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Minimum Degree Conditions for Tilings in Graphs and Hypergraphs

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MINIMUM DEGREE CONDITIONS FOR TILINGS IN GRAPHS AND HYPERGRAPHS

by

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Under the Direction of Dr. Yi Zhao

ABSTRACT

We consider tiling problems for graphs and hypergraphs. For two graphs G and F, an F-tiling of F is a subgraph of G consisting of only vertex disjoint copies of F. By using the absorbing method we give a short proof that in a balanced tripartite graph G, if every vertex is adjacent to $(2/3+\gamma)$ of the vertices in each of the other vertex partitions, the G has a K_3 tiling. Previously Magyar and Martin [14] proved the same result (without γ) by using the Regularity Lemma.
In a 3-uniform hypergraph $\mathcal{H}$, let $\delta_2(\mathcal{H})$ denote the minimum number of edges that contain $\{u, v\}$ for all pairs $\{u, v\}$ of vertices. We show that if $\delta_2(\mathcal{H}) \geq \left(1 - \frac{2}{k(k-2)}\right) n$ there exists a K_k^3-tiling of $\mathcal{H}$ that misses at most k^2 vertices of $\mathcal{H}$. On the other hand, we show that there exist hypergraphs $\mathcal{H}$ such that $\delta_2(\mathcal{H}) = \left(1 - \frac{1}{k}\right) n - 2$ and $\mathcal{H}$ does not have a perfect K_k^3-tiling. These extend the results of Pikhurko [17] on K_4^3-tilings.

INDEX WORDS: Graph tiling, Graph packing, Absorbing method, Hypergraph Codegree

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This thesis is dedicated to Luzy, who stands beside me on all of my adventures.

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Chapter 1

INTRODUCTION

For two graphs G and F, an F-tiling (or F-packing) of G is a subgraph of G consisting of vertex disjoint copies of F. When F is a single (hyper)edge we call an F-tiling a matching. If the F-tiling covers all of the vertices of G we say that the tiling is perfect or refer to the tiling as an F-factor. For a perfect tiling to exist the order of F must divide the order of G.

The purpose of this paper is to determine bounds on the minimum degree necessary to ensure a perfect or near perfect F-tiling. An early result by Dirac [6] proves that any graph on n vertices with minimum degree at least $n/2$ is Hamiltonian. This result allows us to obtain a perfect matching in G by deleting every other edge from the Hamiltonian cycle. For $F = K_h$, the complete graph on h vertices, Hajnal and Szemerédi [8] provide the following result: *If G is a graph with hk vertices and minimum degree at least $(h-1)k$, then G contains k vertex disjoint copies of K_h*. Later, using Szemerédi's Regularity Lemma [22], Alon and Yuster [2, 3] were able to provide minimum degree conditions that guarantee an F-factor for arbitrary F. Kühn and Osthus [12] were able to find the best possible minimum degree conditions for finding an F-factor.

Tiling in multipartite graphs has a shorter history. A graph G is called r-partite if the vertex set $V(G)$ can be partitioned in r sets $V_1, \ldots, V_r$ such any that two vertices $u, v \in V_i$ are not adjacent. The Marriage Theorem by König and Hall (see e.g. [4]) implies that a bipartite graph ($r = 2$) G with partition sets of size n contains a 1-factor if $\delta(G) \geq n/2$. In an r-partite graph G with $r \geq 2$, let $\bar{\delta}(G)$ be the minimum degree from a vertex in one partition set to each other partition set (so $\bar{\delta}(G) = \delta(G)$ when $r = 2$). An r-partite graph is balanced if all partition sets have the same order.

