for each $y \in I(Y_i')$. By direct checking, conditions $(i) \sim (iv)$ of Operation I are satisfied by the pair $(X_i' - V(T_{X_i'}), Y_i' - V(T_{X_i'}))$ and $I(Y_i')$. Then we use Operation I on $(X_i' - V(T_{X_i'}), Y_i' - V(T_{X_i'}))$ and $I(Y_i')$ to get a HIT $T_{Y_i'}$ associated with $Y_i' - V(T_{X_i'})$. Denote

$$X_i^* = X_i' - V(T_{X_i'}) - V(T_{Y_i'}) \quad \text{and} \quad Y_i^* = Y_i' - V(T_{X_i'}) - V(T_{Y_i'}).$$

By using (3.10) in Operation I, we have $|X_i^*| = |Y_i^*| \geq (1 - 2d^2/5 - \varepsilon)N \geq N/2$. By Slicing lemma (Lemma 3.2.5) and Fact 3.4.5, we have the following.

Fact 3.4.7. *For each H-pair (X_i, Y_i), (X_i^*, Y_i^*) is $(4\varepsilon, d - 2\varepsilon - 2d^2/5)$-super-regular with $|X_i^*| = |Y_i^*|$. We call (X_i^*, Y_i^*) a ready H-pair.*

Then for each H-triple (X_i', Y_i', F'), we may assume that $I(X_i') \neq \emptyset$ and $I(F') \neq \emptyset$ (recall that Y_i is assumed to be the dominator of F, so $I(Y_i') = \emptyset$ by the distribution principle of vertices in V_0' from Claim 3.4.2). By Fact 3.4.6, we know that (X_i', Y_i') is $(2\varepsilon, d - 3.1\varepsilon)$-super-regular and (Y_i', F') is $(4.2\varepsilon, d - 3.1\varepsilon - 2d^3)$-super-regular. Notice also that $|X_i'| \geq (1-\varepsilon)N$, $|Y_i'| \geq (1 - 3.1\varepsilon)N$, $|F'| \geq (1 - 2.1\varepsilon - 2d^3)N/2$, and $deg(x, X_i') \geq (d - \varepsilon)|X_i'|$ and $deg(y, F') \geq (d-\varepsilon)|F'|$ for each $x \in I(X_i')$ and each $y \in I(F')$. Since $|I(X_i')|, |I(F')| \leq \frac{d^2 N}{20}$ and $\varepsilon \ll d \ll 1$, the conditions of Operation III are satisfied by (Y_i', F') and $I(F')$ by direct calculations. Let $T_{F'}$ be the insertion forest associated with F'. Then we use Operation II on $(X_i', Y_i' - V(T_{F'}))$ and $I(X_i')$ to get a tree $T_{X_i'}$ associated with X_i'. Denote

$$X_i^* = X_i' - V(T_{X_i'}),\, Y_i^* = Y_i' - V(T_{F'}) - V(T_{X_i'}), \quad \text{and} \quad F^* = F' - V(T_{F'}).$$

By using (3.11) and (3.12) in Operation II and Operation III, respectively, we have $|X_i^*|, |Y_i^*| \geq (1 - 3.1\varepsilon - 9d^2/20)N \geq N/2$ and $|F^*| \geq (1 - 2.1\varepsilon - 2d^3)N/2 - 3d^2N/20 \geq (1 - 2.1\varepsilon - 2d^3 - 3d^2/10)N/2$. By Slicing lemma and Fact 3.4.6, we have the following.

Fact 3.4.8. *For each H-triple (X_i, Y_i, F), (X_i^*, Y_i^*) is $(4\varepsilon, d - 3.1\varepsilon - 9d^2/20)$-super-regular and (Y_i^*, F^*) is $(8.4\varepsilon, d - 2.1\varepsilon - 3d^2/10 - 2d^3)$-super-regular. We call (X_i^*, Y_i^*, F^*) a ready H-triple.*

Step 5. Apply the Blow-up lemma to find a HIT within each ready H-pair and among each ready H-triple.

In order to apply the Blow-up Lemma, we first give two lemmas which assure the existence of a given subgraph in a complete bipartite graph.

Lemma 3.4.4. *Suppose $0 < \varepsilon \ll d \ll 1$ and N is a large integer. If $G(A, B)$ is a balanced complete bipartite graph with $(1 - \varepsilon - d^2/2)N \leq |A| = |B| \leq N$, then $G(A, B)$ contains a HIST T_{pair} with $\Delta(T_{pair}) \leq \lceil 2/d^3 \rceil$ and $||L(T_{pair}) \cap A| - |L(T_{pair}) \cap B|| = \ell$ for any given non-negative integer ℓ with $\ell \leq d^2N$.*

Proof. By the symmetry, we only show that we can construct a HIST T such that

$|L(T) \cap A| - |L(T) \cap B| = \ell$. Let $\Delta' = \lceil d^3 N \rceil$. We choose distinct $a_1, a_2, \cdots, a_{\Delta'} \in A$ and distinct $b_1, b_2, \cdots, b_{\Delta'-1} \in B$. Then we decompose all vertices in B into $B_1, B_2, \cdots, B_{\Delta'}$ such that $3 \leq |B_i| \leq 1/d^3$, $B_i \cap B_{i+1} = \{b_i\}$ for $1 \leq i \leq \Delta' - 1$, and $B_i \cap B_j = \emptyset$ for $|i - j| > 1$. Now we choose $\ell + 1$ distinct vertices $b_{\Delta'}, b_{\Delta'+1}, \cdots, b_{\Delta'+\ell}$ from $B - \{b_i : 1 \leq i \leq \Delta' - 1\}$. As $\Delta' = \lceil d^3 N \rceil$, $\ell + \Delta' \leq (d^2 + d^3)N + 1$, and thus

$$2(\ell + \Delta') \leq (2d^2 + 2d^3)N + 2 \leq (1 - d^2/2 - \varepsilon)N - \lceil d^3 N \rceil \leq |A| - \lceil d^3 N \rceil.$$

Thus we can use all of the vertices in $\{b_i : 1 \leq i \leq \Delta' + \ell\}$ to cover all vertices in $A - \{a_i \,|\, 1 \leq i \leq \Delta' - 1\}$ such that each b_i can be adjacent to at least two distinct vertices. We partition $A - \{a_i \,|\, 1 \leq i \leq \Delta' - 1\}$ arbitrarily into $A_1, A_2, \cdots, A_{\ell+\Delta'}$ such that $2 \leq |A_i| \leq 1/d^3$. Now let T be a spanning subgraph of $G(A, B)$ such that

$$E(T) = \{a_i b \,|\, b \in B_i, 1 \leq i \leq \Delta'\} \cup \{b_j a \,|\, a \in A_j, 1 \leq j \leq \Delta' + \ell\}.$$

Clearly, $\Delta(T) \leq \lceil 2/d^3 \rceil$. As $|A| = |B|$, $|S(T) \cap A| = \Delta'$, and $|S(T) \cap B| = \Delta' + \ell$, we then have that $|L(T) \cap A| - |L(T) \cap B| = \ell$. We denote T as T_{pair}. □

Lemma 3.4.5. *Suppose $0 < \varepsilon \ll d \ll 1$ and N is a large integer. Let $G = G(A, B, F)$ be a tripartite graph with $V(G)$ partitioned into $A \cup B \cup F$ such that both $G[A \cup B]$ and $G[B \cup F]$ are complete bipartite graphs. If (i) $(1 - 4\varepsilon - d^2/2)N \leq |A|, |B| \leq N$, (ii) $(1/2 - 2.1\varepsilon - 3d^2/20 - d^3)N \leq |F| \leq (1/2 - d^3)N$, and (iii) for any given non-negative integer $l \leq 3d^2N/10$, we have $|B| - 2(|A \cup F| - |B| - l) \geq 3d^3N/2$ holds, then G contains a HIST T_{triple} and a path P_{triple} spanning on a subset of $L(T_{triple})$ such that*

(a) *T_{triple} is a HIST of G with $\Delta(T_{triple}) \leq \lceil 3/d^3 \rceil$;*

(b) *$|L(T_{triple}) \cap B| = |L(T_{triple}) \cap (A \cup F)| - l$.*

(c) *P_{triple} is a (b, f)-path on $L(T_{triple}) \cap F$ and any $|L(T_{triple}) \cap F|$ vertices from $L(T_{triple}) \cap B$, and $|V(P_{triple}) \cap F| \leq 5d^2N/6$.*

Proof. Let $\Delta' = \lceil d^3N/2 \rceil$. We choose distinct $b_1, b_2, \cdots, b_{\Delta'} \in B$ and partition all vertices in F into $F_1, F_2, \cdots, F_{\Delta'}$ such that $3 \le |F_i| \le 1/d^3$. Then we choose distinct $a_1, a_2, \cdots, a_{\Delta'-1} \in A$ and decompose all vertices in A into $A_1, A_2, \cdots, A_{\Delta'}$ such that $3 \le |A_i| \le 2/d^3$, $A_i \cap A_{i+1} = \{a_i\}$ for $1 \le i \le \Delta' - 1$, and $A_i \cap A_j = \emptyset$ for $|i-j| > 1$. Choose one more vertex, say $a_{\Delta'} \in A - \{a_i \mid 1 \le i \le \Delta' - 1\}$. Let $l' = |A \cup F| - |B| - l$. Notice that $l' > 0$. Now we choose l' distinct vertices $f_1, f_2, \cdots, f_{l'}$ from $A - \{a_i : 1 \le i \le \Delta'\} \cup F$ (choose as many as possible from F first) and partition any $2l'$ vertices of $B - \{b_i : 1 \le i \le \Delta'\}$ into $B_1, B_2, \cdots, B_{l'}$ such that $|B_i| = 2$. By (iii), we see that there are at least $\lfloor d^3N \rfloor$ vertices left in $B' = B - \{b_i : 1 \le i \le \Delta'\} - \bigcup_{i=1}^{l'} \{B_i\}$. Hence we can partition $B' = B'_1 \cup B'_2 \cup \cdots \cup B'_{\Delta'}$ such that $|B'_{\Delta'}| \ge 2$ and $|B'_j| \ge 1$ for $j \ne \Delta'$. We let T be a subgraph of G on $A \cup B \cup F$ with

$$E(T) = \{b_i f, b_i a, a_i b' : f \in F_i, a \in A_i, b' \in B'_i, 1 \le i \le \Delta'\} \cup \{f_i b : b \in B_i, 1 \le i \le l'\}.$$

By the construction, T is a HIST of G, which clearly satisfies (a). Since $|S(T) \cap B| = \Delta'$ and $|S(T) \cap (A \cup F)| = \Delta' + l' = \Delta' + |A \cup F| - |B| - l$, we then see that T satisfies (b). If $L(T) \cap F \ne \emptyset$, let $f \in L(T) \cap F$ and $b \in L(T) \cap B$, we can then take a (b, f)-path P with $V(P) \cap F = L(T) \cup F$ and $|V(P)| = 2|L(T) \cap F|$. By (i) and (ii), we see that $l' = |A \cup F| - |B| - l \ge (1/2 - 6.1\varepsilon - 4d^2/5 - d^3)N$. Hence $|V(P) \cap F| = |F| - l' \le 5d^2N/6$. Denote T as T_{triple} and P as P_{triple}. □

Now for $1 \le i \le t$ and for each ready H-pair (X_i^*, Y_i^*), suppose, without of loss generality, that $|(L(T_{X'_i}) \cap Y'_i) \cup ((L(T_{Y'_i}) \cap Y'_i)| - |((L(T_{X'_i}) \cap X'_i) \cup ((L(T_{Y'_i}) \cap X'_i)| = l'$, where $T_{X'_i}$ is the insertion HIT associated with X'_i and $T_{Y'_i}$ is the insertion HIT associated with Y'_i. Notice that $l' \le d^2N$ from (3.10) and (3.11). Let $x_a \in S(T_{X'_i}) \cap X'_i$ be a non-leaf of $T_{X'_i}$ and $y_b \in S(T_{Y'_i}) \cap Y'_i$ a non-leaf of $T_{Y'_i}$. Since (X'_i, Y'_i) is $(2\varepsilon, d - 2\varepsilon)$-super-regular by Fact 3.4.5 and $|Y'_i - Y_i^*| \le 2d^2N/5$, we have $deg(x_a, Y_i^*) \ge (d - 2\varepsilon - d^2/2)N \ge dN/2$. Similarly, $deg(y_b, X_i^*) \ge (d - 2\varepsilon - d^2/2)N \ge dN/2$. Also, from Step 2, we have $\Gamma(x_i^*, Y_i), \Gamma(x_i^{**}, Y_i) \ge (d - 3\varepsilon)N$. So, $\Gamma(x_i^*, Y_i^*), \Gamma(x_i^{**}, Y_i^*) \ge (d - 3\varepsilon - d^2/2)N \ge dN/2$. Similarly, we have

$\Gamma(y_i^*, X_i^*), \Gamma(y_i^{**}, X_i^*) \geq (d-3\varepsilon-d^2/2)N \geq dN/2$. Recall that (X_i^*, Y_i^*) is $(4\varepsilon, d-2\varepsilon-8d^2/20)$-super-regular by Fact 3.4.7, and therefore (X_i^*, Y_i^*) is $(4\varepsilon, d/2)$-super-regular. By the the strengthened version of the Blow-up lemma and Lemma 3.4.4 (the conditions are certainly satisfied by X_i^* and Y_i^*), we can find a HIST $T_1^i \cong T_{pair}$ on $X_i^* \cup Y_i^*$ such that there exist $y_a \in S(T_1^i) \cap \Gamma(x_a, Y_i^*)$, $x_b \in S(T_1^i) \cap \Gamma(y_b, X_i^*)$, $y_i' \in S(T_1^i) \cap \Gamma(x_i^*, Y_i)$, $y_i'' \in S(T_1^i) \cap \Gamma(x_i^{**}, Y_i)$, and $x_i' \in S(T_1^i) \cap \Gamma(y_i^*, X_i)$, $x_i'' \in S(T_1^i) \cap \Gamma(y_i^{**}, X_i)$ such that $|L(T_1^i) \cap X_i^*| - |L(T_1^i) \cap Y_i^*| = l'$. Hence $|L(T_1^i) \cap X_i^*| + |L(T_{X_i'}) \cap X_i'| + |L(T_{Y_i'}) \cap X_i'| = |L(T_1^i) \cap Y_i^*| + |L(T_{X_i'}) \cap Y_i'| + |L(T_{Y_i'}) \cap Y_i'|$. Let $T^i = T_1^i \cup T_{X_i'} \cup T_{Y_i'} \cup \{x_a y_a, y_b x_b\} \cup \{x_i^* y_i', x_i^{**} y_i'', y_i^* x_i', y_i^{**} x_i''\}$. It is clear that T^i is a HIST on $X_i' \cup Y_i' \cup I(X_i') \cup I(Y_i')$ such that

$$\{x_i^*, x_i^{**}, y_i^*, y_i^{**}\} \subseteq L(T^i) \quad \text{and} \quad |L(T^i) \cap X_i'| = |L(T^i) \cap Y_i'|.$$

For the ready H-pair (X_0^*, Y_0^*), let $x_a \in S(T_{X_0'}) \cap X_0'$ be a non-leaf of $T_{X_0'}$ and $y_b \in S(T_{Y_0'}) \cap Y_0'$ a non-leaf of $T_{Y_0'}$. By the the strengthened version of the Blow-up lemma and Lemma 3.4.4 (the conditions are certainly satisfied by X_0^* and Y_0^*), we can find a HIST $T_1^0 \cong T_{pair}$ on $X_0^* \cup Y_0^*$ such that there exist $y_0' \in S(T_1^0) \cap \Gamma(x_0^*, Y_0)$, $y_0'' \in S(T_1^0) \cap \Gamma(x_0^{**}, Y_0)$, $x_{t+1}' \in S(T_1^0) \cap \Gamma(y_{t+1}^*, Y_0)$, $x_{t+1}'' \in S(T_1^0) \cap \Gamma(y_{t+1}^{**}, Y_0)$, and $x_0' \in S(T_1^0) \cap \Gamma(y_0^*, X_0)$, $x_0'' \in S(T_1^i) \cap \Gamma(y_0^{**}, X_0)$ such that $|L(T_1^0) \cap X_0^*| + |L(T_{X_0'}) \cap X_0'| + |L(T_{Y_0'}) \cap X_0'| = |L(T_1^0) \cap Y_0^*| + |L(T_{X_0'}) \cap Y_0'| + |L(T_{Y_0'}) \cap Y_0'| + 2$. Let $T^0 = T_1^0 \cup T_{X_0'} \cup T_{Y_0'} \cup \{x_a y_a, y_b x_b\} \cup \{x_0^* y_0', x_0^{**} y_0'', y_0^* x_0', y_0^{**} x_0'', y_{t+1}^* x_{t+1}', y_{t+1}^{**} x_{t+1}''\}$. It is clear that T^0 is a HIST on $X_0' \cup Y_0' \cup I(X_0') \cup I(Y_0')$ such that

$$\{x_0^*, x_0^{**}, y_0^*, y_0^{**}, y_{t+1}^*, y_{t+1}^{**}\} \subseteq L(T^0) \quad \text{and} \quad |L(T^0) \cap X_0'| = |L(T^i) \cap Y_0'| + 2.$$

For each ready triple (X_i^*, Y_i^*, F^*), we know that (X_i^*, Y_i^*) is $(4\varepsilon, d - 3.1\varepsilon - 9d^2/20)$-super-regular and (Y_i^*, F^*) is $(8.4\varepsilon, d - 2.1\varepsilon - 3d^2/10 - 2d^3)$-super-regular by Fact 3.4.8. Notice that $(1 - 4\varepsilon - 9d^2/20)N \leq |X_i^*|, |Y_i^*| \leq N$ and $(1/2 - 2.1\varepsilon - 3d^2/30 - d^3)N \leq |F^*| \leq (1/2 - d^3)N$. Let $|I(X_i')| = l'$ and $|I(F')| = l/6$ for some integer l. By Operation II we have $|V(T_{X_i'}) \cap X_i'| \leq 3l'$ and $|V(T_{X_i'}) \cap Y_i'| \leq 2l' + 1$. By Operation III we have $|V(T_{F'}) \cap F_i'| = l/2$

and $|V(T_{F'}) \cap Y_i'| = l$. Notice that $|L(T_{X_i'}) \cap X_i'| = |l(T_{X_i'}) \cap Y_i'|$. Hence,

$$\begin{aligned} |Y_i^*| - 2(|X_i^* \cup F^*| - |Y_i^*| - l) &\geq 3(|Y_i'| - 2l' - l - 1) - 2(|X_i'| - 3l') - 2(|F'| - l/2) + 2l \\ &= 3|Y_i'| - 2|X_i'| - 2|F'| - 3 \\ &\geq 3(1 - 3.1\varepsilon)N - 2N - N + 2d^3N - 3 > 3d^3N/2. \end{aligned}$$

By the weak version of the Blow-up lemma (Lemma 3.2.2) and Lemma 3.4.5, we then can find a HIT $T_1^i \cong T_{triple}$ on $X_i^* \cup Y_i^* \cup F^*$ and a path $P_i \cong P_{triple}$ spanning on $L(T_1^i) \cap F^*$ and other $|L(T_1^i) \cap F^*|$ vertices from Y_i^*. Let $y_a \in S(T_{X_i'}) \cap Y_i'$ be a non-leaf of $T_{X_i'}$ (take y_a as the degree 2 vertex if $T_{X_i'}$ has one) and $y_a' \in S(T_{F'}) \cap Y_i'$ a non-leaf of $T_{F'}$. Then as (Y_i', F') is $(4.1\varepsilon, d - 2.1\varepsilon - 2d^3)$-super-regular, we have $|\Gamma(y_a, F')|, |\Gamma(y_a', F')| \geq (d - 2.1\varepsilon - 2d^3)N/2$. Since $|F' - F^*| \leq 3d^2N/20$, we then know that $|\Gamma(y_a, F^*)|, |\Gamma(y_a', F^*)| \geq (d - 2.1\varepsilon - 3d^2/10 - 2d^3)N/2$. Since $|F^* \cap L(T_1^i)| = |V(P_i) \cap F^*| \leq 5d^2N/6 < (d - 2.1\varepsilon - 3d^2/10 - 2d^3)N/2$, there exist $f_a \in (S(T_1^i) \cap F^*) \cap \Gamma(y_a, F^*)$ and $f_a' \in (S(T_1^i) \cap F^*) \cap \Gamma(y_a', F^*)$. For each $x \in I(F')$, since $deg(x, F') \geq (d - \varepsilon)|F'| \geq (d - \varepsilon)(1 - 2.1\varepsilon - d^3)N/2$, we know there exists $f' \in (S(T_1^i) \cap F^*) \cap \Gamma(x, F^*)$. From Step 2, we have $|\Gamma(x_i^*, Y_i) \cap \Gamma(x_i^{**}, Y_i)| \geq (d - \varepsilon)^2N$ and $|\Gamma(y_i^*, X_i) \cap \Gamma(y_i^{**}, X_i)| \geq (d - \varepsilon)^2N$. Hence $|\Gamma(x_i^*, Y_i') \cap \Gamma(x_i^{**}, Y_i')| \geq ((d - \varepsilon)^2 - 3.1\varepsilon)N$. Since $|S(T_1^i \cup T_{X_i'} \cup T_{F'}) \cap X_i'| < d^2N/2$, we see that there exists $y' \in \Gamma(x_i^*, Y_i) \cap \Gamma(x_i^{**}, Y_i) \cap L(T_1^i \cup T_{X_i'} \cup T_{F'})$. Similarly, there exists $x' \in \Gamma(y_i^*, X_i) \cap \Gamma(y_i^{**}, X_i) \cap L(T_1^i \cup T_{X_i'} \cup T_{F'})$. Let $T^i = T_1^i \cup T_{X_i'} \cup T_{F'} \cup \{xf' : x \in I(F'), f' \in (S(T_1^i) \cap F^*) \cap \Gamma(x, F^*)\} \cup \{y_a f_a, y_a' f_a'\} \cup \{y' x_i^*, y' x_i^{**}, x' y_i^*, x' y_i^{**}\}$. It is clear that T^i is a HIST on $X_i' \cup Y_i' \cup F' \cup I(X_i') \cup I(F')$ such that

$$\{x_i^*, x_i^{**}, y_i^*, y_i^{**}\} \subseteq L(T^i) \quad \text{and} \quad |L(T^i) \cap X_i'| = |L(T^i) \cap Y_i'|.$$

Let $H^i = T^i \cup P_i$. We call P_i the accompany path of T^i.

Step 6. Apply the Blow-up Lemma again on the regular-pairs induced on the leaves of each HIT obtained in Step 5 to find two vertex-disjoint paths covering all the leaves. Then connect all the HITs into a HIST of G and connect the disjoint paths into a cycle using the edges initiated in Step 2.

Suppose $1 \le i \le t$. For each H-pair (X_i, Y_i), let $X_i^L = X_i' \cap L(T^i) - \{x_i^*, x_i^{**}\}$ and $Y_i^L = Y_i' \cap L(T^i) - \{y_i^*, y_i^{**}\}$, and for each H-triple (X_i, Y_i, F), let $X_i^L = X_i' \cap L(T^i \cup P_i) - \{x_i^*, x_i^{**}\}$ and $Y_i^L = Y_i' \cap L(T^i \cup P_i) - \{y_i^*, y_i^{**}\}$, where T^i is the HIST found in Step 5, and P_i is the accompany path of T^i. By Operations I, II and III, and the proofs of the Lemmas 3.4.4 and 3.4.5, we have $I(X_i') \cup I(Y_i') \subseteq S(T_i)$ and $F' \cup I(F') \subseteq S(T_i \cup P_i)$. Thus, $X_i^L \cup Y_i^L = L(T^i) - \{x_i^*, x_i^{**}, y_i^*, y_i^{**}\}$ for each H-pair and $X_i^L \cup Y_i^L = L(T^i \cup P_i) - \{x_i^*, x_i^{**}, y_i^*, y_i^{**}\}$ for each H-triple. Furthermore, we have $|X_i^L| = |Y_i^L|$. For the H-pair (X_0, Y_0), let $X_0^L = X_0' \cap L(T^0) - \{x_0^*, x_0^{**}, y_{t+1}^*, y_{t+1}^{**}\}$ and $Y_0^L = Y_0' \cap L(T^0) - \{y_0^*, y_0^{**}\}$. We have $X_0^L \cup Y_0^L = L(T^0) - \{x_0^*, x_0^{**}, y_{t+1}^*, y_{t+1}^{**}\}$ and $|X_0^L| = |Y_0^L|$ since from Step 5 we have $|L(T^0) \cap X_0'| = |L(T^0) \cap Y_0'| + 2$. By the construction of T_{pair} and H_{triple}, we see that $|S(T_i) \cap X_i'|, |S(T_i) \cap Y_i'| \le d^2 N$. Since each H-pair (X_i', Y_i') is $(2\varepsilon, d-2\varepsilon)$-super-regular, and each pair (X_i', Y_i') from an H-triple (X_i', Y_i', F') is $(2\varepsilon, d-3.1\varepsilon)$-super-regular, by Slicing Lemma, we then know that (X_i^L, Y_i^L) is $(4\varepsilon, d - 3.1\varepsilon - d^2)$-super-regular and hence is $(4\varepsilon, d/2)$-super-regular.

For each $1 \le i \le t$, by the choice of $x_i^*, x_i^{**}, y_i^*, y_i^{**}$, we have $|\Gamma(x_i^*, Y_i)|, |\Gamma(x_i^{**}, Y_i)| \ge (d-\varepsilon)N$ and $|\Gamma(y_i^*, X_i)|, |\Gamma(y_i^{**}, X_i)| \ge (d-\varepsilon)N$. Hence, $|\Gamma(x_i^*, Y_i^L)|, |\Gamma(x_i^{**}, Y_i^L)| \ge (d - \varepsilon - 3.1\varepsilon - d^2)N > dN/2$ and $|\Gamma(y_i^*, X_i^L)|, |\Gamma(y_i^{**}, X_i^L)| \ge (d - \varepsilon - 3.1\varepsilon - d^2)N > dN/2$. Similar results hold for the vertices $x_0^*, x_0^{**}, y_{t+1}^*, y_{t+1}^{**}$. For each $0 \le i \le t$, we choose distinct vertices $y_i' \in \Gamma(x_i^*, Y_i^L)$, $y_i'' \in \Gamma(x_i^{**}, Y_i^L)$ and $x_i' \in \Gamma(y_i^*, X_i^L)$, $x_i'' \in \Gamma(y_i^{**}, X_i^L)$. If T^i does not have the accompany path, then by the strengthened version of the Blow-up lemma, we can find an (x_i', y_i')-path P_1^i and an (x_i'', y_i'')-path P_2^i such that $P_1^i \cup P_2^i$ is spanning on $X_i^L \cup Y_i^L$. If T^i has the accompany (b, f)-path P_i, we see that $deg(b, X_i^L), deg(f, Y_i^L) \ge dN/2$ as (X_i', Y_i') is $(2\varepsilon, d-3.1\varepsilon)$- super-regular, and (Y_i', F') is $(4.2\varepsilon, d-3.1\varepsilon-2d^3)$-super-regular. Applying the strengthened version of the Blow-up lemma, we can find an (x_i', y_i')-path P_{11}^i and an (x_i'', y_i'')-path P_2^i such that $P_{11}^i \cup P_2^i$ is spanning on $X_i^L \cup Y_i^L$, and two consecutive internal vertices $a', b' \in V(P_{11}^i)$ with $b' \in \Gamma(f, Y_i^L)$, and $a' \in \Gamma(b, X_i^L)$. Let $P_1^i = P_{11}^i \cup P_i \cup \{fb', ba'\} - \{a'b'\}$. Notice that for the H-pair (X_0, Y_0), the two vertices y_{t+1}^*, y_{t+1}^{**} are not used in this step, but we will connect them to y_0^* and y_0^{**}, respectively, in next step.

We now connect the small HITs and paths together to find an SGHG of G. In Case A, for $1 \leq i \leq t-1$, we have $|S(T^i) \cap Y_i| \geq d^3N/2 > \varepsilon N$ and $|S(T^{i+1}) \cap X_{i+1}| \geq d^3N/2 > \varepsilon N$. Since (Y_i, X_{i+1}) is an ε-regular pair with density d, we see that there is an edge e_i connecting $S(T^{i+1}) \cap X_{i+1}$ and $S(T^{i+1}) \cap X_{i+1}$. Let

$$T = \bigcup_{i=1}^{t} T^i \cup \{e_i \,|\, 1 \leq i \leq t-1\}.$$

Then T is a HIST of G. Let C be the cycle formed by all the paths in $\bigcup_{i=1}^{t}(P_1^i \cup P_2^i)$ and all edges in the following set

$$\{x_i^* y_i', x_i^{**} y_i'', y_i^* x_i', y_i^{**} x_i'' \ : 1 \leq i \leq t\} \cup \{y_i^* x_{i+1}^*, y_i^{**} x_{i+1}^{**} \ : 1 \leq i \leq t-1\} \cup \{y_t^* x_1^{**}, y_t^{**} x_1^*\},$$

notices that the edges in $\{y_i^* x_{i+1}^*, y_i^{**} x_{i+1}^{**} \ : 1 \leq i \leq t-1\} \cup \{y_t^* x_1^{**}, y_t^{**} x_1^*\}$ above are guaranteed in Step 2. It is easy to see that C is a cycle on $L(T)$. Hence $H = T \cup C$ is an SGHG of G.

In Case B, for $1 \leq i \leq t-1$, we have $|S(T^i) \cap Y_i| \geq d^3N/2 > \varepsilon N$ and $|S(T^{i+1}) \cap X_{i+1}| \geq d^3N/2 > \varepsilon N$. Since (Y_i, X_{i+1}) is an ε-regular pair with density d, we see that there is an edge e_i connecting $S(T^{i+1}) \cap X_{i+1}$ and $S(T^{i+1}) \cap X_{i+1}$. Similarly, there is an edge e_0 connecting $S(T_0) \cap X_0$ and $S(T^1) \cap X_1$. Let

$$T = \bigcup_{i=1}^{t} T^i \cup \{e_i \,|\, 0 \leq i \leq t-1\}.$$

Then T is a HIST of G. Let C be the cycle formed by all paths in $\bigcup_{i=1}^{t}(P_1^i \cup P_2^i)$ and all edges in the set $\{y_0^* y_{t+1}^*, y_0^{**} y_{t+1}^{**}, y_{t+1}^* x_1^*, y_{t+1}^{**} x_1^{**}, x_0^* y_t^{**}, x_0^{**} y_t^*\}$ and in the following set

$$\{x_i^* y_i', x_i^{**} y_i'', y_i^* x_i', y_i^{**} x_i'' \ : 0 \leq i \leq t\} \cup \{y_i^* x_{i+1}^*, y_i^{**} x_{i+1}^{**} \ : 1 \leq i \leq t-1\}.$$

It is easy to see that C is a cycle on $L(T)$. Hence $H = T \cup C$ is an SGHG of G.

The proof of Theorem 3.4.3 is now finished. $\square$

3.4.3.2 Proof of Theorem 3.4.4 By the assumption that $deg(v_1, V_2) \leq 2\beta n$ for each $v_1 \in V_1$ and the assumption that $\delta(G) \geq (2n+3)/5$ in Extremal Case 1, we see that

$$\delta(G[V_1]) \geq (2n+3)/5 - 2\beta n. \tag{3.13}$$

Then (3.13) implies that

$$|V_1| \geq (2n+3)/5 - 2\beta n \quad \text{and} \quad |V_2| \leq 3n/5 + 2\beta n. \tag{3.14}$$

Also, by $|V_2| \geq (2/5 - 4\beta)n$ in the assumption,

$$|V_1| \leq (3/5 + 4\beta)n. \tag{3.15}$$

We will construct an SGHG of G following several steps below.

Step 1. Repartitioning

Set $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = \alpha^{2/3}$. Let

$$V_1' = V_1 \quad \text{and} \quad V_2' = \{v \in V_2 \,|\, deg(v, V_1) \leq \alpha_1 |V_1|\}.$$

Then by $d(V_1, V_2) \leq \alpha$, we have

$$\alpha_1 |V_1||V_2 - V_2'| \leq e(V_1, V_2') + e(V_1, V_2 - V_2') = e(V_1, V_2) \leq \alpha |V_1||V_2|.$$

This gives that

$$|V_2 - V_2'| \leq \alpha_2 |V_2|. \tag{3.16}$$

Denote $V_{12}^0 = V_2 - V_2'$. Then by the definition of V_2', we have

$$\delta(V_{12}^0, V_1') > \alpha_1 |V_1'| \quad \text{and} \quad \delta(G[V_2']) \geq (2n+3)/5 - \alpha_1 |V_1'| \geq (2/5 - \alpha_1(3/5 + 4\beta))n, \tag{3.17}$$

where the last inequality follows from (3.15).

Let $n_i = |V_i'|$ for $i = 1, 2$. Then by (3.13) and (3.15),

$$\delta(G[V_1']) \geq (2n+3)/5 - 2\beta n \geq \frac{2/5 - 2\beta}{3/5 + 4\beta} n_1 \geq (2/3 - 8\beta)n_1, \tag{3.18}$$

and by (3.14) and the second inequality in (3.17),

$$\delta(G[V_2']) \ \ \geq \ \ (2/5 - \alpha_1(3/5 + 4\beta))n \geq \frac{(2/5 - \alpha_1(3/5 + 4\beta))}{3/5 + 2\beta} n_2 \geq (2/3 - 1.1\alpha_1)n_2,$$

provided that $\beta \leq \frac{0.3\alpha_1}{9\alpha_1 + 20/3}$.

Step 2. Finding three connecting edges

AS G is 3-connected, there are 3 independent edges $x_L^1 y_L^1, x_L^2 y_L^2$ and $x_N y_N$ connecting $V_1' \cup V_{12}^0$ and V_2' such that $x_L^1, x_L^2, x_N \in V_1' \cup V_{12}^0$ and $y_L^1, y_L^2, y_N \in V_2'$. In the remaining steps, we will find a HIST T_1 in $G[V_1' \cup V_{12}^0]$ with x_N as a non-leaf and x_L^1, x_L^2 as leaves, and a HIST T_2 of $G[V_2']$ with y_N as a non-leaf and y_L^1, y_L^2 as leaves. Then $T = T_1 \cup T_2 \cup \{x_N y_N\}$ is a HIST of G. By finding a hamiltonian (x_L^1, x_L^2)-path P_1 on $L(T_1)$, and a hamiltonian (y_L^1, y_L^2)-path on $L(T_2)$, we see that

$$C := P_1 \cup P_2 \cup \{x_L^1 y_L^1, x_L^2 y_L^2\}$$

forms a cycle on $L(T)$. Hence $H := T \cup C$ is an SGHG of G.

Step 3. Initiating two HITs

In this step, we first initiate a HIT in $G[V_1' \cup V_{12}^0]$ containing X_N as a non-leaf and x_L^1 and x_L^2 as leaves. Then, we initiate a HIT in $G[V_2']$ containing y_N as a non-leaf and y_L^1 and y_L^2 as leaves.

For $x_L^1, x_L^2, x_N \in V_1' \cup V_{12}^0$, by (3.13) and (3.17), each of them has at least $\alpha_1 |V_1'| \geq 9$ neighbors in V_1'. Thus, we choose distinct $z_L^1, z^1, z_L^2, z^2, z_N^1, z_N^2, z_N^3 \in V_1'$ such that

$$x_L^1 \sim z_L^1, z^1, \quad x_L^2 \sim z_L^2, z^2, \quad x_N \sim z_N^1, z_N^2, z_N^3.$$

(Note that x_L^1 and x_L^2 may be from V_{12}^0, and therefore they may not have too many neighbors in V_1', we then choose z_L^1 and z_L^2 from V_1' as their neighbors, respectively.)

By (3.18), we see that any two vertices in $G[V_1']$ have at least $(1/3 - 16\beta)n_1 \geq 14$ neighbors in common. Thus, we can choose distinct vertices $z^{11}, z^{22}, z^{12}, v_1^R \in V_1' - \{x_L^1, x_L^2, x_N, z_L^1, z^1, z_L^2, z_N^1, z_N^2, z_N^3\}$ such that

$$z^{11} \sim z_L^1, z^1, \quad z^{22} \sim z_L^2, z^2, \quad z^{12} \sim z^{11}, z^{22}, \quad v_1^R \sim z^{12}, z_N^1.$$

Furthermore, by (3.18) again, we have $\delta(G[V_1']) \geq (2/3 - 8\beta)n_1 \geq 17$. Choose $z_1^1, z_2^2, z_N^{11} \in V_1'$ not chosen above such that

$$z_1^1 \sim z^1, z_2^2 \sim z^2, z_N^{11} \sim z_N^1.$$

Let T_{11} be the graph with

$$V(T_{11}) = \{x_L^1, x_L^2, x_N, z_N^1, z_L^1, z^1, z_L^2, z^{11}, z^{12}, z^{22}, z^2, z_N^2, z_N^3, v_1^R, z_1^1, z_2^2, z_N^{11}\}$$

and with edges indicated above except the edges $x_L^1 z_L^1$ and $x_L^2 z_L^2$. We see that T_{11} is a tree with v_1^R as the only degree 2 vertex, and $|V(T_{11})| = 17$ and $|L(T_{11})| = 9$. Notice that in T_{11}, z_L^1, x_L^1 and z_L^2, x_L^2 are leaves, and x_N is a non-leaf. Figure 3.6 gives a depiction of T_{11}.

