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Sign Pattern Matrices That Require Almost Unique Rank

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by

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Under the Direction of Drs. Frank J. Hall and Zhongshan Li

Abstract

A *sign pattern matrix* is a matrix whose entries are from the set $\{+, -, 0\}$. For a real matrix B, $\operatorname{sgn}(B)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-$, 0). For a sign pattern matrix A, the *sign pattern class of* A, denoted $Q(A)$, is defined as $\{ B : \operatorname{sgn}(B) = A \}$. The *minimum rank* $\operatorname{mr}(A)$ (*maximum rank* $\operatorname{MR}(A)$) of a sign pattern matrix A is the minimum (maximum) of the ranks of the real matrices in $Q(A)$. Several results concerning sign patterns A that *require almost unique rank*, that is to say, the sign patterns A such that $\operatorname{MR}(A) = \operatorname{mr}(A) + 1$ are established. In particular, a complete characterization of these sign patterns is obtained. Further, the results on sign patterns that require almost unique rank are extended to sign patterns A for which the *spread* is $d = \operatorname{MR}(A) - \operatorname{mr}(A) \geq 2$.

Keywords: Sign pattern matrix; Minimum rank; Maximum rank; L-matrix; Requires unique rank; Requires almost unique rank; Spread

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1. Introduction

In qualitative and combinatorial matrix theory, we study properties of a matrix based on combinatorial information, such as the signs of entries in the matrix. Such approach originated from the work in the 1940's of the Nobel Economics Prize winner P.A. Samuelson, as described in his seminal book Foundations of Economic Analysis [12] in 1947. Due to its theoretical importance and applications in economics, biology, chemistry, sociology and computer science, qualitative and combinatorial matrix analysis flourished in the past few decades. R. Brualdi and B. Shader summarized and organized some of the research in this area in their 1995 book Matrices of Sign-solvable Linear Systems [3].

A matrix whose entries come from the set $\{+, -, 0\}$ is called a *sign pattern matrix*. We denote the set of all $n \times n$ sign pattern matrices by Q_n, and more generally, the set of all $m \times n$ sign pattern matrices by $Q_{m,n}$. For a real matrix B, $sgn(B)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-$, 0). If $A \in Q_{m,n}$, then the *sign pattern class of* A is defined by

$$Q(A) = \{B \ : \ sgn(B) = A\}.$$

For $A \in Q_{m,n}$, the *minimum rank* of A, denoted as $mr(A)$, is defined by

$$mr(A) = \min \{\text{rank } B : B \in Q(A)\}.$$

The *maximum rank* of A, $MR(A)$, is given by

$$MR(A) = \max \{\text{rank } B : B \in Q(A)\}.$$

The minimum rank of a sign pattern is not only of interest theoretically, it is also of practical value. For instance [5] is devoted to the question of constructing real $m \times n$ matrices of low rank under the constraint that each entry is nonzero and has a given sign. This problem arises from an interesting topic in neural networks or, more specifically, multilayer perceptrons. In this application, the rank of a realization matrix can be interpreted as the number of elements in a hidden layer, which motivates a search for low rank solutions.

The characterization of the $mr(A)$ (or finding $mr(A)$) for a general $m \times n$ sign pattern matrix A is difficult and is a long outstanding problem. However, the $MR(A)$ is easily described (see Chapter 2).

A sign pattern matrix S is called a *permutation pattern* if exactly one entry in each row and column is equal to $+$, and all the other entries are 0. A product of the form S^TAS, where S is a permutation pattern, is called a *permutational similarity.* We say that A and S^TAS are *permutationally similar*. Two sign pattern matrices A_1 and A_2 are said to be *permutationally equivalent* if there are permutation patterns S_1 and S_2 such that $A_1 = S_1A_2S_2$.

A diagonal sign pattern D is called a *signature sign pattern* if each of its diagonal entries is either $+$ or $-$. For a signature sign pattern D and a sign pattern A of the same order, we say that DAD and A are signature similar. Two sign patterns A_1 and A_2 are said to be *signature equivalent* if $A_1 = D_1A_2D_2$ for some signature sign patterns matrices D_1 and D_2.

If $A = [a_{ij}]$ is an $n \times n$ sign pattern matrix, then a formal product of the form $\gamma = a_{i_1i_2}a_{i_2i_3}\ldots a_{i_ki_1}$, where each of the elements is nonzero and the index set $\{i_1, i_2, \ldots, i_k\}$ consists of distinct indices, is called a *simple cycle* of length k, or a k-cycle, in A. A *composite cycle* γ in A is a product of simple cycles, say $\gamma = \gamma_1\gamma_2\ldots\gamma_m$, where the index sets of the γ_i's are mutually disjoint. If the length of γ_i is l_i, then the length of γ is $\sum_{i=1}^m l_i$. If we say a cycle γ is an *odd* (respectively *even) cycle*, we mean that the length of the simple or composite cycle γ is odd (even). In this thesis, the term cycle always refers to a composite cycle (which as a special case could be a simple cycle).

Let $A = (a_{ij})$ be an $n \times n$ sign pattern matrix. The digraph of A, denoted $D(A)$, is the directed graph with vertex set $\{1, 2, \ldots, n\}$ such that (i, j) is an arc of $D(A)$ iff $a_{ij} \neq 0$. The (undirected) graph of A, denoted $G(A)$, is the graph with vertex set $\{1, 2, \ldots, n\}$ such that $\{i, j\}$ is an edge of $G(A)$ iff at least one of the entries a_{ij} and a_{ji} is nonzero.

An undirected graph G is a tree if it is connected and has no cycles (thus G is minimally connected). For a symmetric $n \times n$ sign pattern A, by $G(A)$ we mean the undirected graph of A, with vertex set $\{1, \ldots, n\}$ and $\{i, j\}$ is an edge iff $a_{ij} \neq 0$. A sign pattern A is a *symmetric tree sign pattern* if A is symmetric and $G(A)$ is a tree, possibly with loops.

