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The Minimum Rank of Sign Pattern Matrices with a 1-Separation

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THE MINIMUM RANK OF SIGN PATTERN MATRICES WITH A 1-SEPARATION

by

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Under the Direction of Dr. Marina Arav and Dr. Hein van der Holst

ABSTRACT

Given a sign pattern matrix M composed of two sub-patterns A and B connected by a 1-separation, we provide a formula that relates the minimum rank of M to the minimum rank of some small variations of A and B.

INDEX WORDS: Minimum rank, Sign pattern, Separation

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

Since the introduction of sign pattern matrices (or sign patterns) by the economist P. Samuelson [12], their study has become an important topic of research in matrix analysis [6]. In particular, the study of the minimum rank of a sign pattern attempts to determine the minimum rank among all real matrices corresponding to a given sign pattern matrix. This research has found important applications in areas such as the study of communication complexity in computer science. For example, Forster [5] established a linear lower bound on the complexity of unbounded error probabilistic communication protocols, using a lower bound of the minimum rank of $(+,-)$-symmetric sign patterns.

The determination of the minimum rank of a sign pattern matrix, however, is not an easy task (see, for example, [1, 4, 9, 11]). Due to the many connections between graph theory and sign pattern matrices, one strategy could be to extend results of graph theory to the study of sign patterns [7]. However, the results of graph theory may not be readily applicable and may require modifications to fit the different constrains posed by sign pattern matrices. We will show that for some special type of sign pattern matrices, namely sign patterns with a 1-separation, we may find their minimum ranks by studying the minimum ranks of their submatrices. Specifically, we extend the results of the minimum rank of a simple graph with a 1-separation discovered, independently, by Hsieh [8] and by Barioli, Fallat, and Hogben[3] to the study of the minimun rank of sign pattern matrices. In the study of the minimum rank of a simple graph with a 1-separation, the matrices we are dealing with are symmetric and their off-diagonal entries are distinguished only on the basis of zero and non-zero. In the study of the minimum rank of sign pattern matrices, however, the matrices are not necessarily symmetric; and in addition to zero/non-zero pattern, non-zero entries in a sign pattern matrix may be further distinguished as "positive" or "negative".

1.1 Basic Definition and Terminology

A *sign pattern matrix* (or a *sign pattern*) is a matrix whose entries are from the set $\{+, -, 0\}$. We define a *submatrix of a sign pattern* A or a *subpattern of a sign pattern* A to be the matrix formed by the entries from a selected subset of the rows and columns of A in their same relative positions.

A *real matrix* is a matrix whose entries are real numbers. For a real matrix B, $\text{sgn}(B)$ denotes the sign pattern matrix whose entries are the signs of the corresponding entries in B (i.e., replacing positive entries by $+$ and negative entries by $-$). If A is a sign pattern matrix, the *sign pattern class* of A, denoted $Q(A)$, is the set of all real matrices whose entries have a sign pattern corresponding to A:

$$Q(A) = \{B : B \text{ is a real matrix and } \text{sgn}(B) = A\}$$

A *permutation pattern* is a square sign pattern with entries from the set $\{0, +\}$ such that there is exactly one $+$ in each row and each column of the matrix. In other words, it is the sign pattern of a permutation matrix. A sign pattern matrix B is called *permutationally equivalent* to a sign pattern matrix A if $B = P_1AP_2$, where P_1 and P_2 are permutation patterns. Moreover, if $B = P^TAP$, where P is a permutation pattern, then we say that B is *permutationally similar* to A.

Similarly, terms such as "diagonal pattern", "triangular pattern", and "identity" refer to the sign pattern matrices of real matrices associated with the corresponding terms (i.e., "diagonal matrix", "triangular matrix", and "the identity matrix"). Specifically, a *diagonal pattern* is a sign pattern matrix all of whose off-diagonal entries are zero. An $n \times n$ diagonal pattern all of whose diagonal entries are $+$ is called the *identity* of order n, denoted I_n.

A *signature pattern* is a diagonal sign pattern with $+$ or $-$ diagonal entries. If for some square sign patterns A and B, we have $B = SAS$, where S is a signature pattern, then we say that B is *signature similar* to A.

The *minimum rank* of a sign pattern matrix A, denoted $\text{mr}(A)$, is defined by

$$\text{mr}(A) = \min\{\text{rank } B : B \text{ is a real matrix and } B \in Q(A)\}.$$

In other words, the minimum rank of a sign pattern A is established by studying the ranks of all real matrices in the sign pattern class of A and finding the one whose rank is smallest. Some progress has been made in characterizing sign patterns with minimum rank 2, as well as giving the upper bound for some special types of sign pattern matrices. For example, researchers in [11] noted that a sign pattern matrix A has minimum rank 2 if and only if (a) its condensed sign pattern A_c (meaning that their is no zero row or column, and that no two rows (columns) are identical or negative of each other) has at least two rows and two columns, (b) each row and column of A_c has at most one zero entry, and (c) there are signature sign patterns D_1 and D_2 and permutation sign patterns P_1 and P_2 such that each row and each column of $P_1D_1A_cD_2P_2$ is non-decreasing. They further noted that a condensed sign pattern matrix A with at least two columns will have minimum rank 2 if and only if each row of A has no more than one zero entry and there exist a permutation sign pattern P and a signature sign pattern D such that each row of ADP is neither decreasing nor increasing. In studies of communication complexity, the upper bound of the minimum rank of $(+,-)$ sign pattern matrix with at most k sign changes in each row was found to be $k+1$ (cited in [11]).

Recently, there have been some papers concerning the rational realization of the minimum rank of a sign pattern [1, 2, 10, 11]. The *rational minimum rank*, denoted $\text{mr}_Q(A)$, is defined by

$$\text{mr}_Q(A) = \min\{\text{rank } B : B \text{ is a rational matrix and } B \in Q(A)\}$$

In this study, we will focus on the minimum rank of a sign pattern matrix over the real.

