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Calculus Students' Understanding of the Derivative in Relation to the Vertex of a Quadratic Function

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CALCULUS STUDENTS' UNDERSTANDING OF THE DERIVATIVE IN RELATION TO THE VERTEX OF A QUADRATIC FUNCTION

by

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Under the Direction of Draga Vidakovic

ABSTRACT

The purpose of this study was to gain insight into students' understanding of the vertex of the quadratic function in connection to the concept of derivative by use of the think-aloud method as a means of data collection. Thirty students enrolled in a *Calculus of One Variable I* course participated. By analyzing students' comprehension of the vertex of a quadratic function and the derivative not only during think-aloud sessions, but also during follow up interviews and on written work, this contributed to a better understanding of how students relate the concepts.

Several different theoretical frameworks were used to analyze student comprehension. This gave multiple viewpoints to further explore students' thoughts as they worked either aloud individually or in a group setting. First, APOS theory (Asiala et al., 1996) was used to analyze students' understanding of the concept of vertex of the quadratic function in relation to the derivative on certain tasks. Students' personal meaning of the vertex and its impact on the understanding of the derivative was noted as well as students' lack of connection between explicit and real world problems. Obstacles of misconception of the vertex, trouble with the free fall formula, and problems with graphing due to a weak schema of quadratic functions were all identified as barriers to student understanding of real world problems.

Next, Skemp's (1976) relational and instrumental understanding framework was used to explain how students think-aloud individually. Trends in the thought process while working alone as well as students' ability to identify and correct mistakes were analyzed. Lastly, Vygotsky's (1978) concept of zone of proximal development was used to describe the difference in ability of students working by themselves versus in a group setting. In a group setting, some students worked within their zone of proximal development as they were influenced by peers to fix incorrect solutions.

Based on APOS, several suggested activities pertaining to the quadratic function and its derivative were developed for implementation in the classroom to help students overcome misconceptions and obstacles. Future research is suggested as a continuation to improve student understanding of quadratic functions and the derivative.

INDEX WORDS: Quadratic function, Vertex, Derivative, APOS, Student understanding, Think-aloud

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For

Jayden and Austin

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

1 INTRODUCTION

One of the most important concepts in mathematics is the concept of function. Most students are introduced to the idea at a very young age, but it is not until possibly middle school or high school that a formal definition is introduced. While the study of functions is one of the backbones of mathematics, many students have trouble with understanding what exactly is a function (Abdullah, 2010; Akkoc & Tall, 2005; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Clement, 2001; Cooney & Wilson, 1996; Dreyfus & Eisenberg, 1983; Dubinsky & Harel, 1992; Dubinsky & Wilson, 2013; Leinhardt, Zaslavsky, & Stein, 1990; Markovits, Eylon, & Bruckheimer, 1988; Monk, 1992; Monk, 1994; Oehrtman, Carlson, & Thompson, 2008; Sajka, 2003; Tall, 1992; Tall & Bakar, 1992; Vinner, 1983; Vinner & Dreyfus, 1989). This comes as no surprise because as the concept of function has evolved over time, the definition of function has evolved and changed as well. Mathematicians at one time did not agree on how to define a function, and the concept took many years to evolve (Jones, 2006; Kleiner, 1989; Ponte, 1992).

Not only do students have trouble grasping the ideas behind the concept of function, but in particular, students have trouble learning quadratic functions (Afamasaga-Fuata'i, 1992; Eraslan, 2005; Eraslan 2008; Kotsopoulos, 2007; Metcalf, 2007; Zaslavsky, 1997). Common struggles for students include transitioning between graphical and algebraic representations of a quadratic function, the relationship between the different expressions of the algebraic forms of a quadratic function, and variable misconceptions. Several studies have shown that the use of a graphing calculator may help students overcome some of their obstacles (McCulloch, 2011; Mesa, 2007). However, the problem still seems to exist in the classroom. Involved in some of these struggles are misconceptions of the vertex of the graph of a quadratic function (Borgen &

Manu, 2002; Ellis & Grinstead, 2008). For example, Ellis and Grinstead (2008) found that students believed that changing the coefficient of x^2 does not change the location of the vertex, an inaccurate understanding. The vertex of a quadratic function is an important concept not only in a college algebra course, but also in a calculus course, as well as in fields other than mathematics such as physics, chemistry, economics, etc.

In a calculus course, the derivative of a quadratic function can be used to find the vertex, which is most often referred to as a maximum or minimum point. First, one has to identify what is known as a *critical point*. Critical points for f are the interior points c of the domain of f for which $f'(c) = 0$ or $f'(c)$ does not exist (Salas, Hille, & Etgen, 2007). Then, the maximum or minimum is found by the application of what is often referred to as the *first-derivative test*:

> Suppose that c is a critical point for f and f is continuous at c. If there is a positive number δ such that:
>
> (i) $f'(x) > 0$ for all x in $(c-\delta, c)$ and $f'(x) < 0$ for all x in $(c, c+\delta)$, then $f(c)$ is a local maximum.
> (ii) $f'(x) < 0$ for all x in $(c-\delta, c)$ and $f'(x) > 0$ for all x in $(c, c+\delta)$, then $f(c)$ is a local minimum.
> (iii) $f'(x)$ keeps constant sign on $(c-\delta, c) \cup (c, c+\delta)$, then $f'(c)$ is not a local extreme value.

Figure 1.1 The first-derivative test (Salas et al., 2007, p. 170)

Being able to apply the first-derivative test is a critical step in being able to graph functions by use of the derivative. In particular, since a quadratic function only has one critical point, either a maximum or minimum, this may simplify the graphing process. The connection between student understanding of the derivative and of its corresponding graph has been the topic of several studies (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Baker, Cooley, & Trigueros, 2000a; Berry & Nyman, 2003; Orhun, 2012), however none have a focus on quadratic functions. Research has also shown that many students struggle with the conceptual understanding of the derivative (Bingolbali and Monaghan, 2008; Clark et al., 1997; Habre and Abboud, 2006; Maharaj, 2013; Orton, 1983; Siyepu, 2013; Uygur, & Ozdas, 2005; White & Mitchelmore, 1996), again; however, none have a focus on quadratic functions. It is important to study how students understand quadratic functions since a quadratic function is the most basic polynomial function of degree greater than one, is studied in the majority of algebra courses, and the concepts and properties are building blocks to students' understanding of the concept of function (Eraslan, 2005; Even, 1990; Metcalf, 2007; Zaslavsky, 1997). If a student understands quadratic functions and its properties, applications, etc. in turn it may be easier for them to build and develop a good understanding of more complex and different types of functions and concepts (Even, 1990; Metcalf, 2007). The purpose of this study is to fill this gap in the literature and to gain insight into student understanding of the connection between the vertex of a quadratic function and the concept of derivative.

As an important part of this study, a think-aloud technique will be used as part of the data collection. To think-aloud means that each participant will "give immediate verbal expression to their thoughts" while working on the given problems (Ericsson, 2006, p. 224). The technique has roots in cognitive psychology, and the idea that thought is mediated by language (Yair, Liat, & Baruch, 2000). Vygotsky's (1962) idea of inner speech can be related to thinking aloud, as the speech is usually not expressed in complete, reasoned sentences, and often may be fragmented and dominated by predicates (Charters, 2003). Verbalization as think-aloud data is a thoroughly reliable and valid source of data (Ericsson, 2006; Ericsson & Simon, 1980). It does not change the structure of the cognitive task process, and instead only may slow down the speed of performance (Ericsson & Simon, 1980). As Watson (1920) states, by making subjects think-aloud we "thereby can observe a large part of the process of thinking" (p. 93).

During the Spring 2013 semester, a pilot study was conducted with 11 undergraduate students taking the course *Calculus of One Variable I*. Nine of the 11 students participated in three separate groups of problem solving sessions and follow up interviews based on questions involving the application of the derivative. The students worked in groups on the given problems for about 30 minutes, and then participated in an audio recorded semi-structured group interview immediately following which lasted about 30 minutes. During the semi-structured group interview, students took turns going first, second, etc. to explain and elaborate on their solutions to each problem. A different procedure was used with one group of two students. For these two students, the same questions were used, only this time they were instructed to work while thinking aloud in a room by themselves. Each of them was audio recorded as they talked out loud while working on the problems. After about 30 minutes, the two students came together in the same room for an audio recorded semi-structured group interview which lasted about 30 minutes. When the think-aloud data was compared to the data gathered by the written work from the group problem solving sessions, the think-aloud method provided much richer data. It provided a much more in depth look into what the students' were thinking as he or she worked through the given problem. Based on this result, the think-aloud protocol as a method of data collection is important for this study as it will help instigate further depth and understanding into how students connect the relationship between the derivative and quadratic functions.

1.1 Background of the Study

As a mathematics instructor, I have encountered student struggles with many different mathematical concepts, including difficulties with quadratic functions and derivatives. This does not come as a surprise as much of the literature confirms students difficulties with these concepts (Afamasaga-Fuata'i, 1992; Bingolbali & Monaghan, 2008; Clark et al., 1997; Eraslan, 2005;

Eraslan 2008; Habre & Abboud, 2006; Kotsopoulos, 2007; Maharaj, 2013; Metcalf, 2007; Orton, 1983; Siyepu, 2013; Tall, 1993; Uygur & Ozdas 2005; White & Mitchelmore, 1996; Zaslavsky, 1997). I have taught classes and worked with students in mathematics courses ranging from *Elementary Algebra* to *Calculus II*. My students' mathematics backgrounds, therefore, have ranged from little understanding of mathematical concepts to more sophisticated ways of thinking of mathematical concepts. Even though there are differences in abilities of my students, one issue seems to be common to all. Students have trouble and difficulties generalizing a mathematical concept in order to apply it to real world problems when they are not explicitly instructed regarding the specific tools they need to use (Dubinsky & Wilson, 2013; Rebello, Cui, Bennet, Zollman, & Ozimek; 2007). That is, problems for which a student would have to use his or her knowledge of a concept to know when, where, and how to use that particular concept to reach a solution.

I have also noted that students particularly have trouble with quadratic functions, another commonality supported in research (Afamasaga-Fuata'i, 1992; Eraslan, 2005; Eraslan 2008; Kotsopoulos, 2007; Metcalf, 2007; Zaslavsky, 1997). For example, the students are able to find the vertex given the problem "find the vertex", however when they are asked to "maximize the revenue of the company" they do not realize that this question also means find the vertex, just in words. When I taught *Calculus of One Variable I*, I found the same problem with my calculus students. They could find the critical point of a given quadratic function using the derivative in an explicit problem, but when the question asked to find the maximum height of a rocket, they were not able to generalize the concept of finding critical points using the derivative to know how to use it. Calculus students are continuing to not be able to generalize concepts to solve non routine problems (J. Selden, Mason, & A. Selden, 1989; A. Selden, J. Selden, Hauk, & Mason, 2000; J. Selden, A. Selden, & Mason, 1994). I believe it is very important to explore and research the cause of disconnect for these students. By researching the obstacles that these students have, I hope to be able to make some pedagogical suggestions for the way these concepts might be taught.

1.2 Purpose of the Study

The purpose of this study was to gain insight into student understanding of the connection between the concept of the vertex of a quadratic function and the concept of derivative. By analyzing students' comprehension during think-aloud sessions, follow up semi-structured group interviews, and written work this can possibly contribute to a better understanding of how students understand the concept of the derivative, and in particular, in relation to the quadratic function. This knowledge can be used to give suggestions on how to possibly revise course curriculum, which could include more emphasis on the application of the derivative with respect to quadratic functions.

1.2.1 Research Questions

(1) How does the understanding of the vertex of a quadratic function shape the understanding of the derivative?

(2) How do students perceive and understand the concept of the vertex of a quadratic function in relation to the derivative in an explicit and real world problem?

(3) What are obstacles that students face when using the concept of vertex of quadratic functions in relation to the derivative in different real world problems?

1.2.1.1 Sub-Research Questions

- What does the think-aloud method reveal about student understanding?

- How does the influence of peers impact the ability of students to understand questions relating the quadratic function and its derivative?

1.3 Significance and Implications of the Study

While there is literature on student understanding of quadratic functions and derivatives, there is no study that links the concepts with a focus on the concept of the vertex of a quadratic function. There is also no study involving these concepts that uses the think-aloud method as a means of data collection. The purpose of this study was to fill this gap, and to gain information about how students understand the relation between these concepts.

In addition, there is a need to further research why the lowest recognition of a function is the application of the function to real world situations (Dubinsky & Wilson, 2013). Tall (1992) similarly identified that one of the difficulties students encounter in calculus is “difficulties in translating real world problems into calculus formulation.” This study addresses this by analyzing how students understand the concept of quadratic function and the derivative in explicit and real world problems. This study also supports and builds on Movshovitz-Hadar’s (1993) study in that it extends her study to college students and the use of the derivative in connection with properties of quadratic functions such as increasing, decreasing, and the maximum or minimum point.

In collaboration with several frameworks, I used a phenomenological methodology and detailed coding to analyze the data (see data analysis; Litzinger et al.1994). This enabled me to analyze the students’ thought processes of concepts, and in particular, the thought process of approaching real world problems where one can apply the use of the derivative in context to quadratic functions. By analyzing students’ comprehension through think-aloud sessions, this contributed to a better understanding of how students understand the concept of the derivative, and in particular, in relation to the vertex of a quadratic function. This knowledge can be used to possibly revise course curriculum to include more emphasis on the application of the derivative with respect to quadratic functions. This knowledge can also be used to include more emphasis on students' particular difficulties and misconceptions of the concepts that I have found some of my students have struggled with. Since functions play a central role in pre-calculus and calculus curriculum, if we are able to help students develop a better understanding of these concepts, then it is hoped that they will be in a better position to apply these concepts to their collegiate studies and further to their employment post collegiate education (Maharaj, 2013; Tall, 1997).

1.4 Summary

The current idea of "what is a function" took many years to evolve (Jones, 2006; Kleiner, 1989; Ponte, 1992). What started out in the seventeenth century as representing a geometric idea has transformed over the years to become one of the most foundational concepts in mathematics. It is of vital importance that students learn and come to a clear understanding of functions in order to succeed in mathematics courses. However, studies have shown that it is one of the more challenging and difficult concepts for students to understand (Abdullah, 2010; Akkoc & Tall, 2005; Breidenbach et al. 1992; Carlson, 1998; Clement, 2001; Cooney & Wilson, 1996; Dreyfus & Eisenberg, 1983; Dubinsky & Harel, 1992; Dubinsky & Wilson, 2013; Leinhardt et al., 1990; Markovits et al., 1988; Monk, 1992; Monk, 1994; Oehrtman et al., 2008; Sajka, 2003; Tall, 1992; Tall & Bakar, 1992; Vinner & Dreyfus, 1989). In particular, the understanding of quadratic functions has shown to be problematic (Afamasaga-Fuata'i, 1992; Eraslan, 2005; Eraslan 2008; Kotsopoulos, 2007; Metcalf, 2007; Zaslavsky, 1997). Involved in some of these struggles is the concept of the vertex of a quadratic function (Borgen & Manu, 2002; Ellis & Grinstead, 2008).

In calculus, the first derivative test can be used to find the vertex of a quadratic function, often referred to as a maximum or minimum. Having a clear understanding of this concept, quadratic functions, and functions in general is crucial to the study of the concept of derivative, one of the most central concepts in calculus (Breidenbach et al., 1992; Monk, 1994; Oehrtman et al., 2008). Research has also shown that many students have difficulties with the graphical understanding of the derivative (Asiala et al., 1997; Baker et al., 2000a; Berry & Nyman, 2003; Orhun, 2012) as well as with the conceptual understanding of the derivative (Bingolbali & Monaghan, 2008; Clark et al., 1997; Habre & Abboud, 2006; Maharaj, 2013; Orton, 1983; Siyepu, 2013; Tall, 1993; Uygur, & Ozdas, 2005; White & Mitchelmore, 1996). However, there is no study that pinpoints the concept of the vertex of a quadratic function in relation to the understanding of the derivative.

The purpose of this study was to gain insight into student understanding and obstacles of the concept of the vertex of quadratic functions in relation to the concept of the derivative by the use of the think-aloud method as a means of data collection. The think-aloud method gives “a simple and powerful way to get such [rich] data” (Larkin & Rainard, 1984, p. 236). The research questions address the issue of obstacles that students have in connecting the concept of derivative with the concept of the vertex of the quadratic function in real world situations. They also address how students relate the two concepts and how the understanding of the vertex of quadratic functions may influence the understanding of the derivative. Sub-research questions address potential revelations in students’ thought processes based on the use of the think-aloud method, as well as how students’ understanding may be influenced by peers in a group setting.

2 LITERATURE REVIEW

The focus of the literature review is based on student understanding of mathematics and the think-aloud protocol as a means of data collection. First, several theoretical frameworks that deal with student understanding of mathematics are summarized. Then the following three sections outline several relevant studies in relation to the understanding of functions in general, the understanding of quadratic functions in particular, and the understanding of the derivative. After that, background of the think-aloud protocol is laid out, along with how it is relevant to be a valuable means of data collection for research in mathematics education.

2.1 Student Understanding in Mathematics

> "One…fact must astonish us, or rather would astonish us if we were not too much accustomed to it. How does it happen that there are people who do not understand mathematics? If the science invokes only the rules of logic, those accepted by all well-formed minds…how does it happen that there are so many people who are entirely impervious to it?" (Poincare, 1952).

This century old question of why one does not understand mathematics is one that is still prevalent today. Many students still claim "I don't understand" or "I don't get it" in mathematics classrooms around the world. In response to this, many different theoretical perspectives have been developed as a means to cope with how students make meaning from mathematics (for example Asiala et al., 1996; Godino, Batanero, & Font, 2007; Pirie & Kieren, 1994; Sfard, 1991; Sierpinska, 1990; Skemp, 1976; Tall &Vinner, 1981; Vygotsky, 1978). It is important as a researcher in mathematics education to be familiar with the existing frameworks and how or if they may be relevant to a particular study.

For this study, three different frameworks are used to explain the data as "the availability of multiple explanatory theories and the use of multiple layers of analysis can, depending on the research, provide a richer set of constructs for accounting for observed phenomena" (Simon, 2009, p.484). The idea of providing a richer set of constructs was applied to this study as each section of the results is explained with a different theoretical framework which corresponds most closely to the research question being analyzed (see data analysis). This parallel analysis of different theories offers "separate 'truths', providing different lenses through which to attain a more complete reciprocal embodied view of mathematics education" (Kieren, 2000, p. 228).

The frameworks used for each major subsection of the results are outlined here briefly as follows: (1) Dubinsky's (1991) action-process-object-schema framework based on Piaget's ideas of reflective abstraction; (2) Skemp's (1976) relational and instrumental understanding; and (3) Vygotsky's (1978) zone of proximal development (ZPD). For more about each framework, see data analysis and results.

Dubinsky's (1991) action-process-object-schema framework (Asiala et al., 1996), commonly known as APOS theory, is based on Piaget's ideas of reflective abstraction. According to this framework, an *action* is an externally driven transformation of a mathematical object (or objects). For an individual, an action could be recalling a fact from memory, or needing a step by step algorithm or formula to follow. After the actions are interiorized and reflected upon, then it becomes a *process*. A process, unlike an action, is internally driven. At a process level, an individual is able to reflect and describe all the steps in a transformation, without having to actually perform them. Once a process is constructed by an individual the process can be reversed, coordinated or linked. When the individual reflects on the actions applied to a particular process, becomes aware of the process as a totality, realizes that transformations can act on it, and can actually construct these transformations, then the process has been encapsulated into an *object*. Sometimes it is important and often necessary to de-encapsulate the object back to the process from which it came once constructed objects and processes can be coordinated and linked. This collection of processes and objects forms a *schema*, which is closely related to what Piaget and Garcia (1989) call *schemata*. The idea of the action-process-object understanding of a mathematical concept which can all be organized in schemas is illustrated in the figure below.

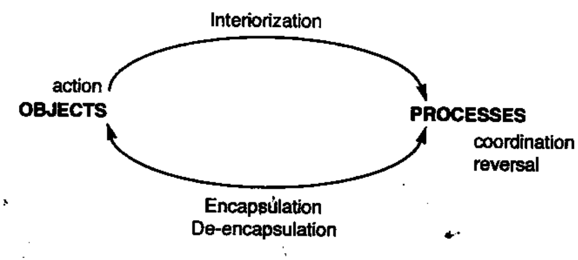


Figure 2.1 APOS constructions for mathematical knowledge
(Asiala et al., 1996)

The second framework is what Skemp (1976) coined as relational understanding and instrumental understanding. According to Skemp (1976), relational understanding is "knowing both what to do and why" (p. 2). Instrumental understanding is the application of rules and the ability to use it, but not necessarily knowing the reasons behind the steps. While instrumental understanding may be easier to achieve initially, and may give a quicker answer, it lacks the depth and breadth involved in fully understanding mathematics. Relational understanding is ideal in that it is more adaptable to new tasks, easier to remember because the student now knows why and is not relying on memorization, and it has long term benefits on the understanding of mathematics.

The last framework discussed is Vygotsky's (1978) idea of the ZPD. This approach suggests that one can develop a deeper understanding of a concept with guidance by a more capable peer than one can by themselves. The ZPD is this “distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers” (Vygotsky, 1978, p. 86). The idea is that what one is capable of doing with assistance will then be able to do it by his or herself, illustrated in the figure below.

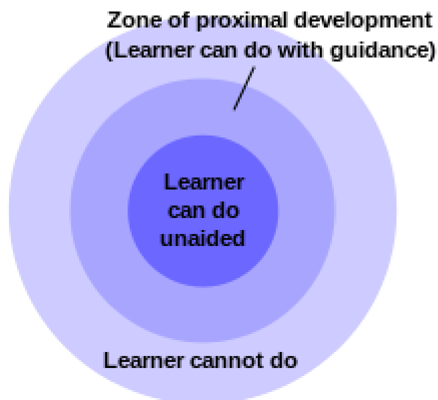


Figure 2.2 Vygotsky’s ZPD (Zone of Proximal Development, 2014)

In summary, Asiala et al. (1996) provided a framework to explain mathematical understanding based on Piaget’s ideas of reflective abstraction, Skemp (1976) looked at the understanding of mathematics with bi-polar explanations: relational/instrumental understanding, and Vygotsky (1978) suggested that development of understanding occurs under guidance within the learner’s ZPD.

2.2 Understanding the Concept of Function

Student understanding of the concept of function is a topic of many studies (Abdullah, 2010; Akkoc & Tall, 2005; Borba & Confrey, 1996; Breidenbach et al. 1992; Carlson, 1998; Clement, 2001; Cooney & Wilson, 1996; Dreyfus & Eisenberg, 1982; Dreyfus & Eisenberg, 1983; Dubinsky & Harel, 1992; Dubinsky & Wilson, 2013; Even, 1998; Kalchman, 2001; Leinhardt et al., 1990; Markovits, Eylon, & Bruckheimer,1986; Markovits et al.,1988; Monk, 1992; Monk, 1994; Oehrtman et al., 2008; Sajka, 2003; Tall, 1992; Tall & Bakar, 1992; Vinner, 1983; Vinner & Dreyfus, 1989; Zazkis, Liljedahl, & Gadowsky, 2003) which presents several themes in the literature on student difficulties with functions. First, there seems to be student difficulty with the definition of a function (Markovits et al., 1986; Sierpinska, 1988; Tall, 1992). Markovits et al. (1986) compared students' difficulties with the modern day definition of a function and the older definition of a function. The modern day definition is related to set theory and is largely the result of the rise of abstract algebra and Bourbaki, a pseudonym given for a group of 20th century mostly French mathematicians who are credited for major contributions towards modern advanced mathematics (Luzin, 1998). The older definition of a function focused more on the relationship between variables. The modern day definition of a function is used in the majority of textbooks today and is more abstract than the older definition as the relation between variables goes almost unnoticed. Their study points out the challenges to students today who now must consider a wider context of functions. As Sierpinska (1988) notes, "introducing functions to young students by their elaborate modern definition is a didactical error- an antididactical inversion" (p. 572).

Vinner and Dreyfus (1989) also doubted that the Dirichlet-Bourbaki approach should be taught in courses if it is not needed. They note that "the Dirichlet-Bourbaki approach defined as functions many correspondences that were not recognized as functions by previous generations of mathematics" (Vinner & Dreyfus, 1989, p. 357). Although this approach is what is most often used in textbooks and classrooms today, the examples used to illustrate the function concept are usually "functions whose rule of correspondence is given by a formula" (Vinner & Dreyfus, 1989, p. 357). Therefore, there is a disconnection in what the student is understanding, which may potentially cause conflicting schemes.

Clement (2001) also found misconceptions with the definition of a function that led students to incorrectly use the vertical line test. Based on their use of the vertical line test, students came to the conclusion that piecewise functions are not functions because they are not continuous. This false assumption is credited to a misinterpretation of the function definition. Clement suggests that time should be spent in the classroom discussing functions in different ways and in different expressions.

Tall and Bakar (1992) also supported the notion that students struggle with conceptual understanding of the function. During their study, they found that while the quadratic expression $y = x^2$ was commonly recognized by almost all as a function, the constant $y = 4$ was not. They also identified a misconception of three quarters of the participants who thought that a circle is a function. This study revealed the gap in concepts that are perceived to be taught and what students are actually learning. Akkoc and Tall (2005) attribute this to a "mismatch between the curriculum design and the students' cognitive structures" (p. 7).

There has also been research on students' intuition on the concept of function (Dreyfus & Eisenberg, 1982; Dreyfus & Eisenberg, 1984). Dreyfus and Eisenberg (1982) gave questionnaire booklets consisting of 42 multiple choice questions to 443 students ranging from sixth grade to ninth grade in twelve different schools. None of the participants had studied the concept of function yet, so the idea was to truly get a sense of what a students' intuition of a function may be. They found that questions about the concept of image were answered best, while questions about the concept of slope were answered worse.

This study was followed by another similar Dreyfus and Eisenberg study in 1984. This time, they gave 127 junior high students from grades 7^{th} and 8^{th} a questionnaire with 48 questions relating to functions. Again, the students' knowledge was tested independent of previous knowledge as intuition was a key factor in the study. As would be expected, they found that high ability students scored consistently higher. They also found that high ability students preferred a graphical approach. This contrasts the preference of the low ability students, whom they found preferred pictorial and tabular presentations. These two large scale studies were important for understanding student intuition of functions without prior exposure to the concept; however, the studies were on functions in general and did not have a focus on quadratic functions.

Another recurring theme is student difficulty moving between representations of functions (Dubinsky & Wilson, 2013; Goldenberg, 1988; Graham & Ferrini-Mundy, 1990; Janvier, 1987; Kabael, 2011; Markovits et al., 1986; Markovits et al., 1988; Martinez-Planell & Gaisman, 2012). Representations of functions include verbal descriptions, arrow diagrams, algebraic, tables and graphical representations. Much of the literature focuses on moving between algebraic and graphical representation. Kabael (2011) and Markovits et al. (1986) both discovered that most students prefer to start with an algebraic approach. Starting with an algebraic approach is easier for students and the primary difficulties arise when a student has to represent the function algebraically from a given graph. These studies are insightful into student difficulties moving between the algebraic and graphical representations, however, they lack information on other types of representations of functions.

Along with difficulty moving between representations of functions, there is also literature that supports the struggle students have with translation of a function (Baker, Hemenway, & Trigueros, 2000b; Borba & Confrey,1996; Even, 1998; Zazkis, et al., 2003). Zazkis et al. (2003) explored 15 preservice secondary teachers and ten 11^{th} and 12^{th} grade students' explanations of translation of a function. In one task, they were to predict, check, and explain the relationship between the graph of the quadratic function $y = x^2$ and the graph of the translation, $y = (x - 3)^2$. Zazkis et al. (2003) accredited troubles of this relationship to epistemological obstacles (Sierpinska, 1994) and the instructional sequence in which the learning of transformations of functions takes place.