Fischer [7] conjectured the following r-partite version of the Hajnal-Szemerédi Theorem and

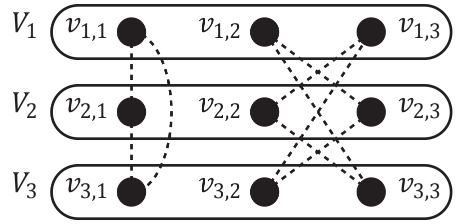

Figure 1.1. Representation of Γ_3 with dotted lines corresponding to non-edges

proved it asymptotically for $r = 3, 4$: *if G is an r-partite graph with n vertices in each partition set and $\bar{\delta}(G) \geq \frac{r-1}{r}n$, then G contains a K_r-factor.* Magyar and Martin [14] used the following theorem to show that Fischer's conjecture is slightly wrong for $r = 3$ (off by only 1): *For G a balanced tripartite graph on $3N$ vertices with $\bar{\delta}(G) \geq (2/3)N + 1$ then G contains a perfect K_3-tiling.* As written, this is a weaker form of the actual theorem, as they prove that G can be perfectly tiled with triangles when $\bar{\delta}(G) \geq (2/3)N$ as long as it is not the graph $\Gamma_3(N/3)$. The case when G is $\Gamma_3(N/3)$ is what disproves Fischer's conjecture and necessitates the extra edge to complete the tiling. Notice in Figure 1 that there can be no K_3-tiling of Γ_3. To form $\Gamma_3(N/3)$, replace each vertex with a cluster of $N/3$ vertices and each edge with the complete bipartite graph $K_{N/3,N/3}$. Since Γ_3 cannot be perfectly tiled by triangles, neither can the blown up version $\Gamma_3(N/3)$ unless you add a single edge. Martin and Szemerédi [15] showed that Fischer's conjecture is true for $r = 4$. Note that in general, a tiling result for multipartite graphs does not follow from a corresponding result for arbitrary graphs. On the other hand, given a graph G of order nr, we can easily obtain (by taking a random partition) an r-partite balanced spanning subgraph G' such that $\bar{\delta}(G') \geq \delta(G)/r - o(n)$. Therefore a tiling result for multipartite graphs immediately gives a slightly weaker tiling result for arbitrary graphs.

The next chapter will focus on a tripartite graph and will provide a lower bound on $\bar{\delta}(G)$, for balanced G, in order to obtain a perfect K_3-tiling, often referring to K_3 as a triangle. Here we use the absorbing lemma, though previously Magyar and Martin [14], by using Szemerédi's Regularity Lemma, were able to avoid γ. The advantage in using the absorbing method is that we will achieve a much smaller order graph than is necessary with the Regularity Lemma.

Theorem 1.1. *For any $\gamma > 0$, there exists n_0 such that for all $n > n_0$ the following holds: Let G be a balanced tripartite graph on $n = 3N$ vertices with $\bar{\delta}(G) \geq (2/3 + \gamma)N$, then G contains a K_3-factor.*

The last chapter focuses on tiling problems in hypergraphs. We say that a hypergraph $\mathcal{H}$ is k-uniform, also called a k-graph, if every edge in $E(\mathcal{H})$ contains exactly k vertices. We denote the complete k-graph on n vertices by K_n^k. For a set T of size $l < k$ in $\mathcal{H}$, we define $\deg(T)$ to be the number of edges in $\mathcal{H}$ that contain T and $\delta_l(\mathcal{H})$ be the minimum l-degree of $\mathcal{H}$. For $l = k-1$, we say that $\delta_{k-1}(\mathcal{H})$ is the minimum vertex codegree of $\mathcal{H}$. All hypergraphs in this chapter will be 3-graphs.

Definition 1.2. Let $t_l^k(n, F)$, for all integers $k > l \geq 1$ and $n \in k\mathbb{Z}$, denote the minimum t such that every k-uniform hypergraph $\mathcal{H}$ on n vertices satisfying $\delta_l(\mathcal{H}) \geq t$ contains a perfect F-tiling.

In their survey on the subject, Rödl and Ruciński [18] point out this result from Kühn and Osthus [10]:

$$t_2^3(n, C_4^{(3,1)}) \sim n/4,$$

where the graph $C_4^{(3,1)}$ is the $(3,1)$-cycle graph on 4 vertices.

When $k = 2$ this is exactly the graph case and has been discussed above. For $k \geq 3, l = k-1$ Kühn and Osthus [11], as well as Rödl et. al. [19–21], investigated the number $t_{k-1}^k(n, F)$. Notably, Rödl, Ruciński and Szemerédi [20] determined $t_{k-1}^k(n, F)$ for arbitrary $k \geq 3$ and sufficiently large n, showing $t_{k-1}^k(n, F) = n/2 - k + c_{k,n}$ where $c_{k,n} \in \{3/2, 3, 5/2, 3\}$ based on the parities of k and n. Continuing this work, Pikhurko [17] provided the bounds

$$\frac{3}{4}n - 2 \leq t_l^k(n, K_4^3) \leq \frac{2 + \sqrt{10}}{6}n + O(\sqrt{n \log N}),$$

where the upper bound was also proved, independently by Keevash and Zhao (unpublished).