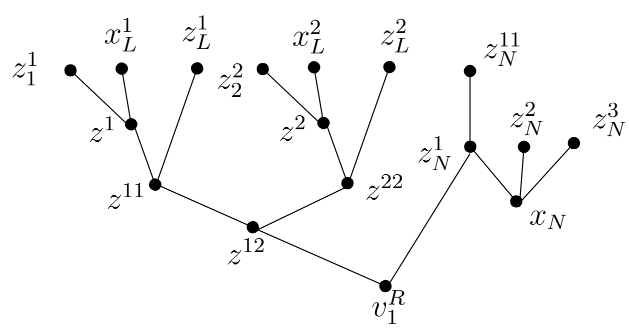


Figure (3.6) The tree T_{11}

Notice that the edges $x_L^1 z_L^1$ and $x_L^2 z_L^2$ are not used in T_{11}. We will first construct a HIST T_1 in $G[V_1^1 \cup V_{12}^0]$ containing T_{11} as a subgraph, then find a hamiltonian (z_L^1, z_L^2)-path on $L(T_1) - \{x_L^1, x_L^2\}$ by Lemma 3.2.6, finally by adding $x_L^1 z_L^1$ and $x_L^2 z_L^2$ to the path, we get a hamiltonian (x_L^1, x_L^2)-path on $L(T_1)$. The reason that we avoid using x_L^1 and x_L^2 is that when $x_L^1, x_L^2 \in V_{12}^0$, we may not be able to have the condition of Lemma 3.2.6 on $G[L(T_1)]$ in our final construction.

Then we initiate a HIT in $G[V_2']$ containing y_L^1, y_L^2 as leaves, and y_N as a non-leaf.

As $y_L^1, y_L^2, y_N \in V_2'$, by (3.19) and the fact that each two vertices from V_2' have at least $(1/3 - 2.2\alpha_1)n_2 \geq 7$ common neighbors implied from (3.19), we can choose distinct vertices

$$y^{12}, y_N^1, y_N^2, y_N^3, v_2^R \in V_2' - \{y_L^1, y_L^2, y_N\}$$

such that

$$y^{12} \sim y_L^1, y_L^2, \quad y_N \sim y_N^1, y_N^2, y_N^3, \quad v_2^R \sim y^{12}, y_N. \tag{3.19}$$

Let T_{21} be the graph with $V(T_{21}) = \{y_L^1, y_L^2, y_N, y^{12}, y_N^1, y_N^2, y_N^3, v_2^R\}$ and with $E(T_{21})$ described as in (3.19).

We see that T_{21} is a tree with v_2^R the only degree 2 vertex and $y_L^1, y_L^2 \in L(T_{21})$, $y_N \in S(T_{21})$ and

$$|V(T_{21}) \cap V_2'| = 8, \quad |L(T_{21}) \cap V_2'| = 5. \tag{3.20}$$

Denote

$$U_1 = V_1' - V(T_{11}), \quad U_2 = V_2' - V(T_{21}), \quad \text{and} \quad V_{12} = V_{12}^0 - V(T_{11}).$$

Step 4. Absorbing vertices in V_{12}^0

We may assume that $V_{12}^0 \neq \emptyset$. For otherwise, we skip this step. Let $|V_{12}| = n_{12}$ and $V_{12}^0 = \{x_1, x_2, \cdots, x_{n_{12}}\}$.

Since $|V(T_{11})| = 17$, by (3.17), we get

$$\delta(V_{12}^0, U_1) > \alpha_1 |V_1'| - 17 \geq 3\alpha_2 |V_2| \geq 3|V_2 - V_2'| \geq 3|V_{12}^0|.$$

Thus, there is a claw-matching M_c from V_{12}^0 to U_1 centered in V_{12}^0. For $i = 1, 2, \cdots, n_{12}$, let x_{i1}, x_{i2} and x_{i3} be the three neighbors of x_i in M_c. If $n_{12} = 1$, let $T_a = M_c$, and we finish this step. Thus we assume $n_{12} \geq 2$.

By (3.18), each two vertices in in V_1' have at least

$$(1/3 - 16\beta)n_1 \geq 6\alpha_2|V_{12}^0| + 17 \tag{3.21}$$

neighbors in common. The above inequality holds as $n_1 \geq 2n/5 - 2\beta n$, $|V_2| \leq 3n/5 + 2\beta n$ by (3.14), and we can assume that $18\alpha_2/5 + 106\beta/15 + 12\alpha_2\beta + 18/n - 32\beta^2 \leq 2/15$.

Thus, for each $i = 1, 2, \cdots, n_{12} - 1$, we can find distinct vertices $x_{13}^i, x_{23}^i, x_{i3}^3, x_{i+1,1}^3$ in $U_1 - V(M_c)$ such that

$$x_{13}^i \sim x_{i3}, x_{i+1,1}, \quad x_{23}^i \sim x_{13}^i, \quad x_{i3}^3 \sim x_{i3}, \quad x_{i+1,1}^3 \sim x_{i+1,1}. \tag{3.22}$$

Let T_a be the graph with $V(T_a) = V(M_c) \cup \{x_{13}^i, x_{23}^i, x_{i3}^3, x_{i+1,1}^3 \ : \ 1 \leq i \leq n_{12} - 1\}$, and $E(T_a)$ including all edges indicated in (3.22) for all i and all edges in M_c. It is easy to see, by the construction, that T_a is a HIT with

$$|V(T_a) \cap U_1| = 7n_{12} - 4 \quad \text{and} \quad |L(T_a) \cap U_1| = 4n_{12} - 1.$$

Using (3.21) again, we can find $x_N^{11} \in U_1 - V(T_a)$ such that $x_N^{11} \sim v_1^R, x_{11}$, where $v_1^R \in V(T_{11})$ and $x_{11} \in V(T_a)$. By (3.18),

$$\delta[G[V_1']] \geq (2n+3)/5 - 2\beta n \geq 6\alpha_2|V_{12}^0| + 20,$$

since $|V_2| \leq 2n/5 + 2\beta n$, and we can assume that $2\beta - 12\alpha_2\beta - 18\alpha_2/5 - 21/n \leq 2/5$. So we can find distinct vertices $x_N^{12}, x_{11}^1 \in U_1 - V(T_a) - \{x_N^{11}\}$ such that $x_N^{12} \sim x_N^{11}, x_{11}^1 \sim x_{11}$.

Let T_1^1 be the graph with

$$V(T_1^1) = V(T_{11}) \cup V(T_a) \cup \{x_N^{11}, x_N^{12}, x_{11}^1\} \quad \text{and} \quad E(T_1^1) = E(T_{11}) \cup E(T_a) \cup \{x_N^{11}v_R^1, x_N^{11}x_{11}, x_N^{12}x_N^{11}, x_{11}^1x_{11}\}.$$

Then T_1^1 is a HIT such that

$$|V(T_1^1) \cap U_1| = 7n_{12} + 16 \quad \text{and} \quad |S(T_1^1) \cap U_1| = 3n_{12} + 7. \tag{3.23}$$

Denote $U_1' = U_1 - V(T_1^2)$ and $U_2' = U_2 - V(T_1^2)$.

Step 5. Completion of HITs T_1 and T_2

In this step, we construct a HIST T_i in $G[V_i'] \, (i = 1, 2)$ containing T_1^i as an induced subgraph.

The following lemma guarantees the existence of a specified HIST in a graph with n vertices and minimum degree at least $(2/3 - \alpha')n$ for some $0 < \alpha' \ll 1$.

Lemma 3.4.6. *Let H be an n-vertex graph with $\delta(H) \geq (2/3 - \alpha')n$ for some constant $0 < \alpha' \ll 1$. Then H has a HIST T_H satisfying*

(i) T_H has a vertex v_R of degree at least $(2/3 - \alpha')n - 1$, and v_R can be chosen arbitrarily from $V(H)$;

(ii) $|S(T_H)| \leq (1/6 + \alpha'/2)n + 2$.

Proof. Let $v_R \in V(H)$ be an arbitrary vertex. If $n(mod\, 2) \equiv deg(v_R) + 1(mod\, 2)$, then we let $N_R = N_H(v_R)$. For otherwise, let N_R be a subset of $N(v_R)$ with $|N_H(v_R)| - 1$ elements. Let T_{v_R} be the star with $V(T_{v_R}) = \{v_R\} \cup N_R$ and $E(T_R) = E(\{v_R\}, N_R)$. Let $V_0 = V(H) - V(T_{v_R})$. By $\delta(H) \geq (2/3 - \alpha')n$, we have $|V_0| \leq (1/3 + \alpha')n + 1$. By the choice of N_R, we have $|V_0| \equiv 0(mod\, 2)$. If $V_0 = \emptyset$, then let $T_H = T_{v_R}$. For otherwise, we claim as follows.

Claim 3.4.3. *Let $V_1 \subseteq V(H)$ be a subset with $|V_1| \geq (2/3 - \alpha')n - 1$ and $|V_1|(mod\, 2) \equiv n(mod\, 2)$. Then there exist two vertices from $V_0 = V(H) - V_1$ such that they have a common neighbor in V_1.*

Proof of Claim 3.4.3. We assume that $|V_1| \leq (2/3 + 2\alpha')n$. For otherwise, $|V_0| < (1/3 - 2\alpha')n$. Since $\delta(H) \geq (2/3 - \alpha')n$, any two vertices of H have at least $(1/3 - 2\alpha')n$ neighbors in common. By $|V_0| < (1/3 - 2\alpha')n$, any two vertices from V_0 have a common neighbor from

V_1. We are done. Thus $|V_1| \le (2/3 + 2\alpha')n$, and hence $|V_0| \ge (1/3 - 2\alpha')n \ge 3$. By the assumption that $|V_1| \ge (2/3 - \alpha')n - 1$, we have $|V_0| \le (1/3 + \alpha')n + 1$. This implies that $\deg(v_0, V_1) \ge (1/3 - 2\alpha')n - 2$ for each $v_0 \in V_0$. As $|V_0| \ge 3$ and $3((1/3 - 2\alpha')n - 2) > (2/3 + 2\alpha')n >\ge |V_1|$ (provided that $8\alpha' + 6/n < 1/3$), we see that there must be two vertices from V_0 such that they have a neighbor in common in V_1. $\square$

By Claim 3.4.3, there exist two vertices $v_0^{11}, v_0^{12} \in V_0$ such that they have a common neighbor in T_{v_R}. Adding v_0^{11} and v_0^{12} to T_{v_R} and two edges connecting them to one of their common neighbor in $V(T_{v_R})$. Let $T_{v_R}^1$ be the resulting graph. Then we see that $T_{v_R}^1$ is a HIT with $|V(T_{v_R}^1)| = |V(T_{v_R})| + 2$, and hence $(|V(T_{v_R})| + 2)(mod\, 2) \equiv n(mod\, 2)$. Also $|V(T_{v_R}^1)| \ge |V(T_{v_R})| \ge (2/3 - \alpha')n - 1$. So we can use Claim 3.4.3 again to find another pair of vertices from $V_0 - \{v_0^{11}, v_0^{12}\}$ such that they have a common neighbor in $V(T_{v_R}^1)$. Adding the new pair of vertices and two edges connecting them to one of their common neighbor in $V(T_{v_R}^1)$ into $T_{v_R}^1$, we get a new HIT $T_{v_R}^2$. By repeating the above process another $l_0 = (|V_0| - 4)/2$ times, we get a HIT $T_{v_R}^{l_0}$. Let $T_H = T_{v_R}^{l_0}$. We claim that T_H has the required properties in Lemma 3.4.6. Notice first that $d_{T_H}(v_R) \ge (2/3 - \alpha')n - 1$. Then since T_H has v_R and at most $|V_0|/2$ distinct vertices as non-leaves and $|V_0| \le (1/3 + \alpha')n + 1$, we see that $|S(T_H)| \le (1/6 + \alpha'/2)n + 2$. $\square$

Let $H_1 = G[U_1' \cup \{v_R^1\}]$. Recall that v_R^1 is a non-leaf in T_1^1. By (3.18) and (3.23), and by noticing that $n_{12} \le |V_2 - V_2'| \le \alpha_2 |V_2| \le 3\alpha_2 n_1/2$ (by (3.14)), we see that

$$\begin{aligned} \delta(H_1) &\ge (2/3 - 8\beta)n_1 - (7n_{12} + 19) \\ &\ge (2/3 - 8\beta)n_1 - 21\alpha_2 n_1/2 - 19 \\ &\ge (2/3 - 11\alpha_2)|V(H_1)|. \end{aligned} \tag{3.24}$$

Let $\alpha' = 11\alpha_2 \ll 1$ (by assuming $\alpha \ll 1$). By Lemma 3.4.6, we can find a HIT T_1' in H_1 with v_R^1 as the prescribed vertex in condition(i). It is easy to see that $T_1 := T_1^1 \cup T_1'$ is a

HIST of $G[V_1' \cup V_{12}^0]$ and

$$\begin{aligned} s_1 &= |S(T_1) \cap V_1'| = |S(T_1^1) \cap V_1'| + |S(T_1') \cap V_1'| \\ &\leq 3n_{12} + 7 + (1/6 + 5.5\alpha_2)|V(H_1)| + 2 \text{ (by (3.23) and Lemma 3.4.6)} \\ &\leq 3n_{12} + 9 + (1/6 + 5.5\alpha_2)n_1 \\ &\leq (1/6 + 10.5\alpha_2)n_1 \text{ (by } n_{12} \leq 3\alpha_2 n_1/2). \end{aligned} \tag{3.25}$$

Let $H_2 = G[U_2' \cup \{v_R^2\}]$. By(3.19) and (3.20), we see that

$$\delta(H_2) \geq (2/3 - 1.1\alpha_1)n_2 - 8 \geq (2/3 - 1.2\alpha_1)|V(H_2)|.$$

By letting $\alpha' = 1.2\alpha_1$, we can find a HIT T_2' in H_2 with v_R^2 as the prescribed vertex in condition (i) of Lemma 3.4.6. Then $T_2 := T_1^2 \cup T_2'$ is a HIST of $G[V_2']$. Also, notice that

$$\begin{aligned} s_2 &= |S(T_2) \cap V_2'| = |S(T_1^2) \cap V_2'| + |S(T_2') \cap V_2'| \\ &\leq 3 + (1/6 + 0.6\alpha_1)|V(H_2)| + 2 \\ &\leq (1/6 + 0.7\alpha_2)n_2, \end{aligned} \tag{3.26}$$

where the last inequality holds by assuming $5/n_2 \leq 0.1\alpha_2$.

Step 6. Finding two long paths

In this step, we first find a hamiltonian (z_L^1, z_2^2)-path P_1^1 in $G[L(T_1) - \{x_L^1, x_L^2\}]$; then find a hamiltonian (y_L^1, y_L^2)-path P_2 in $G[L(T_2)]$. Let $G_{11} = G[L(T_1) - \{x_L^1, x_L^2\}]$ and $n_{11} = |V(G_{11})|$. We will show that $\delta(G_{11}) > \frac{1}{2}n_{11}$. We may assume $s_1 \geq (1/6 - 8\beta)n_1 - 2$. For otherwise, if $s_1 < (1/6 - 8\beta)n_1 - 2$, then by (3.18), we get

$$\begin{aligned} \delta(G_{11}) &\geq \delta(G[V_1']) - s_1 - 2 \\ &\geq (2/3 - 8\beta)n_1 - ((1/6 - 8\beta)n_1 - 1 - 2) - 2 \\ &\geq \frac{1}{2}n_1 + 1 \geq \frac{1}{2}n_{11} + 1. \end{aligned}$$

Hence, $s_1 \geq (1/6 - 8\beta)n_1 - 2$, implying that

$$n_{11} \leq (5/6 + 8\beta)n_1 + 2 \quad \text{and thus} \quad n_1 \geq \frac{n_{11} - 2}{5/6 + 8\beta}. \tag{3.27}$$

Hence, by (3.25)

$$\begin{aligned} \delta(G_{11}) &\geq \delta(G[V_1']) - s_1 - 2 \geq (2/3 - 8\beta)n_1 - (1/6 + 10.5\alpha_2)n_1 - 2 \\ &\geq (1/2 - 8\beta - 11\alpha_2)n_1 \geq \frac{1/2 - 2\beta - 11\alpha_2}{5/6 + 2\beta}(n_{11} - 2) > n_{11}/2, \end{aligned}$$

the last inequality holds by assuming $3\beta + 11\alpha_2 + 2/n_{11} < 1/12$. By applying Lemma 3.4.6 on G_{11}, we find a hamiltonian (z_L^1, z_L^2)-path P_1^1 in G_{11}. Let $P_1 = P_1^1 \cup \{z_L^1 x_L^1, z_L^2 x_L^2\}$. We see that P_1 is a a hamiltonian (x_L^1, x_L^2)-path on $L(T_1)$.

Let $G_{22} = G[L(T_2)]$ and $n_{22} = |V(G_{22})|$. We will show that $\delta(G_{22}) > n_{22}/2$. We may assume that $s_2 \geq (1/6 - 1.1\alpha_1)n_2 - 2$. For otherwise, if $s_2 < (1/6 - 1.1\alpha_1)n_2 - 2$, then by (3.19), we see that

$$\begin{aligned} \delta(G_{22}) &\geq \delta(G[V_2']) - s_2 - 2 \\ &> (2/3 - 1.1\alpha_1)n_2 - ((1/6 - 1.1\alpha_1)n_2 - 2) - 2 \\ &> n_2/2 \geq n_{22}/2. \end{aligned}$$

Thus, $s_2 \geq (1/6 - 1.1\alpha_1)n_2 - 2$, implying that

$$n_{22} \leq n_1 - s_2 \leq (5/6 + 1.1\alpha_1)n_2 + 2 \quad \text{gives that} \quad n_2 \geq \frac{n_{22} - 2}{5/6 + 1.1\alpha_1}.$$

By (3.19) and (3.26),

$$\begin{aligned}
\delta(G_{22}) &\geq \delta(G[V_2']) - s_2 - 2 \\
&\geq (2/3 - 1.1\alpha_1)n_2 - (1/6 + 0.7\alpha_1)n_2 - 2 \\
&\geq (1/2 - 1.9\alpha_1)n_2 \geq \frac{1/2 - 1.9\alpha_2}{5/6 + 1.1\alpha_2}(n_{22} - 2) \\
&> n_{22}/2.
\end{aligned}$$

The last inequality follows by assuming that $2.45\alpha_1 + 2/n_{11} < 1/12$. Hence, by Lemma 3.4.6, there is a hamiltonian (y_L^1, y_L^2)-path P_2 in G_{22}.

Step 7. Forming an SGHG

Let $T = T_1 \cup T_2 \cup \{x_N y_N\}$ and $C = P_1 \cup P_2 \cup \{x_L^1 y_L^1, x_L^2 y_L^2\}$. We see that T is a HIST of G with $L(T) = V(P_1) \cup V(P_2)$ and C is a cycle spanning on $L(T)$. Hence $H = T \cup C$ is an SGHG of G.

3.4.3.3 Proof of Theorem 3.4.5 Notice that the assumption of Extremal Case 2 implies that

$$|V_1| > (3/5 - \alpha)n \quad \text{and} \quad |V_2| \geq (2/5 - 2\beta)n.$$

We may assume that the graph G is minimal with respective to the number of edges. This implies that no two adjacent vertices both have degree larger than $(2n+3)/5$. (For otherwise, we could delete any edges incident to two vertices both with degree larger than $(2n+3)/5$.) We construct an SGHG in G step by step.

Step 1. Repartitioning

Set $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = \alpha^{2/3}$. Let

$$\begin{aligned}
V_2' &= \{v \in V_2 \mid deg(v, V_1) \geq (1 - 3\alpha_1)|V_1|\}, \\
V_0' &= \{v \in V_2 - V_2' \mid deg(v, V_1) \leq \alpha_1 |V_2|/6\}, \\
V_1' &= V_1 \cup V_0', \quad V_{12}^0 = V_2 - V_2' - V_0'.
\end{aligned}$$

As $d(V_1, V_2) \geq 1 - 3\alpha$, the following holds,

$$\begin{aligned}(1-3\alpha)|V_1||V_2| &\leq e_G(V_1, V_2) = e_G(V_1, V_2') + e_G(V_1, V_2 - V_2') \\ &\leq |V_1||V_2'| + (1-3\alpha_1)|V_1||V_2 - V_2'|.\end{aligned}$$

The inequality implies that

$$|V_2 - V_2'| \leq \alpha_2 |V_2|. \tag{3.28}$$

As a consequence of moving vertices in $V_2 - V_2'$ out from V_2, by (3.28) we get

$$\begin{aligned}\delta(V_1, V_2') &\geq (2n+3)/5 - 2\beta n - \alpha_2|V_2| \\ &\geq (2n+3)/5 - 6\beta|V_2| - \alpha_2|V_2| \\ &\geq (2n+3)/5 - 2\alpha_2|V_2|, \end{aligned} \tag{3.29}$$

provided that $6\beta \leq \alpha_2$. And as a consequence of moving vertices in V_0' to V_1,

$$\begin{aligned}\delta(V_0', V_2') &\geq \delta(G) - \Delta(V_0', V_1) - \Delta(V_0', V_2 - V_2') \\ &\geq (2n+3)/5 - \alpha_1|V_2|/6 - \alpha_2|V_2| \\ &\geq (2n+3)/5 - \alpha_1|V_2|/3 \,(\text{provided that } \; \alpha_2 \leq \alpha_1/6), \end{aligned} \tag{3.30}$$

and

$$\alpha_1|V_2|/6 < \delta(V_{12}^0, V_1') < (1-3\alpha_1)|V_1|. \tag{3.31}$$

From (3.29) and (3.30), we have

$$\delta(V_1', V_2') \geq (2n+3)/5 - \alpha_1|V_2|/3. \tag{3.32}$$

As

$$\delta(V_2', V_1') \geq (1-3\alpha_1)|V_1| \geq (1-3\alpha_1)(3/5-\alpha)n > \lceil (2n+3)/5 \rceil, \tag{3.33}$$

we get that

$$deg(v_1') = \lceil (2n+3)/5 \rceil \tag{3.34}$$

for each $v_1' \in V_1'$, by the minimality assumption of $e(G)$. Hence (3.32) and (3.34) give that

$$\Delta(G[V_1']) \leq \alpha_1 |V_2|/3. \tag{3.35}$$

Step 2. Finding a vertex v_2^* from V_2' with large degree in V_1'

Let

$$e_{in} = e(G[V_1']) \tag{3.36}$$

be the number of edges within V_1', notice that e_{in} maybe 0. Then

$$e_G(V_1', V_2' \cup V_{12}^0) = |V_1'| \lceil (2n+3)/5 \rceil - 2e_{in}. \tag{3.37}$$

Let

$$d_{in} = e_{in}/|V_1'| \quad \text{and} \quad |n_0| = |\,|V_2' \cup V_{12}^0| - \lceil (2n+3)/5 \rceil|. \tag{3.38}$$

By (3.35) and the definition of d_{in} in (3.38), we have

$$\lfloor d_{in} \rfloor \leq \alpha_1 |V_2|/6.$$

In fact, since $\Delta(V_1, V_1') \leq \Delta(V_1, V_1) + \Delta(V_1, V_0') \leq 2\beta n + |V_0'| \leq 2\beta n + \alpha_2 |V_2|$, and $\Delta(V_0', V_1') \leq \alpha_1 |V_2|/6 + \alpha_2 |V_2|$, more precisely, we have

$$\begin{aligned}
2d_{in} &= 2e_{in}/|V_1'| \leq (2\beta n + \alpha_2|V_2|)|V_1|/|V_1'| + (\alpha_1|V_2|/6 + \alpha_2|V_2|)|V_0'|/|V_1'| \\
&\leq (2\beta n + \alpha_2|V_2|) + \alpha_2(\alpha_1|V_2|/6 + \alpha_2|V_2|) \text{ (as } |V_0'| \leq \alpha_2|V_2| \text{ and } |V_1|, |V_2| \leq |V_1'|) \\
&\leq (6\beta + \alpha_2 + \alpha/6 + \alpha_2^2)|V_2| \text{ (as } \beta n \leq 3\beta|V_2|) \\
&\leq 2\alpha_2|V_2| \text{ (provided that } 6\beta + \alpha/6 + \alpha_2^2 \leq \alpha_2).
\end{aligned} \tag{3.39}$$

Set

Case A. $\lceil(2n+3)/5\rceil - |V_2' \cup V_{12}^0| = n_0 \geq 0$;

Case B. $|V_2' \cup V_{12}^0| - \lceil(2n+3)/5\rceil = n_0 \geq 1$.

We have

$$n_0 = \begin{cases} \lceil(2n+3)/5\rceil - |V_2' \cup V_{12}^0| \leq 2\beta n + \alpha_2|V_2| \leq (6\beta+\alpha_2)|V_2| \leq 2\alpha_2|V_2|, & \text{Case A,} \\ \\ |V_2' \cup V_{12}^0| - \lceil(2n+3)/5\rceil \leq (2/5+\alpha)n - \lceil(2n+3)/5\rceil \leq \alpha n, & \text{Case B.} \end{cases} \tag{3.40}$$

Then in case A,

$$\begin{aligned} e_G(V_1', V_2' \cup V_{12}^0) &= |V_1'|\lceil(2n+3)/5\rceil - 2e_{in} \quad \text{(by (3.34))} \\ &= |V_1'|(|V_2' \cup V_{12}^0| + n_0 - 2d_{in}) \\ &\geq |V_2' \cup V_{12}^0|(|V_1'| + 1.4n_0 - 3.2d_{in}), \end{aligned}$$

as $1.4|V_2' \cup V_{12}^0| \leq 1.4((2n+3)/5 + \alpha n) \leq (3/5-\alpha)n < |V_1'|$ and $1.6|V_2' \cup V_{12}^0| \geq 1.6((2n+3)/5 - 2\beta - \alpha_2)n \geq (3/5 + 2\beta + \alpha_2)n) > |V_1'|$ provided that $2.4\alpha < 1/25$ and $5.2\beta + 2.6\alpha_2 \leq 1/25$ respectively. Since $e_G(V_1', V_2' \cup V_{12}^0) \leq |V_2' \cup V_{12}^0||V_1'|$, we have $|V_1'| + 1.4n_0 - 3.2d_{in} \leq |V_1'|$, and thus $1.4n_0 \leq 3.2d_{in}$.

In Case B,

$$\begin{aligned} e_G(V_1', V_2' \cup V_{12}^0) &= |V_1'|\lceil(2n+3)/5\rceil - 2e_{in} \quad \text{(by (3.34))} \\ &= |V_1'|(|V_2' \cup V_{12}^0| - n_0 - 2d_{in}) \\ &\geq |V_2' \cup V_{12}^0|(|V_1'| - 1.6n_0 - 3.2d_{in}), \end{aligned}$$

as $1.6|V_2' \cup V_{12}^0| \geq 1.6((2n+3)/5 - 2\beta - \alpha_2 n) \geq (3/5 + 2\beta + \alpha_2)n > |V_1'|$ provided that $5.2\beta + 2.6\alpha_2 \leq 1/25$.

Let

$$d_l = \begin{cases} \lfloor 3.2d_{in} - 1.4n_0 \rfloor, & \text{if Case A,} \\ \\ \lfloor 1.6n_0 + 3.2d_{in} \rfloor, & \text{if Case B.} \end{cases} \tag{3.41}$$

By (3.39) and (3.40), we see that

$$d_l \leq \begin{cases} 3.2\alpha_2|V_2|, & \text{if Case A,} \\ 6.4\alpha_2|V_2|, & \text{if Case B.} \end{cases} \tag{3.42}$$

Then there is a vertex v_2^* in $V_2' \cup V_{12}^0$ of degree at least $|V_1'| - d_l$. We will fix this vertex in what follows. In fact, such a vertex v_2^* is in V_2' by the facts that

$$\delta(V_{12}^0, V_1') < (1-3\alpha_1)|V_1| \quad \text{and} \quad |V_1'| - d_l \geq (1-3\alpha_1)|V_1|, \tag{3.43}$$

where $|V_1'| - d_l \geq (1-3\alpha_1)|V_1|$ holds because of (3.42).

Step 3. Finding a matching M within $G[\Gamma(v_2^*, V_1')]$

In this step, if $e_{in} \geq 1$, we first find a matching within $G[V_1']$ of size at least $e_{in}/(2 \triangle (G[V_1']))$. We assume this by giving the following lemma.

Lemma 3.4.7. *If G is a graph with maximum degree Δ, then G contains a matching of size at least $\frac{|E(G)|}{2\Delta}$.*

Proof. We use induction on $|V(G)|$. We may assume that the graph is connected. For otherwise, we are done by the induction hypothesis. Let $e = xy \in E(G)$ be an edge and $G' = G - \{x, y\}$. Since $|N_G(x) \cup N_G(y)| - |\{x, y\}| \leq 2(\Delta - 1)$, we have

$$e(G') \geq e(G) - 2(\Delta - 1) - 1 \geq e(G) - 2\Delta.$$

Hence, by the induction hypothesis, G' has a matching of size at least $\frac{e(G)-2\Delta}{2\Delta} = \frac{e(G)}{2\Delta} - 1$. Adding e to the matching obtained in G'gives a matching of size at least $\frac{e(G)}{2\Delta}$ in G. $\square$

In case A, we take a matching in $G[V_1']$ of size at least $\max\{\lfloor 11d_{in} \rfloor, 11n_0\}$. This is possible because

$$\frac{e_{in}}{2 \triangle (G[V_1'])} \geq \frac{e_{in}}{2\alpha_1|V_1'|/3} = \frac{3d_{in}}{2\alpha_1} \geq 11d_{in}$$

provided that $\alpha \leq (\frac{3}{22})^3$, and

$$\begin{aligned}
2e_{in} &\geq |V_1'|\lceil (2n+3)/5\rceil - |V_1'||V_2'| - (1-3\alpha_1)|V_1||V_{12}^0| \\
&\geq |V_1'|\lceil (2n+3)/5\rceil - |V_1'|(\lceil (2n+3)/5\rceil - n_0 - |V_{12}^0|) - |V_1||V_{12}^0| + 3\alpha_1|V_1||V_{12}^0| \\
&\geq |V_1'|n_0 + 3\alpha_1|V_1||V_{12}^0| \qquad (3.44)
\end{aligned}$$

implying that

$$\frac{e_{in}}{2\bigtriangleup(G[V_1'])} \geq \frac{e_{in}}{2\alpha_1|V_1'|/3} \geq \frac{|V_1'|n_0/2}{2\alpha_1|V_1'|/3} \geq \frac{3n_0}{4\alpha_1} \geq 11n_0$$

provided that $\alpha \leq (\frac{3}{44})^3$.

By (3.41), $|V_1'| - \Gamma(v_2^*, V_1') \leq d_l \leq \lfloor 3.2d_{in}\rfloor$, we can then choose a matching M from $\Gamma(v_2^*, V_1')$ such that

$$|M| = \max\{\lfloor 7d_{in}\rfloor, 7n_0\}. \qquad (3.45)$$

In case B, we take a matching in $G[V_1']$ of size at least $\lfloor 8d_{in}\rfloor$. This is possible as

$$\frac{e_{in}}{\bigtriangleup(G[V_1'])} \geq \frac{e_{in}}{2\alpha_1|V_1'|/3} = \frac{3d_{in}}{2\alpha_1} \geq \lfloor 8d_{in}\rfloor$$

provided that $\alpha \leq (\frac{3}{16})^3$.

By the second equality of (3.41), $|V_1'| - \Gamma(v_2^*, V_1') \leq \lfloor 3.2d_{in} + 1.6n_0\rfloor$. If $n_0 < 2d_{in}$, then $\lfloor 3.2d_{in} + 1.6n_0\rfloor \leq \lfloor 7d_{in}\rfloor$. Thus, there is a matching M within $\Gamma(v_2^*, V_1')$ such that

$$|M| = \begin{cases} \lfloor d_{in}, \rfloor & \text{if } n_0 < 2d_{in}, \\ 0, & \text{if } n_0 \geq 2d_{in}. \end{cases} \qquad (3.46)$$

We fix M for denoting the matching we defined in this step hereafter.

Step 4. Insertion

In this step, we insert vertices in V_{12}^0 into $V_1' - V(M)$. Let $I = V_{12}^0 = \{x_1, x_2, \cdots, x_I\}$,

$U_1 = \Gamma(v_2^*, V_1') - V(M)$, and $U_2 = V_2'$. Then (i)

$$
\begin{aligned}
\delta(I, U_1) &\geq \delta(I, V_1') - |V(M)| - |V_1' - \Gamma(v_2^*, V_1')| \\
&\geq \alpha_1|V_2|/6 - \max\{\lfloor 7d_{in} \rfloor, 7n_0\} - \lfloor 1.6n_0 + 3.2d_{in} \rfloor, \\
&\geq \alpha_1|V_2|/6 - 20.4\alpha_2|V_2| \quad \text{(by (3.39) and (3.40))} \\
&\geq 3\alpha_2|V_2| \geq 3|I| \quad \text{(provided that } 23.4\alpha_2 \leq \alpha_1/6\text{)},
\end{aligned}
$$

and from (3.32), we have (ii)

$$\delta(U_1, U_2 - \{v_2^*\}) \geq \lceil (2n+3)/5 \rceil - \alpha_1|V_2|/3 - 1 > \alpha_2|V_2| \geq |I|.$$

By condition (i), there is a claw-matching M_1 between I and U_1 centered in I. Suppose that $\Gamma(x_i, M_1) = \{x_{i0}, x_{i1}, x_{i2}\}$. We denote by P_{x_i} the path $x_{i1}x_ix_{i2}$. By (ii), there is a matching M_2 between $\{x_{i0} \,|\, 1 \leq i \leq |I|\}$ and $U_2 - \{v_2^*\}$ covering $\{x_{i0} \,|\, 1 \leq i \leq |I|\}$. So far, we get two matchings M_1 and M_2.

Next we delete three types of edges not contained in

$$(\bigcup_{i=1}^{|I|} E(P_{x_i})) \cup \{x_ix_{i0} \,:\, 1 \leq i \leq |I|\}.$$

Those edges include edges incident to a vertex in I, edges incident to a vertex in

$$\bigcup_{i=1}^{|I|} ((\Gamma(x_{i1}) - \Gamma(x_{i2})) \cup (\Gamma(x_{i2}) - \Gamma(x_{i1}))),$$

and one edge from the two edges connecting a vertex in $\Gamma(x_{i1}) \cap \Gamma(x_{i2})$ to both x_{i1} and x_{i2}, for each $i = 1, 2, \cdots, |I|$.

For the resulting graph after the deletion of edges above, we contract each path P_{x_i} ($1 \leq i \leq |I|$) into a single vertex v_{x_i}. We call each v_{x_i} a *wrapped vertex* and call P_{x_i} the *preimage* of v_{x_i}. Denote by G^* the graph obtained by deleting and contracting the same edges as above, and let $U_2^* = V_2'$ and $U_1^* = V(G^*) - U_2^*$. (We will need the following degree condition in the end of this proof.) Since $|U_2^*| = |V_2'| \le (2/5+\alpha)n$, combining with (3.32), we have

$$deg(v_{x_i}, U_2^*) \ge |\Gamma(x_{i1}, U_2^*) \cap \Gamma(x_{i2}, U_2^*)| - 1 \ge 2n/5 - \alpha_1|V_2|.$$

By the above inequality and (3.32), we get the first inequality below in (3.47). Since one edge from the two edges connecting a vertex in $\Gamma(x_{i1}) \cap \Gamma(x_{i2})$ to both x_{i1} and x_{i2} is deleted in G^* for each $i = 1, 2, \cdots, |I|$, combining with (3.33), we have the second inequality as follows.

$$\begin{aligned} \delta(U_1^*, U_2^*) &\ge 2n/5 - \alpha_1|V_2|, \\ \delta(U_2^*, U_1^*) &\ge \delta(V_2', V_1') - 1 \ge (1-3\alpha_1)|V_1| - 1. \end{aligned} \tag{3.47}$$

Let U_1' and U_2' be the corresponding sets of U_1 and U_2, respectively, after the contraction. Let T_W be the graph with

$$V(T_W) = \{x_{i0}, v_{x_i} : 1 \le i \le |I|\} \cup (V(M_2) \cap U_2) \quad \text{and} \quad E(T_W) = \{x_{i0}v_{x_i} : 1 \le i \le |I|\} \cup E(M_2).$$

By the construction,

$$|V(T_W) \cap U_1'| = |\{x_{i0}, v_{x_i} : 1 \le i \le |I|\}| = 2|I|, \ |L(T_W) \cap U_1'| = |\{v_{x_i} : 1 \le i \le |I|\}| = |I|, \quad \text{and}$$

$$|V(T_W) \cap U_2'| = |L(T_I) \cap U_2'| = |V(M_2) \cap U_2'| = |I|.$$

Notice that T_W is a forest with $|I|$ components and each vertex $x_{i0}\,(1 \le i \le |I|)$ has degree 2 in T_W. (We will make T_W connected in the end by connecting each x_{i0} to v_2^*.) See a depiction of this operation with $|I| = 1$ in Figure 3.7 below.