Another major difficulty students' encounter with the concept of function is the misconception of variable (Abdullah, 2010; Oehrtman et al., 2008; Sajka, 2003; Trigueros & Ursini, 2003). A group of 12 Malaysian above average secondary students showed trouble with understanding the symbolism $f(x)$, and recognizing that the symbol x is a variable in $f(x)$ (Abdullah, 2010). A case study of a 16 year old average student also gave an example of misinterpretation of the symbols used in the functional notation (Sajka, 2003). When given the task: "Give an example of a function f such that for any real numbers x, y in the domain of f the following equation holds: $f(x + y) = f(x) + f(y)$", the students' greatest difficulty was the interpretation of the function symbolism (Sajka, 2003, p. 233). It took much prompting from the interviewer for the student to finally understand the problem. Trigueros and Ursini's (2003) analysis of 164 first year college students who answered a questionnaire of 65 open-ended questions found that "students' understanding of the concept of variable lacks the flexibility that is expected" at the university level (p. 18). Their data analysis further supports the idea that students do not have a solid understanding of the concept of variable.

For one of the most recent studies, Dubinsky and Wilson (2013) implemented instructional treatment for the function concept to low achieving and low social and economic status students during their junior year in high school. Their instructional design was strongly successful, and proved that underrepresented students are capable of developing a conceptual understanding of mathematics with proper pedagogy. However, it still came out in this study that the lowest recognition of a function was the application of the function to real world situations. It is still not clear why this happens or why these students face obstacles when trying to apply the function to a real world problem.

Once a student has learned linear functions, the next step in the classroom is usually to move to quadratic functions. Most of the time, linear functions and quadratic functions are taught separately in two distinct chapters; however, this separation may not be necessary. In fact, according to Movshovitz-Hadar (1993) this separation can hinder a students' conceptual understanding of the concept of function. Teaching the two in combination of one another, and showing important relationships between the two may establish a nice transition moving from linear to quadratic functions.

2.3 Understanding the Concept of Quadratic Function

While the literature on functions in general is fairly rich, surprisingly the literature on quadratic functions is not as prevalent. The literature on student difficulties with quadratic functions does, however, support much of the literature on student difficulties with functions. Many of the same themes appear including misconception of variable and moving between representations (Ellis & Grinstead, 2008; Eraslan, 2005; Eraslan, 2008; Eraslan, Aspinwall, Knott, & Evitts, 2007; Kotsopoulos, 2007; Vaiyavutjamai & Clements, 2006; Zaslavsky, 1997).

Vaiyavutjamai and Clements (2006) gave 231 9th grade students from six different math classes in Chiang Mai, Thailand the same 18 quadratic equations to solve. They then followed up with interviews from 36 of the students. They found that there were several difficulties of the students including variable misconception and ability to check solutions. This study focused on students' ability to solve quadratic equations, but did not include a component on finding the vertex of quadratic functions.

Ellis and Grinstead (2008) did have a small component in their study relating to finding the vertex of a quadratic function. They looked for students' connections between the coefficients a, b, and c in the standard form of the quadratic function, $y = ax^2 + bx + c$, and the graphical representation of the quadratic function. By use of video observation from an *Algebra II/Trigonometry* class of 34 students, and two sets of semi-structured interviews, Ellis and Grinstead were able to gain descriptions of what students perceive the coefficients a, b, and c do to the graph. Many responses were incorrect, and there was an unexpected generalization from the students of the coefficient a representing the slope of the quadratic function. Given that a quadratic function does not have a constant slope, this was an unexpected finding. However, given that the study of linear functions preceded the study of quadratic functions, this focus on slope did make some sense. There were also students who believed the vertex was impacted by the coefficient c and not impacted by the coefficient a, both false understandings.

One of the most thorough studies on quadratic functions is Orit Zaslavsky's (1997) study. During her study, she collected data through classroom observations and written responses to problems from over 800 10^{th} and 11^{th} grade students in 25 mathematics classes from eight high schools in well-established areas in Israel. The problems presented were mostly multiple-choice, and the majority of the tasks dealt with the translation between graphical and algebraic representations of quadratic functions.

Based on the analysis of students' responses, Zaslavsky identified five main obstacles in student understanding of a quadratic function: graphical interpretation, relation between a quadratic function and a quadratic equation, analogy between a quadratic function and a linear function, change in form of a quadratic function, and the over-emphasis on only one coordinate as a special point. Over-emphasis on only one coordinate as a special point, for example, would be finding the vertex as the coordinate $x = 2$, rather than as the point $(2, 4)$. Besides the five obstacles identified, students also displayed a preference for translating from algebraic equations to graphs, rather than from graphs to algebraic equations. Students also seemed to have a tendency of preference toward the standard form of a quadratic function, $y = ax^2 + bx + c$, which sometimes hindered the ability of students to visualize the graph as a whole.

With such a large sample size, Zaslavsky was able to analyze many different trends in the data. Zaslavsky did not, however, use any student interviews as data collection, which may have been beneficial in obtaining student reasoning behind their responses.

Several other studies have focused on student challenges relating to nonstandard form of a quadratic function. Eraslan (2008) noticed a cognitive obstacle of reducing the level of abstraction with Colin, a tenth grade honor student, when he tried to identify the vertex of a quadratic function by incorrectly attempting to relate the vertex form of a quadratic function, $y = a(x - h)^2 + k$ to a transformation of the standard form of a quadratic function, $y = x(ax + b) + c$. Kotsopoulos (2007) suggests that in order to overcome the difficulty with variations of quadratic forms, instructors need to pay attention to ways in which the brain creates long-term semantic memory. She suggests that this can enrich student learning by linking the way the brain constructs cognitive representations with quadratic relations.

Even those that perform well in class and appear to have some understanding of quadratic functions in reality may not. In the article *What do students really understand?*, Borgen and Manu (2002) illustrate this idea by videotaping two students working together to find a solution to a problem of finding the stationary point for a quadratic function, and determining if the point is a maximum or minimum. The students gave the correct answer to the problem; however, it was clear that their understanding of the concepts were weak. One student was reliant on the calculator, which led to improper imagining of the function. There was also confusion between the standard form, $y = ax^2 + bx + c$, and the vertex form, $y = a(x - h)^2 + k$, of a quadratic function. Even though the paper answer was correct, the student's schema lacked connection of related concepts.

While there are some studies on quadratic functions and the vertex, none in particular link quadratic functions to the concept of derivative. The studies on student understanding of the concept of derivative focus on the derivative in particular, without necessarily making an explicit connection to quadratic functions or to the vertex of quadratic functions.

2.4 Understanding the Concept of Derivative

There are several studies that focus on students' graphical understanding of the derivative (Asiala et al., 1997; Baker et al., 2000a; Berry & Nyman, 2003; Orhun, 2012) as well as students' conceptual understanding of the derivative (Bingolbali & Monaghan, 2008; Clark et al., 1997; Habre & Abboud, 2006; Hahkioniemi, 2004; Maharaj, 2013; Monk, 1994; Orton, 1983; Siyepu, 2013; Tall, 1993; Uygur & Ozdas, 2005; White & Mitchelmore, 1996). Asiala et al. (1997) interviewed 41 engineering, science and mathematics students, who had previously taken at least two semesters of single variable calculus, eleven questions on the concept of derivative. Student performance was also compared from those who took a *Calculus, Concepts, Computers, and Cooperative Learning*, or C^4L, reformed calculus course (Schwingendorf, Mathews, & Dubinsky, 1996) to those who took a traditional lecture calculus course. The C^4L course consisted of a combination of computer Activities, Classroom tasks without computers followed by discussion, and group Exercises (in C^4L courses this is referred to as *ACE* cycle). The students' understanding was analyzed based on the action-process-object-schema framework (Asiala et al., 1996). Based on the results, a revised genetic decomposition for the graphical understanding of the derivative was suggested, and it was also concluded that those who took the C^4L course exhibited more conceptual understanding of the concepts than those who took the traditional course (Asiala et al., 1997). This study had a focus on the graphical understanding of the derivative, but did not make specific connections of the conceptual understanding of the derivative to the quadratic function and real world situations.

Even those students who appear to have a conceptual understanding of calculus and the derivative by being able to pass a calculus course are not necessarily able to solve nonroutine calculus problems (J. Selden et al., 1989; A. Selden et al., 2000; J. Selden et al., 1994). Selden et al. (1989) found that out of 17 students who had completed the first quarter of calculus with a C average, not one student could solve one of the five nonroutine calculus problems presented correctly. According to Selden et al. (1989), a nonroutine problem is different than an exercise, which is what appears at the end of sections in most calculus textbooks. A nonroutine problem is a problem for which a student has not been taught how to find the solution.

In 1994, Selden et al. carried out the same study of students working the same nonroutine calculus problems, only this time the participants were students who had completed the first year of calculus, 10 having earned an A average, and 10 having earned a B average. While they performed better than the C average students, two-thirds still failed to solve a single nonroutine problem completely. Then, Selden et al. (2000) presented the same nonroutine calculus problems, only this time to 28 students who had for the majority completed a calculus III course, and were enrolled in a differential equations course. The students' performance was better on the nonroutine problems than in the other studies, but still, more than half were unable to solve even one nonroutine problem completely. Even though students may appear to have an adequate knowledge base, they are not able to solve nonroutine problems in calculus. These studies suggest the importance of exposing students to nonroutine problems in the classroom.

One of the most important studies on student understanding of the derivative is Orton's 1983 investigation. He proposed 21 tasks related to differentiation to 110 students. His study found that while the application of the derivative was relatively easy to the participants, the understanding of differentiation and graphical approaches related to rate of change was much more difficult. Students made algebraic errors, limit errors, and symbolic errors in their reasoning and computation. Orton suggested the aid of calculators, a more lively approach to the teaching of ratio, and more application to real life situations as a means of helping students come to a more coherent understanding. His study was the first large study to identify student obstacles related to understanding the derivative. Out of Orton's 21 tasks, none of them were explicitly related to quadratic functions or the vertex in particular.

White and Mitchelmore (1996) followed with a study on student understanding of introductory calculus by collecting responses to word problems involving rate of change. They collected data from 40 first year full time university mathematics students. The students were separated into four groups of 10 students each based on their performance in a previous algebra course. Each group was given four tests before, during, immediately after, and then six weeks after the calculus course. They also conducted interviews from four students in each of the four groups of 10 within three days of each of the written data collections. The results of the study indicated that students had an "underdeveloped concept of a variable" (White & Mitchelmore, 1996, p. 88). The students tried to manipulate and maneuver the variables without any conceptual meaning or understanding behind them. They had a concept of variable that was limited to algebraic symbols without any regard to their possible conceptual meaning. This study highlighted the importance of student conceptual understanding of the concept of a variable when working with calculus. This was also a major difficultly and obstacle for students with understanding the function concept.

To help overcome the obstacles students face in calculus, Habre and Abboud (2006) tested an experimental calculus course. The experimental course focused on the use of calculators and the calculus software *Autograph*. Tasks in the class often required use of the technology and many exercises required a written component. The authors believed that as a result of the emphasis on visualization in the course, the students would come out of the class with a better understanding of the function and its derivative. However, out of the 89 students who were originally enrolled in the course, 33 dropped out, and 12 failed. The course set up was unpopular and difficult for many. Those that were higher achievers in the class seemed to have the least problem. One surprising result of the study was that although this experimental course was centered around an emphasis on visualization, most students at the end of the semester still relied on the algebraic representation as the dominant way of thinking.

Even with a different approach to calculus, students are not grasping the ideas behind the concepts, derivative in particular. In many of these studies, student written work, student tests, or interviews are the primary sources of data. None include the think-aloud method, which has shown to be a very useful and valid source of data collection because it provides some insight into students' thinking during the problem solving process (Ericsson, 2006; Ericsson & Simon, 1980).

2.5 Background of the Think-Aloud Method

The think-aloud method as a type of verbalization is a method in which the inspector gives a subject a task and instructs him or her to "say aloud everything that comes to mind as they are performing it" (Wade, 1990, p.444). Although this method is now widely accepted, this was not always the case. Some may argue that failure of subjects to report some information or incompleteness in a report could make some information unavailable. However, this does not invalidate the reports that are given (Ericsson & Simon, 1980). In fact, one of the pioneers of the think-aloud method, Karl Duncker (1945) stated that "a protocol is relatively reliable only for what it positively contains, but not for which it omits" (p. 11).

The use of the think-aloud method as a means of data collection dates back to John B. Watson (1920) who "often felt a good deal more can be learned about the psychology of thinking by making subjects think-aloud about definite problems" (p. 91). Watson believed that by making subjects think-aloud, we can gain insight into a great part of the thinking process. Twenty-five years later, Karl Duncker (1945) used the same method and idea of instructing his subjects, who were mostly college or university students, to think-aloud while working on various word problems in which no specialized knowledge was necessary. He, along with Watson, made clear the difference in the instruction to think-aloud and the instruction to introspect. Introspection was common as a method of gaining insight into the thought process. Duncker (1945) made clear that "while the introspecter makes himself as thinking the object of his attention, the subject who is thinking aloud remains immediately directed to the problem, so to speak allowing his activity to become verbal" (p. 2). Introspection requires that the subject reports and theorizes about the process, while the think-aloud method requires that the subject reports, however, the theorizing is left up to the researcher (Afflerbach & Johnston, 1984; Ericsson & Crutcher, 1991).

After Watson and Duncker, in 1980, Ericsson and Simon, two pioneers in the argument for verbal reports as valid data, published the article *Verbal Reports as Data*. This article was instrumental in outlining and backing up the validity of verbal reports as data, which includes the think-aloud protocol. Other types of concurrent verbal reports (information verbalized during the task) as data that are related are probing the subject for information while they are working on a task, or probing the subject after the task is completed, which is referred to as retrospective verbalization (Ericsson & Simon, 1980). Retrospective verbalization is sometimes referred to as think afters since the verbalization is after the task is completed (Branch, 2000). According to Ericsson and Simon (1980), when subjects are given instructions to think-aloud, they are drawing on thoughts that are in short-term memory. Since it is believed by Ericsson and Simon that all cognitive developments go through short-term memory, the thoughts of the subject can be verbalized and conveyed as they are processed, resulting in the thinking aloud. Thus, given the task to think-aloud does not change the structure of the thought process (Ericsson & Simon, 1980). In fact, based on their theoretical analysis, Ericsson and Simon argued that "the closest connection between actual thoughts and verbal reports is found when people verbalize thoughts that are spontaneously attended during task completion" (Ericsson, 2006, p. 227). By vocalizing inner thoughts by thinking aloud, information is gained that would otherwise have not been accessible.

There are some advantages and disadvantages to the think-aloud method as a means of data collection. As an advantage, Ericsson and Simon (1993) found no evidence of differences in the sequence of thoughts when they compared individuals who thought aloud to individuals who thought silently. However, although there is evidence of an advantage that thinking aloud does not change the thought process, it may slow down the task performance due to the fact that the subject is required to speak while they are thinking, and that additional time is required to verbalize thoughts (Ericsson, 2006).

One disadvantage that may occur is if the subject stops verbalizing or verbalizes incompletely. This could be attributed to the feeling of a high cognitive load when carrying out a problem solving task (Ericsson & Simon, 1980; Branch, 2000). Branch (2000) found this to be true in her study of interaction with CD-ROM encyclopedias of five 12-15 year old students analyzed based on the think-aloud and the think after method. Even though she identified that cognitive load of the problem and articulation may have been too difficult for some, overall she attributed the use of the think-aloud method to providing “the richness of data associated with verbal protocol analysis” (Branch, 2000, p. 384). Branch (2000) went on to say that the think-alouds “provided the most complete and detailed description of the information-seeking process” and that they “allowed a glimpse into the affective nature of the information-seeking process as well” (p. 382).

Other important indicators to look for along with the verbalization of the thought process are “reaction times, error rates, patterns of brain activation, and sequences of eye fixations” (Ericsson, 2006, p. 229). These indicators are important when analyzing the thought process that occurs during the think-aloud method. There is also a need to account for features of spoken discourse, which could be lost during transcription, such as intonation, inflection, pauses, variation in the rate of speech along with sarcasm, rhetorical questioning, and utterances (Afflerbach & Johnston, 1984). Pauses are important as they may disclose individual differences into the thought process of participants (Charters, 2003).

2.6 Think-Aloud Method in Connection to Student Understanding

The think-aloud method has been used in various studies (Afflerbach & Johnston, 1984; Brown & Day, 1983; Chi, Bassok, Lewis, Reimann, & Glaser, 1989; Chi, de Leeuw, Chiu, & LaVancher, 1994; Cooper, 1999; Johnson, 1964; Litzinger, et al., 2010; Neuman & Schwarz

1998; J. Selden & A. Selden, 2008; A. Selden & J. Selden, 2014; Shepherd, A. Selden, & J. Selden, 2012; Wade; 1990; Yair et al., 2000). It has successfully been used by National Center on Educational Outcomes as a method to detect design problems in large-scale assessments given to a variety of students, including students with learning disabilities, English language learners, and students without disabilities who were proficient in English (Johnstone, Bottford-Miller, & Thompson, 2006). It is also a popular method of data collection in reading and writing research on student understanding (Afflerbach & Johnston, 1984; Brown & Day, 1983; Charters, 2003; Cooper, 1999; Wade, 1990). Elizabeth Charters (2003) used the think-aloud method along with interview to study adult ESL learners thought processes to solve sentence combining problems. She found this approach very effective in her study in that it "not only provided a detailed picture of my participants' thought processes, but also helped to highlight individual differences in response" (Charters, 2003, p. 69).

The think-aloud method has also shown to be a useful method of data collection in problem solving research (Henjes, 2007; Johnson, 1964; Litzinger et al., 2010). Litzinger et al. (2010) analyzed in detail twelve students engineering problem solving process in statics by the use of think-aloud sessions. The students thought aloud while they worked on typical textbook problems. Their results showed that even the best students rely heavily on memory and many struggled with unfamiliar connections.

Henjes (2007) collected data over a five week period from 13 sixth grade mathematics students on the use of the think-aloud method while solving word problems. She found that when the students were given structured instruction to think-aloud during problem solving, the more successful they were. Henjes viewed the think-aloud strategy as a way to help students overcome struggles with solving word problems in middle school mathematics classes.

Several other studies have shown that the think-aloud strategy, sometimes referred to as self-explanation, improves student performance (Chi et al., 1989; Chi et al. 1994; Neuman & Schwarz 1998; Yair et al., 2000). Chi et al. (1989) found several differences in the self-explanations of ten college students with varying ability. The students were instructed to give self-explanations as they studied three worked out examples of problems dealing with the application of Newton's laws of motion. Their findings indicated that "good" students generate significantly greater number of ideas and explanations, raise specific inquiries about the physical situation described, and use examples for a specific reference. On the other hand, "poor" students may not realize that they don't understand the material and thus they are not learning. Other results from the poor students was that they may state their lack of understanding in a general way, restate an equation that they don't understand, or reread examples as if to search for a solution. While this article has many findings differentiating trends, it is unclear what defines a "good" student and what defines a "poor" student. Also, the ten students that were recruited were not mathematics students, and instead had varying majors.

Yair et al. (2000) followed this idea of good and poor students by identifying patterns of self-explanations to distinguish between good and poor problem solvers. Thirty two 9th grade students solved three mixture problems while given the instructions to think-aloud. The results were analyzed using statics sequential analysis to identify patterns. They found that good problem solvers produced more inferences and clarifications and were more likely to produce justifications after regulations. Again, it is unclear what defines a "good" student versus a "poor" student. Also, none of the participants had taken a calculus course, and the focus of the mixture problems was algebra word problems.

While the think-aloud method is prevalent in reading and writing research, along with problem solving research, it has not been the focus of many mathematics studies. J. Selden and A. Selden (2008) conducted a study about behavioral schemas for solving simple mathematical problems. They asked several undergraduate and graduate students along with a professor to think-aloud while calculating $(10/5\ ) + 7$. Based on their informal observation, they suggested a theoretical view of the enactment and origin of behavioral schemas.

A. Selden and J. Selden (2014) conducted another study in which sixteen undergraduates after taking a transition-to-proof course were asked to think-aloud while validating four proofs of a single number theory theorem. The students thought aloud while the interviewer was in the room. Results suggested that taking the transition-to-proof course did not seem to improve the ability of students to validate proofs. While this study also used the think-aloud protocol, again it did not have a focus on the derivative, and instead looked at undergraduate students enrolled in an introductory proof course.

As there are a few mathematical studies that use the think-aloud protocol, none are calculus based, particularly on the concept of derivative or on its relationship to the vertex of a quadratic function. This technique has shown to be a valuable means of data collection as a way to gain insight into the students' thought processes as they are solving problems (Ericsson & Simon, 1980). It is appropriate for this research, as most of the literature on student understanding of the derivative focuses on interviews, and there is a lack of the think-aloud method as a means of data collection in the literature on calculus concepts in mathematics. Also based on results of the pilot study, use of this method provides a new outlook on understanding students' thoughts as they work aloud on calculus problems.

2.7 Overview

Over the past 50 years, students' conceptual difficulties in understanding the concept of function have been a focus of many studies. Some of the conceptual difficulties identified include difficulties with representation, misunderstanding of the definition of function, and misconception of variables (Abdullah, 2010; Borba & Confrey, 1996; Breidebach et al., 1992; Carlson, 1998; Clement, 2001; Cooney & Wilson, 1996; Dreyfus & Eisenberg, 1983; Dubinsky & Harel, 1992; Dubinsky & Wilson, 2013; Even, 1998; Leinhardt et al., 1990; Markovits et al., 1988; Monk, 1992; Monk, 1994; Oehrtman et al., 2008; Sajka, 2003; Tall, 1992; Tall & Bakar, 1992; Vinner & Dreyfus, 1989; Zazkis et al., 2003). These difficulties students encounter with the concept of function carry over into the understanding of the concept of quadratic functions, and even further into the understanding of the derivative. However, none of the literature specifically studies students' understanding of the connection between the quadratic function, the vertex, and the derivative. Along with the missing literature in linking these concepts, there is also a shortage of literature in mathematics that uses the think-aloud method as a means of data collection.

The objective of this study was to fill this gap, and to determine how students perceive and understand the concept of vertex relating to the derivative of quadratic functions. Sub research objectives included the following: to determine how the understanding of a vertex of a quadratic function shapes the understanding of the derivative, to determine obstacles students may have when applying their knowledge of the concept of vertex of quadratic functions in relation to the derivative in real world problems, to seek insight into students' thought processes as they think-aloud, and to determine how the influence of peers can impact the ability of students to conceptually understand.

3 METHODOLOGY

This is a qualitative study (Bogdan & Biklen, 2003; Lincoln & Guba, 1985). A phenomenological study design was used. Phenomenology emphasizes a focus on understanding people's subjective experiences and interpretations of the world from the way they see it (Moustakas, 1994, Van Manen, 1990). Phenomenology was chosen because this approach encourages the participants, in this case students, to describe the mathematics of the problems as they experience it. Along with a phenomenological approach, several theoretical frameworks and detailed coding were used to analyze the way the students thought and approached the concepts (see data analysis; Litzinger et al., 2010).

3.1 Research Setting

Data collection took place at a large public research university in the southeast during the Fall 2013 semester. Student think-aloud sessions, along with follow up group interviews took place in conference rooms in the mathematics department. Student written work was collected during class time of one section of the *Calculus of One Variable I* course. *Calculus of One Variable I* met three times a week: Monday, Wednesday, and Friday from 12:00pm to 1:10pm. There were 46 students enrolled in the course, and 30 of them participated in the study.

3.1.1 Course Description

The course *Calculus of One Variable I* is an undergraduate course required by students who are majoring in math, computer science, physics, geology, or actuarial science. For students who are majoring in biology, chemistry, or neuroscience, there is an alternative calculus course, *Calculus for the Life Sciences I,* that is strongly recommended to be taken in place of *Calculus of One Variable I* to fulfill the calculus requirement. Thus, because of the option for some to take another alternative course, the majority of students in *Calculus of One Variable I* have a background in the required major areas.

The instruction design for the course *Calculus of One Variable I* consisted of a variety of different methods. For the majority of the classes, during the first 30 minutes discussion or lecture would take place. Then, the class would break up for the remaining 40 minutes into their assigned groups and work on problems together that dealt with the topic being covered. The instructor would walk around the room checking on the groups as they worked, listening to their conversations to give input, and answering any questions. There was a homework assignment given each week. There were four tests given during the semester, along with a final exam. Before each test and before the final exam, written assignments were worked on in groups where the students had to write and explain about concepts and ideas that were learned.

As a departmental policy, calculators were not allowed to be used on tests or on the final, and thus were discouraged for use in the class. The course also did not have access to a computer lab, and thus the use of computer activities could not be implemented. While the use of calculators and technology may be beneficial and give new insight into concepts (Breidenbach et al., 1992; Leng, 2011; Park & Travers, 1996; Tall, Smith, & Piez, 2008), for this study, because of limitations, calculators and technology were not included.

Topics covered in *Calculus of One Variable I* include limits and continuity, the derivative, applications of the first and second derivative, and integration. The textbook used for the course was *Calculus One and Several Variables* by Salas et al. (2007). Chapter 4, The Mean-Value Theorem; Applications of the First and Second Derivatives, outlines the importance of the derivative in connection to real world problems and finding the vertex (maximum or minimum) of a function. The problems used for data collection were based off of this chapter.

The chapter has twelve sections, however, only the following ten are taught in the course in the following order (Salas et al., 2007, p. xvi):

4.1 The Mean-Value Theorem

4.2 Increasing and Decreasing Functions

4.3 Local Extreme Values

4.4 Endpoint Extreme Values; Absolute Extreme Values

4.5 Some Max-Min Problems

4.6 Concavity and Points of Inflection

4.7 Vertical and Horizontal Asymptotes; Vertical Tangents and Cusps

4.8 Some Curve Sketching

4.9 Velocity and Acceleration; Speed

4.10 Related Rates of Change Per Unit Time

Sections 4.4 Endpoint Extreme Values; Absolute Extreme Values and 4.9 Velocity and Acceleration; Speed contain the most information relevant to the data collection. Section 4.4 emphasizes how to find extreme values, or critical points using the derivative, and section 4.9 emphasizes the relativity of this concept in connection with real world application of velocity, acceleration, and speed. Since for a quadratic function, the vertex is related to the extreme value, these two sections were important for the data collection of this study.

3.2 Participants

Participants were recruited from students who were enrolled in one section of the course *Calculus of One Variable I*. All students were enrolled in an undergraduate program, with the majority requiring *Calculus of One Variable I* as part of their program. The pre-requisite required to enroll in *Calculus of One Variable I* is a C or higher in the course *Pre-Calculus*. A student, however, is also able to enroll in *Calculus of One Variable I* if they score high enough on a mathematics placement test upon entering the university. Even if a student is able to bypass taking the course *Pre-Calculus* by scoring high enough on the placement test to enroll in *Calculus of One Variable I*, the student is still expected to have pre-calculus knowledge. Since knowledge of pre-calculus is required by all, students enrolled in *Calculus of One Variable I* are expected to be familiar with the pre-calculus content.

