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Weak Primary Decomposition of Modules Over a Commutative Ring

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by

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Under the Direction of Dr. Yongwei Yao

ABSTRACT

This paper presents the theory of weak primary decomposition of modules over a commutative ring. A generalization of the classic well-known theory of primary decomposition, weak primary decomposition is a consequence of the notions of weakly associated prime ideals and nearly nilpotent elements, which were introduced by N. Bourbaki. We begin by discussing basic facts about classic primary decomposition. Then we prove the results on weak primary decomposition, which are parallel to the classic case. Lastly, we define and generalize the Compatibility property of primary decomposition.

INDEX WORDS: Primary decomposition, Weak primary decomposition, Associated primes, Weakly associated primes, Nilpotent, Nearly nilpotent, Weak primary submodules, Compatibility property

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This thesis is dedicated to

Richard, Sharon, Cam, and Brandi.

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Chapter 1

INTRODUCTION

The subject of primary decomposition is an essential topic in any graduate text on commutative algebra. Although it is commonly first introduced in the context of ideals, the theory of primary decomposition is also seen in modules. Primary decomposition is rooted in the notions of associated prime ideals and nilpotency. In the classic sense, if a prime ideal is equal to the annihilator of a nonzero element from a given module, then the prime ideal is said to be "associated" with the module. In [2], N. Bourbaki, however, introduces a more general associated prime ideal − if a prime ideal is minimal over the annihilator of an element from a given module, then the prime ideal is said to be "weakly associated" with the module. Moreover, Bourbaki generalizes the notion of nilpotency to "nearly" nilpotency. As a result, the primary decompositions are called "weak." In this paper, we will explore the consequences of such generalizations.

In Chapter 1, we begin with a review of familiar facts and terminology relating to classic ring theory, with a focus on results regarding minimal ideals, as they will be necessary to proceed to Chapter 2. Also, in Chapter 1, we will define terms relating to classic primary decomposition, so that we may distinguish it from weak primary decomposition. The purpose of Chapter 2 is to form the foundation of the theory of weak primary decomposition by defining precisely what we mean by a "weakly associated prime ideal" and a "nearly nilpotent element". Then we will show how these definitions alter the classic theory of primary decomposition. Indeed, there are assumptions on the characteristics of the ring and modules that would make the notions of weak and classic primary decomposition equivalent. Thus we will show which assumptions must be disregarded in order to make the theory surrounding weak primary decomposition truly more general. Finally, in Chapter 3, we will generalize Y. Yao's result on the Compatibility property seen in [5].

1.1 Preliminary Facts

The results in this section are standard in any text on commutative algebra, such as [2], [3], and [4]. We will omit the definitions of a ring, ideal, and module, and begin with the definition of a prime ideal.

Note: All rings throughout this paper are commutative with unity. It is to be understood that anything denoted by R is such a ring.

Definition 1.1.1. An ideal P of R is said to be a *prime ideal* of R if $P \subsetneq R$ and one of the following equivalent statements hold:

(i) If $a, b \in R$ such that $ab \in P$, then $a \in P$ or $b \in P$.

(ii) If $a, b \in R$ such that $ab \in P$ with $b \notin P$, then $a \in P$.

(iii) If $a, b \in R$ such that $a \notin P$ and $b \notin P$, then $ab \notin P$.

Notation 1.1.2. In some cases, we will refer to the set of all prime ideals of R as $\text{Spec}(R)$.

It is important to note that if P is an ideal of R and $P \notin \text{Spec}(R)$, then $P = R$ or there exists $a, b \in R$ such that $ab \in P$ with $a \notin P$ and $b \notin P$.

Fact 1.1.3. *Let* $P \in \text{Spec}(R)$. *If* $a_1, a_2, \ldots, a_n \in R$ *such that* $a_1 a_2 \cdots a_n \in P$, *then* $a_i \in P$ *for some* $1 \leq i \leq n$. *Therefore, if* $a^n \in P$ *for some* $n \in \mathbb{N}$, *then* $a \in P$.

Definition 1.1.4. A subset S of R is said to be *multiplicatively closed*, or *multiplicative*, if the following hold:

(i) $1 \in S$; and

(ii) if $s_1, s_2 \in S$, then $s_1 s_2 \in S$.

Definition 1.1.5. Let V be a non-empty set.

(i) A relation $\preceq$ on V is said to be a *partial order* on V if the following properties hold:

- reflexive: $u \preceq u$ for all $u \in V$;

- transitive: if $u \preceq v$ and $v \preceq w$ for some $u, v, w \in V$, then $u \preceq w$; and
- antisymmetric: if $u \preceq v$ and $v \preceq u$ for some $u, v \in V$, then $u = v$.

It is in this case that we say $(V, \preceq)$ is a *partially ordered* set.

(ii) The partially ordered set $(V, \preceq)$ is said to be *totally ordered* if for all $u, v \in V$, at least one of $u \preceq v$, $v \preceq u$ holds.

(iii) For a non-empty subset W of the partially ordered set $(V, \preceq)$, an element $u \in V$ is said to be an *upper bound* of W if $w \preceq u$ for all $w \in W$.

(iv) For a partially ordered set $(V, \preceq)$, an element $u \in V$ is said to be *maximal* in V if there does not exist $v \in V$ such that $u \preceq v$ and $u \neq v$.

Lemma 1.1.6 (Zorn's Lemma)**.** *Let $(V, \preceq)$ be a non-empty partially ordered set such that every non-empty totally ordered subset of V has an upper bound in V. Then V has at least one maximal element.*

Theorem 1.1.7 ([4], page 50, Theorem 3.44)**.** *Let S be a multiplicative subset of R and let I be an ideal of R. If $I \cap S = \emptyset$, then there exists $P \in \operatorname{Spec}(R)$ such that $P \cap S = \emptyset$ and $I \subseteq P$.*

Proof. Let $I \cap S = \emptyset$, and define

$$\Omega := \{J \mid J \text{ is an ideal of } R \text{ such that } J \supseteq I \text{ and } J \cap S = \emptyset\},$$

which is clearly non-empty since $I \in \Omega$. Moreover, Ω is partially ordered by $\subseteq$. Let Θ be a non-empty totally ordered subset of Ω. Then $Q := \bigcup_{J \in \Theta} J$ is an ideal of R such that $Q \supseteq I$ and $Q \cap S = \emptyset$. Thus Q is an upper bound for Θ in Ω. By applying Zorn's Lemma 1.1.6, we have that Ω contains at least one maximal element.

Let P be a maximal element of Ω. Since $P \in \Omega$, $P \cap S = \emptyset$. We claim $P \in \operatorname{Spec}(R)$. Let $a, a' \in R$ such that $a \notin P$ and $a' \notin P$. We aim to show that $aa' \notin P$. It is true that

$$I \subseteq P \subsetneq P + Ra \quad \text{and} \quad I \subseteq P \subsetneq P + Ra'.$$

By the maximality of $P \in \Omega$, we have that

$$(P + Ra) \cap S \neq \emptyset \quad \text{and} \quad (P + Ra') \cap S \neq \emptyset.$$

Then there exist $s, s' \in S$, $r, r' \in R$, and $u, u' \in P$ such that

$$s = u + ra \quad \text{and} \quad s' = u' + r'a'.$$

Since S is multiplicative, we have $ss' \in S$. But

$$ss' = (u + ra)(u' + r'a') = (uu' + rau' + r'a'u) + rr'aa',$$

where $uu' + rau' + r'a'u \in P$. If $aa' \in P$, then $rr'aa' \in P$, forcing $ss' \in P$, which is false, since $P \cap S = \emptyset$. Thus $aa' \notin P$, proving $P \in \operatorname{Spec}(R)$. □

As we will see in Chapter 2, the study of weak primary decomposition depends on the notion of minimal prime ideals. Thus it is important that we now establish the definitions and results on this concept that will be valuable for the purpose of this paper.

Definition 1.1.8. Let Ω be a collection of subsets of R. An element (set) S of Ω is said to be *minimal* in Ω if there exists no $S' \in \Omega$ such that $S \supsetneq S'$.

Proposition 1.1.9 ([2], page 73, II.2.6, Proposition 12)**.** *Let P be minimal in* Spec(R)*. Then for all $a \in P$, there exists $s \in R \setminus P$ such that $a^n s = 0$ for some $n \in \mathbb{N}$.*

Proof. Let $a \in P$. The set

$$S := \{a^m s \mid s \in R \setminus P \text{ and } m \geq 0\}$$

is a multiplicative subset of R. Clearly $a \in S$ so that $P \cap S \neq \emptyset$. We aim to prove $0 \in S$. Suppose $0 \notin S$. Then $\{0\} \cap S = \emptyset$. By 1.1.7, there exists $P' \in \operatorname{Spec}(R)$ such that $P' \cap S = \emptyset$. This and the fact that $R \setminus P \subseteq S$ imply $P' \cap (R \setminus P) = \emptyset$, implying $P' \subseteq P$. In fact, $P' \subsetneq P$, since $a \in P$ and $a \notin P'$, because $a \in S$ and $P' \cap S = \emptyset$. Thus we have a contradiction of the minimality of P in $\operatorname{Spec}(R)$. Therefore $0 \in S$, completing our proof. □

Definition 1.1.10. Let I be an ideal of R. A prime ideal P of R is said to be *minimal over* I if the following hold:

(i) $P \supseteq I$; and

(ii) there exists no prime ideal P' of R such that $P \supsetneq P' \supseteq I$.

Notation 1.1.11. In some cases, we will refer to the set of all minimal primes over I as $\operatorname{Min}(I)$.

Fact 1.1.12 ([4], page 53, Theorem 3.52). *Let I be a proper ideal of R. Then there exists $P \in \operatorname{Spec}(R)$ such that P is minimal over I. That is, $\operatorname{Min}(I) \neq \emptyset$.*

Definition 1.1.13. Let I be an ideal of R. The *radical* of I, denoted $\sqrt{I}$, is the set

$$\{a \in R \mid \text{there exists } n \in \mathbb{N} \text{ such that } a^n \in I\}.$$

Fact 1.1.14. *Let I be an ideal of R.*

(i) *$\sqrt{I}$ is an ideal of R.*

(ii) *$I \subseteq \sqrt{I} = \sqrt{\sqrt{I}}$.*

Fact 1.1.15 ([4], pages 52 and 54, Lemma 3.48 and Corollary 3.54). *Let I be an ideal of R. Then*

$$\sqrt{I} = \bigcap_{P \in \operatorname{Min}(I)} P = \bigcap_{P \supseteq I} P,$$

where P runs through the prime ideals of R containing I.