Suppose P is a property referring to a real matrix. A sign pattern A is said to *require* P if every matrix in $Q(A)$ has property P; A is said to *allow* P if some real matrix in $Q(A)$ has property P. A sign pattern $A \in Q_n$ is said to be *sign nonsingular* (SNS for short) if every matrix $B \in Q(A)$ is nonsingular. It is well known that, A is sign nonsingular if and only if det A $= +$ or det$A = -$, that is, in the standard expansion of det A into $n!$ terms, there is at least one nonzero term, and all the nonzero terms have the same sign. Note that a nonzero term in such expansion of det A corresponds to a cycle of length n in A. It is also known that if all the diagonal entries of A are negative, then A is sign nonsingular iff every simple cycle in A has negative weight (namely, the product of the entries in the simple cycles is negative).

An $m \times n$, where $m \leq n$, sign pattern matrix A is said to be an *L-matrix* if every real matrix $B \in Q(A)$ has linearly independent rows (see [3]). It is known (see Theorem 3.5 (i)) that A is an L-matrix iff for every nonzero diagonal pattern D, DA has a *unisigned column* (that is, a nonzero column that is nonnegative or nonpositive).

In this thesis, we establish several results concerning sign patterns A that *require almost unique rank*, that is to say, the sign patterns A such that $\mathrm{MR}(A) = \mathrm{mr}(A)+1$. In particular, we obtain a complete characterization of these sign patterns. Further, we extend the results on sign patterns that require almost unique rank to sign patterns A for which the *spread* is $d = \mathrm{MR}(A) - \mathrm{mr}(A) \geq 2$.

2. Some Basic Results

Let H and K be $m \times n$ matrices, with rank $K = 1$. Then

$$\text{rank}\,(H+K) \;\le\; \text{rank}\; H \;+\; \text{rank}\; K \;=\; \text{rank}\; H + 1,$$

so that

$$\text{rank}\,(H+K) \;\le\; \text{rank}\; H + 1.$$

Next, since $H = (H+K) - K$, we have that

$$\text{rank}\,H \;= \text{rank}\,[(H+K)-K] \;\le\; \text{rank}\,(H+K) + \text{rank}\,(-K) \;=\; \text{rank}\;(H+K) + 1.$$

So, $\text{rank}\,H\; - 1 \;\le\; \text{rank}(H+K)$. Thus,

$$\text{rank}\,H\; - 1 \;\le\; \text{rank}\,(H+K) \;\le\; \text{rank}\; H + 1.$$

that is to say, a rank 1 perturbation of a matrix H does not change the rank of H by more than 1.

Now, let A be an $m \times n$ sign pattern, with $B, C \in Q(A)$, where rank $B = mr(A)$, rank $C = MR(A)$. By successively replacing only one column of B by the corresponding column of C, we obtain a sequence of matrices

$$B_0 = B,\; B_1,\; B_2,\; \ldots,\; B_n = C$$

in $Q(A)$, where

$$\text{rank}\,(B_{j-1}) - 1 \;\le\; \text{rank}\,B_j \;\le\; \text{rank}\,B_{j-1} \,+\, 1.$$

Hence, the set of ranks

$$\{\text{rank}\,B_1,\, \text{rank}\,B_2,\, \ldots,\, \text{rank}\,B_t\}$$

covers all the ranks between rank B_1 and rank B_t. Thus, we have the following result, which was proved first by C.R. Johnson.

Proposition 2.1. Let A be an $m \times n$ sign pattern matrix. Then all of the intermediate ranks between mr(A) and MR(A) can be achieved by suitable matrices in $Q(A)$.

Unlike the minimum rank of a sign pattern matrix, the maximum rank is conceptually clear. We next give some characterizations of the maximum rank.

Proposition 2.2. Let A be an $m \times n$ sign pattern matrix. Then MR(A) is the maximum number of nonzero entries of A with no two of the nonzero entries in the same row or in the same column.

Proof. Let s be the maximum number of nonzero entries of A with no two of the nonzero entries in the same row or in the same column, $t = MR(A)$ and $q = \min\{m, n\}$. By assigning values of q or $-q$ to entries on some generalized diagonal of length s, while assigning values of 1 or -1 to the other nonzero entries of A, we obtain a matrix $B \in Q(A)$ with an $s \times s$ submatrix that has a strictly dominant generalized diagonal. (A generalized diagonal yields a composite cycle length s.) Since this matrix must be nonsingular, we have

$$s \;\le\; \operatorname{rank} B \;\le\; MR(A) \;= t,$$

so that $s \;\le\; t$.

Next, let $C \in Q(A)$ with $t = MR(A) = \operatorname{rank} C$. Then C has a nonsingular $t \times t$ submatrix and hence a generalized diagonal of length t. This means that C (and hence A) has t nonzero entries with no two of the nonzero entries in the same row or in the same column. Hence, $t \;\le\; s$. Combined with $s \le t$, we get $s = t$. ■

The maximum number of nonzero entries of A with no two of the nonzero entries in the same row or column is also known as the *term rank* of A. This leads to the famous fundamental minimax theorem of Konig (1936). This theorem has a long history and many ramifications. The theorem deals exclusively with properties of a $(0,1)$-matrix that remain invariant under arbitrary permutations of the lines of the matrix. The following statement of Konig's theorem and its proof are adapted from [2]. Note that the maximum number of nonzero entries of A with no two of the nonzero entries in the same row or column is the same as the maximal number of 1's in A with no two of the 1's on a line.

Theorem 2.3. Let A be a $(0,1)$-matrix of size $m \times n$. The minimal number of lines in A that cover all of the 1's in A is equal to the maximal number of 1's in A with no two of the 1's on a line.

Proof. We use induction on the number of lines in A. The theorem is valid for $m = 1$ or $n = 1$. Hence we take $m > 1$ and $n > 1$. We let p' equal the minimal number of lines in A that cover all of the 1's in A, and let p equal the the maximal number of 1's in A with no two of the 1's on a line. We may conclude at once from the definitions of p and p' that $p \leq p'$. Thus it suffices to prove that $p \geq p'$. A minimal covering of the 1's of A is called proper provided that it does not consist of all m rows of A or of all n columns of A. The proof the theorem splits in to two cases.

In the first case we assume that A does not have a proper covering. It follows that we must have $p' = \min\{m, n\}$. We permute the lines of A so that the matrix has a 1 in the $(1,1)$ position. We delete row 1 and column 1 of the permuted matrix and denote the resulting matrix of size $(m-1) \times (n-1)$ by A'. The matrix A' cannot have a covering composed of fewer than $p' - 1 = \min\{m-1, n-1\}$ lines because such a covering composed of A' plus the deleted lines would yield a proper covering for A. We now apply the induction hypothesis to A' and this allows us to conclude that A' has p' -1 1's with no two of the 1's on aline. But then A has p' 1's with no two of the 1's on a line and it follows that $p \geq p'$.