Let $U_I^1 = (V_1' - U_1) \cup U_1' - V(T_W)$, $U_I^2 = U_2' - V(T_W)$, and G_I the resulting graph with

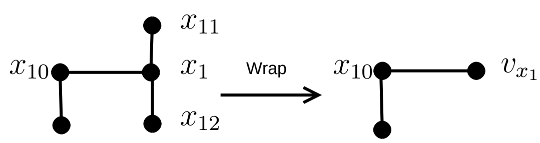


Figure (3.7) T_W with $|I| = 1$

$V(G_I) = U_I^1 \cup U_I^2$. We have that

$$\begin{aligned} |U_I^1| &= |V_1'| - 3|I| = |V_1'| - 3n_{12}^0, \quad |U_I^2| = |V_2'| - |I| = |V_2'| - n_{12}^0, \\ \delta(U_I^1, U_I^2) &\geq \delta(V_1', V_2') - n_{12}^0 \geq \lceil (2n+3)/5 \rceil - \alpha_1 |V_2|/3 - n_{12}^0, \\ \delta(U_I^2, U_I^1) &\geq \delta(V_2', V_1') - 3n_{12}^0 \geq (1 - 3\alpha_1)|V_1| - 3n_{12}^0. \end{aligned} \tag{3.48}$$

Step 5. Matching Extension

In this step, in the graph G_I, we do some operation on the matching M found in Step 3. Notice that the vertices in M are unused in Step 4. Recall that $|M| \leq \max\{7n_0, \lfloor 7d_{in} \rfloor\}$. By $\lfloor d_{in} \rfloor \leq \alpha_2 |V_2|$ from (3.39) and $n_0 \leq 2\alpha_2 |V_2|$ from (3.40), we get

$$|M| \leq 14\alpha_2 |V_2|. \tag{3.49}$$

Hence, $\delta(U_I^1, U_I^2 - \{v_2^*\}) \geq \lceil (2n+3)/5 \rceil - \alpha_1 |V_2|/3 - n_{12}^0 - 1 \geq |M|$. Let V_M be the set of vertices containing exactly one end of each edge in M. Then there is a matching M' between V_M and $U_2 - \{v_2^*\}$ covering V_M. Let F_M be a forest with

$$V(F_M) = V(M) \cup (V(M') \cap U_2) \quad \text{and} \quad E(F_M) = E(M) \cup E(M').$$

Notice that

$$|V(F_M) \cap U_1| = 2|M|, \quad |L(F_M) \cap U_1| = |V(M) - V_M| = |M|,$$

$$|V(F_M) \cap U_2| = |L(F_M) \cap U_2| = |M|.$$

Notice that F_M has $|M|$ components, and all vertices in V_M has degree 2. (We will make F_M a HIT later on by connecting each vertex in V_M to the vertex $v_2^* \in U_2$) See Figure 3.8 for a depiction of F_M with $|M| = 3$.

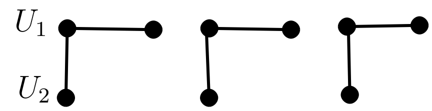


Figure (3.8) F_M with $|M| = 3$

Let

$$U_M^1 = U_I^1 - V(F_M) \quad \text{and} \quad U_M^2 = U_I^2 - V(F_M).$$

Notice that

$$\begin{aligned} |U_M^1| &= |U_I^1| - 2|M| = |V_1'| - 3n_{12}^0 - 2|M|, \\ |U_M^2| &= |U_I^2| - |M| = |V_2'| - n_{12}^0 - |M|, \end{aligned} \tag{3.50}$$

and

$$\begin{aligned} \delta(U_M^1, U_M^2) &= \lceil (2n+3)/5 \rceil - \alpha_1 |V_2|/3 - n_{12}^0 - |M|, \\ \delta(U_M^2, U_M^1) &\geq (1 - 3\alpha_1)|V_1| - 3n_{12}^0 - 2|M|. \end{aligned} \tag{3.51}$$

Step 6. Distribute Remaining vertices in $U_M^1 - \Gamma(v_2^*, V_1')$

Let

We may assume $n_l \geq 1$. For otherwise, we skip this step. By (3.42), we have

$$n_l \leq \begin{cases} 3.2\alpha_2|V_2|, & \text{Case A,} \\ 6.4\alpha_2|V_2|, & \text{Case B.} \end{cases} \tag{3.52}$$

By $n_{12}^0 \le \alpha_2|V_2|$ from (3.28) and $|M| \le 14\alpha_2|V_2|$ from (3.49), we have (i)

$$\begin{aligned}
\delta(U_M^1, U_M^2) &\ge \lceil(2n+3)/5\rceil - \alpha_1|V_2|/3 - n_{12}^0 - |M| \ge \lceil(2n+3)/5\rceil - \alpha_1|V_2|/3 - 15\alpha_2|V_2| \\
&\ge (1-3\alpha)|V_2| - \alpha_1|V_2|/3 - 15\alpha_2|V_2| \quad (\text{as } \lceil(2n+3)/5\rceil \ge (1-3\alpha)(2/5+\alpha)n) \\
&\ge (1-3\alpha-\alpha_1/3-15\alpha_2)|V_2|) \ge (1-\alpha_1)|V_2| \quad (\text{provided}\, 3\alpha+15\alpha_2 \le 2\alpha_1/3) \\
&\ge (1-\alpha_1)|U_M^2|. \qquad (3.53)
\end{aligned}$$

By (3.50) and (3.52), we have (ii)

$$\begin{aligned}
|U_M^2| - 10\alpha_1|V_2| - \lceil n_l/10\rceil - 1 &\ge |V_2'| - n_{12}^0 - |M| - 16\alpha_1|V_2| - 0.64\alpha_2|V_2| - 2 \\
&\ge (1-\alpha_2-14\alpha_2-10\alpha_1-0.64\alpha_2-|V_2|/2)|V_2| \\
&\ge (1-11\alpha_1)|V_2| \quad (\text{provided}\, 15.64\alpha_2 + |V_2|/2 \le \alpha_1) \\
&> 0 \quad (\text{provided}\, 11\alpha_1 < 1).
\end{aligned}$$

Let $U_R = U_M^1 - \Gamma(v_2^*, V_1')$ and denote $\left\lceil \frac{|U_R|}{10} \right\rceil = l$. Suppose first that $|U_R| \ge 2$. We partition $U_R = U_{R_1} \cup U_{R_2} \cup \cdots \cup U_{R_l}$ arbitrary such that each set contains at least 2 and at most $|U_R|/10$ vertices. Then by the conditions (i) and (ii), for each $1 \le i \le l$, there is a vertex $y_i \in U_2 - \{v_2^*\}$ which is common to all vertices in U_{R_i}, and is not used by any other U_{R_j} with $j \ne i$. Let T_R be the graph with

$$V(T_R) = U_R \cup \{y_i \,:\, 1 \le i \le l\} \quad \text{and} \quad E(T_R) = \{xy_i \,:\, x \in U_{R_i}, 1 \le i \le l\}.$$

Suppose now $|U_R| = 1$, let $U_R = \{x_R\}$. Choose $x_R' \in U_M^1 - U_R$ and $y_R \in U_M^2 - \{v_2^*\}$ be a vertex common to x_R and x_R'. Let T_R be a tree with

$$V(T_R) = \{x_R, x_R', y_R\} \quad \text{and} \quad E(T_R) = \{x_R y_R, x_R' y_R\}.$$

By the construction,

$$|V(T_R) \cap U_M^1| = |L(T_R \cap U_M^1| = \max\{|U_R|, 2\}, \quad |V(T_R) \cap U_M^2| = l, \quad \text{and} \quad |L(T_R \cap U_M^2| = 0.$$

Notice that T_R is not connected when $|U_R| \geq 17$ and that T_R may have degree 2 vertices in $V(T_R) \cap U_M^2$. Later on, by joining each vertex in $T_R \cap U_M^2$ to a vertex of a tree, we will make the resulting graph connected, and thereby eliminating the possible degree 2 vertices in T_R. Let

$$U_R^1 = U_M^1 - V(T_R) \quad \text{and} \quad U_R^2 = U_M^2 - V(T_R).$$

Then we have

$$\begin{aligned} |U_R^1| &= |U_M^1| - n_l = |V_1'| - 3n_{12}^0 - 2|M| - \max\{2, n_l\}, \\ |U_R^2| &= |U_M^2| - \lceil n_l/10 \rceil = |V_2'| - n_{12}^0 - |M| - \lceil n_l/10 \rceil, \end{aligned} \tag{3.54}$$

and

$$\begin{aligned} \delta(U_R^1, U_R^2) &\geq \lceil (2n+3)/5 \rceil - \alpha_1 |V_2|/3 - n_{12}^0 - |M| - \lceil n_l/10 \rceil, \\ \delta(U_R^2, U_R^1) &= (1 - 3\alpha_1)|V_1| - 3n_{12}^0 - 2|M| - max\{2, n_l\}. \end{aligned} \tag{3.55}$$

Let G_R be the subgraph of G induced on $U_R^1 \cup U_R^2$.

Step 7. Completion of a HIST in G_R

In this step, we find a HIST T_{main} in G_R such that

$$|L(T_{main}) \cap U_R^1| = |L(T_W)|/2 + |L(F_M) \cap U_I^1| + |L(T_R) \cap U_M^1| \;\; =$$
$$|L(T_{main} \cap U_R^2)| = |L(T_W)|/2 + |L(F_M) \cap U_I^2| + |L(T_R) \cap U_M^1|.$$

By the construction of F_M and T_R, we have $|L(F_M) \cap U_I^1| = |L(F_M) \cap U_I^2|$ and $|L(T_R) \cap$

$U_M^1| - |L(T_R) \cap U_M^2| = \max\{2, n_l\}$, respectively. So

$$|L(T_{main}) \cap U_R^2| - |L(T_{main}) \cap U_R^1| = \max\{2, n_l\}. \tag{3.56}$$

Notice that $v_2^* \in U_R^2$, v_2^* is adjacent to each vertex in U_R^1, and $V_1' - \Gamma(v_2^*, V_1') \subseteq V(U_R^1)$. We now construct T_{main} step by step.

Step 7.1

Let T_{main}^1 be the graph with

$$V(T_{main}^1) = \{v_2^*\} \cup U_R^1 \quad \text{and} \quad E(T_{main}^1) = \{v_2^* x \mid x \in U_R^1\}.$$

To make the requirement of (3.56) possible, we need to make at least

$$\begin{aligned} d_3 &= |U_R^1| - |U_R^2| + \max\{2, n_l\}, \\ &= |V_1'| - |V_2'| - 2n_{12}^0 - |M| + \lceil n_l/10 \rceil \end{aligned} \tag{3.57}$$

vertices in U_R^1 with degree at least 3 in T_{main}, where the last inequality above follows from (3.54). Hereinafter, we assume that $\max\{2, n_l\} = n_l$. Since the proof for $\max\{2, n_l\} = 2$ follows the same idea, we skip the details.

Since all vertices in U_R^1 are included in T_{main}^1 and T_{main}^1 is connected, each vertex in T_{main}^1 needs to join to at least two distinct vertices from $U_R^2 - \{v_2^*\}$ to have degree no less than 3. Hence, to make a desired HIST T_{main}, it is necessary that

$$\begin{aligned} d_{f*} &= |U_R^2| - 1 - 2d_3 \\ &= |V_2'| - n_{12}^0 - e_M - \lceil n_l/10 \rceil - 1 - 2d_3 \\ &= 3|V_2'| - 2|V_1'| + 3n_{12}^0 + |M| - 3\lceil n_l/10 \rceil - 1 \\ &\geq 0. \end{aligned} \tag{3.58}$$

We show (3.58) is true, separately, for each of Case A and Case B. For Case A, notice that

$$|V_1'| = n - \lceil (2n+3)/5 \rceil + n_0 \quad \text{and} \quad |V_2'| = \lceil (2n+3)/5 \rceil - n_0 - n_{12}^0.$$

Hence,

$$3|V_2'| = 3\lceil (2n+3)/5 \rceil - 3n_0 - 3n_{12}^0 \quad \text{and} \quad 2|V_1'| = 2n - 2\lceil (2n+3)/5 \rceil + 2n_0.$$

Thus,

$$\begin{aligned}
d_{f*} &= 5\lceil (2n+3)/5 \rceil - 2n - 5n_0 - 3n_{12}^0 + 3n_{12}^0 + |M| - 3\lceil n_l/10 \rceil - 1 \\
&\geq 2 - 5n_0 + |M| - 3\lceil \lfloor 3.2d_{in} \rfloor /10 \rceil \quad (\text{by } n_l \leq d_l \lfloor 3.2d_{in} - 1.4n_0 \rfloor \text{ from (3.41)}) \\
&= 2 - 5n_0 + \max\{7n_0, \lfloor 7d_{in} \rfloor\} - 3\lceil \lfloor 3.2d_{in} \rfloor /10 \rceil \\
&\geq 0.
\end{aligned}$$

Now we show (3.58) is true for case B. Notice that

$$|V_1'| = n - \lceil (2n+3)/5 \rceil + n_0 \quad \text{and} \quad |V_2'| = \lceil (2n+3)/5 \rceil + n_0 - n_{12}^0.$$

So

$$3|V_2'| = 3\lceil (2n+3)/5 \rceil + 3n_0 - 3n_{12}^0 \quad \text{and} \quad 2|V_1'| = 2n - 2\lceil (2n+3)/5 \rceil + 2n_0.$$

Recall that $n_0 \geq 1$ in this case. We have

$$\begin{aligned}
d_{f*} &= 5\lceil (2n+3)/5 \rceil - 2n + n_0 - 3n_{12}^0 + 3n_{12}^0 + |M| - 3\lceil n_l/10 \rceil - 1 \\
&\geq 2 + n_0 + |M| - 3\lceil n_l/10 \rceil \\
&= 2 + n_0 + |M| - 3\lceil \lfloor 3.2d_{in} + 1.6n_0 \rfloor /10 \rceil \quad (\text{by } n_l \leq \lfloor 3.2d_{in} + 1.6n_0 \rfloor \text{ from (3.41)}) \\
&\geq \begin{cases} 2 + n_0 + \lfloor d_{in} \rfloor - \lfloor 9.2d_{in}/10 \rfloor - \lfloor 4.8n_0/10 \rfloor - 1 \geq 0, & \text{if } n_0 < 2d_{in}; \\ 2 + n_0 - \lfloor 9.2d_{in}/10 \rfloor - \lfloor 4.8n_0/10 \rfloor - 1 \geq 0, & \text{if } n_0 \geq 2d_{in}. \end{cases}
\end{aligned}$$

We now in Step 2 below show that there is a way to make exactly d_{f*} vertices in T^1_{main} with degree 3 by joining each to two distinct vertices from $U^2_R - \{v^*_2\}$.

Step 7.2

We first take $2d_3$ vertices from $U^2_R - \{v^*_2\}$. For those $2d_3$ vertices, pair them up into d_3 pairs. We show that for each pair of vertices, they have at least d_3 common neighbors in U^1_R. Using (3.55), $|M| \leq 14\alpha_2|V_2|$ from (3.49), $n_l \leq d_l \leq 6.4\alpha_2|V_2|$ from (3.42), we have

$$\begin{aligned}\delta(U^2_R, U^1_R) &\geq (1-3\alpha_1)|V_1| - 3n^0_{12} - 2|M| - \max\{2, n_l\} \\ &\geq |V_1| - 3\alpha_1|V_1| - 3\alpha_2|V_2| - 28\alpha_2|V_2| - 6.4\alpha_2|V_2| \\ &\geq |V'_1| - |V_1 - V_1| - 37.4\alpha_2|V_2| - 3\alpha_1|V_1|. \end{aligned} \tag{3.59}$$

Since $|U^1_R| \leq |V'_1|$, we know that any two vertices in U^2_R have at least

$$\begin{aligned} n_c &= |V_1| - 2|V'_1 - V_1| - 74.8\alpha_2|V_2| - 6\alpha_1|V_1| \\ &\geq (3/5-\alpha)n - 76.8\alpha_2|V_2| - 6\alpha_1|V_1| \quad (\text{by } |V'_1 - V_1| = |V'_0| \leq |V_2 - V'_2| \leq \alpha_2|V_2|) \\ &\geq 3n/5 - 10\alpha_1|V_1| \quad (\text{provided that } 76.8\alpha_2 + 3\alpha \leq 4\alpha_1) \end{aligned}$$

common neighbors in U^1_R. On the other hand,

$$\begin{aligned} d_3 &= |V'_1| - |V'_2| - 2n^0_{12} - |M| - \lceil n_l/10 \rceil \\ &\leq (3/5-\alpha)n - (2n/5 - 2\beta n - |V_2 - V'_2|) + (1.6n_0 + 3.2\lfloor d_{in} \rfloor)/10 + 1 \\ &= n/5 - \alpha n + 2\beta n + |V_2 - V'_2| + (3.2\alpha_2|V_2| + 3.2\alpha_2|V_2|)/10 \quad (\text{by (3.39) and (3.40)}) \\ &\leq n/5 - \alpha n + 2\beta n + \alpha_2|V_2| + 0.64\alpha_2|V_2| \\ &\leq n/5 + 2\alpha_1|V_2| < n_c \, (\text{provided } 12\alpha_1 < 2/5). \end{aligned}$$

Denote by $\{u^1_1, u^2_1\}, \{u^1_2, u^2_2\}, \cdots, \{u^1_{d_3}, u^2_{d_3}\}$ the d_3 pairs of vertices from $U^2_R - \{v^*_2\}$. Then by the above argument, we can choose d_3 distinct vertices say $v_1, v_2, \cdots, v_{d_3}$ from $L(T^1_{main})$ such that $v_i \sim u^1_i, u^2_i$ for all $1 \leq i \leq d_3$.

Let T^2_{main} be the graph with

$$V(T^2_{main}) = V(T^1_{main}) \cup \{u_i^1, u_i^2 \,:\, 1 \le i \le d_3\} \quad \text{and} \quad E(T^2_{main}) = E(T^1_{main}) \cup \{v_i u_i^1, v_i u_i^2 \,:\, 1 \le i \le d_3\}.$$

If $V(G_R) - V(T^2_{main}) = \emptyset$, we let $T_{main} = T^2_{main}$. For otherwise, we need one more step to finish constructing T_{main}.

Step 7.3

For the remaining vertices in $U_R^2 - V(T^2_{main})$, we show that each of them has a neighbor in $S(T^2_{main}) \cap U_R^1$; that is, a neighbor in U_R^1 of degree 3 in $V(T^2_{main})$. This is clear, as by (3.59), we have

$$\begin{aligned}
\delta(U_R^2, U_R^1) &\ge |V_1'| - |V_1' - V_1| - 37.4\alpha_2|V_2| - 3\alpha_1|V_1| \\
&\ge |U_R^1| - 38.4\alpha_2|V_2| - 3\alpha_1|V_1| \quad (\text{by } |V_1' - V_1| \le |V_2 - V_2'| \le \alpha_2|V_2|).
\end{aligned}$$

Since $|S(T^2_{main}) \cap U_R^1| = d_3$, and

$$\begin{aligned}
d_3 &= |V_1| - |V_2'| - 2n_{12}^0 - 2|M| + \lceil n_l/10 \rceil \\
&\ge (3/5 - \alpha)n - (2/5 + \alpha)n - 2\alpha_2|V_2| - 28\alpha_2|V_2| + 0.64\alpha_2|V_2| \\
&\ge n/5 - 2\alpha n - 29.36\alpha_2|V_2| \\
&> 38.4\alpha_2|V_2| + 3\alpha_1|V_1| \quad (\text{provided } 2\alpha + 67.76\alpha_2 + 3\alpha_1 < 1/5).
\end{aligned}$$

Now, we join an edge between each vertex in $U_R^2 - V(T^2_{main})$ and a neighbor of the vertex in $S(T^2_{main}) \cap U_R^1$. Let T_{main} be the resulting tree. By the construction procedure, it is easy to verify that T_{main} is a HIST of G_R.

Step 8. Connecting T_W, F_M, T_R, and $V(T_{main})$ into a connected graph

In this step, we connect T_W, F_M, T_R, and $V(T_{main})$ into a connected graph. Recall that each degree 2 vertex in T_W and F_M is a neighbor of v_2^*. We join an edge connecting v_2^* in $V(T_{main})$ and each degree 2 vertex in T_W and F_M. By the argument in step 7.3 above, we know each vertex in $V(T_R) \cap U_M^2$ has a neighbor in $S(T_{main}) \cap U_R^1$. Thus, we join an edge between each vertex in $V(T_R)\cap U_M^2$ to exactly one of its neighbor in $S(T_{main})\cap U_R^1$. Let T^* be the final resulting graph. Notice that $I = V_{12}^0 = \{x_1, x_2, \cdots, x_I\} \subseteq L(T^*)$ is the set of the wrapped vertices from Step 4. Recall that G^* is the graph obtained from G be deleting and contracting edges from Step 4. Then by the constructions of T_W, F_M, T_R, and T_{main}, we see that T^* is a HIST of G^* with $|L(T^*)\cap U_1^*| = |L(T^*)\cap U_2^*|$.

Step 9. Finding a cycle on $L(T^*)$

Denote

$$U_L^1 = L(T^*)\cap U_1^*,\ U_L^2 = L(T^*)\cap U_2^* \quad \text{and} \quad G_L = G[E_G(U_L^1, U_L^2)].$$

Notice that G_L is a balanced bipartite graph. And

$$\begin{aligned}
|S(T^*)\cap U_1^*| &= d_3 \le n/5 + 2\alpha_1|V_2| \quad (\text{by } (3.60)) \\
|S(T^*)\cap U_2^*| &= 1 + \lceil n_l/10\rceil \le 2 + 0.64\alpha_2|V_2| \quad (\text{by } n_l \le d_l \le 6.4\alpha_2|V_2| \text{ from } (3.42)).
\end{aligned}$$

Thus by (3.47),

$$\begin{aligned}
\delta_{G^*}(U_L^1, U_L^2) &\ge 2n/5 - \alpha_1|V_2| - (2 + 0.64\alpha_2|V_2|) > 3n/10 > |U_L^2|/2 + 1, \\
\delta_{G^*}(U_L^2, U_L^1) &\ge (1 - 3\alpha_1)|V_1| - 1 - (n/5 + 2\alpha_1|V_2|) > n/3 > |U_L^1|/2 + 1.
\end{aligned}$$

By Lemma 3.2.7, G_L contains a hamiltonian cycle C'.

Step 10. Unwrap vertices in $V(C')\cap\{v_{x_1}, v_{x_2}, \cdots, v_{x_{|I|}}\}$

On C', replace each vertex v_{x_i} with its preimage $P_{x_i} = x_{i1}x_ix_{i2}$ for each $i = 1, 2, \cdots, |I|$. Denote the resulting cycle by C. Recall that $x_{i1}, x_{i2} \in \Gamma(v_2^*)$ by the choice of x_{i1} and x_{i2}. Let T be the graph on $V(G)$ with

$$E(T) = E(T^*)\cup\{v_2^*x_{i1}, v_2^*x_{i2} : i = 1, 2, \cdots, |I|\}.$$

Then T is a HIST of G. Let $H = T\cup C$. Then H is an SGHG of G.

The proof of Extremal Case 2 is finished. $\square$

PART 4

A LOWER BOUND ON CIRCUMFERENCES OF 3-CONNECTED GRAPHS WITH BOUNDED MAXIMUM DEGREES

4.1 Introduction

In 1980, Bondy and Simonovits [8] showed that the best general lower bound on the length of a longest cycle in an n-vertex 3-connected cubic graph is between $\exp(c_1\sqrt{\log n})$ and n^{c_2} for some positive constants c_1 and c_2, and they also obtained similar bounds for 3-connected graphs with bounded degrees. The lower bound $\exp(c_1\sqrt{\log n})$ for cubic graphs was improved to $n^{0.69}$ by Jackson [32] and was further improved to $n^{0.8}$ by Liu, Yu and Zhang [39]. In 1993, Jackson and Wormald [33] proved that every 3-connected n-vertex graph with maximum degree at most d has a cycle of length at least $\frac{1}{2}n^{\log_b 2}+1$ with $b = 6d^2$. They also conjectured that for $d \geq 4$ the correct value for b should be $d-1$, and they gave an infinite class of graphs showing that $b = d-1$ is the best possible value that one can hope for.

Recently there has been considerable interest in approximating longest cycles in 3-connected graphs with bounded degrees. Karger, Motwani, and Ramkumar [35] showed that unless $\mathcal{P} = \mathcal{NP}$, it is impossible to find, in polynomial time, a path of length $n - n^{\epsilon}$ (for any $\epsilon < 1$) in an n-vertex Hamiltonian graph. They conjectured that it is hard even for graphs with bounded degrees. On the positive side, Feder, Motwani, and Subi [23] showed that there is a polynomial time algorithm for finding a cycle of length at least $n^{(\log_3 2)/2}$ in any 3-connected cubic n-vertex graph, and they asked the same question for 3-connected graphs with bounded degrees. Chen, Xu, and Yu [14] provided a cubic-time algorithm that, given a 3-connected n-vertex graph with maximum degree at most d, finds a cycle of length at least $n^{\log_b 2} + 3$ with $b = 2(d-1)^2 + 1$. This result was improved to $b = 4d + 1$ by Chen, Gao, Yu, and Zang [12].

Before stating the main result, we introduce some notation. For any graph G, we denote by $|G|$ the number of vertices of G, $G - z$ the graph obtained from G by deleting the vertex $z \in V(G)$, and $N_G(z)$ the set of neighbors of z in G. If G is a path or cycle, then $\ell(G)$ denotes the length of G. Let $S_1, S_2 \subseteq V(G)$ be two disjoint sets. An (S_1, S_2)-path is a path P connecting one vertex in S_1 and one vertex in S_2 such that $|V(P) \cap S_1| = |V(P) \cap S_2| = 1$. When $S_1 = \{x\}$ is a singleton, we simply write as (x, S_2)-path. The main result of this paper is the following:

Theorem (4.1.1). *Let $d \geq 425$ be an integer, and $r = \log_{d-1} 2$. Let G be either a cycle or a 3-connected graph and $e = xy \in E(G)$ be an edge.*

(a) If G is 3-connected, then for any $z \in V(G) - \{x, y\}$ such that $\Delta(G - z) \leq d$ and z has at most t neighbors distinct from x and y, there is a cycle C in $G - z$ through xy such that $\ell(C) \geq \frac{1}{4}\left(\frac{d-2.1}{d-1}\frac{|G|}{t}\right)^r + 2$.

(b) If $\Delta(G) \leq d$, then for any $f \in E(G) - \{e\}$, there is a cycle C in G through e and f such that $\ell(C) \geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|G|\right)^r + 2$.

(c) If $\Delta(G) \leq d$, then there is a cycle C through e in G such that $\ell(C) \geq \frac{1}{4}|G|^r + 2$.

We first note that Theorem (4.1.1) holds trivially when $|G| \leq d$; hence, throughout the rest of this part, we assume $|G| \geq d+1 \geq 426$. Also note that Theorem (4.1.1) holds trivially when G is a cycle. However, we include cycles in the statement of Theorem (4.1.1) for the following reason: cycles occur in our inductive arguments, and their inclusion makes many arguments less cumbersome.

The rest of this part is organized as follows. In Section 2, we recall Tutte decomposition [52] for decomposing a 2-connected graph into 3-connected components and some results from [12] concerning paths in 2-connected graphs. In Section 3, we prove a useful inequality about the function $f(x) = x^{\log_b 2}$. In Section 4, we state lemmas concerning paths in a chain of 3-connected components, and in Section 5, we inductively prove Theorem (1.1) (a) and (b). Section 6 is the most significant part of the paper, where we prove Theorem (4.1.1)(c) inductively.

4.2 Paths in block-chains

We recall Tutte decomposition for decomposing a 2-connected graph into 3-connected components. A detailed description can be found in [14] and [29]. Let $\mathcal{D}$ denote the set of all 3-connected (simple) graphs, $\mathcal{C}$ denote the set of cycles (with at least three vertices), and $\mathcal{B}$ denote the set of *bonds* (a bond is a multigraph with two vertices and at least three edges between them). Tutte [52] proved that every 2-connected graph G can be uniquely decomposed into 3-connected components, which belong to $\mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$. We call such a decomposition as the *Tutte decomposition*. Those 3-connected components are linked together by *virtual edges* to form a tree-like structure. More precisely, if we define a graph whose vertices are the 3-connected components of G obtained from the Tutte decomposition and two vertices are adjacent if the corresponding two 3-connected components share a common virtual edge, then such a graph is a tree, which we call the *block-bond tree* of G. Hopcroft and Tarjan [29] gave a linear time algorithm for decomposing any 2-connected graph into 3-connected components.

Recall that in the *block-cut tree* of a connected graph there is a cut-vertex between two consecutive blocks. However, in a block-bond tree, it is not necessarily true that there is a bond between any two 3-connected components. For example, let G_1 and G_2 be two 3-connected graphs such that each G_i contains two nonadjacent vertices u_i and v_i for each $i = 1, 2$. Let G be obtained from G_1 and G_2 by identifying u_1 with v_1 and u_2 with v_2, respectively. According to Tutte's decomposing algorithm, $G_1 + u_1v_1$ and $G_2 + u_2v_2$ are the only two 3-connected components of G. Clearly, in the *block-bond tree*, they are adjacent but there is no bond between them.

For convenience, 3-connected components that are not bonds are called *3-blocks*, consisting of cycles and simple 3-connected graphs. An *extreme* 3-block is a 3-block that contains at most one virtual edge. That is, either it is the only 3-connected component (in which case G is either a cycle or a 3-connected simple graph), or it corresponds to a degree one vertex in the block-bond tree.

A *block-chain* in G is a sequence $H_1H_2 \dots H_h$ of 3-blocks of G for which there exist $B_1, B_2, \dots, B_{h-1}$ such that for each $1 \le j \le h-1$, $B_j = \emptyset$ or B_j is a bond, and $H_1B_1H_2B_2 \dots B_{h-1}H_h$ is a path in the block-bond tree of G. A detailed description with examples can be found in [14]. For convenience, we sometimes write $\mathcal{H} := H_1H_2 \dots H_h$ for this situation. In the rest of the paper, unless stated otherwise, *we will always assume that each virtual edge in $E(H_i \cap H_{i+1})$ is deleted from $\mathcal{H}$ if at least one of H_i and H_{i+1} is 3-connected and there are exactly two components in $G - V(H_i \cap H_{i+1})$.* Because in this case it is not possible to replace the virtual edge by a path in G outside of $\mathcal{H}$. However, if both H_i and H_{i+1} are cycles, then the virtual edge shared by H_i and H_{i+1} can always be replaced by a path outside of $\mathcal{H}$. Throughout this section, we adopt the convention that an object is empty if it is not defined. For example, if $\mathcal{H} = H_1H_2 \dots H_h$ is a block-chain under consideration, then H_0 and H_{h+1} are both empty graphs.

The following result is proved in the proof of Lemma (3.6) in [14], which will be used to link together long paths from different block-chains. We note that the path stated in the lemma can be found in linear time by using a result from [43].

Lemma (4.2.1). *Let $\mathcal{H} = H_1H_2 \dots H_h$ be a block-chain in a 2-connected graph G, $x \in V(H_1) - V(H_2)$, and $f \in E(H_h) - E(H_{h-1})$ such that f is not incident with x, and let pq, vw be two distinct edges in $E(\mathcal{H}) - \{f\}$. Then there is a path P in $\mathcal{H}$ through f from x to some $z \in \{p, q\} \cup \{v, w\}$ such that if $z \in \{p, q\}$ then $pq \notin E(P)$ and $vw \in E(P)$ and if $z \in \{v, w\}$*

then $vw \notin E(P)$ and $pq \in E(P)$.

Lemma (4.2.2). *Let $\mathcal{H} = H_1H_2\dots H_h$ be a block-chain in a 2-connected graph G, $x \in V(H_1) - V(H_2)$, and $pq, f \in E(H_h) - E(H_{h-1})$ be distinct such that neither pq nor f is incident with x. If H_h is 3-connected and q is not incident with f, then in $\mathcal{H} - q$, there is an (x,p)-path through f.*

Proof. We use induction on h. If $h = 1$, then H_1 is 3-connected and $H_1 - q$ is 2-connected, so $H_1 - q$ contains an (x,p)-path through f. Suppose $h \geq 2$ and let $\{a,b\} = V(H_{h-1} \cap H_h)$. Since $pq \notin E(H_{h-1})$, we may assume $a \notin \{p,q\}$. Let P_h be an (a,p)-path through f in $H_h - q$.

If $ab \notin P_h$, let P_1 be an (x,a)-path in $H_1H_2\dots H_{h-1} - b$; then $P_h := P_1 \cup P_h$ is the desired path. If $ab \in P_h$, let P_1 be an (x,b)-path in $H_1H_2\dots H_{h-1} - a$; then $P := P_1 \cup (P_h - ab)$ is the desired path. □

Lemma (4.2.3). *Let $\mathcal{H} = H_1H_2\dots H_h$ be a block-chain. Let xy, pq, uv be three edges such that $xy \in E(H_1) - E(H_2)$ and $pq, uv \in E(H_h) - E(H_{h-1})$, where $pq \neq xy \neq uv$ but it is possible that $pq = uv$. Then there is a path P in $\mathcal{H}$ from some $z \in \{x,y\}$ to $w \in \{p,q\} \cup \{u,v\}$ such that $\{x,y\} \not\subseteq V(P)$, $uv \in E(P)$ if $w \in \{p,q\}$, and $pq \in E(P)$ if $w \in \{u,v\}$.*

Proof. We first consider $h = 1$, that is $\mathcal{H} = H_1$. The result is trivial if H_1 is a cycle. Suppose that H_1 is 3-connected. Then $H_1 - y$ is 2-connected, and thus contains an $(x, \{p,q\})$- path P through uv.

We now assume $h \geq 2$. Let $\{a,b\} = V(H_h) \cap V(H_{h-1})$. By the same argument as for the case where $h = 1$, there is a path P_H from $z^* \in \{a,b\}$ to $w \in \{p,q\} \cup \{u,v\}$ such that $\{a,b\} \not\subseteq V(P_H)$, $uv \in E(P_H)$ if $w \in \{p,q\}$ and $pq \in E(P_H)$ if $w \in \{u,v\}$. Clearly, there is a path Q in $H_1 \dots H_{h-1} - ab$ from some $z \in \{x,y\}$ to z^* such that $\{x,y\} \not\subseteq V(Q)$. Then $Q \cup P_H$ is the desired path. □

Lemma (4.2.4). *Let $\mathcal{H} = H_1H_2 \dots H_h$ be a block-chain, $u, v \in V(H_h)$ be distinct, and $x \in V(\mathcal{H} - v)$. Then there is a path from x to u avoiding v.*

Proof. We use induction on h. The result is clearly true when $h = 1$. Assume the claim holds for block-chains with fewer than h blocks. Let $\{a, b\} = V(H_{h-1} \cap H_h)$. If $x \in V(H_h)$, let P_h be a path in H_h from x to u avoiding v. If $ab \notin E(P_h)$, let $P := P_h$; if $ab \in E(P_h)$, let P be obtained from P_h by replacing ab by a path in $H_1 \dots H_{h-1}$ from a to b.