Fact 1.1.16. *Let I be an ideal of R. If $\sqrt{I}$ is a prime ideal, then*

$$\operatorname{Min}(I) = \{\sqrt{I}\}.$$

Now we introduce definitions and facts pertaining to modules and their annihilators. It is important to note that ideals are a special type of module, that is, a module is a more general structure. Therefore all results on modules apply to ideals. When important, we will emphasize results in the context of ideals.

Definition 1.1.17. Let M be an R-module. An element $a \in R$ is said to be a *zerodivisor* of M if there exists $0 \neq x \in M$ such that $ax = 0$.

Notation 1.1.18. We denote the set of all zerodivisors of M by $\operatorname{Zdv}_R(M)$.

Definition 1.1.19. Let M be an R-module and let $x \in M$. An element $a \in R$ is said to *annihilate* x if $ax = 0$. In this case, a is called an *annihilator* of x. Moreover, if $ax = 0$ for all $x \in M$, i.e., $aM = 0$, then a is called an *annihilator* of M.

Notation 1.1.20. We denote the set of all annihilators of x and M by $\operatorname{Ann}(x)$ and $\operatorname{Ann}(M)$, respectively.

Fact 1.1.21. *Let M be an R-module. Then*

$$\operatorname{Zdv}_R(M) = \bigcup_{0 \neq x \in M} \operatorname{Ann}(x).$$

Fact 1.1.22. *Let M be an R-module and let $x \in M$. Then $\operatorname{Ann}(x)$, $\sqrt{\operatorname{Ann}(x)}$, $\operatorname{Ann}(M)$, and $\sqrt{\operatorname{Ann}(M)}$ are ideals of R.*

Lemma 1.1.23. *Let $P \in \operatorname{Spec}(R)$. If P is minimal over $\operatorname{Ann}(x)$ for some $0 \neq x \in M$, then P is minimal over $\operatorname{Ann}(rx)$ for all $r \in R \setminus P$.*

Proof. Let P be minimal over $\operatorname{Ann}(x)$ for some $0 \neq x \in M$. First, we show $\operatorname{Ann}(rx) \subseteq P$ for all $r \in R \setminus P$. Let r be an arbitrary element of $R \setminus P$, and let $a \in \operatorname{Ann}(rx)$. Then $arx = 0$, implying $ar \in \operatorname{Ann}(x)$. Since $\operatorname{Ann}(x) \subseteq P$ and $r \notin P$, we have $a \in P$. Thus $\operatorname{Ann}(rx) \subseteq P$ for all $r \in R \setminus P$.

Now, we show P is minimal over $\operatorname{Ann}(rx)$ for all $r \in R \setminus P$. Suppose P is not minimal over $\operatorname{Ann}(rx)$ for some $r \in R \setminus P$. Then there exists $P' \in \operatorname{Spec}(R)$ such that

$$\operatorname{Ann}(x) \subseteq \operatorname{Ann}(rx) \subseteq P' \subsetneq P,$$

contradicting the minimality of P over $\operatorname{Ann}(x)$. Therefore P is minimal over $\operatorname{Ann}(rx)$ for all $r \in R \setminus P$. □

Definition 1.1.24. Let N be a submodule of the R-module M, let $x \in M$, and let $a \in R$. Then we have the following definitions:

(i) $(N :_M a) := \{y \in M \mid ay \in N\}$.

(ii) $(N :_R x) := \{r \in R \mid rx \in N\}$.

Obviously, $(N :_M a) \subseteq M$ and $(N :_R x) \subseteq R$.

Fact 1.1.25 (The Submodule Criterion)**.** *Let N be a subset of the R-module M. Then N is a submodule of M if and only if the following hold:*

(i) *$N \neq \emptyset$; and*

(ii) *if $x, y \in N$ and $a, b \in R$, then $ax + by \in N$.*

Lemma 1.1.26. *Let N be a submodule of the R-module M, and let $a \in R$.*

(i) *$N \subseteq (N :_M a)$ and $(N :_M a)$ is a submodule of M.*

(ii) *$a \in \operatorname{Ann}(M/N)$ if and only if $(N :_M a) = M$.*

Proof. (i) Since $aN \subseteq N$, we have $N \subseteq (N :_M a)$. By definition, $(N :_M a) \subseteq M$, and $(N :_M a) \neq \emptyset$, since $0_M \in (N :_M a)$. Let $x, y \in (N :_M a)$ and $b, c \in R$. Then $ax \in N$ and $ay \in N$. Hence $bax \in N$ and $cay \in N$, implying $bax + cay \in N$. But $N \subseteq (N :_M a)$. Thus $bax + cay \in (N :_M a)$. Therefore, by 1.1.25, $(N :_M a)$ is a submodule of M.

(ii)($\Rightarrow$) Let $a \in \operatorname{Ann}(M/N)$. Then $aM = 0_{M/N}$, i.e., $aM \subseteq N$, implying $M \subseteq (N :_M a)$. Therefore $(N :_M a) = M$.

($\Leftarrow$) Let $(N :_M a) = M$. Then $aM \subseteq N$, implying $a \in \operatorname{Ann}(M/N)$. $\square$

Proposition 1.1.27. *Let N be a submodule of the R-module M, and let $x \in M$.*

(i) *$(N :_R x)$ is an ideal of R and $\operatorname{Ann}(x) \subseteq (N :_R x) = \operatorname{Ann}(x + N)$, where $x + N \in M/N$.*

(ii) *$x \in N$ if and only if $(N :_R x) = R$.*

Proof. (i) By definition, $(N :_R x) \subseteq R$, and $(N :_R x) \neq \emptyset$, since $0_R \in (N :_R x)$. Let $a, b \in (N :_R x)$ and let $c \in R$. Then $ax \in N$ and $bx \in N$, implying $ax + bx \in N$, i.e., $(a + b)x \in N$. Moreover, $cax \in N$. Thus $a + b \in (N :_R x)$ and $ca \in (N :_R x)$. Therefore $(N :_R x)$ is an ideal of R.

Let $a \in \operatorname{Ann}(x)$. Then $ax = 0 \in N$. Thus $a \in (N :_R x)$ and $\operatorname{Ann}(x) \subseteq (N :_R x)$.

Now, $a \in (N :_R x) \Leftrightarrow ax \in N \Leftrightarrow 0 + N = ax + N = a(x + N) \Leftrightarrow a \in \operatorname{Ann}(x + N)$. Therefore $(N :_R x) = \operatorname{Ann}(x + N)$.

(ii)($\Rightarrow$) Let $x \in N$. Then $Rx \subseteq N$, implying $R \subseteq (N :_R x)$. Therefore $(N :_R x) = R$.

($\Leftarrow$) Let $(N :_R x) = R$. Then $Rx \subseteq N$, implying $ax \in N$ for all $a \in R$. In particular, $1 \cdot x \in N$, as desired. $\square$

Fact 1.1.28 ([4], pages 30 and 107, Exercises 2.33 and 6.18)**.** *Let $(N_\lambda)_{\lambda \in \Lambda}$ be a family of submodules of the R-module M, and let $a \in R$ and $x \in M$.*

(i) $\left(\bigcap_{\lambda \in \Lambda} N_\lambda :_M a \right) = \bigcap_{\lambda \in \Lambda} (N_\lambda :_M a)$.

(ii) $\left(\bigcap_{\lambda \in \Lambda} N_\lambda :_R x \right) = \bigcap_{\lambda \in \Lambda} (N_\lambda :_R x)$.

Fact 1.1.29 (The First Isomorphism Theorem)**.** *Let M and N be R-modules and let $\varphi : M \to N$ be an R-linear mapping. Then $M/\operatorname{Ker}\varphi$ is isomorphic to $\operatorname{Im}\varphi$, denoted $M/\operatorname{Ker}\varphi \cong \operatorname{Im}\varphi$.*

Proposition 1.1.30. *If $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ is a direct sum of R-modules, then for each $\lambda \in \Lambda$, M_λ is isomorphic to a submodule of M.*

Proof. For each $\lambda \in \Lambda$, define

$$\varphi \;:\; M_\lambda \longrightarrow M$$

by $\varphi(x) = (x_\lambda)_{\lambda \in \Lambda}$ such that $x_{\lambda'} = 0_{M_{\lambda'}}$ when $\lambda' \neq \lambda$ for all $x_\lambda \in M_\lambda$. It is clear that φ is R-linear and $\operatorname{Ker}\varphi = \{0_{M_\lambda}\}$. Thus $M_\lambda \cong \operatorname{Im}\varphi \subseteq M$ for each $\lambda \in \Lambda$, as desired. $\square$

Fact 1.1.31 ([1], page 260, Chapter 14, Section 3, Problem 2)**.** *Let N and L be submodules of the R-module M. Then*

$$\frac{N}{N \cap L} \cong \frac{N + L}{L}.$$

1.2 Associated Prime Ideals, Nilpotent Elements, Primary Submodules, and Primary Decomposition

In this section, we state the definitions and results regarding the classic theory of primary decomposition of modules. These definitions and results are consistent with the definitions and results found in [3] and [4].

Definition 1.2.1. Let M be an R-module. A prime ideal P of R is said to be *associated* with M if there exists $0 \neq x \in M$ such that $\operatorname{Ann}(x) = P$.

Notation 1.2.2. We denote the set of all prime ideals associated with M by $\operatorname{Ass}(M)$.

Definition 1.2.3. Let M be an R-module. An element $a \in R$ is said to be *nilpotent* on M if there exists $n \in \mathbb{N}$ such that $a^n M = 0$, that is, $a^n x = 0$ for all $x \in M$.

Notation 1.2.4. We denote the set of all elements nilpotent on M by $\operatorname{Nil}(M)$.