1.2 Definition: Separation

Let $m, n, r, s \in \mathbb{N}, 0 \leq k \leq \min(m, n, r, s)$ and let

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$$

be $m \times n$ and $r \times s$ real matrices, respectively, where $A_{2,2}$ and $B_{1,1}$ are $k \times k$. Then the *k-subdirect sum* of A and B, denoted by $A \bigoplus_k B$, is the matrix

$$A \bigoplus_k B = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & A_{2,2} + B_{1,1} & B_{1,2} \\ 0 & B_{2,1} & B_{2,2} \end{bmatrix}$$

A *separation* (A, B) of a matrix M is a pair of submatrices A and B of M such that $A \bigoplus_k B = M$. The *order* of a separation equals k. Correspondingly, we call $\operatorname{sgn}(M = A \bigoplus_k B)$ a sign pattern matrix with a $k-$separation. In other words, a sign pattern matrix A with a 1-separation is an $m \times n$ sign pattern matrix that, after necessary permutations of lines, can be expressed in the following form:

$$A_{m\times n} = \left[\begin{array}{c|c|c} A_1 & & \mathbf{0} \\ \hline & B_{1\times 1} & \\ \hline \mathbf{0} & & A_2 \end{array}\right]$$

, where $A_1 = [\alpha_{ij}]$ is a $p \times q$ matrix, $A_2 = [\beta ij]$ is an $r \times s$ matrix, with the conditions that $p + r - 1 = m, q + s - 1 = n$, and $\alpha_{pq} = \beta_{11}$ (i.e., A_1 and A_2 share the element $B_{1\times 1}$).

Similarly, a sign pattern matrix A with a 2-separation is an $m \times n$ sign pattern matrix that, after necessary permutations of lines, can be expressed in the following form:

$$A_{m\times n} = \left[\begin{array}{c|c|c} A_1 & & \mathbf{0} \\ \hline & B_{2\times 2} & \\ \hline \mathbf{0} & & A_2 \end{array}\right]$$

, where $A_1 = [\alpha_{ij}]$ is a $p \times q$ matrix, $A_2 = [\beta ij]$ is an $r \times s$ matrix, with the conditions that $p + r - 2 = m, q + s - 2 = n$, and A_1 and A_2 share the submatrix $B_{2\times 2}$, which is a 2×2 matrix.

1.3 Submatrix Notations

Let $B \in M_{m,n}(\mathbb{R})$ be an $m \times n$ real matrix with row indices in set M and column indices in set N. We use $B[\overline{m}, n]$ to denote an $(m-1) \times 1$ submatrix with row indices in $M - \{m\}$ and column index n; $B[m, \overline{n}]$ denotes a $1 \times (n-1)$ submatrix with row index m and column indices in $N - \{n\}$; $B(i, j)$ denotes an $(m-1) \times (n-1)$ submatrix with row indices in $M - \{i\}$ and column indices in $N - \{j\}$. When $i = j$, we write $B(i)$ instead of $B(i, i)$.

CHAPTER 2

THE MINIMUM RANK OF SIGN PATTERN MATRICES WITH A 1-SEPARATION

2.1 Some Lemmas

In the following lemmas, we will use the well-known fact that for any $m \times n$ matrix A and any $n \times p$ matrix B,

$$\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}.$$

Lemma 1. *Let*

$$C = \begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} D_{1,1} & D_{1,2} \\ D_{2,1} & D_{2,2} \end{bmatrix},$$

where $C_{1,1}$ is an $m \times m$ matrix, $D_{2,2}$ is an $n \times n$ matrix, and $C_{2,2}$ and $D_{1,1}$ are $k \times k$ matrices. Then rank$(C \oplus_k D) \leq$ rank$(C \oplus D)$.

Proof. Let

$$P = \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_k & 0 \\ 0 & I_k & 0 \\ 0 & 0 & I_n \end{bmatrix}.$$

A calculation shows that $P^T(C \oplus D)P = C \oplus_k D$. Hence $\operatorname{rank}(C \oplus_k D) \leq \operatorname{rank}(C \oplus D)$. □

The proof of the following lemma is clear.

Lemma 2. *If the matrix B is obtained from C by deleting one row, then*

$$rank(C) \leq rank(B) + 1$$

.

From the previous lemma one easily obtains:

Lemma 3. *For each $m \times n$ matrix C, where $m, n \geq 1$, $rank(C) \leq rank(C(m,n)) + 2$.*

Lemma 4. *For any $m \times n$ real matrix B with $m, n \geq 1$, and any nonzero real numbers a and c,*

$$rank(\begin{bmatrix} 0 & a & 0 \\ c & b_{1,1} & B[1,\overline{1}] \\ 0 & B[\overline{1},1] & B(1) \end{bmatrix}) = rank(B(1)) + 2.$$

Proof. Let

$$P = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{b_{1,1}}{2a} & 1 & 0 \\ -\frac{B[\overline{1},1]}{a} & 0 & I_{m-2} \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} \frac{1}{c} & -\frac{b_{1,1}}{2c} & -\frac{B[1,\overline{1}]}{c} \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix}.$$

Then

$$P \begin{bmatrix} 0 & a & 0 \\ c & b_{1,1} & B[1,\overline{1}] \\ 0 & B[\overline{1},1] & B(1) \end{bmatrix} Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & B(1) \end{bmatrix}.$$

From this the lemma easily follows. □

Lemma 5. *Let $A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$ be a real matrix, where $A_{1,1}$ is $m_1 \times n_1$, $A_{1,2}$ is $m_1 \times n_2$, $A_{2,1}$ is $m_2 \times n_1$, and $A_{2,2}$ is $m_2 \times n_2$. If $x \in \ker(A_{2,2}^T)$ and $y \in \ker(A_{2,2})$, then*

$$rank \begin{bmatrix} 0 & x^T A_{2,1} & 0 \\ A_{1,2}y & A_{1,1} & A_{1,2} \\ 0 & A_{2,1} & A_{2,2} \end{bmatrix} = rank\, A.$$

Proof. Let

$$P = \begin{bmatrix} 0 & x^T \\ I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & I_{n_1} & 0 \\ y & 0 & I_{n_2} \end{bmatrix}.$$