Suppose $x \notin V(H_h)$. Assume, without loss of generality, that $a \neq v$, and let P_h be a path in H_h from a to u avoiding v. By induction, let P_1 be an (x, a)-path avoiding b in $H_1 \dots H_{h-1}$ if $ab \notin E(P_h)$, and let P_1 be an (x, b)-path avoiding a in $H_1 \dots H_{h-1}$ if $ab \in E(P_h)$. Then $P := P_1 \cup (P_h - ab)$ is the desired path. $\square$

Lemma (4.2.5). *Let $\mathcal{H} = H_1H_2 \dots H_h$ be a block-chain, and let $xx' \in E(H_1) - E(H_2)$ and $uv \in E(H_h)$ be two edges in $\mathcal{H}$. Then there is an $(x', \{u, v\})$-path in $\mathcal{H} - x$.*

Proof. We use induction on h. The statement is clearly true when $h = 1$ as $H_1 - x$ is connected. So we assume $h \geq 2$, and let $V(H_1) \cap V(H_2) = \{a, b\}$. Suppose first that $x \notin \{a, b\}$. By induction, we let P_1 be an $(x', \{a, b\})$-path, say (x', a)-path, in $H_1 - x$, and let P_2 be an $(a, \{u, v\})$-path in $H_2H_2 \cdots H_h - b$. Then $P_1 \cup P_2$ is the desired path. Then, suppose, without loss of generality, that $x = a$. Let P_1 be an (x', b)-path in $H_1 - x$, and by induction, let P_2 be a $(b, \{u, v\})$-path in $H_2H_2 \cdots H_h - a$. Then $P_1 \cup P_2$ is the desired path. $\square$

We conclude this section by recalling two graph operations from [14]. Let G be a graph and let e, f be distinct edges of G. An *H-transform* of G *at* $\{e, f\}$ is an operation that subdivides e and f by vertices x and y, respectively, and then adds the edge xy. Let $x \in V(G)$ such that x is not incident with e. A *T-transform* of G *at* $\{x, e\}$ is an operation that subdivides e with a vertex y and then adds the edge xy. Let G' be a graph obtained from a 3-connected graph G by an H-transform or a T-transform. It is easy to see that G' is also 3-connected (see, e.g., Lemma (3.3) in [14] for a proof).

4.3 Lower bounds of $m^{\log_b 2} + n^{\log_b 2}$

Fix $b = d-1$ hereinafter, where $d \geq 4$ is an integer and let $r = \log_b 2$. Clearly, $0 < r < 1$, which in turn gives that $m^r + n^r \geq (m+n)^r$. In this section we improve this inequality under different situations. The new inequalities will be used to show that the union of some long paths has the desired length. The first one is a strengthening of Lemmas (3.1) and (3.2) in [12].

Lemma (4.3.1). *Let m and n be two positive real numbers such that $m \geq b^\beta n > 0$ for some real number β. Then,*

$$m^r + n^r \geq \left(m + b^\beta \left(b^{\log_2\left(1+2^{-\beta}\right)} - 1\right) n\right)^r \geq \left(b^\beta b^{\log_2\left(1+2^{-\beta}\right)} \, n\right)^r. \tag{4.1}$$

Proof. Define $f(t) = \frac{1}{t}\left((1+t^r)^{1/r} - 1\right)$. It is easy to verify that

$$m^r + n^r = (m + f(n/m)n)^r \quad \text{and} \quad f'(t) = \frac{1}{t^2}\left(1 - (1+t^r)^{(1-r)/r}\right).$$

Since $b \geq 3$, we have $0 < r < 1$, and hence $f'(t) < 0$ when $t > 0$. Therefore $f(t)$ is a decreasing function on the interval $(0, \infty)$. For $m \geq b^\beta n > 0$, we have $n/m \leq b^{-\beta}$, and so (since $b^r = 2$)

$$f(n/m) \geq f(b^{-\beta}) = b^\beta \left(\left(1 + 2^{-\beta}\right)^{1/r} - 1\right) = b^\beta \left(b^{\log_2\left(1+2^{-\beta}\right)} - 1\right).$$

Here, the first inequality in (4.1) follows from $m^r + n^r \geq (m + f(n/m)n)^r$, and the 2nd inequality in (4.1) follows from $m \geq b^\beta n$. □

Taking $\beta = 0, \log_b 1.1, 1, 2, -1, -2$, we get the following inequalities from (4.1). It is straightforward to verify (4.1a) - (4.1f)

$$m^r + n^r \geq \begin{cases} (m+(b-1)n)^r, & \text{if } m \geq n; \quad (4.1a) \\ (m+1.1(b^{\log_2(1+2^{-\log_b 1.1})}-1)n)^r, & \text{if } m \geq 1.1n; \quad (4.1b) \\ (m+b(b^{\log_2 3/2}-1)n)^r, & \text{if } m \geq bn; \quad (4.1c) \\ (m+b^2(b^{\log_2 5/4}-1)n)^r, & \text{if } m \geq b^2 n; \quad (4.1d) \\ (b^{\log_2 3/2} n)^r, & \text{if } m \geq n/b; \quad (4.1e) \\ (b^{\log_2 5/4} n)^r, & \text{if } m \geq n/b^2. \quad (4.1f) \end{cases}$$

In the proofs, the following elementary inequality will be used frequently for any two positive real numbers x and y,

$$x^r + y^r \geq 2\sqrt{x^r y^r} = \left((d-1)^2 xy\right)^{r/2}. \tag{4.2}$$

Lemma (4.3.2). *The following inequalities hold:*

$$x^r + 1 \quad \geq \quad (x+d-1)^r \qquad \textit{provided } x \geq 1.$$

Lemma (4.3.3). *Let $b \geq 23$ be an integer. If m and n are two positive real numbers such that $m \geq 1.1n$, then $m^r + n^r \geq (m+bn)^r$.*

Proof. Applying Lemma (4.3.1) for $\beta = \log_b 1.1 \leq \log_{23} 1.1$, we have $m^r + n^r \geq (m + 1.1(b^{\log_2(1+2^{-\log_b 1.1})} - 1)n)^r$. So, we only need to show that $1.1(b^{\log_2(1+2^{-\log_b 1.1})} - 1) \geq b$ provided $b \geq 23$. For any $x \geq 1.1$, let $\tau := \tau(x) = \log_x 1.1$, $\phi := \phi(\tau) = \log_2(1+2^{-\tau})$ and $f(x) = x^{\phi(\tau(x))}$. It is clearly that $\lim_{x\to\infty}(1.1f(x) - x) = \infty$. It is sufficient to show that $1.1f(x) - x$ is an increasing function for $x \geq 23$, which is equivalent to $\frac{d}{dx}f(x) \geq 10/11$ for $x \geq 23$.

Simple calculations show that $\frac{d}{dx}\tau(x) = -\frac{\ln 1.1}{x\ln^2 x} = -\frac{\tau(x)}{x\ln x}$ and $\frac{d}{d\tau}\phi(\tau) = -\frac{1}{1+2^\tau}$. So,

$$\frac{d}{dx}f(x) = f(x)\left((\ln x)\frac{d\phi}{d\tau}\frac{d\tau}{dx} + \phi\frac{d\ln x}{dx}\right) = \frac{f(x)}{x}\left(\frac{\tau}{1+2^\tau} + \phi(\tau)\right).$$

Since $\lim_{x\to\infty}\tau(x) = 0$ and $\lim_{\tau\to 0}\phi(\tau) = 1$, $\lim_{x\to\infty}\frac{d}{dx}f(x) = 1$. It is sufficient to show that $\frac{d}{dx}f(x)$ is decreasing as x increasing. Writing $\frac{d}{dx}f(x)$ in terms of τ, we have

$$\frac{d}{dx}f(x) = x^{\phi(\tau)-1}(\frac{\tau}{1+2^\tau} + \phi(\tau)) = e^{\frac{(\phi(\tau)-1)\ln 1.1}{\tau}}(\frac{\tau}{1+2^\tau} + \phi(\tau)).$$

We only need to show $\frac{d}{dx}f(x)$ is increasing as τ increasing when $\tau \le \tau(23)$, which is equivalent to $\frac{d}{d\tau}\frac{df}{dx} > 0$. Taking derivative, we obtain

$$\frac{d}{d\tau}\frac{df}{dx} = e^{\frac{(\phi(\tau)-1)\ln 1.1}{\tau}}\left(\frac{\ln 1.1(1-\phi-\frac{\tau}{1+2^\tau})(\phi+\frac{\tau}{1+2^\tau})}{\tau^2} - \frac{\tau 2^\tau \ln 2}{(1+2^\tau)^2}\right).$$

So, we only need to show that

$$g(\tau) = \frac{\ln 1.1(1-\phi-\frac{\tau}{1+2^\tau})(\phi+\frac{\tau}{1+2^\tau})}{\tau^2} - \frac{\tau 2^\tau \ln 2}{(1+2^\tau)^2} > 0 \quad \text{if } \tau \le \tau(23).$$

We define the value of $g(\tau)$ at $\tau = 0$ as

$$g(0) = \lim_{\tau\to 0} g(\tau) = \frac{\ln 1.1 \ln 2}{8}.$$

Then $g(\tau)$ is a continuous function on the closed interval $[0, \tau(23)]$. To show $g(\tau) > 0$ within $[0, \tau(23)]$, as $g(0) > 0$, by the *intermediate zero theorem*, instead, we show that $g(\tau)$ has no zero in $[0, \tau(23)]$. To do so, using the *bisection method*, with tolerance as 1×10^{-10}, a numerical search within $[0, 1]$ interval gives 0.04765221 as the root of $g(\tau)$. Since $\tau(23) < 0.0304 < 0.04765221$, we conclude that $g(\tau) > 0$ when $0 \le \tau \le \tau(23)$. The proof is completed. $\square$

4.4 Long paths in block-chains

In this section, we will give a few lower bounds of long paths connecting special vertices in a block-chain. Throughout this section, we assume that $n \geq 4$, Theorem (4.1.1) holds for graphs with at most $n-1$ vertices, and $\mathcal{H} := H_1 H_2 \ldots H_h$ is a block-chain such that $|\mathcal{H}| \leq n-1$ such that

- $\Delta(H_i) \leq d$ for each $1 \leq i \leq h$.
- As of a subgraph of G, $\mathcal{H}$ contains at most $2d-1$ vertices of degree 2.

Recall, for convention, we also denote by $\mathcal{H}$ the graph with vertex set $\bigcup V(H_i)$ and edge set $\bigcup E(H_i)$ with the deletion of virtual edges.

Lemma (4.4.1). *For any edge $uv \in E(H_1) - E(H_2)$, there is a (u,v)-path P in $\mathcal{H}$ such that*

$$\ell(P) \geq \frac{1}{4}\left(\frac{d-2.1}{d-1}(|\mathcal{H}|+1)\right)^r + 1,$$

provided that $d \geq 23$.

Proof. Since $|\mathcal{H}| \leq n-1$, it follows from our assumption that Theorem (4.1.1) holds for each H_i. We proceed with induction on h. Suppose $h = 1$. Then $\mathcal{H}$ is either a cycle or a 3-connected graph. Since the case $|\mathcal{H}| \leq d$ is trivial, we may assume $|\mathcal{H}| \geq d+1$, and hence

$$|\mathcal{H}| > \frac{d-2.1}{d-1}(|\mathcal{H}|+1).$$

By Theorem (4.1.1)(c), $\mathcal{H} = H_1$ contains a cycle C through uv such that

$$\ell(C) \geq \frac{1}{4}|\mathcal{H}|^r + 2 \geq \frac{1}{4}\left(\frac{d-2.1}{d-1}(|\mathcal{H}|+1)\right)^r + 2.$$

Hence $P := C - \{uv\}$ gives the desired path.

Therefore, assume $h \geq 2$. Let $\mathcal{H}' := H_2 \cdots H_h$ and $\{a_1, b_1\} := V(H_1 \cap H_2)$. We consider two cases.

First, assume $|H_1| \geq \frac{d-2.1}{d-1}(|\mathcal{H}| + 1)$. By Theorem (4.1.1)(c), we can find a cycle C_1 in H_1 through uv such that

$$\ell(C_1) \geq \frac{1}{4}|H_1|^r + 2 \geq \frac{1}{4}\left(\frac{d-2.1}{d-1}(|\mathcal{H}| + 1)\right)^r + 2.$$

If C_1 does not contain $a_1 b_1$, then $P := C_1 - \{uv\}$ is the desired path. If C_1 contains $a_1 b_1$, then let C_2 be a cycle in $\mathcal{H}'$ through $a_1 b_1$. It is clear that $P := (C_1 \cup C_2) - \{a_1 b_1, uv\}$ is the desired path.

Now assume $|H_1| < \frac{d-2.1}{d-1}(|\mathcal{H}| + 1)$. Then

$$|\mathcal{H}'| + 1 = |\mathcal{H}| - |H_1| + 3 > \frac{d-1}{d-2.1}|H_1| - |H_1| + 2 > \frac{1.1|H_1|}{d-2.1} > \frac{1.1|H_1|}{d-1}.$$

Applying Theorem (4.1.1)(b), we find a cycle C_1 in H_1 through uv and $a_1 b_1$ such that

$$|C_1| \geq \frac{1}{4}\left(\frac{(d-2.1)|H_1|}{(d-1)^2}\right)^r + 2.$$

By induction, we find a path P' in $\mathcal{H}'$ between a_1 and b_1 such that

$$\ell(P') \geq \frac{1}{4}\left(\frac{d-2.1}{d-1}(|\mathcal{H}'| + 1)\right)^r + 1.$$

Hence $P := (C_1 \cup P') - \{uv, a_1 b_1\}$ is a path between u and v in $\mathcal{H}$ such that

$$\begin{aligned}
\ell(P) &\geq \frac{1}{4}\left(\frac{(d-2.1)|H_1|}{(d-1)^2}\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}(|\mathcal{H}'| + 1)\right)^r + 1 \\
&> \frac{1}{4}\left(\frac{d-2.1}{d-1}\left((|\mathcal{H}'| + 1) + (d-1)\frac{|H_1|}{d-1}\right)\right)^r + 1 \quad \text{(by Lemma (4.3.3))} \\
&> \frac{1}{4}\left(\frac{d-2.1}{d-1}(|\mathcal{H}| + 1)\right)^r + 1,
\end{aligned}$$

where in the 2nd inequality above, the inequality $|\mathcal{H}'|+1 \geq \frac{1.1|H_1|}{d-1}$ is used. $\square$

Lemma (4.4.2). *Let $x \in V(H_h) - V(H_{h-1})$ such that $d_{\mathcal{H}}(x) = d_{H_h}(x) \leq d-1$ and $uv \in E(H_1) - E(H_2)$ such that $x \notin \{u, v\}$ when $h = 1$. Then there exists a path P in $\mathcal{H} - v$ from u to x such that*

$$\ell(P) \;\geq\; \frac{1}{4}\sum_{i=1}^{h}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r + \frac{1}{2} \geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|\mathcal{H}|\right)^r + \frac{1}{2}.$$

Moreover, if H_1 is 3-connected, we can improve the constant $1/2$ to 1:

$$\ell(P) \geq \frac{1}{4}\sum_{i=1}^{h}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r + 1 \geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|\mathcal{H}|\right)^r + 1.$$

Proof. Note that the second inequality in each of the lower bounds above for $\ell(P)$ is a simple application of Lemma (4.3.1). So we only show the first part of the lower bounds.

We apply induction on h. Suppose $h = 1$. If H_1 is a cycle, then $|H_1| \leq 2d - 1$, which in turn gives $\frac{1}{4}(\frac{d-2.1}{(d-1)^2}|H_1|)^r < 1/2$ and $\frac{1}{4}(\frac{d-2.1}{(d-1)^2}|H_1|)^r + \frac{1}{2} < 1$. On the other hand, since $x \notin \{u, v\}$, there is an (x, u)-path P in $H_1 - v$ with $\ell(P) \geq 1$. Hence, the assertion holds. Now assume $H_1 = \mathcal{H}$ is 3-connected. Note that $\Delta(H_1 + xu - v) \leq d$ and v has at most $d-1$ neighbors in $H_1 - v$. By applying Theorem (4.1.1)(a) to $H_1 + xu$ we find a cycle C through xu in $(H_1 + xu) - v$ such that $\ell(C) \geq \frac{1}{4}\left((d-2.1)|H_1|/(d-1)^2\right)^r + 2$. Hence $P := C - \{xu\}$ gives the desired path.

Now we assume $h \geq 2$. Let $\{a_1, b_1\} := V(H_1 \cap H_2)$ and $\mathcal{I} := H_2H_3\cdots H_h$. By induction, in $H_1 - v$, there exists a path P_1 from u to some vertices in $\{a_1, b_1\}$, say to a_1 (notice that a_1b_1 may be on P_1) such that $\ell(P_1) \geq \frac{1}{4}\left((d-2.1)|H_1|/(d-1)^2\right)^r + \frac{1}{2}$ unless H_1 is a cycle and $u \notin \{a_1, b_1\}$ (in this case, P_1 may only contain one vertex). Moreover, $\ell(P_1) \geq \frac{1}{4}\left((d-2.1)|H_1|/(d-1)^2\right)^r + 1$ when H_1 is 3-connected. We will consider the case that H_1 is a cycle and $u \in \{a_1, b_1\}$ at the end.

Applying induction again we find a path P_2 in $\mathcal{I} - b_1$ from x to a_1 such that

$$\ell(P_2) \geq \frac{1}{4} \sum_{i=2}^{h} \left(\frac{(d-2.1)|H_i|}{(d-1)^2} \right)^r + \frac{1}{2}.$$

Moreover, when H_2 is 3-connected,

$$\ell(P_2) \geq \frac{1}{4} \sum_{i=2}^{h} \left(\frac{(d-2.1)|H_i|}{(d-1)^2} \right)^r + 1.$$

If $a_1b_1 \in E(\mathcal{H})$ or $a_1b_1 \notin E(P_1)$, $P := P_1 \cup P_2$ is the desired path. Thus, we may assume that a_1b_1 is a virtual edge in H_1, $a_1b_1 \in E(P_1)$ and $a_1b_1 \notin E(\mathcal{H})$.

If H_1 is a cycle, then H_2 must be 3-connected since $a_1b_1 \notin E(\mathcal{H})$. Let P_2 in $\mathcal{I} - a_1$ from x to b_1 such that

$$\ell(P_2) \geq \frac{1}{4} \sum_{i=2}^{h} \left(\frac{(d-2.1)|H_i|}{(d-1)^2} \right)^r + 1.$$

Then, $P := (P_1 - a_1) \cup P_2$ satisfying

$$\ell(P) \geq \ell(P_2) \geq \frac{1}{4} \sum_{i=2}^{h} \left(\frac{(d-2.1)|H_i|}{(d-1)^2} \right)^r + 1 \geq \frac{1}{4} \sum_{i=1}^{h} \left(\frac{(d-2.1)|H_i|}{(d-1)^2} \right)^r + \frac{1}{2},$$

so P is the desired path.

We may assume that H_1 is 3-connected. In this case, we have $\ell(P_1) \geq \frac{1}{4}(\frac{(d-2.1)|H_1|}{(d-1)^2})^r + 1$. Let P_2 be an (x, b_1)-path in $\mathcal{I} - a_1$ such that $\ell(P_2) \geq \frac{1}{4} \sum_{i=2}^{h} \left(\frac{(d-2.1)|H_i|}{(d-1)^2} \right)^r + \frac{1}{2}$. Moreover, in this case, we can find the desired path if H_2 is 3-connected. So, we may additionally assume that H_2 is a cycle. Since P_1 is a (u, a_1)-path, $a_1b_1 \in E(P_1)$, and $\ell(P_1) > 1$, $u \notin \{a_1, b_1\}$.

We now, under the assumption that H_1 is 3-connected, H_2 is a cycle, $u \notin \{a_1, b_1\}$, and $a_1b_1 \in E(P_1)$, construct a path P according to the following two cases.

Suppose first that $h = 2$. If H_2 is a triangle, that is, $V(H_2) = \{a_1, b_1, x\}$. Applying

Theorem (4.1.1)(a) to $(\mathcal{H}+ux)-v$, we obtain the desired (x,u)-path P in $\mathcal{H}-v$. So we may assume that $|V(H_2)| \geq 4$, which in turn shows that H_2 contains a path P_2 with $\ell(P_2) \geq 2$, which is either an (x,a_1)-path or an (x,b_1)-path. Assume, without not loss of generality, P_2 is an (x,a_1)-path and Q_2 is the (x,b_1)-path in $H_2 - a_1b_1$. Let Q_1 be a (u,b_1)-path in $H_1 - v$ such that $\ell(Q_1) \geq \frac{1}{4}(\frac{(d-2.1)|H_1|}{(d-1)^2})^r + 1$. If $a_1b_1 \notin E(Q_1)$, then $P := Q_2 \cup Q_1$ is the desired path. If $a_1b_1 \in E(Q_1)$, then $P_2 \cup (Q_1 - a_1)$ is the desired path since $|H_2| \leq 2d-1$.

We now assume that $h \geq 3$ and let $\{a_2, b_2\} := V(H_2 \cap H_3)$ and $\mathcal{H}'' := H_3H_4 \cdots H_h$. Applying induction, there is a (u,a_2)-path P_1 in H_1H_2 avoiding v such that

$$\ell(P_1) \geq \frac{1}{4}\sum_{i=1}^{2}\left(\frac{(d-2.1)|H_i|}{(d-1)^2}\right)^r + 1.$$

If $a_2b_2 \notin E(P_1)$, by the induction hypothesis, we find an (a_2,x)-path P' in $\mathcal{H}'' - b_2$ such that

$$\ell(P') \geq \frac{1}{4}\sum_{i=3}^{h}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r + \frac{1}{2}.$$

Then $P := P_1 \cup P'$ gives the desired path. Thus, we assume $a_2b_2 \in E(P_1)$. If H_3 is 3-connected, then by induction there is a (b_2,x)-path P' avoiding a_2 in $\mathcal{H}''$ such that $\ell(P') \geq \frac{1}{4}\sum_{i=3}^{h}\left(\frac{(d-2.1)|H_i|}{(d-1)^2}\right)^r + 1$ in $\mathcal{H}''$. Hence, $P := (P_1 - a_2) \cup P'$ gives the desired path. Thus, we have $a_2b_2 \in E(P_1)$ and H_3 is a cycle. Since both H_2 and H_3 are cycles, $a_2b_2 \in E(\mathcal{H})$. We find an (a_2,x)-path P' in $\mathcal{H}'' - b_2$ such that $\ell(P') \geq \frac{1}{4}\sum_{i=3}^{h}\left(\frac{(d-2.1)|H_i|}{(d-1)^2}\right)^r + \frac{1}{2}$. Then, $P := P_1 \cup P'$ is the desired path. $\square$

Let U and W be two vertex sets. By definition, an (U,W)-path P is a (u,w)-path for some $u \in U$ and $w \in W$, and $|V(P) \cap U| = 1$ and $|V(P) \cap W| = 1$. We call P *a path from U to W* if P is a (u,v)-path from some $u \in U$ and $w \in W$ while $V(P) \cap U$ or $V(P) \cap W$ may not be singleton.

Lemma (4.4.3). *Suppose that $|\mathcal{H}| \leq n-2$ and Theorem (4.1.1) holds for graphs with less than n vertices. Let $x \in V(H_h) - V(H_{h-1})$ such that $d_{\mathcal{H}}(x) \leq d-1$, $f \in E(H_1) - E(H_2)$*

and $pq \in E\left(\cup_{i=1}^h H_i\right) - \{f\}$. *Then there exists a path* P *in* $\mathcal{H}$ *from* x *to* $z \in \{p,q\}$ *through* f *such that* $pq \notin E(P)$ *and* $\ell(P) \geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|\mathcal{H}|\right)^r$.

Proof. We use induction on h and consider the base case $h = 1$ first. In this case, if H_1 is a cycle, then there exists a path P from x to $\{p,q\}$ through f. Since $|H_1| < 2d - 1$, $\ell(P) \geq 1 \geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|\mathcal{H}|\right)^r$. We may assume that H_1 is 3-connected and consider two cases according to whether $x \in \{p,q\}$.

If $x \notin \{p,q\}$, let H_1' be the graph obtained from H_1 by a T-transform at $\{x, pq\}$, and let x' be the new vertex. Since $|V(H_1')| \leq |V(\mathcal{H})| + 1 \leq n - 1$, we use Theorem (1.1)(b) to find a cycle C in H_1' through xx' and f such that $\ell(C) \geq \frac{1}{4}((d-2.1)|H_1'|/(d-1)^2)^r + 2$. It is clear that $C - x'$ gives a desired path. If $x \in \{p,q\}$, we use Theorem (4.1.1)(b) to find a cycle in H_1 through pq and f. Then $C_1 - \{pq\}$ is the desired path.

Assume $h \geq 2$. Let $\mathcal{H}' = H_2H_3\cdots H_h$ and $\{a_1, b_1\} := V(H_1) \cap V(H_2)$. We consider the following three cases.

Case 1. $pq \notin E(H_1)$. We use Theorem (4.1.1)(b) to find a cycle C in H_1 through f and a_1b_1 such that

$$\ell(C) \geq \frac{1}{4}\left(\frac{(d-2.1)|H_1|}{(d-1)^2}\right)^r + 2.$$

We apply induction to find a path P' in $\mathcal{H}'$ from x to $\{p,q\}$ such that $a_1b_1 \in E(P')$, $pq \notin E(P')$, and

$$\ell(P') \geq \frac{1}{4}\left(\frac{(d-2.1)|\mathcal{H}'|}{(d-1)^2}\right)^r.$$

Then $(C - \{a_1b_1\}) \cup P'$ is also the desired path.

Case 2. $\{p,q\} = \{a_1, b_1\}$. We use Lemma (4.4.2) to find a path P' in $\mathcal{H}' - q$ from x to p avoiding q such that $\ell(P') \geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|\mathcal{H}'|\right)^r + \frac{1}{2}$ and let C be the cycle in H_1 as in Case 1. Then $P := P' \cup (C - \{pq\})$ is the desired path.

Case 3. $pq \in E(H_1) - \{a_1b_1\}$. We assume, without loss of generality, that $a_1 \notin V(\cup_{i\geq 3} H_i)$. By induction, there is a path P_1 in H_1 from a_1 to $\{p,q\}$ such that $f \in E(P_1)$, $pq \notin E(P_1)$, and $\ell(P_1) \geq \left(\frac{d-2.1}{(d-1)^2}|H_1|\right)^r$. Since $x \in V(H_h) - V(H_{h-1})$ and $\{a_1, b_1\} = V(H_1) \cap V(H_2)$, we have $x \notin \{a_1, b_1\}$. By Lemma (4.4.2), there is a path P' in $\mathcal{H}' - b_1$ from x to a_1 such that

$$\ell(P') \geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|\mathcal{H}'|\right)^r + \frac{1}{2}.$$

Moreover, if H_2 is 3-connected then

$$\ell(P') \geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|\mathcal{H}'|\right)^r + 1.$$

Hence $P := P_1 \cup P'$ is the desired path in $\mathcal{H}$ if $a_1b_1 \notin E(P_1)$, $a_1b_1 \in E(\mathcal{H})$, or H_2 is 3-connected. So, we assume $a_1b_1 \in E(P_1)$, a_1b_1 is a virtual edge in H_1, $a_1b_1 \notin E(\mathcal{H})$, and H_2 is a cycle.

If $h = 2$ and $|V(H_2)| \geq 4$, then in H_2, we can find an $(x, \{a_1, b_1\})$-path P' such that $\ell(P') \geq 2$. If P' is an (x, b_1)-path, then $P := (P_1 - a_1) \cup P'$ is the desired path by noting that $|H_2| \leq 2d - 1$. So assume that P' is an (x, a_1)-path. In H_1, let P_1 be a path from b_1 to $\{p,q\}$ such that $f \in E(P_1)$, $pq \notin E(P_1)$ and $\ell(P_1) \geq \frac{1}{4}\left((d-2.1)|H_1|/(d-1)^2\right)^r$. We may assume that $a_1b_1 \in E(P_1)$ (otherwise, let P' be an (x, b_1)-path in $H_2 - a_1$, then $P := P_1 \cup P'$ is the desired path, as $x \in V(H_2) - V(H_1)$, $\ell(P') \geq 1$). Then $P := (P_1 - b_1) \cup P'$ is the desired path. So we assume $|V(H_2)| = 3$ or $V(H_2) - V(H_1) = \{x\}$. Let H^* be the graph obtained from H_1H_2 by a T-transform at $\{x, pq\}$, and let x' be the new vertex. Since $|V(H^*)| \leq n-1$, we can then apply Theorem (1.1)(b) to find a cycle C in H^* through xx' and f such that $\ell(C) \geq \frac{1}{4}((d-2.1)|H_1'|/(d-1)^2)^r + 2$. It is clear that $C - x'$ gives the desired path.

If $h \geq 3$, let $\{a_2, b_2\} := V(H_2) \cap V(H_3)$ and $\mathcal{H}'' := H_3H_4 \cdots H_h$. Assume, without loss of generality, that $a_2 \in V(H_2) - V(H_1)$. Applying induction, there is a path P_1 in H_1H_2

from a_2 to $\{p, q\}$ such that $f \in E(P_1)$ and $pq \notin E(P_1)$, and

$$\ell(P_1) \geq \frac{1}{4}\sum_{i=1}^{2}\left(\frac{(d-2.1)|H_i|}{(d-1)^2}\right)^r.$$

If $a_2b_2 \notin E(P_1)$, by Lemma (4.4.2), we find an (a_2, x)-path P' in $\mathcal{H}'' - b_2$ such that $\ell(P') \geq \frac{1}{4}\sum_{i=3}^{h}\left(\frac{(d-2.1)|H_i|}{(d-1)^2}\right)^r + \frac{1}{2}$. Then $P := P_1 \cup P'$ gives the desired path. Thus, we assume $a_2b_2 \in E(P_1)$. If H_3 is 3-connected, then there is a (b_2, x)-path P' in $\mathcal{H}''$ such that $a_2 \notin V(P')$ and $\ell(P') \geq \frac{1}{4}\sum_{i=3}^{h}\left(\frac{(d-2.1)|H_i|}{(d-1)^2}\right)^r + 1$ by Lemma (4.4.2). Hence, $P := (P_1 - a_2) \cup P'$ gives the desired path. Thus, we have $a_2b_2 \in E(P_1)$ and H_3 is a cycle. Recall that H_2 is a cycle by our earlier assumption. We use Lemma (4.4.2) to find an (a_2, x)-path P' of desired length in $\mathcal{H}'' - b_2$. Let $P := P_1 \cup P'$ (since $a_2b_2 \in E(\mathcal{H})$ in this case). Then P is the desired path. □

Lemma (4.4.4). *Assume that Theorem (4.1.1) holds for graphs with less than n vertices. Let $\mathcal{H} = H_1H_2 \dots H_h$ be a block-chain in $G - y$ such that $|\mathcal{H}| < n$, $x \in V(H_1) - V(H_2)$ with $d_{\mathcal{H}}(x) \leq d - 1$, $w \in V(H_k) - V(H_{k-1}) - \{x\}$ for some k with $d_{\mathcal{H}}(w) \leq d - 1$, and let $1 \leq m \leq h$ be fixed. Then there is a (w, x)-path P in $\mathcal{H}$ such that*

$$\ell(P) \geq \frac{1}{4}|H_m|^r + \frac{1}{4}\sum_{i=1,\neq m}^{\max\{k,m\}}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r; \tag{4.3}$$

particularly, when $k = h$,

$$\ell(P) \geq \frac{1}{4}|H_m|^r + \frac{1}{4}\sum_{i\neq m}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r. \tag{4.4}$$

Proof. Let $V(H_i \cap H_{i+1}) = \{a_i, b_i\}$ for $i = 1, 2, \dots, h-1$ such that each $H_i - a_ib_i - a_{i-1}b_{i-1}$ contains two vertex-disjoint paths connecting a_{i-1} to a_i and b_{i-1} to b_i, respectively. We consider the following cases.

Case 1. $m = k = 1$

If H_m is 3-connected, then by Theorem (4.1.1) (c), $H_m + wx$ contains a cycle C_m through wx such that $|C_m| \geq \frac{1}{4}|H_m|^r + 2$. So, $C_m - \{wx\}$ is the desired path. If H_m is a cycle, the (x, w)-path in $H_m - a_m b_m$ with length at least 1 is the desired path (as $x \neq w$).

Case 2. $m = 1$ **and** $k > 1$

If H_m is 3-connected, we perform a T-transform on $(x, a_m b_m)$ and let z be the resulting new vertex. Then, by Theorem (4.1.1) (c), there is a cycle C_m in the T-transformation through xz such that $|C_m| \geq \frac{1}{4}|H_m|^r + 2$. Let $P_m = C_m - z$. Clearly, $a_m b_m \notin E(P_m)$ and $\ell(P_m) \geq \frac{1}{4}|H_m|^r$. Assume P_m is from x to a_m. If H_m is a cycle, let P_m be a path from x to $\{a_m, b_m\}$, say to a_m, which has length at least 1. Let Q be a (w, a_m)-path in $H_{m+1} H_{m+2} \ldots H_k - b_m$ given by Lemma (4.4.2). Then $P := P_m \cup Q$ is the desired path.

Case 3. $m > 1$ **and** $k = m$

Then $w \in H_m$ and $w \notin \{a_{m-1}, b_{m-1}\}$. If H_m is 3-connected, we do a T-transformation on $(w, a_{m-1} b_{m-1})$ and, use Thorem (4.1.1) (c) to obtain a path P_m from $\{a_{m-1}, b_{m-1}\}$, say a_{m-1}, to w with $\ell(P_m) \geq \frac{1}{4}|H_m|^r$; if H_m is a cycle, let P_m be a path from w to $\{a_{m-1}, b_{m-1}\}$, say a_{m-1} of length at least 1. Let P_1 be an (x, a_{m-1})-path in $H_1 \ldots H_{m-1} - b_{m-1}$ with $\ell(P_1) \geq \frac{1}{4} \sum_{i<m} \left(\frac{d-2.1}{(d-1)^2} |H_i| \right)^r$ given by Lemma (4.4.2). Then, $P := P_1 \cup P_m$ is the desired path.

Case 4. $m > 1$ **and** $k < m$

Applying Theorem (4.1.1)(c), we find an (a_{m-1}, b_{m-1})-path P_m in H_m with $\ell(P_m) \geq \frac{1}{4}|H_m|^r + 1$. In case that $a_m b_m \in E(P_m)$ and $a_m b_m \notin E(G)$, the edge $a_m b_m$ on P_m is replaced by an (a_m, b_m)-path in $H_{m+1} H_{m+2} \cdots H_h$.

For each i with $k < i < m$, we use Theorem (4.1.1) (b) to find a cycle C_i in H_i through $a_{i-1} b_{i-1}$, and $a_i b_i$ such that $\ell(C_i) \geq \frac{1}{4}(\frac{(d-2.1)|H_i|}{(d-1)^2})^r + 2$. Let P_i and Q_i be the two components of $C_i - \{a_{i-1} b_{i-1}, a_i b_i\}$.

If $w \in \{a_k, b_k\}$, say $w = a_k$, applying Lemma (4.4.2), we find an (x, b_k)-path P_1 in $H_1 \ldots H_k - a_k$ with $\ell(P_1) \geq \frac{1}{4}\sum_{i\leq k}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r$. Clearly, $P_1 \cup (\cup_{i=k}^{m-1}(P_i \cup Q_i)) \cup P_m$ gives the desired path. So assume $w \notin \{a_k, b_k\}$. If H_k is 3-connected, we do a T-transformation on $(w, a_{k-1}b_{k-1})$ and let w' denote the new vertex. Applying Theorem (4.1.1) (b), we find a cycle C_k in H_k through $w'w$ and a_kb_k such that $\ell(C_k) \geq \frac{1}{4}(\frac{(d-2.1)|H_k|}{(d-1)^2})^r + 2$. Let P_k and Q_k be the two components of $C_k - \{w', a_kb_k\}$. Without loss of generality, we may assume that P_k is a (w, a_{k-1})-path. Note that, $\ell(P_k) + \ell(Q_k) \geq \frac{1}{4}(\frac{d-2.1}{(d-1)^2}|H_k|)^r - 1$. If H_k is a cycle, let P' be the path from w to $\{a_{k-1}, b_{k-1}\}$, say a_{k-1}, in $H_k - a_{k-1}b_{k-1}$ through a_kb_k. Let P_k and Q_k be the two components of $P' - \{a_kb_k\}$. Then $\ell(P_k) + \ell(Q_k) \geq \frac{1}{4}(\frac{d-2.1}{(d-1)^2}|H_k|)^r$ as $w \notin \{a_{k-1}, b_{k-1}, a_k, b_k\}$ and $|H_k| \leq 2d - 1$.

Applying Lemma (4.4.2), we find an (x, a_{k-1})-path P_1 in $H_1 \ldots H_{k-1} - b_{k-1}$ with $\ell(P_1) \geq \frac{1}{4}\sum_{i<k}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r$. Clearly, $P_1 \cup (\cup_{i=k}^{m-1}(P_i \cup Q_i)) \cup P_m$ gives the desired path.