For the upper bound on t for K_k^3-tilings we extend an argument from Fischer [7] by introducing a weight function to handle the added complexity of the hypergraph.

Theorem 1.3. *Let $\mathcal{H}$ be a 3-graph of order n with $\delta_2(\mathcal{H}) \geq \left(1 - \frac{2}{k(k-2)}\right) n$ and $k|n$. Then there exists a tiling of vertex disjoint copies of K_k^3 in $\mathcal{H}$ covering all but at most k^2 vertices.*

Lo and Markström [13] have a proof that extends this proof to all K_k^t-tilings, obtaining the same bound.

To show the lower bound on t we we extend a construction from Pikhurko [17] to show that $\mathcal{H}$ may not contain a K_k^3-factor.

Proposition 1.4. *Let $\mathcal{H}$ be 3-graph on $n = 2kq + r$ for integers $k, q \geq 0$ and $r \in \{0, k\}$, we have*

$$\delta_2(\mathcal{H}) \geq 2(k-1)q + r - 2 \geq \left(1 - \frac{1}{k}\right) n - 2.$$

Lo and Markström [13] also extended this construction to all K_k^t and achieved an improved bound.

Chapter 2

PROOF OF THEOREM 1.1

Let $\gamma > 0$ and $n_0(\gamma)$ be the minimum positive integer satisfying the following two conditions:

(i) $2\gamma^2 n_0^2 + \frac{5}{3}\gamma n_0^2 + 1 \geq 3\gamma n_0 + n_0$

(ii) $6\gamma^2 n_0^2 + 2 \geq 7\gamma n_0 + \frac{2}{3} n_0$

Also let $G = (V_1, V_2, V_3, E)$ be a balanced tripartite graph of order $n = 3N$ with $\bar{\delta} \geq (2/3 + \gamma)N$. We prove Theorem 1.1 in three steps. First we show that for an arbitrary $T = \{v_1, v_2, v_3\}, v_i \in V_i$, there are many absorbing 6-sets. Next we show that G will have a near perfect tiling that misses only six vertices. Last, we will show that the final six vertices can be absorbed into the tiling.

2.1 Absorbing Sets

We use Proposition 2.1 to establish an absorbing structure in G and prove that the edge density provides enough absorbing 6-sets for an arbitrary T to be added to a partial tiling. The proof follows from Lemma 10 (Absorbing Lemma) by Hán et. al. [9].

Proposition 2.1. *For G, as in the theorem, there exists a tiling M in G of size $|M| \leq \frac{1}{2}\gamma^2 N$ such that for every set $W \subset V \setminus V(M)$ of size at most $\frac{1}{2}\gamma^6 N$ there exists a tiling covering exactly the vertices in $V(M) \cup W$.*

Proof. In G we say that a set $A = A_1 \cup A_2 \cup A_3, A_i \in \binom{V_i}{2}$, is an absorbing 6-set for T if A spans a tiling of size 2 and $A \cup T$ spans a tiling of size 3. Lemma 2.2 determines how many such A exist for arbitrary T.

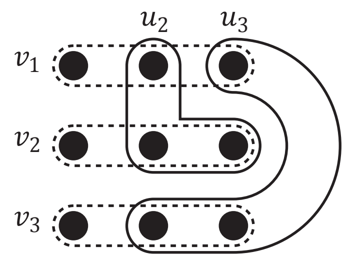

Figure 2.1. An Absorbing Structure

Lemma 2.2. *For every T in G, there are at least $\frac{2}{9}\gamma^2 N^6$ absorbing 6-sets for T.*

Proof. Fix a set T. We wish to build the structure in Figure 2.1, so we begin by finding a triangle containing v_1 but not v_2 or v_3. By the degree condition, v_1 has at least $(2/3+\gamma)N-1$ vertices in V_2 that are not v_2. Let $u_2 \neq v_2$ be a neighbor of v_1 and consider $N_{V_3}(v_1) \cap N_{V_3}(u_2)$. The shared neighborhood of v_1 and u_2 that avoids v_3 must be at least

$$(2/3+\gamma)N + (2/3+\gamma)N - N - 1 = (1/3+2\gamma)N - 1$$

vertices $u_3 \neq v_3$. Thus, we have in total

$$((2/3+\gamma)N-1)((1/3+2\gamma)N-1) \geq \frac{2}{9}N^2 \tag{2.1}$$

triangles that contain v_1 and not v_2 or v_3, as $N \to \infty$.