Since students differ in ability on these topics, purposeful sampling (Patton, 1990) was used. Purposeful sampling allowed selection of "information-rich cases whose study will illuminate the questions under study" (Patton, 1990, p. 169). Out of the 46 students initially enrolled in one section of *Calculus of One Variable I,* five were minors, which excluded them from the study and three withdrew from the course. This left 38 students enrolled in one section of *Calculus of One Variable I* who were asked for permission to use some of their written work as data (questionnaire, Test 3 questions, Final Exam questions). From those 38 students, 34 consented to use their written work as data and from those 34, 30 volunteered to participate in the think-aloud and follow up interview session. This many participants ensured a wide range of knowledge, some gender difference, and some differences in ethnic background. All students were given pseudonyms.

3.3 Data Collection

Data collection was based on student written work, think-aloud sessions, and follow up interviews. Since think-aloud expressions vary in quantity and quality, there was a need for this triangulation (Charters, 2003). Triangulation in qualitative research is used to enhance credibility by using two or more sources of data, theoretical frameworks, or types of data collected (Denzin, 1978).

3.3.1 Student Written Work

The student written work consisted of a questionnaire based on quadratic functions and the derivative (Appendix A), two questions from Test 3 (Appendix B), and questions from the Final Exam (Appendix C). The questionnaire based on quadratic functions and the derivative served to gain insight into student background and understanding of quadratic functions. First, the questionnaire asked for personal meaning of the quadratic function and personal meaning of the vertex. Then, the questionnaire asked if there is a relationship between the vertex and the derivative, as well as how the vertex can be found. The worksheet was given and completed by the students prior to any think-aloud or interview session.

As another component of the written work data collection, two questions were used for data analysis from Test 3. Question 1 consisted of an explicit algebraic approach to finding critical points, intervals of increase and decrease, and graphing a quadratic function. Question 2 was a real world application problem of the same concepts used in Question 1. The students were to find the time and maximum height for debris from an explosion, as well as the time and speed at which the debris hit the ground. These two questions from Test 3 were also given before any think-aloud or interview session took place. The purpose of these questions was to gain insight into patterns students may have in answering different types of questions pertaining to the derivative and the quadratic function.

The last piece of written work collected was the Final Exam questions. The Final Exam questions served as a follow up to the previous written data collected as well as a follow up to the think-aloud and interview sessions. These questions served to gain better understanding into how students connect the quadratic function and its corresponding derivative, if at all. Students were to graph a position function, which was also a quadratic function, graph its corresponding derivative, and then answer a series of questions relating the quadratic function and the derivative function.

Thirty-eight students who were eligible to participate were asked permission to use their written work for data collection. Out of the 38 asked, only 34 voluntarily gave consent. From these 34, only 30 students written work was analyzed, as these 30 students were the same ones who voluntarily participated in the verbalization sessions.

3.3.2 Think-Aloud Sessions

Thirty students voluntarily participated in the think-aloud sessions. These sessions took place after Chapter 4 on the application of the derivative had been taught, and after Test 3 had taken place. For each session, the goal was to have two students working separately at the same time; however two students did not show up for two different groups. Thus, there were 14 different groups of two students each, and two different groups with just one student. For the first thirty minutes of the session, each student was placed in a separate room and were individually recorded as they thought aloud and worked on two problems alone (Appendix D), which were over topics seen and studied in chapter 4 of *Calculus One and Several Variables* (Salas et al., 2007) during the *Calculus of One Variable I* course. The first problem, Question 1, asked the students to answer questions about the quadratic function and the derivative. It also asked them to explicitly find the critical points, local extreme values, and intervals of increase and decrease of the quadratic function, as well as to sketch the graph. Question 2 asked the students to find the time a rocket reached maximum height, maximum height of the rocket, when the rocket hit the ground and at what speed, as well as to sketch the graph. Question 2 also pertained to a quadratic function and the derivative, just in a real world situation. The idea and purpose of the think-aloud session was to better understand students' thought processes as they worked through problems alone. All of the think-aloud sessions were audio recorded and all data was transcribed.

3.3.3 Follow Up Group Interviews

Immediately following the individual think-aloud sessions, the two students who were in separate rooms then came together in the same room for a follow up semi-structured group interview. Follow up group interviews were used to expand on think-aloud results to "add depth of information about the participant's thought processes" (Charters, 2003, p. 73). For those particularly who have difficulties with the think-aloud method, follow up questioning was helpful (Rankin, 1988). Follow up questioning, also referred to as clinical interviews, is a technique that was pioneered by Piaget (1975) to study knowledge structures and reasoning processes. The strength of this technique is "the ability to collect and analyze data on mental processes at the level of a subject's authentic ideas and meanings, and to expose hidden structures and processes in the subjects thinking that could not be detected by less open-ended techniques" (Clement, 2000, p.341).

For the group follow up interviews, there were 14 separate groups of two students each, and two groups with just one student. The follow up interviews lasted approximately 30-40 minutes long and involved both the two students and the interviewer, except for the two groups with one student, which involved just the student and the interviewer. The interviewer asked the two (or one) students as a group to elaborate how they approached and answered each question during their think-aloud session (Appendix E). For consistency of data collection, we developed and used a protocol for the follow up interviews. Also, in answering questions, both were asked to elaborate on each question; however, the students took turns in answering first or second. This enabled clarification of the results on their paper, as the semi-structured interview consisted of discussion about their solutions, concepts used, and strategies involved. The sessions/interviews were audio recorded and all data was transcribed.

3.4 Data Analysis

Data gathered from student written work, think-aloud sessions, and follow up interviews were the basis for data analysis. Data analysis was ongoing throughout the collection of the data. All of the written work was copied, analyzed, and coded based on the way students responded to the given questions. Once the think-aloud sessions and follow up interviews were completed, all of the verbal data was transcribed in detail. During transcription, pauses, utterances, and inflection, along with other non-verbal cues were noted. After transcription, several theoretical frameworks in collaboration with a phenomenological methodology and detailed coding were used to analyze and interpret the data collected (Litzinger et al., 2010).

For this study, since “the use of a single theory can unnecessarily limit the observations that are made and the types of explanations that can be generated”, several theoretical frameworks were used to describe students’ ability and understanding (Simon, 2009, p.484). Each framework gave a different perspective and approach for further insight to the data. First, the action-process-object-schema framework was used to analyze student performance on specific tasks (Asiala et al., 1996). Based on the students’ ability to work different problems, their level of understanding according to APOS was explored. This framework was most appropriate to analyze student perception and understanding of the concept of vertex of the quadratic function in relation to the derivative and obstacles that students may face because of the theories ability to describe student understanding on different levels of capability. By characterizing data and coding based on the action, process, and object levels, this framework gave a very specific methodology for which to interpret student performance.

Next, Skemp's (1976) relational understanding and instrumental understanding framework was used to explore students' individual thought processes as they talked aloud while working. This framework illustrated the trends of the students' ability to think out loud instrumentally versus relationally. This framework was most appropriate to analyze the thought processes of students as they thought aloud because it provided a means to discuss the competence of students as they thought by themselves. It also gave insight into students' capability to identify and correct mistakes while working, as well as their ability to conceptually explain concepts.

Lastly, the individual think-aloud results were compared to that of the group interview. Students' influences on each other when in a group setting were analyzed based on Vygotsky's (1978) concept of ZPD. The ZPD is the difference in a students' ability to work independently by themselves and the ability to work under guidance by others. This framework was important in examining how the influence of peers can impact the understanding of students as it provides a method of describing how a students' explanation of a correct response during the group interview in turn can influence the other student to change their answer.

Along with the three frameworks mentioned used for data analysis, a phenomenological methodology and detailed coding were used to analyze and interpret the data. Coding categories were similar to those used by Litzinger et al (2010). Litzinger et al. coded videotapes for instances of self-explanation and the use of meta-cognitive control processes. Self-explanation is classified as either principle-based or anticipative. Principle-based self-explanation refers to a students' explanation of how a specific concept applies to part of the problem. For example, a student knowing to use the concept of derivative to find the maximum when not explicitly told to do so would be coded as principle-based self-explanation. Anticipative self-explanation refers to a students' explanation of how a current step in the problem solving process will affect future steps. For example, if a student reorders the powers of a quadratic algebraic expression to put them in descending order because it seems to make it easier for future steps. Meta-cognitive control processes are classified as either monitoring or evaluation. Monitoring refers to students' explanation and awareness, either verbal or nonverbal, that he or she is experiencing difficulty. Metacognitive monitoring includes error detection, confusion, and reworking an earlier portion of the problem. For example, long pauses or expressions such as "I'm not sure" or "I don't understand". Metacognitive evaluation refers to a student checking whether the final answer is correct (Litzinger et al., 2010).

3.5 Ethical Considerations and Trustworthiness

For this study, since I was an instructor of *Calculus of One Variable I*, I did not lead or conduct any group interview session. This avoided any bias from me, and eliminated any hesitancy from the students. Trustworthiness was addressed by credibility, transferability, dependability, and confirmability (Lincoln & Guba, 1985). Credibility was addressed by triangulation. Students' written work, think-aloud sessions, and follow up interviews were three different methods that data was collected from the subjects. Credibility and confirmablity were addressed by peer debriefing from Dr. Vidakovic, who is an expert in the field of collegiate mathematics. She is also an expert on APOS theory, and has published in conjunction with Ed Dubinsky, who is the founder of APOS theory (Dubinsky, 1991; Cottrill et al., 1996; Czarnocha, Dubinsky, Loch, Prabhu, & Vidakovic, 2001; Czarnocha, Dubinsky, Prabhu, & Vidakovic, 1999; Hagelgans et al., 1995). Transferability was addressed by purposeful sampling and detailed description about the data collection, participants, and data analysis. Lastly, dependability was met by keeping a reflexive journal on self-awareness related to research methods and personal biases through the data collection process.

4 RESULTS

Chapter 4 presents the results in several sections which identify different categories and trends that arose during data analysis. Each section gives insight to the purpose and different research questions. There are three major sections of results to help organize the information. The first section is student performance on specific tasks. There were several trends in how students answered certain questions or tasks when relating the vertex to the derivative. Common obstacles were noted as well as similar responses that were grouped and analyzed based on APOS (Asiala, et al., 1996).

The second section looks at the results of the think-aloud method in particular. The transcripts of these students talking out loud by themselves gave a wealth of information and insight into each student's thoughts as they worked through problems in general, and in particular, insight into each student's thought process of the two problems they were given connecting the vertex of a quadratic function to the concept of derivative. There were clear tendencies in the students' performances when thinking out loud and how the students were able to incorrectly or correctly work through the problems. Skemp's (1976) relational understanding versus instrumental understanding framework is used in this section to describe students' ability to work through the problems.

Lastly, in the third section, comparisons are made of responses of students during the think-aloud session to their responses during the group interview. Changes in thought and influences group members had on each other are noted. Vygotsky's (1978) ZPD is used in this section as the basis for a theoretical framework identifying and analyzing influences and pattern in thought when in a group setting.

4.1 Student Performance on Tasks Relating the Vertex to the Derivative

There were several trends and reoccurrences in the way students performed on certain tasks. Based on these tendencies, categories were created. These included the relationship between the personal meaning of the vertex and its impact on the understanding of the derivative, students' inability to carry the concept of vertex from an explicit problem to a real world problem, and obstacles to student understanding of the real world problems. These categories give depth and meaning to how the understanding of the vertex of a quadratic function shapes the understanding of the derivative, students' perception and understanding of the relationship between the vertex of a quadratic function and the derivative in an explicit and real world problem, and obstacles students may have when using the concept of vertex of quadratic functions in relation to the derivative in world problems. The responses of students' performance were analyzed based on the action-process-object level of understanding according to APOS theory (Asiala et al., 1996).

APOS theory is an extension of Piaget's ideas of reflective abstraction, and is based on an action-process-object-schema understanding of mathematical knowledge. An *action* is an externally driven transformation of a mathematical object (or objects). For an individual, an action could be recalling a fact from memory, or in another case, needing an example or step by step algorithm to follow. For example, when dealing with the quadratic function, an individual would need the formula $x = -\frac{b}{2a}$ in order to calculate the vertex. Actions are the first step toward understanding a mathematical concept.

Once the actions have been interiorized and reflected upon, then it becomes a *process*. A process, unlike an action, is internally driven. At a process level, an individual is able to reflect and describe all the steps in a transformation, without having to actually perform them. Once a process is constructed by an individual the process can be reversed, coordinated or linked. For example, the process in the example above would be that an individual now can explain how to find the vertex of any quadratic function, without the need of an explicit formula to follow or the need of a specific example to find it.

When the individual then reflects on the actions applied to a particular process, becomes aware of the process as a totality, realizes that transformations can act on it, and can actually construct these transformations, then the process has been encapsulated into an *object*. Sometimes it is important and often necessary to de-encapsulate the object back to the process from which it came. An individual that is capable of explaining the position of a vertex of a parabola when certain transformations are applied is at an object level of understanding for that particular concept. For example, if a horizontal shift is applied, the vertex will move to the right or left along the line that is parallel to the x-axis. In this case, we can say that in order to even conceive this problem, a student needs to think of a vertex as an object on which a horizontal transformation is applied. Then, to actually perform this transformation, he/she would need to de-encapsulate this object into a process and explain, for example, that in this case the y-coordinate of a new vertex will stay the same, while the x-coordinate will change to reflect the shift to the left or right along the x-axis.

A *schema* is formed by the collection of actions, processes, objects, and other schemas, along with the linkage between them brought to a mathematical problem situation. These schemas evolve and are constructed and reconstructed as new relations between actions, processes, and object conceptions form (Asiala et al., 1996.; Cottrill et al., 1996). Each student's ability was coded based on whether they moved step by step through problems with no conceptual understanding (action), could explain in words the meaning behind these steps (process), or could create linkage between concepts and were aware of the process as a totality (object).

4.1.1. Personal Meaning of the Vertex and its Impact on the Understanding of the Derivative

For the first part of the results, student performance is given through the lens of their corresponding personal meaning of the vertex. Several misconceptions of the vertex arose, which appeared to possibly have negatively impacted the ability of some students to be able to conceptually perform and explain its relationship to the derivative. However, for others, the misconceptions did not have an effect on the ability to complete the problems correctly, a sign of action level of understanding according to APOS. Besides the misconceptions, there was also a strong association of personal meaning of the vertex as a maximum or minimum, as well as some students relating the vertex to symmetry. Those who had a stronger connection to the correct meaning of the vertex appeared to have a stronger understanding of the derivative, and a better ability to perform on the questions pertaining to finding the maximum of a quadratic function. These students who could conceptually explain and talk about the meaning behind the concepts appeared to have a process level of understanding according to APOS, and some even possibly an object level of understanding.

4.1.1.1 Personal Meaning: Misconceptions of the Vertex

There were many misconceptions of the vertex that were apparent in the personal meaning given by the students on the in class questionnaire. In fact, twenty three of the thirty students' written work that was analyzed contained some type of inaccurate response for one of the seven questions on the questionnaire. Only seven students correctly answered all questions. However, despite these misconceptions and misunderstandings, during the think-aloud session, many were still able to procedurally compute and find the maximum for the quadratic function in Question 1 by use of the derivative. Nonetheless, some of the misconceptions did hinder the ability of students to carry this concept of the vertex relating to the derivative over to the real world problem in Question 2 during the think-aloud session.

4.1.1.1.1 Misconception 1: Vertex as Intercept

The most frequent misconception of students responses on the in class questionnaire was the idea of the vertex as an intercept. A majority of participants somewhere in their work insinuated that the vertex is the same as an intercept. One of the common occurrences was that during the think-aloud session, those whose personal meaning was that the vertex is an intercept were able to correctly find the derivative, critical point and extreme value, and intervals of increase and decrease step by step for Question 1, part c, however for Question 2, instead of following the same method to find when the rocket reached maximum height, they would instead find zeros of the original free fall function, $f(t) = 0$. This was a frequent occurrence throughout the data analysis, which could have been due to the erroneous personal meaning of the vertex as an intercept and inadequate pre-requisite knowledge of the quadratic function.

One of the several students who followed this pattern was Nick. Nick's personal meaning was that "the vertex of a quadratic function is wherever $x = 0$. Another name for a vertex is x-intercept."

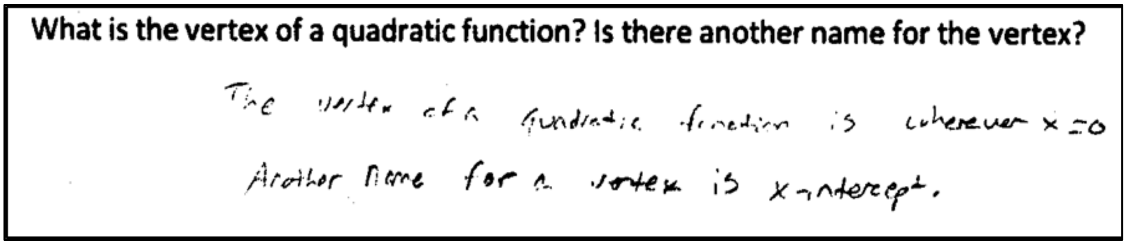


Figure 4.1 Nick's misconception of the vertex

Nick also wrote that you can find the vertex by "solving for the x-intercepts." When asked if there is more than one way to find the vertex he responded that "yes, by solving for the y-intercepts as well."

Although Nick struggled with the conceptual meaning of vertex, he was able to correctly find the maximum using the derivative on Question 1 during the think-aloud session. His only mistake was an arithmetic error made when computing the y-coordinate of the maximum.

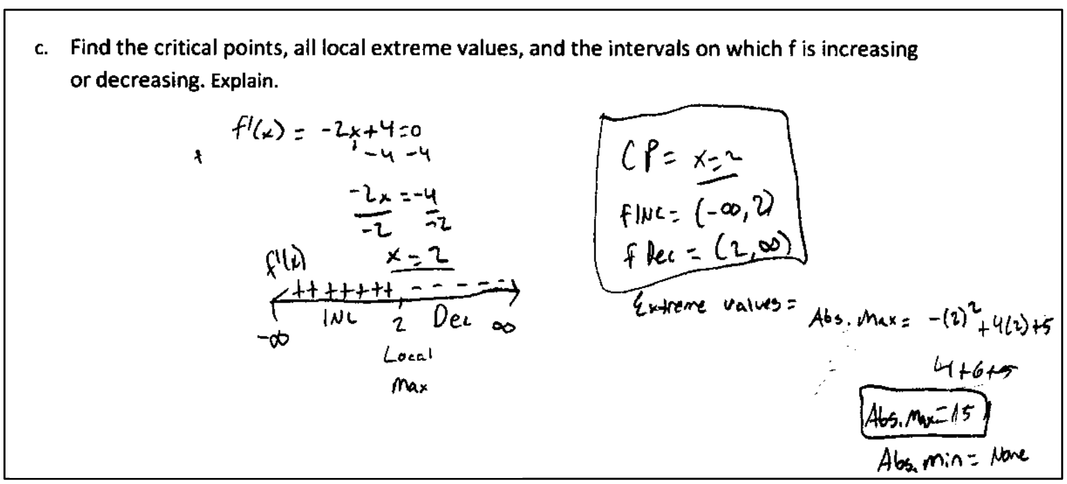
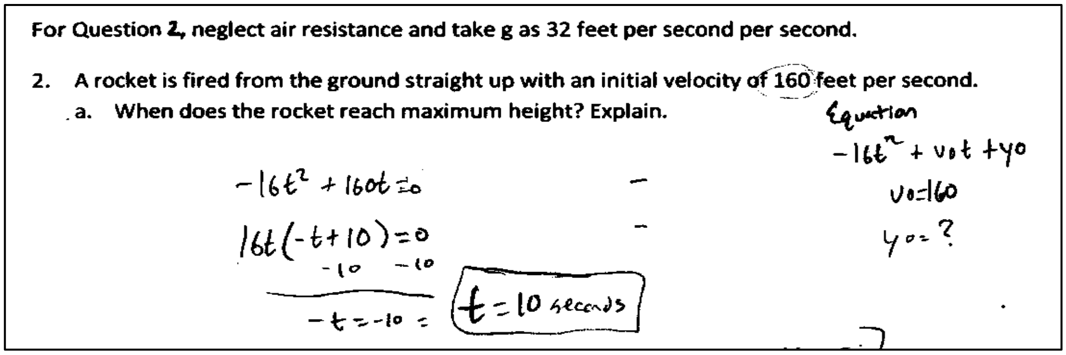


Figure 4.2 Nick's solution during think-aloud, Question 1

Nick did, however, get stuck on Question 2, as he tried to find the time the rocket reached maximum height by solving for the zeros from the original free fall function, rather than taking the derivative. This is consistent with his personal meaning of the vertex.



For Question 2, neglect air resistance and take g as 32 feet per second per second.

2. A rocket is fired from the ground straight up with an initial velocity of 160 feet per second.
   a. When does the rocket reach maximum height? Explain.

Equation
$-16t^2 + v_0t + y_0$
$v_0 = 160$
$y_0 = ?$

$-16t^2 + 160t = 0$
$16t(-t+10) = 0$
$-10 \quad -10$
$-t = -10$
$\boxed{t = 10 \text{ seconds}}$

Figure 4.3 Nick’s solution during think-aloud, Question 2

Although the concept is the same, just presented in a different way, Nick was not able to carry the idea over into the real world problem. This is confirmation that he might be at most at the action level of understanding the critical points, local extreme values, and intervals of increase and decrease for the function f.

N: Oh you’re trying to find um, the time it reaches the maximum height so all you gotta do is $-16t^2 + 160t$ and you um, solve for t, so set that equal to 0.

Nick continued to think out loud as he went through the steps of solving for t to find the time when the maximum height occurred. Nick’s misconception of vertex may have kept him from moving forward past the action level into a more conceptual understanding of the vertex in relation to the derivative in different situations. This same pattern occurred for him on his solutions to Test 3’s questions. He was able to answer the explicit question correctly, yet could not figure out the real world problem. The ability to accurately answer Question 1 without conceptual meaning suggests he has an action level of understanding of how to find critical points when explicitly asked.

While some students’ personal meaning of the vertex as an intercept possibly interfered with their ability to answer Question 2, the real world problem, other students’ personal meaning of the vertex as an intercept interfered with their ability to graph the quadratic function. Joey, the other member in Nick's group, during his think-aloud session, for Question 1, part d, was supposed to sketch the graph of $f(x) = -x^2 + 4x + 5$. While sketching the graph of the quadratic function, Joey said that "if I plug in a zero for every x in the original function that will give me the y-intercept, which is 5. So, $(0, 5)$ is the vertex of the parabola." The point $(0, 5)$ is the y-intercept of the graph, it is not also the vertex. Joey's misconception of the vertex as the y-intercept prevented him from being able to correctly sketch the graph.

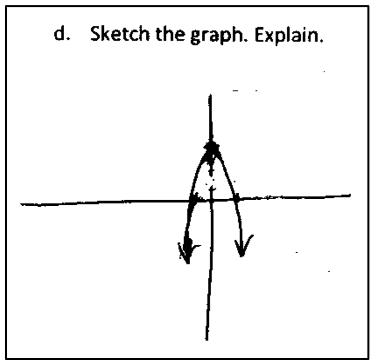


Figure 4.4 Joey's misconception of the vertex

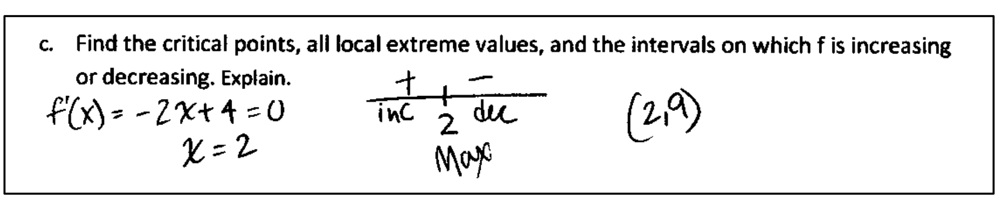
Joey was, however, still capable of finding $x = 2$ as the critical value by solving for x in $f'(x) = 0$. He even wrote next to $x = 2$ that it was a local extreme max, yet this value did not appear on his graph.

Joey was able to follow steps very procedurally and correctly to calculate the critical value and intervals of increase and decrease using the derivative. However, Joey's misconception of the vertex kept him from being able to accurately sketch the graph for the quadratic function. He could not relate the two concepts of the vertex of a quadratic function and the derivative in relation to finding the extreme value of a function. This suggests that Joey may be at an action level for finding critical points, extreme values, and intervals of increase and decrease for the explicit question, part c of Question 1. He could come up with the right answer; however, there was no conceptual understanding behind his responses, as in his work he provided only algorithmic work without any verbal explanation. This might be an indication that he did not have any meaning behind procedures that he most likely memorized.

When both Nick and Joey were in the group interview session, the interviewer asked them if they could have found the maximum of the quadratic function without knowing the derivative, and neither believed at first that one could. The interviewer continued prompting both of them several minutes until they finally came up with another way of finding the maximum by using the property of symmetry of the parabola. Even then, however, Nick and Joey did not mention the notion of finding the vertex as another way. This could be because neither one of them had a correct understanding of the vertex in the first place. By being stuck on an action level of understanding, they were not able to relate and connect between concepts and ideas.

Sara, like Nick and Joey, had a personal meaning and misconception of the vertex as an intercept. On the in class questionnaire, when asked if there is more than one way to find the vertex, she wrote "by plugging in zero in place of x." Plugging in 0 in place of x finds the y-intercept, not the vertex. This inaccurate perception was carried over into her graph for Question 1 during her think-aloud session as her graph was almost identical to Joeys (see Figure 4.4).

Sara also struggled with what it meant to find the critical points and all local extreme values. She interpreted finding the critical points as a completely different process then finding the extreme values. For finding the critical point, Sara first solved for x in $f'(x) = 0$, and came up with a maximum at the value $x = 2$, with the corresponding point $(2, 9)$.


c. Find the critical points, all local extreme values, and the intervals on which f is increasing or decreasing. Explain.

$f'(x) = -2x + 4 = 0$

$x = 2$

+ −

inc 2 dec

Max

(2,9)

Figure 4.5 Sara's solution during think-aloud, finding the critical point

Then, after finding the critical point, it was evident during her think-aloud session that she was stumped on how to find the extreme value, and could not relate the concepts.

S: The next part is to find the local extreme values. Hmmm. Local extreme values, so it would be to find the max or min. To find the max, to find the minimum and max value so that would be hm, hmm [Long Pause] Hmmm.

This debate and uncertainty with herself was after the fact that she had already found the critical point. Clearly, there was no connection for her between the concept of critical point and the concept of an extreme value. She decided to move on to Question 2, and then went back to find what she believed to be local extreme values.

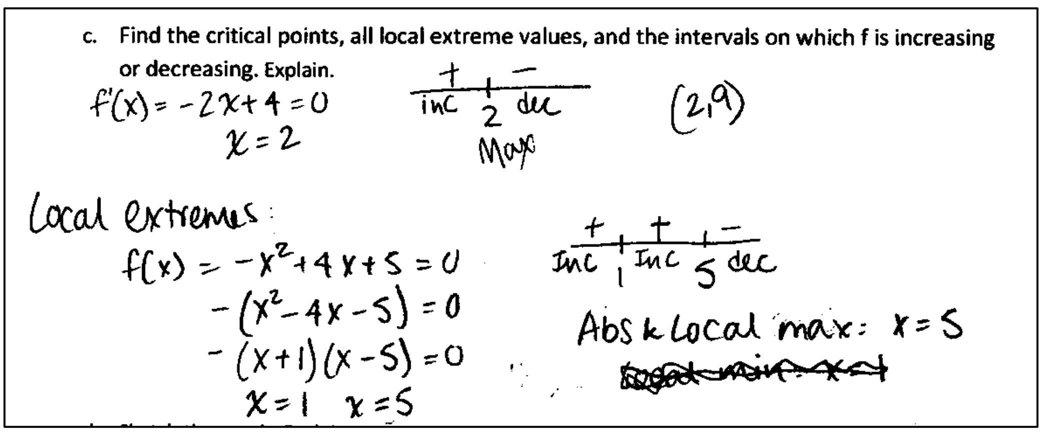


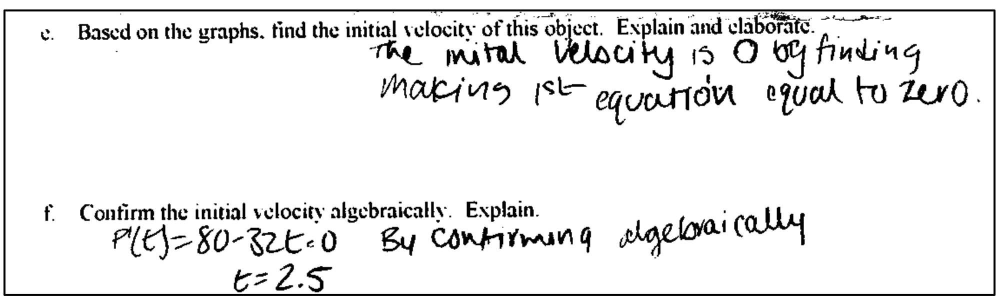
Figure 4.6 Sara's solution during think-aloud, finding local extreme values

To find what she believed were the local extreme values, Sara tried to solve for the zeros of the original quadratic function.