In the second case we assume that A has a proper covering composed of e rows and f columns where $p' = e + f$. We permute lines of A so that these e rows and

f columns occupy the initial positions. Then our permuted matrix assumes the following form

$$\begin{bmatrix} * & A_1 \\ A_2 & 0 \end{bmatrix}.$$

In this decomposition 0 is a zero matrix of size $m - e \times n - f$. The matrix A_1 has e rows and cannot be covered by fewer than e lines and the matrix A_2 has columns and cannot covered fewer than f lines. This is the case because otherwise we contradict the fact that $p' = e + f$ is the minimal number of lines in A that cover all of the 1's on A. We may apply the induction hypothesis to both A_1 and A_2 and this allows us to conclude that $p \geq p'$. ■

Theorem 2.4. Let A be a sign pattern with $r = \mathrm{MR}(A)$. Then there exist permutation patterns P_1 and P_2 such that

$$P_1 A P_2 = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r - k)$, for some k with $0 \leq k \leq r$. Furthermore, $\mathrm{MR}(Y) = k$, $\mathrm{MR}(Z) = r - k$, $\mathrm{MR}([X\ Y]) = k$, and $\mathrm{MR}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right) = r - k$.

Proof. Since $r = \mathrm{MR}(A)$, by Theorem 2.3 we know that there are r lines that cover all nonzero entries of A. So, we can say that there are k rows and $r - k$ columns that cover all of the nonzero entries of A, for some k with $0 \leq k \leq r$. We can then permute these k rows up and the $r - k$ columns to the left. Thus, there exist permutation patterns P_1 and P_2 such that

$$P_1 A P_2 = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r - k)$, for some k with $0 \leq k \leq r$. Since Y has k rows, $\mathrm{MR}(Y) \leq k$. Assume that $\mathrm{MR}(Y) < k$ Then $\mathrm{rank}(B_2) < k$ for any matrix $B_2 \in Q(Y)$. Hence, any matrix

$$\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix} \in Q(P_1 A P_2)$$

has rank $< (r-k)+k = r$. But, $r = \mathrm{MR}(A) = \mathrm{MR}(P_1AP_2)$. We then have a contradiction. Thus, $\mathrm{MR}(Y) = k$.

The proofs of the other three parts are similar. $\blacksquare$

3. L-matrices

Let A be an $m \times n$ sign pattern matrix. Recall that A is an *L- matrix* if and only if every matrix in the qualitative class $Q(A)$ has linearly independent rows. If A is an L-matrix, then every matrix obtained from A by appending column vectors is also an L-matrix. If A is an L-matrix and each of the m by $n-1$ matrices obtained from A by deleting a column is not an L-matrix, then A is called a *barely L-matrix.* Thus a barely L-matrix is an L-matrix in which every column is essential. If A is an L-matrix, then we can obtain a barely L-matrix by deleting certain columns of A. An *SNS-matrix*, that is, a square L-matrix, is a barely L-matrix. But there are barely L-matrices which are not square.

As we shall see throughout this thesis research, L-matrices form a rich and difficult class of matrices. The subclass of L-matrices for which one can assert the linear independence of rows solely on the basis of the zero pattern has a simple characterization. Clearly, an $m \times n$ matrix which has an invertible (namely SNS) triangular submatrix of order m is an L-matrix. These matrices and their permutations are the only matrices A for which one can conclude that A is an L-matrix knowing only the zero pattern of A.

Example 3.1 Let

$$A = \begin{bmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \end{bmatrix}.$$

Then A is an L-matrix. Let B be a matrix in $Q(A)$. The sign pattern A implies that no row of B is a multiple of another row. Every 3×1 $(+, -)$ sign pattern is the sign pattern of some column of A or its negative. It follows that no nontrivial linear combination of the rows of B equals zero, hence the rows of B are linearly independent.

More generally, it is shown in [3] that an $m \times n$ $(+,-)$ sign pattern A is an L-matrix iff for every $m \times 1$ $(+,\ -)$ sign pattern vector x, at least one of x and $-x$ is a column in A (and thus $n \geq 2^{m-1}$).

If A is an L-matrix, then A^T is an L-matrix if and only if A is square. A *signing* of order k is a nonzero diagonal sign pattern matrix of order k. A *strict signing* is a signing that is invertible (namely, SNS). Let $D = \mathrm{diag}(d_1, d_2, \ldots, d_k)$ be a signing of order k with diagonal entries $d_1, d_2, \ldots, d_k$. If $k = m$, then the matrix DA is a *row signing* of the matrix A, and if D is a strict signing , then DA is a strict row signing of A. If $k = n$ then the matrix AD is a *column signing* of the matrix A, and if D is a strict signing, then AD is a strict column signing of A. A signing $\tilde{D} = \mathrm{diag}(d'_1, d'_2, \ldots, d'_k)$ is an *extension* of the signing D provided that $d_i \neq 0 \Rightarrow d'_i = d_i$. A vector is said to be *balanced* provided either it is a zero vector or it has both a positive entry and a negative entry. A vector v is said to be *unisigned* provided that it is not balanced. Thus v is unisigned iff $v \neq 0$ and the nonzero entries of v have the same sign. A *balanced row signing* of the matrix A is a row signing of A in which all columns are balanced. A *balanced column signing* of A is a column signing of A in which all rows are balanced.

Let $v_1, v_2, \ldots, v_k$ be $n \times 1$ (or $1 \times n$) sign pattern matrices (which may be called sign pattern vectors). Let S $= \{v_1, v_2, \ldots, v_k\}$. We say that S is *weakly dependent* if

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k \overset{c}{\longleftrightarrow} 0,$$

where $c_i \in \{+,-,0\}, (1 \leq i \leq k)$ and at least one of the c_i is nonzero, and $\overset{c}{\leftrightarrow}$ denotes compatibility of two generalized sign patterns, see [6]. The set S of $(+,-,0)$ vectors is said to be *strongly independent* if it is not weakly dependent.

Theorem 3.2. An $m \times n$ sign pattern has m strongly independent rows if and only if A is an L-matrix.