Fix one such triangle $\{v_1, u_2, u_3\}$ and let $U_1 = \{u_2, u_3\}$. Now suppose we are able to choose a set U_2 such that it is disjoint to $U_1 \cup T$ and both $U_2 \cup \{u_2\}$ and $U_2 \cup \{v_2\}$ are triangles in G. Suppose further that we are able to choose a set U_3 such that it is disjoint to $U_1 \cup U_2 \cup T$ and both $U_3 \cup \{u_3\}$ and $U_3 \cup \{v_3\}$ are triangles in G. Then we call such a choice for U_2 and U_3 good, motivated by $U_1 \cup U_2 \cup U_3$ being an absorbing 6-set for T, which describes the structure shown in Figure 2.1.

Focus on the number of good sets for U_2. The shared neighborhood of u_2 and v_2 in V_1 is at least $(1/3+2\gamma)N-1$ vertices avoiding v_1. Fix a vertex $x_1 \neq v_1$ and count how many of its neighbors in V_3 are also adjacent to both v_2 and u_2, while avoiding v_3. The vertices x_1, v_2 and u_2

will have at least $(1/3+2\gamma)N+(2/3+\gamma)N-N-2=3\gamma N-2$ common neighbors in V_3 that avoid v_3 and u_3. We have in all at least

$$((1/3+2\gamma)N-1)(3\gamma N-2)\geq \gamma N^2 \tag{2.2}$$

good choices for U_2. The same analysis hold for the number of choices for U_3.

Using equations (2.1) and (2.2), we see that the total number of absorbing 6-sets for T is

$$\frac{2}{9}N^2\times(\gamma N^2)^2=\frac{2}{9}\gamma^2N^6.$$

□

To continue the proof of Proposition 2.1, we let $\mathcal{L}(T)$ denote the family of all the 6-sets that can absorb the T fixed in Lemma 2.2. We know that $|\mathcal{L}(T)|\geq \frac{2}{9}\gamma^2N^6$, again from Lemma 2.2. Choose a family $\mathcal{F}$ of 6-sets by selecting each of the $\binom{N}{2}^3$ possible 6-sets independently with probability

$$p=\frac{\gamma^3}{N^5}.$$

Then we can use the following result by Chernoff (see [1]) to determine how big $\mathcal{F}$ is likely to be.

Proposition 2.3. *If $X_i, 1\leq i\leq n$, be mutually independent random variables with*

$$Pr[X_i=+1]=Pr[X_i=-1]=\frac{1}{2}$$

and set

$$S_n=X_1+\cdots+X_n.$$

Let $a>0$. Then

$$Pr[S_n>a]<e^{-a^2/2n}.$$

Therefore, with probability $1-o(1)$, as $N\to\infty$ the family $\mathcal{F}$ fulfills the following properties:

$$|\mathcal{F}|\leq 2\mathrm{E}(|\mathcal{F}|)\leq 2\frac{\gamma^3}{N^5}\binom{N}{2}^3\leq\frac{1}{4}\gamma^3N \tag{2.3}$$

and

$$|\mathcal{L}(T) \cap \mathcal{F}| \geq \frac{1}{2}\mathrm{E}(|\mathcal{L}(T) \cap \mathcal{F}|) \geq \frac{1}{2}\left(\frac{\gamma^3}{N^5}\right) \times \frac{2}{9}\gamma^2 N^6 \geq \frac{1}{9}\gamma^5 N \tag{2.4}$$

Moreover we can bound the expected number of intersecting 6-sets by choosing a 6-set, a vertex in the 6-set, a second vertex in same partition and a pair of vertices from each of the other two partitions:

$$\binom{N}{2}^3 \times 6(N-1)\binom{N}{2}^2.$$

Then, the probability of choosing both sets is

$$p^2\binom{N}{2}^3 \times 6(N-1)\binom{N}{2}^2 \leq \frac{1}{4}\gamma^6 N \tag{2.5}$$

Now, in order to upper bound the number of intersecting sets we use Markov's bound (also in [1]).