S: So the local min would be 5, $x = 5$. Local max would be $x = 1$? 1? I'm not sure if hmmm ok so absolute min hmm so local and absolute max would be at $x = 1$, local min would be $x = 5$.

Sara did not even find one of the zeros of the quadratic function correctly, as she concluded that it was $x = 1$, not $x = -1$. She did find a maximum point of $(2, 9)$, however, she also claimed there was an absolute maximum at $x = 1$, and a local minimum at $x = 5$. For her, zeros of a function are the same as local extreme values. She even made a number line with the x-values of 1 and 5 and tested points in between. She did not even notice any contradiction when she tested the sign of the function with respect to the zeros. That is, at 1, she had that the function did not change a sign, while on the other number line, at 2, it did. This is impossible for a quadratic function. This is all indication that in what she did correctly was at most at the action level, or even pre-action level of understanding the concept of the derivative to find maximums and minimums, as well as the concept of the quadratic function. Her work clearly indicates confusion about the concepts involved and their relationships.

On the Final Exam, it did not come as a surprise when her answers had no conceptual meaning and she could not even find the initial velocity correctly.



e. Based on the graphs, find the initial velocity of this object. Explain and elaborate.

The initial Velocity is 0 by finding making 1st equation equal to zero.

f. Confirm the initial velocity algebraically. Explain.

$P'(t) = 80 - 32t = 0$ By confirming algebraically

$t = 2.5$

Figure 4.7 Sara’s misconception on the Final Exam

Even further, during the group interview, she struggled with conceptual meaning, in particular of the derivative. When asked to explain what is the meaning of the derivative in the given example, she responded “I have no idea…I mean I just know how to do it, I didn’t think I would knowww…I have no idea.”

A few minutes later, when asked by the interviewer why we set the first derivative equal to 0 to find critical values, she again responded with an “Mmm no idea.” Her struggles once more suggest she is not even at an action level of understanding as she could not correctly identify the maximum of the quadratic function, nor could she accurately sketch the graph. She had no connection between the concept of vertex and the derivative. Her inaccuracies pertaining to the vertex may have hindered her ability to relate the two concepts, as well as her inability to describe what the derivative means.

4.1.1.1.2 Misconception 2: Vertex as the Origin

A few students' personal meanings associated the vertex as the origin. One of these, Jerry, had a personal meaning that the vertex of a quadratic function is the "origin." Even further, he claimed that to find the vertex you "set $x = 0$", another misconception, as setting $x = 0$ finds the y-intercept. Jerry could not conceptually relate the vertex to the derivative other than the fact that "the derivative of a quadratic function can be determined" and he also did not believe that there is more than one way to find the vertex. Jerry's misconceptions could have prevented him from being able to correctly answer the real world problem, Question 2, during his think-aloud session. He struggled with the free fall formula and solved the equation $-64t + 160 = 0$ for the variable t for both finding when the rocket reached maximum height (part a) and for finding when the rocket hit the ground (part c). He used the same procedure for two questions asking two completely different things. When prompted by the interviewer to talk about what he did to find when the rocket hit the ground, he replied:

J: Ok, so this is where everything got like really confusing….I was gonna set um either the derivative or y' or um the actual equation to 0 and solve for it, but um, I don't know, I just got really really confused…I just took the derivative and set it equal to 0 and then I got $\frac{5}{2}$ for seconds… and then I put $\frac{5}{2}$ into the um original.

Jerry's mistake of $\frac{5}{2}$ seconds for the time the rocket reached the ground also created graphing trouble. When he sketched the graph, he made his t-intercepts at $t = 0$ and at $t = \frac{5}{2}$, yet the time he had found for when the rocket reached its maximum height was at $t = 5$ seconds. This is not possible with the way he drew his graph.

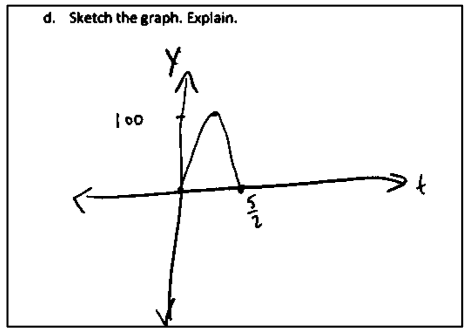


Figure 4.8 Jerry's graphing mistake during think-aloud, Question 2

Jerry did, however, correctly find the critical points, local extreme values, and intervals of increase and decrease for part c of Question 1 during the think-aloud session, except for an arithmetic error in finding the corresponding y-coordinate for the maximum point. This suggests that Jerry is at an action level of understanding for part c of Question 1 as he could perform, yet this action level possibly keeps him from being able to apply the same concept in a different situation.

Although he proceeded through part c of Question 1 during the think-aloud session for the most part correctly, Jerry struggled on the same type question on Test 3 and the Final Exam questions. He was the only one to get all parts of both questions wrong on Test 3, including graphing the wrong shape of the quadratic function.

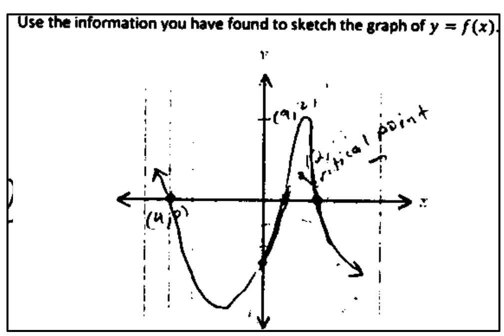
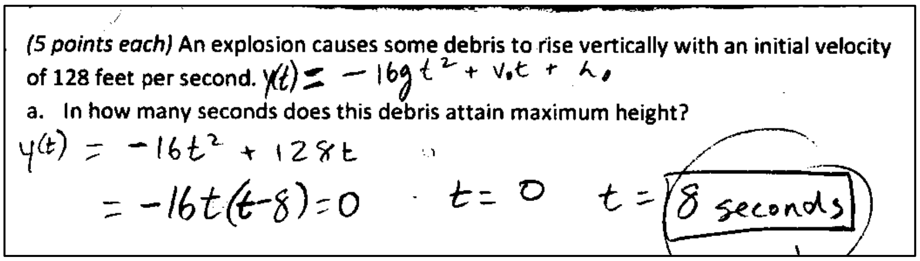


Figure 4.9 Jerry's graphing mistake, Test 3

Jerry's inability to graph the quadratic function suggests that he has a weak schema of quadratic functions and their corresponding graph. Furthermore, for the second question on Test 3, to find when the debris reached its maximum height, Jerry again resorted to solving for the variable t from the free fall formula set equal to zero.



(5 points each) An explosion causes some debris to rise vertically with an initial velocity of 128 feet per second. $y(t) = -16g\,t^2 + v_0 t + h_0$

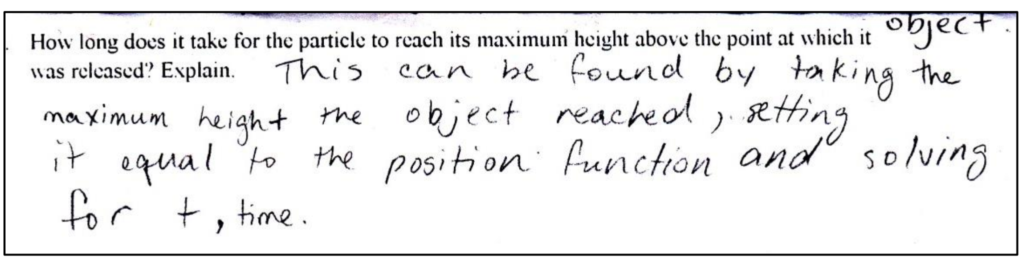
a. In how many seconds does this debris attain maximum height?

$y(t) = -16t^2 + 128t$

$= -16t(t-8) = 0 \quad t = 0 \quad t = \boxed{8 \text{ seconds}}$

Figure 4.10 Jerry's solution to the real world problem, Test 3

Not only were his solutions for Test 3 inaccurate, but on his Final Exam, he also had some erroneous responses, especially to finding the maximum of the particle. His confusing and unclear response is illustrative of the confusion and misconception he had about the maximum of the particle.

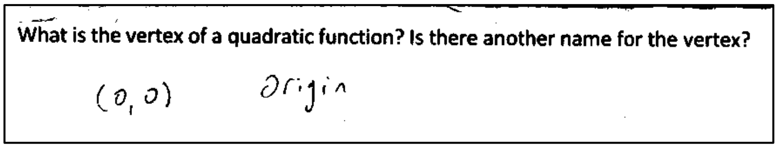


How long does it take for the particle to reach its maximum height above the point at which it was released? Explain. object. This can be found by taking the maximum height the object reached, setting it equal to the position function and solving for t, time.

Figure 4.11 Jerry's Final Exam response

Jerry does not even appear to be at an action level, other than his correct response to Question 1 part c during the think-aloud, and even then he made an arithmetic mistake on calculating the y-coordinate of the maximum. Jerry's misconceptions relate all the way back to his misinterpretation of his personal meaning of the vertex of a quadratic function. His weak schema of quadratic functions possibly disables him from being able to accurately work through problems relating the vertex to the derivative in different context problems.

Nathan was another of the several students who associated the vertex as the origin.

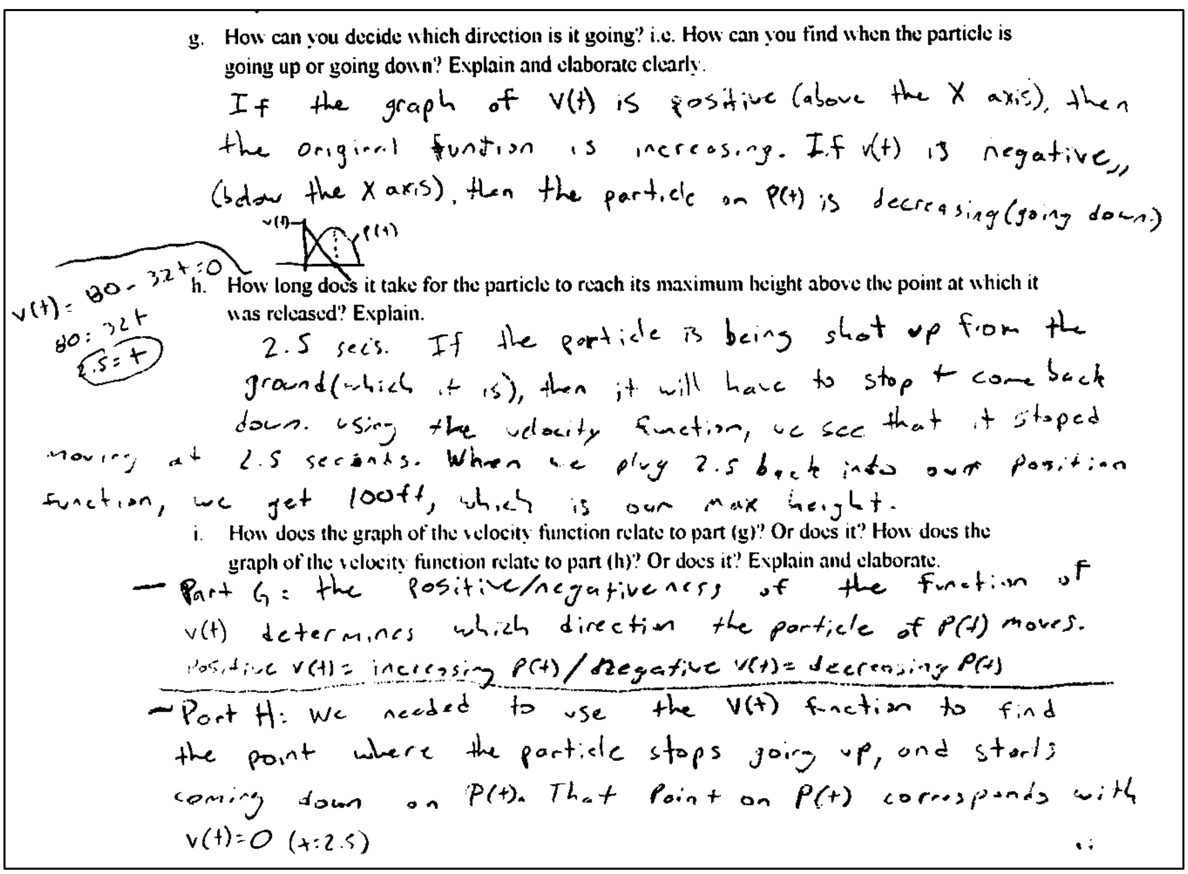


What is the vertex of a quadratic function? Is there another name for the vertex?

(0, 0) origin

Figure 4.12 Nathan's misconception of the vertex

When asked to explain what a quadratic function is, Nathan wrote the quadratic formula. These two misconceptions, however, did not impact his ability to perform on Test 3, the think-aloud questions, or the Final Exam questions. Nathan correctly answered both questions during the think-aloud session and both questions on Test 3. Furthermore, Nathan had very strong conceptual responses on the Final Exam questions. In fact, on the final exam, Nathan had some of the best conceptual responses out of all thirty participants, especially on parts g through i.



g. How can you decide which direction is it going? i.e. How can you find when the particle is going up or going down? Explain and elaborate clearly.

If the graph of V(t) is positive (above the X axis), then the original funtion is increasing. If v(t) is negative, (below the X axis), then the particle on P(t) is decreasing (going down.)

$v(t) = 80 - 32t = 0$
$80 = 32t$
$2.5 = t$

h. How long does it take for the particle to reach its maximum height above the point at which it was released? Explain.

2.5 secs. If the particle is being shot up from the ground (which it is), then it will have to stop + come back down. Using the velocity function, we see that it stoped moving at 2.5 seconds. When we plug 2.5 back into our position function, we get 100ft, which is our max height.

i. How does the graph of the velocity function relate to part (g)? Or does it? How does the graph of the velocity function relate to part (h)? Or does it? Explain and elaborate.

— Part G: the positive/negativeness of the function of v(t) determines which direction the particle of P(t) moves.
Positive v(t) = increasing P(t) / Negative v(t) = decreasing P(t)

— Part H: We needed to use the velocity function to find the point where the particle stops going up, and starts coming down on P(t). That point on P(t) corresponds with v(t) = 0 (t = 2.5)

Figure 4.13 Nathan's Final Exam responses

During Nathan's group interview, when both participants were asked to explain the meaning of the derivative, Nathan was even able to connect concepts and relate the graph of the quadratic function to the graph of its derivative.

N: When on the derivative, when it's above the x-intercepts, 0, then on the original function it's increasing and when it's below its decreasing.

I: Ok, so you are showing me with your pen. So, it ah yeah, so if the derivative function is above the x-axis you're saying the original function is increasing.

N: is increasing

I: Ok, and when the derivative is below the x-axis, then that means-

N: -decreasing

A few minutes later, when discussing responses to the real world problem, Question 2, Nathan was also able to explain within the context of the problem why we set the derivative equal to 0 in order to find the maximum point of a function.

N: That point where it changes from increasing to decreasing at x-value 5, $t = 5$, that's where like on the left is going up and on the right of it it's going down so it's changing.

I: Uh huh and what happens to the rocket at that point?

N: It comes down, like I don't know, what do you mean? It goes up then it's slowing down and then it stops then it comes back down.

I: Ok, so you are saying it stops at $t = 5$?

N: Yeah, yeah it stops at $t = 5$.

Even though Nathan had two initial misconceptions of the quadratic function and the vertex, this did not impact his conceptual understanding of the derivative or his ability to perform on either problem during the think-aloud session or on Test 3. His ability to explain and articulate the meanings behind the derivative concepts suggests he may be at least at a process level of understanding of the derivative.

4.1.1.1.3 Misconception 3: Vertex as Point of Inflection

The last misconception of personal meaning identified was that of the vertex as the point of inflection. According to Salas et al. (2007), the definition of a point of inflection is as follows:

Let f be a function continuous at c and differentiable near c. The point $(c, f(c))$ is called a point of inflection if there exists a $\delta > 0$ such that the graph of f is concave in one sense on $(c - \delta, c)$ and concave in the opposite sense on $(c, c + \delta)$.

Figure 4.14 Definition of point of inflection (Salas et al., 2007, p.191)

Identifying a point of inflection is different than identifying a critical point. Whereas with a critical point, the first derivative can be used to find the value, with a point of inflection, it is the second derivative.

If the point $(c, f(c))$ is a point of inflection, then $f''(c) = 0$ or $f''(c)$ does not exist.

Figure 4.15 Theorem 4.6.4 point of inflection (Sala et al., 2007, p.192)

Despite the differences in these concepts, a few students' personal meaning related the vertex to the point of inflection.

Adam was one of these. His personal meaning was that the vertex is "$-\frac{b}{2a}$, [a] point of inflection." While $-\frac{b}{2a}$ is the correct formula for finding the x-coordinate of the vertex, he added another name for this is a point of inflection. Adam had the ability to perform on certain question parts, however, his misconceptions became more evident when talking during the think-aloud session about the shape of the quadratic function. During the think-aloud session, Adam said that the shape of the graph of a quadratic function is "not a parabola but hmmm I'm gonna draw the shape of the graph, it's more like a large squiggly line more than likely." He then drew what he believed to be the shape.

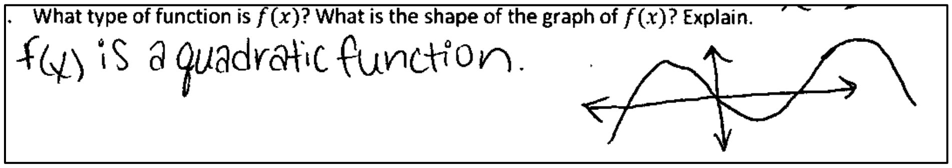


Figure 4.16 Adam's misconception of the graph of a quadratic function

While Adam did identify the function correctly as a quadratic function, he had a total erroneous idea about the shape of the graph.

Adam's responses to the in class questionnaire further supported his inaccurate conceptual understanding of the quadratic function and vertex. When asked to give an example of an algebraic expression of a quadratic function he wrote "$y = 4a + 2b + 3c$", followed by "$4x^2 + 3$." While the first expression is not a quadratic function, the second is. This suggests he has different meanings associated with a quadratic function. When asked if the vertex relates to the derivative he wrote that "the graph of the derivative runs through the vertex and does not exist at that point." Conceptually, this makes no sense. Even further, he wrote that another way to find the vertex is to "look at the graph and look at the point where the derivative does not exist." Again, he has a complete misconception and inability to relate between the vertex and the derivative.

Ironically though, Adam was able to correctly find the critical point and intervals of increase and decrease for Question 1, part c, and he answered all of Question 2 correctly during his think-aloud session. One would think that without conceptually knowing what a certain concept means, that they would in turn not be able to accurately solve problems pertaining to it, however, this was not the case. Because of Adam's ability to find critical points, intervals of increase and decrease, and even because of his ability to work through the real world problem of

Question 2 correctly, this suggests that he has memorized how to work these problems and is at a total action level of understanding. According to APOS, the concepts have not been interiorized, and he is working purely off of step by step procedure. This memorization was even more evident as he described what he believed to be the meaning of the derivative:

I: Ok, um what do you mean by instantaneous rate of change?

A: Um, when I had high school calculus, it was just my teacher would be like when you have y' or $f'(x)$, just always remember, I might say derivative, I might say instantaneous rate of change, remember it's always uh, something going on or something is changing at some point like nothing unless it's like a horizontal slope of 0, you know something is always going on you know it's like something is always changing, just remember it like that.

I: Ok, is there a difference between rate of change and instantaneous rate of change?

A: Hm, I would like-that like I knew what's the difference I really really don't even have a clue to be truthful.

Adam based his responses off of memorizing what a high school teacher had told him, and when prompted, admitting to having no clue about the meaning. His ability to perform procedurally at the action level was also apparent in his capability to correctly answer both questions on Test 3, and his inability to form conceptual meaning was also apparent on his final exam responses.

Cade was another student who kept referring to the point of inflection as the vertex. Initially, he wrote his personal meaning of the vertex as an inflection point and critical point, two completely different concepts. Then, during the think-aloud session, while Cade was able to correctly and procedurally find the critical point, intervals of increase and decrease, and sketch of the graph, he still had an association of a critical point as an inflection point, which was evident as he spoke his thoughts out loud. “It changes point of inflection at 2 cause that’s the critical point when x is 2.” Again, this misconception of the vertex as a critical point and a point of inflection prevented him from being able to use the correct terminology to conceptually talk about the concepts.

While Cade performed on Question 1, he was not able to carry this concept over to the real world problem in Question 2, as he resorted to solving for the variable t in his free fall formula set equal to 0. He exhibited a lack of confidence throughout Question 2, as he made remarks such as “oh gosh, I’m not good at this problem” and “uh, if we had uh I don’t know, I’m not too sure.” Cade lacked the connection between problems as he could not relate that Question 2 was also asking for the critical point, just in a word problem. Cade also appeared to be missing any conceptual meaning of the derivative as well, exhibited in the group interview.

C: And the derivative means the slope of $f(x)$ so that's, that's just the derivative and it gives us the rate of change I'm pretty sure and graphically it's the tangent of the line.

When probed about what he meant by “tangent of the line”, Cade at first said that the derivative, $f'(x) = -2x + 4$ is the tangent line. When probed further about how this could be true he was stumped.

I: Ok so how, how are those tangents connected to this $-2x + 4$?

C: Uhhh I don't know I thought all the tangents together I don't know make that or something.

I: Ok let me rephrase my question, so you said that, how did you connect the derivative with the tangent line, how did you connect, what did you say?

C: I said graphically, the derivative is the tangent to the line, and I don't know how it's connected, I don't know.

Cade still could not conceptually give meaning to what he meant by “tangent to the line.” He also struggled with connecting meaning from the graph of the original function to the graph of the derivative. He did not think there was a connection between the original function and the derivative, which could be influenced by his weak conceptual understanding of the derivative.

On his Final Exam questions, Cade again associated the vertex with the point of inflection. On part h, which asked how long it took for the particle to reach maximum height, Cade’s response was incorrect. He wrote that “It takes one second for the particle to reach maximum height, $P(1)$ gives the highest possible value. Looking at a graph makes it even more obvious as it’s an point of inflection.”

Not only was his time for when the rocket reached maximum height wrong, but he was still incorrectly associating the maximum point as being a point of inflection. Thus, Cade’s weak personal meaning of the vertex could possibly have influenced his inability to work through the real world problems, and his inability to express any conceptual meaning to the derivative. Even more, Cade never once mentioned the term quadratic function as the type of function he was working with. He always associated it as just a polynomial. But, because of his ability to find critical points and intervals of increase and decrease on the explicit Question 1 during the think-aloud, and his ability to perform on the explicit Question 1 on Test 3, he was capable of an action level of understanding for that type of explicit problem. For the real world problems, however, he is not even at an action level as he could not correctly progress through any parts of the question. His misunderstanding of the vertex of the quadratic function could be hindering his capability of understanding the derivative.

4.1.1.2 Personal Meaning: Vertex as a Maximum or Minimum

Despite all the misconceptions of the vertex that many students carried with them conceptually, there were some students who had a very strong conceptual understanding of not only the vertex, but also its relationship to the derivative. These students had no trouble corresponding between different types of problems, and no difficulty in explaining or describing what different concepts mean. They portrayed more of a process level of understanding of the vertex of a quadratic function and the derivative, and some even possibly an object level of understanding of the vertex of a quadratic function and the derivative as they were able to perform at a high level of ability and made connections between concepts.

Henry was one of those students. He was pretty much flawless in all his answers and responses. On his in class worksheet, he exhibited strong connections between the vertex and the derivative. His personal meaning of the vertex was that it is "the point at which the function changes from inc [increasing] to dec [decreasing] or vice versa. AKA the critical point." He actually related the meaning of vertex to the term critical point, a term associated with the derivative. He went on to mention that "the vertex is the point of which the derivative equals zero." He was able to make a connection between the two concepts.

Not only were his responses on the in class questionnaire accurate, everything he worked on during his think-aloud session was correct, as well as both Test 3 questions and even his Final Exam questions. All of this proved his ability to perform, yet, it was his ability to conceptually answer and the ability to think-aloud more conceptually that separated him from others. During his think-aloud session, Henry did not just give a procedural answer for finding the critical point, as most others did, but he gave a reason why.

H: What I'm going to do is take $f'(x)$, which I found in the last problem and set it equal to 0 because that is how you find the critical points, it's where there maximums and minimums when the slope [of the tangent line] is 0.

Even further, in his group interview, he explained about the meaning of the derivative:

H: I said it gives the slope of each value of x of $f(x)$. Graphically, it shows us the intervals of increasing and decreasing so I didn't use the term tangent line cause that just didn't come to mind, but that's what I was trying to say was that each $f'(x)$ at whatever point of x you plug in gives you the slope of the tangent line of $f(x)$.

When asked if there are any other ways to find the maximum point without using the derivative, Henry immediately suggested knowing the properties of quadratics, and even remembered the formula $x = -\frac{b}{2a}$ to find the x-coordinate of the vertex.

Henry's strong association and understanding of the quadratic function possibly enriched his schema to better understanding the concept of derivative and enhanced his ability to relate the concept of vertex to the derivative. Although Henry could perform to perfection, it was his conceptual explanations and meanings behind what he was doing that puts him at least at a process level of understanding of the vertex of a quadratic function and the derivative. Even more, because of his ability to connect ideas between the two concepts and the two questions to see the process as a totality, Henry could maybe even be at an object level of understanding of the vertex, quadratic function, and the derivative.

Anna, like Henry, exhibited a higher level of understanding than the action level. Anna described the vertex as "where the two sides meet of the graph. Another name for the vertex is minimum or maximum (based on the concavity of the graph)." She also remembered the formula $x = -\frac{b}{2a}$ for finding the x-coordinate of the vertex as another way other than the derivative.

Anna also appeared to have understanding of the meaning of the derivative. During the think-aloud session she said that the derivative "gives us the slope of the tangent line of x, of any point on the graph." During the group interview, she was able to explain why we set the derivative equal to 0 to find critical points by stating that "well the rate of change would be 0 like he said. It stops [referring to the rocket in Question 2] so there would be no change, so it would be 0, um, the tangent line is horizontal so it [the slope] has to be 0."