Case 5. $k > m > 1$

We start by finding a desired path in H_m and first consider the case that H_m is 3-connected. Let H'_m be obtained by an H-transform of H_m over $(a_{m-1}b_{m-1}, a_mb_m)$ and let c_{m-1} and c_m be new vertices. By Theorem (4.1.1) (c), we find a cycle C_m in H'_m through $c_{m-1}c_m$ such that $|C_m| \geq \frac{1}{4}|H_m|^r + 2$. Then $C_m - \{c_m, c_{m-1}\}$ gives a path P_m from $\{a_{m-1}, b_{m-1}\}$ to $\{a_m, b_m\}$, say from a_{m-1} to b_m, such that $\ell(P_m) \geq \frac{1}{4}|H_m|^r - 1$. If H_m is a cycle, let P_m be a nontrivial path from $\{a_{m-1}, b_{m-1}\}$ to $\{a_m, b_m\}$, say from a_{m-1} to b_m, not containing the edges $a_{m-1}b_{m-1}$ and a_mb_m.

Applying Lemma (4.4.2), we find an (x, a_{m-1})-path P_1 in $H_1H_2 \ldots H_{m-1} - b_{m-1}$ with $\ell(P_1) \geq \frac{1}{4}\sum_{i<m}(\frac{d-2.1}{(d-1)^2}|H_i|)^r + \frac{1}{2}$; and find a (b_m, w)-path in $H_{m+1}H_{m+2} \ldots H_k - a_m$ with $\ell(P_2) \geq \frac{1}{4}\sum_{m<i<k}(\frac{d-2.1}{(d-1)^2}|H_i|)^r + \frac{1}{2}$. Clearly, $P_1 \cup P_m \cup Q$ is the desired path. □

Let H be a 3-connected graph and $ab \in E(H)$. Then $H - ab$ is a block-chain, and by a simple argument, each of a and b belongs to exactly one block in $H - ab$. In the following lemma, we will say $H - ab$ is a block-chain even in the case that H is a cycle. We include this trivial case just for notational convenience.

Lemma (4.4.5). *Let $\mathcal{H} = H_1 H_2 \dots H_h$ be a block-chain in $G - y$ such that $|\mathcal{H}| \leq n - 1$, and $xx' \in E(H_1) - E(H_2)$ and $y' \in V(H_h) - V(H_{h-1}) - \{x, x'\}$. Suppose $H_k = \max\{|H_i| : H_i \in \mathcal{H}\}$. Let $\{a, b\} = V(H_k) \cap V(H_{k-1})$, where $a = x'$ and $b = x$ if $k = 1$. Let $H_0 := H_{k1} H_{k2} \cdots H_{kk_0} \cdots H_{kk_1}$ be the block-chain $H_k - ab$ (when H_k is a cycle, H_{ki} is a copy of K_2 for each $1 \leq i \leq k_1$) such that $a \in H_{k1}$, $b \in H_{kk_1}$, and $|H_{kk_0}| = \max\{|H_{ki}| : 1 \leq i \leq k_1\}$. Then there is a path P in $\mathcal{H} - x$ from x' to y' with*

$$\ell(P) \geq \frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}\sum_{i=1}^{k_0-1}\left(\frac{d-2.1}{(d-1)^2}|H_{ki}|\right)^r + \frac{1}{4}\sum_{i=k+1}^{h}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r - \frac{1}{2}, \tag{4.5}$$

and a path Q in $\mathcal{H}$ from x to x' with

$$\ell(Q) \geq \frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}\sum_{i \neq k_0}\left(\frac{d-2.1}{(d-1)^2}|H_{ki}|\right)^r + \frac{1}{4}\sum_{i<k}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r; \tag{4.6}$$

moreover, if H_1 is a cycle,

$$\ell(Q) \geq \frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}\sum_{i \neq k_0}\left(\frac{d-2.1}{(d-1)^2}|H_{ki}|\right)^r + \frac{1}{4}\sum_{i<k}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r + \frac{1}{2}. \tag{4.7}$$

Proof. We prove the first statement first.

Case 1. $h = k = 1$

If H_1 is a cycle, then since $y' \notin \{x, x'\}$, we can find an (x', y')-path P in $H_1 - x$ such that $\ell(P) \geq 1 \geq \frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}\sum_{i=1}^{k_0-1}\left(\frac{d-2.1}{(d-1)^2}|H_{ki}|\right)^r$ (using $|H_1| \leq 2d - 1$). So assume H_1 is 3-connected, then apply Lemma (4.4.4) on $H_0 = H_k - ab$, there is a path P from x' to y' such that $\ell(P) \geq \frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}\sum_{i=1}^{k_0-1}\left(\frac{d-2.1}{(d-1)^2}|H_{ki}|\right)^r$.

Case 2. $h > 1$ **and** $k < h$

Let $\{a_k, b_k\} = V(H_k \cap H_{k+1})$. Suppose $k = 1$. If H_1 is a cycle, then let P_1 be a path in $H_1 - x$ from x' to $\{a_k, b_k\}$, say to a_k, such that $\ell(P_1) \geq 1$ (notice that $a_k b_k$ may be on P_1). If $a_k b_k \notin E(P_1)$, then let P_2 be an (a_k, y')-path in $H_2 H_3 \cdots H_h - b_k$ given by Lemma (4.4.2) such that $\ell(P_2) \geq \frac{1}{4}\sum_{i\geq 2}(\frac{d-2.1}{(d-1)^2}|H_i|)^r + \frac{1}{2}$. Then $P := P_1 \cup P_2$ gives the desired path. Hence we assume that $a_k b_k \in E(P_1)$. If H_2 is also a cycle, then $a_k b_k \in E(G)$. We let $P := P_1 \cup P_2$ as in the previous case. So assume H_2 is 3-connected. Let P_2 be a (b_k, y')-path in $H_2 H_3 \cdots H_h - a_k$ given by Lemma (4.4.2) such that $\ell(P_2) \geq \frac{1}{4}\sum_{i\geq 2}(\frac{d-2.1}{(d-1)^2}|H_i|)^r + 1$. Then $(P_1 - \{a_k b_k\}) \cup P_2$ gives the desired path. Suppose H_1 is 3-connected. Since $\{a, b\} = V(H_k) \cap V(H_{k-1})$ and $\{a_k, b_k\} = V(H_k) \cap V(H_{k+1})$, we have $\{a, b\} \neq \{a_k, b_k\}$. Assume that $a \neq a_k$. By applying Lemma (4.4.4) on $H_0 = H_1 - ab$, there is a path P_1 from a to a_k such that $\ell(P_2) \geq \frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}\sum_{i=1}^{k_0-1}\left(\frac{d-2.1}{(d-1)^2}|H_{ki}|\right)^r$. If $a_k b_k \notin E(P_1)$, let P_2 be an (a_k, y')-path in $H_2 H_3 \cdots H_h - b_k$ given by Lemma (4.4.2) such that $\ell(P_2) \geq \frac{1}{4}\sum_{i\geq 2}(\frac{d-2.1}{(d-1)^2}|H_i|)^r + \frac{1}{2}$. Then $P := P_1 \cup P_2$ gives the desired path. Hence we assume that $a_k b_k \in E(P_1)$. Let P_2 be a (b_k, y')-path in $H_2 H_3 \cdots H_h - a_k$ given by Lemma (4.4.2) such that $\ell(P_2) \geq \frac{1}{4}\sum_{i\geq 2}(\frac{d-2.1}{(d-1)^2}|H_i|)^r + \frac{1}{2}$. Then $(P_1 - \{a_k b_k\}) \cup P_2$ gives the desired path.

If $k \geq 2$, let P_1 be an $(x', \{a, b\})$-path, say (x', a)-path, in $H_1 H_2 \cdots H_{k-1} - x$ given by Lemma (4.2.5). Again assume that $a \neq a_k$. By applying Lemma (4.4.4) on $H_0 = H_k - ab$, there is a path P_2 from a to a_k such that $\ell(P_2) \geq \frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}\sum_{i=1}^{k_0-1}\left(\frac{d-2.1}{(d-1)^2}|H_{ki}|\right)^r$. If $a_k b_k \notin E(P_2)$, then by Lemma (4.4.2) we find a path P_3 in $H_{k+1} H_{k+2} \cdots H_h - b_k$ from a_k to y' such that $\ell(P_3) \geq \frac{1}{4}\sum_{i\geq k+1}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r + \frac{1}{2}$. Then $P := P_1 \cup P_2 \cup P_3$ is the desired path. Hence assume $a_k b_k \in E(P_2)$. By Lemma (4.4.2), let P_3 in $H_{k+1} H_{k+2} \cdots H_h - a_k$ from b_k to y' such that $\ell(P_3) \geq \frac{1}{4}\sum_{i\geq k+1}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r + \frac{1}{2}$. Then $P := P_1 \cup (P_2 - a_k b_k) \cup P_3$ is the desired path.

Case 3. $h = k > 1$

Since $y' \in V(H_h) - V(H_{h-1})$ and $\{a, b\} = V(H_k) \cap V(H_{k-1})$, we have $y' \notin \{a, b\}$. Applying Lemma (4.4.4) on $H_0 = H_k - ab$, we obtain an (a, y')-path P_2 such that $\ell(P_2) \geq \frac{1}{4}|H_{kk_0}|^r +$

$\frac{1}{4}\sum_{i=1}^{k_0-1}\left(\frac{d-2.1}{(d-1)^2}|H_{ki}|\right)^r$. Then $P := P_1 \cup P_2$ is the desired path.

For the second statement, we first apply Lemma (4.4.4) to $H_0 = H_k - ab$ to find an (a, b)-path P_k of length at least $\frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}\sum_{i\neq k_0}\left(\frac{d-2.1}{(d-1)^2}|H_{ki}|\right)^r$ (the virtual edge may contained in $E(H_k \cap H_{k+1})$ will be replaced by a path in $H_{k+1}H_{k+2}\cdots H_h$). Denote $\{a_0, b_0\} := \{x, x'\}$. Then for each $1 \leq i \leq k-1$ we use Theorem (4.1.1) (b) to find a cycle C_i in H_i through $a_{i-1}b_{i-1}$, and a_ib_i such that $\ell(C_i) \geq \frac{1}{4}(\frac{(d-2.1)|H_i|}{(d-1)^2})^r + 2$. Let P_i and Q_i be the two components of $C_i - \{a_{i-1}b_{i-1}, a_ib_i\}$. Then $Q := \bigcup_{i=1}^{k-1}(P_i \cup Q_i) \cup P_k$ gives the desired path.

Particularly, suppose H_1 is a cycle. If $k = 1$, we can find an (x, x')-path Q in H_1 such that $\ell(Q) \geq 2$ (the virtual edge may contained in $E(H_1 \cap H_2)$ will be replaced by a path in $H_2H_2\cdots H_h$). Then $\ell(Q) \geq 2 \geq \frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}\sum_{i\neq k_0}\left(\frac{d-2.1}{(d-1)^2}|H_{ki}|\right)^r + 1/2$. If $k > 1$, then we apply Lemma (4.4.4) to $H_0 = H_k - ab$ to find an (a, b)-path P_k of length at least $\frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}\sum_{i\neq k_0}\left(\frac{d-2.1}{(d-1)^2}|H_{ki}|\right)^r$ (the virtual edge may contained in $E(H_k \cap H_{k+1})$ will be replaced by a path in $H_{k+1}H_{k+2}\cdots H_h$). Denote $\{a_0, b_0\} := \{x, x'\}$. Then for each $1 \leq i \leq k-1$ we use Theorem (4.1.1) (b) to find a cycle C_i in H_i through $a_{i-1}b_{i-1}$, and a_ib_i such that $\ell(C_i) \geq \frac{1}{4}(\frac{(d-2.1)|H_i|}{(d-1)^2})^r + 2$. Let P_i and Q_i be the two components of $C_i - \{a_{i-1}b_{i-1}, a_ib_i\}$. As H_1 is a cycle, in particular, $\ell(P_1) + \ell(Q_1) \geq 1 \geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|H_1|\right)^r + 1/2$. Then $Q := \bigcup_{i=1}^{k-1}(P_i \cup Q_i) \cup P_k$ gives the desired path. □

Lemma (4.4.6). *Assume that Theorem (4.1.1) holds for graphs with less than n vertices. Let G be a 3-connected graph with $\Delta(G) \leq d$, $|G| < n$ and $xy \in E(G)$. Suppose $\mathcal{H} = H_1H_2\ldots H_h$ and $\mathcal{L} = L_1L_2\ldots L_\ell$ are two block-chains in $G - y$ such that (a) $x \in (V(H_1) - V(H_2)) \cap (V(L_1) - V(L_2))$; (b) $xw \in E(H_1) - E(H_2)$ and $xw' \in E(L_1) - E(L_2)$; and (c) $\{x\} = V(\mathcal{H}) \cap V(\mathcal{L})$ when $w \neq w'$, and $\{x, w\} = V(\mathcal{H}) \cap V(\mathcal{L})$ otherwise. Let $y' \in V(H_h) - V(H_{h-1}) - \{x, w\}$ and $y'' \in V(L_l) - V(L_{l-1}) - \{x, w'\}$. Then, provided that $d \geq 25$, either there is a path P_H from w to y' in $\mathcal{H} - x$, and a path P_L from w' to x in $\mathcal{L}$ or there is a path P_H from w to x in $\mathcal{H}$, and a path P_L from w' to y' in $\mathcal{L} - x$ such that $\ell(P_H) + \ell(P_L) \geq \frac{1}{4}|\mathcal{H}|^r + \frac{1}{4}|\mathcal{L}|^r - 1/2$; moreover, if H_1 is a cycle and L_1 is a cycle, we can* *have* $\ell(P_H)+\ell(P_L)\geq \frac{1}{4}|\mathcal{H}|^r+\frac{1}{4}|\mathcal{L}|^r$.

Proof. Let $1\leq k\leq h$ and $1\leq p\leq \ell$ such that $|H_k|=\max\{|H_i|:1\leq i\leq h\}$ and $|L_p|=\max\{|L_i|:1\leq i\leq \ell\}$. Let $\{a,b\}=V(H_k)\cap V(H_{k-1})$, where $\{a,b\}=\{x,w\}$ when $k=1$; and $\{c,d\}=V(L_p)\cap V(L_{p-1})$, where $\{c,d\}=\{x,w'\}$ when $p=1$. Let $H_0:=H_{k1}H_{k2}\cdots H_{kk_0}\cdots H_{kk_1}$ be the block-chain H_k-ab and $L_0:=L_{p1}L_{p2}\cdots L_{pp_0}\cdots P_{pp_1}$ be the block-chain L_p-cd, such that (i) $|H_{kk_0}|=\max\{|H_{ki}|:H_{ki}\in H_0\}$ and $|L_{pp_0}|=\max\{|L_{pi}|:L_{pi}\in L_0\}$, and (ii) $a\in H_{k1}, b\in H_{kk_1}$ and $c\in L_{p1}, d\in L_{pp_1}$ be distinct. Denote

- $h^+=\sum_{i>k}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r$, $\quad h^-=\sum_{i<k}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r$;
- $h_0^+=\sum_{i>k_0}\left(\frac{d-2.1}{(d-1)^2}|H_{ki}|\right)^r$, $\quad h_0^-=\sum_{i<k_0}\left(\frac{d-2.1}{(d-1)^2}|H_{ki}|\right)^r$;
- $l^+=\sum_{i>p}\left(\frac{d-2.1}{(d-1)^2}|L_i|\right)^r$, $\quad l^-=\sum_{i<p}\left(\frac{d-2.1}{(d-1)^2}|L_i|\right)^r$;
- $l_0^+=\sum_{i>p_0}\left(\frac{d-2.1}{(d-1)^2}|L_{pi}|\right)^r$, $\quad l_0^-=\sum_{i<p_0}\left(\frac{d-2.1}{(d-1)^2}|L_{pi}|\right)^r$.

By symmetry between $\mathcal{H}$ and $\mathcal{L}$ (both H_1 and L_1 are cycles for the statement in the "moreover" part), we may assume

$$h^++l^-+l_0^+ \;\;\geq\;\; l^++h^-+h_0^+.$$

Let P_H be a path in $\mathcal{H}-x$ from w to y' given by Lemma (4.4.5) (see (4.5)) such that

$$\ell(P_H)\geq \frac{1}{4}|H_{kk_0}|^r+\frac{1}{4}h^++\frac{1}{4}h_0^- -1/2,$$

and P_L be a path in $\mathcal{L}$ from w' to x given by Lemma (4.4.5) (see (4.6)) such that

$$\ell(P_L)\geq \frac{1}{4}|L_{pp_0}|^r+\frac{1}{4}l_0^++\frac{1}{4}l_0^-+\frac{1}{4}l^-.$$

In particular, if L_1 is a cycle, then

$$\ell(P_L) \geq \frac{1}{4}|L_{pp_0}|^r + \frac{1}{4}l_0^+ + \frac{1}{4}l_0^- + \frac{1}{4}l^- + 1/2.$$

Since $h^+ + \ell^- + \ell_0^+ \geq \ell^+ + h^- + h_0^+$,

$$h^+ + l^- + l_0^+ \geq 1/2(h^+ + l^- + l_0^+) + 1/2(l^+ + h^- + h_0^+).$$

Using $(d-1)^{\log_2 5/4} - 1 \geq \frac{d-1}{d-2.1}$ when $d \geq 25$ and $x^r + y^r \geq (x + (d-1)^2((d-1)^{\log_2 5/4} - 1)y)^r$ if $x \geq (d-1)^2 y$ (equality (1d)), we have

$$\frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}(\frac{1}{d-1})^r h^+ + \frac{1}{4}h_0^- + \frac{1}{4}(\frac{1}{d-1})^r h^- + \frac{1}{4}(\frac{1}{d-1})^r h_0^+ \geq \frac{1}{4}|\mathcal{H}|^r,$$

and,

$$\frac{1}{4}|L_{pp_0}|^r + \frac{1}{4}(\frac{1}{d-1})^r l^+ + \frac{1}{4}l_0^- + \frac{1}{4}(\frac{1}{d-1})^r l^- + \frac{1}{4}(\frac{1}{d-1})^r l_0^+ \geq \frac{1}{4}|\mathcal{L}|^r.$$

Hence,

$$\begin{aligned}
& \ell(P_H) + \ell(P_L) \\
\geq\ & \frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}h^+ + \frac{1}{4}h_0^- + \frac{1}{4}|L_{pp_0}|^r + \frac{1}{4}l_0^+ + \frac{1}{4}l_0^- + \frac{1}{4}l^- - \frac{1}{2} \\
=\ & \frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}h_0^- + \frac{1}{4}|L_{pp_0}|^r + \frac{1}{4}l_0^- + (\frac{1}{4}h^+ + \frac{1}{4}l^- + \frac{1}{4}l_0^+) - \frac{1}{2} \\
\geq\ & \frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}h_0^- + \frac{1}{4}|L_{pp_0}|^r + \frac{1}{4}l_0^- + \frac{1}{4}(h^+ + \ell^- + \ell_0^+ + \ell^+ + h^- + h_0^+) - \frac{1}{2} \\
\geq\ & \frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}h_0^- + \frac{1}{4}|L_{pp_0}|^r + \frac{1}{4}\ell_0^- + \frac{1}{4}(\frac{1}{d-1})^r h^+ + \frac{1}{4}(\frac{1}{d-1})^r h_0^- + \\
& + \frac{1}{4}(\frac{1}{d-1})^r l_0^+ + \frac{1}{4}(\frac{1}{d-1})^r h_0^+ + \frac{1}{4}(\frac{1}{d-1})^r l^+ + \frac{1}{4}(\frac{1}{d-1})^r l^-) - \frac{1}{2} \quad \left(\text{since } (\tfrac{1}{d-1})^r = \tfrac{1}{2})\right) \\
=\ & \frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}(\frac{1}{d-1})^r h^+ + \frac{1}{4}h_0^- + \frac{1}{4}(\frac{1}{d-1})^r h^- + \frac{1}{4}(\frac{1}{d-1})^r h_0^+ + \\
& \frac{1}{4}|L_{pp_0}|^r + \frac{1}{4}(\frac{1}{d-1})^r l^+ + \frac{1}{4}l_0^- + \frac{1}{4}(\frac{1}{d-1})^r l^- + \frac{1}{4}(\frac{1}{d-1})^r l_0^+ - \frac{1}{2} \\
\geq\ & \frac{1}{4}|\mathcal{H}|^r + \frac{1}{4}|\mathcal{L}|^r - \frac{1}{2};
\end{aligned}$$

and when both H_1 and L_1 are cycles, $\ell(P_H) + \ell(P_L) \geq \frac{1}{4}|\mathcal{H}|^r + \frac{1}{4}|\mathcal{L}|^r$. $\square$

4.5 Proofs of Theorem (4.1.1) (a) and (b)

The following two lemmas state that parts (a) and (b) of Theorem (4.1.1) can be reduced to Theorem (4.1.1) for smaller graphs. The proof of (a) is essentially the same as that in [12], but the proof of (b) needs more work.

Lemma (4.5.1). *Let $n \geq 4$ be an integer. If Theorem (4.1.1) holds for graphs with at most $n-1$ vertices, then Theorem (4.1.1)(a) holds for graphs with n vertices.*

Proof. Let G be an arbitrary 3-connected graph with n vertices, let $xy \in E(G)$ and $z \in V(G) - \{x, y\}$, and assume that $\Delta(G - z) \leq d$. Let t denote the number of neighbors of z in $G - \{x, y\}$. Since G is 3-connected, $t \geq 1$.

Let $\mathcal{H} = H_1 \ldots H_h$ be a block-chain in $G - z$ such that $xy \in E(H_1) - E(H_2)$ and, subject to this, $|\mathcal{H}|$ is maximum. Therefore, H_h is an extreme 3-block of $G - z$. Since each extreme 3-block of $G - z$ must contain a neighbor of z, there are at most $t - 1$ extreme 3-blocks of $G - z$ different from H_h, and hence $V(G - z)$ is covered by at most t block-chains starting from H_1 and ending with an extreme 3-block of $G - z$. It then follows that $|\mathcal{H}| \geq (n-1)/t$.

Note that $\Delta(G - z) \leq d$ implies that $\Delta(H_i) \leq d$ for $1 \leq i \leq h$. By Lemma (4.4.1), there is a path P in $\mathcal{H}$ from x to y such that

$$\ell(P) \geq \frac{1}{4}\left(\frac{d-2.1}{d-1}(|\mathcal{H}|+1)\right)^r + 1 \geq \frac{1}{4}\left(\frac{d-2.1}{d-1} \cdot \frac{n}{t}\right)^r + 1.$$

Then $C := P + xy$ is a cycle through xy in $G - z$ with $\ell(C) = \ell(P) + 1$, giving the desired cycle. $\square$

Lemma (4.5.2). *Let $n \geq 4$ be an integer. Suppose Theorem (4.1.1) holds for graphs with* *at most $n-1$ vertices. Then Theorem (4.1.1)(b) holds for graphs with n vertices.*

Proof. We note that by Lemma (4.5.1), Theorem (4.1.1)(a) holds for graphs with n vertices. Also note that Theorem (4.1.1)(b) holds trivially for cycles. So it suffices to show that Theorem (4.1.1)(b) holds for 3-connected graphs. Let G be a 3-connected graph with n vertices, let $e = xy, f$ be two distinct edges of G, and assume $\Delta(G) \leq d$.

Suppose e and f share a common vertex. Let $f = yz$. We note that $G' := G + xz$ is 3-connected, $\Delta(G' - y) \leq d$, and that y has at most $d-2$ neighbors distinct from x and z. By employing Theorem (4.1.1)(a) to G', which has n vertices, there is a cycle C' through xz in $G' - y$ such that

$$\ell(C') \geq \frac{1}{4}\left(\frac{(d-2.1)n}{(d-1)(d-2)}\right)^r + 2 \geq \frac{1}{4}\left(\frac{(d-2.1)n}{(d-1)^2}\right)^r + 2.$$

Then $C := (C' - \{xz\}) \cup \{e, f\}$ gives a cycle through e and f such that

$$\ell(C) \geq \frac{1}{4}\left(\frac{(d-2.1)n}{(d-1)^2}\right)^r + 3.$$

Therefore, we may assume that e and f are not adjacent. Let $\mathcal{H} := H_1 \dots H_h$ be a block-chain in $G - y$ such that $x \in V(H_1) - V(H_2)$ and $f \in E(H_h) - E(H_{h-1})$. Note that the degree of x is at most $d-1$ in $\mathcal{H}$ and $\Delta(H_i) \leq d$ for all $1 \leq i \leq h$. Suppose $V(\mathcal{H}) = V(G - y)$. If $\mathcal{H}$ is a cycle, then every vertex of $\mathcal{H}$ is adjacent to y. Let x' be a neighbor of x in $\mathcal{H}$ and P be the path in $\mathcal{H}$ from x to x' through f. Then $P \cup \{yx, yx'\}$ is the desired cycle for Theorem (4.1.1)(b).

Now assume that $\mathcal{H}$ is not a cycle. If H_h is a cycle, we choose x' to be an endvertex of f which has degree 2 in $\mathcal{H}$; otherwise, let $x' \in (V(H_h) - V(H_{h-1})) \cap N_G(y)$ such that x' is incident with f whenever possible (for the choice of f' in the following). Let H' be the graph obtained from $\mathcal{H}$ by joining x to x', and then suppressing all the remaining degree

2 vertices. It is clear that H' is 3-connected, $|H'| \geq n-1-(d-1)$, and $\Delta(H') \leq d$. Let $f' = f$ if $f \in E(H')$, otherwise let f' denote the new edge incident with x' in H'. We use Theorem (4.1.1)(b) to find a cycle C' in H' through xx' and f' such that $\ell(C') \geq \frac{1}{4}(\frac{d-2.1}{(d-1)^2}|H'|)^r + 2$. Then $(C' - \{xx'\}) \cup \{yx, yx'\}$ (adding back the suppressed vertices if necessary) gives a cycle C in G through xy and f such that

$$\ell(C) \geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|H'|\right)^r + 3 \geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}n\right)^r + 2 \quad \text{(by Lemma (4.3.2))}.$$

So we may assume that $V(\mathcal{H}) \neq V(G-y)$. Then there is a 3-block B of $G-y$ such that $|V(B) \cap V(\mathcal{H})| = 2$. Let $\{p, q\} := V(B) \cap V(\mathcal{H})$ and G_1 be the graph obtained from G by deleting those components of $G - \{y, p, q\}$ containing a vertex of $\mathcal{H}$. We choose $\{p, q\}$ so that $|G_1|$ is maximum. Then,

$$(d-1)|G_1| + |\mathcal{H}| \geq n. \tag{4.8}$$

If $V(G) = V(G_1 \cup \mathcal{H})$, we let $G_2 = \emptyset$. Otherwise, there is a 3-block B' of $G-y$ such that $V(B') \cap V(\mathcal{H} \cup G_1) = \{v, w\}$ for some $\{v, w\} \neq \{p, q\}$. (Note that $\{v, w\} \subseteq V(\mathcal{H})$.) Define G_2 as the graph obtained from G by deleting those components of $G - \{y, v, w\}$ containing a vertex of $G_1 \cup \mathcal{H}$. We choose G_2 such that $|G_2|$ is maximum. Then

$$(d-2)|G_2| + |G_1| + |\mathcal{H}| \geq n. \tag{4.9}$$

Clearly, $|G_1| \geq |G_2|$. Let G_1' be the graph obtained from G_1 by adding the edges yp, yq, and pq if they are not already in G_1. Define G_2' similarly from G_2. We note that G_1' and G_2' (if nonempty) are both 3-connected. We shall find the desired cycle for Theorem (4.1.1)(b) by combining long paths in the two largest graphs among $\mathcal{H}$, G_1, and G_2. Let $t_i := |N(y) \cap V(G_i) - (\{p, q\} \cup \{v, w\})|$ for $i = 1$, 2, respectively. We divide the remaining proof into two cases.

4.5.1 Case 1: $t_1 \geq 2$ or $t_2 \geq 2$.

In this case, inequalities (4.8) and (4.9) can be improved (by exactly the same reasons) to

$$(d-2)|G_1| + |\mathcal{H}| \geq n \quad \text{and} \tag{4.10}$$

$$(d-3)|G_2| + |G_1| + |\mathcal{H}| \geq n. \tag{4.11}$$

Suppose $|\mathcal{H}| \geq |G_2|$. Then from (4.11), we have

$$|G_1| + (d-2)|\mathcal{H}| \geq n. \tag{4.12}$$

If $pq \neq f$, we use Lemma (4.4.3) to find a path P in $\mathcal{H}$ from x to $z \in \{p, q\}$, say $z = p$, through f such that $pq \notin E(P)$, and

$$\ell(P) \geq \frac{1}{4}\left(\frac{(d-2.1)|\mathcal{H}|}{(d-1)^2}\right)^r.$$

Since $e = xy$ is not adjacent to f, $x \notin \{p, q\}$. Hence if $pq = f$, we can apply Lemma (4.4.2) to find a path P' in $\mathcal{H} - p$ from x to q such that

$$\ell(P') \geq \frac{1}{4}\left(\frac{(d-2.1)|\mathcal{H}|}{(d-1)^2}\right)^r + \frac{1}{2},$$

and set $P := P' \cup \{pq\}$ in this case. Since $\Delta(G_1' - q) \leq d$, we may use Theorem (4.1.1)(a) to find a cycle C_1 through py in $G_1' - q$ such that

$$\ell(C_1) \geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|G_1|\right)^r + 2.$$

Then $P \cup (C_1 - \{py\}) \cup \{xy\}$ gives a cycle C in G through xy and f such that

$$\begin{aligned}
\ell(C) &\geq \frac{1}{4}\left(\frac{(d-2.1)|G_1|}{(d-1)^2}\right)^r + \frac{1}{4}\left(\frac{(d-2.1)|\mathcal{H}|}{(d-1)^2}\right)^r + 2 \\
&\geq \begin{cases} \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}((d-2)|G_1| + |\mathcal{H}|)\right)^r + 2, & \text{if } |\mathcal{H}| \geq |G_1|; \\ \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}(|G_1| + (d-2)|\mathcal{H}|)\right)^r + 2, & \text{if } |\mathcal{H}| < |G_1|; \end{cases} \qquad \text{(by (1a))} \\
&\geq \frac{1}{4}\left(\frac{(d-2.1)n}{(d-1)^2}\right)^r + 2 \quad \text{(by (4.10) and (4.12))}.
\end{aligned}$$

Now assume $|\mathcal{H}| < |G_2|$, and hence $G_2 \neq \emptyset$. Let P_1 be a path in $\mathcal{H}$ from x to $z \in \{p,q\} \cup \{v,w\}$ through f as given by Lemma (4.2.1) such that (i) exactly one of pq and vw is in $E(P_1)$; (ii) if $pq \in E(P_1)$ then $z \in \{v,w\}$; and (iii) if $vw \in E(P_1)$ then $z \in \{p,q\}$. Assume, without loss of generality, that $pq \in E(P_1)$ and $z = v$. Let P_2 be a (p,q)-path in $G_1' - y$ given by Theorem (4.1.1)(a), and let P_3 be a (v,y)-path in $G_2' - w$ given by Theorem (4.1.1)(a). Then $\ell(P_2) \geq \frac{1}{4}\left(\frac{(d-2.1)}{(d-1)^2}|G_1'|\right)^r + 1$ and $\ell(P_3) \geq \frac{1}{4}\left(\frac{(d-2.1)}{(d-1)^2}|G_2|\right)^r + 1$. Now $C := (P_1 - \{pq\}) \cup P_2 \cup P_3 \cup \{xy\}$ is a cycle through xy and f in G such that

$$\begin{aligned}
\ell(C) &\geq \frac{1}{4}\left(\frac{(d-2.1)|G_1|}{(d-1)^2}\right)^r + \frac{1}{4}\left(\frac{(d-2.1)|G_2|}{(d-1)^2}\right)^r + 2 \\
&\geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}(|G_1| + (d-2)|G_2|)\right)^r + 2 \quad \text{(by (1a) and } |G_1| \geq |G_2|) \\
&\geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}(|G_1| + (d-3)|G_2| + |\mathcal{H}|)\right)^r + 2 \quad \text{(since } |G_2| > |\mathcal{H}|) \\
&\geq \frac{1}{4}\left(\frac{(d-2.1)n}{(d-1)^2}\right)^r + 2 \quad \text{(by (10))}.
\end{aligned}$$

4.5.2 **Case 2.** $t_1 = t_2 = 1$.

Let P be a path in $\mathcal{H}$ from x to $z \in \{p,q\} \cup \{v,w\}$ through f as given by Lemma (4.2.1) such that if $pq \in E(P)$ then $z \in \{v,w\}$ and if $vw \in E(P)$ then $z \in \{p,q\}$.

Suppose $|\mathcal{H}| \leq (d-3)|G_2|$. If $vw \in E(P)$ and $z \in \{p,q\}$, say $z = p$, then let P_1 be a longest path in $G_1' - q$ from p to y and P_2 a longest (v,w)-path in $G_2' - y$. By Theorem (4.1.1) (a) for P_1 and Lemma (4.4.1) for P_2 (using $t_2 = 1$, in this case $G_2' - y$ is a block-chain), we have the following lower bounds for $\ell(P_1)$ and $\ell(P_2)$.

$$\ell(P_1) \geq \frac{1}{4}\left(\frac{(d-2.1)|G_1|}{(d-1)^2}\right)^r + 2, \tag{4.13}$$
$$\ell(P_2) \geq \frac{1}{4}\left(\frac{(d-2.1)|G_2|}{d-1}\right)^r + 1. \tag{4.14}$$

Let $C := (P - \{vw\}) \cup P_1 \cup P_2 \cup \{xy\}$. Then C is a cycle in G though xy and f such that

$$\ell(C) \geq \begin{cases} \frac{1}{4}\left(\frac{(d-2.1)}{(d-1)^2}((d-2)|G_1| + (d-1)|G_2|)\right)^r + 3 \geq \frac{1}{4}\left(\frac{(d-2.1)}{(d-1)^2}n\right)^r + 3, & \text{if } (d-1)|G_2| \geq |G_1| \\ \frac{1}{4}\left(\frac{(d-2.1)}{(d-1)^2}(|G_1| + (d-2)(d-1)|G_2|)\right)^r + 3 \geq \frac{1}{4}\left(\frac{(d-2.1)}{(d-1)^2}n\right)^r + 3, & \text{if } (d-1)|G_2| < |G_1|; \end{cases}$$

If $pq \in E(P), z \in \{v,w\}$, say $z = w$, let P_2 be a longest path in $G_2' - v$ from w to y, and P_1 be a longest (p,q)-path in $G_1' - y$. Using Theorem (4.1.1) (a) for P_2 and Lemmas (4.4.1) for P_1 (using $t_1 = 1$, then in this case $G_1' - y$ is a block-chain), we have the following lower bounds for $\ell(P_1)$ and $\ell(P_2)$.

$$\ell(P_1) \geq \frac{1}{4}\left(\frac{(d-2.1)|G_1|}{d-1}\right)^r + 1 \tag{4.16}$$
$$\ell(P_2) \geq \frac{1}{4}\left(\frac{(d-2.1)|G_2|}{(d-1)^2}\right)^r + 2. \tag{4.17}$$

Let $C := (P - \{pq\}) \cup P_1 \cup P_2 \cup \{xy\}$. Since $(d-2)|G_2| \geq |\mathcal{H}|$, by (7), $(d-1)|G_1| + (d-2)|G_2| \geq n$. Then as $(d-1)|G_1| \geq |G_2|$ always holds, by (1a) we have

$$\ell(C) \geq \frac{1}{4}\left(\frac{(d-2.1)}{(d-1)^2}((d-1)|G_1| + (d-2)|G_2|)\right)^r + 3 \geq \frac{1}{4}\left(\frac{(d-2.1)}{(d-1)^2}n\right)^r + 3.$$

So we may assume $|\mathcal{H}| > (d-3)|G_2|$. If $pq \neq f$, we use Lemma (4.4.3) to find a path P in $\mathcal{H}$ from x to $z \in \{p,q\}$, say $z = p$, through f such that $pq \notin E(P)$ and

$$\ell(P) \geq \frac{1}{4}\left(\frac{(d-2.1)|\mathcal{H}|}{(d-1)^2}\right)^r.$$

Again $x \notin \{p,q\}$. Hence if $pq = f$, we can apply Lemma (4.4.2) to find a path P' in $\mathcal{H} - p$ from x to q such that

$$\ell(P') \geq \frac{1}{4}\left(\frac{(d-2.1)|\mathcal{H}|}{(d-1)^2}\right)^r,$$

and set $P := P' \cup \{pq\}$. Since $\Delta(G_1' - q) \leq d$, we may use Theorem (4.1.1)(a) to find a cycle C_1 through py in $G_1' - q$ such that

$$\ell(C_1) \geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|G_1|\right)^r + 2.$$

Then $P \cup (C_1 - \{py\}) \cup \{xy\}$ gives a cycle C in G through xy and f. If $|\mathcal{H}| \geq (d-4)|G_1|$, then by inequality (1c),

$$\begin{aligned}
|C| &\geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}(|\mathcal{H}| + (d-4)((d-1)^{\log_2 3/2} - 1)|G_1|)\right)^r + 2 \\
&\geq \frac{1}{4}\left(\frac{(d-2.1)}{(d-1)^2}(|\mathcal{H}| + (d-1)|G_1|)\right)^r + 2 \ \big(\text{when } d \geq 8,\ (d-1)^{\log_2(3/2)} \geq 3 \text{ and } 2(d-4) \geq (d-1)\big) \\
&\geq \frac{1}{4}\left(\frac{(d-2.1)}{(d-1)^2}n\right)^r + 2 \quad (\text{by } (4.8)).
\end{aligned}$$

So $(d-3)|G_2| < |\mathcal{H}| < (d-4)|G_1|$. Then from (4.8) and (4.9), we have

$$|G_1| + (d-2)|\mathcal{H}| \geq n \text{ and } (d-2)|G_1| + |\mathcal{H}| \geq |G_1| + (d-2)|G_2| + |\mathcal{H}| \geq n,$$

where the second inequality follows from $(d-3)|G_1| > (d-2)|G_2|$ as $(d-4)|G_1| > (d-3)|G_2|$. By using the inequalities,

$$|G_1|^r + |\mathcal{H}|^r \geq \min\{(|G_1| + (d-2)|\mathcal{H}|)^r, ((d-2)|G_1| + |\mathcal{H}|)^r\},$$

we obtain that $\ell(C) \geq \frac{1}{4}((d-2.1)n/(d-1)^2)^r + 2$. □

4.6 Reduction of Theorem (4.1.1)(c)

In this section, we prove the following result which reduces Theorem (4.1.1)(c) to Theorem (4.1.1) for smaller graphs. The part of proof is long and tedious, but contains a few crucial new ideas in estimating the lower bound of special paths.