Proof. ($\Longrightarrow$) Assume that A is not an L-matrix. Then there exists $B \in Q(A)$ whose rows are linearly dependent. Permuting the rows of B (denoted $B_1, \ldots, B_m$) if necessary, we may assume that the rows have a dependence relation of the form

$$d_1 B_1 + d_2 B_2 + \cdots + d_m B_m = 0,$$

where $d_1 > 0$. Setting $c_i = \mathrm{sgn}(d_i)$ and $A_i = \mathrm{sgn}(B_i)$ for all $1 \le i \le m$, we then have

$$c_1 A_1 + c_2 A_2 + \cdots + c_m A_m \xleftrightarrow{c} 0,$$

namely, the m rows of A are weakly dependent.

($\Longleftarrow$) Assume that the rows of A are weakly dependent. Then

$$c_1 A_1 + c_2 A_2 + \cdots + c_m A_m \xleftrightarrow{c} 0,$$

where $c_i \in \{+, -, 0\}$. By permuting the rows of A if necessary, we may assume that $c_1 = c_2 = \cdots = c_p = +$, $c_{p+1} = \cdots = c_{p+q} = -$, and $c_{p+q+1} = \cdots = c_m = 0$, where $1 \le p \le m$ and $0 \le q \le m - p$. Thus we have

$$(c_1 A_1 + c_2 A_2 + \cdots + c_p A_p) + (c_{p+1} A_{p+1} + \cdots + c_{p+q} A_{p+q}) \xleftrightarrow{c} 0.$$

By concentrating on each component individually, we can find $B_i \in Q(A_i)$ for $1 \le i \le p + q$ such that

$$(B_1 + B_2 + \cdots + B_p) - (B_{p+1} + \cdots + B_{p+q}) = 0.$$

Thus every matrix $B \in Q(A)$ with the first $p+q$ rows satisfying the above equation have linearly dependent rows. Therefore, A is not an L-matrix. ■

Corollary 3.3. Let A be a sign pattern matrix. Let M be any submatrix of A of size $m_1 \times n_1$ such that the rows of M are strongly independent. Then $m_1 \leq mr(A)$.

In particular, the *maximum number of strongly independent rows* of A, denoted $m_R(A)$, satisfies $m_R(A) \leq mr(A)$. Similarly, the *maximum number of strongly independent columns* of A, denoted $m_C(A)$, satisfies $m_C(A) \leq mr(A)$.

Corollary 3.4. Let A be a square sign pattern. Then the rows of A are strongly independent $\Longleftrightarrow$ the columns of A are strongly independent $\Longleftrightarrow$ A is sign nonsingular (SNS).

Theorem 3.5. Let A be an $m \times n$ sign pattern matrix. Then

(i). A is an L-matrix if and only if every row signing of A contains a unisigned column.

(ii). A is a barely L-matrix if and only if A is an L-matrix and for each $i = 1, 2, \ldots, n$, there is a row signing of A for which column i is the only unisigned column.

Proof. First assume that there is a signing D such that every column of DA is balanced. This implies that there exists a matrix B in $Q(A)$ such that each of the column sums of DB equals zero. Hence the rows of B are linearly dependent and A is not an L-matrix. Now assume that A is not an L-matrix. Then there is a matrix $B \in Q(A)$ whose rows are linearly dependent. Hence there exists a nonzero diagonal matrix $E = \text{diag}(e_1, e_2, \ldots, e_m)$ such that $(e_1, e_2, \ldots, e_m)B = 0$. Let E' be the signing obtained from E by replacing e_i by their signs. Then each column of $E'A$ is balanced. Therefore (i) holds.

We now prove that (ii) holds. Assume that A is barely L-matrix. Then A is an L-matrix and by (i) every row signing of A contains a unisigned column. Let i be an integer with $1 \leq i \leq n$. The matrix A_i obtained from A by deleting column i is not an L-matrix and hence by (i) there is a balanced row signing DA_i of A_i. It follows that the column i is the only unisigned column of DA. Conversely, assume that A is an L-matrix and for each $i = 1, 2, \ldots, n$, there is a row signing of A for which column i is the only unisigned column. Then for each i, the matrix A_i obtained from A by deleting column i has a balanced row signing, and it follows from (i) that A_i is not an L-matrix. Thus A is a barely L-matrix. ■

Let

$$A = \begin{bmatrix} B_1 & 0 \\ B_3 & B_2 \end{bmatrix}.$$

be an $m \times n$ matrix. If B_1 and B_2 are L-matrices, then A is also an L-matrix. Conversely, if A is an L-matrix, then B_1 is an L-matrix but B_2 is not necessarily an L-matrix. If A is a barely L-matrix and B_1 and B_2 are L-matrices, then B_1 and B_2 are barely L-matrices.

For example,

$$A = \left[\begin{array}{ccc|cc} + & + & + & 0 & 0 \\ \hline - & + & + & + & + \\ 0 & - & + & + & + \end{array}\right]$$

is an L-matrix (it has an SNS submatrix of order 3) but $A[\{2,3\},\{4,5\}]$ is not an L-matrix.

4. Sign Patterns That Require a Fixed Rank

In this section we investigate the sign patterns A such that the minimum rank and the maximum rank are equal, that is to say, $\mathrm{mr}(A) = \mathrm{MR}(A)$. In this case, we say that A *requires a fixed rank*. More specifically, if $\mathrm{mr}(A) = \mathrm{MR}(A) = k$, we say that A requires the fixed rank k. Obviously, if A or A^T is an L-matrix, then A requires a fixed rank.

Example 4.1. Let A be an $m \times n$ L-matrix with $m < n$. Then the sign pattern

$$\tilde{A} = \begin{bmatrix} A^T & 0 \\ 0 & A \end{bmatrix}$$

requires the fixed rank $2m$. However, neither $\tilde{A}$ nor $\tilde{A}^T$ is an L-matrix.

The symmetric tree sign patterns which require unique inertia are characterized in [8]. These are precisely the symmetric tree sign patterns A which require fixed rank. In particular, when a symmetric tree sign pattern A has zero diagonal (G(A) has no loops), we have the following result which may be found in [4].

Theorem 4.2. A symmetric tree sign pattern A with zero diagonal requires the unique rank $2t$, where t is the maximum number of independent edges in the tree $G(A)$.