Anna also had correct answers to all parts of the think-aloud session, and answered both Test 3 questions correctly. However, when asked if the vertex relates to the derivative in any way, she did not think so. She wrote that "you can find the derivative of a quadratic function, but it's not directly related to derivatives." Even more, on this same questionnaire, when asked how to find the vertex, she did not mention being able to use the derivative. She instead wrote down the vertex formula. It was if she had a strong schema of quadratics and the concept of vertex, as well as a strong schema of the concept of derivative. During the group interview, however, when she was asked again if there is any other way to find the maximum besides the derivative, she immediately said "the vertex, $-\frac{b}{2a}$" and connected the two concepts. This suggests she may be at an object level of understanding of the vertex and the derivative as she was aware of the connection between them and of the process as a totality.

4.1.1.3 Personal Meaning: Vertex as Symmetry

The last subcategory of students' personal meaning of the vertex is its relationship to the concept of symmetry. Some students had this strong connection between their meaning of the vertex, its relationship to the symmetry of the graph, and how this relates to the derivative. Fay wrote that the "vertex of a quadratic function is the line that splits the function in half. It's also called the axis of symmetry." She even drew a picture of a concave up parabola with a dotted line going through the middle indicating the symmetry of the graph.

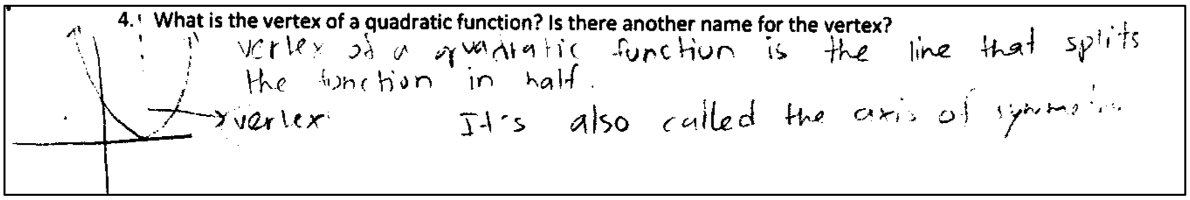


Figure 4.17 Fay's connection of the vertex to symmetry

Fay's personal meaning was that the vertex is strongly associated with the line of symmetry. She related this idea of the vertex as a line of symmetry to the derivative by writing that "the vertex goes through the middle of the curve x^2. So, wherever there is a vertex, the derivative at that point is $= 0$, so these might be local max/min at that point." She connected the concept of finding the vertex to that of finding where the derivative is equal to 0.

Fay was able to answer all of the explicit Question 1 during the think-aloud session correctly. She had the procedure correct to Question 2, however, she incorrectly took the derivative of the original free fall function, which caused problems for her as she worked through her step by step procedure. Fay also got both questions on Test 3 correct. Fay's ability to work through the different types of problems suggests she is at least at an action level of understanding of finding critical points, extreme values, and intervals of increase and decrease in an explicit and real world problem. Fay, however, was also able to explain and give meaning behind her steps, which was evident in the group interview.

F: So it's if um, the derivative is positive, that means the original function is increasing and so here-

I: And graphically that means?

F: That um, that it's above the x-axis, meaning it's positive.

I: the derivative

F: The derivative, and so the original function from that interval is increasing and at $x = 2$, since it's um, critical point, we have the horizontal slope so there's, there must be a local max or minimum, but in this case we have a max and then after that um, since the derivative is negative, meaning it's um, below the x-axis, so it means the, the curve um, $f(x)$ is um going down or decreasing.

When she was asked why we find the critical point by setting the first derivative equal to 0, she responded that "to find where the derivative crosses the x-axis, and whenever it crosses the x-axis, we know that the derivative at that point is 0, and since we know derivative, whenever the derivative's 0, the original function has horizontal slope, so there must be a maximum or minimum at that point." It is suggested by Fay's terminology of "horizontal slope" along with her other explanations that she understands it is the horizontal tangent line that has slope zero. She was able to explain why we set the derivative equal to 0 to find the maximum or minimum point of a function.

On the Final Exam, Fay also gave strong conceptual explanations. She related the graph of the position function $P(t)$ to the graph of its derivative, $P'(t)$ by writing that the "derivative of a function predicts how the original functions graph should look like. Whenever the derivative is negative, it means that the original function is decreasing at that x value." She continued to explain that "when the derivative $P'(t) = 0$, that means that there is a horizontal [tangent] line in the original function $P(t)$ (slope$= 0$). When the derivative $P(t)$ is positive number, the sign [value] of function is increasing."

Since Fay could not only compute, but could also explain, this suggests that not only is Fay at an action level of understanding, but that these actions have been interiorized and reflected upon, and thus she is at a process level of understanding. Her understanding is internally driven as she is able to reflect and describe her steps in the process.

Another student, Amy, associated the vertex with symmetry; however, unlike Fay, she had a weak connection of the concept and its relation to the derivative. She wrote that the vertex of a quadratic function is "where each side of the parabola meets." When asked explicitly how to find the vertex, she did not mention the derivate, but instead wrote the vertex form of a quadratic function. Even more, when asked if the vertex relates to the derivative in any way, she left it blank. This suggests that she does not connect between the concept of vertex of a quadratic function and the concept of derivative.

During the think-aloud session, however, on Question 1, part c, she correctly found the maximum point and the intervals of increase and decrease of the quadratic function using the derivative and she answered the entire real world Question 2 correctly. Being able to complete the steps suggests she may be at an action level of understanding, especially since she struggled to make connections between concepts, and could not give conceptual meaning behind her steps.

While Amy used the derivative to find the maximum for both questions during the think-aloud, she still could not connect the relationship between the vertex and the derivative. When prompted in the group interview, she could not think of any other way besides the derivative to find the maximum besides testing numbers, a guess and check strategy. Later in the group interview, when prompted again, she was still stumped. The term vertex did not even come to mind.

A: I mean, I guess to find the max height you could use the same whatever the formula is to find the origin of a parabola if I could remember what it is but-

I: The origin of a parabola? What do you mean by origin of a parabola?

A: The lowest point of the parabola, where the two sides come together, I don't- I'm not sure it- it would be called an origin or not.

Instead of being able to connect that finding the vertex of a quadratic function is another way, she used the term origin, a misconception. Again, however, she used the idea of symmetry in her description as she mentioned this point being "where the two sides come together." In fact, she used almost the exact wording that she used for her personal meaning of the vertex, yet there was still a disconnection between the two.

Interestingly enough, on Amy's Final Exam questions, she mentioned and used the concept of symmetry again to find when the object reached its maximum height. For part a, she graphed the position function, $P(t) = 80t - 16t^2$ by plugging in values for t to find the corresponding y-coordinate.

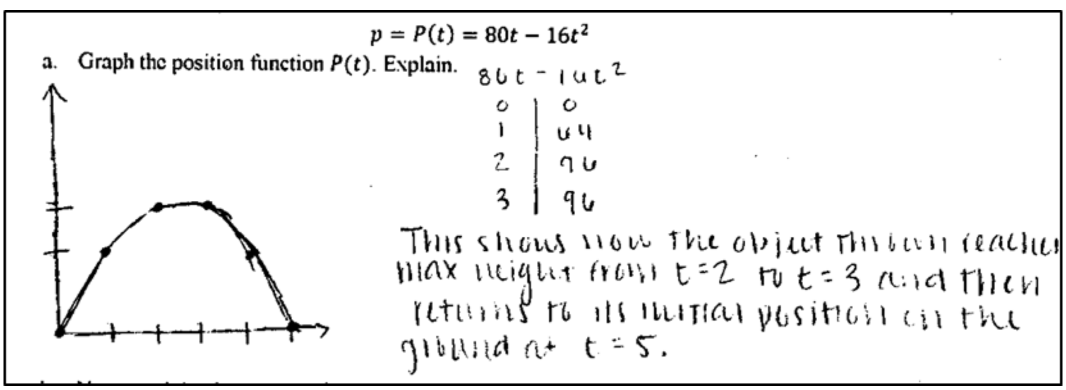


Figure 4.18 Amy's graph of the position function, Final Exam

Amy noted, based on her table, that "this shows how the object thrown reaches max height from $t = 2$ to $t = 3$." To find how long exactly it took the particle to reach maximum height, she again used the concept of symmetry as she wrote "this can also be shown by calculating the maximum height, which is $t = 5/2$, approximately halfway between 2 and 3." Amy's strong association to symmetry of the parabola enabled her to find the maximum time without having to use the derivative. Her abundant schema of the quadratic function enhanced her ability to relate the question to her previous knowledge and the property of symmetry.

Sam, like Fay and Amy, related the vertex to symmetry and wrote that the "vertex of a quadratic function is the axis that splits the graph in even half." She then wrote "vertex = axis of symmetry", almost insinuating that the vertex is the same as the axis of symmetry, when they actually have two different meanings. The vertex is the point, while the axis of symmetry is the line. Sam did, however, connect the vertex to the derivative. She mentioned that the derivative can be used to "find the absolute max or absolute min. It would be the same as vertex."

During the think-aloud session, Sam correctly answered Question 1, and the majority of Question 2. She used the derivative, and during the group interview, even spoke about how the property of symmetry could have been used instead to find when the rocket hit the ground.

S: You don't really need to set it equal to 0. We can kind of find it from here because where it take[s] the max will be the high point and since it's an even function it will be like at 5 more seconds.

I: Ah, ok so you are using symmetry, the property of parabola, right?

S: Yeah, it can do both.

She continued to explain why the initial velocity and the speed when the rocket hit the ground are related based on the symmetry.

S: They're, they have to be the same because it's symmetrical. If the velocity was different, it would be, it wouldn't be symmetrical, it would be something different.

Sam's personal meaning of the vertex as a strong association to symmetry enabled her to reason through her think-aloud session without having to rely on the derivative as the only way to find the maximum of a quadratic function. It is possible that Sam's understanding of the vertex in connection with the line of symmetry enabled her and shaped her understanding of the derivative in a more sophisticated way, as she was able to relate and connect between concepts. Sam's ability to reflect and describe the steps associated with her process, as well as her ability to coordinate and link between ideas suggests she may be at a process level of understanding according to APOS.

4.1.2 Lack of Connection between an Explicit and Real World Problem

One of the biggest overall trends that was noted in data analysis was that of a student being able to correctly answer an explicit question pertaining to finding a maximum of a quadratic function, such as that of Question 1 during the think-aloud session and Question 1 on Test 3, but not being able to transfer this concept over into a real world situation problem, such as that of Question 2 for the think-aloud session and Question 2 on Test 3. This lack of connection between the explicit problem and the real world problem is illustrated in the table below which represents responses from the thirty participants on Test 3, Questions 1 and 2. T1, T2, T3, T4, T5, and T6 represent Trend 1, Trend 2, Trend 3, Trend 4, Trend 5 and Trend 6 correspondingly. A correct answer is denoted by a C and an incorrect answer is denoted by I.

Table 4.1 Trends of Student Responses to Question 1 and Question 2 on Test 3

	Question 1a.	Question 1b.	Question 1c.	Question 2a.	Question 2b.	Question 2c.	Total Students
T1	C	C	C	C	C	C	14
T2	C	C	C	I	I	I	11
T3	I	I	I	I	I	I	2
T4	I	I	I	C	C	C	1
T5	I	C	I	I	I	I	1
T6	C	C	C	C	C	I	1

Based on this table, it is clear to see that there are two major trends in student responses. The first trend, T1, represents a student being able to answer both questions correctly. This comprised 14 of the 30 total students' responses on Test 3. The second major trend, T2, represents a student answering all parts of Question 1 correctly, yet not able to answer any part of Question 2 correctly. This accounted for 11 of the 30 students, about 1/3 of the total. The remaining five students' answers did not fit into the top two trends, and comprised Trend 3 through Trend 6.

A common mistake of these 11 students who correctly answered Question 1, yet incorrectly answered the real world problem, Question 2, was that many of them tried to solve for the variable t by setting the original free fall formula equal to 0, $f(t) = 0$. Solving for t in this case finds the t-intercepts, the time when the debris was on the ground, not the time it took the debris to reach maximum height. These students wrote down the free fall formula, plugged in numbers for gravity, initial velocity, and initial height, set it equal to zero and then immediately started factoring to solve for the variable t. These students were instantly searching for a solution, and did not make the connection of the concept of finding the maximum as a critical point from Question 1 to the real world application problem in Question 2. Table 4.2 illustrates the same data in a different way. Again, C represents a correct answer and I, an incorrect answer.

Table 4.2 Total Student Responses to Question 1 and Question 2 on Test 3

	C	I
Question 1a.	26	4
Question 1b.	27	3
Question 1c.	26	4
Question 2a.	16	14
Question 2b.	16	14
Question 2c.	15	15

From Table 4.2, it is again clear that a majority of students could answer Question 1 correctly; however, many fell short on Question 2. The students had difficulties in making this jump from a specific algebraic procedural approach to an application problem. While many students could correctly apply the concept of finding the maximum of the quadratic function using the derivative in an algebraic situation, many had trouble applying the concept with the real world problem in which the most common error was the resort to trying to solve for t in the quadratic free fall formula set equal to zero.

Similar to Test 3's questions, during the think-aloud sessions, each student worked on an explicit algebraic approach problem and a real world problem pertaining to the vertex of a quadratic function and the derivative. The data collected and analyzed during the think-aloud session further supported the notion that students are able to perform on an explicit algebraic problem, yet many are unable to perform and carry the concepts over to a real world problem. For Tables 4.3 and 4.4, Question 1, part c, in which the task was to find all critical points, all local extreme values, and the intervals of increase and decrease is compared to Question 2, parts a and b, in which the student was to find the time the rocket reached maximum height, along with the maximum height of the rocket. For both parts of Question 1 and Question 2, the task was to find the maximum of a quadratic function; the questions were just presented in a different way, one an explicit problem and one a real world problem. The patterns of correct and incorrect solutions during the think-aloud sessions is illustrated, with C denoting a correct solution and I, an incorrect solution. As before T1 denotes Trend 1, T2 denotes Trend 2, T3 denotes Trend 3, and T4 denotes Trend 4.

Table 4.3 Trends of Student Responses to Question 1 and Question 2 during Think-aloud Session

	Question 1c.	Question 2a.	Question 2b.	Total Students
T1	C	C	C	16
T2	C	I	I	12
T3	I	C	C	1
T4	I	C	I	1

Again, there were two top patterns of students' responses. The first, T1, is students answering both Question 1c and Questions 2a and 2b correctly. This means that the student correctly found the critical point, extreme value, and intervals of increase and decrease on the explicit question, and found the time the rocket reached maximum height, along with the maximum height reached correctly. Even though this accounts for about ½ of all participants, this does not imply that all 16 of these students have developed conceptual meaning of the concepts involved. Many are actually at an action level of understanding, memorizing and following a step by step procedure to come up with the correct answer.

The next top pattern, T2, is a student able to explicitly find the critical point, extreme value, and intervals of increase and decrease, but not able to relate this to the real world problem, thus getting parts a and b of Question 2 wrong. Like for Test 3, the problem for many of these students was the urge to solve for the variable t from the original free fall function set equal to zero. Others had trouble with the free fall formula, and the meaning behind it. These 12 students who fell into this category could possibly be at an action level of understanding for Question 1, which prevents them from being able to understand and apply the same routine to Question 2. A weak schema of quadratic functions and misconception of the vertex also created obstacles to student understanding.

Table 4.4 shows the same data, just in a different presentation. Here, total student responses for Question 1c, and Questions 2a and 2b are compared for their results during the think-aloud session. This table makes it even more clear the tendency in a student being able to explicit perform to find the maximum of the quadratic function, but not being able to take the idea and carry it over to a real world problem.

Table 4.4 Total Student Responses to Question 1 and Question 2 during Think-Aloud Session

	C	I
Question 1c.	28	2
Question 2a.	18	12
Question 2b.	17	13

The results in this subsection clearly present that there is a disconnection for some students in their ability to work an explicit problem versus a real world problem. So why are students struggling with this concept that presented one way, is done correctly, yet presented another, is done wrong? For one thing, it is possible that for those who cannot make the connection between problem types that they are at an action level of understanding of an explicit problem according to APOS. They may be memorizing the steps involved to find critical points and intervals of increase and decrease. This memorization may hinder them from being able to make the connection that a problem asking for when does a rocket reach its maximum height, is the same idea, just put in words.

It could also be possible that the transfer of learning (Singley & Anderson, 1989), the ability to apply concepts in different situations is not occurring. In particular, vertical transfer, in which “a learner recognizes features of the situation that intuitively activate elements of her/his prior knowledge” appears to be problematic for these students (Rebello, Cui, Bennett, Zollman,

& Ozimek, 2007, p. 231). Most real world problems require vertical transfer in that the learner must recognize similarity between two contexts or apply previously learned knowledge in a prior context to learn to solve problems in a new context (Rebello et al., 2007). Some studies have found that transfer is difficult and often times not achieved (Gick & Holyoak, 1980; Rebello et al., 2007; Reed, Ernst, & Banerji, 1974). For this study, the gap in ability of some of these students to transfer the knowledge from the explicit problem to the real world problem correctly further indicates the incapability of vertical transfer.

Also, it is possible that a weak schema of quadratic functions and erroneous personal meanings of the vertex hinders students' capability from seeing the connection and relationship between the concepts. And lastly, it is possible that there are other obstacles to the students' understanding which need to be addressed and overcome before they can move from an action level into a process level of understanding.

4.1.3 Obstacles to Student Understanding of the Real World Problem

While many students have the procedural steps to correctly compute explicit questions, there are potentially several obstacles that prevent students from overcoming the disconnection in being able to perform on real world problems. These obstacles include misconceptions of the vertex, which was discussed in *4.1.1.1 Personal Meaning: Misconceptions of the Vertex*, trouble with the free fall formula, and problems with graphing possibly resulting from a weak schema of quadratic functions. Each obstacle prevents students from being able to correctly develop meanings of mathematical concepts and to transfer ideas back and forth between different contexts.

4.1.3.1 Obstacle 1: Misconception of the Vertex

Described in further detail in the subsection *4.1.1.1 Personal Meaning: Misconceptions of the Vertex,* there were three major misconceptions of the vertex that appeared in data analysis. These included the misconception of the vertex as an intercept, the misconception of the vertex as the origin, and the misconception of the vertex as a point of inflection. These misconceptions and lack of understanding of the vertex created obstacles to the understanding of quadratic functions in general, which in turn can hinder the understanding of the derivative and the relationship between the derivative and the quadratic function. Some of the misconceptions of the vertex led to trouble in graphing, as well as trouble carrying over the ideas and concepts from the explicit problem to the real world problem. Because of a lack of a strong schema in quadratic functions, some students failed to realize that the real world problem of Question 2 on the think-aloud was also asking for the maximum, only in words.

In particular, the students who exhibited a weak graphical schema of the quadratic function lacked the ability to correctly identify the vertex as a maximum or minimum and instead identified it as an intercept, origin, or point of inflection. This could be attributed to a limited *procept* of a quadratic function. A procept consists of three components: “a *process* that produces a mathematical *object*, and a *symbol* that represents either the process or the object” (Gray & Tall, 1994, p. 121). The fact that some of these students who had an irrational personal meaning of a vertex were still able to perform suggests that these students could be procedural thinking rather than proceptual thinking.

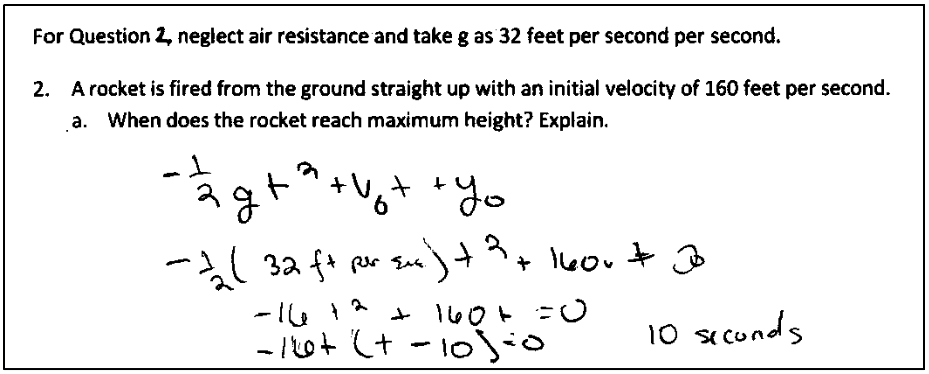
While the misconception of the vertex did not create an obstacle for being able to compute step by step answers for some students, it created an obstacle for conceptual meaning behind the steps. Some students could perform at an action level of understanding, without the ability to explain why. These students could still come up with right answers, yet the conceptual understanding behind the answers was missing. Without having the prerequisite knowledge, some students were not capable of reaching a higher level of understanding to put meaning behind concepts.

4.1.3.2 Obstacle 2: Trouble with the Free Fall Formula

Another common obstacle to students being able to transfer the idea of finding the maximum using the derivative in an explicit situation over to real world problems had to do with trouble with the free fall formula. Question 2 on Test 3 and Question 2 during the think-aloud session both required knowledge of the free fall formula. However, during the think-aloud session, the free fall formula in its general form, $y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$, was given to the students on a sheet of paper in case they did not have it memorized. Thus, if they could not recall it by memory, it would not prevent them from being able to carry on with the problem. The students were, however, expected to recognize the free fall formula, along with the representation of g as gravity, v_0 as initial velocity, and y_0 as initial height as these types of problems were done in class and on homework. Still, even though the formula was given to students as a backup, many struggled with the formula and many mistakes were made, creating an obstacle for connection between the explicit problem and the real world situation. All students with this particular obstacle correctly answered the explicit Question 1 during the think-aloud session, yet because of trouble with the free fall formula, could not carry over the concept and connection to the real world problem on Question 2.

Several students' troubles with the free fall formula came from not knowing the meaning of y_0. In the free fall formula, y_0 represents the initial height, however some did not understand this. During Cindy's think-aloud session, when she first reasoned with the free fall formula on

Question 2, she did not realize that y_0 represents initial height and thus left her formula as $-\frac{1}{2}(32)t^2 + 160t + y_0$. When she tried to find when the rocket reached its maximum height, she resorted to just dropping the y_0 from the equation all together, with no reasoning behind it. She then erroneously solved for the variable t to find when the rocket reached maximum height.



For Question 2, neglect air resistance and take g as 32 feet per second per second.

2. A rocket is fired from the ground straight up with an initial velocity of 160 feet per second.

a. When does the rocket reach maximum height? Explain.

$-\frac{1}{2}gt^2 + v_0t + y_0$

$-\frac{1}{2}(32 \text{ ft per sec})t^2 + 160t + 0$

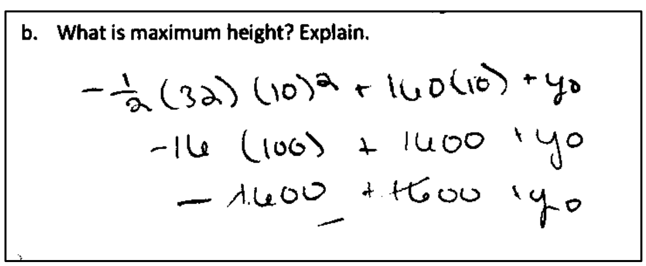
$-16t^2 + 160t = 0$

$-16t(t - 10) = 0$

10 seconds

Figure 4.19 Cindy's solution during think-aloud, Question 2

Next, when trying to find not the time, but the actual maximum height the rocket reached, she left y_0 in her equation, causing her again to get an erroneous solution.



b. What is maximum height? Explain.

$-\frac{1}{2}(32)(10)^2 + 160(10) + y_0$

$-16(100) + 1600 + y_0$

$-1600 + 1600 + y_0$

Figure 4.20 Cindy's trouble with the free fall formula

Cindy described her arithmetic steps during her think-aloud session, and became stumped at the end as she asked herself "We got -1600 plus 1600 plus y_0? Those cancel each other out so I just got y_0?"

By not knowing that y_0 represents initial height, which in this case is 0, along with her wrong answer of 10 seconds for the time the rocket reached maximum height from solving the original function set equal to 0 in part a, Cindy came up with an incorrect solution for part b.

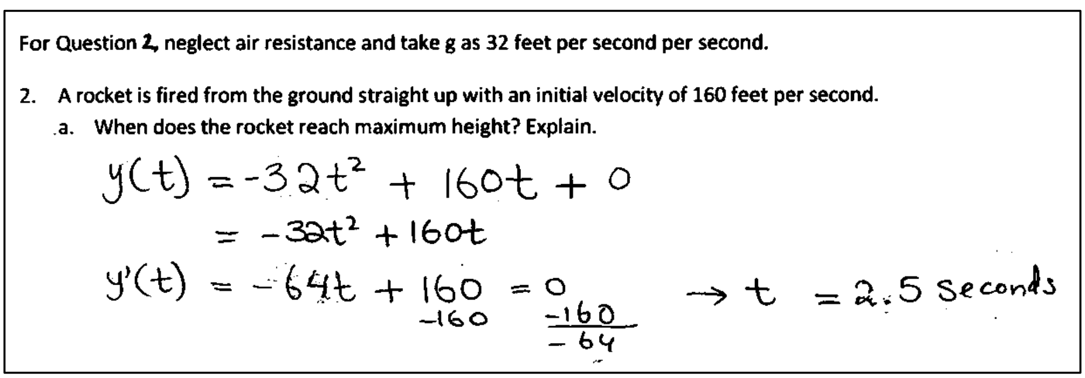
This formula trouble was an obstacle in her understanding of how to proceed through the real world problem. This kept her from even being at an action level of understanding for the real world problem as she did not even know steps to a correct solution, nor could she give meaning behind the free fall formula.

Jake also had trouble with the representation of y_0 and the free fall formula on Question 2, yet answered Question 1 correctly. After initially reading Question 2 during his think-aloud session, he looked at the piece of paper given for the free fall formula, and began to try to reason through what the formula means. "Ok, so $y(t)$ is $-\frac{1}{2}$, so it's $32t^2$ plus uhhhh $160t + y_0$. What is y_0? Oh when it's at 0. So when it's at 0 wait wait wait uhhhh this is confusing." While Jake initially had the right idea that y_0 is 0, he could not explain why. He continued to hum and take long pauses, both signs of confusion, as he tried to figure the formula out. "What is y_0? Is it the height when it's 0? …why the y_0 is? I know v_0 is, I think v_0 is initial velocity cause it's 0 and that's where it start[s]. Is y_0 initial height?"

As Jake struggled and questioned the meaning of y_0, his uncertainty and unfamiliarity with the formula created an obstacle for him. He finally resorted to "I'm just making assumptions so I'm thinking, whew, I feel like I need to take the derivative of this cause it's asking for maximum so I'm just gonna take the derivative of it just in case." The word maximum triggered him to know what to do, a sign of pure action level of understanding. He ended up replacing y_0 with 0 and taking the derivative to find the time the rocket reached maximum height, however he was never sure why, as during the group interview he explained "cause I remember in calc class, we were trying to find maximum size or minimum size of some question, we would always take derivative of the equation, and then plug in and find the max or minimum so I thought since it’s asking for maximum height, so I thought I should take derivative of that equation and plug it in, that’s how I thought.”

Jake’s ability to eventually work through his obstacle of the free fall formula to come up with the correct maximum time and height of the rocket, suggests he has an action level of understanding for this type of real world problem, especially because of his inability to explain why his steps make sense and reliance on memorization.

While some students had trouble with the representation of y_0, a few others had trouble with the coefficient g in the free fall formula. It was given in the problem to take g as 32 feet per second per second, where g represents gravity. When plugging values in the free fall formula, one student, Urray, did not take the $\frac{1}{2}$ of 32 and thus her free fall function was $-32t^2 + 160t$ rather than $-16t^2 + 160t$. She then took the derivative, $f'(t) = -64t + 160$, and set it equal to 0 to solve for t.