Then

$$PAQ = \begin{bmatrix} 0 & xA_{2,1} & 0 \\ A_{1,2}y & A_{1,1} & A_{1,2} \\ 0 & A_{2,1} & A_{2,2} \end{bmatrix}.$$

Hence, rank $\begin{bmatrix} 0 & x^TA_{2,1} & 0 \\ A_{1,2}y & A_{1,1} & A_{1,2} \\ 0 & A_{2,1} & A_{2,2} \end{bmatrix} \leq \operatorname{rank} A$. The other inequality is clear. □

Lemma 6. *Let* $A = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & a_{2,2} & A_{2,3} \\ 0 & A_{3,2} & A_{3,3} \end{bmatrix}$ *be an* $m \times n$*, where* $A_{1,1}$ *is* $m_1 \times n_1$ *and* $A_{3,3}$ *is* $m_2 \times n_2$*, (and so* $m = m_1 + m_2 + 1$ *and* $n = n_1 + n_2 + 1$*). Then at least one of the following holds:*

(i) *There exist vectors* $v \in \mathbb{R}^{m_1}$ *and* $z \in \mathbb{R}^{n_1}$ *such that*

$$rank(\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & v^TA_{1,1}z \end{bmatrix}) + rank(\begin{bmatrix} a_{2,2} - v^TA_{1,1}z & A_{2,3} \\ A_{3,2} & A_{3,3} \end{bmatrix}) = rank(A).$$

(ii) $rank(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + rank(\begin{bmatrix} A_{2,3} \\ A_{3,3} \end{bmatrix}) + 1 = rank(A).$

(iii) $rank([A_{1,1}\ A_{1,2}]) + rank([A_{3,2}\ A_{3,3}]) + 1 = rank(A).$

(iv) $rank(A_{1,1}) + rank(A_{3,3}) + 2 = rank(A).$

Proof. Suppose first that $[A_{2,1}\ A_{2,3}]x = 0$ for all $x \in \ker(A_{1,1} \oplus A_{3,3})$ and that $y^T \begin{bmatrix} A_{1,2} \\ A_{3,2} \end{bmatrix} = 0$ for all $y \in \ker((A_{1,1} \oplus A_{3,3})^T)$. Then there exist a vector $v \in \mathbb{R}^{m_1}$ such that $v^TA_{1,1} = A_{2,1}$

and a vector $z \in \mathbb{R}^{n_1}$ such that $A_{1,1}z = A_{1,2}$. Let

$$P = \begin{bmatrix} I_{m_1} & 0 & 0 \\ v^T & 0 & 0 \\ -v^T & 1 & 0 \\ 0 & 0 & I_{m_2} \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} I_{n_1} & z & -z & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix}.$$

A calculation shows that

$$PAQ = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & v^T A_{1,1} z \end{bmatrix} \oplus \begin{bmatrix} a_{2,2} - v^T A_{1,1} z & A_{2,3} \\ A_{3,2} & A_{3,3} \end{bmatrix}.$$

Hence,

$$\text{rank}(A) \geq \text{rank}(\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & v^T A_{1,1} z \end{bmatrix}) + \text{rank}(\begin{bmatrix} a_{2,2} - v^T A_{1,1} z & A_{2,3} \\ A_{3,2} & A_{3,3} \end{bmatrix}).$$

By Lemma 1, also the opposite inequality holds.

Suppose next that $[A_{2,1}\ A_{2,3}]x = 0$ for all $x \in \ker(A_{1,1} \oplus A_{3,3})$ and that there exists a vector $y \in \ker((A_{1,1} \oplus A_{3,3})^T)$ such that $y^T \begin{bmatrix} A_{1,2} \\ A_{3,2} \end{bmatrix} = e \neq 0$. By Lemma 5,

$$\text{rank}(\begin{bmatrix} 0 & 0 & e & 0 \\ 0 & A_{1,1} & A_{1,2} & 0 \\ 0 & A_{2,1} & a_{2,2} & A_{2,3} \\ 0 & 0 & A_{3,2} & A_{3,3} \end{bmatrix}) = \text{rank}(A).$$

Hence,

$$1 + \text{rank}(\begin{bmatrix} A_{1,1} & 0 \\ A_{2,1} & A_{2,3} \\ 0 & A_{3,3} \end{bmatrix}) = \text{rank}(A).$$

Since

$$\text{nullity}(\begin{bmatrix} A_{1,1} & 0 \\ A_{2,1} & A_{2,3} \\ 0 & A_{3,3} \end{bmatrix}) = \text{nullity}(\begin{bmatrix} A_{1,1} & 0 \\ 0 & A_{3,3} \end{bmatrix}),$$

we obtain

$$\text{rank}(\begin{bmatrix} A_{1,1} & 0 \\ A_{2,1} & A_{2,3} \\ 0 & A_{3,3} \end{bmatrix}) = \text{rank}(\begin{bmatrix} A_{1,1} & 0 \\ 0 & A_{3,3} \end{bmatrix}).$$

Hence $\text{rank}(A) = \text{rank}(A_{1,1}) + \text{rank}(A_{3,3}) + 1$. From $[A_{2,1}\ A_{2,3}]x = 0$ for all $x \in \ker(A_{1,1} \oplus A_{3,3})$, it follows that $\text{rank}(\begin{bmatrix} A_{1,1} \\ A_{1,2} \end{bmatrix}) = \text{rank}(A_{1,1})$ and $\text{rank}(\begin{bmatrix} A_{2,3} \\ A_{3,3} \end{bmatrix}) = \text{rank}(A_{3,3})$. Thus $\text{rank}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \text{rank}(\begin{bmatrix} A_{2,3} \\ A_{3,3} \end{bmatrix}) + 1 = \text{rank}(A)$.