Lemma (4.6.1). *Let $n \geq 4$ be an integer. If Theorem (4.1.1) holds for graphs with at most $n-1$ vertices, then Theorem (4.1.1)(c) holds for graphs with n vertices.*

To prove Lemma (4.6.1), let G be a 3-connected graph with n vertices and $\Delta(G) \leq d$, and let $xy \in E(G)$. It is easy to see that when $n \geq 5$, G contains a cycle through xy of length at least $5 = \frac{1}{2}\left((d-1)^{\log_2 6}\right)^r + 2$. Hence, Theorem (4.1.1)(c) holds when $n \leq (d-1)^{\log_2 6}$. So we assume $n > (d-1)^{\log_2 6}$ hereafter.

Let $\mathcal{H} := H_1 H_2 \cdots H_h$ be a block-chain in $G - y$ such that $x \in V(H_1) - V(H_2)$ and subject to this, $|\mathcal{H}|$ is maximum. We note that $\mathcal{H}$ may contain only one block H_1. In this case, all 3-blocks attached to H_1 contain x and H_1 may not be an extreme block. However, when $h \geq 2$, H_h must be an extreme 3-block in $G - y$ and there is a vertex $x' \in (V(H_h) - V(H_{h-1})) \cap N_G(y)$.

Claim (4.6.0.1). *We may assume $V(G-y) \neq V(\mathcal{H})$.*

Proof. Suppose $V(G-y) = V(\mathcal{H})$. Since G is 3-connected, there exists a vertex $x' \in$

$(V(H_h) - V(H_{h-1})) \cap (N_G(y) - x)$. Let H' be obtained from H by joining x' and x and then suppressing all remaining degree 2 vertices. Clearly, H' is 3-connected with $\Delta(H') \leq d$ and $n > |H'| \geq (n-1) - (d-2) = n - d + 1$. Let C' be a longest cycle in H' through xx'. By Theorem (4.1.1) (c), we have $|C'| \geq \frac{1}{4}|H'|^r + 2$. Let $C = (C' - \{xx'\}) \cup \{xy, x'y\}$. Then by Lemma (4.3.2), we have $|C| \geq |C'| + 1 \geq \frac{1}{4}(n-d+1)^r + 3 \geq \frac{1}{4}n^r + 2$, so C is the desired cycle. □

A block-chain $\mathcal{L} := L_1 L_2 \ldots L_\ell$ different from $\mathcal{H}$ is called an *$\mathcal{H}$-leg* if $\mathcal{H} \cap \mathcal{L} \subseteq L_1 - L_2$ and L_ℓ is an extreme block. Note that, for each extreme block L not in $\mathcal{H}$, there is a unique $\mathcal{H}$-leg containing L.

Since $\mathcal{H} \neq G - y$ and G is 3-connected, there are $\mathcal{H}$-legs. Let $\mathcal{L} := L_1 L_2 \ldots L_\ell$ be an $\mathcal{H}$-leg with $|\mathcal{L}|$ maximum. Suppose further that $V(\mathcal{H}) \cap V(\mathcal{L}) = V(H_t) - V(H_{t-1}) \cap V(L_1) = \{p, q\}$ for some $1 \leq t \leq h$. Since each $\mathcal{H}$-leg contains an extreme block and each extreme block contains a neighbor of y, there are at most $d-1$ $\mathcal{H}$-legs. Hence, $(d-1)(|\mathcal{L}|-2) + |\mathcal{H}| \geq n - 1$, that is,

$$(d-1)|\mathcal{L}| + |\mathcal{H}| \geq n + 2d - 3. \tag{4.18}$$

We will use the following parameters (which approach 0 as $d \to \infty$):

$$\epsilon_1 := \frac{d-1}{(d-2.1)((d-1)^{\log_2(3/2)} - 1)}, \qquad \epsilon_2 := \frac{1}{(d-1)^{\log_2(5/4)} - 1}.$$

Claim (4.6.0.2). *We may assume $|\mathcal{H}| \leq (\epsilon_1 + \epsilon_2)n$.*

Proof. Suppose $|\mathcal{H}| > (\epsilon_1 + \epsilon_2)n$. Let $H_m \in \mathcal{H}$ such that $|H_m|$ is maximum, $\mathcal{H}' := H_1 H_2 \ldots H_{m-1}$ and $\mathcal{H}'' := H_{m+1} H_{m+2} \ldots H_h$. For each $2 \leq i \leq h$, let $\{a_i, b_i\} = V(H_i) \cap V(H_{i-1})$.

If the vertex $x' \in (V(H_h) - V(H_{h-1})) \cap (N_G(y) - x)$ is well defined, we let P be a longest (x, x')-path in $\mathcal{H}$ and $C := P \cup \{xy, x'y\}$. Clearly, C is a cycle containing edge xy.

We will show that C is the desired cycle by estimating lower bounds of $|C|$ in different cases accordingly.

Suppose $|\mathcal{H}'| + |\mathcal{H}''| \geq \epsilon_1 n > 0$. Then, $h \geq 2$ and H_h is an extreme block of $G - y$, so the vertex $x' \in (V(H_h) - V(H_{h-1})) \cap (N_G(y) - x)$ is well defined. Applying Lemma (4.4.4) (see (4)), we obtain a lower bound of $|C|$ below.

$$\begin{aligned}
|C| &\geq \frac{1}{4}|H_m|^r + \frac{1}{4}\sum_{i \neq m}\left(\frac{d-2.1}{(d-1)^2}|H_i|\right)^r + 2 \\
&\geq \frac{1}{4}\left(|H_m| + \left((d-1)^{\log_2(3/2)} - 1\right)\frac{d-2.1}{d-1}(|\mathcal{H}'| + |\mathcal{H}''|)\right)^r + 2 \quad \text{(by } |H_m| \geq |H_i| \text{ and (4.1c))} \\
&\geq \frac{1}{4}\left(\frac{d-2.1}{d-1}\left((d-1)^{\log_2(3/2)} - 1\right)(|\mathcal{H}'| + |\mathcal{H}''|)\right)^r + 2 \\
&\geq \frac{1}{4}n^r + 2.
\end{aligned}$$

Thus, we assume $|\mathcal{H}'| + |\mathcal{H}''| < \epsilon_1 n$. Then, $|H_m| > \epsilon_2 n$ as $|\mathcal{H}| > (\epsilon_1 + \epsilon_2)n$.

We distinguish two cases by considering which one is bigger between $|\mathcal{L}|$ and $|\mathcal{H}''|$. Suppose first that $|\mathcal{L}| \leq |\mathcal{H}''|$. Then, using $|H_m| \geq |H_i| = \frac{(d-1)^2}{d-2.1}\frac{d-2.1}{(d-1)^2}|H_i|$ for each $1 \leq i \leq m-1$ and (4.1d), we have a lower bound of $|C|$ below.

$$|C| \geq \frac{1}{4}\left(|H_m| + \left((d-1)^{\log_2(5/4)} - 1\right)|\mathcal{H}'|\right)^r + \frac{1}{4}\left(\frac{(d-2.1)|\mathcal{H}''|}{(d-1)^2}\right)^r + 2.$$

If $\frac{(d-2.1)|\mathcal{H}''|}{(d-1)^2} \geq \frac{|H_m| + ((d-1)^{\log_2(5/4)} - 1)|\mathcal{H}'|}{(d-1)^2}$, then using inequality (4.1f) of Lemma (4.3.1) and the inequality $(d-1)^{\log_2(5/4)} \geq \frac{1}{\epsilon_2}$,

$$|C| \geq \frac{1}{4}\left((d-1)^{\log_2(5/4)}(|H_m| + ((d-1)^{\log_2(5/4)} - 1)|\mathcal{H}'|)\right)^r + 2 \geq \frac{1}{4}n^r + 2.$$

Thus we may assume $\frac{(d-2.1)|\mathcal{H}''|}{(d-1)^2} < \frac{|H_m| + \left((d-1)^{\log_2(5/4)} - 1\right)|\mathcal{H}'|}{(d-1)^2}$. Using inequality (4.1d) in

Lemma (4.3.1), by noting that $(d-1)^{\log_2(5/4)} > 2$ if $d > 10$, we have

$$\begin{aligned}
|C| &\geq \frac{1}{4}\left(|H_m| + \big((d-1)^{\log_2(5/4)}-1\big)|\mathcal{H}'| + \big((d-1)^{\log_2(5/4)}-1\big)\frac{(d-1)^2(d-2.1)|\mathcal{H}''|}{(d-1)^2}\right)^r + 2,\\
&\geq \frac{1}{4}\Big(|H_m| + |\mathcal{H}'| + |\mathcal{H}''| + (d-4)((d-1)^{\log_2(5/4)}-1)|\mathcal{H}''|\Big)^r + 2\\
&\geq \frac{1}{4}(|\mathcal{H}| + (d-1)|\mathcal{L}|)^r + 2 \quad \big(\text{since } |\mathcal{H}''| \geq |\mathcal{L}| \text{ and } (d-4)((d-1)^{\log_2(5/4)}-1) \geq d-1\big)\\
&\geq \frac{1}{4}n^r + 2 \ \ (\text{by (4.18)}).
\end{aligned}$$

We now consider the case $|\mathcal{L}| > |\mathcal{H}''|$. Since $|\mathcal{H}|$ is maximum subject to $x \in V(H_1) - V(H_2)$, we have either $m \geq 2$ and $1 \leq t \leq m-1$ or $m = t = 1$ and $x \in \{p, q\}$.

If $m \geq 2$, let P_m be a path in H_m between a_{m-1} and b_{m-1} as given by Theorem (4.1.1)(c). If $\{a_{m-1}, b_{m-1}\} \neq \{p, q\}$, let P' be a path in $\mathcal{H}'$ through $a_{m-1}b_{m-1}$ from x to $\{p, q\}$ as given by Lemma (4.4.3), and let the notation be chosen so that P' is from x to p; otherwise, $\{a_{m-1}, b_{m-1}\} = \{p, q\}$, let P'' be a path in $\mathcal{H}' - p$ from x to q given by Lemma (4.4.2), and let $P' := P'' \cup \{pq\}$. Let $P_{\mathcal{L}}$ be a path in $\mathcal{L} - q$ from p to $y' \in (V(L_l) - V(L_{l-1})) \cap N_G(y)$ as given by Lemma (4.4.2). Then $C := P_{\mathcal{L}} \cup P' \cup P_m \cup \{yx, yy'\} - \{a_{m-1}b_{m-1}\}$ is a cycle through xy in G such that

$$|C| \ \geq \ \frac{1}{4}|H_m|^r + \frac{1}{4}\Big(\frac{(d-2.1)|\mathcal{H}'|}{(d-1)^2}\Big)^r + \frac{1}{4}\Big(\frac{(d-2.1)|\mathcal{L}|}{(d-1)^2}\Big)^r + 2.$$

By the same argument as above and using (4.1d) and (4.1f) depending on whether $\frac{(d-2.1)|\mathcal{H}'|}{(d-1)^2} \geq \frac{|H_m|}{(d-1)^2}$, we have

$$|C| \geq \frac{1}{4}\Big((d-1)^{\log_2(5/4)}|H_m|\Big)^r + \frac{1}{4}\Big(\frac{(d-2.1)|\mathcal{L}|}{(d-1)^2}\Big)^r + 2 \geq \frac{1}{4}n^r + 2 \quad \text{or}$$

$$\begin{aligned}|C| &\geq \frac{1}{4}\Big(|H_m| + (d-1)^2\big((d-1)^{\log_2(5/4)} - 1\big)\frac{(d-2.1)|\mathcal{H}'|}{(d-1)^2}\Big)^r + \frac{1}{4}\Big(\frac{(d-2.1)|\mathcal{L}|}{(d-1)^2}\Big)^r + 2 \\ &\geq \frac{1}{4}\Big(|H_m| + |\mathcal{H}'|\Big)^r + \frac{1}{4}\Big(\frac{(d-2.1)|\mathcal{L}|}{(d-1)^2}\Big)^r + 2.\end{aligned}$$

If $|H_m| + |\mathcal{H}'| \geq (d-1)^2 \frac{(d-2.1)|\mathcal{L}|}{(d-1)^2}$, then

$$\begin{aligned}&\frac{1}{4}\Big(|H_m| + |\mathcal{H}'|\Big)^r + \frac{1}{4}\Big(\frac{(d-2.1)|\mathcal{L}|}{(d-1)^2}\Big)^r + 2 \\ \geq\ &\frac{1}{4}\Big(|H_m| + |\mathcal{H}'| + (d-1)^2((d-1)^{\log_2(5/4)} - 1)\frac{(d-2.1)|\mathcal{L}|}{(d-1)^2}\Big)^r + 2 \\ \geq\ &\frac{1}{4}\Big(|H_m| + |\mathcal{H}'| + |\mathcal{H}''| + (d-1)|\mathcal{L}|)^r + 2 \quad (\text{since } |\mathcal{L}| > |\mathcal{H}''|) \\ \geq\ &\frac{1}{4}n^r + 2.\end{aligned}$$

Otherwise, $|H_m| + |\mathcal{H}'| < (d-1)^2 \frac{(d-2.1)|\mathcal{L}|}{(d-1)^2}$, then we get

$$\begin{aligned}&\frac{1}{4}\Big(|H_m| + |\mathcal{H}'|\Big)^r + \frac{1}{4}\Big(\frac{(d-2.1)|\mathcal{L}|}{(d-1)^2}\Big)^r + 2 \\ \geq\ &\frac{1}{4}\Big((d-1)^{\log_2(5/4)} - 1)(|H_m| + |\mathcal{H}'|)\Big)^r + 2 \\ \geq\ &\frac{1}{4}n^r + 2 \quad (\text{since } |H_m| > \epsilon_2 n).\end{aligned}$$

We now assume $m = t = 1$ and, without loss of generality, $x = p$. Let P_m be a longest path from x to q in $\mathcal{H}$ given by Lemma (4.4.4) and $P_{\mathcal{L}}$ a longest path in $\mathcal{L} - p$ from q to $y' \in (V(L_l) - V(L_{l-1})) \cap N_G(y)$ given by Lemma (4.4.2). Then $C := P_m \cup P_{\mathcal{L}} \cup \{xy, y'y\}$ is a cycle of length $\ell(C) \geq \frac{1}{4}|H_m|^r + \frac{1}{2}\Big(\frac{(d-2.1)|\mathcal{L}|}{(d-1)^2}\Big)^r + 2 \geq \frac{1}{4}n^r + 2$, where the last inequality follows from a similar argument as above for $m \geq 2$ and $|\mathcal{L}| > |\mathcal{H}''|$. $\square$

An $\mathcal{H}$-leg $\mathcal{M}$ is called a *minor-leg* of $\mathcal{H}$ if $V(\mathcal{M} \cap \mathcal{H}) \begin{cases} \neq \{p, q\} & \text{if } x \notin \{p, q\} \\ \not\ni x & \text{if } x \in \{p, q\} \end{cases}$; or there is another $\mathcal{H}$-leg $\mathcal{L}^*$ such that both $\mathcal{M}$ and $\mathcal{L}^*$ intersect $\mathcal{H}$ on $\{p, q\}$, $V(\mathcal{L}^*) \cap (\mathcal{M}) \neq \{p, q\}$, and $|\mathcal{L}^* - \mathcal{M}| \leq \epsilon_2 n/(d-2.1)$. We call the minor-leg $\mathcal{M}$ defined in the first case an A-type minor-leg; and in the later case a B-type minor-leg.

Note that if $\mathcal{H}$ has an A-type minor-leg, then $h \geq 2$. For an A-type minor-leg $\mathcal{M}$ of $\mathcal{H}$, we have the following claim.

Claim (4.6.0.3). *If $\mathcal{M} := M_1 \cdots M_m$ is an A-type minor-leg of $\mathcal{H}$, then $|\mathcal{M}| \leq \dfrac{\epsilon_2 n}{d-2.1}$.*

Proof. Suppose $|\mathcal{M}| > \frac{\epsilon_2 n}{d-2.1}$. By our choice of H_t, t is the smallest positive integer such that $\{p,q\} \subseteq H_t$. Similarly, let s be the smallest integer such that $V(\mathcal{M}) \cap V(\mathcal{H}) = V(\mathcal{M}) \cap V(H_s)$. Let $\{u,v\} = V(H_s) \cap V(\mathcal{M})$ and $g := \max\{s,t\}$. Moreover, let $\mathcal{H}' = H_1 H_2 \cdots H_g$ and $\mathcal{H}'' = H_{g+1} H_{g+2} \cdots H_h$. By the maximality of $|\mathcal{H}|$ and $|\mathcal{L}|$ and the existence of $\mathcal{M}$, we have $g < h$, $|\mathcal{H}''| \geq |\mathcal{M}|$, and $|\mathcal{L}| \geq |\mathcal{M}|$. Recall $V(H_g) \cap V(H_{g+1}) = \{a_g, b_g\}$. Since $x \in V(H_1) - V(H_2)$ and $a_g, b_g \in V(H_g) \cap V(H_{g+1})$, we know x is not incident to $a_g b_g$. Hence, we can apply Lemma (4.2.1) to find a path P' in $\mathcal{H}'$ through $a_g b_g$ from x to $z \in \{p,q\} \cup \{u,v\}$. Let P'' be a path in $\mathcal{H}''$ between a_g and b_g as given by Lemma (4.4.1) such that $\ell(P'') \geq \frac{1}{4}(\frac{d-2.1}{d-1}(|\mathcal{H}''|+1))^r + 1$. If $z \in \{u,v\}$ (say, $z = u$), let $P_{\mathcal{M}}$ be a path in $\mathcal{M} - v$ from u to $y' \in N_G(y) \cap (V(M_m) - V(M_{m-1}) - \{u,v\})$ (the vertex y' exists by the 3-connectivity of G) as given by Lemma (4.4.2) with $\ell(P_{\mathcal{M}}) \geq \frac{1}{4}(\frac{d-2.1}{(d-1)^2}|\mathcal{M}|)^r + \frac{1}{2}$, and $P_{\mathcal{L}}$ be a path in $\mathcal{L}$ between p and q as given by Lemma (4.4.1) with $\ell(P_{\mathcal{L}}) \geq \frac{1}{4}(\frac{d-2.1}{d-1}(|\mathcal{L}|+1))^r + 1$. The case $z \in \{p,q\}$ is treated similarly. Since $|\mathcal{M}| \leq |\mathcal{L}|$ and $|\mathcal{M}| \leq |\mathcal{H}''|$, the paths $P', P'', P_{\mathcal{L}}, P_{\mathcal{M}}$, and edges yx, yy' give rise to a cycle C in G through xy such that

$$
\begin{aligned}
|C| &\geq \frac{1}{4}\left(\frac{d-2.1}{d-1}|\mathcal{M}|\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}|\mathcal{M}|\right)^r + \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|\mathcal{M}|\right)^r + 2 && (4.19)\\
&= \frac{1}{4}\left((d-2.1)|\mathcal{M}|\right)^r + \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|\mathcal{M}|\right)^r + 2 && (4.20)\\
&\geq \frac{1}{4}\left((d-1)^{\log_2(5/4)}(d-2.1)|\mathcal{M}|\right)^r \quad \text{((1f) in Lemma (4.3.1))}\\
&\geq \frac{1}{4}n^r + 2.
\end{aligned}
$$

□

Recall that t is the minimum positive integer such that $\{p, q\} \subseteq V(H_t)$. Let $\mathcal{I} := H_1 H_2 \cdots H_t$ and $\mathcal{J} := H_{t+1} H_{t+2} \cdots H_h$. We have a similar claim for a B-type $\mathcal{H}$-minor-leg.

Claim (4.6.0.4). *Let $\mathcal{L}^*$ and $\mathcal{L}^{**}$ be two $\mathcal{H}$-legs with attachments $\{p, q\}$. If $\mathcal{L}^* \cap \mathcal{L}^{**} \neq \{p, q\}$, then we may assume that one of $\mathcal{L}^*$ and $\mathcal{L}^{**}$ is a B-type $\mathcal{H}$-minor-leg.*

Proof. Assume, without loss of generality, that $|\mathcal{L}^*| \leq |\mathcal{L}^{**}|$. Let $\mathcal{L}_0 = \mathcal{L}^* \cap \mathcal{L}^{**}$. Let $\{u^*, v^*\} = V(\mathcal{L}_0) \cap V(L_1^*)$ and $\{u^{**}, v^{**}\} = V(\mathcal{L}_0) \cap V(L_1^{**})$.

By Lemma (4.2.3), we may assume that there is a (p, u^*)-path P_0 through edge $u^{**}v^{**}$ in $\mathcal{L}_0$, but not the edge u^*v^*. Let $\omega^* = |\mathcal{L}^* - \mathcal{L}_0| + 2$. Let P_I be a path in $\mathcal{I} - p$ from x to q such that $\ell(P_I) \geq \frac{1}{4}(\frac{(d-2.1)|\mathcal{I}|}{(d-1)^2})^r$ given by Lemma (4.4.2), P_J a (q, p)-path in $\mathcal{J}$ such that $\ell(P_J) \geq \frac{1}{4}(\frac{(d-2.1)|\mathcal{J}|}{d-1})^r \geq \frac{1}{4}(\frac{(d-2.1)\omega^*}{d-1})^r$ given by Lemma (4.4.1) (note that $|\mathcal{J}| \geq |\mathcal{L}| \geq \omega^*$), P^{**} be a path in $\mathcal{L}^{**} - (\mathcal{L}_0 - \{u^{**}, v^{**}\})$ from u^{**} to v^{**} given by Lemma (4.4.1) with $\ell(P^{**}) \geq \frac{1}{4}(\frac{(d-2.1)|\mathcal{L}^{**}|}{(d-1)})^r + 1$, and P^* a path in $G[\mathcal{L}^* - (\mathcal{L}_0 - \{u^*, v^*\})]$ from x' to u^* avoiding v^* given by Lemma (4.4.2), where x' is a neighbor of y in the last block of $\mathcal{L}^*$. Then we obtain a cycle $C := P^* \cup (P_0 - \{u^{**}v^{**}\}) \cup P^{**} \cup P_{\mathcal{J}} \cup P_{\mathcal{I}} \cup \{yx', xy\}$ through xy such that

$$\ell(C) \geq \frac{1}{4}\left(2(\frac{(d-2.1)\omega^*}{d-1})^r + (\frac{(d-2.1)\omega^*}{(d-1)^2})^r\right) + 2 \geq \frac{1}{4}((d-2.1)(d-1)^{\log_2 5/4}\omega^*)^r + 2,$$

where the last inequality follows from (4.1f). Noticing $\epsilon_2 = \frac{1}{(d-1)^{\log_2(5/4)}-1}$, we have $(d - 2.1)(d-1)^{\log_2 5/4}\omega^* \geq n$ if $\omega^* \geq \epsilon_2 n/(d-2.1)$. So, we may assume $\omega^* < \epsilon_2 n/(d-2.1)$. □

Let G_0 be the subgraph of G obtained by deleting the components of $G - \{y, p, q\}$ that contain a vertex in $\mathcal{H}$. By adding a few special edges to G_0, we define G_0' as follows:

$$G_0' := \begin{cases} G_0 \cup \{py, qy\} & \text{if } H_t \text{ is a cycle and } \{p, q\} \neq \{a_t, b_t\}; \\ G_0 \cup \{y\}] \cup \{py, qy, pq\} & \text{if } pq \notin E(G) \text{ and the above case false.} \end{cases}$$

Note that the difference is whether the edge pq is forced to be added.

Suppose that there are exactly ς $\mathcal{H}$-minor-legs. Then $0 \leq \varsigma \leq d-3$ (as y is adjacent to at least two vertices in $\mathcal{H}$ and at least one vertex in $\mathcal{L}$, there are at most $d-3$ neighbors of y contained in $\mathcal{H}$-minor-legs). Let $\mathcal{M}$ be one of the largest minor-legs if there is one. Then, the following inequalities hold.

$$\begin{aligned} |G_0| &\geq n-|\mathcal{H}|-\varsigma|\mathcal{M}| \geq n-|\mathcal{H}|-\frac{\varsigma\epsilon_2 n}{d-2.1} \geq (1-\epsilon_1-2\epsilon_2)n, \\ |\mathcal{H}| &\geq \frac{n-\varsigma|\mathcal{M}|}{d-1-\varsigma}, \quad \text{and} \\ |\mathcal{J}| &\geq |\mathcal{L}| \geq |\mathcal{M}|. \end{aligned}$$

Since $|\mathcal{H}| < (\epsilon_1+\epsilon_2)n$, we have $|\mathcal{L}| \geq \frac{n-|\mathcal{H}|}{d-1} > \frac{(1-\epsilon_1-\epsilon_2)n}{d-1}$. To complete our proof of Lemma (4.6.1), we consider two cases according to whether $x \in \{p,q\}$ or not.

4.6.1 **Case 1** $x \notin \{p,q\}$.

In this case, by the maximality of $|\mathcal{H}|$, we have $|\mathcal{H}| \geq |\mathcal{J}| \geq |\mathcal{L}|$ and $1 \leq t \leq h-1$. Consequently, we have $h \geq 2$ and the vertex $x' \in (V(H_h)-V(H_{h-1})) \cap (N_G(y)-x)$ is well defined.

Claim (4.6.1.1). *$\Delta(G_0') \leq d$ and G_0' is 3-connected.*

Proof. Since $d_{G_0'}(v) = d_G(v)$ for every $v \in V(G_0)-\{y,p,q\}$, we only need to verify that degrees $d_{G_0'}(p)$, $d_{G_0'}(q)$, and $d_{G_0'}(y)$ are not bigger than d. Since $|\mathcal{J}| \geq |\mathcal{L}| > 0$, H_{t+1} exists. Then both p and q have at least two neighbors in $G-V(G_0)$, and thus $d_{G_0'}(p) \leq d$ and $d_{G_0'}(q) \leq d$. Furthermore, $d_{G_0'}(y) \leq d_G(y)+|\{p,q\}|-|\{x,x'\}| \leq d$.

For the connectivity, it is clear that if there exist at least three internally vertex-disjoint (p,q)-path, then G_0' is 3-connected. As G_0 is connected, there is a (p,q)-path using only vertices of G_0; pyq is another (p,q)-path which intersects $V(G_0)$ only on $\{p,q\}$. If $pq \in E(G_0')$, the edge pq gives the third (p,q)-path. Hence G_0' is 3-connected if $pq \in E(G_0')$. So, we only need to show that G_0' is 3-connected when $G_0' = G_0 \cup \{yp, yq\}$ and $pq \notin E(G_0')$. We suppose on the contrary that G_0' has exactly two internally vertex-disjoint (p,q)-paths (as $G_0' + pq$ is 3-connected). As y connecting p and q, G_0 contains exactly one (p,q)-path. Denote by $P[p,q]$ a shortest (p,q)-path in G_0. Then in $G - y$, $(H_t - pq) \cup P[p,q]$ is an induced cycle. According to Tutte's decomposition algorithm, $(H_t - pq) \cup P[p,q]$ forms a 3-block. This gives a contradiction to that H_t is a 3-block. □

Claim (4.6.1.2). *There is a path P_0 in G_0' with two endvertices in $\{y,p,q\}$ such that $\ell(P_0) \geq \frac{1}{4}(|G_0|+1)^r$ and $(\{py, qy, pq\} - E(G)) \cap E(P_0) = \emptyset$ (when $pq \in E(G)$, we allow $pq \in E(P_0)$). Moreover, if H_t is a cycle, given $z \in \{p,q\}$, we can choose P_0 such that one of the endvertices of P_0 is z.*

Proof. Since $\Delta(G_0') \leq d$ and G_0' is 3-connected, G_0' contains a (p,y)-path P_0' such that $\ell(P_0') \geq \frac{1}{4}(|G_0|+1)^r + 1$ by Theorem (4.1.1)(c). If $qy \in P_0'$, then $P_0 := P_0' - y$ is the desired (p,q)-path. Since $|G_0| \geq (1 - \epsilon_1 - 2\epsilon_2)n \geq (1 - \epsilon_1 - 2\epsilon_2)(d-1)^{\log_2 6} > (d-1)^2$, we have $\ell(P_0') \geq 3$. Hence if $pq \in P_0'$, then $qy \notin E(P_0')$. So $P_0 := P_0' - p$ is the desired path.

When H_t is a cycle, we use Theorem (4.1.1) (c) to find a (z,y)-path P_0 in G_0' such that $\ell(P_0) \geq \frac{1}{4}(|G_0|+1)^r + 1$. If $V(P_0) \cap (\{p,q\} - z) = \emptyset$, then P_0 itself is the desired path. So assume $\{p,q\} - z \subseteq V(P_0)$. If $pq \notin E(P_0)$, then $P_0 - y$ is the desired path. Hence, assume that $pq \in E(P_0)$, and so $pq \in E(G_0')$. We may assume $pq \notin E(G)$; otherwise P_0 is the desired path. By the definition of G_0', we have $\{p,q\} = \{a_t, b_t\}$ in this case. Let $P_{\mathcal{J}}$ be an (a_t, b_t)-path in $\mathcal{J}$ given by Lemma (4.4.1). Then $P_0 := (P_0 - \{pq\}) \cup P_{\mathcal{J}}$ is the desired path with $\ell(P_0) \geq \frac{1}{4}(|G_0|+1)^r + \frac{1}{4}(\frac{(d-2.1)|\mathcal{J}|}{d-1})^r + 1$. □

4.6.1.1 Subcase1.1. $\{p,q\} \neq \{a_t, b_t\} = V(H_t \cap H_{t+1})$.

Using the inequalities $\max\{|\mathcal{I}|, |\mathcal{J}|\} \geq |\mathcal{J}| \geq |\mathcal{L}|\frac{n-|\mathcal{H}|}{d-1} \geq \frac{(1-\epsilon_1-\epsilon_2)n}{d-1}$, we will consider a few cases to show that there exists a cycle C through xy such that

$$
\begin{aligned}
|C| &\geq \frac{1}{4}\left((|G_0|+1)^r + (\frac{(d-2.1)\cdot \max\{|\mathcal{I}|,|\mathcal{J}|\}}{(d-1)^2})^r\right)+2 \\
&\geq \frac{1}{4}\left((1-\epsilon_1-2\epsilon_2)^r + \left(\frac{(1-\epsilon_1-\epsilon_2)(d-2.1)}{(d-1)^3}\right)^r\right)n^r+2 \\
&\geq \frac{1}{4}\Big((1-\epsilon_1-2\epsilon_2)+(1-\epsilon_1-2\epsilon_2)((d-1)^{\log_2(1+2^{-\beta})}-1)\Big)^r n^r+2 \\
&\geq \frac{1}{4}n^r+2,
\end{aligned}
$$

where we let $\beta = \log_{d-1}\left(\frac{(d-1)^3(1-\epsilon_1-2\epsilon_2)}{(d-2.1)(1-\epsilon_1-\epsilon_2)}\right)$ for Lemma (4.3.1), which is greater than 1 but less than 2 when $d \geq 42$. We also use the inequalities $1-\epsilon_1-2\epsilon_2 > 0$ when $d \geq 43$, and $(1-\epsilon_1-2\epsilon_2)+(1-\epsilon_1-2\epsilon_2)((d-1)^{log_2(1+2^{-\beta})}-1) > 1$ when $d \geq 68$.

We first consider the case that there is a (p,q)-path P_0 in $G_0 - y$ such that $\ell(P_0) \geq \frac{1}{4}(|G_0|+1)^r$. Let P_I be an $(\{a_t,b_t\},x)$, say (a_t,x)- path in $\mathcal{I}$ through pq given by Lemma (4.4.3) such that $\ell(P_I) \geq \frac{1}{4}\left(\frac{(d-2.1)|\mathcal{I}|}{(d-1)^2}\right)^r$ (as $\{p,q\} \neq \{a_t,b_t\}$), and P_J be an (a_t,x') path in $\mathcal{J} - b_t$ given by Lemma (4.4.2) such that $\ell(P_J) \geq \frac{1}{4}\left(\frac{(d-2.1)|\mathcal{J}|}{(d-1)^2}\right)^r$, where $x' \in (V(H_h)-V(H_{h-1})) \cap N_G(y)$ (as H_{t+1} exists and G is 3-connected, $x' \notin \{a_t,b_t\}$). Then, $C := (P_I - \{pq\}) \cup P_J \cup P_0 \cup \{xy\}$ is the desired path.

Suppose that H_t is 3-connected. By Claim (4.6.1.2) and the discussion above, we may assume that there is a path P_0 in G_0 from p to y avoiding q such that $\ell(P_0) \geq \frac{1}{4}(|G_0|+1)^r$. If $|\mathcal{I}| \geq |\mathcal{J}|$, by Lemma (4.4.2), let P_H be a path in $\mathcal{I} - q$ from x to p such that $\ell(P_H) \geq \frac{1}{4}(\frac{(d-2.1)|\mathcal{I}|}{(d-1)^2})^r + 1$. If $a_tb_t \in E(P_H)$, then we replace a_tb_t by a path in $\mathcal{J}$ from a_t to b_t. Then, $C := P_H \cup P_0 \cup \{xy\}$ is the desired cycle. If $|\mathcal{I}| \leq |\mathcal{J}|$, let P_I be a path in $\mathcal{I} - q$ from x to p through a_tb_t given by Lemma (4.4.3) and P_J be a path in $\mathcal{J}$ from a_t to b_t such that $\ell(P_J) \geq \frac{1}{4}(\frac{(d-2.1)|\mathcal{J}|}{(d-1)})^r + 1$ given by Lemma (4.4.1). Then $C := (P_I - \{a_tb_t\}) \cup P_J \cup P_0 \cup \{xy\}$ is the desired cycle.

Finally, we assume that H_t is a cycle and P_0 given by Claim (4.6.1.2) is a (p,y)-path. Since $|\mathcal{J}| \geq |\mathcal{L}|$ in this case, the edge a_tb_t exists. As $\{a_t,b_t\} \neq \{p,q\}$, we can assume, without loss of generality, that $a_{t-1}\dots a_t b_t \dots p \dots b_{t-1}$ lie in this order along $H_t - a_{t-1}b_{t-1}$. Let $\mathcal{I}^* := H_1 H_2 \dots H_{t-1}$. Applying Lemma (4.4.2), we find a path P_H^* in $\mathcal{I}^* - b_{t-1}$ from x to a_{t-1} such that $\ell(P_H^*) \geq \frac{1}{4}(\frac{(d-2.1)|\mathcal{I}^*|}{(d-1)^2})^r + \frac{1}{2}$. Extending this path along H_t and $\mathcal{J}$, we obtain a path P_H in $\mathcal{H}$ from x to p avoiding q such that $\ell(P_H) \geq \ell(P_H^*) + 2$. Since the number of degree 2 vertices in $\mathcal{H}$ is no more than $2d-1$, we have $\ell(P_H) \geq \frac{1}{4}(\frac{(d-2.1)|\mathcal{I}|}{(d-1)^2})^r + 1$. Let P_J be a path in $\mathcal{J}$ from a_t to b_t such that $\ell(P_J) \geq \frac{1}{4}(\frac{(d-2.1)|\mathcal{J}|}{(d-1)})^r + 1$ given by Lemma (4.4.1). Note that in this case, we can choose P_0 to be a (p, y)-path such that $\ell(P_0) \geq \frac{1}{4}(|G_0| + 1)^r$. Then, $C := (P_H - \{a_t b_t\}) \cup P_J \cup P_0 \cup \{xy\}$ is the desired cycle.