Fact 1.2.5. *Let M be an R-module. Then*

$$\operatorname{Nil}(M) = \sqrt{\operatorname{Ann}(M)}.$$

Proof. Let $a \in R$. Then $a \in \operatorname{Nil}(M) \Leftrightarrow a^n M = 0$ for some $n \in \mathbb{N} \Leftrightarrow a^n \in \operatorname{Ann}(M)$ for some $n \in \mathbb{N}$ $\Leftrightarrow a \in \sqrt{\operatorname{Ann}(M)}$. □

Proposition 1.2.6. *Let $M \neq 0$ be an R-module. Then*

$$\operatorname{Nil}(M) \subseteq \operatorname{Zdv}_R(M).$$

Proof. Let $a \in \operatorname{Nil}(M)$. Then there exists a minimal $n \in \mathbb{N}$ such that $0 = a^n M = a \cdot a^{n-1} M$ and $a^{n-1} M \neq 0$. Thus $a \in \operatorname{Zdv}_R(M)$. □

Definition 1.2.7. A submodule Q of the R-module M is said to be a *primary submodule* of M (or *primary* in M) if the following hold:

(i) $Q \subsetneq M$, i.e., $M/Q \neq 0$; and

(ii) $\operatorname{Zdv}_R(M/Q) = \operatorname{Nil}(M/Q)$, or equivalently, $\operatorname{Zdv}_R(M/Q) \subseteq \operatorname{Nil}(M/Q)$, in light of 1.2.6.

Fact 1.2.8. *Let Q be a primary submodule of the R-module M. Then* $\operatorname{Nil}(M/Q)$ *is a prime ideal.*

Definition 1.2.9. Let Q be a primary submodule of the R-module M. By 1.2.8, $\operatorname{Nil}(M/Q)$ is a prime ideal. If we denote $\operatorname{Nil}(M/Q)$ by P, then we say Q is a *P-primary submodule* of M, or *P-primary* in M. If $M = R$ and the previous conditions hold, then Q is a *P-primary ideal* of R.

Fact 1.2.10. *Let Q be a P-primary submodule of the R-module M.*

(i) $\sqrt{\operatorname{Ann}(M/Q)} = P$.

(ii) *If $M = R$ so that Q is a P-primary ideal of R, then $\sqrt{Q} = P$.*

Proof. (i) By definition, $\mathrm{Nil}(M/Q) = P$, and by 1.2.5, $\mathrm{Nil}(M/Q) = \sqrt{\mathrm{Ann}(M/Q)}$. Therefore $\sqrt{\mathrm{Ann}(M/Q)} = P$.

(ii) Let $M = R$. Then $\mathrm{Ann}(M/Q) = Q$, implying $\sqrt{\mathrm{Ann}(M/Q)} = \sqrt{Q}$. But $\sqrt{\mathrm{Ann}(M/Q)} = P$, by (i). Therefore $\sqrt{Q} = P$. □

Definition 1.2.11. Let $N \subsetneq M$ be R-modules. We say N is a *decomposable submodule* of M if it can be written as an intersection of finitely many primary submodules of M. Such an intersection

$$N = Q_1 \cap \cdots \cap Q_n \quad \text{with } Q_i \ P_i\text{-primary in } M \ (i = 1, 2, \ldots, n)$$

is called a *primary decomposition* of N in M.

Chapter 2

WEAK PRIMARY DECOMPOSITION OF MODULES

In this chapter, we will define the terminology and develop the theory surrounding weak primary decomposition. Any reader familiar with primary decomposition will notice that most of these results are parallel to the theory in classic primary decomposition. The proofs, however, are more intricate because of the terminology's underlying definitions. In Section 2.1, we will define the notions of weakly associated prime ideals and nearly nilpotent elements, which were introduced by N. Bourbaki ([2], 289, IV.2, Exercise 17.) Then we will develop the theory that characterizes and relates these two notions. In Section 2.2, we will demonstrate how altering the definitions of associated prime ideals and nilpotent elements to weakly associated prime ideals and nearly nilpotent elements, respectively, creates a new kind of primary submodule, namely, a weak primary submodule. In Section 2.3, we will develop the definitions and theory of weak primary decomposition, including the first and second uniqueness theorems.

2.1 Weakly Associated Prime Ideals and Nearly Nilpotent Elements

Definition 2.1.1. Let M be an R-module. A prime ideal P of R is said to be *weakly associated* with M if there exists $0 \neq x \in M$ such that P is minimal over $\mathrm{Ann}(x)$.

Notation 2.1.2. We denote the set of all prime ideals weakly associated with M by $\mathrm{Ass_f}(M)$.

Lemma 2.1.3. *Let M be an R-module. Then* $\mathrm{Ass}(M) \subseteq \mathrm{Ass_f}(M)$.

Proof. Let $P \in \mathrm{Ass}(M)$. Then there exists $0 \neq x \in M$ such that $\mathrm{Ann}(x) = P$. Clearly P is minimal over $\mathrm{Ann}(x)$. Thus $P \in \mathrm{Ass_f}(M)$. Therefore $\mathrm{Ass}(M) \subseteq \mathrm{Ass_f}(M)$. □

Before continuing with the study of weakly associated prime ideals of M and their consequences, it is natural to ask when $\text{Ass}(M) = \text{Ass}_{\text{f}}(M)$. This equality is achieved when R is Noetherian. (Recall that R is said to be *Noetherian* if every non-empty set of ideals of R contains a maximal element.)

Theorem 2.1.4. *Let M be an R-module. If R is Noetherian, then* $\text{Ass}(M) = \text{Ass}_{\text{f}}(M)$.

Proof. We have already shown in 2.1.3 that $\text{Ass}(M) \subseteq \text{Ass}_{\text{f}}(M)$, and it does not rely on R being Noetherian. Now, let R be Noetherian, and let $P \in \text{Ass}_{\text{f}}(M)$. Then P is minimal over $\text{Ann}(x)$ for some $0 \neq x \in M$. Define

$$\Theta := \{\text{Ann}(rx) \mid r \in R \text{ such that } \text{Ann}(rx) \subseteq P\}.$$

Since R is Noetherian, Θ contains a maximal element, say $\text{Ann}(sx)$, where $s \in R$ such that $\text{Ann}(sx) \subseteq P$. We claim $\text{Ann}(sx)$ is a prime ideal. Suppose $\text{Ann}(sx)$ is not a prime ideal. Then there exists $a, b \in R$ such that $ab \in \text{Ann}(sx)$ with $a \notin \text{Ann}(sx)$ and $b \notin \text{Ann}(sx)$. Then $absx = 0$, i.e., $b \in \text{Ann}(asx)$. It is clear that $\text{Ann}(sx) \subseteq \text{Ann}(asx)$. In fact, since $b \in \text{Ann}(asx)$ and $b \notin \text{Ann}(sx)$, we have $\text{Ann}(sx) \subsetneq \text{Ann}(asx)$. By the maximality of $\text{Ann}(sx)$ in Θ, $\text{Ann}(asx) \notin \Theta$. Thus $\text{Ann}(asx) \nsubseteq P$, implying there exists $t \in \text{Ann}(asx)$ such that $t \notin P$. Thus $tasx = 0$, i.e., $a \in \text{Ann}(tsx)$. Since $a \notin \text{Ann}(sx)$, we have $\text{Ann}(sx) \subsetneq \text{Ann}(tsx)$, and by the maximality of $\text{Ann}(sx)$ in Θ, $\text{Ann}(txs) \notin \Theta$. Thus $\text{Ann}(txs) \nsubseteq P$, implying there exists $u \in \text{Ann}(tsx)$ such that $u \notin P$. Then $utsx = 0$, i.e., $ut \in \text{Ann}(sx) \subseteq P$, forcing $u \in P$ or $t \in P$, which is a contradiction. Thus our original supposition that $\text{Ann}(sx)$ is not a prime ideal is false.

Now, we have

$$\text{Ann}(x) \subseteq \text{Ann}(sx) \subseteq P$$

with $\text{Ann}(sx)$ a prime ideal. Since P is a minimal prime ideal over $\text{Ann}(x)$, it is forced that $\text{Ann}(sx) = P$. Therefore, $P \in \text{Ass}(M)$. □

Remark 2.1.5. For the purpose of this paper, the reader may assume that all rings are not necessarily Noetherian, unless explicitly stated otherwise. Therefore this thesis deals with a more general case than the classic theory.

Lemma 2.1.6. *Let M be an R-module. Then $\text{Ass}_{\text{f}}(M) = \emptyset$ if and only if $M = 0$.*

Proof. ($\Leftarrow$) Suppose $M = 0$. Then clearly $\mathrm{Ass_f}(M) = \emptyset$.

($\Rightarrow$) Suppose $M \neq 0$. Then there exists $x \in M$ such that $x \neq 0$. By 1.1.22, $\mathrm{Ann}(x)$ is an ideal of R, and by 1.1.27, $\mathrm{Ann}(x) \neq R$. Thus, by 1.1.12, there exists $P \in \mathrm{Spec}(R)$ such that P is minimal over $\mathrm{Ann}(x)$, i.e., $\mathrm{Ass_f}(M) \neq \emptyset$. □

Lemma 2.1.7. *Let M_1 and M_2 be R-modules. If $M_1 \cong M_2$, then* $\mathrm{Ass_f}(M_1) = \mathrm{Ass_f}(M_2)$.

Proof. Let $M_1 \cong M_2$ and let $P \in \mathrm{Ass_f}(M_1)$. There exists an R-linear bijection

$$\begin{array}{ccc} M_1 & \longrightarrow & M_2 \\ x & \longmapsto & \varphi(x) \end{array}$$

for all $x \in M_1$. Since $P \in \mathrm{Ass_f}(M_1)$, there exists $0 \neq x \in M_1$ such that P is minimal over $\mathrm{Ann}(x)$. We claim $\mathrm{Ann}(x) = \mathrm{Ann}(\varphi(x))$.

Let $a \in \mathrm{Ann}(x)$. Then $ax = 0_{M_1}$, and $\varphi(ax) = 0_{M_2}$. But $\varphi(ax) = a\varphi(x)$. Thus $a\varphi(x) = 0_{M_2}$, and $a \in \mathrm{Ann}(\varphi(x))$. Hence $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(\varphi(x))$.