The case that there exists an $x \in \ker(A_{1,1} \oplus A_{3,3})$ such that $[A_{2,1}\ A_{2,3}]x$ is nonzero and $y^T \begin{bmatrix} A_{1,2} \\ A_{3,2} \end{bmatrix} = 0$ for all $y \in \ker((A_{1,1} \oplus A_{3,3})^T)$ yields $\text{rank}([A_{1,1}\ A_{1,2}]) + \text{rank}([A_{3,2}\ A_{3,3}]) + 1 = \text{rank}(A)$.

Hence, we are left with the case that there exist an $x \in \ker(A_{1,1} \oplus A_{3,3})$ such that $f = [A_{2,1}\ A_{2,3}]x$ is nonzero and there exists a $y \in \ker((A_{1,1} \oplus A_{3,3})^T)$ such that $e = y^T \begin{bmatrix} A_{1,2} \\ A_{3,2} \end{bmatrix}$ is nonzero. Then, by Lemma 5,

$$\text{rank}(\begin{bmatrix} 0 & 0 & e & 0 \\ 0 & A_{1,1} & A_{1,2} & 0 \\ f & A_{2,1} & a_{2,2} & A_{2,3} \\ 0 & 0 & A_{3,2} & A_{3,3} \end{bmatrix}) = \text{rank}(A).$$

By Lemma 4,

$$\text{rank}(\begin{bmatrix} A_{1,1} & 0 \\ 0 & A_{3,3} \end{bmatrix}) + 2 = \text{rank}(A).$$

Thus, $\text{rank}(A_{1,1}) + \text{rank}(A_{3,3}) + 2 = \text{rank}(A)$. □

2.2 Four Inequalities

Let

$$M = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & g & B_{1,2} \\ 0 & B_{2,1} & B_{2,2} \end{bmatrix}$$

be a sign pattern matrix, where $A_{1,1}$ is $m \times n$, $A_{1,2}$ is $m \times 1$, $A_{2,1}$ is $1 \times n$, g is 1×1, $B_{1,2}$ is $1 \times q$, $B_{2,1}$ is $p \times 1$ and $B_{2,2}$ is $p \times q$.

Let $R \in Q(M)$ such that $\text{rank}(R) = \text{mr}(M)$, and

$$R = \begin{bmatrix} C_{1,1} & C_{1,2} & 0 \\ C_{2,1} & r & D_{1,2} \\ 0 & D_{2,1} & D_{2,2} \end{bmatrix},$$

where $C_{i,j} \in Q(A_{i,j}), D_{i,j} \in Q(B_{i,j})$, $\text{rank}(C_{i,j}) = \text{mr}(A_{i,j})$, $\text{rank}(D_{i,j}) = \text{mr}(B_{i,j}), (i = 1, 2; j = 1, 2)$, and $\text{sgn}(r) = g$.

Lemma 7. *$mr(A_{1,1}) + mr(B_{2,2}) + 2 \geq mr(M)$.*

Proof. Let $C_{1,1} \in Q(A_{1,1})$ and $D_{2,2} \in Q(B_{2,2})$ such that $\text{rank}(C_{1,1}) = \text{mr}(A_{1,1})$ and $\text{rank}(D_{2,2}) = \text{mr}(B_{2,2})$. Let $C_{1,2} \in Q(A_{1,2})$, $C_{2,1} \in Q(A_{2,1})$, $D_{1,2} \in Q(B_{1,2})$, $D_{2,1} \in Q(B_{2,1})$, and $\text{sgn}(r) = g$. By Lemma 3,

$$\text{mr}(A_{1,1}) + \text{mr}(B_{2,2}) = \text{rank}(\begin{bmatrix} C_{1,1} & 0 \\ 0 & D_{2,2} \end{bmatrix}) + 2 \geq \text{rank}(\begin{bmatrix} C_{1,1} & C_{1,2} & 0 \\ C_{2,1} & r & D_{1,2} \\ 0 & D_{2,1} & D_{2,2} \end{bmatrix}) \geq \text{mr}(M),$$

which concludes the proof. □

Lemma 8. *$mr([A_{1,1}\ A_{1,2}]) + mr([B_{2,1}\ B_{2,2}]) + 1 \geq mr(M)$*

Proof. Let $[C_{1,1}\ \ C_{1,2}] \in Q([A_{1,1}\ \ A_{1,2}])$ and $[D_{2,1}\ \ D_{2,2}] \in Q([B_{2,1}\ \ B_{2,2}])$ be such that $\text{rank}([C_{1,1}\ C_{1,2}]) = \text{mr}([A_{1,1}\ A_{1,2}])$ and $\text{rank}([D_{2,1}\ D_{2,2}]) = \text{mr}([B_{2,1}\ B_{2,2}])$. Clearly,

$$\text{rank}([C_{1,1}\ C_{1,2}]) + \text{rank}([D_{2,1}\ D_{2,2}]) \geq \text{rank}(\begin{bmatrix} C_{1,1} & C_{1,2} & 0 \\ 0 & D_{2,1} & D_{2,2} \end{bmatrix}).$$

By Lemma 2,

$$\text{rank}(\begin{bmatrix} C_{1,1} & C_{1,2} & 0 \\ 0 & D_{2,1} & D_{2,2} \end{bmatrix}) + 1 \geq \text{rank}(\begin{bmatrix} C_{1,1} & C_{1,2} & 0 \\ C_{2,1} & r & D_{1,2} \\ 0 & D_{2,1} & D_{2,2} \end{bmatrix}).$$

Hence $\text{mr}([A_{1,1}\ A_{1,2}]) + \text{mr}([B_{2,1}\ B_{2,2}]) + 1 \geq \text{mr}(M)$. □

Lemma 9. $mr(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + mr(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1 \geq mr(M)$.

Proof. The proof of this lemma is similar to the proof of Lemma 8. □

In order to express our last inequalities in an easy way, we need to extend the definition of a sign pattern matrix. (Only the entry that is shared by both parts of the 1-separation needs to be extended.)