Thus, corresponding to every tree T, there exist tree sign patterns A such that A requires a fixed rank and $G(A) = T$.

The above examples of sign patterns that require a fixed rank indicate that the structure of sign patterns that require a fixed rank can be very diverse. However, a beautiful and unifying characterization of sign patterns that require a fixed rank is the following result by Hershkowitz and Schneider [9].

Theorem 4.3. Let A be an $m \times n$ sign pattern matrix. Then A requires the fixed rank r [namely, $\mathrm{mr}(A) = \mathrm{MR}(A) = r$] if and only if there exist nonnegative integers e and f with $e + f = r$ and permutation sign patterns P and Q such that PAQ has the form

$$\begin{bmatrix} X & 0 \\ Z & Y \end{bmatrix}, \tag{*}$$

where Z is an $e \times f$ matrix and X^T and Y are L-matrices.

Proof. First assume that A requires the fixed rank r. Then r is the term rank of A, and by Konig's theorem (see Theorem 2.3) we may assume that A has the form (*) where X is an $(m-e) \times f$ matrix with term rank f, Y is an $e \times (n-f)$ matrix with term rank e, and $r = e + f$. Since Y is $e \times (n-f)$, each matrix in $Q(Y)$ has at most e linearly independent columns, and since each matrix in $Q(A)$ has exactly r linearly independent columns, each matrix in $Q(Y)$ has exactly e linearly independent columns. It follows that the rows of each matrix in $Q(Y)$ are linearly independent and hence that Y is an L-matrix. It can be shown similarly that X^T is an L-matrix.

Conversely, assume that A has the form (*) where Z is an $e \times f$ matrix with $e + f = r$, and X^T and Y are L-matrices. Then

$$\mathrm{MR}(A) \leq \mathrm{MR}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right) + \mathrm{MR}(Y) \leq f + e = r.$$

On the other hand, it can be seen that

$$\mathrm{mr}(A) \geq \mathrm{mr}(X) + \mathrm{mr}(Y) = f + e = r.$$

Therefore, $\mathrm{mr}(A) = \mathrm{MR}(A) = r$. ■

5. Sign Patterns That Require Almost Unique Rank

We now investigate sign patterns A that require almost unique rank, namely, $\mathrm{MR}(A) = \mathrm{mr}(A) + 1$. Let $r = \mathrm{MR}(A)$. In view of Theorem 2.4, without loss of generality, we may assume that A has the block form

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r-k)$ for some k, $0 \leq \ k \leq r$. Therefore, in the remainder of this thesis, we will assume that A has the above block form.

Theorem 5.1. Suppose that A is a sign pattern with $r = \mathrm{MR}(A)$ and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r-k)$ for some k, $0 \leq \ k \leq r$. Then

(a). If Z^T is an L-matrix and $\mathrm{MR}(Y) = \mathrm{mr}(Y) + 1$, then $\mathrm{MR}(A) = \mathrm{mr}(A) + 1$.

(b). If Y is an L-matrix and $\mathrm{MR}(Z) = \mathrm{mr}(Z) + 1$, then $\mathrm{MR}(A) = \mathrm{mr}(A) + 1$.

Proof. We prove only (a); the proof of (b) is similar and is omitted. It is clear from $r = \mathrm{MR}(A)$ and X is $k \times (r-k)$ that $\mathrm{MR}(Y) = k$, (see Theorem 2.4), so that $\mathrm{mr}(Y) = k - 1$. Since Z^T is an L-matrix, $\mathrm{mr}(Z) = r - k$. Hence,

$$\mathrm{mr}(A) \geq \mathrm{mr}(Z) + \mathrm{mr}(Y) = r - 1.$$

By taking a real matrix $B_2 \in Q(Y)$ with $\mathrm{rank}(B_2) = k - 1$, combined with any $B_1 \in Q(X)$ and $B_3 \in Q(Z)$, we obtain a matrix

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}$$

in $Q(A)$ with $\text{rank}(B) \leq (r-k)+(k-1) = r-1$. Since $\text{mr}(A) \geq r-1$, we must have $\text{rank}(B) = r-1$ and hence, $\text{mr}(A) = r-1$. ■

A natural question arises. Suppose that A is a sign pattern with $\text{MR}(A) = \text{mr}(A)+1 = r$, and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r-k)$ for some k with $0 \leq k \leq r$. Does it follow that either Y or Z^T is an L-matrix? Examples show that the answer to this question is no. For instance,

$$A = \left[\begin{array}{cc|cc} + & 0 & + & + \\ 0 & + & + & + \\ \hline + & + & 0 & 0 \\ + & + & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} X & Y \\ \hline Z & 0 \end{array}\right]$$

satisfies $\text{MR}(A) = \text{mr}(A)+1 = 4$, yet neither Y nor Z^T is an L-matrix.

Theorem 5.2. Let A be a sign pattern with $r = \text{MR}(A) = \text{mr}(A)+1$ and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r-k)$, for some k with $0 \leq k \leq r$. Then

$$\text{MR}(Y) \leq \text{mr}(Y)+1, \tag{1}$$

and

$$\text{MR}(Z) \leq \text{mr}(Z)+1. \tag{2}$$

Proof. Assume that $\text{MR}(Y) > \text{mr}(Y)+1$. Note that $\text{MR}(Y) = k$ holds since $\text{MR}(A) = r$. Then there exists a real matrix B_2 in $Q(Y)$ with $\text{rank}(B_2) \leq k-2$. For any $B_1 \in Q(X)$ and $B_3 \in Q(Z)$, we get

$$\text{rank}\left(\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}\right) \leq (r-k)+(k-2) = r-2.$$

This contradicts $\text{mr}(A) = r-1$. Thus, $\text{MR}(Y) \leq \text{mr}(Y)+1$. Similarly, we have $\text{MR}(Z) \leq \text{mr}(Z)+1$. ■

In other words, if A requires almost unique rank, then each of the two blocks Y and Z requires unique rank or requires almost unique rank. The converse of Theorem 5.2 is not true. For example, with

$$A = \left[\begin{array}{cc|cc} + & + & + & + \\ + & + & + & + \\ \hline + & + & 0 & 0 \\ + & + & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} X & Y \\ \hline Z & 0 \end{array}\right],$$

we have $\mathrm{MR}(Y) \leq \mathrm{mr}(Y) + 1$, $\mathrm{MR}(Z) \leq \mathrm{mr}(Z) + 1$, and yet $\mathrm{MR}(A) \neq \mathrm{mr}(A) + 1$ ($4 \neq 2 + 1$). As a result of the above, we raise a question as to what further conditions beyond (1) and (2) do we need to guarantee that $\mathrm{MR}(A) = \mathrm{mr}(A) + 1$?