Let $a \in \mathrm{Ann}(\varphi(x))$. Then $0_{M_2} = a\varphi(x) = \varphi(ax)$. But $\varphi(0_{M_1}) = 0_{M_2}$. Thus $\varphi(ax) = \varphi(0_{M_1})$. By the injectivity of φ, $ax = 0_{M_1}$. Hence $a \in \mathrm{Ann}(x)$, and $\mathrm{Ann}(\varphi(x)) \subseteq \mathrm{Ann}(x)$.

Therefore $\mathrm{Ann}(x) = \mathrm{Ann}(\varphi(x))$. Since P is minimal over $\mathrm{Ann}(x)$, we have that P is minimal over $\mathrm{Ann}(\varphi(x))$. Thus $P \in \mathrm{Ass_f}(M_2)$, and $\mathrm{Ass_f}(M_1) \subseteq \mathrm{Ass_f}(M_2)$.

In the same manner as above, we can show that $\mathrm{Ass_f}(M_2) \subseteq \mathrm{Ass_f}(M_1)$. Therefore, if $M_1 \cong M_2$, then $\mathrm{Ass_f}(M_1) = \mathrm{Ass_f}(M_2)$. □

Lemma 2.1.8. (i) *Let $N \subseteq M$ be R-modules. Then*

$$\mathrm{Ass_f}(N) \subseteq \mathrm{Ass_f}(M) \subseteq \mathrm{Ass_f}(N) \cup \mathrm{Ass_f}(M/N).$$

(ii) *Consider the short exact sequence*

$$0 \longrightarrow M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3 \longrightarrow 0,$$

where M_1, M_2, and M_3 are R-modules. Then

$$\mathrm{Ass_f}(M_1) \subseteq \mathrm{Ass_f}(M_2) \subseteq \mathrm{Ass_f}(M_1) \cup \mathrm{Ass_f}(M_3).$$

(iii) *Consider the exact sequence*

$$0 \longrightarrow M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3,$$

where M_1*,* M_2*, and* M_3 *are* R*-modules. Then*

$$\mathrm{Ass_f}(M_1) \subseteq \mathrm{Ass_f}(M_2) \subseteq \mathrm{Ass_f}(M_1) \cup \mathrm{Ass_f}(M_3).$$

Proof. (i) It is clear that $\mathrm{Ass_f}(N) \subseteq \mathrm{Ass_f}(M)$. Let $P \in \mathrm{Ass_f}(M)$. Then P is minimal over $\mathrm{Ann}(x)$ for some $0 \neq x \in M$. Let

$$X = \{rx \mid r \in R \setminus P\}.$$

If $X \cap N \neq \emptyset$, then there exists $y \in X \cap N$, and, by 1.1.23, P is minimal over $\mathrm{Ann}(y)$. Therefore $P \in \mathrm{Ass_f}(N)$. Consider the case where $X \cap N = \emptyset$. We claim $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(x+N) \subseteq P$. Indeed, by 1.1.27(i), $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(x+N)$. Now let $b \in \mathrm{Ann}(x+N)$, and suppose $b \notin P$. Then $bx \in X \cap N$. But $X \cap N = \emptyset$. Hence we have a contradiction, and $b \in P$. Thus we have shown $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(x+N) \subseteq P$, which proves that P is minimal over $\mathrm{Ann}(x+N)$. Therefore $P \in \mathrm{Ass_f}(M/N)$. This completes the proof of (i).

(ii) Because $M_1 \cong \mathrm{Im}\,\varphi$ and $M_3 \cong M_2/\mathrm{Im}\,\varphi$, it suffices to prove $\mathrm{Ass_f}(\mathrm{Im}\,\varphi) \subseteq \mathrm{Ass_f}(M_2) \subseteq \mathrm{Ass_f}(\mathrm{Im}\,\varphi) \cup \mathrm{Ass_f}(M_2/\mathrm{Im}\,\varphi)$, which follows directly from (i).

(iii) Consider $0 \longrightarrow M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi'} \mathrm{Im}\,\psi \longrightarrow 0$, which is exact at M_1 and M_2. Moreover, since $\mathrm{Im}\,\psi' = \mathrm{Im}\,\psi$, the sequence is exact at $\mathrm{Im}\,\psi$. From (ii), $\mathrm{Ass_f}(M_1) \subseteq \mathrm{Ass_f}(M_2) \subseteq \mathrm{Ass_f}(M_1) \cup \mathrm{Ass_f}(\mathrm{Im}\,\psi) \subseteq \mathrm{Ass_f}(M_1) \cup \mathrm{Ass_f}(M_3)$, because $\mathrm{Im}\,\psi \subseteq M_3$, as modules. This completes the proof of (iii) and Lemma 2.1.8.

□

Lemma 2.1.9. *If* $M = \bigoplus_{i=1}^{n} M_i$ *is a direct sum of* R*-modules, then*

$$\mathrm{Ass_f}(M) = \bigcup_{i=1}^{n} \mathrm{Ass_f}(M_i).$$

Proof. ($\supseteq$) By 1.1.30, for each i with $1 \leq i \leq n$, M_i is isomorphic to a submodule of M. Hence

$\mathrm{Ass_f}(M_i) \subseteq \mathrm{Ass_f}(M)$ for each $1 \le i \le n$. Thus $\bigcup_{i=1}^{n} \mathrm{Ass_f}(M_i) \subseteq \mathrm{Ass_f}(M)$.

($\subseteq$) Consider the exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_1 \oplus M_2 \longrightarrow M_2.$$

By 2.1.8(iii),

$$\mathrm{Ass_f}(M_1 \oplus M_2) \subseteq \mathrm{Ass_f}(M_1) \cup \mathrm{Ass_f}(M_2).$$

Proceeding by induction,

$$\mathrm{Ass_f}(M) \subseteq \bigcup_{i=1}^{n} \mathrm{Ass_f}(M_i),$$

completing our proof. □

Theorem 2.1.10. *If $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ is a direct sum of R-modules, then*

$$\mathrm{Ass_f}(M) = \bigcup_{\lambda\in\Lambda} \mathrm{Ass_f}(M_\lambda).$$

Proof. ($\supseteq$) This is by the same reasoning as ($\supseteq$) in 2.1.9.

($\subseteq$) Let $P \in \mathrm{Ass_f}(M)$. Then P is minimal over $\mathrm{Ann}(x)$ for some $0 \neq x \in M$. We may write $x = (x_\lambda)_{\lambda\in\Lambda}$, where $x_\lambda \in M_\lambda$ for each $\lambda \in \Lambda$ and only finitely many x_λ's are nonzero. Let the nonzero x_λ's be $x_{\lambda_1}, \ldots, x_{\lambda_n}$ and let $x' = (x_{\lambda_i})_{i=1}^n \in \bigoplus_{i=1}^{n} M_{\lambda_i}$. Then $\mathrm{Ann}(x) = \mathrm{Ann}(x')$. Thus P is minimal over $\mathrm{Ann}(x')$, implying

$$P \in \mathrm{Ass_f}\left(\bigoplus_{i=1}^{n} M_{\lambda_i}\right) \overset{2.1.9}{=} \bigcup_{i=1}^{n} \mathrm{Ass_f}(M_{\lambda_i}) \subseteq \bigcup_{\lambda\in\Lambda} \mathrm{Ass_f}(M_\lambda).$$

That is,

$$P \in \bigcup_{\lambda\in\Lambda} \mathrm{Ass_f}(M_\lambda).$$

Therefore

$$\mathrm{Ass_f}(M) \subseteq \bigcup_{\lambda\in\Lambda} \mathrm{Ass_f}(M_\lambda),$$

completing our proof. □

Lemma 2.1.11. *Let M be an R-module. Then*

$$\mathrm{Zdv}_R(M) = \bigcup_{P \in \mathrm{Ass_f}(M)} P.$$

Proof. ($\subseteq$) Let $a \in \mathrm{Zdv}_R(M)$. Then $a \in \mathrm{Ann}(x)$ for some $0 \neq x \in M$. Since $x \neq 0$, it follows that $\mathrm{Ann}(x) \neq R$. Thus by 1.1.12, there exists a prime ideal P of R such that P is minimal over $\mathrm{Ann}(x)$. Hence we have $a \in \mathrm{Ann}(x) \subseteq P \in \mathrm{Ass_f}(M)$, implying $a \in \bigcup_{P \in \mathrm{Ass_f}(M)} P$, which proves the first inclusion.

($\supseteq$) Let $a \in P$ for some $P \in \mathrm{Ass_f}(M)$, and say P is minimal over $\mathrm{Ann}(x)$ for some $0 \neq x \in M$. Denote $I := \mathrm{Ann}(x)$ and consider the ring R/I. Noticing that $P/I \in \mathrm{Spec}(R/I)$, it is clear that P/I is minimal in $\mathrm{Spec}(R/I)$. Thus by 1.1.9, there exists $b + I \in (R/I) \setminus (P/I)$ such that $a^n b + I = 0 + I$ for some minimal $n \in \mathbb{N}$. Hence $a^n b \in I$ and $a^{n-1} b \notin I$. That is, $a^n bx = 0$ and $a^{n-1}bx \neq 0$, implying $a \in \mathrm{Zdv}_R(M)$, which proves the reverse inclusion. □

Now, we define the notion of nearly nilpotency and establish the notable results regarding nearly nilpotent elements.

Definition 2.1.12. An element $a \in R$ is said to be *nearly nilpotent* on the R-module M if for every $x \in M$, there exists $n(x) \in \mathbb{N}$ such that $a^{n(x)}x = 0$.

Notation 2.1.13. We denote the set of all elements nearly nilpotent on M by $\mathrm{Nil_f}(M)$.

Theorem 2.1.14. *Let M be an R-module.*

(i) $\mathrm{Nil}(M) \subseteq \mathrm{Nil_f}(M)$.

(ii) *If M is finitely generated, then* $\mathrm{Nil}(M) = \mathrm{Nil_f}(M)$.

Proof. (i) This is trivially true.