A *generalized sign pattern matrix* $A = [a_{i,j}]$ is a matrix whose entries are nonempty subsets of $\{+, -, 0\}$. The *sign pattern class* of a generalized sign pattern matrix A, denoted by $Q(A)$, is defined as the set of all real matrices $B = [b_{i,j}]$ with the same size as A such that $\text{sgn}(b_{i,j}) \in a_{i,j}$ for all entries $b_{i,j}$ of B. The *minimum rank* of a generalized sign pattern matrix, denoted $\text{mr}(A)$, is defined by

$$\text{mr}(A) = \min\{\text{rank}\, B :\ B \text{ is a real matrix and } B \in Q(A)\}.$$

We say that a sign pattern matrix $A = [a_{i,j}]$ belongs to a generalized sign pattern matrix $C = [c_{i,j}]$, whose size is the same as A if $a_{i,j} \in c_{i,j}$ for each entry $a_{i,j}$. The minimum rank of a generalized sign pattern matrix C can be expressed as the minimum of the minimum ranks of all sign pattern matrices belonging to C.

Let

$$M = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & m & B_{1,2} \\ 0 & B_{2,1} & B_{2,2} \end{bmatrix}.$$

For $a \in \{+, -, 0\}$, we define the sign pattern matrix

$$M_a^1 = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & a \end{bmatrix}$$

and the generalized sign pattern matrix

$$M_a^2 = \begin{bmatrix} m - a & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix},$$

where $m - a$ is defined based on the following rules:

(i) $(+) - (+) = \{+, -, 0\}, (0) - (0) = \{0\}, (-) - (-) = \{+, -, 0\}$,

(ii) $(+) - (0) = \{+\}, (0) - (+) = \{-\}, (-) - (0) = \{-\}, (0) - (-) = \{+\}$,

(iii) $(+) - (-) = \{+\}$,

(iv) $(-) - (+) = \{-\}$.

Lemma 10. *For each $a \in \{+, -, 0\}$, $mr(M_a^1) + mr(M_a^2) \geq mr(M)$.*

Proof. Let

$$C = \begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & c \end{bmatrix} \in Q(M_a^1) \quad \text{and} \quad D = \begin{bmatrix} d & D_{1,2} \\ D_{2,1} & D_{2,2} \end{bmatrix} \in Q(M_a^2)$$

be such that $\text{rank}(C) = \text{mr}(M_a^1)$ and $\text{rank}(D) = \text{mr}(M_a^2)$.

We now do a case-checking.

Suppose first that $m - a = \{0\}$. Then $a = 0$ and $m = 0$. Hence, $c = 0$ and $d = 0$. Then $C \oplus_1 D \in Q(M)$, and, by Lemma 1, $\mathrm{mr}(M) \leq \mathrm{rank}(C \oplus_1 D) \leq \mathrm{rank}(C) + \mathrm{rank}(D) = \mathrm{mr}(M_a^1) + \mathrm{mr}(M_a^2)$.

Suppose next that $m - a = \{+\}$. Then one of the following holds:

(i) $a = -$ and $m = 0$,

(ii) $a = 0$ and $m = +$, or

(iii) $a = -$ and $m = +$.

Suppose $a = -$ and $m = 0$. By scaling D by a positive scalar, we may assume that $d = -c$. Then $C \oplus_1 D \in Q(M)$, and, by Lemma 1, $\mathrm{mr}(M) \leq \mathrm{rank}(C \oplus_1 D) \leq \mathrm{rank}(C) + \mathrm{rank}(D) = \mathrm{mr}(M_a^1) + \mathrm{mr}(M_a^2)$. Suppose $a = 0$ and $m = +$. Then $C \oplus_1 D \in Q(M)$, and, by Lemma 1, $\mathrm{mr}(M) \leq \mathrm{rank}(C \oplus_1 D) \leq \mathrm{rank}(C) + \mathrm{rank}(D) = \mathrm{mr}(M_a^1) + \mathrm{mr}(M_a^2)$. Suppose $a = -$ and $m = +$. By scaling D by a positive scalar, we may assume that $c + d > 0$. Then $C \oplus_1 D \in Q(M)$, and, by Lemma 1, $\mathrm{mr}(M) \leq \mathrm{rank}(C \oplus_1 D) \leq \mathrm{rank}(C) + \mathrm{rank}(D) = \mathrm{mr}(M_a^1) + \mathrm{mr}(M_a^2)$.

The case where $m - a = \{-\}$ is similar.

Suppose finally that $m - a = \{+, -, 0\}$. Then one of the following holds:

(i) $a = +$ and $m = +$, or

(ii) $a = -$ and $m = -$.

Suppose $a = +$ and $m = +$. Then $C \oplus_1 D \in Q(M)$, and, by Lemma 1, $\mathrm{mr}(M) \leq \mathrm{rank}(C \oplus_1 D) \leq \mathrm{rank}(C) + \mathrm{rank}(D) = \mathrm{mr}(M_a^1) + \mathrm{mr}(M_a^2)$. The case, where $a = -$ and $m = -$, is similar. $\square$

2.3 Minimum Rank of Sign Pattern with a 1-Separation

Theorem 2.3.1. *Let*

$$M = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & m & B_{1,2} \\ 0 & B_{2,1} & B_{2,2} \end{bmatrix}$$

Then

$$\begin{aligned} mr(M) = \min\{&mr(A_{1,1}) + mr(B_{2,2}) + 2, \\ &mr([A_{1,1}\ A_{1,2}]) + mr([B_{2,1}\ B_{2,2}]) + 1, \\ &mr(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + mr(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1, \\ &mr(M_+^1) + mr(M_+^2), \\ &mr(M_0^1) + mr(M_0^2), \\ &mr(M_-^1) + mr(M_-^2)\}. \end{aligned} \tag{2.3.1.1}$$

Proof. By the previous section,

$$\begin{aligned} \mathrm{mr}(M) \leq \min\{&\mathrm{mr}(A_{1,1}) + \mathrm{mr}(B_{2,2}) + 2, \\ &\mathrm{mr}([A_{1,1}\ A_{1,2}]) + \mathrm{mr}([B_{2,1}\ B_{2,2}]) + 1, \\ &\mathrm{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \mathrm{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1, \\ &\mathrm{mr}(M_+^1) + \mathrm{mr}(M_+^2), \\ &\mathrm{mr}(M_0^1) + \mathrm{mr}(M_0^2), \\ &\mathrm{mr}(M_-^1) + \mathrm{mr}(M_-^2)\}. \end{aligned} \tag{2.3.1.2}$$

We now show that at least one of the terms in the minimum on the right-hand side of (2.3.1.2) equals $\mathrm{mr}(M)$.