For Question 2, neglect air resistance and take g as 32 feet per second per second.

2. A rocket is fired from the ground straight up with an initial velocity of 160 feet per second.

a. When does the rocket reach maximum height? Explain.

$y(t) = -32t^2 + 160t + 0$

$= -32t^2 + 160t$

$y'(t) = -64t + 160 = 0 \rightarrow t = 2.5$ seconds

$-160 \quad \frac{-160}{-64}$

Figure 4.21 Urray’s solution to think-aloud, Question 2

While Urray had the right procedure to find the time the rocket reached maximum height by taking the derivative, she could not come up with the right answer because of her obstacle with the free fall formula and confusion over g. During the group interview, when Question 2 was brought up, she knew something was wrong as her equation was different from her group members’.

U: Oh ok well I don't know, I put -32 instead, I got the formula wrong, but, um, I did the same thing that Henry said, find the derivative, set it equal to 0, and then found t which gave me the maximum- I mean the time when it reaches the maximum height.

While Urray knew the procedures, she did not know why she had the formula wrong. Being able to complete the procedure with no meaning behind the steps suggests that she is at an action level of understanding for the real world problems. Her obstacle of the free fall formula, however, prevented her procedure from working as she stumbled with the coefficient g of t^2.

Many of the students who struggled with the free fall formula were at an action level of understanding on the explicit algebraic context, which could have prevented connection to the real world problem. Some of the other problems with the free fall formula came from not knowing the representation of y_0, or making a mistake plugging in for the gravity, g. While many had an obstacle with the meaning of the free fall formula and its relationship to the real world problem, others had trouble with sketching the graph, possibly influenced by a weak schema of quadratic functions.

4.1.3.3 Obstacle 3: Graphing; Weak Schema of Quadratic Functions

The last obstacle identified in student understanding of the real world situations was problems with graphing the quadratic function, possibly resulting from a weak schema of quadratic functions. Students struggled with sketching the graph of the quadratic function for the real world problem of Question 2, yet were able to sketch the graph of the quadratic function in the explicit Question 1 correctly. Others had misinterpretations of the graph of a quadratic function all together, and sketched erroneous graphs for both questions.

Randy, during his think-aloud session, answered everything for Question 1 correctly, including sketching the graph. When sketching, Randy even found the x-intercepts using the quadratic formula, and the y-intercept by plugging in 0 for x. He also found the correct maximum of the quadratic function at the point $(2, 9)$, and drew the correct shape of an upside down parabola.

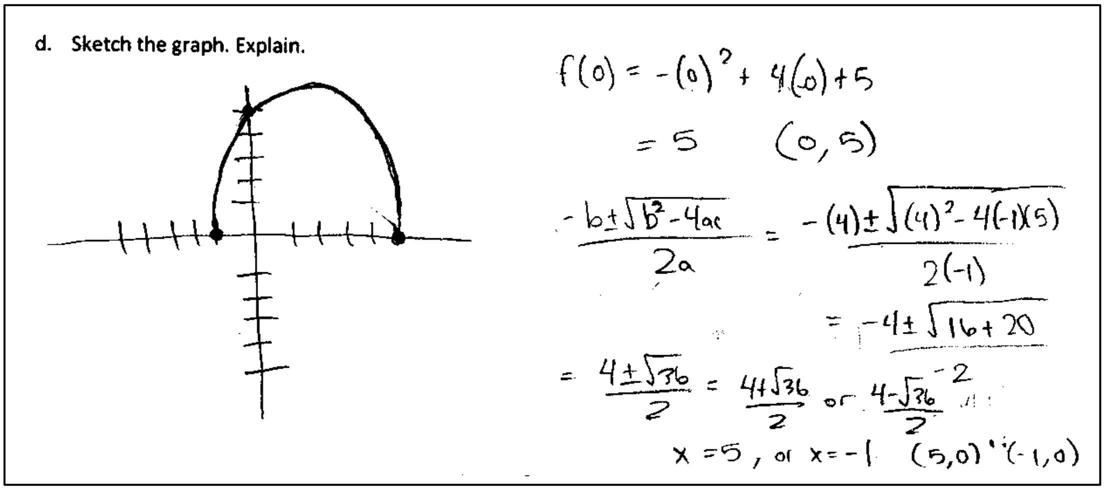


Figure 4.22 Randy’s graph during think-aloud, Question 1

As Randy moved to Question 2, he found the time the rocket reached maximum height, the maximum height, when the rocket hit the ground, and at what speed all correctly. Yet, when it came time for him to sketch the graph of the function $y(t) = -16t^2 + 160t$, he said “I’m not sure how to sketch the graph”, and left part d blank.

Even though he had an action level of understanding of sketching the graph of the quadratic function on the explicit problem, he was not able to connect that for the real world problem, he was also sketching a quadratic function. Even though he had all the correct information to use for the sketch from parts a through c, his inability to recognize the free fall function as a quadratic function kept him from reaching an action level of understanding of sketching the graph of a quadratic function in a real world situation.

Adam’s work during his think-aloud session was just the opposite of Randy. Instead of sketching the graph of the quadratic function in the explicit problem correctly, he drew the maximum on his graph as a pointy peak, almost like the top of a triangle, with incorrect x-intercepts. During his think-aloud session, he noted that his graph was "kind of like a downward shaped V."

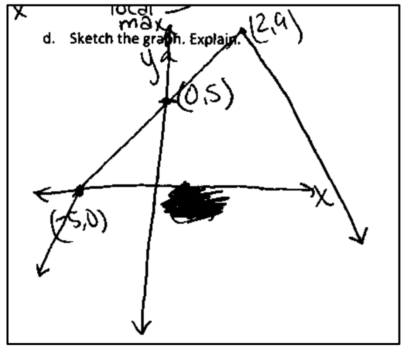


Figure 4.23 Adam's graph during think-aloud, Question 1

Since the function he was sketching is a quadratic function, the graph should be rounded at the top, a characteristic of a parabola. Even more so, during the think-aloud session, he initially thought that the shape of the graph of a quadratic function was a squiggly line (see Figure 4.16). These problems with the shape of the graph of the quadratic function suggest that Adam has a weak pre-requisite schema of the quadratic function and its corresponding graph.

Despite Adam's inability to sketch the graph correctly on Question 1, and his initial thought that the graph of a quadratic function is a squiggly line, he was still able to sketch the general picture of the quadratic function on the real world problem, Question 2, correctly.

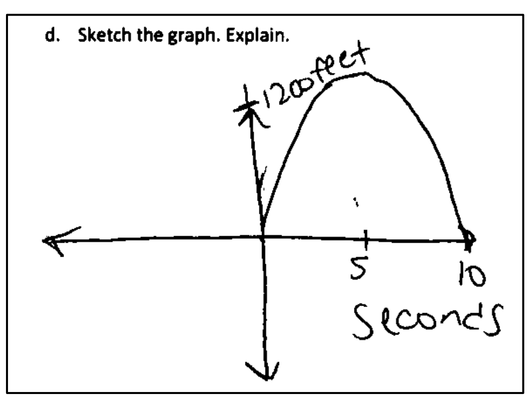


Figure 4.24 Adam's graph during think-aloud, Question 2

For Question 2, immediately after reading the directions to sketch the graph, he said "basically it's like an upside parabola." The only mistake made was when he plugged in the time of 5 seconds to find the maximum height the rocket reached, he made an arithmetic error and left out a negative sign, which gave him a maximum height of 1200 feet instead of 400 feet.

Adam went from the idea of the graph of a quadratic function as being a squiggly line to drawing a pointy peak to finally, an upside parabola. It is possible that the context of the problem helped Adam to sketch the graph in Question 2 correctly. Because of Adam's ability to complete step by step on the real world problem, with possibly not even knowing what exactly a graph of a quadratic function should look like, suggests that Adam is at an action level of understanding of graphing a quadratic function on a real world problem.

Andy, rather than initially thinking that the shape of the graph of a quadratic function was a squiggly line, believed it to be a straight line. Andy said that "the shape of this [the quadratic function] is actually just a line." He even drew a picture for what he believed to be the sketch of a quadratic function.

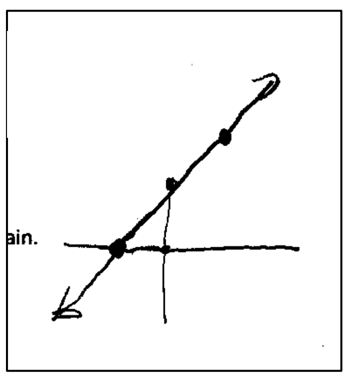


Figure 4.25 Andy's sketch of a quadratic function

When he sketched the graph for the quadratic function in Question 1 during the think-aloud, however, rather than a line, he drew a pointy mountain peak, just as Adam had.

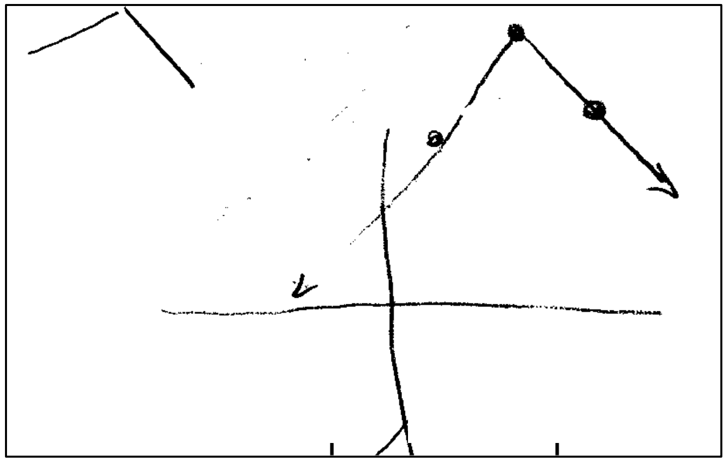


Figure 4.26 Andy's graph during think-aloud, Question 1

Even more so, his idea of a pointy top of the graph carried over into the real world problem as one of the first thoughts and things Andy did was draw a triangle. "So we're gonna draw a picture of a rocket kind of shooting- actually I'm gonna draw a triangle." His graph for part d of Question 2, was the same pointy peak sketch that was drawn on the previous question.

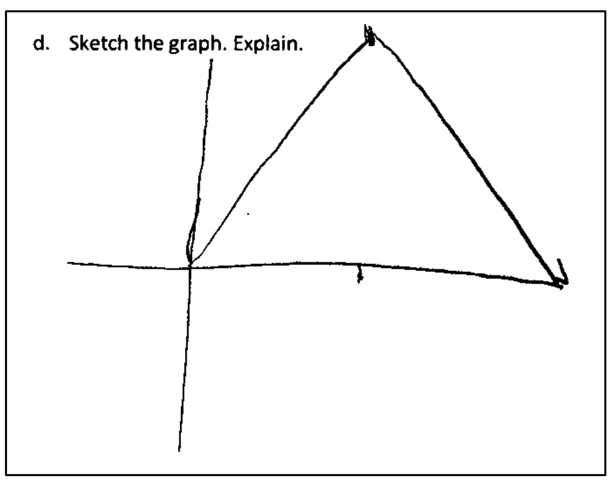


Figure 4.27 Andy's graph during think-aloud, Question 2

Andy's trouble with graphing even continued on the Final Exam. On his Final Exam, for his graph of the derivative function $P'(t) = 80 - 32t$ he drew a straight line coming up from the origin.

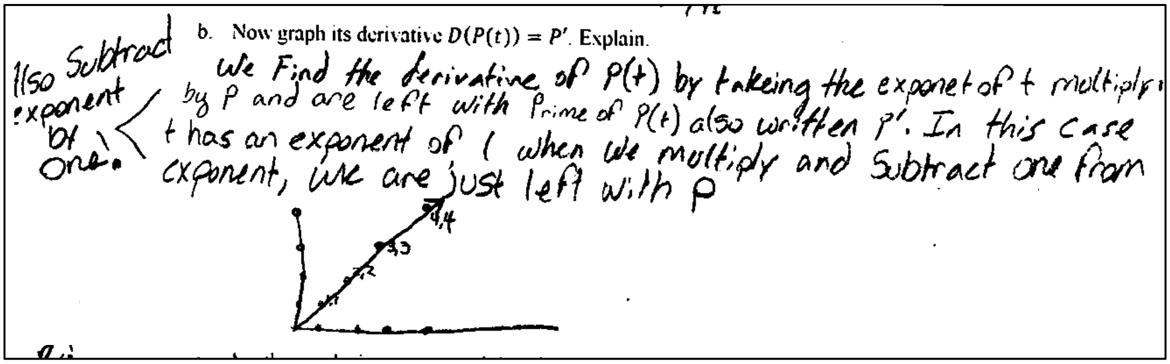


Figure 4.28 Andy’s graph of the derivative, Final Exam

Andy’s explanation of his graph was based on how he found the derivative step by step. Andy could not sketch the graph of a quadratic function and could not correctly sketch the graph of a linear function as well. He was also not able to form a relationship between the original function and the derivative. His troubling connection in graphical representation was even more obvious as he did not notice that the derivative does not exist at the peak point of a function.

Andy’s irrational conceptual meaning of the derivative was also evident during the group interview. When he was asked to give the meaning of the derivative, he answered that “I think of the derivative as … let’s say you don’t know $f(x)$ for sure, what the points are on $f(x)$, then the derivative is something that runs parallel to it …if the slope of this parallel line is increasing then the slope of the original line is increasing. … we’re talking what we know about the original function, we’re looking at the slope of let’s say something parallel to it and then saying well since that’s increasing the original function’s increasing.” Andy’s erroneous meaning of the derivative and the idea of the derivative as a parallel line continued as he explained how he found his critical value.

A: What I did is I plugged in 0 to find my critical point, then I took points on both sides of my critical point to determine whether the derivative was positive or negative so whether the tangent line at that particular point is increasing or decreasing which tells me whether the function's slope is increasing or decreasing.

I: The function's slope is increasing or decreasing?

A: At that point, yeah.

I: What does it mean for the function's slope to be increasing?

A: Well again, and I may have this wrong, I always thought of the derivative as a line running parallel to the actual line, so if the slope of the derivative is increasing or decreasing at that particular point then the actual slope of the original line is increasing or decreasing at the particular point, slope by meaning whether it's going up or coming down.

I: Ok if it's going up, does that mean the slope is increasing?

A: If the slope is getting larger.

I: If the graph is going up, does that mean the slope is increasing?

A: Yeah, if the graph, you're asking me if the graph is moving upwards along the x-axis, so moving up along the y-axis actually running, yes it's increasing.

I: The slope is increasing, is that what you're saying?

After a long pause Andy then confirmed that "yes, that's what I'm saying. I feel confident with that answer." Andy had an irrational impression that the slope of a linear function, a line, can increase or decrease. This discussion continued with the interviewer trying to prompt him to realize this inaccurate perception, however, he remained adamant with his ideas. This could be in part from not only a weak and irrational pre-requisite schema of a quadratic function, but from a weak and irrational pre-requisite schema of a linear function as well.

Andy's troubles graphing a quadratic function along with his inability to explain conceptually about the meaning of the derivative created an obstacle to his understanding of the concept of vertex in relation to the derivative in a real world problem. His inaccurate pre-requisite schemas of quadratic functions and linear functions also contributed to Andy's problems. He did not even demonstrate an action level of understanding as he was not capable of even computing step by step or graphing, nor was he anywhere close to putting the correct conceptual meaning behind his steps.

4.2 The Think-Aloud Method

> "The closest connection between actual thoughts and verbal reports is found when people verbalize thoughts that are spontaneously attended during task completion" (Ericsson, 2006, p.227).

Verbalization of thoughts during task completion took place during the think-aloud sessions. This valuable method of data collection gave insight into students' thoughts as they worked (Ericsson & Simon, 1993). While analyzing the transcripts of the participants as they thought aloud, one theme seemed to be common. Many of them as they spoke out loud while working only gave procedural explanations. The majority did not ever mention a reason why they were taking the steps they did. Skemp's (1976) instrumental understanding and relational understanding framework is used in this section to describe students' thoughts and ability to work through problems out loud.

4.1.1 Instrumental vs. Relational Understanding

According to Skemp (1976), there are two types of understanding, relational and instrumental. Instrumental understanding is when a student has "rules without reasons" (p. 2). The student can achieve a page of correct answers; however, there is no understanding as to why the certain steps are done. According to Skemp (1976), "instrumental understanding necessitates memorizing which problems a method works for and which not, and also learning a different method for each new class of problems" (p. 9). Instrumental understanding has no depth or conceptual understanding of the concepts involved, and thus makes it difficult to relate concepts to new tasks. Relational understanding, on the other hand, is "knowing both what to do and why" (Skemp, 1976, p. 2). It is the understanding that is more adjustable to new tasks, more lasting and has more long term rewards (Skemp, 1976).

During the think-aloud sessions, for Question 1, part c, when the students were to find critical points, extreme values, and intervals of increase and decrease, 27 of the 30 students went immediately as their first thought to set the derivative, $f'(x) = -2x + 4$, equal to 0. Out of the three that did not have this immediate thought, two of them first found intercepts, and then set the derivative equal to 0, and one of them initially thought you were to plug in 0 for $f(x)$, but then corrected his thought and set the derivative equal to 0 as well. Eventually, every single student set the derivative of the quadratic function equal to 0 for this question. No mistakes were made on finding the derivative of the quadratic function; however, some arithmetic errors were made when finding the y-coordinate of the maximum, and a few errors were made with the arithmetic for solving for x from the derivative, $f'(x) = 0$.

While all students who participated in the study at some point during their think-aloud, and for the majority, immediately after reading the question, set the derivative equal to 0, only three of them as they thought aloud gave an indication of conceptual meaning as to why they were setting the derivative equal to 0 to find the critical point. Besides these three students who gave a reason why they were taking the steps they did, the rest of them thought out loud instrumentally. They were following the rules with no explanation of why.

Alex and Emma were two students who during their individual think-aloud sessions thought instrumentally out loud to find the critical value. Emma's explanation was almost the exact same as Alex's.

Alex: To find the critical points, you have to set the derivative equal to 0. By doing that you get $-2x = -4, -2x = -4$ so divide by -2, $-2, x = 2$. Now let's plug in 0 to see if that works out and that will not equal 0, so the only x the only uh zero on the graph is $x = 2$ and um so that's the critical point, $x = 2$.

Alex and Emma's explanations were both instrumental as they explained a step by step arithmetic procedure that they followed, giving no reasoning behind their steps. When they were both prompted by the interviewer in the group to explain why we set the derivative equal to 0 to find critical points, neither of them could conceptually answer. Emma answered that "because, it's just like the first problem. To find the critical points, like where the maximum or the minimum is, you set the derivative to 0." Alex answered that "cause it starts from 0". When further prompted, Alex again answered "cause, ummm I was gonna say cause you start from the ground, but you do that for every every parabola so it's not that uhhh", and Emma resorted to "I don't know." Neither Alex nor Emma ever mentioned any reasoning based on the slope of the tangent line. Both of them exhibited instrumental understanding, which was further evident in their thought process during the think-aloud session, and their ability to come up with the correct critical point, yet not knowing why they were doing so.

Nick was another student who instrumentally thought aloud during his individual think-aloud session. In particular, for finding the critical value, his procedure was the exact same as Alex and Emma.

Nick: To find critical points, you basically, you find $f(x)$ which equals $-2x + 4$ which is the derivative and then to find critical points you set your derivative equal to 0, and solve for x, so you get $-2x = -4$ and then you divide by -2, so $x = 2$.

He was able to come up with the correct critical value of $x = 2$, however, he too, could not give any conceptual meaning to the derivative being set equal to 0, a sign of instrumental understanding for not knowing why.

I: How did you know that you need to set it up to 0, and why, why does it make sense to set it up equal to 0?

N: I mean that's what we learned in class, so I've just been following that.

Nick's suggestion of remembering what was taught in class further illustrates his instrumental understanding as memorization of how to work a certain problem is a typical aspect of this type of understanding.

When Nick was prompted for a reason why we set the derivative equal to 0 other than remembering from class, after a long pause, he could not give one. The interviewer then asked Nick to sketch a tangent line at the point $(2, 9)$ on his graph. He drew a line with a positive slope going through the maximum point $(2, 9)$.

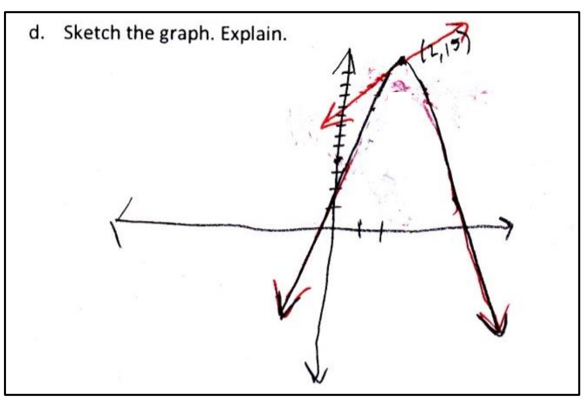


Figure 4.29 Nick's tangent line at (2,9)

The interviewer then proceeded to prompt Nick until he realized that it would be a horizontal line, not a slanted line, and that the slope of that horizontal line is 0.

When Nick found the critical point, he initially had no clue as to why he was doing so. It took a lot of prompting from the interviewer for him to finally make the connection of the horizontal tangent line with a slope of 0. Even after this supposed realization, it did not stick with him as on the Final Exam, many of Nick's responses did not make sense conceptually. He implied that the graph of velocity "shows that the graph [of the position function] is always decreasing, velocity is always negative." He also wrote that finding maximum time "doesn't [relate to velocity] because time has to do with acceleration." Nick was still not able to relate concepts or make any type of connections. His instrumental understanding prevented him from being able to give any accurate explanation of why.

Other students individually thought out loud instrumentally, yet when prompted by the interviewer they were able to give a relational meaning to why they took the steps they did. For example, Kevin, for both Question 1 and Question 2 during his think-aloud, spoke procedurally to find the maximum for the explicit problem and the real world problem.

K (Question 1): So to find the to find the critical points, you have to make your derivative equal to 0, so what we're gonna do let's see $-2x + 4 = 0$ then you have $-2x = -4$, $x = 2$. So your only critical point right now is 2.

K (Question 2): And then you find the derivative which is like $-32t + 160t$, wait a minute just a 160, and then you gotta find increasing, so where it's increasing and decreasing to find the maximum, and what you do, you say t is equal to $160/32$ which is 5, yeah, $t = 5$. Ok, at 5.

Kevin appeared to have an instrumental understanding of the concepts as he thought out loud because he never mentioned why he set the derivative equal to 0 to find the maximum. However, when he was asked during the group interview, he was able to immediately give a reason.

I: Ok, um, so setting the first derivative equal to 0 and solving for t would tell you-

K: Um, when the tangent line equals 0 basically.

I: When the tangent line equals 0?

K: Or when the tangent line slope is equal to 0, and um, and that tells you whether it stops increasing or decreasing.

While it appeared during Kevin's think-aloud session that there was no meaning behind his method, he was able to right away answer during the group interview that the reason we set the derivative equal to 0 is to find when the slope of the tangent line is 0. Kevin had the technique of finding the maximum down and he also knew the reason as to why. This suggests that Kevin has relational understanding of the concept. Even though Kevin and several others could explain when prompted, this does not take away the fact that they, like most, explained instrumentally during the think-aloud session.

Out of the thirty participants who spoke out loud during the think-aloud sessions, only three during their think-aloud session gave any indication as to why they were setting the derivative equal to 0 to find the critical point. Henry, during his think-aloud, said that "what I'm going to do is take $f'(x)$, which I found in the last problem, and set it equal to 0, because that is how you find the critical point. It's where there maximums and minimum, when the slope [of the tangent line] is 0. So I'm just going to factor that, 2, $x = 2$." He did not give the detail explanation of the arithmetic as the others did. He even mentioned that it is "when the slope [of the tangent line] is 0." Henry also displayed relational understanding during the group interview as he described the derivative as at "each $f'(x)$ at whatever point of x you plug in gives you the slope of the tangent line of $f(x)$."

Henrys' ability to think-aloud with not only procedure, but with correct reasoning, illustrates that he had relational understanding of the involved concepts. His relational knowledge could have also contributed to the fact that he was able to answer all questions during the think-aloud and Test 3 correctly, as well as his Final Exam explanations.

While 27 of the 30 participants during the think-aloud went immediately to set the derivative equal to 0 to find the critical point for Question 1, part c, all following the same technique, only 18 of the 30 on the real world problem for part a, Question 2, went immediately to set the derivative equal to 0. For part a of Question 2, they were to find the time the rocket reached maximum height. So even though both Question 1 and Question 2 involved finding a maximum, there was a stark difference in the ability of students to recognize what the question was asking. The students' instrumental understanding of following a fixed plan for Question 1 could have prevented them from being able to adapt this knowledge to different problem types, even though they involved the same concept.

Even more, out of all the students who set the derivative equal to zero to find the critical point for Question 1, only three of them suggested during the think-aloud any reason as to why they did so. The thought process for the students was instrumental. It was as if the students had an "increasing number of fixed plans", in which they were able to "find their way from particular starting points (the data) to required finishing points (the answers to the question)" (Skemp, 1976, p. 14). They were able to arrive at the correct answer, yet many did not know why they got there, which became more evident in the group interviews.

However, some students did exhibit relational understanding when prompted in the group interview; yet, their thought process during the think-aloud was instrumental. For some students, despite an instrumental thought process to finding critical points, their understanding capability enabled them to correct mistakes as they worked during the think-aloud session.

4.2.2 Ability to Correct Mistakes

One of the biggest benefits to the think-aloud method as a means of data collection was the opportunity to hear students' thought processes as they worked. While many of the students thought out loud instrumentally, some, when prompted were able to explain relationally. These students who were able to give more conceptual meaning to concepts were able to identify and correct mistakes as they worked. These same students who had the ability to identify and correct mistakes also performed to a much higher degree than the others, answering mostly both Question 1 and Question 2 correctly after fixing mistakes.

Chuck was one of the students whose conceptual meaning enabled him to identify and fix his mistake. On Question 1, when he solved to find the critical value of the quadratic function, he made an arithmetic error. He set the derivative equation, $f'(x) = -2x + 4 = 0$, and then said "from that we know that x is -2, so that is our critical point." He did the mental math in his head and miscalculated. Chuck continued and found intervals of increase and decrease by plugging in a number less than -2 and a number greater than -2 into the derivative function. This gave him the sign of the derivative, which based on whether it is positive or negative, tells whether the original function is increasing or decreasing. If the sign of the derivative is positive, then for that interval, the original function is increasing. If the sign of the derivative is negative, then for that interval the original function is decreasing.

Chuck first plugged -3, a number less than -2, into the derivative function. This gave him $-2 * -3 + 4$, which is positive 10. Then, he plugged in 0, a number bigger than -2, into the derivative function. This gave him $-2 * 0 + 4$ which equals 4, another positive number. This was Chuck's first realization that he arrived to some contradiction. "Let me see. I miscalculated somewhere…For there to be an extreme value, there has to be a sign change from one interval to another, but if $f'(x)$ is positive on both sides of -2, then there is no extreme value." He tried to recalculate by plugging values into the derivative once more, but then just settled with an answer of "so f is increasing on $(-\infty, -2]$, and f is decreasing on $[-2, +\infty)$." He had not yet figured out his mistake in the arithmetic of solving for the critical point, however, he recognized the critical point to be a maximum, evident of his intervals of increase and decrease. He knew something was not right as the sign of the derivative should have changed and that the function should be decreasing after the critical point.