(ii) Let M be finitely generated by $J = \{x_1, \ldots, x_k\} \subseteq M$ and let $a \in \mathrm{Nil_f}(M)$. Then for every $x_i \in J$ where $i = 1, \ldots, k$, there exists $n(x_i) \in \mathbb{N}$ such that $a^{n(x_i)}x_i = 0$. Let $n = \max\{n(x_1), \ldots, n(x_k)\}$, and let x be an arbitrary element in M. Then $x = r_1x_1 + \cdots + r_kx_k$ where

$r_i \in R$ for each $i = 1, \cdots, k$, and

$$\begin{aligned} a^n x &= a^n r_1 x_1 + \cdots + a^n r_k x_k \\ &= r_1(a^n x_1) + \cdots + r_k(a^n x_k) \\ &= 0\,. \end{aligned}$$

Hence $a \in \mathrm{Nil}(M)$, and $\mathrm{Nil}_{\mathrm{f}}(M) \subseteq \mathrm{Nil}(M)$. Therefore, if M is finitely generated, then $\mathrm{Nil}(M) = \mathrm{Nil}_{\mathrm{f}}(M)$. □

Remark 2.1.15. For the purpose of this paper, the reader may assume that all modules are not necessarily finitely generated, unless explicitly stated otherwise.

Lemma 2.1.16. *Let M be an R-module. Then*

$$\mathrm{Nil}_{\mathrm{f}}(M) = \bigcap_{x \in M} \sqrt{\mathrm{Ann}(x)}.$$

Proof. ($\subseteq$) Let $a \in \mathrm{Nil}_{\mathrm{f}}(M)$. Then for each $x \in M$, there exists $n(x) \in \mathbb{N}$ such that $a^{n(x)}x = 0$, implying $a^{n(x)} \in \mathrm{Ann}(x)$. Then $a \in \sqrt{\mathrm{Ann}(x)}$ for all $x \in M$. Thus $a \in \bigcap_{x \in M} \sqrt{\mathrm{Ann}(x)}$.

($\supseteq$) Let $a \in \bigcap_{x \in M} \sqrt{\mathrm{Ann}(x)}$. Then $a \in \sqrt{\mathrm{Ann}(x)}$ for all $x \in M$. Thus for each $x \in M$, there exists $n(x) \in \mathbb{N}$ such that $a^{n(x)}x = 0$, i.e., $a \in \mathrm{Nil}_{\mathrm{f}}(M)$. □

Remark 2.1.17. If $M \neq 0$, then $\mathrm{Nil}_{\mathrm{f}}(M) = \bigcap_{0 \neq x \in M} \sqrt{\mathrm{Ann}(x)}$.

Lemma 2.1.18. *Let $M \neq 0$ be an R-module. Then*

$$\mathrm{Nil}_{\mathrm{f}}(M) \subseteq \mathrm{Zdv}_R(M).$$

Proof. Let $a \in \mathrm{Nil}_{\mathrm{f}}(M)$, and let $0 \neq x \in M$. Then there exists a minimal $n(x) \in \mathbb{N}$ such that $0 = a^{n(x)}x = a \cdot a^{n(x)-1}x$ and $a^{n(x)-1}x \neq 0$. Thus $a \in \mathrm{Zdv}_R(M)$. □

2.2 Weak Primary Submodules

Definition 2.2.1. A submodule Q of the R-module M is said to be a *weak primary submodule* of M (or *weakly primary* in M) if the following hold:

(i) $Q \subsetneq M$, i.e., $M/Q \neq 0$; and

(ii) $\mathrm{Zdv}_R(M/Q) = \mathrm{Nil_f}(M/Q)$, or equivalently, $\mathrm{Zdv}_R(M/Q) \subseteq \mathrm{Nil_f}(M/Q)$, in light of 2.1.18.

When M happens to be finitely generated, the weak primary submodules of M agree with the primary submodules of M, in light of 2.1.14(ii) and 1.2.7. So that we can study primary submodules in a more general form, namely weak primary submodules, the reader should continue to assume that all modules are not necessarily finitely generated, unless explicitly stated otherwise.

Lemma 2.2.2. *Let Q be a weak primary submodule of the R-module M. Then* $\sqrt{\mathrm{Ann}(x)} = \sqrt{\mathrm{Ann}(y)}$ *for all nonzero $x, y \in M/Q$.*

Proof. Let x and y be nonzero in M/Q, and let $a \in \sqrt{\mathrm{Ann}(x)}$. Then $a^n x = 0$ for some minimal $n \in \mathbb{N}$, that is, $a \cdot a^{n-1}x = 0$ with $a^{n-1}x \neq 0$. Thus $a \in \mathrm{Zdv}_R(M/Q)$. Since Q is weakly primary in M, $\mathrm{Zdv}_R(M/Q) = \mathrm{Nil_f}(M/Q)$. Hence $a \in \mathrm{Nil_f}(M/Q)$, implying $a^{n(y)}y = 0$ for some $n(y) \in \mathbb{N}$. Thus $a \in \sqrt{\mathrm{Ann}(y)}$, and $\sqrt{\mathrm{Ann}(x)} \subseteq \sqrt{\mathrm{Ann}(y)}$ for all nonzero $x, y \in M/Q$. Similarly, $\sqrt{\mathrm{Ann}(y)} \subseteq \sqrt{\mathrm{Ann}(x)}$. Therefore $\sqrt{\mathrm{Ann}(x)} = \sqrt{\mathrm{Ann}(y)}$ for all nonzero $x, y \in M/Q$. □

Theorem 2.2.3. *Let Q be a submodule of the R-module M. Then the following statements are equivalent:*

(i) $\mathrm{Ass_f}(M/Q)$ *is a singleton set;*

(ii) *Q is weakly primary in M;*

(iii) $\mathrm{Nil_f}(M/Q)$ *is a prime ideal of R, and* $\mathrm{Ass_f}(M/Q) = \{\mathrm{Nil_f}(M/Q)\}$.

Proof. (i)⇒(ii) Let $\mathrm{Ass_f}(M/Q)$ be a singleton set; say $\mathrm{Ass_f}(M/Q) = \{P\}$, that is, P is the only prime ideal minimal over $\mathrm{Ann}(x)$ for all $0 \neq x \in M/Q$. Then by 2.1.11 and 1.1.15, $\mathrm{Zdv}_R(M/Q) = P = \sqrt{\mathrm{Ann}(x)}$ for all $0 \neq x \in M/Q$. Thus for every $a \in \mathrm{Zdv}_R(M/Q)$ and every $x \in M/Q$, there exists $n(x) \in \mathbb{N}$ such that $a^{n(x)}x = 0$. Hence $\mathrm{Zdv}_R(M/Q) \subseteq \mathrm{Nil}_{\mathrm{f}}(M/Q)$. Moreover, notice that since x is nonzero in M/Q, we have $M/Q \neq 0$. Therefore Q is weakly primary in M.

(ii)⇒(iii) Let Q be weakly primary in M. Then $\mathrm{Zdv}_R(M/Q) = \mathrm{Nil}_{\mathrm{f}}(M/Q)$. First we show $\mathrm{Nil}_{\mathrm{f}}(M/Q)$ is a prime ideal. To do this, it suffices to show $\mathrm{Zdv}_R(M/Q)$ is a prime ideal. Let $a, b \in R$ such that $ab \in \mathrm{Zdv}_R(M/Q)$ and $b \notin \mathrm{Zdv}_R(M/Q)$. Then there exists $0 \neq x \in M/Q$ such that $abx = 0$. Moreover, $0 \neq bx \in M/Q$. Hence $a \in \mathrm{Zdv}_R(M/Q)$, and $\mathrm{Zdv}_R(M/Q)$ is a prime ideal, i.e., $\mathrm{Nil}_{\mathrm{f}}(M/Q)$ is a prime ideal.

Now, we show $\mathrm{Ass}_{\mathrm{f}}(M/Q) = \{\mathrm{Nil}_{\mathrm{f}}(M/Q)\}$. By 2.2.2, $\sqrt{\mathrm{Ann}(x)} = \sqrt{\mathrm{Ann}(y)}$ for all nonzero $x, y \in M/Q$. Thus we have by 2.1.17 that $\mathrm{Nil}_{\mathrm{f}}(M/Q) = \sqrt{\mathrm{Ann}(x)}$ for all $0 \neq x \in M/Q$, implying $\sqrt{\mathrm{Ann}(x)}$ is a prime ideal. Then by 1.1.16, $\sqrt{\mathrm{Ann}(x)}$ is the only prime ideal minimal over $\mathrm{Ann}(x)$ for all $0 \neq x \in M/Q$, i.e., $\mathrm{Nil}_{\mathrm{f}}(M/Q)$ is the only prime ideal minimal over $\mathrm{Ann}(x)$ for all $0 \neq x \in M/Q$. This proves $\mathrm{Ass}_{\mathrm{f}}(M/Q) = \{\mathrm{Nil}_{\mathrm{f}}(M/Q)\}$.

(iii)⇒(i) Clear. □

Definition 2.2.4. Let Q be a weak primary submodule of the R-module M. By Theorem 2.2.3, $\mathrm{Nil}_{\mathrm{f}}(M/Q)$ is a prime ideal of R. If we denote $\mathrm{Nil}_{\mathrm{f}}(M/Q)$ by P so that $\mathrm{Ass}_{\mathrm{f}}(M/Q) = \{P\}$, then we say Q is a *weak P-primary submodule* of M, or *weakly P-primary* in M.

Observation 2.2.5. Let Q be a weak P-primary submodule of the R-module M.

(i) If M is finitely generated, then Q is a P-primary submodule of M.

(ii) Specifically, if $M = R$, then Q is a P-primary ideal of R. Thus, if Q is a weak P-primary ideal of R, then it is in fact P-primary.