Let

$$R = \begin{bmatrix} C_{1,1} & C_{1,2} & 0 \\ C_{2,1} & r & D_{1,2} \\ 0 & D_{2,1} & D_{2,2} \end{bmatrix} \in Q(M)$$

be such that $\text{rank}(R) = \text{mr}(M)$. Then, by Lemma 6,

(i) There exist a vector v and a vector z such that

$$\text{rank}(\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & v^T C_{1,1} z \end{bmatrix}) + \text{rank}(\begin{bmatrix} r - v^T C_{1,1} z & D_{1,2} \\ D_{2,1} & D_{2,2} \end{bmatrix}) = \text{rank}(R).$$

(ii) $\text{rank}(\begin{bmatrix} C_{1,1} \\ C_{2,1} \end{bmatrix}) + \text{rank}(\begin{bmatrix} D_{1,2} \\ D_{2,2} \end{bmatrix}) + 1 = \text{rank}(R).$

(iii) $\text{rank}([C_{1,1}\ C_{1,2}]) + \text{rank}([D_{2,1}\ D_{2,2}]) + 1 = \text{rank}(R).$

(iv) $\text{rank}(C_{1,1}) + \text{rank}(D_{2,2}) + 2 = \text{rank}(R).$

Suppose first that (ii) holds. Then

$$\text{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \text{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1 \leq \text{rank}(\begin{bmatrix} C_{1,1} \\ C_{2,1} \end{bmatrix}) + \text{rank}(\begin{bmatrix} D_{1,2} \\ D_{2,2} \end{bmatrix}) + 1 = \text{rank}(R) = \text{mr}(M).$$

Case (iii) is similar to (ii).

Suppose next that (iv) holds. Then

$$\text{mr}(A_{1,1}) + \text{mr}(B_{2,2}) + 2 \leq \text{rank}(C_{1,1}) + \text{rank}(D_{2,2}) + 2 = \text{rank}(R) = \text{mr}(M).$$

Suppose finally that (i) holds. If $v^T C_{1,1} z > 0$, then

$$\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & v^T C_{1,1} z \end{bmatrix} \in Q(M_+^1) \quad \text{and} \quad \begin{bmatrix} r - v^T C_{1,1} z & D_{1,2} \\ D_{2,1} & D_{2,2} \end{bmatrix} \in Q(M_+^2).$$

Hence,

$$\mathrm{mr}(M) = \mathrm{rank}(R) = \mathrm{rank}(\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & v^T C_{1,1} z \end{bmatrix}) + \mathrm{rank}(\begin{bmatrix} r - v^T C_{1,1} z & D_{1,2} \\ D_{2,1} & D_{2,2} \end{bmatrix}) \geq \mathrm{mr}(M_+^1) + \mathrm{mr}(M_+^2).$$

The cases, where $v^T C_{1,1} z = 0$ and $v^T C_{1,1} z < 0$, are similar. □

CHAPTER 3

EXAMPLES

In this chapter, we exhibit several examples illustrating the utility of our formula.

(i) $\mathrm{mr}(M) = \mathrm{mr}(A_{1,1}) + \mathrm{mr}(B_{2,2}) + 2$

Let

$$M = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & m & B_{1,2} \\ 0 & B_{2,1} & B_{2,2} \end{bmatrix} = \begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix}.$$

Observe that $\mathrm{mr}(M) = 2$. Note that $\mathrm{mr}(A_{1,1}) + \mathrm{mr}(B_{2,2}) + 2 = 0 + 0 + 2 = 2, \mathrm{mr}([A_{1,1}\ A_{1,2}]) + \mathrm{mr}([B_{2,1}\ B_{2,2}]) + 1 = 1 + 1 + 1 = 3, \mathrm{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \mathrm{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1 = 1 + 1 + 1 = 3, \mathrm{mr}(M_+^1) + \mathrm{mr}(M_+^2) = 2 + 2 = 4, \mathrm{mr}(M_0^1) + \mathrm{mr}(M_0^2) = 2 + 2 = 4, \mathrm{mr}(M_-^1) + \mathrm{mr}(M_-^2) = 2 + 2 = 4.$

Hence,

$$\begin{aligned} \text{mr}(M) = 2 &= \text{mr}(A_{1,1}) + \text{mr}(B_{2,2}) + 2 \\ &= \min\{\text{mr}(A_{1,1}) + \text{mr}(B_{2,2}) + 2, \\ &\text{mr}([A_{1,1}\ A_{1,2}]) + \text{mr}([B_{2,1}\ B_{2,2}]) + 1, \\ &\text{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \text{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1, \\ &\text{mr}(M_+^1) + \text{mr}(M_+^2), \\ &\text{mr}(M_0^1) + \text{mr}(M_0^2), \\ &\text{mr}(M_-^1) + \text{mr}(M_-^2)\}. \end{aligned}$$

Our formula yields the correct result.