It was not until Chuck started to sketch the graph that he caught his mistake. When he tried to find the corresponding y-coordinate for $x = -2$, he realized again that he was getting some contradicting results. "Let's see let's see so it's a local max. But that can't be if uh hmm I miscalculated somewhere." He went back over his answers to the first parts of the question, and found his mistake. "Ohhhh oh actually yeah my critical point is actually 2 and not -2, so I made a mistake here…makes more sense." Based on his understanding of the derivative and quadratic functions, Chuck was able to first acknowledge that he obtained conflicting results. Then after his acknowledgement of a mistake, he was even able to find and correct it. He went on to complete the rest of Question 1 and Question 2 correctly. Chuck's relational understanding with the concepts of the quadratic function and the derivative helped him to correct and fix his computational mistake.

Khan was another student who made the same exact computational mistake and found $x = -2$ as the critical value. He too, was able to catch his mistake, only a bit quicker than Chuck did. He knew that at the maximum point the derivative changes sign and this was not happening when he checked points on both sides of -2. Khan also was able to identify and fix another mistake that he made during the think-aloud session on Question 2. To find the time the rocket reached maximum height, Khan correctly took the derivative, set it equal to 0, and solved to find a time of 5 seconds. Then, for part b, when he needed to find the maximum height the rocket reached, he accidently left out a t from the original free fall function. He mistakenly plugged 5 for t into $y(t) = -16t^2 + 160$, rather than $y(t) = -16t^2 + 160t$. This gave him a maximum height of -240 feet, something he knew was impossible as height cannot be negative. "$-400 + 160 = -240$, but I'm not too sure how that would be maximum height." After contemplating and a long pause, he realized his mistake. "Oh I know, I didn't see that last t…so that's $-16 * 5^2 + 160 * 5$ and that gives us $-400 + 800 = 400$, so that's the maximum height." Based on the context of the problem, Khan knew that a maximum height of -240 did not make sense, and thus something was wrong. His relational understanding of the derivative, the quadratic function, and of the context of the problem enabled him to be able to identify and correct his mistakes, preventing him from coming up with a final wrong answer. Khan, like Chuck, after overcoming the errors, answered everything correctly.

Amy also made a mistake with Question 2 that she was able to correct. Amy, like many others, for part a, initially started to solve for t from the original free fall function $-16t^2 + 160t = 0$ to find the time the rocket reached maximum height. When she came up with values of $t = 0$ seconds and $t = 10$ seconds for the time the rocket reached maximum height, she noted that "it wouldn't make sense for it to reach maximum height at 0 seconds, so just this one. So it would reach maximum height at 10 seconds." She did not mention that it would also not make sense to have two different times that the rocket reached maximum height. Instead, she continued on to the next part of the question to find the maximum height of the rocket.

To find the maximum height of the rocket, rather than initially recognizing to plug into the original free fall function, Amy now set the derivative equal to 0 and solved for t, what she should have done in part a. When she found $t = 5$, she knew something was wrong, especially because t represents time and not height in feet. "Wait. I think I just did problem, wait, ohhh ok I did the problem backwards so x out. You would reach that [maximum height] at $t = 5$ based on this. This is this answer. I did that wrong." She not only realized that she made a mistake in part b, finding the maximum height, she also realized that she had made a mistake for part a, finding the time of the maximum height. She connected the concepts, and was able to fix all mistakes. After fixing the time of maximum height to $t = 5$ seconds, she then correctly plugged in 5 seconds for t into the original free fall function to find a maximum height of 400 feet. She went on to correctly answer parts c and d as well.

Amy's relational understanding gave her the ability to catch her mistakes and to recognize contradicting answers. The context of the problem also helped her to understand her mistakes in order to fix her errors. After working through her errors, she, like Chuck and Khan, answered all parts to both Question 1 and Question 2 during the think-aloud correctly.

The relational understanding of these students enabled them to identify contradicting answers, and the ability to fix their mistakes to have correct final solutions. For other students, however, this was not the case. Some students realized a mistake, but did not have the capability to fix it, and then others did not recognize a mistake or erroneous solution at all.

For example, Nick, for part b of Question 2, found that the rocket reached a maximum height at $-3{,}400$ feet. Rather than going back and checking his work for a mistake, he just changed the sign with no explanation. "You get $-3{,}400$ feet, but since it's it can't be negative, it's just $3{,}400$ feet." He had no justification for this except that he knew by common sense that height can't be negative. His lack of conceptual understanding was also evident as he was not able to find the time of maximum height, or the maximum height correctly, and was representative of the ones who resorted to solving for t from the original free fall function set equal to zero.

Jerry was another student who changed the sign of the maximum height of the rocket with no justification. For part b of Question 2, he found the maximum height of the rocket to be -100 feet. Instead of going back through his work to find what was wrong, he took the absolute value to make it 100 feet. No justification, no reason why, he just knew from common sense that the height should not be negative. His weak understanding of the concepts prevented him from being capable of finding and fixing his mistakes.

Cade and Alex were two other students who changed the sign of an answer on Question 2 with no validation. Both of them, based on the context of the problem, knew that their time for the maximum height should not be negative, so they just changed their answer from negative to positive without figuring out why their answer was negative in the first place. While coming up with a negative time should be an indication that something is wrong in the work, neither one took any steps to try to identify their mistake.

These students' weak understanding of the concepts prevented them to overcome their incorrect solutions. Even more, all of these students who did not have the ability to correct mistakes also struggled and had a lot of other inaccuracies during their think-aloud session.

The think-aloud method results emphasize the importance of relational understanding for students. Those that had a relational understanding of the concepts performed at a higher level, and were able to relate, connect and see concepts as a whole. The students who could only instrumentally understand, if even understand at all lacked the adaptability to new tasks and the development of richer schemas for the different concepts.

4.2 Students' Think-Aloud vs. Group Responses

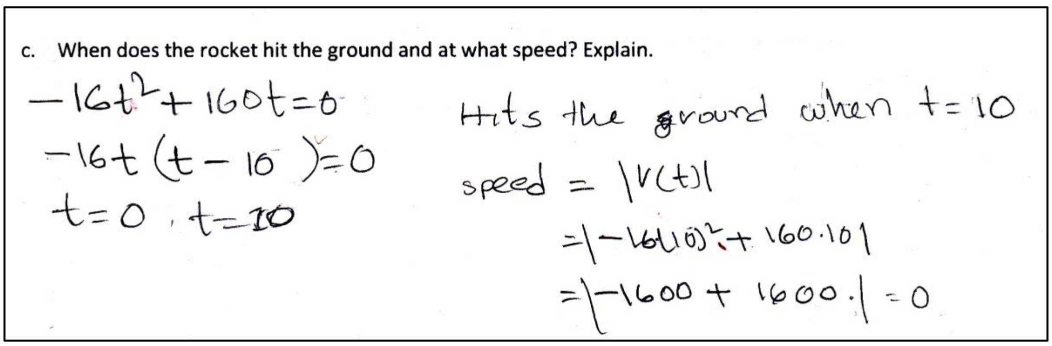
The think-aloud sessions not only provided insight into the students' thought processes as they worked, but they also gave a means of comparison for responses and thoughts of students by themselves during the think-aloud session to the responses and thoughts in collaboration with others during the group interviews. During the think-aloud sessions, as each student worked alone, there was no possibility of influence from others. However, after the think-aloud sessions, when the two students came together as a group with the interviewer, it became evident that some students were highly influenced by the responses of their peer.

4.3.1 Influence from Peers; Vygotsky's Zone of Proximal Development

The influence of peer responses sometimes led to changes from incorrect solutions to correct solutions and possibly a deeper level of understanding of the concepts. Vygotsky (1978) believed that guidance by peers was desirable in that "what a child can do with assistance today she will be able to do by herself tomorrow" (p. 87). He coined this area of learning by social interaction the ZPD. The ZPD is the difference in what a student is capable of individually without help, and what a student is capable of with help from others.

During the group interviews, there was evidence of students' influence on each other as they discussed the different problems. This was mostly evident in groups whose two members were of different levels of capability, one student at a higher level of understanding than the other. This was the case for Ralph and Sam.

Sam, during her think-aloud session was stumped on how to find the speed at which the rocket hit the ground for part c of Question 2. Even though she knew that the absolute value of velocity equals speed, this was only a sign of memorization as she did not know which function was velocity, and could not reason through her mistake. Sam plugged 10, her correct time for when the rocket hit the ground into her original free fall function, rather than plugging it into the derivative.



c. When does the rocket hit the ground and at what speed? Explain.

$-16t^2+160t=0$

$-16t(t-10)=0$

$t=0,\ t=10$

Hits the ground when $t=10$

speed $= |v(t)|$

$= |-16(10)^2 + 160 \cdot 10|$

$= |-1600 + 1600 \cdot| = 0$

Figure 4.30 Sam's solution for speed during think-aloud

When she came up with a speed of 0, she knew something was wrong as she questioned herself "why the speed is 0"? She continued to make comments such as "how do I find that", "I'm not sure about speed", and "how do I find the speed?" Finally, she gave up as she could not figure the solution out. "How do I find the speed then? 90? I don't know. Ok. Speed speed speed. Mmm I don't know. Ok. Done." Sam contemplated and struggled with this concept for several minutes by herself during her think-aloud session before finally giving up.

If not for the think-aloud session, this struggle would not have been apparent as during the group interview, she didn't mention having much trouble and immediately changed her response based on hearing Ralph's explanation. Ralph went first to explain what he did for part c of Question 2, including finding the speed of the rocket as it hit the ground. To find speed, he described that you plug the 10 seconds, the time when the rocket hit the ground, into "the absolute value of the derivative because you'll get a negative value, and if you plug in 10 you'll get 160 feet per second and that's the speed when it hits the ground."

After hearing Ralph's response, when Sam was asked what she did for the same question, she knew, from Ralph's influence, what she had done was wrong. Immediately, she said "I kind of messed up here. I plugged in 10 into the original function, it gave me 0, obviously it will give me 0. I need to plug it into the derivative to get the speed." Sam's ZPD was expanded by Ralph's correct explanation of how he achieved his solution. She then was able to recognize her mistake and correct it, something she was not able to do individually by herself.

Elroy was another student who had trouble with finding the speed the rocket hit the ground. He took the second derivative of the free fall function, and came up with a speed of -32 feet per second when the rocket hit the ground.

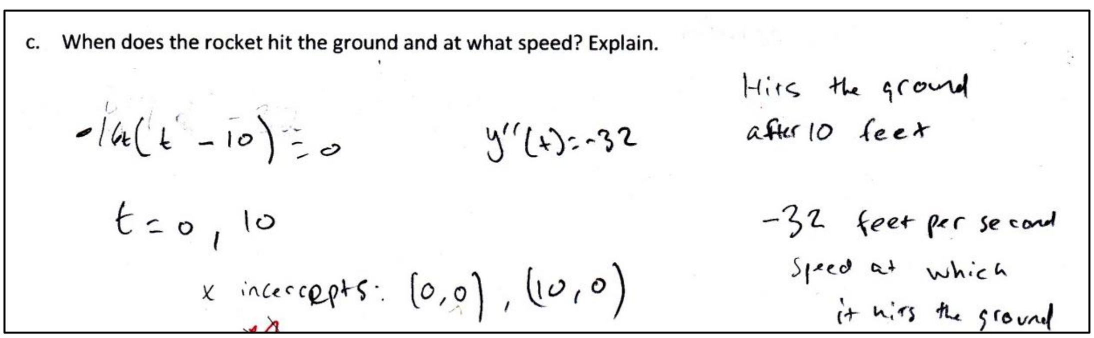
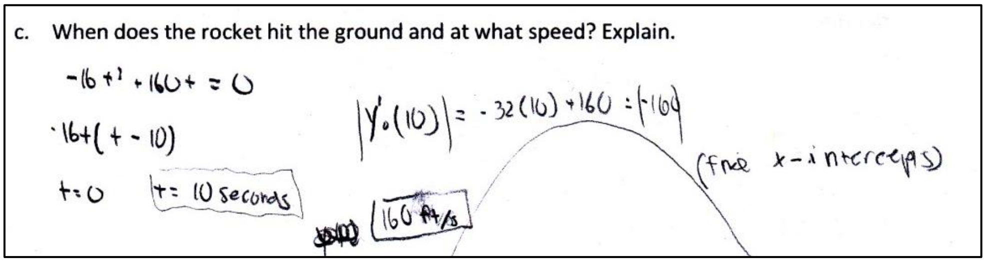


Figure 4.31 Elroy's solution for speed during think-aloud

Elroy did not have the capability to find speed correctly during his individual think-aloud session, yet afterwards, during the group follow up interview, he was influenced by his group member Khan.

When Elroy joined the interviewer and Khan, he knew something was contradicting based on Khan's response for how he found the speed of the rocket when it hit the ground. Khan described how he plugged in 10 seconds, the time the rocket hit the ground, into the absolute value of the velocity function.


c. When does the rocket hit the ground and at what speed? Explain.

$-16t^2+160t=0$

$-16t(t-10)$

$t=0$ $\boxed{t=10\text{ seconds}}$

$|y_0'(10)| = -32(10)+160 = |-160|$

(find x-intercepts)

$\boxed{160\text{ ft/s}}$

Figure 4.32 Khan's solution for speed during think-aloud

He explained his answer during the interview before Elroy did, thus influencing Elroy that his solution was not correct as he said "I got mine a completely different answer, maybe cause I have no idea what I was doing." Based on Khan's response, within his ZPD, Elroy was able to strengthen his understanding of how to find speed by the guidance of his peers' correct solution.

Urray and Henry were two others whose explanations of solutions had an influence on each other. Urray, individually during her think-aloud session, for Question 2, came up with the wrong free fall function for the problem. Her function was $-32t^2+160t$, when it should have been $-16t^2+160t$. Thus, her answers for all parts of Question 2 were incorrect based on her trouble with the free fall formula. However, she had the correct procedure for each part as she worked through them independently.

Then, during the group interview, she was influenced by Henry as she heard Henry's explanations for Question 2 before having to give her own. After Henry correctly explained his solution, she in turn knew there was a mistake with her formula. "I put -32 instead, I got the formula wrong, but,um, I did the same thing that Henry said, find the derivative, set it equal to 0, and then found t, which gave me the maximum, I mean the time when it reaches the maximum height." Based on Henry's explanation, she knew she had the right process, yet the wrong formula. Henry then, chimed in to show Urray where she had made a mistake.

H: I think, I, if you don't mind if I like, like correct it all, I don't want to be one of those jerks who like oh I know what you did wrong, but like um no I think all you did and I actually forgot, like, this was-

U: Yeah, divide that by 2, right, so basically it's just 16.

H: Right, right exactly

U: So everything else is the same.

H: Is the same right.

Within Urray's ZPD, she was able to see her mistake with guidance from Henry, something she was not able to catch on her own. After Henry's guidance during the group interview, it is then possible that she will be capable of doing such problems on her own.

Sara was the student who had a misconception of the critical point for part c of Question 1. She thought that finding the critical point and finding the extreme values were two completely different processes. She pondered out loud and decided to find the x-intercepts as extreme values. She even mentioned that the "local and absolute max would be at $x = 1$, local min would be $x = 5$". Conceptually, this does not even make sense, as a quadratic function cannot have both a maximum and a minimum in the first place. However, this misconception was not even evident during the group interview, as she immediately changed her thoughts after hearing her group peer, Ace, describe what he did to find the critical point. To find the critical point, Ace did as most other students and set the derivative equal to 0 to solve for x.

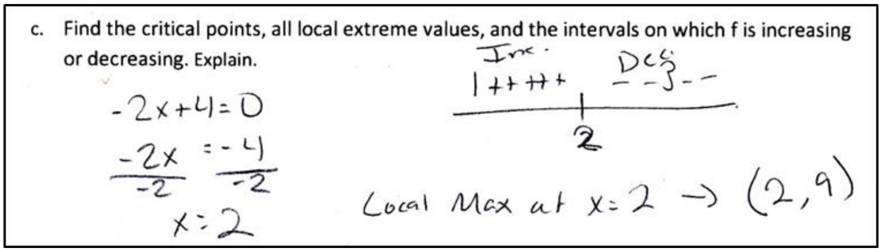


Figure 4.33 Ace's solution, Question 1

He also explained during the group interview how he found the intervals of increase and decrease, and how he found the local maximum to be at the point $(2, 9)$. After hearing all of this, Sara totally changed her response. What she had struggled with so bad conceptually during the think-aloud, was now not apparent.

I: "Ok, ok let me hear what you said [referring to her solution for Question 1, part c, finding critical points, local extreme values, and intervals of increase and decrease]"

S: "Um, I have the same thing. I found my local max at 2. I was kind of, this was a horrible test, so I, I had problems with max and min but… I found my max was $x = 2$…I forgot to find the pair for it, but in order to do that you have to plug it into the original equation".

Sara went from finding the extreme values as the x-intercepts and erroneously identifying the graph of a quadratic function as having both a maximum and minimum to "I have to same thing", referring to Ace's correct solution of a maximum at $(2, 9)$.

Sara's influence by Ace was clear as her explanation of the concept completely changed. She went from not being able to perform individually to changing her incorrect solution after hearing Ace's solution. It is obvious that Ace's answers were within Sara's ZPD since she was able to understand his solution and correct her own solutions.

For many of the students, the follow up interviews also acted as learning sessions, especially for those who had misconceptions or weak understanding of certain concepts. Even more, the students whose interviewing peers offered explanations that were within their ZPD were able to enrich their conceptual understanding of the vertex of a quadratic function and the derivative.

5 DISCUSSION

In this qualitative study, 30 students enrolled in one section of a *Calculus of One Variable I* course participated in a think-aloud session and follow up group interview to identify how students perceive and understand the concept of the vertex of a quadratic function in relation to the derivative. Written work from in class was also collected from these students for further data analysis. The students' responses were analyzed based on three different theoretical frameworks which gave the most insight into each research question. The action-process-object-schema framework (Asiala et al., 1996) was used to analyze how the understanding of the vertex of a quadratic function shapes the understanding of the derivative, how students perceive and understand the concept of the vertex of a quadratic function in relation to the derivative in an explicit and real world problem, and obstacles that students face when using the concept of vertex of quadratic functions in relation to the derivative in different real world problems. Skemp's (1976) relational and instrumental understanding framework was used to distinguish the thoughts of students during the think-aloud sessions, and Vygotsky's (1978) concept of ZPD was used to analyze how the influence of peers impacts the ability of students to understand questions relating the derivative and the quadratic function. This chapter presents the discussion of these results for each research question analyzed by the different frameworks, implications for teaching, limitations of the study, and suggestions for future research.

5.1 Discussion of Results

The results of this study provided understanding and insight for each research question based on the three different theoretical frameworks. First, the study revealed how the understanding of the vertex of a quadratic function shaped the understanding of the derivative. The majority of students had a weak schema of the vertex of a quadratic function, a commonality to Borgen and Manu (2002) and Ellis and Grinstead (2008). Twenty-three of the 30 students' responses that were analyzed contained some type of misconception. These misconceptions further support the idea that students have difficulties with quadratic functions (Afamasaga-Fuata'i, 1992; Eraslan, 2005; Eraslan, 2008; Kotsopoulos, 2007; Metcalf, 2007; Zaslavsky, 1997). The more a student displayed a misconception of the vertex, the more a student was not capable of connecting the concept to the derivative to correctly answer the questions during the think-aloud session. Also, for some, even though there was a misconception of the vertex, they were still able to perform at an action level of understanding with no conceptual understanding. Borgen and Manu (2002) found the same to be true of a student who presented a picture-perfect solution, yet was not aware of the relationships among the components of the question.

Out of the misconceptions identified, the majority had a misconception of the vertex as an intercept. These students were more likely to have a weak schema of the derivative. A common occurrence for these students who had a misconception of the vertex as an intercept was a student performing at an action level of understanding to find the critical point for the explicit problem of Question 1, yet not being able to transfer the concepts to the real world problem in Question 2. Many resorted to solving for the variable t from the original free fall function $f(t) = 0$ to find the time the rocket reached maximum height. The misconception of the vertex as an intercept is suggested to have hindered the ability of students to conceptually understand the derivative. These students who are less capable of performing are likely to need special treatment (Tall & Razali, 1993), and a common suggestion to help learners overcome these misconceptions is to "engage them in situations where their initial conceptions and intuitions fail" (Zazkis et al., 2003 p. 449).

Other misconceptions of the understanding of the vertex of a quadratic function that were identified included the vertex as the origin and the vertex as a point of inflection, possibly resulting from a weak graphical schema of a quadratic function. Some of these students also exhibited a weak schema of the derivative and an incapability to find the maximum for the real world problems. The students' lack of understanding of the vertex could have prevented their ability to relate concepts. This emphasizes the importance of understanding quadratic functions and functions in general as a pre-requisite to understanding calculus. In fact, according to Oehrtman et al. (2008), "a process view of functions is crucial for developing rich conceptual understandings of the content in an introductory calculus course" (p. 11). To foster this, several recommendations are offered including asking the students to explain basic function facts in terms of input and output, as well as to have students make and compare conclusions about functions across multiple representations. It is also suggested that spending time at the beginning of a calculus course exploring, assessing, and strengthening students' function understanding can prove valuable to build conceptual understanding of the major calculus concepts (Oehrtman et al., 2008).

While some students who had a misconception of the vertex could not conceptually explain meaning behind their steps, others who had a misconception of the vertex were capable of conceptually understanding the derivative. Based on correct conceptual responses during the follow up group interview, this suggests that some students may be at a process level of understanding of the derivative, despite the misconception of the vertex. These students' initial misconception did not impact their ability to conceptually understand the derivative. However, their weak schema of the vertex of a quadratic function may have hindered their ability to connect and relate concepts to see the process as a totality, which prevents them from reaching an object level of understanding.

Despite the misconceptions of the understanding of the vertex, which in some cases may or may not have led to an underdeveloped concept of the derivative, some students exhibited a strong knowledge base of the vertex of a quadratic function as the maximum or minimum. Those students who were able to appropriately give meaning to the vertex in turn appeared to have a stronger conceptual understanding of the derivative. The more a student was able to talk and describe about the vertex accordingly, the better the student performed on the think-aloud questions and the more conceptually he or she could talk about the concepts and the relationship between the two. According to Clark, Moore, and Carlson (2008) “there is a link between an individual’s ability to speak with meaning and their content knowledge” (p. 309). They associate a student’s conceptual knowledge with their ability to speak with meaning and communicate meaningfully with others. They suggest that those who are more capable of speaking with meaning present deeper conceptual knowledge. According to APOS, those students who could speak with meaning exhibited at least a process level of understanding, as they could reflect and describe the reasons behind the steps, and some even exhibited an object level of understanding as they recognized the process as a totality.

The last personal meaning of the vertex identified was the vertex and its relationship to the symmetry of the graph. This supports the literature as students strong association of a quadratic function and symmetry was also apparent while preservice teachers and 11^{th} and 12^{th} grade students worked on a translation task involving a quadratic function (Zazkis et al, 2003). About half of the participating preservice teachers found the vertex of the parabola and then described that the rest of the points are symmetric around it. Similarly, for this study, some of the students had a very strong association of the vertex to the symmetric property of the parabola. The students who understood the vertex this way in turn used the property to answer parts of the questions pertaining to the derivative. Their understanding of the vertex enabled them to connect the concept of symmetry to verify or answer questions pertaining to the derivative. Their strong background possibly enabled them to have a more sophisticated understanding of how to approach problems relating to the derivative.

In summary, the understanding of the vertex of a quadratic function does shape the understanding of the derivative in several ways. First, for some students who had a misunderstanding of the vertex as an intercept, origin, or point of inflection, they had a weaker conceptual understanding of the derivative. Many performed on an action level of understanding to the explicit problem, yet could not answer the real world problem and could not relate concepts. Second, for some of those who had a misunderstanding of the vertex as an intercept, origin, or point of inflection, this did not impact the ability of them to perform. Some were capable of correct solutions, based possibly on memorization and an action level of understanding of critical points, extreme values, and intervals of increase and decrease. Next, those who had a stronger understanding of the vertex of a quadratic function as a maximum or minimum, or who related the vertex to the concept of symmetry, were better able to conceptually talk about and understand the derivative, a sign of a process level of understanding. Lastly, those who were able to conceptually understand the vertex of a quadratic function and the derivative, and who were able to make connections between the two to see the process as a totality performed at possibly an object level of understanding.

Next, the study also revealed how students' perceived and understood the concept of the vertex of a quadratic function in relation to the derivative in an explicit and real world problem.

Further supporting the literature on students' inability to solve nonroutine problems (J. Selden et al., 1989; A. Selden et al., 2000; J. Selden et al., 1994) this study discovered an overall lack of connection between the explicit and real world problem. Almost all of the students were able to find the critical points, extreme values, and intervals of increase and decrease when explicitly asked for part c of Question 1 on the think-aloud questions and on Test 3. However, the ability of students to then generalize this concept to find the maximum, or critical point in a real world problem dropped to about half of the students for parts a and b, of Question 2 on the think-aloud questions and on Test 3. Students were not able to recognize that the real world problem of Question 2 was asking for the same thing as Question 1, just in words. This could be because of difficulties in coordinating information (Baker et al., 2000a) or a weak conception of a function (Bingolbali & Monaghan; Breidenbach, et al., 1992; Thompson, 1994). It could also be attributed to an inability of students to vertically transfer information in different situations (Rebello et al., 2007). Students were not able to choose the "most productive internal representation from several representations" for the real world problem (Rebello et al., 2007, p. 231). For this study, several obstacles were identified that students face when using the concept of vertex of quadratic functions in relation to the derivative in different real world problems.

The first obstacle identified was the misconception of the vertex. The misconception of the vertex prevented some students from being able to appropriately connect the ideas from Question 1 to the real world problem of Question 2. By not being able to identify meaning to the vertex correctly, this led some of the students to solve the original function set equal to 0 or to graph the vertex as an intercept incorrectly. The next obstacle identified was trouble with the free fall formula. Many students struggled with the meaning of the formula, and the coefficients involved. This prevented some students from being able to proceed through the problem as many pondered and questioned what certain parts of the formula meant. As students have difficulty with misconception of variable (Abdullah, 2010; Oehrtman et al., 2008; Sajka, 2003; Trigueros & Ursini, 2003) this does not come as a surprise. According to Trigueros & Ursini (2003), "Students' understanding of the concept of variable lacks the flexibility that is expected" at the collegiate level (p. 18). They suggest that it is "necessary to reconsider the way the concept of variable is taught in elementary algebra" and that it is necessary to design curriculum with emphasis to "foster students' understanding of the concept of variable" (Trigueros & Ursini, 2003, p.19).

Lastly, an obstacle of graphing based on a weak schema of quadratic functions was identified. Several students were not capable of graphing a quadratic function, even after finding the maximum point using the derivative. These students lacked a conceptual understanding of quadratic functions and their corresponding graph, and thus, were not able to relate ideas. This suggests that one may need to have an object level of understanding of a parabola and its vertex in order to think of and represent its derivative as the slope of the tangent line at the vertex. The student needs to think of the graph of the parabola as an object to which an action, drawing a tangent line at the vertex, could be applied. The students who faced obstacles with the real world problem could not even perform at an action level, as these obstacles prevented them from reaching even a basic understanding.