Proposition 2.4. *Suppose that Y is an arbitrary nonnegative random variable, $\alpha > 0$. Then*

$$Pr[Y > \alpha E[Y]] < 1/\alpha.$$

Therefore, with probability at least 1/2

$$\mathcal{F} \text{ contains at most } \frac{1}{2}\gamma^6 N \text{ intersecting pairs.}$$

Therefore, with positive probability the family $\mathcal{F}$ has the properties stated in (2.3), (2.4) and (2.5). Since some of the 6-sets will not absorb any T and some will intersect each other, we delete all of these undesired 6-sets in the family $\mathcal{F}$ to get a subfamily $\mathcal{F}'$ consisting of pairwise disjoint absorbing 6-sets which satisfies

$$|\mathcal{L}(T) \cap \mathcal{F}'| \geq \frac{1}{9}\gamma^5 N - \frac{1}{2}\gamma^6 N \geq \frac{1}{2}\gamma^6 N.$$

Finally, the thinned out family $\mathcal{F}'$ consists of pairwise disjoint absorbing 6-sets and $G[V(\mathcal{F}')]$ contains a perfect tiling M of size at most $\frac{1}{2}\gamma^3 N$. Also, for any subset $W \subset V \setminus V(M)$ of size $\frac{1}{2}\gamma^6 N$

we can partition W into sets of size 3 and successively absorb them using a different absorbing 6-set each time. This gives us a tiling that covers exactly the vertices in $V(\mathcal{F}') \cup W$. □

2.2 Complete Tiling

To complete the proof of the theorem, we find in G an absorbing family M guaranteed by Proposition 2.1. We let $G' = G - V(M)$ and observe that

$$\bar{\delta}(G') \geq (2/3+\gamma)N - \frac{3}{2}\gamma^3 N \geq \frac{2}{3}N \geq \frac{2}{3}N'$$

where N' is the number of vertices in each partition set of G'. Notice further that G' is still balanced and we can apply Proposition 3.2 in Fischer [7] to find an incomplete tiling in G'.

Proposition 2.5. *If G is a tripartite graph with vertex partitions V_1, V_2 and V_3 of size N, such that each vertex in any partition has at least $\frac{2}{3}N$ neighbors in each of the other partitions, then G contains $N-2$ disjoint triangles.*

This proposition gives us an almost perfect tiling of G', leaving only a set W containing 6 vertices uncovered. By Proposition 2.1 we can divide W into sets of 3 and use M to absorb each triple and complete the perfect tiling on G.

Chapter 3

PROOFS ON 3-GRAPHS

In this chapter we provide a minimum degree condition that guarantees an almost perfect tiling of a 3-graph $\mathcal{H}$ that misses at most k^2 vertices. Next we will provide a construction that shows that if the minimum degree condition is too small, we cannot guarantee a perfect tiling of $\mathcal{H}$.

3.1 Proof of Theorem 1.3

This proof is adapted from the proof of Lemma 6.1 by Pikhurko [17] which adapts the proof of Theorem 2.1 by Fischer [7].