Proof. This is clear, in light of 2.1.14(ii) and 1.2.9. □

Lemma 2.2.6. *Let $Q_1, Q_2, \ldots, Q_n$ ($n \in \mathbb{N}$) be weak P-primary submodules of the R-module M. Then $Q := \bigcap_{i=1}^{n} Q_i$ is weakly P-primary in M.*

Proof. Define

$$\varphi \;:\; M \longrightarrow \bigoplus_{i=1}^{n} \frac{M}{Q_i}$$

by $\varphi(x) = (x + Q_1, \dots, x + Q_n)$ for all $x \in M$. It is clear that φ is R-linear and $\operatorname{Ker}\varphi = \bigcap_{i=1}^{n} Q_i$, i.e., $\operatorname{Ker}\varphi = Q$. Thus

$$\frac{M}{Q} \cong \operatorname{Im}\varphi \subseteq \frac{M}{Q_1} \oplus \cdots \oplus \frac{M}{Q_n},$$

implying

$$\operatorname{Ass_f}\left(\frac{M}{Q}\right) \subseteq \operatorname{Ass_f}\left(\bigoplus_{i=1}^{n} \frac{M}{Q_i}\right) = \bigcup_{i=1}^{n} \operatorname{Ass_f}\left(\frac{M}{Q_i}\right) = \{P\}.$$

Since $Q \neq M$, we have $M/Q \neq 0$, implying $\operatorname{Ass_f}(M/Q) \neq \emptyset$. Therefore $\operatorname{Ass_f}(M/Q) = \{P\}$, and Q is weakly P-primary in M. $\square$

Lemma 2.2.7. *Let Q be a weak P-primary submodule of the R-module M (i.e., $\operatorname{Ass_f}(M/Q) = \{P\}$), and let $a \in R$.*

(i) *If $a \notin \operatorname{Ann}(M/Q)$, then $(Q :_M a)$ is a weak P-primary submodule of M.*

(ii) *If $a \notin P$, then $(Q :_M a) = Q$.*

(iii) *If $a \in P$, then $\bigcup_{t=1}^{\infty} (Q :_M a^t) = M$.*

Proof. (i) Suppose $a \notin \operatorname{Ann}(M/Q)$. By 1.1.26, $(Q :_M a)$ is a submodule of M. Let us denote $N := (Q :_M a)$. It remains to show that N is weakly P-primary.

Consider the R-linear mapping

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & M/Q \\ x & \longmapsto & ax + Q \,. \end{array}$$

It is clear that $\operatorname{Ker}\varphi = N$. Thus $M/N \cong \operatorname{Im}\varphi \subseteq M/Q$, implying $\operatorname{Ass_f}(M/N) \subseteq \operatorname{Ass_f}(M/Q) = \{P\}$. But $\operatorname{Ass_f}(M/N) \neq \emptyset$, because $M \neq N$; otherwise $M = N$ and, by Fact 1.1.26, $a \in \operatorname{Ann}(M/Q)$, which is false. Thus $\operatorname{Ass_f}(M/N) = \{P\}$, which means N is weakly P-primary, i.e., $(Q :_M a)$ is weakly P-primary.

(ii) Clearly $Q \subseteq (Q :_M a)$. Suppose $a \notin P$. Then $a \notin \operatorname{Nil_f}(M/Q)$ and $a \notin \operatorname{Zdv}_R(M/Q)$. Let $x \in (Q :_M a)$. Then $ax \in Q$, forcing $x \in Q$, since $a \notin \operatorname{Zdv}_R(M/Q)$. Therefore $(Q :_M a) = Q$.

(iii) Clearly $\bigcup_{t=1}^{\infty} (Q :_M a^t) \subseteq M$. Let $a \in P$, and let $x \in M$. Since $a \in P$, a is nearly nilpotent on M/Q. Thus for some $n(x) \in \mathbb{N}$, $a^{n(x)}x = 0_{M/Q}$, implying $a^{n(x)}x \in Q$, i.e., $x \in (Q :_M a^{n(x)})$. Hence $x \in \bigcup_{t=1}^{\infty} (Q :_M a^t)$. Therefore $\bigcup_{t=1}^{\infty} (Q :_M a^t) = M$. $\square$

Lemma 2.2.8. *Let Q be a weak P-primary submodule of the R-module M, and let $x \in M$. If $x \notin Q$, then $(Q :_R x)$ is a P-primary ideal of R.*

Proof. Suppose $x \notin Q$. By 1.1.27, $(Q :_R x)$ is an ideal of R. Denote $I := (Q :_R x)$. It remains to show that I is weakly P-primary.

Consider the R-linear mapping

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & M/Q \\ a & \longmapsto & ax + Q\,. \end{array}$$

It is clear that $\operatorname{Ker}\varphi = I$. Thus $R/I \cong \operatorname{Im}\varphi \subseteq M/Q$, implying $\operatorname{Ass_f}(R/I) \subseteq \operatorname{Ass_f}(M/Q) = \{P\}$. But $\operatorname{Ass_f}(R/I) \neq \emptyset$, because $R \neq I$; otherwise $R = I$ and, by Fact 1.1.27, $x \in Q$, which is false. Thus $\operatorname{Ass_f}(R/I) = \{P\}$, which means I is weakly P-primary, i.e., $(Q :_R x)$ is weakly P-primary. Then, by 2.2.5, $(Q :_R x)$ is a P-primary ideal of R. □

2.3 Weak Primary Decomposition

In this section, we establish the results on weak primary decomposition, which are parallel to the results on the classic theory of primary decomposition.

Definition 2.3.1. Let $N \subsetneq M$ be R-modules. We say N is a *weakly decomposable submodule* of M if it can be written as an intersection of finitely many weak primary submodules of M. Such an intersection

$$N = Q_1 \cap \cdots \cap Q_n \quad \text{with } Q_i \text{ weakly } P_i\text{-primary in } M\ (i = 1, 2, \ldots, n)$$

is called a *weak primary decomposition* of N in M.

Definition 2.3.2. Let N be a weakly decomposable submodule of the R-module M. In particular, let

$$N = Q_1 \cap \cdots \cap Q_n \quad \text{with } Q_i \text{ weakly } P_i\text{-primary in } M\ (i = 1, 2, \ldots, n).$$

We say this weak primary decomposition is *minimal* if

(i) $P_1, \ldots, P_n$ are all distinct; and

(ii) for all $j = 1, 2, \ldots, n$, $Q_j \not\supseteq \bigcap_{i \neq j} Q_i$.

Definition 2.3.3. Let N be a weakly decomposable submodule of the R-module M. A weak primary submodule Q of M is said to be a *weak primary component* of N in M if it appears in some minimal weak primary decomposition of N in M. In particular, if Q is weakly P-primary and it appears in some minimal weak primary decomposition of N in M, then Q is said to be a *weak P-primary component* of N in M.

Lemma 2.3.4. *Let $N \subsetneq M$ be R-modules such that N is weakly decomposable in M. Then N has a minimal weak primary decomposition.*

Proof. Since N is weakly decomposable in M, we may write N as

$$N = Q_1 \cap \cdots \cap Q_n \quad \text{with } Q_i \text{ weakly } P_i\text{-primary in } M \ (i = 1, 2, \ldots, n).$$

If this expression is not minimal, then at least one of the following is true:

(i) $P_j = P_k$ for some $j \neq k$ with $1 \leq j \leq n$ and $1 \leq k \leq n$; or

(ii) $Q_j \supseteq \bigcap_{i \neq j} Q_i$ for some j with $1 \leq j \leq n$.

Case(i) If $P_j = P_k$ for some $j \neq k$ with $1 \leq j \leq n$ and $1 \leq k \leq n$, then reorder the Q_i's so that $P_1 = P_2$; denote $P_1 = P_2 = P$. By 2.2.6, $Q_1 \cap Q_2$ is weakly P-primary. By denoting $Q = Q_1 \cap Q_2$, we obtain another weak primary decomposition of N with $n-1$ terms. If necessary, we may repeat this process until each submodule in the decomposition is weakly primary to a distinct prime ideal.

Case(ii) If $Q_j \supseteq \bigcap_{i \neq j} Q_i$ for some j with $1 \leq j \leq n$, then reorder the Q_i's so that $Q_j = Q_n$. Then $Q_n \supseteq \bigcap_{i=1}^{n-1} Q_i$. It is easy to verify that $\bigcap_{i=1}^{n-1} Q_i = \bigcap_{i=1}^{n} Q_i$. Thus we may discard Q_n. If necessary, we may repeat this process until no submodule can be removed without changing the weak primary decomposition.

Therefore N has a minimal primary decomposition. □

Remark 2.3.5. Throughout this paper, the reader may assume all weak primary decompositions to be minimal, unless stated otherwise.

Lemma 2.3.6. *Let $a \in R$ and let N be a weakly decomposable submodule of the R-module M. In particular, let*

$$N = Q_1 \cap \cdots \cap Q_n \quad \textit{with } Q_i \textit{ weakly } P_i\textit{-primary in } M \ (i = 1, 2, \ldots, n)$$

be a minimal weak primary decomposition of N in M. Then $\displaystyle\bigcup_{t=1}^{\infty}(N :_M a^t) = \bigcap_{a \notin P_i} Q_i$.

Proof. We have that

$$\bigcup_{t=1}^{\infty}(N :_M a^t) = \bigcup_{t=1}^{\infty}\left(\bigcap_{i=1}^{n} Q_i :_M a^t\right) \overset{1.1.28}{=} \bigcup_{t=1}^{\infty}\left[\bigcap_{i=1}^{n}(Q_i :_M a^t)\right].$$

We claim

$$\bigcup_{t=1}^{\infty}\left[\bigcap_{i=1}^{n}(Q_i :_M a^t)\right] = \bigcap_{i=1}^{n}\left[\bigcup_{t=1}^{\infty}(Q_i :_M a^t)\right].$$

Let $x \in \displaystyle\bigcup_{t=1}^{\infty}\left[\bigcap_{i=1}^{n}(Q_i :_M a^t)\right]$. Then $x \in \displaystyle\bigcap_{i=1}^{n}(Q_i :_M a^{t_0})$ for some $t_0 \geq 1 \implies x \in (Q_i :_M a^{t_0})$ for all $1 \leq i \leq n \implies x \in \displaystyle\bigcup_{t=1}^{\infty}(Q_i :_M a^t)$ for all $1 \leq i \leq n \implies x \in \displaystyle\bigcap_{i=1}^{n}\left[\bigcup_{t=1}^{\infty}(Q_i :_M a^t)\right]$. Thus

$$\bigcup_{t=1}^{\infty}\left[\bigcap_{i=1}^{n}(Q_i :_M a^t)\right] \subseteq \bigcap_{i=1}^{n}\left[\bigcup_{t=1}^{\infty}(Q_i :_M a^t)\right].$$