These obstacles that students face may also be contributed to the lack of understanding of the function when students first enter a calculus course (Monk, 1994; Tall, 1992). As Monk (1994) states, "Most students come to a calculus course neither equipped with it [understanding of a function] nor on the verge of acquiring it" (p. 27). As the function concept is complex, many high performing undergraduates also possess a weak understanding of the function (Breidenbach et al., 1992). Instructional treatment based on context, technology, and theory-driven activities was implemented by Kalchman (2001) to an experimental group of students in grades 6, 8, and 10. The experimental group showed improved understanding of the function concept and demonstrated a better understanding of how the graphic, tabular, and algebraic representations of functions are connected than the control group who learned from a textbook approach. Instructional treatment in algebra and pre-calculus courses should emphasize the importance of conceptually understanding functions as students ideally need to have a process level of function upon entering a calculus course (Oehrtman et al., 2008)

This study also revealed answers to the sub-research questions. First, the think-aloud method revealed several things about how students understand and think based on Skemp's (1976) relational and instrumental framework. The most common thought process for students as they thought aloud by themselves was instrumental. Students gave a step by step procedure through their work without any reason why they were doing so. Only three students exhibited any relational understanding during the think-aloud session. However, some of the students who thought out loud instrumentally did display relational understanding when probed in the group interview. The students who could explain relationally when probed performed better on the questions, however, this does not take away from the fact that most all thought out loud instrumentally.

The think-aloud method also revealed the ability of students to correct their mistakes while working. Many students made errors in their work as they thought aloud, yet some were able to recognize that a mistake was made, identify it, and then correct it. Those who were capable of identifying and then taking the steps to correct mistakes performed at a much higher level than the others. During the group interview, these students were more likely to conceptually explain and were more capable of relating concepts, demonstrating relational understanding. The students who were not even capable of recognizing a mistake, and who changed answers with no justification had a very weak understanding of concepts which prevented them to recognize and overcome incorrect solutions.

The think-aloud method revealed the importance of relational understanding as those who were capable of performing at a higher level of understanding were in turn able to work through and correct mistakes to see concepts as a whole. On the other hand, those who were not able to form relational understanding lacked the flexibility to perform different tasks pertaining to the same concept and lacked the ability to identify and correct mistakes, an important skill in mathematics.

The last discovery of the study analyzed how the influence of peers impacted the ability of students to understand questions relating the derivative and the quadratic function. For some students, their ZPD was influenced by the other group member, who usually had a higher level of capability and understanding. Based on listening to other students' solutions on given problems before explaining their own caused several students to change their answers and to describe what they did and thought differently. Struggles for some students would not have even been apparent if not for the think-aloud data. In some groups, students' responses impacted the other student to better explain and to realize their mistakes. The idea is that since a student has learned from the other, in turn, they should be able to then perform alone (Blanton, 1998). Small groups may be beneficial to help students overcome their difficulties (Vidakovic, 1993;Vidakovic & Martin, 2004).

5.2 Implications for Teaching

Several implications for teaching are suggested based on the discussion and results. First, since many students are performing at an action level of understanding, it is important to reassess how students are taught. Rather than a traditionally lecture style classroom, it is suggested that students should be pushed past the action level of understanding by requiring students to talk and explain reasons behind their procedures. A pedagogical approach based on APOS theory, *ACE Teaching Cycle* (Asiala et al., 1996; Dubinsky, Schwingendorf, & Mathews, 1995), may potentially enable students to reflect on their work and to interiorize their actions into a process. *ACE Teaching Cycle* consists of three components: team activities, in class discussion, and exercises. Team activities take place in a computer lab where the students work on computer programming tasks, class discussion gives students the opportunity to reflect on their work and connect ideas and concepts, and exercises outside of class provide reinforcement to what is being learned (Asiala et al., 1996; Cottrill et al., 1996). *ACE Teaching Cycle* has shown to be a valuable instructional tool as studies have shown students with this kind of instruction perform at least as well if not better than those who complete a traditional lecture classroom (Weller et al., 2003). However, *ACE Teaching Cycle* is based in a computer lab, something which many schools do not have. In this case, a suggested alternative to this component is in class worksheets or homework sheets in combination with applets/mathlets that are available.

Applets/Mathlets are small interactive web based tools designed for the teaching and learning of mathematics. They require very little knowledge of syntax and thus “are perhaps more effective [than computer algebra systems such as *Maple* or *Mathematica*] because the students are less likely to be distracted by the technology” (Holdener, 2001). Tom Leathrum’s (2002) Mathlets: Java Applets for Math Explorations found at http://cs.jsu.edu/~leathrum/Mathlets/ has multiple interactive learning tools for pre-calculus and calculus concepts. There is a mathlet to graph quadratic functions called Parabolas, which allows the user to change parameters to see how the changes move or shift the graph. There is also a mathlet, Tangent Lines, which allows the user to enter a function $f(x)$ for which the mathlet graphs. It then displays the tangent line to the graph of $f(x)$ for a given x-value. These activities could help students to develop a relationship between the graph of the quadratic function, the vertex, and the derivative.

Also, MIT Mathlets (2009) found at http://mathlets.org/courses/ and Flash & Math (2007) found at http://www.flashandmath.com/mathlets/calc/index.html have mathlets for differentiation and practice sketching the derivative. At http://flashandmath.com/mathlets/calc/index.html, the Derivative Plotter allows the user to enter a function in a box, displays the graph of the function, and then allows the user to attempt to sketch the graph of the corresponding derivative based on the graph of the original function entered. These activities could help students develop a connection between the graph of the quadratic function and the graph of its corresponding derivative. As a supplement to these activities, in class worksheets could be developed to further improve students' understanding of the quadratic function and the derivative, and to help them make connections between the graphical and algebraic representations through discussion and small group work.

Based on this study, most of the students had at most an action level of understanding of quadratic functions and its derivative as considered in different contexts/representations such as: analytical or symbolic, graphical, verbal, and physical or real life. For each of these two concepts (quadratic function and its derivative), it is necessary for students to develop a process or object conception. In particular, in order for students to be able to solve real life problems of falling objects, for example, in an analytic sense they need to be able to think of the derivative of a quadratic function as another (linear) function that is ‘derived’ from the original quadratic function; in a graphical sense they need to think of the derivative as the slope of the tangent line to the quadratic function at a point; in a verbal sense they need to think of the derivative as the instantaneous rate of change; and in physical, as a speed or velocity. As this study showed that students have many obstacles with both of these two concepts (quadratic function and its derivative), a sample set of activities was developed that addresses misconceptions and may help students overcome their obstacles (Appendix F)

The lack of connection between an explicit and real world problem pertaining to the derivative should be addressed by possibly emphasizing the connection between different problem types, and allowing more opportunities for students to work and reflect on these problem types in and out of class. Obstacles that students face when working on real world problems should be addressed in the classroom, and can be incorporated into lessons to help students overcome, or at least minimize these problems. Providing a contextually rich environment by allowing students to discover connections between real world problems and the flexibility of mathematical conceptions may also be beneficial (Schwalbach & Dosemagen, 2000).

It is also important for teachers to understand the significance of students’ potential in developing relational understanding of concepts. The students who develop relational understanding perform better and are more adaptable to new tasks. Assessing relational understanding possibly requires discussion and asking students for out loud explanations. According to Sfard, Nesher, Streefland, Cobb and Mason (1998), “classroom discussion provokes a lot of reflection and gives an opportunity to compare, criticize, refute, complete, reject, and so on” (p. 48). The idea that discussion is like reflection allows students to proactively move forward to a fuller conceptual understanding of concepts. Requiring writing assignments such as journal entries is another way that this conceptual understanding can be assessed by teachers (Hodges & Conner, 2011).

It is also important to take into consideration the positive impact that peers can have on each other in a group setting. Small group work can strengthen the ability of students to explain and perform through discussion with peers (Vidakovic, 1993; Vidakovic & Martin, 2004). While students work within their ZPD with guidance from one another, they are strengthening their capability of being able to perform problems by themselves. By encouraging students to speak to each other with meaning, they may in turn deeper develop their content knowledge (Clark et al., 2008).

The last implication is the idea that the inquiry methods used in this study can be adapted and generalized to be used in the classroom (Silver, 1990). While there may not be the time for detailed interviews and sessions, a general inquiry such as asking students probing questions and to think-aloud through what they are doing can help the teacher to understand what the student is thinking. In fact, thinking aloud may be beneficial for students as they work alone on out of class activities (Chi et al., 1989; Chi et al., 1994; Neuman & Schwarz, 1998; Yair et al., 2000).

5.3 Limitations of the Study

The purpose of this study was to gain insight into student understanding of the concept of the vertex of a quadratic function and its relationship to the derivative. Thirty students participated from one section of a *Calculus of One Variable I* course. Data was collected and analyzed based on their written work, think-aloud session, and follow up group interview. Since students from only one section of the course participated, this limited the quantity of students in the study. Potentially adding a larger population to the study by incorporating more sections of the course could have helped overcome this. For those that did participate, data collection was largely dependent on each student's written work and thoughts.

Another limitation of the study was the lack of longevity. This study took place during one semester. If the study did not have time consideration, then it is possible that follow up data could have been collected, and implementation of the suggested activities could have been studied with the same group.

Lastly, because of a departmental policy of no calculators to be used on tests, calculator use was discouraged in the classroom. Also, the class did not have access to a computer lab, and thus computer activities could not be implemented. These limitations excluded the use of calculators and technology to be incorporated into the study, which may have been beneficial.

5.4 Future Research

Future research is needed as a continuation to improve student understanding of the connection between quadratic functions and the derivative. A study with more participants may reveal more about students' thought processes and could reveal the potential of more obstacles that were not identified in this study. Also, a study with implementation of the activities suggested and their impact on students' learning is suggested. More curriculum development could be designed and tested to figure out the best way to further develop and understand students' conceptual understanding of the quadratic function and its relation to the derivative.

During this study, many students had to be probed for a lengthy period to acknowledge a relationship between the graph of the original quadratic function and the graph of its derivative. While not a focus of this study, the following questions may be worthwhile to future research:

- How do students perceive the relationship of the graph of the original quadratic function in connection to the graph of its derivative?

- Are students able to make conceptual connections between the vertex of the quadratic function and the critical point of the derivative?

Lastly, this study did not include incorporation of technology influence. It would be valuable to include the use of graphing calculators and/or computers in future studies to see if this impacts student understanding.

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APPENDICES

Appendix A

In Class Questionnaire

1. What is a quadratic function?

2. Give an example of and algebraic expression of a quadratic function.

3. What does a quadratic function look like graphically? What is the shape?

4. What is the vertex of a quadratic function? Is there another name for the vertex?

5. Does this relate to the derivative in any way? If so, how?

6. How can you find the vertex?

7. Is there more than one way?

Appendix B **Test 3 Questions**

1. For the function $f(x) = -x^2 + 4x - 3$:
    a. Find the domain of the function and all x and y intercepts of f. Explain.

b. Find the critical points, all local extreme values, and the intervals on which f is increasing or decreasing. Explain.
c. Use the information you have found to sketch the graph of $y = f(x)$.

2. An explosion causes some debris to rise vertically with an initial velocity of 128 feet per second.
   a. In how many seconds does this debris attain maximum height? Explain.
   b. What is this maximum height? Explain.
   c. When does the debris hit the ground and at what speed? Explain.

Appendix C **Final Exam Questions**

For the following questions, you must show all work and explain with words for credit.

1. Consider an object projected directly upward. Because of gravity, we may consider that it is constrained to move in a straight line. It will move directly upward, and then turn around to complete its journey downward. Suppose that the position of the object t seconds after it is projected is given by the formula

$$p = P(t) = 80t - 16t^2$$

   a. Graph the position function $P(t)$. Explain.

   b. Now graph its derivative $D(P(t)) = P'$. Explain.

   c. Are the graphs in parts (a) and (b) related? If so, explain and elaborate in words how.

   d. What does the derivative represent in terms of the particle position? Explain.

   e. Based on the graphs, find the initial velocity of this object. Explain and elaborate.

   f. Confirm the initial velocity algebraically. Explain.

   g. How can you decide which direction is it going? i.e. How can you find when the particle is going up or going down? Explain and elaborate clearly.

   h. How long does it take for the particle to reach its maximum height above the point at which it was released? Explain.

   i. How does the graph of the velocity function relate to part (g)? Or does it? How does the graph of the velocity function relate to part (h)? Or does it? Explain and elaborate.

Appendix D **Think-Aloud Session Questions**

Instructions: The following questions consist of content from chapter 4 over the derivative. As you work through them, say out loud everything that comes to mind. It is not about right or wrong, but just speaking out loud what comes to you as you work. Please speak as loud and clear as you can. Feel free to ask any questions. Thank you!!

1. Given $f(x) = -x^2 + 4x + 5$, answer the following:
   a. What type of function is $f(x)$? What is the shape of the graph of $f(x)$? Explain.

   b. Find $f'(x)$. What does $f'(x)$ mean? What does it give us? Graphically? Explain.

   c. Find the critical points, all local extreme values, and the intervals on which f is increasing or decreasing. Explain.

   d. Sketch the graph. Explain.

For Question 3, neglect air resistance and take g as 32 feet per second per second.

2. A rocket is fired from the ground straight up with an initial velocity of 160 feet per second.
   a. When does the rocket reach maximum height? Explain.

   b. What is maximum height? Explain.

   c. When does the rocket hit the ground and at what speed? Explain.

   d. Sketch the graph. Explain.

To ensure that we obtain consistent and comparable data, each student interviewed over the same problems that they have completed in the think-aloud session will be presented with the same questions in the same order. However, if we need to further clarify the student's reasoning or obtain deeper insight into their solution, we may ask additional questions.

Prior to the think-aloud session, we will introduce ourselves and give a brief overview of what we are doing. We will then ask the student a few questions to "break the ice."

1. What is your favorite course at GSU? Why?
2. How do you feel about math? What are your favorite and least favorite things to study in your math course?

After the introductory questions, we will ask them to take no more than 30 minutes to answer the questions provided on the paper. Then, we will tell the student that they may quit at any time, if they wish. For those who decide to proceed, we will ask them to work on the problems alone and to talk out loud as they are working everything that comes to mind until they are finished or cannot go any farther, but no longer than 30 minutes. They will be reminded that it is important to always remember to keep talking. It is not about being right or wrong, but about speaking out loud what comes to mind while working. While they are working, we will observe their progress and methods.

If the student is stuck, we will ask them questions to get them thinking in the right direction.

1. Can you explain this problem in your own words?
2. What are you supposed to do here? What is happening?
3. Are there any words that might tell you what to do?

When the student can go no further or has answered all questions, we will have them walk us through each step of their work.

1. What is your answer to the first problem/question?
2. Will you give me your detailed explanation?
3. What made you choose that approach?

If necessary, we may ask additional questions for clarification or to probe deeper into the student's methods.

At the end of the each problem, we will ask some wrap-up questions:

1. What did you think about this problem? Hard? Easy?
2. Did you have any problem in understanding this problem? Can you elaborate?
3. What do you think might have helped to develop a better understanding of these kind of problems? Elaborate.

This process will be repeated for all questions that are included in the instrument. Upon completion of all questions, we will ask them some concluding questions:

1. Do you believe that mathematical literacy is important for the learning of mathematics?
2. Do you believe that mathematical literacy is important for teaching mathematics?

Appendix F Suggested Activities to Help Students Overcome Obstacles

Activity 1 and Activity 2 can be given as homework assignments and discussed in class. Students can use the library to complete them if they don't have their own computer. In class with one desk computer, the instructor can call on different students to share their solutions and then develop a discussion around it. Activity 3 can be worked on in small groups during class with discussion.

Activity 1

Note to the Instructor: Objective of this activity is to explore the relationship of the sign of the derivative and where the function is increasing or decreasing by looking at the relationship of the sign of the slope of the tangent lines and the values of the quadratic function.

Go to the website http://cs.jsu.edu/~leathrum/Mathlets/ . Open up the mathlet titled Tangent Lines.

1. Enter the function x^2 in the text input field marked "$f(x) =$".

2. Click the "Graph" button.

What is the name of this type of graph? What are the coordinates of the vertex? Is the vertex a maximum or minimum? Is the graph symmetric? If yes, what is the line of symmetry?

3. Enter the derivative $f'(x)$ of the function $f(x)$ in the text input field marked "$f'(x) =$".

4. Select three x-values to the left of the minimum by either clicking and dragging the mouse on the graph or by entering in values for x in the text input field marked "$x =$"

Observe the tangent lines. Are the slopes of the tangent lines increasing, decreasing, horizontal, or vertical? Find the slopes of the tangent lines. What is happening with the values of the function f at your selected point as you move from the smallest to the largest x-value?

5. Click the “Graph” button to refresh. Now select three x-values to the right of the minimum by either clicking and dragging the mouse on the graph or by entering in values for x in the text input field marked “$x =$”

Observe the tangent lines. Are the slopes of the tangent lines increasing, decreasing, horizontal, or vertical? Find the slopes of the tangent lines. What is happening with the values of the function f at your selected point as you move from the smallest to the largest x-value?

6. Click the graph button to refresh. Now select the x-value at the minimum by either clicking and dragging the mouse on the graph or by entering in the x-value of the vertex in the text input field marked “$x =$”

Observe the tangent line. Is the tangent line increasing, decreasing, horizontal, or vertical? What is the slope of the tangent line?

What is the relationship between the sign of the slope of the tangent line on the intervals where the function is increasing or decreasing? What is the relationship between the slope of the tangent line and where the function has a minimum? Explain.

7. Click the “Clear” button and now enter the function $x^2 - 4$ in the text input field marked “$f(x) =$”.

8. Click the “Graph” button.

What is the name of this type of graph? What are the coordinates of the vertex? Is the vertex a maximum or minimum? Is the graph symmetric? If yes, what is the line of symmetry? How does the graph compare to the graph of the function x^2? How does the vertex of $x^2 - 4$ compare to the vertex of the function x^2?

9. Enter the derivative $f'(x)$ of the function $f(x)$ in the text input field marked "$f'(x) =$".

4. Select three x-values to the left of the minimum by either clicking and dragging the mouse on the graph or by entering in values for x in the text input field marked "$x =$"

Observe the tangent lines. Are the slopes of the tangent lines increasing, decreasing, horizontal, or vertical? Find the slopes of the tangent lines. What is happening with the values of the function f at your selected point as you move from the smallest to the largest x-value?

5. Click the "Graph" button to refresh. Now select three x-values to the right of the minimum by either clicking and dragging the mouse on the graph or by entering in values for x in the text input field marked "$x =$"

Observe the tangent lines. Are the slopes of the tangent lines increasing, decreasing, horizontal, or vertical? Find the slopes of the tangent lines. What is happening with the values of the function f at your selected point as you move from the smallest to the largest x-value?

6. Click the graph button to refresh. Now select the x-value at the minimum by either clicking and dragging the mouse on the graph or by entering in the x-value of the vertex in the text input field marked "$x =$"

Observe the tangent line. Is the tangent line increasing, decreasing, horizontal, or vertical? What is the slope of the tangent line?

What is the relationship between the sign of the slope of the tangent line on the intervals where the function is increasing or decreasing? What is the relationship between the slope of the tangent line and where the function has a minimum? Explain.

13. Click the "Clear" button and now enter the function $-x^2 + 4x - 5$ in the text input field marked "$f(x) =$".

14. Click the "Graph" button.

What is the name of this type of graph? What are the coordinates of the vertex? Is the vertex a maximum or minimum? Is the graph symmetric? If yes, what is the line of symmetry? How does the graph compare to the graph of the function x^2? How does the graph compare to the graph of the function $x^2 - 4$? How does the vertex of $-x^2 + 4x - 5$ compare to the vertex of the function x^2? How does the vertex of $-x^2 + 4x - 5$ compare to the vertex of the function $x^2 - 4$?

Now, before using the mathlet, guess what the relationship between the slope of the tangent line on the intervals where the function is increasing or decreasing will be. Can you guess what the slope of tangent line will be at the maximum?
Write your guess here:

Now, check your guess by following the steps:

15. Enter the derivative $f'(x)$ of the function $f(x)$ in the text input field marked "$f'(x) =$".

16. Select three x-values to the left of the maximum, three x-values right of the maximum, and the x-value of the maximum by either clicking and dragging the mouse on the graph or by entering in a value for x in the text input field marked "$x =$"

Observe how the tangent line changes each time.

In general, what is the relationship between the slope of the tangent line on the intervals where the function is increasing or decreasing? What can you conjecture about the tangent line and the slope of the tangent line at a maximum or minimum? Explain.

Activity 2

Note to the Instructor: Objective of this activity is to explore the relationship of the graph of the quadratic function and the graph of its derivative.

Go to the website http://flashandmath.com/mathlets/calc/index.html . Open up the mathlet titled Derivative Plotter. Click on the screen shot to open the mathlet in a new window.

1. Click on the dropdown menu and select 'User defined function'.
2. Enter in the box the function x^2 using the syntax x^2.
3. Press the GRAPH button.

What is the name of this type of graph? What are the coordinates of the vertex? Is the vertex a maximum or minimum? Is the graph symmetric? If yes, what is the line of symmetry?

4. Find the derivative of the function x^2.
5. Now, with your cursor, draw the corresponding derivative. If you mess up, you can press the ERASE button to start over drawing the derivative function.
6. When you are finished, drag the slider (located under the graph) to display the derivative function.

What is the zero of the derivative function (where does the graph cross the x-axis)? How is that point related to the vertex of the quadratic function? Would this always be the case? Why, or why not? Explain. What is the relationship of the intervals of increase and decrease of the quadratic function and the sign of the derivative function? Or is there one? Explain.

7. Now press the RESET button, and enter the function $x^2 + 2x - 4$ in the box using the syntax x^2+2*x-4.

8. Press the GRAPH button.

What is the name of this type of graph? What are the coordinates of the vertex? Is the vertex a maximum or minimum? Is the graph symmetric? If yes, what is the line of symmetry? How does the graph of this quadratic function compare to the graph of the function x^2? How does the vertex of this quadratic function compare to the vertex of the function x^2?

9. Find the derivative of the function $x^2 + 2x - 4$.

8. With your cursor, draw the corresponding derivative.

9. When you are finished, drag the slider (located under the graph) to display the derivative function.

What is the zero of the derivative function (where does the graph cross the x-axis)? How is that point related to the vertex of the quadratic function? Would this always be the case? Why, or why not? Explain. What is the relationship of the intervals of increase and decrease of the quadratic function and the sign of the derivative function? Or is there one? Explain.

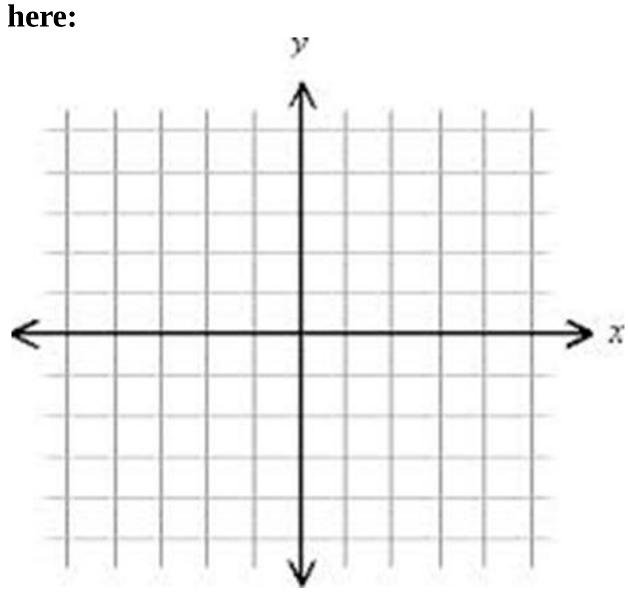
10. Now press the RESET button, and enter the function $-2x^2 + 6x + 4$ in the box using the syntax -2*x^2+6*x+4.

11. Press the GRAPH button.

What is the name of this type of graph? What are the coordinates of the vertex? Is the vertex a maximum or minimum? Is the graph symmetric? If yes, what is the line of symmetry? How does the graph of this quadratic function compare to the graph of the function x^2? How does the graph of this quadratic function compare to the graph of the function $x^2 + 2x - 4$? How does the vertex of this quadratic function compare to the vertex of the function x^2? How does the vertex of this quadratic function compare to the vertex of the function $x^2 + 2x - 4$?

12. Find the derivative of the function $-2x^2 + 6x + 4$.

Now, before using the mathlet, observing the graph of f, can you sketch the graph of the derivative function? Can you guess where the derivative will cross the x-axis?
Sketch the graph of f' here:



Now, check your guess by following the steps:

12. With your cursor, draw the corresponding derivative.

13. When you are finished, drag the slider (located under the graph) to display the derivative function.

What is the zero of the derivative function (where does the graph cross the x-axis)? How is that point related to the vertex of the quadratic function? Would this always be the case? Why, or why not? Explain. What is the relationship of the intervals of increase and decrease of the quadratic function and the sign of the derivative function? Or is there one? Explain.

14. Now create several of your own quadratic functions. Graph each one, and then graph the corresponding derivative function. Check your answers by following the steps above.

In general, what can you conjecture about the relationship between the graph of the quadratic function and the graph of its derivative? How are they related? Or are they?

Activity 3

Note to the Instructor: Objective of this activity is to explore the application of the quadratic function and its derivative in a real world context using the free fall formula.

Now consider an object projected directly upward. Because of gravity, we may consider that it is constrained to move in a straight line. It will move directly upward, and then turn around to complete its journey downward. We will assume that the object is in free fall; that the gravitational pull on the object is constant throughout the fall and that there is no air resistance. Galileo's formula for the free fall and height of the object at each time t of the fall is given: $y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$, where g is a positive constant the value of which depends on the units used to measure time and the units used to measure distance. For this formula, the point of reference is the ground level and the positive y direction is up.

Find $y(0)$. What can you infer that this constant represents in terms of the object at time $t = 0$?

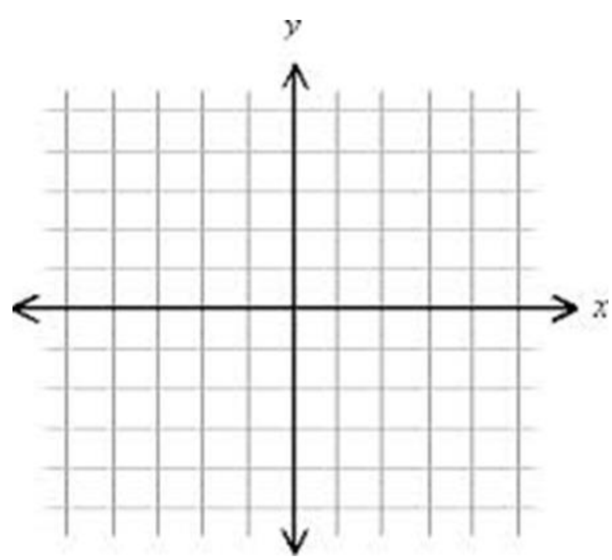
Find $y'(t)$ and $y'(0)$. What can you infer that this constant represents in terms of the object at time $t = 0$?

Find $y''(t)$. What can you infer that this constant represents in terms of the object at time $t = 0$?

Now, for the following, neglect air resistance and take g as 32 feet per second per second. Suppose that an object is shot from the ground straight up with an initial velocity of 112 feet per second.

1. Find the expression for the position function, $y(t)$, of the object t seconds after it is projected.

2. Graph the position function $y(t)$.



What is the name of this type of graph? What are the coordinates of the vertex? Is the vertex a maximum or minimum? Is the graph symmetric? If yes, what is the line of symmetry? How is this graph similar to the graphs in the previous tasks? Or is it?

3. Based on the previous activities what do you predict to be true about the tangent lines and slope of tangent lines of the graph for different t-values? What do you predict to be true about the tangent line and slope of the tangent line at the maximum point?

4. Find the derivative, $y'(t)$. What does the derivative represent in terms of the particle position? Based on the previous activities, predict where the graph of the derivative will cross the x-axis?

6. Graph the derivative of $y(t) = y'(t)$ on the same coordinate plane.

7. How does the graph of the derivative function relate to the direction of the particle? Or does it? How does the graph of the derivative function relate to the time it takes for the particle to reach its maximum height? Or does it?