(ii) $\text{mr}(M) = \text{mr}(\begin{bmatrix} A_{1,1} & A_{1,2} \end{bmatrix}) + \text{mr}(\begin{bmatrix} B_{2,1} & B_{2,2} \end{bmatrix}) + 1$

Let

$$M = \begin{bmatrix} A_{1,1_{2\times 2}} & A_{1,2_{2\times 1}} & 0_{2\times 2} \\ A_{2,1_{1\times 2}} & m & B_{1,2_{1\times 2}} \\ 0_{1\times 2} & B_{2,1_{1\times 1}} & B_{2,2_{1\times 2}} \end{bmatrix} = \begin{bmatrix} + & + & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 \\ 0 & + & + & + & 0 \\ 0 & 0 & + & 0 & + \end{bmatrix}.$$

Observe that $\text{mr}(M) = 3$. Note that $\text{mr}(A_{1,1}) + \text{mr}(B_{2,2}) + 2 = 1 + 1 + 2 = 4$, $\text{mr}([A_{1,1}\ A_{1,2}]) + \text{mr}([B_{2,1}\ B_{2,2}]) + 1 = 1 + 1 + 1 = 3, \text{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \text{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1 = 2 + 2 + 1 = 5$, $\text{mr}(M_+^1) + \text{mr}(M_+^2) = 2 + 2 = 4$, $\text{mr}(M_0^1) + \text{mr}(M_0^2) = 2 + 2 = 4$, $\text{mr}(M_-^1) + \text{mr}(M_-^2) = 2 + 2 = 4$.

Hence,

$$
\begin{aligned}
\operatorname{mr}(M) = 1 = \operatorname{mr}(M) = \operatorname{mr}(\begin{bmatrix} A_{1,1} & A_{1,2} \end{bmatrix}) + \operatorname{mr}(\begin{bmatrix} B_{2,1} & B_{2,2} \end{bmatrix}) + 1 \\
= \min\{\operatorname{mr}(A_{1,1}) + \operatorname{mr}(B_{2,2}) + 2, \\
\operatorname{mr}([A_{1,1}\ A_{1,2}]) + \operatorname{mr}([B_{2,1}\ B_{2,2}]) + 1, \\
\operatorname{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \operatorname{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1, \\
\operatorname{mr}(M_+^1) + \operatorname{mr}(M_+^2), \\
\operatorname{mr}(M_0^1) + \operatorname{mr}(M_0^2), \\
\operatorname{mr}(M_-^1) + \operatorname{mr}(M_-^2)\}.
\end{aligned}
$$

Our formula yields the correct result.

(iii) $\operatorname{mr}(M) = \operatorname{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \operatorname{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1$

Let

$$
M = \begin{bmatrix} A_{1,1_{2\times 2}} & A_{1,2_{2\times 1}} & 0_{2\times 1} \\ A_{2,1_{1\times 2}} & m & B_{1,2_{1\times 1}} \\ 0_{2\times 2} & B_{2,1_{2\times 1}} & B_{2,2_{2\times 1}} \end{bmatrix} = \begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & 0 \\ + & + & + & + \\ 0 & 0 & + & 0 \\ 0 & 0 & + & + \end{bmatrix}.
$$

Observe that $\operatorname{mr}(M) = 3$. Note that $\operatorname{mr}(A_{1,1}) + \operatorname{mr}(B_{2,2}) + 2 = 1 + 2 + 2 = 5$, $\operatorname{mr}([A_{1,1}\ A_{1,2}]) + \operatorname{mr}([B_{2,1}\ B_{2,2}]) + 1 = 2 + 2 + 1 = 5, \operatorname{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \operatorname{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1 = 1 + 1 + 1 = 3$, $\operatorname{mr}(M_+^1) + \operatorname{mr}(M_+^2) = 2 + 2 = 4$, $\operatorname{mr}(M_0^1) + \operatorname{mr}(M_0^2) = 2 + 2 = 4$, $\operatorname{mr}(M_-^1) + \operatorname{mr}(M_-^2) = 2 + 2 = 4$.

Hence,

$$\begin{aligned}\operatorname{mr}(M) = 3 = \operatorname{mr}(M) = \operatorname{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \operatorname{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1 = \min\{&\operatorname{mr}(A_{1,1}) + \operatorname{mr}(B_{2,2}) + 2, \\ &\operatorname{mr}([A_{1,1}\ A_{1,2}]) + \operatorname{mr}([B_{2,1}\ B_{2,2}]) + 1, \\ &\operatorname{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \operatorname{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1, \\ &\operatorname{mr}(M_+^1) + \operatorname{mr}(M_+^2), \\ &\operatorname{mr}(M_0^1) + \operatorname{mr}(M_0^2), \\ &\operatorname{mr}(M_-^1) + \operatorname{mr}(M_-^2)\}.\end{aligned}$$

Our formula yields the correct result.

(iv) $\operatorname{mr}(M) = \operatorname{mr}(M_+^1) + \operatorname{mr}(M_+^2)$

Let

$$M = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & m & B_{1,2} \\ 0 & B_{2,1} & B_{2,2} \end{bmatrix} = \begin{bmatrix} + & + & 0 \\ + & - & - \\ 0 & + & + \end{bmatrix}.$$

Observe that $\operatorname{mr}(M) = 2$. Note that $\operatorname{mr}(A_{1,1}) + \operatorname{mr}(B_{2,2}) + 2 = 1 + 1 + 2 = 4, \operatorname{mr}([A_{1,1}\ A_{1,2}]) + \operatorname{mr}([B_{2,1}\ B_{2,2}]) + 1 = 1 + 1 + 1 = 3, \operatorname{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \operatorname{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1 = 1 + 1 + 1 = 3, \operatorname{mr}(M_+^1) + \operatorname{mr}(M_+^2) = 1 + 1 = 2, \operatorname{mr}(M_0^1) + \operatorname{mr}(M_0^2) = 2 + 1 = 3, \operatorname{mr}(M_-^1) + \operatorname{mr}(M_-^2) = 2 + 1 = 3.$

Hence,

$$\begin{aligned}\mathrm{mr}(M) = 2 = \mathrm{mr}(M_+^1) + \mathrm{mr}(M_+^2) = \min\{&\mathrm{mr}(A_{1,1}) + \mathrm{mr}(B_{2,2}) + 2,\\ &\mathrm{mr}([A_{1,1}\ A_{1,2}]) + \mathrm{mr}([B_{2,1}\ B_{2,2}]) + 1,\\ &\mathrm{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \mathrm{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1,\\ &\mathrm{mr}(M_+^1) + \mathrm{mr}(M_+^2),\\ &\mathrm{mr}(M_0^1) + \mathrm{mr}(M_0^2),\\ &\mathrm{mr}(M_-^1) + \mathrm{mr}(M_-^2)\}.\end{aligned}$$

Our formula yields the correct result.