We next strengthen the necessary conditions given in Theorem 5.2.

Theorem 5.3. Let A be a sign pattern with $r = \mathrm{MR}(A)$ and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r - k)$, for some k with $0 \leq k \leq r$. If A requires almost unique rank, then Y or Z requires almost unique rank.

Proof. Suppose A is a sign pattern matrix with $\mathrm{MR}(A) = \mathrm{mr}(A) + 1$ and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r - k)$, for some k with $0 \leq k \leq r$. Then by Theorem 5.2, we have

$$\mathrm{MR}(Y) \leq \mathrm{mr}(Y) + 1$$

and

$$\mathrm{MR}(Z) \leq \mathrm{mr}(Z) + 1.$$

If we have both

$$\mathrm{MR}(Y) = \mathrm{mr}(Y)$$

and

$$\mathrm{MR}(Z) = \mathrm{mr}(Z),$$

then $\mathrm{MR}(A) = \mathrm{mr}(A)$ by Theorem 4.3, which contradicts $\mathrm{MR}(A) = \mathrm{mr}(A)+1$. So, either $\mathrm{MR}(Y) = \mathrm{mr}(Y) + 1$ or $\mathrm{MR}(Z) = \mathrm{mr}(Z) + 1$. ■

Theorem 5.4. Let A be sign pattern with $r = \mathrm{MR}(A) = \mathrm{mr}(A) + 1$, and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r-k)$, for some k with $0 \leq \ k \leq r$. Then

(a). If $\mathrm{MR}(Y) = \mathrm{mr}(Y) + 1$, then $\mathrm{MR}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right) = \mathrm{mr}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right)$.

(b). If $\mathrm{MR}(Z) = \mathrm{mr}(Z) + 1$, then $\mathrm{MR}\left(\begin{bmatrix} X \ Y \end{bmatrix}\right) = \mathrm{mr}\left(\begin{bmatrix} X \ Y \end{bmatrix}\right)$.

Proof. To prove (a), observe that clearly, $\mathrm{MR}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right) \geq \mathrm{MR}(Z) = r - k$. Then $\mathrm{MR}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right) = r - k$ (as seen in Theorem 2.4).

Suppose that $\mathrm{mr}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right) < r-k$. Then, there exists a matrix $\begin{bmatrix} B_1 \\ B_3 \end{bmatrix} \in Q\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right)$ with $\mathrm{rank}\left(\begin{bmatrix} B_1 \\ B_3 \end{bmatrix}\right) \leq r - k - 1$.

Since $\mathrm{mr}(Y) = k - 1$, there is a matrix $\begin{bmatrix} B_2 \\ 0 \end{bmatrix} \in Q\left(\begin{bmatrix} Y \\ 0 \end{bmatrix}\right)$ with $\mathrm{rank}\left(\begin{bmatrix} B_2 \\ 0 \end{bmatrix}\right) = k - 1$. Then,

$$\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix} \in Q(A)$$

satisfies

$$\mathrm{rank}\left(\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}\right) \leq (r-k-1) + (k-1) = r - 2,$$

which contradicts $\mathrm{mr}(A) = r - 1$. The proof of (b) is similar. ■

Theorems 5.3 and 5.4 yield the following result.

Theorem 5.5. Let A be sign pattern with $r = \mathrm{MR}(A) = \mathrm{mr}(A) + 1$, and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r-k)$, for some k with $0 \leq \; k \leq r$. Then either

(a). $\mathrm{MR}(Y) = \mathrm{mr}(Y) + 1$ and $\mathrm{MR}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right) = \mathrm{mr}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right)$, or

(b). $\mathrm{MR}(Z) = \mathrm{mr}(Z) + 1$ and $\mathrm{MR}\left(\begin{bmatrix} X \; Y \end{bmatrix}\right) = \mathrm{mr}\left(\begin{bmatrix} X \; Y \end{bmatrix}\right)$.

We now establish further sufficient conditions for a sign pattern to require almost unique rank.

Theorem 5.6. Let A be a sign pattern with $r = \mathrm{MR}(A)$ and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r-k)$, for some k with $0 \leq \; k \leq r$. Suppose

(a). $\mathrm{MR}(Y) = \mathrm{mr}(Y) + 1$, $\begin{bmatrix} X \\ Z \end{bmatrix}^T$ is an L-matrix, and

$\mathrm{col}(B) \bigcap \mathrm{col}(C) = \{0\}$, for all $B \in Q\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right)$ and $C \in Q\left(\begin{bmatrix} Y \\ 0 \end{bmatrix}\right)$,

or

(b). $\mathrm{MR}(Z) = \mathrm{mr}(Z) + 1$, $[X \; Y]$ is an L-matrix, and

$\mathrm{row}(B) \bigcap \mathrm{row}(C) = \{0\}$, for all $B \in Q\left(\begin{bmatrix} X \; Y \end{bmatrix}\right)$ and $C \in Q\left(\begin{bmatrix} X \; 0 \end{bmatrix}\right)$.

Then $\mathrm{MR}(A) = \mathrm{mr}(A) + 1$.

Proof. Assume that (a) holds. Every matrix $B \in Q\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right)$ has $r-k$ linearly independent columns. Also, every matrix $C \in Q\left(\begin{bmatrix} Y \\ 0 \end{bmatrix}\right)$ has at least $k-1$ linearly independent columns, since $k = \mathrm{MR}(Y) = \mathrm{mr}(Y) + 1$. Then for every matrix $[B \; C] \in Q(A)$, we have

$$\begin{aligned} \dim(\mathrm{col}\left(\begin{bmatrix} B \; C \end{bmatrix}\right) &= \dim(\mathrm{col}(B) + \; \mathrm{col}(C)) \\ &= \dim(\mathrm{col}(B)) + \dim(\mathrm{col}(C)) - \dim(\mathrm{col}(B) \cap \mathrm{col}(C)) \\ &\geq (r-k) + (k-1) - 0 \\ &= r-1. \end{aligned}$$

Therefore, $\mathrm{mr}(A) \geq r-1$.