Proof. Let $\mathcal{H}$ be a 3-graph on n vertices with $\delta_2(\mathcal{H}) \geq \left(1 - \frac{2}{k(k-2)}\right) n$ and $k|n$. Begin with a partition $\mathcal{P}$ of the vertex set $V(\mathcal{H})$ into sets of size k, $V_1, \ldots, V_{n-k}$. Let G_i be the largest complete graph in V_i. If V_i is an independent set, we define $|G_i| = 2$. Denote by $w : \{2, \ldots, k\} \to \mathbb{R}$ the function defined by $w(2) = 0$ and $w(j+1) - w(j) = 1 - \frac{1}{k^j}$ for $2 \leq j \leq k-1$. We say that $w(\mathcal{P})$, the weighting of $\mathcal{P}$, is $\sum_{1 \leq j \leq n/k} w(|G_j|)$. Assume that $\mathcal{P}$ is chosen such that $w(\mathcal{P})$ is maximal. We will now show that for each weight class $2 \leq i \leq k-1$ there are at most $k-1$ sets V_j in $\mathcal{P}$ with $|G_j| = i$. Suppose, for a contradiction, that $|G_1| = \cdots = |G_k| = i < k$. Since $|G_j| < k$ for $1 \leq j \leq k$ we can find at least one $v_j \in V_j \setminus G_j$. Now, for $1 \leq j \leq k$ and vertex $v \notin V_j$, we say the pair (v, j) is a connection if and only if $\{v\} \cup G_j$ spans a complete hypergraph. If there are any connections (v, j) with $v \in V_1 \cup \cdots \cup V_k$ then switching v with any vertex v_j will result in a new partition $\mathcal{P}'$. Note that since

$$1 - \frac{1}{k^i} \geq 1 - \frac{1}{k^{i-1}}$$

we have

$$w(i+1) - w(i) \geq w(i) - w(i-1)$$

which is

$$w(i+1) + w(i-1) \geq 2w(i)$$

and we immediately provide a contradiction to $w(\mathcal{P})$ being maximal. Thus, we can assume there are no connections with $v \in V_1 \cup \cdots \cup V_k$ and $1 \leq j \leq k$.

Using the condition on $\delta_2(\mathcal{H})$, for $1 \leq j \leq k$ we can determine a lower bound on the number of connections there are by double counting the number of adjacencies among the G_j's. An arbitrary pair of vertices in G_j is adjacent to at least $\delta_2(\mathcal{H})$ vertices. If we let c be the number of connections to G_j then

$$\binom{i}{2}\delta_2(\mathcal{H}) \leq \binom{i}{2}c + \left(\binom{i}{2} - 1\right)(n-c)$$

and

$$c \geq \binom{i}{2}\delta_2(\mathcal{H}) - \left(\binom{i}{2} - 1\right)n \geq \frac{(k-i)n}{k}$$

where the last inequality is true since $i < k$.

Now there are at at least $(k-i)n$ connections (v, j) with $v \notin V_1 \cup \cdots \cup V_k$ and $1 \leq j \leq k$. Since $n > k$ we can choose V_j' such that there are more than $k(k-i)$ connections (v', j) for $v' \in V_j'$ and $1 \leq j \leq k$. Consider the bipartite graph B with parts $\{G_1, \ldots, G_k\}$ and V_j' whose edge set consists of those pairs that make a connection. Since B has at least $k(k-i)$ edges, the König-Egerváry Theorem (see [4] Theorem 8.32) shows that B contains a matching of size at least $k-i+1$. Now by moving v_j' to V_j for $1 \leq j \leq k-i+1$ and $\{v_1, \ldots, v_{k-i+1}\}$ to V_j', see Figure 3.1, $w(\mathcal{P})$ increases by

$$\begin{aligned}(k-i+1)(w(i+1) - w(i)) - (w(|G_j'|) - w(\max\{2, |G_j'| - k + 1 + i\})) \\ \geq (k-i+1)\left(1 - \frac{1}{k^i}\right) - \left(k + 1 - i - \frac{k-i+1}{k}\right) \\ = \frac{(k^i - 1)(k-i+1)}{k^{i+1}} > 0\end{aligned}$$

a contradiction.

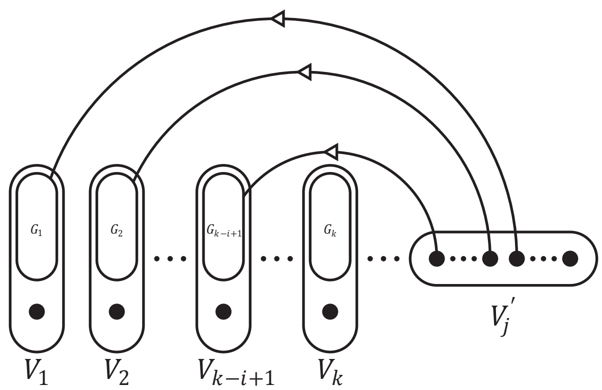

Figure 3.1. Vertices making a connection from V_j'

□

3.2 Proof of Proposition 1.4

We now provide a construction that proves that the codegree of $\mathcal{H}$ must be larger than $(1-1/k)n-2$ if we are to be guaranteed a perfect tiling.