4.6.1.2 Case $\{p, q\} = \{a_t, b_t\} = V(H_t \cap H_{t+1})$

In this case, $G_0' := G_0 \cup \{py, qy, pq\}$. Assume, without loss of generality, that $d_{G_0}(p) \leq d_{G_0}(q)$. Let $t_p = |N_{G_0'}(p) - \{q, y\}|$. Clearly, $t_p \leq d_G(p) - 2 \leq d - 2$. Let P_I be an (x, p)-path in $\mathcal{I} - q$ given by Lemma (4.4.2), P_J a (p, q)-path in $\mathcal{J}$ given by Lemma (4.4.1), and P_0 a (q, y)-path in $G_0' - p$ given by Theorem (4.1.1) (a). Let $C := P_I \cup P_J \cup P_0 \cup \{xy\}$. Then we have

$$\ell(C) \geq \frac{1}{4}\left((\frac{(d-2.1)|G_0|}{(d-1)t_p})^r + (\frac{(d-2.1)|\mathcal{J}|}{d-1})^r + (\frac{(d-2.1)|\mathcal{I}|}{(d-1)^2})^r \right) + 2. \tag{4.21}$$

Claim (4.6.1.3). *$|\mathcal{J}|^r + (\frac{|\mathcal{I}|}{d-1})^r \geq (|\mathcal{J}| + |\mathcal{I}|)^r \geq |\mathcal{H}|^r$ provided $d \geq 61$.*

Proof. By Lemma (4.3.3), we only need to show that $|\mathcal{J}| \geq \frac{1.1|\mathcal{I}|}{d-1}$. Otherwise, using $|\mathcal{L}| \geq \frac{(1-\epsilon_1-\epsilon_2)n}{d-1}$, we have

$$(\epsilon_1 + \epsilon_2)n > |\mathcal{H}| \geq |\mathcal{I}| > \frac{(d-1)|\mathcal{J}|}{1.1} \geq \frac{(d-1)|\mathcal{L}|}{1.1} \geq \frac{(1-\epsilon_1-\epsilon_2)n}{1.1}.$$

However, when $d \geq 61$, $\epsilon_1 + \epsilon_2 \leq 0.47$ and $\frac{(1-\epsilon_1-\epsilon_2)}{1.1} > 0.47$, showing a contradiction. ∎

Consequently, by Lemma **??** we have

$$\begin{aligned}\ell(C) &\geq \frac{1}{4}\left(\left(\frac{(d-2.1)|G_0|}{(d-1)t_p}\right)^r + \left(\frac{(d-2.1)|\mathcal{H}|}{d-1}\right)^r\right) + 2 \\ &\geq \frac{1}{4}\left(\left(\frac{(d-2.1)^2}{t_p}(1-|\mathcal{H}|/n - \varsigma\epsilon_2/(d-2.1))(|\mathcal{H}|/n)\right)^{r/2} n^r + 2.\end{aligned} \tag{4.22}$$

Clearly, C is the desired cycle if

$$\frac{(d-2.1)^2(1-|\mathcal{H}|/n - \varsigma\epsilon_2/(d-2.1))(|\mathcal{H}|/n)}{t_p} \geq 1. \tag{4.23}$$

Assuming this is not the case, we will show that there are very few minor-legs of $\mathcal{H}$, which reveals some properties of $\mathcal{H}$.

Claim (4.6.1.4). *We may assume $\varsigma \leq 3$ (provided $d \geq 195$).*

Proof. By plugging $|\mathcal{H}|/n \geq \frac{1-\varsigma\epsilon_2/(d-2.1)}{d-1-\varsigma}$ and $t_p \leq d-2$ in (4.23), we get

$$\begin{aligned}&(d-2.1)^2(1-|\mathcal{H}|/n - \varsigma\epsilon_2/(d-2.1))\cdot(|\mathcal{H}|/n)/t \\ \geq\ &(d-2.1)^2\left(1-\frac{1-\varsigma\epsilon_2/(d-2.1)}{d-1-\varsigma} - \frac{\varsigma\epsilon_2}{d-2.1}\right)\left(\frac{1-\varsigma\epsilon_2/(d-2.1)}{d-1-\varsigma}\right)/(d-2) \\ =\ &\frac{(d-2.1)^2(d-2.1-\varsigma\epsilon_2)^2(d-2-\varsigma)}{(d-2.1)^2(d-1-\varsigma)^2(d-2)} \geq 1,\end{aligned}$$

provided $d \geq 195$ and $\varsigma \geq 4$. □

We now refine the legs of $\mathcal{H}$ contained in G_0'. Let $\mathcal{L}_1$, $\mathcal{L}_2$, …, $\mathcal{L}_\ell$ attach $\mathcal{H}$ at $\{p,q\}$ such that $V(\mathcal{L}_i)\cap V(\mathcal{L}_j) = \{p,q\}$ for any $i \neq j$ and, subject to this constraint, $\sum_i |V(\mathcal{L}_i)|$ is maximum. We name them *major-legs* of $\mathcal{H}$. Clearly, all other $\mathcal{H}$-legs remained in G_0' are B-type minor-legs.

Claim (4.6.1.5). *We may assume $\ell \geq d-5$ provided that $d \geq 195$.*

Proof. Since $\mathcal{L}_1$, $\mathcal{L}_2$, …, $\mathcal{L}_\ell$ are all possible non-minor-legs, $|\mathcal{H}| \geq (1 - \frac{\varsigma\epsilon_2}{d-2.1})n/(\ell+1)$. Plugging this inequality in (4.23) and assuming $\ell \leq d-6$, we get the following

$$\begin{aligned}
\ell(C) &\geq \frac{1}{4}\left((\frac{(d-2.1)^2}{t}(1 - \frac{1-\frac{\varsigma\epsilon_2}{d-2.1}}{\ell+1} - \varsigma\epsilon_2/(d-2.1))(\frac{1-\frac{\varsigma\epsilon_2}{d-2.1}}{\ell+1})\right)^{r/2} n^r + 2 \\
&\geq \frac{1}{4}\left(\frac{\ell(d-2.1-\varsigma\epsilon_2)^2}{(\ell+1)^2 t}\right)^{r/2} n^r + 2 \\
&\geq \frac{1}{4}n^r + 2,
\end{aligned}$$

for each $\varsigma = 0, 1, 2, 3$ when $d \geq 195$; where we used $t \leq d-2$ and $\ell \leq d-6$. □

For each $\mathcal{L}_i$, let G_i be induced by the component of $G - \{p, q, y\}$ containing $\mathcal{L}_i - \{p, q\}$, and including two vertices p and q, that is, the union of $\mathcal{L}_i$ and all B-type minor-legs sharing a vertex with $\mathcal{L}_i - \{p, q\}$. Let $G'_i = G[V(G_i) \cup \{y\}] \cup \{pq, py, qy\}$. Clearly, each G'_i is 3-connected with $\Delta(G'_i) \leq d$. Let $t_i(p) = d_{G'_i}(p) - 2$, $t_i(q) = d_{G'_i}(q) - 2$, and $t_i = \frac{1}{2}(t_i(p) + t_i(q))$. By counting the neighbors of p and q, respectively, we have

$$\begin{aligned}
t_1(p) + t_2(p) + \cdots + t_\ell(p) &\leq d_G(p) - |N_{\mathcal{H}}(p) - \{q\}| \leq d-2, \\
t_1(q) + t_2(q) + \cdots + t_\ell(q) &\leq d_G(q) - |N_{\mathcal{H}}(q) - \{p\}| \leq d-2, \\
t_1 + t_2 + \cdots + t_\ell &= \sum_{1 \leq i \leq \ell} (t_i(p) + t_i(q)) \leq d-2.
\end{aligned}$$

We note that, for each i, $t_i(p) \geq 1$, $t_i(q) \geq 1$, and $t_i \geq 1$. Assume, without loss of generality, $\frac{|G_1|}{t_1} = \max_{1 \leq i \leq \ell} \frac{|G_i|}{t_i}$.

Claim (4.6.1.6). *We may assume $t_1 = 1$ provided that $d \geq 194$.*

Proof. Otherwise, we have $t_1 \geq 3/2$ since both $t_1(p)$ and $t_1(q)$ are positive integers. Then either p or q has degree at least 2 in G_1; and consequently, $\ell \leq d - 2 - 1 - \varsigma$ (each $\mathcal{L}_i$ has a neighbor of p and a neighbor of q). By Claim (4.6.1.5) that $\ell \geq d-5$, we may assume that $\varsigma \leq 2$ in this case.

Let $T_1 := \{i \ : \ t_i = t_i(p) = t_i(q) = 1\}$ and $T_2 := \{i \ : \ t_i > 1\}$ and let $\ell_1 := |T_1|$ and $\ell_2 := |T_2|$. By Claim (4.6.1.5), we have $\ell_1 + \ell_2 = \ell \geq d-5$. On the other hand, we have $\ell_1 + 3/2\ell_2 \leq \sum_{1\leq i\leq \ell} t_i \leq d-2$, which in turn gives $\ell_2 \leq 12$. So, $\ell_1 \geq d-17$. As $d \geq 180$, $2\ell_1/3 \geq \ell_2$.

For each $i = 1, 2, \ldots, \ell$, let $\omega_i = |V(G_i) - V(\mathcal{L}_i)|$. Clearly, $\sum_{i=1}^{\ell} \omega_i < \sum_{\mathcal{M} \text{ is an } \mathcal{H}\text{-minor-leg}} |\mathcal{M}| \leq \varsigma\epsilon_2 n/(d-1)$.

For each $i \in T_1$, by the maximality of $|G_1|/t_1$, we have

$$|G_i| = |G_i|/t_i \leq |G_1|/t_1 \leq (2/3)|G_1| \leq (2/3)(|\mathcal{H}| + \omega_1) \text{ (since } |\mathcal{L}_1| \leq |\mathcal{H}| \text{ when } x \notin \{p, q\}).$$

For each $i \in T_2$, we have

$$|G_i| = |\mathcal{L}_i| + \omega_i \leq |\mathcal{H}| + \omega_i.$$

A simple calculation gives the following inequalities.

$$\begin{aligned}
\sum_{1\leq i\leq \ell} |G_i| &\leq (\frac{2\ell_1}{3} + \ell_2)|\mathcal{H}| + \frac{2\ell_1}{3}\omega_1 + \sum_{i\in T_2}\omega_i \\
&\leq (\frac{2\ell}{3} + \frac{\ell_2}{3})|\mathcal{H}| + \frac{2\ell_1}{3}(\omega_1 + \max_{i\in T_2}\{\omega_i\}) \quad \text{(since } \ell_2 \leq 2\ell_1/3) \\
&\leq (\frac{2\ell}{3} + \frac{\ell_2}{3})|\mathcal{H}| + \frac{2\ell_1}{3}\frac{\varsigma\epsilon_2 n}{d-2.1} \quad \text{(since } \textstyle\sum_i \omega_i \leq \frac{\varsigma\epsilon_2 n}{d-2.1}) \\
&\leq \frac{2(d+3)}{3}|\mathcal{H}| + \frac{2\varsigma(d-3)\epsilon_2 n}{3(d-2.1)} \quad \text{(since } \ell \leq d-3, \ell_2 \leq 12\text{)}
\end{aligned}$$

Since $|\mathcal{H}| + \sum_{1\leq i\leq \ell}|G_i| \geq n - \frac{\varsigma\epsilon_2 n}{d-2.1}$, we get the following inequality

$$|\mathcal{H}| \geq \frac{3 - \frac{2(d-1.5)\varsigma\epsilon_2}{d-2.1}}{2(d+4.5)} n.$$

When $d \geq 194$, for each $\varsigma = 0, 1, 2$, $\frac{3-\frac{2(d-1.5)\varsigma\epsilon_2}{d-2.1}}{2(d+4.5)} n > \frac{n - \frac{\varsigma\epsilon_2 n}{d-2.1}}{d-5}$. Recall that $\frac{n-\frac{\varsigma\epsilon_2 n}{d-2.1}}{d-5}$ is the lower bound on $|\mathcal{H}|$ used in the proofs of both Claim (4.6.1.4) and Claim (4.6.1.5), and so we are done by the previous conclusions. $\square$

Let H_k be a block of $\mathcal{J}$ with maximum number of vertices, that is, $|H_k| = \max\{|H_i| : t+1 \le i \le h\}$. Let $\mathcal{L}_1 := L_1L_2\dots L_s$ and let L_m be a block of $\mathcal{L}_1$ with maximum number of vertices. Since $t_1 = 1$, L_1 is a cycle.

Claim (4.6.1.7). *Let $z' \in (V(L_s) - V(L_{s-1})) \cap N_G(y)$. We may assume that there is a (p, z')-path P_1 in $\mathcal{L}_1 - q$ such that*

$$\ell(P) \ge \frac{1}{4}\left(|L_m|^r + \sum_{i \neq m}(\frac{d-2.1}{(d-1)^2}|L_i|)^r\right) - 1 \ge \frac{1}{4}|\mathcal{L}_1|^r - 1.$$

Proof. Assume $V(L_1 \cap L_2) = \{a, b\}$. In $\mathcal{L}_1$, we replace L_1 by a triangle $zabz$ and apply the particular part of Lemma (4.4.4) to get a (z, z')-path. Replacing either the edge za or zb by a path from p to $\{a, b\}$ (we can fix p as L_1 is a cycle), and denote the resulted path by P. We obtain the desired path; in case that $p \in \{a, b\}$, we may have the lower bound $\ell(P)$ above 1 unit less than the bound given in Lemma (4.4.4). $\square$

Claim (4.6.1.8). *$d_{\mathcal{H}}(p) - 1 \le 2$, so both H_t and H_{t+1} are cycles.*

Proof. Otherwise, we have $\ell \le d - 3$, which in turn shows that

$$|\mathcal{H}| \ge \frac{n - \varsigma\epsilon_2 n/(d-2.1)}{\ell + 1}.$$

Let P_I be an (x, q)-path in $\mathcal{I} - p$ given by Lemma (4.4.2), P_J be a (p, q)-path in $\mathcal{J}$ given by Lemma (4.4.1), and P_1 be a (z', p)-path given by Claim (4.6.1.7). Let $C := P_I \cup P_J \cup P_1 \cup$

$\{yz', xy\}$. Then C is a cycle through xy and

$$
\begin{aligned}
\ell(C) &\geq \frac{1}{4}|\mathcal{L}_1|^r - 1 + \frac{1}{4}(\frac{(d-2.1)|\mathcal{I}|}{(d-1)^2})^r + \frac{1}{4}(\frac{(d-2.1)|\mathcal{J}|}{d-1})^r + 1 + 2 \\
&\geq \frac{1}{4}\left(|\mathcal{L}_1|^r + (\frac{(d-2.1)|\mathcal{H}|}{d-1})^r\right) + 2 \quad \text{(by Claim (4.6.1.3))} \\
&\geq \frac{1}{4}\left((\frac{n - |\mathcal{H}| - \frac{\varsigma\epsilon_2 n}{d-2.1}}{\ell})^r + (\frac{(d-2.1)|\mathcal{H}|}{d-1})^r\right) + 2 \\
&\geq \frac{1}{4}\left(\frac{(d-1)^2(n - \frac{n-\frac{\varsigma\epsilon_2 n}{\ell+1}}{d-2.1} - \frac{\varsigma\epsilon_2 n}{d-2.1})(d-2.1)(n - \frac{\varsigma\epsilon_2 n}{d-2.1})}{\ell(d-1)(\ell+1)}\right)^{r/2} + 2 \\
&= \frac{1}{4}\left(\frac{(d-1)(d-2.1-\varsigma\epsilon_2)^2}{(d-2.1)(\ell+1)^2}\right)^{r/2} n^r + 2 \\
&\geq \frac{1}{4}\left(\frac{(d-1)(d-2.1-\varsigma\epsilon_2)^2}{(d-2.1)(d-\varsigma-1)^2}\right)^{r/2} n^r + 2 \quad \text{(by } \ell + \varsigma \leq d-2) \\
&\geq \frac{1}{4}n^r + 2,
\end{aligned}
$$

when $d \geq 41$ and $\varsigma \geq 1$. Thus, we assume $\varsigma = 0$. Then by $\ell \leq d-3$, we get

$$
\begin{aligned}
\ell(C) &\geq \frac{1}{4}\left(\frac{(d-1)(d-2.1-\varsigma\epsilon_2)^2}{(d-2.1)(\ell+1)^2}\right)^{r/2} n^r + 2 \\
&\geq \frac{1}{4}\left(\frac{(d-1)(d-2.1)^2}{(d-2.1)(d-2)^2}\right)^{r/2} n^r + 2 \\
&\geq \frac{1}{4}n^r + 2.
\end{aligned}
$$

□

Let P_I be an (x,p)-path in $\mathcal{I}-q$ given by Lemma (4.4.2) such that $\ell(P_I) \geq \frac{1}{4}\frac{(d-2.1)|\mathcal{I}|}{(d-1)^2})^r + \frac{1}{2}$. Applying Lemma (4.4.6) on $\mathcal{J}$ and $\mathcal{L}$, with $\mathcal{J}$ taking the role of $\mathcal{H}$, p taking the role of x and q taking the role of both w and w' in the lemma, respectively. Let $y' \in (V(H_h) - V(H_{h-1})) \cap N_G(y)$. Then we can find a (q,y')-path P_J in $\mathcal{J}-p$ and a (p,q)-path P_L in $\mathcal{L}$ such that $\ell(P_J) + \ell(P_L) \geq \frac{1}{4}|\mathcal{J}|^r + \frac{1}{4}|\mathcal{L}|^r - 1/2$. Then $C := P_I \cup P_J \cup P_L \cup \{xy, yy'\}$ is a cycle through xy such that

$$\begin{aligned}\ell(C) &\geq \frac{1}{4}\frac{(d-2.1)|\mathcal{I}|}{(d-1)^2})^r + \frac{1}{2} + \frac{1}{4}|\mathcal{J}|^r + \frac{1}{4}|\mathcal{L}|^r - \frac{1}{2} + 2 \\ &\geq \frac{1}{4}|\mathcal{H}|^r + \frac{1}{4}|\mathcal{L}|^r + 2 \quad \text{(By Claim (4.6.1.3))} \\ &\geq \frac{1}{4}\left((d-1)^2 \cdot \frac{n}{d-1} \cdot \frac{n-n/(d-1)}{d-2}\right)^{r/2} + 2 \\ &= \frac{1}{4}n^r + 2,\end{aligned}$$

since in this case, $|\mathcal{L}| \geq \frac{n-|\mathcal{H}|}{d-2}$, and as $|\mathcal{H}| > |\mathcal{J}| \geq |\mathcal{L}|$ gives that $|\mathcal{H}| \geq \frac{n}{d-1}$.

4.6.2 **Case 2** $x \in \{p, q\}$.

Let the notation be chosen so that $x = p$. In this case, the notation $\mathcal{L} = L_1 L_2 \cdots L_m$ is used to indicate an arbitrary $\mathcal{H}$-leg. We note that $|\mathcal{H}| \geq |\mathcal{L}|$ may no longer hold because it is possible that $x \in V(L_2)$. An $\mathcal{H}$-leg $\mathcal{L}$ is *proper* if $x \in V(L_1) - V(L_2)$. For a proper $\mathcal{H}$-leg $\mathcal{L}$, $|\mathcal{L}| \leq |\mathcal{H}|$ still holds.

If $\{x, v\}$ is a 2-cut of $G - y$ for some $v \in V(H_1)$, let G_v be obtained from $G - y$ by deleting all components of $G - \{y, x, v\}$ containing a vertex of $\mathcal{H}$ and adding the edge xv when $xv \notin E(G)$. Let $v_0 \in V(H_1)$ such that $\frac{|G_{v_0}|}{d_{G_{v_0}}(x)-1}$ is maximum. Let $G'_v = G_v \cup \{xy, vy\}$.

Claim (4.6.2.1). $\frac{|G_{v_0}|}{d_{G_{v_0}}(x)-1} > \frac{\epsilon_2 n}{d-2.1}$ *provided* $d \geq 93$.

Proof. Notice that all $\mathcal{H}$-legs not containing x are A-type minor-legs, and there are at most three $\mathcal{H}$-minor-legs by Claim (4.6.1.4). Hence $|G_{v_0}| + \sum\limits_{v \neq v_0} |G_v| + |\mathcal{H}| + \frac{3\epsilon_2 n}{d-2.1} \geq n$. Thus we have

$$\begin{aligned}
\frac{|G_{v_0}|}{d_{G_{v_0}}(x)-1} &\geq \frac{\sum\limits_{v}|G_v|}{\sum\limits_{v}(d_{G_v}(x)-1)} \geq \frac{n-|\mathcal{H}|-\frac{3\epsilon_2 n}{d-2.1}}{d-2} \\
&\geq \frac{n-(\epsilon_1+\epsilon_2)n-\frac{3\epsilon_2 n}{d-2.1}}{d-2} > \frac{\epsilon_2 n}{d-2.1}.
\end{aligned}$$

□

Let $\mathbb{T}$ denote the block-bond tree resulted in from $G-y+xv_0$ by the Tutte decomposition. We treat $\mathbb{T}$ as a rooted tree with the root at the bond B_0 containing the edge xv_0 (notice that as $xv_0 \in E(G-y+xv_0)$, and $G-y+xv_0-\{x,v_0\}$ has at least two components, the bond B_0 exists). Except B_0, we assume that all bonds are removed from $\mathbb{T}$ and two 3-blocks are adjacent if either one is the parent of the other one in the original tree or there is a bond B between them such that one is the parent of B and the other one is a child of B. We will follow the *partial order* $\prec$ of $\mathbb{T}$ generalized naturally by the parent-child relationship of the tree, that is $B_1 \prec B_2$ if B_1 is a *descendent* of B_2.

A block of $\mathbb{T}$ is called an *x-block* if it contains x. Let $\mathcal{X}$ be the union of all x-blocks. A block-chain $Y_1Y_2\ldots Y_m$ such that $Y_1 \cap \mathcal{X}$ is a virtual edge not incident with x is called a *y-chain*. The following definition will play a key role in our proof.

Claim (4.6.2.2). *If Y_1 and Y_2 are two distinct y-chains attached to the same x-block $B \notin \mathcal{H}$ with $|Y_1| \geq |Y_2|$, then $|Y_2| < \frac{\epsilon_2 n}{d-2.1}$.*

Proof. Suppose to the contrary that $|Y_2| \geq \frac{\epsilon_2 n}{d-2.1}$. Suppose there are x-blocks $B_1, B_2, \cdots, B_t$ such that $BB_1B_2\cdots B_t$ is a block-chain and $B_t \cap H_1 = \{x, v\}$, $x_iy_i = E(Y_i \cap B)$ for each $i = 1, 2$, and $B \cap B_1 = \{x, u\}$.

By Lemma (4.2.1), we may assume that there is a path (u, x_1)-path P_B in B containing edge x_2y_2. Let P_1 be a longest (x_1, z')- path in $Y_1 - y_1$ given by Lemma (4.4.2), where z' is a vertex in the extreme block in Y_1 which is adjacent to y. Let P_2 be a longest (x_2, y_2)-path in Y_2 given by Lemma (4.4.1), and P_H be a longest (x, v)-path in $\mathcal{H}$ given by Lemma (4.4.1). Since $BB_1B_2 \cdots B_t$ is 2-connected, $BB_1B_2 \cdots B_t - x$ is connected. Let P_3 be a (u, v)-path in $BB_1B_2 \cdots B_t - x$. Let $C = P_H \cup P_3 \cup (P_B - x_2y_2) \cup P_2 \cup P_1 \cup \{xy, yz'\}$. Since $x \notin Y_1$ and $x \notin Y_2$, $|\mathcal{H}| \geq |Y_1| \geq |Y_2|$. Then

$$\begin{aligned} \ell(C) &\geq \frac{1}{4}\left(\frac{d-2.1}{d-1}|\mathcal{H}|\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}|Y_2|\right)^r + \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|Y_1|\right)^r + 2 \\ &\geq \frac{1}{4} \cdot 2.5 \left(\frac{d-2.1}{d-1}|Y_2|\right)^r + 2 \geq \frac{1}{4}n^r + 2, \end{aligned}$$

since $2.5 = ((d-1)^{\log_2 5/2})^r$. □

We call a y-chain Y a small chain if $|Y| < \frac{\epsilon_2 n}{d-2.1}$.

Definition (4.6.2). *A block $B \in \mathcal{X}$ is called a giant block (GB) if*

- $\frac{|B|}{d_B(x)-1} \geq \frac{\epsilon_2 n}{d-2.1}$, *or if there is a y-chain Y attached to B such that*
- $|Y| \geq \frac{\epsilon_2 n}{d-2.1}$, *or*
- $\frac{|BY|}{d_{BY}(x)-1} \geq \frac{2\epsilon_2 n}{d-2.1}$.

If B is not a GB, we call B a small block (SB).

Let B be an x-block. If there exist y-chains attached to B, let Y be one of the y-chains with largest cardinality. Then BY is called a y-extension of B. Notice that BY is a proper $\mathcal{H}$-leg, and so $|BY| \leq |\mathcal{H}|$.

Following the notation in the above definition, we have the following observation.

Claim (4.6.2.3). *Let B be an x-block and BY a y-extension of B. Suppose that xb and xb' are the virtual edges of B corresponding to its parent and one of its children, respectively.*

Then, there is a (b, b')-path P in $BY - x$ of length $\ell(P) \geq \frac{1}{4}(\frac{d-2.1}{d-1}|Y|)^r + 1$ and there is a (b, y)-path Q in $G[V(BY) \cup \{y\}] - x$ of length $\ell(Q) \geq \frac{1}{4}(\frac{d-2.1}{d-1}|Y|)^r + 1$.

Proof. Let $\{u, v\} = V(B) \cap V(Y)$. If B is a cycle, let P_1 be the unique path from b to b' through uv. If B is 3-connected, then $B - x$ is 2-connected. There is a path P_1 from b to b' through uv. By Lemma (4.4.1),there is a (u, v)-path path P_2 in Y such that $\ell(P_2) \geq \frac{1}{4}(\frac{d-2.1}{d-1}|Y|)^r + 1$. Then $P := (P_1 - \{uv\}) \cup P_2$ gives the desired (b, b')-path.

To prove the second statement, if B is a cycle, let P_1 be the unique path in $B - x$ from b to $z \in \{u, v\}$, say u, avoiding v; if B is 3-connected, $B - x - v$ is connected, there is a path P_1 from b to u. We may assume that $d_Y(v) \geq 3$. For otherwise, let Y^* be the graph obtained from $G[(V(Y) \cup \{y\}] \cup \{yu, yv, uv\}$ by suppressing all degree 2 vertices. Then, applying Theorem (4.1.1) (a) to $Y^* - v$, we can find a (u, y)-path P_2 not containing uv such that $\ell(P_2) \geq \frac{1}{4}(\frac{d-2.1}{d-1}|Y|)^r + 1$. Thus, $P_2 \cup P_1$ is the desired path. Hence, $d_Y(v) \geq 3$. This implies that the first block of Y is 3-connected. Let Y^* be the graph obtained from $G[(V(Y) \cup \{y\}] \cup \{yu, uv\}$ by suppressing all degree 2 vertices. If $d_{Y^*}(y) \geq 3$, then Y^* is 3-connected. Applying Theorem (4.1.1) (c) on Y^*, we find a (u, y)-path P_2 of $\ell(P_2) \geq \frac{1}{4}(\frac{d-2.1}{d-1}|Y|)^r + 2$. (We may assume that $uv \notin E(P_2)$. As otherwise we can choose P_1 to be a (b, v)-path avoiding u and $P := P_1 \cup (P_2 - \{uv\})$ gives the desired path.) Otherwise, $d_{Y^*}(y) = 2$; and thus Y^* is a block-chain with edge uy in one end-block and v at the other end. Applying Lemma (4.4.1) on Y^*, we find a (u, y)-path P_2 of $\ell(P_2) \geq \frac{1}{4}(\frac{d-2.1}{d-1}|Y|)^r + 1$. In any case, $P_2 \cup P_1$ is the desired path. $\square$

We need to distinguish three different types of degrees of x in B for each x-block: $d_B(x)$ is the degree of x in B, $d_{(G,B)}(x)$ is the number of edges of G incident with x in B, and $d_{(V,B)}(x)$ is the number of virtual edges in B, that is, the degree of B, as a vertex in the subtree of $\mathbb{T}$ induced by all x-blocks. We have $d_B(x) \leq d_{(G,B)}(x) + d_{(V,B)}(x)$ and the strick inequality may hold (for example, an edge may be counted in both $d_{(G,B)}(x)$ and $d_{(V,B)}(x)$).

For each x-block, we now associate it with a number $t(B)$:

$$t(B) = d_B(x) - 2.$$

This number is in correspondence of the parameter t in Theorem (4.1.1) (a), and it is used when applying Theorem (4.1.1) (a) on B. We will consider the ration $|B|/t(B)$. For convention, we define $|B|/t(B) = 0$ when $t(B) = 0$. (In this case B is a cycle.)

Claim (4.6.2.4). *All GBs form an anti-chain (a set of vertices in the block-bond tree forms an anti-chain if, pairwise, they don't have the parent-child relationship) in* $\mathbb{T}$.

Proof. Suppose there is an $\mathcal{H}$-leg $B = B_1 \cdots B_{k-1}B_k\mathcal{L}$, where each $B_j, 1 \le j \le k$, is a 3-block containing x, and $\mathcal{L}$ is the largest block-chain attached to B_k (so $B_k\mathcal{L}$ contains an extreme block, and so has a neighbor of y). Let $\{x, b_0\} := V(H_1) \cap V(B_1)$, and $V(B_i) \cap V(B_{i+1}) = \{x, b_i\}$ for $1 \le i \le k-1$. Suppose there are indices i and m with $i < m$ such that both B_i and B_m are GBs, and $m = k$ if B_m is an external GB.

For each $1 \le j \ne m \le k-1$, if B_j is a cycle then let P_j be the path in $B_j - x$ from b_{j-1} to b_j; otherwise let P_j be a path in $B_j - x$ from b_{j-1} to b_j as given by Theorem (4.1.1)(a). Let Y_i and Y_m be the largest y-chains (if exist) attached to B_i and B_m, respectively.

In the case $k \ne m$, let P_m be a longest (b_{m-1}, b_m)-path in $B_m - x$ if $\frac{|B_m|}{d_{B_m}(x)-2} \ge \frac{\epsilon_2 n}{d-2.1}$, and let P_m be a longest (b_{m-1}, b_m)-path as guaranteed by Claim (4.6.2.3) if $|Y_m| \ge \frac{\epsilon_2 n}{d-2.1}$; let P_k be a longest path in $G[V(B_k\mathcal{L}) \cup \{y\}] + b_{k-1}y - x$ from b_{k-1} to y as given by Theorem (4.1.1)(a) (as $B_k\mathcal{L}$ has an extreme block which contains a neighbor of y, $G[V(B_k\mathcal{L}) \cup \{y\}] + b_{k-1}y$ is 3-connected).

If $k = m$, we pick P_k as in the previous case if $\frac{|BY_k|}{t(B_k)} \ge \frac{2\epsilon_2 n}{d-2}$, or $\frac{|B|}{d_B(x)-2} \ge \frac{\epsilon_2 n}{d-2.1}$; if $|Y_k| \ge \frac{\epsilon_2 n}{d-2.1}$, let P_k be the (b_{k-1}, y)-path as guaranteed by Claim (4.6.2.3). Let P_0 be a path in $\mathcal{H}$ from x to b_0 as given by Lemma (4.4.1). So, we have in either case that

$\ell(P_k) \geq \frac{1}{4}(\frac{d-2.1}{d-1}\frac{\epsilon_2 n}{d-2.1})^r$.

Then $C := P_0 \cup (\cup_{i=1}^{k-1} P_i) \cup P_k \cup \{xy\}$ gives a cycle in G through xy. Noting $|\mathcal{H}| \geq |B_j|$ and $|\mathcal{H}| \geq |B_k Y|$, thus

$$\begin{aligned}
\ell(C) &\geq \frac{1}{4}\left(\frac{d-2.1}{d-1}|\mathcal{H}|\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}\frac{|B_i|}{d_{B_i}(x)-2}\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}\frac{\epsilon_2 n}{d-2}\right)^r + 2 \\
&\geq \frac{1}{4}\left(\frac{d-2.1}{d-1}|B_i|\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}\frac{|B_i|}{d_{B_i}(x)-2}\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}\frac{\epsilon_2 n}{d-2.1}\right)^r + 2 \\
&= \frac{1}{4}n^r + 2. \quad (\text{Provided } \tfrac{|B_i|}{d_{B_i}(x)-2} \geq \tfrac{\epsilon_2 n}{d-2.1}.)
\end{aligned}$$

So, we may assume $\frac{|B_i|}{d_{B_i}(x)-2} < \frac{\epsilon_2 n}{d-2}$. Since B_i is a GB, by the definition, it has a y-chain Y_i such that $|Y_i| \geq \frac{\epsilon_2 n}{d-2.1}$. Let P_i be a path in $B_i - x$ from b_{i-1} to b_i as guaranteed by Claim (4.6.2.3) such that $\ell(P_i) \geq \frac{1}{4}(\frac{d-2.1}{d-1}|Y_i|)^r + 1$. All other paths are as defined in the previous argument, we obtain a cycle C in G through xy such that

$$\begin{aligned}
\ell(C) &\geq \frac{1}{4}\left(\frac{d-2.1}{d-1}|\mathcal{H}|\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}|Y_i|\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}\frac{\epsilon_2 n}{d-2}\right)^r + 2 \\
&\geq \frac{1}{4}\left(\frac{d-2.1}{d-1}|Y_i|\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}|Y_i|\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}\frac{\epsilon_2 n}{d-2}\right)^r + 2 \\
&= \frac{1}{4}n^r + 2. \quad (\text{Provided } |Y_i| \geq \tfrac{\epsilon_2 n}{d-2.1}.)
\end{aligned}$$

□

Claim (4.6.2.5). *We may assume* $|\mathcal{H}| \geq \frac{2\epsilon_2 n}{d-2.1}$ *provided that* $d \geq 123$.

Proof. Suppose that $|\mathcal{H}| < \frac{2\epsilon_2 n}{d-2.1}$. Then for each maximal proper $\mathcal{H}$-leg $\mathcal{L}$, we have $|\mathcal{L}| \leq |\mathcal{H}| < \frac{2\epsilon_2}{d-2.1}$. As each maximal proper $\mathcal{H}$-leg either contains an extreme block (and thus has a neighbor of y), or it is an x-block (and thus has a neighbor of x), we then have at most $2(d-1)$ maximal proper $\mathcal{H}$-legs. All those $\mathcal{H}$-legs, together with $\mathcal{H}$, cover all the vertices of $V(G) - y$. However, $\frac{4(d-1)\epsilon_2 n}{d-2.1} < n - 0.1$ when $d \geq 123$, showing a contradiction. □

Claim (4.6.2.6). *For any virtual edge xv with $v \neq v_0$, if $\mathcal{L}$ is an $\mathcal{H}$-leg such that $\mathcal{L}\cap\mathcal{H} = xv$, then $|\mathcal{L}| \leq \frac{\epsilon_2 n}{d-2.1}$ provided $d \geq 123$.*

Proof. We consider two cases according to whether $|H_1| > |H_2H_2\dots H_h|$. If $|H_1| \geq |H_2H_3\dots H_h|$, then $H_1 \geq \frac{\epsilon_2 n}{d-2.1}$ by Claim (4.6.2.5). Let P_L be a longest (x,v)-path in $\mathcal{L}$, P_0 be a longest (v_0,y)-path in $G'_{V_0} - x$ given by Theorem (4.1.1)(a), C_H be a longest cycle in H_1 through two edges xv and xv_0 given by Theorem (4.1.1)(b), and let $P_H = C_H - \{xv_0\}$. From them, we can obtain a cycle C through xy such that

$$\ell(C) \geq \frac{1}{4}\left(\frac{d-2.1}{d-1}|\mathcal{L}|\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}\frac{|G'_{v_0}|}{d_{G'_{v_0}}(x)-2}\right)^r + \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|H_1|\right)^r + 2 \geq \frac{1}{4}n^r + 2,$$

provided $\min\{|\mathcal{L}|, \frac{|G'_0|}{d_{G'_0}(x)-2}, |H_1|\} \geq \frac{\epsilon_2 n}{d-2.1}$. Since the other two already do by Claims (4.6.2.1) and (4.6.2.5), we may assume $|\mathcal{L}|$ does not.