Choose a specific $C \in Q\left(\begin{bmatrix} Y \\ 0 \end{bmatrix}\right)$ with $\mathrm{rank}(C) = k-1$ and let $B \in Q\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right)$. Then, similarly as above, it can be shown that rank $([B\ C]) = r-1$. Hence, $\mathrm{mr}(A) = r-1 = \mathrm{MR}(A)-1$.

By a parallel argument, it can be shown that if (b) holds, then we also have $\mathrm{mr}(A) = r-1 = \mathrm{MR}(A)-1$. ■

Theorems 5.1 through 5.6 have generalizations (as done in Section 6) from sign patterns that require almost unique rank to sign patterns A for which $\mathrm{MR}(A) - \mathrm{mr}(A) = d \geq 2$. The column (row) space conditions in Theorem 5.6 need to be weakened appropriately to obtain necessary and sufficient conditions for a sign pattern A to require almost unique rank.

Theorem 5.7. Let A be sign pattern with $r = \mathrm{MR}(A)$, and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r-k)$, for some k with $0 \leq k \leq r$. Then $\mathrm{MR}(A) = \mathrm{mr}(A)+1$ iff

(a′). $\mathrm{MR}(Y) = \mathrm{mr}(Y)+1$, $\mathrm{MR}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right) = \mathrm{mr}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right)$, $\mathrm{mr}(Z) \geq \mathrm{MR}(Z)-1$, and $\mathrm{col}(B) \bigcap \mathrm{col}(C) = \{0\}$, for all $B \in Q\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right)$ and $C \in Q\left(\begin{bmatrix} Y \\ 0 \end{bmatrix}\right)$ with $\mathrm{rank}(C) = k-1$, or

(b′). $\mathrm{MR}(Z) = \mathrm{mr}(Z)+1$, $\mathrm{MR}\left(\begin{bmatrix} X & Y \end{bmatrix}\right) = \mathrm{mr}\left(\begin{bmatrix} X & Y \end{bmatrix}\right)$, $\mathrm{mr}(Y) \geq \mathrm{MR}(Y)-1$, and $\mathrm{row}(B) \bigcap \mathrm{row}(C) = \{0\}$, for all $B \in Q\left(\begin{bmatrix} X & Y \end{bmatrix}\right)$ and $C \in Q\left(\begin{bmatrix} X & 0 \end{bmatrix}\right)$ with $\mathrm{rank}(C) = k-1$.

Proof. ($\Longrightarrow$). From theorem 5.2, we have both $\mathrm{mr}(Z) \geq \mathrm{MR}(Z)-1$ and $\mathrm{mr}(Y) \geq \mathrm{MR}(Y)-1$. By Theorem 5.5, we have (a) or (b) of Theorem 5.5. Suppose (a) holds and the column condition of (a′) does not hold. Then we have $\mathrm{col}(B) \bigcap\ \mathrm{col}(C) \neq$

$\{0\}$, for some $B \in Q\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right)$ and $C \in Q\left(\begin{bmatrix} Y \\ 0 \end{bmatrix}\right)$ with $\mathrm{rank}(C) = k-1$. Hence,

$$\begin{aligned} \dim(\mathrm{col}\left(\begin{bmatrix} B\ C \end{bmatrix}\right) &= \dim(\mathrm{col}(B) + \ \mathrm{col}(C)) \\ &= \dim(\mathrm{col}(B)) + \dim(\mathrm{col}(C)) - \dim(\mathrm{col}(B) \cap \mathrm{col}(C)) \\ &\leq (r-k) + (k-1) - 1 \\ &= r-2. \end{aligned}$$

Since $\begin{bmatrix} B\ C \end{bmatrix} \in Q(A)$, $\mathrm{mr}(A) \leq r-2$, contradicting $\mathrm{mr}(A) = r-1$. Thus, the column condition in (a′) holds. Similarly, if Theorem 5.5 (b) holds, then we have (b′).

($\Longleftarrow$). Assume (a′). Let $[B\ C] \in Q(A)$ and $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$.

If $\mathrm{rank}(C) = k-1$, then by the intersection condition of (a′), we get $\mathrm{rank}([B\ C])=$ $\mathrm{rank}\ (B)+\ \mathrm{rank}(C) = (r-k) + (k-1) = r-1$.

If $\mathrm{rank}(C) = k$ and $\mathrm{mr}(Z) = r-k-1$, then for any $v \in \mathrm{col}(B)$, v is a linear combination of the columns of $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$. If, in addition, $v \in\ \mathrm{col}\ (C)$, then since the null space of B_2 is of dimension at most 1, the coefficients in the linear combination of the columns of B are all multiples of one fixed vector. Thus, $\dim(\mathrm{col}(B) \bigcap \mathrm{col}(C)) \leq 1$. Hence, $\mathrm{rank}([B\ C])=\ \mathrm{rank}\ (B)+\ \mathrm{rank}(C)-$ $\dim(\mathrm{col}(B) \bigcap \mathrm{col}(C)) \geq \mathrm{rank}(B) + \mathrm{rank}(C) - 1 \geq (r-k) + k - 1 = r-1$.

If $\mathrm{rank}(C) = k$ and $\mathrm{mr}(Z) = r-k$, then since the null space of B_2 has a dimension 0, we see that $\mathrm{col}(B) \bigcap \mathrm{col}(C) = \{0\}$. Thus, $\mathrm{rank}([B\ C])=\ \mathrm{rank}\ (B)+$ $\mathrm{rank}(C) = (r-k) + k = r \geq r-1$.

Combining the above cases, we see that $\mathrm{mr}(A) = r-1 = \mathrm{MR}(A) - 1$.

Similarly, we can show that (b′) implies $\mathrm{mr}(A) = r-1 = \mathrm{MR}(A) - 1$. ■

In [1], it was conjectured that the minimum rank of any sign pattern matrix A can be achieved by a rational matrix $B \in Q(A)$, and several classes of sign patterns that do have this property were exhibited. Even though this conjecture does not hold in general (see [11]), we now give another instance where rational realization of the minimum rank does occur.

Theorem 5.8. Let A be sign pattern with $r = \mathrm{MR}(A)$, and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r-k)$, for some k with $0 \le\ k \le r$. If $\mathrm{MR}(A) \le \mathrm{mr}(A) + 1$, then there is a rational matrix $B \in Q(A)$ attaining the minimum rank of A.