Let $x \in \displaystyle\bigcap_{i=1}^{n}\left[\bigcup_{t=1}^{\infty}(Q_i :_M a^t)\right]$. Then $x \in \displaystyle\bigcup_{t=1}^{\infty}(Q_i :_M a^t)$ for all $1 \leq i \leq n \implies x \in (Q_i :_M a^{t(i)})$ for all $1 \leq i \leq n$ and for some $t(i) \geq 1$. Let $t = \max\{t(1), \ldots, t(n)\}$. Then $x \in (Q_i :_M a^t)$ for all $1 \leq i \leq n \implies x \in \displaystyle\bigcap_{i=1}^{n}(Q_i :_M a^t) \implies x \in \displaystyle\bigcup_{t=1}^{\infty}\left[\bigcap_{i=1}^{n}(Q_i :_M a^t)\right]$. Thus

$$\bigcap_{i=1}^{n}\left[\bigcup_{t=1}^{\infty}(Q_i :_M a^t)\right] \subseteq \bigcup_{t=1}^{\infty}\left[\bigcap_{i=1}^{n}(Q_i :_M a^t)\right].$$

Therefore

$$\bigcup_{t=1}^{\infty}\left[\bigcap_{i=1}^{n}(Q_i :_M a^t)\right] = \bigcap_{i=1}^{n}\left[\bigcup_{t=1}^{\infty}(Q_i :_M a^t)\right].$$

Now,

$$\bigcup_{t=1}^{\infty}(N :_M a^t) = \bigcap_{i=1}^{n}\left[\bigcup_{t=1}^{\infty}(Q_i :_M a^t)\right].$$

If $a \notin P_i$, then $a^t \notin P_i$. Thus, by 2.2.7(ii), $(Q_i :_M a^t) = Q_i$ for all $t \geq 1$ and all i such that $a \notin P_i$. By 2.2.7(iii), $\bigcup_{t=1}^{\infty}(Q_i :_M a^t) = M$ for all $t \geq 1$ and all i such that $a \in P_i$. Let $a \notin P_i$ for $1 \leq i \leq r$ and $a \in P_i$ for $r+1 \leq i \leq n$, then

$$\bigcap_{i=1}^{n}\left[\bigcup_{t=1}^{\infty}(Q_i :_M a^t)\right] = Q_1 \cap \cdots \cap Q_r \cap M \cap \cdots \cap M = \bigcap_{i=1}^{r} Q_i = \bigcap_{a \notin P_i} Q_i.$$

That is, $\bigcup_{t=1}^{\infty}(N :_M a^t) = \bigcap_{a \notin P_i} Q_i$, completing our proof. □

Next, we present the uniqueness theorems in the context of weak primary decomposition. We begin with an important lemma that is necessary to prove the first uniqueness theorem.

Lemma 2.3.7. *Let N be a weakly decomposable submodule of the R-module M. In particular, let*

$$N = Q_1 \cap \cdots \cap Q_n \quad \text{with } Q_i \text{ weakly } P_i\text{-primary in } M \ (i = 1, 2, \ldots, n)$$

be a minimal weak primary decomposition of N in M, and let $P \in \operatorname{Spec}(R)$. Then the following statements are equivalent:

(i) *$P = P_i$ for some i with $1 \leq i \leq n$;*

(ii) *for some $x \in M$, $(N :_R x)$ is a P-primary ideal of R;*

(iii) *for some $x \in M$, $\sqrt{(N :_R x)} = P$;*

(iv) *$P \in \operatorname{Ass_f}(M/N)$.*

Thus $\operatorname{Ass_f}(M/N) = \{P_1, \ldots, P_n\}$.

Proof. (i) ⇒ (ii) Let $P = P_i$ for some i with $1 \leq i \leq n$; without loss of generality, say $P = P_1$. Because $N \neq Q_2 \cap \cdots \cap Q_n$, we have

$$0 \neq \frac{Q_2 \cap \cdots \cap Q_n}{N} = \frac{Q_2 \cap \cdots \cap Q_n}{Q_1 \cap (Q_2 \cap \cdots \cap Q_n)} \overset{1.1.31}{\cong} \frac{Q_1 + (Q_2 \cap \cdots \cap Q_n)}{Q_1} \subseteq M/Q_1,$$

implying

$$\operatorname{Ass_f}\left(\frac{Q_2 \cap \cdots \cap Q_n}{N}\right) \subseteq \operatorname{Ass_f}(M/Q_1) = \{P\}.$$

This forces

$$\mathrm{Ass_f}\left(\frac{Q_2 \cap \cdots \cap Q_n}{N}\right) = \{P\},$$

because $\mathrm{Ass_f}\left(\frac{Q_2 \cap \cdots \cap Q_n}{N}\right) \neq \emptyset$, since $\frac{Q_2 \cap \cdots \cap Q_n}{N} \neq 0$.

Let $x \in Q_2 \cap \cdots \cap Q_n$ such that $x \notin N$, and consider the R-linear mapping

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & \frac{Q_2 \cap \cdots \cap Q_n}{N} \\ a & \longmapsto & ax + N\,. \end{array}$$

It is clear that $\operatorname{Ker} \varphi = (N :_R x)$. Thus, denoting $I := (N :_R x)$, we have

$$R/I \cong \operatorname{Im} \varphi \subseteq \frac{Q_2 \cap \cdots \cap Q_n}{N},$$

implying

$$\mathrm{Ass_f}(R/I) \subseteq \mathrm{Ass_f}\left(\frac{Q_2 \cap \cdots \cap Q_n}{N}\right) = \{P\}.$$

Since $x \notin N$, we have $R \neq I$, by the contrapositive of 1.1.27(ii). Thus $\mathrm{Ass_f}(R/I) \neq \emptyset$, implying $\mathrm{Ass_f}(R/I) = \{P\}$. Hence I is a weak P-primary ideal. Then, by 2.2.5, I is a P-primary ideal, i.e., $(N :_R x)$ is a P-primary ideal.

(ii) $\Rightarrow$ (iii) Clear, from 1.2.10.

(iii) $\Rightarrow$ (iv) By 1.1.27, $(N :_R x) = \operatorname{Ann}(x + n)$. Thus $\sqrt{(N :_R x)} = \sqrt{\operatorname{Ann}(x + N)}$, i.e., $P = \sqrt{\operatorname{Ann}(x + N)}$. Thus $\sqrt{\operatorname{Ann}(x + N)}$ is a prime ideal, and, by 1.1.16, $\sqrt{\operatorname{Ann}(x + N)}$ is minimal over $\operatorname{Ann}(x + N)$, i.e., P is minimal over $\operatorname{Ann}(x + N)$. Therefore $P \in \mathrm{Ass_f}(M/N)$.

(iv) $\Rightarrow$ (i) Consider the R-linear mapping

$$\begin{array}{ccc} M & \xrightarrow{\psi} & M/Q_1 \oplus \cdots \oplus M/Q_n \\ x & \longmapsto & (x + Q_1, \ldots, x + Q_n)\,. \end{array}$$

Since $\operatorname{Ker} \psi = Q_1 \cap \cdots \cap Q_n = N$, we have

$$M/N \cong \operatorname{Im} \psi \subseteq M/Q_1 \oplus \cdots \oplus M/Q_n.$$

Thus

$$\mathrm{Ass_f}(M/N) \subseteq \mathrm{Ass_f}(M/Q_1) \cup \cdots \cup \mathrm{Ass_f}(M/Q_n) = \{P_1, \ldots, P_n\}.$$

Therefore, if $P \in \mathrm{Ass_f}(M/N)$, then $P = P_i$ for some i with $1 \le i \le n$. □

Theorem 2.3.8 (The First Uniqueness Theorem)**.** *Let N be a weakly decomposable submodule of the R-module M. In particular, let*

$$N = Q_1 \cap \cdots \cap Q_n \quad \text{with } Q_i \text{ weakly } P_i\text{-primary in } M \ (i = 1, 2, \ldots, n)$$

and

$$N = Q'_1 \cap \cdots \cap Q'_{n'} \quad \text{with } Q'_i \text{ weakly } P'_i\text{-primary in } M \ (i = 1, 2, \ldots, n')$$

be two minimal weak primary decompositions of N in M. Then $n = n'$ and $\{P_1, P_2, \ldots, P_n\} = \{P'_1, P'_2, \ldots, P'_n\} = \mathrm{Ass_f}(M/N)$.

Proof. This follows directly from 2.3.7. □

Theorem 2.3.9 (The Second Uniqueness Theorem)**.** *Let N be a weakly decomposable submodule of the R-module M. In particular, let*

$$N = Q_1 \cap \cdots \cap Q_n \quad \text{with } Q_i \text{ weakly } P_i\text{-primary in } M \ (i = 1, \ldots, n)$$

and

$$N = Q'_1 \cap \cdots \cap Q'_n \quad \text{with } Q'_i \text{ weakly } P_i\text{-primary in } M \ (i = 1, \ldots, n)$$

be two minimal weak primary decompositions of N in M. (Here we have made use of the First Uniqueness Theorem 2.3.8*.) If P_j is a minimal member of $\{P_1, P_2, \ldots, P_n\}$, then $Q_j = Q'_j$.*

Proof. Without loss of generality, let P_1 be a minimal member of $\{P_1, ..., P_n\}$. Then for all k with $2 \le k \le n$, $P_1 \not\supseteq P_k$. Thus there exists $a_k \in P_k \setminus P_1$ for all $2 \le k \le n$. Let $a = a_2 a_3 \cdots a_n$. Clearly $a \in P_k$ for all $2 \le k \le n$ and $a \notin P_1$. Then by 2.3.6,

$$\bigcup_{t=1}^{\infty} (N :_M a^t) = Q_1 \quad \text{and} \quad \bigcup_{t=1}^{\infty} (N :_M a^t) = Q'_1.$$

Therefore $Q_1 = Q'_1$, as desired. □

It is a well-known fact from the study of the classic theory of primary decomposition that if a module M is finitely generated over a Noetherian ring R, then every proper submodule N of M has a primary decomposition ([3] and [4].) (Here, we omit the term "weak" as it is not necessary in the context of a Noetherian ring and a finitely generated module.) For the more general case in which R is not necessarily Noetherian or M is not necessarily finitely generated, the existence of a weak primary decomposition of N in M is not guaranteed, which is seen in the following examples.