Proof. For $n = 2kq + r$, if $r = k$ let $a_0 = 2q + 1$. Otherwise we let a_0 be either $2q + 1$ or $2q - 1$, with both choices giving the same bound. Partition $V(\mathcal{H}) = A_0 \cup A_1 \cup \cdots \cup A_{k-1}$ into parts of sizes $a_0 + a_1 + \cdots + a_{k-1} = n$, where $a_1, \ldots, a_{k-1}$ are nearly equal, that is $|a_i - a_j| \leq 1$ for $1 \leq i < j \leq k-1$. Let $\mathcal{H}$ be the 3-graph on n vertices whose edge set consists of all triple excluding any that satisfy one of the following (mutually exclusive) properties:

(i) have exactly three vertices in A_0

(ii) have one vertex in A_0 and two vertices in A_i for some $1 \leq i \leq k-1$

(iii) intersect each of A_1, A_2 and A_3.

Figure 3.2 shows examples of edges that are excluded from $\mathcal{H}$. To see why there can be no K_k^3-tiling, consider any K_k^3-subgraph K of $\mathcal{H}$. By Property (i), K cannot intersect A_0 in more than two vertices. Suppose that K intersects A_0 in exactly one vertex and avoids at least one partition. Then by the pigeon hole principle there is a partition A_i for $1 \leq i \leq k-1$ that contains at least

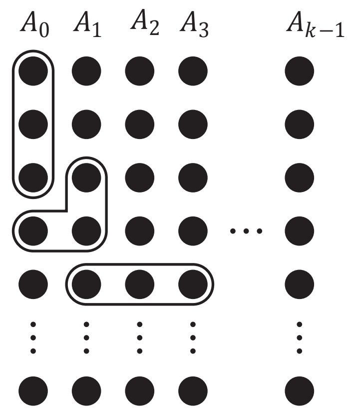

Figure 3.2. Examples of Edges Not Allowed

two vertices of K. Property (ii) forbids the edge spanning the vertex in A_0 along with any pair in A_i. So if K is to intersect A_0 in exactly one vertex, K must also intersect every other partition in exactly one vertex. By property (iii), the edge with a vertex in A_1, A_2 and A_3 is forbidden, so K cannot intersect A_0 in one vertex in this manner either.

Therefore every K_k^3-subgraph of $\mathcal{H}$ has an even number of vertices in A_0. This makes a perfect tiling impossible, since $|A_0| = 2q \pm 1$, which is odd.

A case by case analysis gives the desired bound.

Case 1 Two vertices in A_0 are in an edge with every vertex in A_i for $1 \leq i \leq k-1$, so the codegree is $\frac{k-1}{k}n$;

Case 2 One vertex in A_0 and one vertex in A_i for $1 \leq i \leq k-1$ are in an edge with every other vertex in A_0 and every vertex in A_j for $j \neq i$ and $1 \leq j \leq k-1$, so the codegree is $\frac{k-1}{k}n - 1$;

Case 3 Two vertices in A_i for $1 \leq i \leq k-1$ are in an edge with every other vertex in A_i and every vertex in A_j for $j \neq i$ and $1 \leq j \leq k-1$, so the codegree is $\frac{k-1}{k}n - 2$;

Case 4 One vertex in A_i and one vertex in A_j for $i, j \in [3]$ and $i \neq j$ are in an edge with every vertex in A_0, every other vertex in A_i and A_j and every vertex in A_ℓ for $4 \leq \ell \leq k-1$, so the codegree is $\frac{k-1}{k}n - 2$;

Case 5 One vertex in A_i for $i \in [3]$ and one vertex in A_j for $4 \leq j \leq k-1$ are in an edge with every other vertex of $\mathcal{H}$, so the codegree is $n-2$;

Case 6 Two vertices in A_i for $4 \leq i \leq k-1$ are in an edge withe every other vertex of $\mathcal{H}$, so the codegree is $n-2$.

We take the minimum of these codegrees, which is $\frac{k-1}{k}n-2$.

□

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