(v) $\mathrm{mr}(M) = \mathrm{mr}(M_-^1) + \mathrm{mr}(M_-^2)$

Let

$$M = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & m & B_{1,2} \\ 0 & B_{2,1} & B_{2,2} \end{bmatrix} = \begin{bmatrix} + & - & 0 \\ + & 0 & + \\ 0 & + & + \end{bmatrix}.$$

Observe that $\mathrm{mr}(M) = 2$. Note that $\mathrm{mr}(A_{1,1}) + \mathrm{mr}(B_{2,2}) + 2 = 1 + 1 + 2 = 4, \mathrm{mr}([A_{1,1}\ A_{1,2}]) + \mathrm{mr}([B_{2,1}\ B_{2,2}]) + 1 = 1 + 1 + 1 = 3, \mathrm{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \mathrm{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1 = 1 + 1 + 1 = 3, \mathrm{mr}(M_+^1) + \mathrm{mr}(M_+^2) = 2 + 2 = 4, \mathrm{mr}(M_0^1) + \mathrm{mr}(M_0^2) = 2 + 2 = 4, \mathrm{mr}(M_-^1) + \mathrm{mr}(M_-^2) = 1 + 1 = 2.$

Hence,

$$\begin{aligned}\mathrm{mr}(M) = 2 = \mathrm{mr}(M_-^1) + \mathrm{mr}(M_-^2) = \min\{&\mathrm{mr}(A_{1,1}) + \mathrm{mr}(B_{2,2}) + 2,\\ &\mathrm{mr}([A_{1,1}\ A_{1,2}]) + \mathrm{mr}([B_{2,1}\ B_{2,2}]) + 1,\\ &\mathrm{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \mathrm{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1,\\ &\mathrm{mr}(M_+^1) + \mathrm{mr}(M_+^2),\\ &\mathrm{mr}(M_0^1) + \mathrm{mr}(M_0^2),\\ &\mathrm{mr}(M_-^1) + \mathrm{mr}(M_-^2)\}.\end{aligned}$$

Our formula yields the correct result.

(vi) $\mathrm{mr}(M) = \mathrm{mr}(M_0^1) + \mathrm{mr}(M_0^2)$

Let

$$M = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & m & B_{1,2} \\ 0 & B_{2,1} & B_{2,2} \end{bmatrix} = \begin{bmatrix} + & 0 & 0 \\ + & - & + \\ 0 & + & - \end{bmatrix}.$$

Observe that $\mathrm{mr}(M) = 2$. Note that $\mathrm{mr}(A_{1,1}) + \mathrm{mr}(B_{2,2}) + 2 = 1 + 1 + 2 = 4, \mathrm{mr}([A_{1,1}\ A_{1,2}]) + \mathrm{mr}([B_{2,1}\ B_{2,2}]) + 1 = 1 + 1 + 1 = 3, \mathrm{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \mathrm{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1 = 1 + 1 + 1 = 3, \mathrm{mr}(M_+^1) + \mathrm{mr}(M_+^2) = 2 + 1 = 3, \mathrm{mr}(M_0^1) + \mathrm{mr}(M_0^2) = 1 + 1 = 2, \mathrm{mr}(M_-^1) + \mathrm{mr}(M_-^2) = 2 + 1 = 3$.

Hence,

$$\begin{aligned}\mathrm{mr}(M) = 2 = \mathrm{mr}(M_0^1) + \mathrm{mr}(M_0^2) = \min\{&\mathrm{mr}(A_{1,1}) + \mathrm{mr}(B_{2,2}) + 2,\\ &\mathrm{mr}([A_{1,1}\ A_{1,2}]) + \mathrm{mr}([B_{2,1}\ B_{2,2}]) + 1,\\ &\mathrm{mr}(\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix}) + \mathrm{mr}(\begin{bmatrix} B_{1,2} \\ B_{2,2} \end{bmatrix}) + 1,\\ &\mathrm{mr}(M_+^1) + \mathrm{mr}(M_+^2),\\ &\mathrm{mr}(M_0^1) + \mathrm{mr}(M_0^2),\\ &\mathrm{mr}(M_-^1) + \mathrm{mr}(M_-^2)\}.\end{aligned}$$

Our formula yields the correct result.

CHAPTER 4

FUTURE WORK

Similar method may be extended to find a formula for sign pattern matrices with a 2×1 separation and with a 2-separation.

4.1 2×1 separation

Let a, b, c, d be an element in $\{+, -, 0\}$. Define the sign pattern matrix

$$M = A \bigoplus_{2\times 1} B = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & a+b & B_{1,2} \\ A_{3,1} & c+d & B_{2,2} \\ 0 & B_{3,1} & B_{3,2} \end{bmatrix},$$

where $A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & a \\ A_{3,1} & c \end{bmatrix}$ and $B = \begin{bmatrix} b & B_{1,2} \\ d & B_{2,2} \\ B_{3,1} & B_{3,2} \end{bmatrix}$.

We call (A, B) a 2×1 separation of matrix M.

Can we find a formula similar to Formula 2.3.1.1 that relates the minimum rank of M to some variations of A and B?

4.2 2-separation

Let a, b, c, d, e, f, g, h be elements in $\{+, -, 0\}$. Define the sign pattern matrix

$$M = A \bigoplus_2 B = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & 0 \\ A_{2,1} & a+b & c+d & B_{1,3} \\ A_{3,1} & e+f & g+h & B_{2,3} \\ 0 & B_{3,1} & B_{3,2} & B_{3,3} \end{bmatrix},$$

where $A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & a & c \\ A_{3,1} & e & g \end{bmatrix}$ and $B = \begin{bmatrix} b & d & B_{1,3} \\ f & h & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{bmatrix}$.

We call (A, B) a $2-$separation of matrix M.

Can we find a formula similar to Formula 2.3.1.1 that relates the minimum rank of M to some variations of A and B?

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