If $|H_1| < |H_2H_3\dots H_h|$, then $|H_2H_3\dots H_h| > \frac{\epsilon_2 n}{d-2.1}$. Define P_L and P_0 the same way as above. Let $\mathcal{H}' = H_2H_3\dots H_h$. Let C_1 be a cycle in H_1 through edges xv, xv_0, and ab, where $ab = H_1\cap\mathcal{H}'$, and $P_1 = C_1 - \{xv_0\}$. Let P'_H be a longest (a,b)-path in $\mathcal{H}'$ given by Lemma (4.4.1). Then we have $\ell(P'_H) \geq \frac{1}{4}(\frac{d-2.1}{d-1}|\mathcal{H}'|)^r + 1$. Similarly, we can show that the cycle $C := (P_1 - \{ab, xv\}) \cup P'_H \cup P_L \cup \{xy\}$ passes through xy, and $\ell(C) \geq \frac{1}{4}n^r + 2$ if $|\mathcal{L}| \geq \frac{\epsilon_2 n}{d-2.1}$. □

Notice that there are at most $d-2$ (as G_{v_0} has an extreme block, and thus has a neighbor of y) $\mathcal{H}$-legs $\mathcal{M}$ with $\mathcal{M}\cap\mathcal{H} \neq xv_0$. By Claim (4.6.0.3) and Claim (4.6.2.6), $|\mathcal{M}| \leq \frac{\epsilon_2 n}{d-2.1}$ for each such $\mathcal{H}$-leg, which gives $\sum_{\mathcal{M}} |\mathcal{M}| \leq \frac{d-2}{d-2.1}\epsilon_2 n$. An immediate consequence is that

$$\frac{|G'_{v_0}|}{d_{G'_{v_0}}(x)} \geq \frac{n - |\mathcal{H}| - \frac{d-2}{d-2.1}\epsilon_2 n}{d-2}. \tag{4.24}$$

Claim (4.6.2.7). *We may assume that $|\mathcal{H}| < \frac{(1+1.5\epsilon_2)n}{d-1}$ provided $d \geq 125$.*

Proof. Let P_0 be a longest (v_0, y)-path in $G'_{v_0} - x$ given by Lemma (4.1.1) (a), and P_H be a longest (x, v_0)-path in $\mathcal{H}$ given by Lemma (4.4.1). Then $C := P_0 \cup P_H \cup \{xy\}$ gives a cycle through xy such that

$$\begin{aligned}
\ell(C) &\geq \frac{1}{4}\left(\frac{d-2.1}{d-1}|\mathcal{H}|\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}\frac{|G'_{v_0}|}{d_{G'_{v_0}}(x)-2}\right)^r + 2 \\
&\geq \frac{1}{4}\left((d-1)^2 \cdot \frac{(d-2.1)^2|\mathcal{H}|}{(d-1)^2(d-2)}(1 - \frac{|\mathcal{H}|}{n} - \frac{d-2}{d-2.1}\epsilon_2 n)\right)^{r/2} n^r + 2 \geq \frac{1}{4}n^r + 2,
\end{aligned}$$

provided $|\mathcal{H}| \geq \frac{(1+1.5\epsilon_2)n}{d-1}$ and $d \geq 125$. $\square$

Since all GBs form an anti-chain of $\mathbb{T}$ and the root B_0 is not a GB, there is a subtree $\mathbb{T}_0$ of $\mathbb{T}$ containing the root such that it contains no GB and each branch of $\mathbb{T} - \mathbb{T}_0$ contains at most one GB. (The subtree containing B_0 obtained from $\mathbb{T}$ by deleting all of the GBs has the described property.) We may assume $\mathbb{T}_0$ has this property with maximum cardinality. Let $\mathcal{T}_1$, $\mathcal{T}_2$, $\cdots$, $\mathcal{T}_m$ be the block-trees corresponding to branches of $\mathbb{T} - \mathbb{T}_0$. For each $\mathcal{T}_i$, we call the block of $\mathcal{T}_i$ which is an immediate child of the x-block to which $\mathcal{T}_i$ attaching in $\mathbb{T}$ the first block of $\mathcal{T}_i$. By the maximality of $\mathbb{T}_0$ and the fact that all GBs form an anti-chain in $\mathbb{T}$, we have the following observations:

- Excluding B_0, every x-block which is a leave of $\mathbb{T}_0$ is adjacent to a GB, which is the first block of some $\mathcal{T}_i$ (as if not, we can make $\mathbb{T}_0$ larger by adding the first block of the $\mathcal{T}_i$ to that leaf). Conversely, each GB is attached to an x-block which is a leaf of $\mathbb{T}_0$;
- Each virtual edge in $B \in \mathbb{T}_0$ which is adjacent to neither the parent of B nor the child of B is corresponding to at least one some branch $\mathcal{T}_i$.

For each x-block B, let $\overline{B} := B\mathcal{L}$ be a maximal block-chain containing B as the first block such that $\overline{B}$ has the largest cardinality among all of such block-chains. Then by the maximality of $\overline{B}$, it contains an extreme block.

For each i, let B_i be the GB contained in $\mathcal{T}_i$, we let $\mathcal{L}_i := \overline{B_i}$. Then by Claim (4.6.0.3) and Claim (4.6.0.4), we know each leg of $\mathcal{L}_i$ is contained in an $\mathcal{H}$-minor-leg, and hence has cardinality less than $\frac{\epsilon_2 n}{d-2.1}$. By the construction of $\mathcal{L}_i$, it contains an extreme block. Note that each $\mathcal{L}_i$ and the branch $\mathcal{T}_i$ containing it, have exactly the same predecessors, that is they are connected to B_0 through exactly a same block-chain in $\mathbb{T}_0$. Also, notice that the first block of each $\mathcal{L}_i$ is an x-block. Let

$$\mathbf{L} \;=\; \{\mathcal{L}_1,\ \mathcal{L}_2,\ \ldots, \mathcal{L}_m\}.$$

Correspondingly, for each $\mathcal{L}_i \in \mathbf{L}$, we let $\mathcal{M}_i \subset \mathbb{T}_0$ be the chain of x-blocks which connects $\mathcal{L}_i$ to B_0. Thus, $\mathcal{M}_i\mathcal{L}_i$ is a block-chain. Notice that $\mathcal{M}_i$ may be empty in case that the first block of $\mathcal{L}_i$ is an immediate child of B_0.

We use the partial order $\prec$ generalized by $\mathbb{T}$ naturally, i.e., if B_1 is a child of B_0, we have $B_1 \prec B_0$. For each $\mathcal{L}_i$, if $\mathcal{M}_i \neq \emptyset$, let $\eta(\mathcal{L}_i) = \sum_{B\in\mathcal{M}_i} \frac{|B|}{t(B)}$, and

$$\omega(\mathcal{L}_i) = |\mathcal{L}_i| + \eta(\mathcal{L}_i).$$

Note that by introducing $\eta(\mathcal{L}_i)$, $(d_{(V,B)}(x)-2)\cdot\frac{|B|}{t(B)}$ vertices in B are distributed into $\mathcal{L}_i$ when $d_{(V,B)}(x) \geq 3$ (B is not a cycle). As each virtual edge incident to x in B which is not incident to the parent or the child of B is contained also in some $\mathcal{T}_i$, and there are $(d_{(V,B)}(x) - 2)$ of such virtual edges. Let us see now which portion of vertices of G are not considered into $\sum_i \omega(\mathcal{L}_i)$.

(i) On a cycle-block $B \in \mathbb{T}_0$, degree 2 vertices which are neighbors of y;

(ii) Small y-chains and legs of $\mathcal{L}_i$ contained in the branch $\mathcal{T}_i$;

(iii) For each $B \in \mathbb{T}_0$, we have $(d_{(V,B)}(x)-2)\cdot\frac{|B|}{t(B)}$ vertices in $|B|$ are distributed into $\omega(\mathcal{L}_i)$. So, there are at most $d_{(G,B)}(x)\cdot\frac{|B|}{t(B)}$ vertices in B remained.

We estimate the number of vertices in the above three cases.

- If (i), let δ_2 be the number of such degree 2 vertices.
- If (ii), as each maximal block-chain has an extreme block, and each block-chain can be extended to a maximal one, we suppose there are in total exactly s extreme blocks which are contained in some branch $\mathcal{T}_i$, but not in any one of $\mathcal{L}_j$. Then,

$$\sum_i |\mathcal{T}_i| \leq \sum_i |\mathcal{L}_i| + \frac{s\epsilon_2 n}{d-2.1}.$$

- For each $B \in \mathbb{T}_0$, which is not in case (i), $|B| \leq (d_{(V,B)}(x) - 2) \cdot \frac{|B|}{t(B)} + d_{(G,B)}(x) \cdot \frac{|B|}{t(B)}$. As each $B \in \mathbb{T}_0$ is a SB, we have $|B| \leq \left((d_{(V,B)}(x) - 2) \cdot \frac{|B|}{t(B)} + d_{(G,B)}(x)\right) \frac{\epsilon_2 n}{d-2.1}$.

Let

$$s' = \delta_2 + \sum_{B \in \mathbb{T}_0} d_{(G,B)}(x).$$

For each $\mathcal{L} \in \mathbf{L}$, let $\tau_x(\mathcal{L}) = |N_{\mathcal{L}}(x)| - 1$, $\tau_y(\mathcal{L}) = |N_G(y) \cap \mathcal{L}|$, and $\tau(\mathcal{L}) = \frac{1}{2}(\tau_x(\mathcal{L}) + \tau_y(\mathcal{L}))$. Note that the definition for $\tau_x(\mathcal{L})$ is different from that for $\tau_y(\mathcal{L})$, as when we remove legs of $\mathcal{L}_i$ in $\mathcal{T}_i$, it may be possible that in $\mathcal{L}_i$, x is only incident to virtual edges. However, each virtual edge incident to x correspondences to at least one real edge incident to x in some legs of $\mathcal{L}_i$, so we let $\tau_x(\mathcal{L}) = |N_{\mathcal{L}}(x)| - 1$. The following inequalities hold.

$$\begin{aligned}
\sum_{\mathcal{L} \in \mathbf{L}} \tau_x(\mathcal{L}) &\leq d(x) - 1 \leq d - 1, \\
\sum_{\mathcal{L} \in \mathbf{L}} \tau_y(\mathcal{L}) &\leq d(y) - 1 \leq d - 1, \text{ and} \\
\sum_{\mathcal{L} \in \mathbf{L}} \tau(\mathcal{L}) &\leq \frac{1}{2}(d(x) + d(y) - 2) \leq d - 1.
\end{aligned}$$

We note that for each $\mathcal{L} \in \mathbf{L}$, we have $\tau_x(\mathcal{L}) \geq 1$, $\tau_y(\mathcal{L}) \geq 1$, and $\tau(\mathcal{L}) \geq 1$. By relabeling the branches $\mathcal{L} \in \mathbf{L} - \mathcal{H}$, suppose we have

$$\frac{\omega(\mathcal{L})}{\tau(\mathcal{L})} \geq \frac{\omega(\mathcal{L}')}{\tau(\mathcal{L}')} \geq \cdots \geq \frac{\omega(\mathcal{L}_m)}{\tau(\mathcal{L}_m)}.$$

As $xy \in E(G)$, and also by noticing that $|N_{\mathcal{H}}(y) - \{x\}| \geq 1$ and $d_{\mathcal{H}}(x) - 1 \geq 1$, when $d \geq 425$,

$$\frac{\omega(\mathcal{L})}{\tau(\mathcal{L})} \geq \frac{\sum_i \omega(\mathcal{L}_i)}{\sum_i \tau(\mathcal{L}_i)} \geq \frac{n - |\mathcal{H}| - \frac{(s+s')\epsilon_2 n}{d-2.1}}{d - 1 - \frac{1}{2}(\tau_x(\mathcal{H}) + \tau_y(\mathcal{H})) - (s+s')/2} \geq \frac{n - |\mathcal{H}|}{d-2}, \tag{4.25}$$

since under the assumption that $|\mathcal{H}| \leq \frac{(1+1.5\epsilon_2)n}{d-1}$ and $s + s' \leq 2(d-3)$ ($xy \in E(G)$, and both x and y have at least one neighbor in each of $\mathcal{L}$ and $\mathcal{H}$), $\frac{n-|\mathcal{H}|-\frac{(s+s')\epsilon_2 n}{d-2.1}}{d-2-(s+s')/2}$ is an increasing function of $s + s'$. The notations $\mathcal{L}$ and $\mathcal{L}'$ will be fixed for the above definition hereafter.

Claim (4.6.2.8). *Let $\mathcal{M} := M_1 M_2 \ldots M_m$ be the block-chain connecting some $\mathcal{L}'' \in \mathbf{L}$ to B_0. Suppose $V(\mathcal{M} \cap \mathcal{L}) = V(M_m \cap \mathcal{L}) = \{x, v_m\}$ and $xv_0 \in E(M_1)$. Then in $\mathcal{M}$, there is a (v_0, v_m)-path P_M with $\ell(P_M) \geq \frac{1}{4} \sum_i \left(\frac{d-2.1}{d-1} \frac{|M_i|}{t(M_i)}\right)^r$.*

Proof. For each $i = 1, 2, \cdots, m-1$, let $M_i \cap M_{i+1} = \{x, v_i\}$. Let P_i be an (m_{i-1}, m_i)-path in $M_i - x$ given by Theorem (4.1.1) (a) (when M_i is a cycle, the assertion trivially holds) such that $\ell(P_i) \geq \frac{1}{4}\left(\frac{d-2.1}{d-1} \frac{|M_i|}{t(M_i)}\right)^r$. If P_i contains some virtual edges, which are supposed to be replaced by a path connecting the two ends of the virtual edge in a y-chain of M_i with the ends as attachments (notice that this y-chain is a small-chain; and thus is not contained in any other block-chain in $\mathbf{L} - \{\mathcal{L}''\}$ by the construction of $\mathbf{L}''$). Let $P_M := \cup_i P_i$, which is the desired path. □

Claim (4.6.2.9). *We have $\mathcal{L}$ satisfies $\tau(\mathcal{L}) = 1$. In particular, if let $\mathcal{L} = L_1 L_2 \cdots L_l$, then $x \in V(L_1) - V(L_2)$ and L_1 is a cycle provided that $d \geq 85$.*

Proof. In the proof, we let $\alpha = \tau(\mathcal{L})$. Suppose on the contrary that $\alpha > 1$. Then $\alpha \geq 1.5$ from the definition. So, by (4.25) we have

$$\omega(\mathcal{L}) \geq \frac{1.5(n - |\mathcal{H}|)}{d-2}.$$

Let $\mathcal{M} := M_1 M_2 \ldots M_m$ be the block chain connecting the root B_0 of $\mathbb{T}$ and the block L_1 in $\mathcal{L}$, and suppose $L_1 \cap M_m = \{x, v_m\}$. Let G' be a graph obtained from $G[V(\mathcal{L}) \cup \{y\}] \cup \{yx, yv_m\}$ by suppressing all degree 2 vertices. Then it is 3-connected, and then by Theorem (4.1.1) (a), there is an (v_m, y')-path P_L in $G' - x$ such that $\ell(P_L) \geq \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|\mathcal{L}|\right)^r + 2$, where y' is a neighbor of y in the last block of $\mathcal{L}$. Let P_H be an (x, v_0)-path in $\mathcal{H}$ given by Lemma (4.4.1), and P_M be a (v_0, v_m)-path in $\mathcal{M} - x$ such that $\ell(P_M) \geq \frac{1}{4}\sum_i \left(\frac{d-2.1}{d-1}\frac{|M_i|}{t(M_i)}\right)^r$ given by Claim (4.6.2.8).

Set $C := P_H \cup P_M \cup P_L \cup \{yy', xy\}$, which is a cycle through xy such that

$$\ell(C) \ \geq \ \frac{1}{4}\left(\frac{d-2.1}{d-1}|\mathcal{H}|\right)^r + \frac{1}{4}\sum_i \left(\frac{d-2.1}{d-1}\frac{|M_i|}{t(M_i)}\right)^r + \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}|\mathcal{L}|\right)^r + 2.$$

As $|\mathcal{H}| \geq \frac{n - \frac{(d-2)\epsilon_2 n}{d-2.1}}{d-1}$ and $|\mathcal{L}| \geq \frac{|M_i|}{t(M_i)}$ for each M_i (as $\mathcal{L}$ contains a GB and each M_i is not a GB), by using (4.1c),

$$\begin{aligned}
\ell(C) \ &\geq \ \frac{1}{4}\left(\frac{d-2.1}{d-1}|\mathcal{H}|)\right)^r + \frac{1}{4}\left(\frac{d-2.1}{(d-1)^2}(|\mathcal{L}| + (d-1)\eta(\mathcal{L}))\right)^r + 2 \\
&\geq \ \frac{1}{4}\left(\frac{d-2.1}{d-1}|\mathcal{H}|)\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}\omega(\mathcal{L})\right)^r + 2 \\
&\geq \ \frac{1}{4}\left(\frac{(d-1)^2(d-2.1)^2|\mathcal{H}|\omega(\mathcal{L})}{(d-1)^2}\right)^{r/2} n^r + 2 \\
&= \ \frac{1}{4}\left(\frac{1.5(d-2.1-(d-2)\epsilon_2)(d-2.1+\epsilon_2)}{(d-1)^2}\right)^{r/2} n^r + 2 \\
&\geq \ \frac{1}{4}n^r + 2,
\end{aligned}$$

when $d \geq 85$. $\square$

Claim (4.6.2.10). *We may assume that* $\omega(\mathcal{L}) < \frac{n}{d-2.1}$ *provided* $d \geq 25$.

Proof. Suppose not. Then we have also $|\mathcal{H}| + \eta(\mathcal{L}) \geq \frac{n}{d-2.1}$. Let $\mathcal{M} := M_1 M_2 \ldots M_m$ be the block chain connecting the root B_0 of $\mathbb{T}$ and the block L_1 in $\mathcal{L}$, and suppose $L_1 \cap M_m = \{x, x'\}$. Let G' be a graph obtained from $G[V(\mathcal{L}) \cup \{y\}] + \{yx, yx'\}$ by suppressing all degree 2 vertices. Then it is 3-connected, and by Theorem (4.1.1) (a), there is an (x', y')-path P_L in $G' - x$ such that $\ell(P_L) \geq \frac{1}{4}\left(\frac{d-2.1}{d-1}|\mathcal{L}|\right)^r + 2$, as $|N_G(x) \cap \mathcal{L}| = 2$, where y' is a neighbor of y in the last block of $\mathcal{L}$. Let P_H be an (x, v_0)-path in $\mathcal{H}$ given by Lemma (4.4.1), and P_M be a (v_0, x')-path in $\mathcal{M} - x$ such that $\ell(P_M) \geq \frac{1}{4}\sum_i \left(\frac{d-2.1}{d-1}\frac{|M_i|}{t(M_i)}\right)^r$.

Set $C := P_H \cup P_M \cup P_L \cup \{yy', xy\}$, which is a cycle through xy such that

$$\ell(C) \geq \frac{1}{4}\left(\frac{d-2.1}{d-1}|\mathcal{H}|\right)^r + \frac{1}{4}\sum_i \left(\frac{d-2.1}{d-1}\frac{|M_i|}{t(M_i)}\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}|\mathcal{L}|\right)^r + 2.$$

As $\frac{1}{4}\sum_i \left(\frac{d-2.1}{d-1}\frac{|M_i|}{t(M_i)}\right)^r = \frac{1}{4}\sum_i \left(\frac{d-2.1}{(d-1)^2}\frac{|M_i|}{t(M_i)}\right)^r + \frac{1}{4}\sum_i \left(\frac{d-2.1}{(d-1)^2}\frac{|M_i|}{t(M_i)}\right)^r$, and $|\mathcal{H}| \geq \frac{|M_i|}{t(M_i)}$ and $|\mathcal{L}| \geq \frac{|M_i|}{t(M_i)}$ for each M_i, by using (4.1c), and the fact that $\frac{(d-2.1)(d-1)^{\log_2(3/2)}}{(d-1)} \geq 1$ when $d \geq 25$, we have

$$\begin{aligned}
\ell(C) &\geq \frac{1}{4}\left(\frac{d-2.1}{d-1}(|\mathcal{H}| + \eta(\mathcal{L}))\right)^r + \frac{1}{4}\left(\frac{d-2.1}{d-1}(|\mathcal{L}| + \eta(\mathcal{L}))\right)^r + 2 \\
&\geq \frac{1}{4}\left(\frac{(d-1)^2(d-2.1)^2}{(d-1)^2(d-2.1)^2}\right)^{r/2} n^r + 2 \\
&= \frac{1}{4}n^r + 2.
\end{aligned}$$

$\square$

We now show that there is a cycle C through xy in G such that $\ell(C) \geq \frac{1}{4}n^r + 2$.

Case 1. $H_1 \in \mathcal{H}$ **is a cycle.**

Recall that $|\mathcal{L}| + \eta(\mathcal{L}) \geq \frac{n-|\mathcal{H}|}{d-2}$. This gives that $|\mathcal{H}| + \eta(\mathcal{L}) \geq \frac{n}{d-1}$ by $|\mathcal{H}| \geq |\mathcal{L}|$ (as $\mathcal{L}$ is a proper $\mathcal{H}$-leg).

Let $w := v_0$, and let $\mathcal{M} := M_1 M_2 \ldots M_m$ be the block-chain connecting the root B_0 of $\mathbb{T}$ and the block L_1 in $\mathcal{L}$. Suppose that $V(\mathcal{M} \cap \mathcal{L}) = V(M_m \cap L_1) = \{x, w'\}$. Let y' be a neighbor of y in $\mathcal{H}$ different from x (y' exists by the 3-connectivity of G). Let P_M be a path in $\mathcal{M} - x$ from w to w' such that $\ell(P_M) \geq \frac{1}{4}\sum_i \left(\frac{d-2.1}{d-1}\frac{|M_i|}{t(M_i)}\right)^r$ given by Claim (4.6.2.8). As both H_1 and L_1 are cycles, apply the particular part of Lemma (4.4.6) on $\mathcal{H}$ and $\mathcal{L}$, without loss of generality, assume that we find a (w, y')-path P_H in $\mathcal{H} - x$, and an (x, w')-path P_L in $\mathcal{L}$ such that $\ell(P_H) + \ell(P_L) \geq \frac{1}{4}(|\mathcal{H}|^r + |\mathcal{L}|^r)$. Let $C := P_H \cup P_M \cup P_L \cup \{yy', xy\}$. Then C is a cycle through xy such that $\ell(C) \geq \frac{1}{4}|\mathcal{H}|^r + \frac{1}{4}|\mathcal{L}|^r + \ell(P_M) + 2$. By splitting the value $\ell(P_M)$, we have

$$\begin{aligned}
\ell(C) &\geq \frac{1}{4}|\mathcal{H}|^r + \frac{1}{4}|\mathcal{L}|^r + \ell(P_M) + 2 \\
&\geq \frac{1}{4}\left(|\mathcal{H}| + \eta(\mathcal{L})\right)^r + \frac{1}{4}\left(|\mathcal{L}| + \eta(\mathcal{L})\right)^r + 2 \\
&\geq \frac{1}{4}\left((d-1)^2 \cdot \frac{n}{(d-1)} \cdot \frac{n - n/(d-1)}{d-2}\right)^{r/2} + 2 \\
&= \frac{1}{4}n^r + 2.
\end{aligned}$$

Case 2. $H_1 \in \mathcal{H}$ **is 3-connected.**

In this case, we have $\tau_x(\mathcal{H}) \geq 2$. Hence by (4.25), we have

$$\frac{\omega(\mathcal{L})}{\tau(\mathcal{L})} \geq \frac{n - |\mathcal{H}|}{d - 2.5}. \tag{4.26}$$

As $\tau(\mathcal{L}) \geq 1$, a similar argument as in (4.25) gives that

$$\frac{\omega(\mathcal{L}')}{\tau(\mathcal{L}')} \geq \frac{n - |\mathcal{H}| - \omega(\mathcal{L})}{d - 3.5}. \tag{4.27}$$

If $\eta(\mathcal{H}) \geq \eta(\mathcal{L}')$, then we construct a cycle C_1 through xy using $\mathcal{H}$ and $\mathcal{L}$ and a cycle C_2 through xy using $\mathcal{L}$ and $\mathcal{L}'$, and show that $\ell(C_1) + \ell(C_2) \geq 2(\frac{1}{4}n^r + 2)$. If $\eta(\mathcal{H}) < \eta(\mathcal{L}')$, then we construct a cycle C_1 through xy using $\mathcal{H}$ and $\mathcal{L}'$ and a cycle C_2 through xy using $\mathcal{L}$ and $\mathcal{L}'$, and show that $\ell(C_1) + \ell(C_2) \geq 2(\frac{1}{4}n^r + 2)$. Assume, without loss of generality, that $\eta(\mathcal{H}) \geq \eta(\mathcal{L}')$. Suppose $\mathcal{L} = L_1 L_2 \cdots L_l$ and $\mathcal{L}' = L'_1 L'_2 \cdots L'_{l'}$. Let $\mathcal{M} := M_1 M_2 \ldots M_m$ be the block-chain connecting the root B_0 of $\mathbb{T}$ and the first block L_1 in $\mathcal{L}$, and let $\mathcal{M}' := M'_1 M'_2 \ldots M'_{m'}$ be the block-chain connecting the root B_0 of $\mathbb{T}$ and the first block L'_1 in $\mathcal{L}'$. Furthermore, we suppose

- $M_m \cap L_1 = \{x, b\}$ and $M'_{m'} \cap L'_1 = \{x, b'\}$;
- $L_k = \max\{L_i : L_i \in \mathcal{L}_1\}$ and $L'_p = \max\{L'_i : L'_i \in \mathcal{L}_2\}$;
- $L_k \cap L_{k-1} = \{a, b\}$, $L_k \cap L_{k+1} = \{a_k, b_k\}$, $L'_p \cap L'_{p-1} = \{c, d\}$, and $L'_p \cap L'_{p+1} = \{c_k, d_k\}$;
- $L_0 := L_{k1} L_{k2} \cdots L_{kk_0} \cdots L_{kk_1}$ is the block-chain $L_k - ab$, and
- $L'_0 := L'_{p1} L'_{p2} \cdots L'_{pp_0} \cdots L'_{pp_1}$ is the block-chain $L'_p - cd$ such that
  - (i) $L_{kk_0} = \max\{L_{ki} : L_{ki} \in L_0\}$ and $L'_{pp_0} = \max\{L'_{pi} : L'_{pi} \in L'_0\}$,
  - (ii) $a \in L_{k1}, b \in L_{kk_1}$, $c \in L'_{p1}$, and $d \in L'_{pp_1}$, and
  - (iii) given by Lemma (4.2.5), P_{L1} is a path in $L_1 L_2 \cdots L_{k-1} - x$ from b to a, and $P_{L'1}$ is a path in $L'_1 L'_2 \cdots L'_{k-1} - x$ from b' to c.

We include the trivial case that L_k or L'_p is a cycle in the above notations. Denote

- $l^+ = \sum_{i>k} \left(\frac{d-2.1}{(d-1)^2}|L_i|\right)^r, \quad l^- = \sum_{i<k} \left(\frac{d-2.1}{(d-1)^2}|L_i|\right)^r;$

- $l_0^+ = \sum_{i>k_0} \left(\frac{d-2.1}{(d-1)^2}|L_{ki}|\right)^r, \quad l_0^- = \sum_{i<k_0} \left(\frac{d-2.1}{(d-1)^2}|L_{ki}|\right)^r;$
- $w^+ = \sum_{i>p} \left(\frac{d-2.1}{(d-1)^2}|L'_i|\right)^r, \quad w^- = \sum_{i<p} \left(\frac{d-2.1}{(d-1)^2}|L'_i|\right)^r;$
- $w_0^+ = \sum_{i>p_0} \left(\frac{d-2.1}{(d-1)^2}|L'_{pi}|\right)^r, \quad w_0^- = \sum_{i<p_0} \left(\frac{d-2.1}{(d-1)^2}|L'_{pi}|\right)^r.$

Let y' be a neighbor of y in the last block of $\mathcal{L}$. We now construct a cycle C_1 through xy by using paths in $\mathcal{H}$ and $\mathcal{L}$ as follows:

- Let P_H be a path in $\mathcal{H}$ from x to v_0 given by Lemma (4.4.1) such that $\ell(P_H) \geq \frac{1}{4}(\frac{d-2.1}{d-1}|\mathcal{H}|)^r + 1$,
- P_M be a path from v_0 to b in $\mathcal{M} - x$ such that $\ell(P_M) \geq \frac{1}{4}\sum_i \left(\frac{d-2.1}{d-1}\frac{|M_i|}{t(M_i)}\right)^r$ given by Claim (4.6.2.8), and
- P_L be a path in $\mathcal{L} - x$ from b to y' given by Lemma (4.4.5) such that $\ell(P_L) \geq \frac{1}{4}|L_{kk_0}|^r + \frac{1}{4}\ell_0^- + \frac{1}{4}\ell^+ - \frac{1}{2}$.

Then $C_1 := P_H \cup P_M \cup P_L \cup \{yy', xy\}$ is a cycle through xy. Now we construct a cycle C_2 in $\mathcal{L}$ and $\mathcal{L}'$. Assume, without loss of generality, that the following inequality holds.

$$l^+ + w^- + w_0^+ \quad \geq \quad w^+ + l^- + l_0^+.$$

Let P_L be a path in $\mathcal{L} - x$ from b to y' given by Lemma (4.4.5) such that

$$\ell(P_L) \geq \frac{1}{4}|L_{kk_0}|^r + \frac{1}{4}l^+ + \frac{1}{4}l_0^- - \frac{1}{2},$$

and $P_{L'}$ be a path in $\mathcal{L}' - x$ from b' to x given by (4.6) of Lemma (4.4.5) such that

$$\ell(P_{L'}) \geq \frac{1}{4}|L'_{pp_0}|^r + \frac{1}{4}w_0^+ + \frac{1}{4}w_0^- + \frac{1}{4}w^-.$$

Let P_M be a path in $\mathcal{M} - x$ from b to v_0 such that $\ell(P_M) \geq \frac{1}{4}\sum_i \left(\frac{d-2.1}{d-1}\frac{|M_i|}{t(M_i)}\right)^r$ given by Claim (4.6.2.8), and let $P_{M'}$ be a path in $\mathcal{M}' - x$ from b' to v_0 such that $\ell(P_{M'}) \geq \frac{1}{4}\sum_i \left(\frac{d-2.1}{d-1}\frac{|M_i'|}{t(M_i')}\right)^r$ given by Claim (4.6.2.8). Then $C_2 := P_L \cup P_M \cup P_{M'} \cup P_{L'} \cup \{xy, yy'\}$ contains a cycle through xy of length at least $\ell(P_L) + \ell(P_L') + 2 - \frac{1}{2}$ (notice that P_M and $P_{M'}$ may intersect). Then,

$$\begin{aligned}
\ell(P_H) + \frac{1}{4}\ell(P_M) &= \frac{1}{4}\left(\frac{d-2.1}{d-1}(|\mathcal{H}|)\right) + \frac{1}{4}\sum_i \left(\frac{d-2.1}{(d-1)^3}\frac{|M_i|}{t(M_i)}\right)^r + 1 \\
&\geq \frac{1}{4}\left(\frac{d-2.1}{d-1}(|\mathcal{H}| + \sum_i \frac{(d-2.1)((d-1)^{log_2(5/4)} - 1)}{d-1} \cdot \frac{|M_i|}{t(M_i)})\right)^r + 1 \\
&\geq \frac{1}{4}\left(\frac{d-2.1}{d-1}(|\mathcal{H}| + \eta(\mathcal{L}_1))\right)^r + 1,
\end{aligned}$$

as when $d \geq 12$, $\frac{(d-2.1)((d-1)^{log_2(5/4)}-1)}{d-1} > 1$ and

$$\begin{aligned}
&\ell(P_L) + \tfrac{1}{4} \cdot (l_0^+ + l^- + \ell(P_M)) \\
\geq\ & \tfrac{1}{4}|L_{pp_0}|^r + \tfrac{1}{4}\textstyle\sum_j \left(\frac{d-2.1}{(d-1)^4} \cdot |L_{pj}|\right)^r + \tfrac{1}{4}\sum_{i\neq p}\left(\frac{d-2.1}{(d-1)^4} \cdot |L_i|\right)^r + \tfrac{1}{4}\sum_i \left(\frac{d-2.1}{(d-1)^3} \cdot \frac{|M_i|}{t(M_i)}\right)^r - \tfrac{1}{2} \\
\geq\ & \tfrac{1}{4}(|L_{pp_0}| + \textstyle\sum_j \frac{(d-2.1)((d-1)^{log_2(9/8)}-1)|L_{pj}|}{d-1} + \sum_{i\neq p}\frac{(d-2.1)((d-1)^{log_2(9/8)}-1)|L_i|}{d-1} \\
&+ \textstyle\sum_i \frac{(d-2.1)((d-1)^{log_2(5/4)}-1)}{d-1} \cdot \frac{|M_i|}{t(M_i)})^r - \tfrac{1}{2} \\
\geq\ & \tfrac{1}{4}\omega(\mathcal{L})^r - \tfrac{1}{2},
\end{aligned}$$

as $(d-2.1)((d-1)^{log_2(9/8)} - 1)/(d-1) > 1$ when $d \geq 64$. Therefore,

$$\begin{aligned}
\ell(C_1) + \frac{1}{4}\cdot(l_0^+ + l^-) - \frac{1}{2}\ell(P_M) &\geq \frac{1}{4}\left(\frac{d-2.1}{d-1}(|\mathcal{H}| + \eta(\mathcal{L}))\right)^r + \frac{1}{4}\omega(\mathcal{L})^r + 2 + \frac{1}{2} \\
&\geq \frac{1}{4}\left(\frac{(d-1)(d-2-1.5\epsilon_2)}{(d-1.5)(d-2.5)}\right)^{r/2} n^r + 2 + \frac{1}{2} \\
&\geq \frac{1}{4}n^r + 2 + \frac{1}{2},
\end{aligned}$$

where the last inequality is obtained by using inequality (4.2), $\omega(\mathcal{L}) \geq \frac{n-|\mathcal{H}|}{d-2.5}$ from (4.26), $|\mathcal{H}| + \eta(\mathcal{L}) \geq \frac{n}{d-1.5}$ following from $|\mathcal{H}| + \eta(\mathcal{L}) \geq \omega(\mathcal{L})$, and $|\mathcal{H}| \leq \frac{(1+1.5\epsilon_2)n}{d-1}$ from Claim

(4.6.2.7). Similarly,

$$
\begin{aligned}
& \ell(P_L) + \frac{1}{4}\ell(P_M) + \ell(P_{L'}) + \frac{1}{4}\ell(P_M) - \frac{1}{4}(l_0^+ + l^-) \\
\geq\; & \frac{1}{4}(|\mathcal{L}| + \eta(\mathcal{L}))^r + \frac{1}{4}(|\mathcal{L}'| + \eta(\mathcal{L}'))^r - \frac{1}{2}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\ell(C_2) + \frac{1}{4}\cdot(\ell(P_M) + \ell(P_M)) - \frac{1}{4}(l_0^+ + l^-) \;&\geq\; \frac{1}{4}(|\mathcal{L}_1| + \eta(\mathcal{L}_1))^r + \frac{1}{4}(|\mathcal{L}_2| + \eta(\mathcal{L}_2))^r + 2 - \frac{1}{2} \\
&\geq\; \frac{1}{4}\left(\frac{(d-2-1.5\epsilon_2)(d-2-1.5\epsilon_2-\frac{1}{d-2.1})}{(d-2.5)(d-3.5)}\right)^{r/2} + 2 - \frac{1}{2} \\
&\geq\; \frac{1}{4}n^r + 2 - \frac{1}{2},
\end{aligned}
$$

provided that $d \geq 125$, where the conditions that $\omega(\mathcal{L}) \geq \frac{n-|\mathcal{H}|}{d-2.5}$, $\omega(\mathcal{L}') \geq \frac{n-|\mathcal{H}|-\omega(\mathcal{L})}{d-3.5}$ from (4.27), $\omega(\mathcal{L}) \leq \frac{n}{d-2.1}$ from Claim (4.6.2.10), and $|\mathcal{H}| \leq \frac{(1+1.5\epsilon_2)n}{d-1}$ from Claim (4.6.2.7) are used.

From above, we now can see $\ell(C_1) + \ell(C_2) \geq 2\cdot(\frac{1}{4}n^r + 2)$, this implies that at least one of $\ell(C_1)$ and $\ell(C_2)$ is at least $\frac{1}{4}n^r + 2$. The proof is then completed. $\square$

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