Example 2.3.10. Let R be the ring of all continuous functions defined on $\mathbb{R}$, which is not Noetherian. If we let $M = R$, then $\{0_M\}$ is not a weakly decomposable submodule of M.

Proof. First, we claim $(0 :_R f) = \sqrt{(0 :_R f)}$. Let $g \in \sqrt{(0 :_R f)}$. Then there exists $n \in \mathbb{N}$ such that $g^n(x)f(x) = 0$ for all $x \in \mathbb{R}$. Thus $g^n(x) = 0$ for all $x \in \mathbb{R}$ such that $f(x) \neq 0$, implying $g(x) = 0$ for all $x \in \mathbb{R}$ such that $f(x) \neq 0$. Hence $g(x)f(x) = 0$ for all $x \in \mathbb{R}$, i.e., $g \in (0 :_R f)$. Therefore $(0 :_R f) = \sqrt{(0 :_R f)}$.

Now suppose $\{0_M\}$ is weakly decomposable, and let $P \in \mathrm{Ass_f}(M)$ (the existence of P is guaranteed because $M \neq 0$.) Then by 2.3.7, $\sqrt{(0 :_R f)} = P$ for some $f \in M$, and this implies that $f(x) \neq 0$ for some $x \in \mathbb{R}$. Hence there exists $a \in \mathbb{R}$ such that $f(a) \neq 0$ and there exists $\delta > 0$ such that $f(x) \neq 0$ for all $x \in (a - \delta, a + \delta)$.

We have just shown that $(0 :_R f) = \sqrt{(0 :_R f)}$. Thus $(0 :_R f) = P$ so that $(0 :_R f)$ is a prime ideal. But we can define two functions $g_1, g_2 \in M$ such that $g_1 \cdot g_2 \in (0 :_R f)$, with $g_1 \notin (0 :_R f)$ and $g_2 \notin (0 :_R f)$. Let

$$g_1(x) = \max\left\{\frac{\delta}{2} - \left|x - \left(\frac{2a-\delta}{2}\right)\right| \,,\, 0\right\} \quad \text{and} \quad g_2(x) = \max\left\{\frac{\delta}{2} - \left|x - \left(\frac{2a+\delta}{2}\right)\right| \,,\, 0\right\}.$$

It is easy to verify that $g_1(x) \cdot g_2(x) \cdot f(x) = 0$ for all $x \in \mathbb{R}$, while $g_1(x) \cdot f(x) \neq 0$ for all $x \in (a-\delta, a)$ and $g_2(x) \cdot f(x) \neq 0$ for all $x \in (a, a+\delta)$, which contradicts $(0 :_R f)$ being prime. Therefore $\{0_M\}$ is not weakly decomposable. □

Example 2.3.11. Let $R = \mathbb{Z}$. Let $M = \mathbb{Z}/(2) \oplus \mathbb{Z}/(3) \oplus \mathbb{Z}/(5) \oplus \cdots$, which is not finitely generated. Then $\{0_M\}$ is not a weakly decomposable submodule of M.

Proof. Suppose $\{0_M\}$ is weakly decomposable. Then $\mathrm{Ass_f}(M/\{0_M\}) = \mathrm{Ass_f}(M)$ is finite, by 2.3.7. But

$$M = \bigoplus_{p \text{ prime}} \mathbb{Z}/(p),$$

which implies

$$\begin{aligned} \mathrm{Ass_f}(M) &= \mathrm{Ass_f}\Bigg(\bigoplus_{p \text{ prime}} \mathbb{Z}/(p)\Bigg) \\ &\overset{2.1.10}{=} \bigcup_{p \text{ prime}} \mathrm{Ass_f}\Big(\mathbb{Z}/(p)\Big) \\ &= \{(p) \mid p \text{ is prime in } \mathbb{Z}\}. \end{aligned}$$

Since there are infinitely many prime numbers, we have $|\mathrm{Ass_f}(M)| = \infty$, a contradiction. Therefore $\{0_M\}$ is not weakly decomposable. □

Chapter 3

COMPATIBILITY

3.1 Background and Terminology

If N is a decomposable submodule of the R-module M and Q_i is a weak primary component of N in M for each $1 \leq i \leq n$, then up to this point we could not assume that $N = Q_1 \cap \cdots \cap Q_n$. In fact, Q_i and Q_j being weak primary components of N in M simply means that they appear in some weak primary decomposition of N in M, but not necessarily in the same weak primary decomposition of N in M, which is known as the Compatibility property of primary decomposition.

If we assume that $N \subsetneq M$ are finitely generated modules over a Noetherian ring R, it is guaranteed that there exists a primary decomposition of N in M. It is in this context that Y. Yao, in [5], proved the Compatibility property, which formally says that if $\text{Ass}(M/N) = \{P_1, \ldots, P_n\}$ and Q_i is a P_i-primary component of N in M for each $1 \leq i \leq n$, then $N = Q_1 \cap \cdots \cap Q_n$, which is a minimal primary decomposition.

The purpose of this chapter is to generalize the Compatibility property in the context of weak primary decomposition, by first assuming the existence of a weak primary decomposition of N in M where $N \subsetneq M$ are R-modules that are not necessarily finitely generated and R is not necessarily Noetherian.

Notation 3.1.1. Let N be a weakly decomposable submodule of the R-module M. We denote the set of all weakly P-primary components of N in M by Λ_P.

Definition 3.1.2. Let N be a weakly decomposable submodule of the R-module M. In particular, let $N = Q_1 \cap \cdots \cap Q_n$ and $N = Q'_1 \cap \cdots \cap Q'_n$ be any two weak primary decompositions of N in M with $Q_i, Q'_i \in \Lambda_{P_i}$. Then $\text{Ass}_{\text{f}}(M/N) = \{P_1, \ldots, P_r, P_{r+1}, \ldots, P_n\}$. If $Q_1 \cap \cdots \cap Q_r = Q'_1 \cap \cdots \cap Q'_r$, then we say the weak primary decompositions of N in M are *independent* over $\{P_1, \ldots, P_r\}$.

3.2 The Compatibility Property of Weak Primary Decomposition

Lemma 3.2.1. *Let N be a weakly decomposable submodule of the R-module M and let $a \in R$. Then the weak primary decompositions of N in M are independent over $\{P \in \mathrm{Ass_f}(M/N) \mid a \notin P\}$.*

Proof. Let $N = Q_1 \cap \cdots \cap Q_n$ and $N = Q'_1 \cap \cdots \cap Q'_n$ be two arbitrary weak primary decompositions of N in M with $Q_i, Q'_i \in \Lambda_{P_i}$ for all $1 \leq i \leq n$. Then, by 2.3.6,

$$\bigcup_{t=1}^{\infty}(N :_M a^t) = \bigcap_{a \notin P_i} Q_i \quad \text{and} \quad \bigcup_{t=1}^{\infty}(N :_M a^t) = \bigcap_{a \notin P_i} Q'_i.$$

Thus $\bigcap_{a \notin P_i} Q_i = \bigcap_{a \notin P_i} Q'_i$. Therefore the primary decompositions of N in M are independent over $\{P \in \mathrm{Ass_f}(M/N) \mid a \notin P\}$. □

Theorem 3.2.2 (Compatibility Property of Weak Primary Decomposition)**.** *Let N be a weakly decomposable submodule of the R-module M. If $\mathrm{Ass_f}(M/N) = \{P_1, \ldots, P_n\}$ and $Q_i \in \Lambda_{P_i}$, then $N = Q_1 \cap \cdots \cap Q_n$, which is a minimal weak primary decomposition.*

Proof. We induce on $|\mathrm{Ass_f}(M/N)|$. If $|\mathrm{Ass_f}(M/N)| = 1$, the claim is trivially true.

Now, let $|\mathrm{Ass_f}(M/N)| = n$ so that $\mathrm{Ass_f}(M/N) = \{P_1, \ldots, P_n\}$, and assume the claim is true if $|\mathrm{Ass_f}(M/N)| = n - 1$. By reordering the P_i's, we may assume P_n is a maximal prime ideal in $\mathrm{Ass_f}(M/N)$. We are given that $Q_i \in \Lambda_{P_i}$ for all $1 \leq i \leq n$. Thus for each i, there exists a weak primary decomposition of N in M with Q_i as a component. Let these weak primary decompositions be as follows:

$$\begin{aligned} N &= Q_{(1,1)} \cap Q_{(1,2)} \cap \cdots \cap Q_{(1,n)} \\ N &= Q_{(2,1)} \cap Q_{(2,2)} \cap \cdots \cap Q_{(2,n)} \\ &\vdots \\ N &= Q_{(n,1)} \cap Q_{(n,2)} \cap \cdots \cap Q_{(n,n)}, \end{aligned}$$

where $Q_{(i,i)} = Q_i$ and $Q_{(i,j)} \in \Lambda_{P_j}$ for all $1 \leq i \leq n$ and $1 \leq j \leq n$ such that $i \neq j$. Since P_n is a maximal ideal in $\mathrm{Ass_f}(M/N)$, there exists $a \in P_n \setminus \bigcup_{i=1}^{n-1} P_i$. By 3.2.1, the weak primary decompositions of N in M are independent over $\{P_1, \ldots, P_{n-1}\}$. Thus

$$\begin{aligned} L &:= Q_{(1,1)} \cap \cdots \cap Q_{(1,n-1)} \\ &= Q_{(2,1)} \cap \cdots \cap Q_{(2,n-1)} \\ &\vdots \\ &= Q_{(n-1,1)} \cap \cdots \cap Q_{(n-1,n-1)}. \end{aligned}$$

Now, L is a decomposable submodule of M and $|\mathrm{Ass_f}(M/L)| = n - 1$. Applying the induction hypothesis, we have

$$\begin{aligned} L &= Q_{(1,1)} \cap Q_{(2,2)} \cap \cdots \cap Q_{(n-1,n-1)} \\ &= Q_1 \cap Q_2 \cap \cdots \cap Q_{n-1}. \end{aligned}$$

Clearly, $N = L \cap Q_n$. Therefore $N = Q_1 \cap Q_2 \cap \cdots \cap Q_{n-1} \cap Q_n$, completing our proof. $\square$

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