Proof. If A requires unique rank, then certainly every rational matrix $B \in Q(A)$ attains the minimum rank of A. Suppose that A does not require unique rank, so that $\mathrm{mr}(A) = r - 1$. Then Y or Z^T is not an L-matrix by Theorem 4.3. Assume that Y is not an L-matrix. Then it is well known (see Proposition 2.2 of [1]) that there is a rational matrix $Y' \in Q(Y)$ such that $\mathrm{rank}(Y') \le k-1$. Let $\begin{bmatrix} X' \\ Z' \end{bmatrix}$ be any rational matrix in $Q\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right)$. Then

$$\mathrm{rank}\left(\begin{bmatrix} X' & Y' \\ Z' & 0 \end{bmatrix}\right) \le \mathrm{rank}\left(\begin{bmatrix} X' \\ Z' \end{bmatrix}\right) + \mathrm{rank}\left(\begin{bmatrix} Y' \\ 0 \end{bmatrix}\right) \le (r-k) + (k-1) = r-1.$$

However, since $\mathrm{mr}(A) \ge \mathrm{MR}(A) - 1 = r - 1$, we have

$$\mathrm{rank}\left(\begin{bmatrix} X' & Y' \\ Z' & 0 \end{bmatrix}\right) = r - 1.$$

Thus, we have a rational matrix in $Q(A)$ attaining the minimum rank of A.

Similarly, if Z^T is not an L-matrix, we can show that rational realization of the minimum rank is achieved. ■

An open problem is the following: for any sign pattern matrix A, does the condition $\mathrm{MR}(A) = \mathrm{mr}(A) + 2$ imply that there is a rational matrix $B \in Q(A)$ attaining the minimum rank of A?

6. Sign Patterns with Spread $d > 1$

For sign pattern matrix A, the *spread* of A is defined as $\mathrm{MR}(A) - \mathrm{mr}(A)$. In this section, we present Theorems 6.1–6.6, which generalize Theorems 5.1–5.6, respectively.

By replacing 1 with d in the proofs of Theorems 5.1 and 5.2, we obtain the following two results.

Theorem 6.1. Suppose that A is a sign pattern with $r = \mathrm{MR}(A)$ and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r - k)$ for some k, $0 \leq k \leq r$. Then

(a). If Z^T is an L-matrix and $\mathrm{MR}(Y) = \mathrm{mr}(Y) + d$, then $\mathrm{MR}(A) = \mathrm{mr}(A) + d$.

(b). If Y is an L-matrix and $\mathrm{MR}(Z) = \mathrm{mr}(Z) + d$, then $\mathrm{MR}(A) = \mathrm{mr}(A) + d$.

Theorem 6.2. Let A be a sign pattern with $r = \mathrm{MR}(A) = \mathrm{mr}(A) + d$ and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r - k)$, for some k with $0 \leq k \leq r$. Then,

$$\mathrm{MR}(Y) \leq \mathrm{mr}(Y) + d, \text{ and}$$

$$\mathrm{MR}(Z) \leq \mathrm{mr}(Z) + d.$$

A generalization of Theorem 5.3 is more involved.

Theorem 6.3. Let A be a sign pattern with $r = \mathrm{MR}(A) = \mathrm{mr}(A) + d$ and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r-k)$, for some k with $0 \leq k \leq r$. Let $d_1 = \mathrm{MR}(Y) - \mathrm{mr}(Y)$ and $d_2 = \mathrm{MR}(Z) - \mathrm{mr}(Z)$. Then $d_1 + d_2 \geq d$.

Proof. If $d > d_1 + d_2$, then for every matrix $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix} \in Q(A)$, we have $\mathrm{rank}\left(\begin{bmatrix} B_1 & B_2 \\ B_3 & 0 \end{bmatrix}\right) \geq \mathrm{rank}(B_2) + \mathrm{rank}(B_3) \geq (k-d_1)+(r-k-d_2) = r-(d_1+d_2) > r-d$, contradicting $\mathrm{mr}(A) = r-d$. ■

Remark: By Theorem 6.2, $d_1 \leq d$ and $d_2 \leq d$.

Theorem 6.4. Let A be sign pattern with $r = \mathrm{MR}(A) = \mathrm{mr}(A) + d$, and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r-k)$, for some k with $0 \leq k \leq r$.

(a). If $\mathrm{MR}(Y) = \mathrm{mr}(Y) + d$, then $\mathrm{MR}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right) = \mathrm{mr}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right)$.

(b). If $\mathrm{MR}(Z) = \mathrm{mr}(Z) + d$, then $\mathrm{MR}\left(\begin{bmatrix} X & Y \end{bmatrix}\right) = \mathrm{mr}\left(\begin{bmatrix} X & Y \end{bmatrix}\right)$.

Proof. The proof is similar to the proof of Theorem 5.4. ■

The following generalization of Theorem 6.4 can also be viewed as a generalization of Theorem 5.5. The proof is straightforward and hence is omitted.

Theorem 6.5. Let A be sign pattern with $r = \mathrm{MR}(A) = \mathrm{mr}(A) + d$, and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r-k)$, for some k with $0 \leq k \leq r$. Let $d_1 = \mathrm{MR}(Y) - \mathrm{mr}(Y)$ and $d_2 = \mathrm{MR}(Z) - \mathrm{mr}(Z)$. Then

(a). $\mathrm{MR}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right) \leq \mathrm{mr}\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right) + (d - d_1)$, and

(b). $\mathrm{MR}\left(\begin{bmatrix} X & Y \end{bmatrix}\right) \leq \mathrm{mr}\left(\begin{bmatrix} X & Y \end{bmatrix}\right) + (d - d_2)$.

Note that when $d = 1$, then $d = d_1 = 1$ or $d = d_2 = 1$, so that Theorem 5.5 is a special case of Theorem 6.5.

Theorem 6.6. Let A be sign pattern with $r = \mathrm{MR}(A)$ and

$$A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r-k)$, for some k with $0 \leq \ k \leq r$. Suppose

(a). $\mathrm{MR}(Y) = \mathrm{mr}(Y) + d$, $\begin{bmatrix} X \\ Z \end{bmatrix}^T$ is an L-matrix, and

$\mathrm{col}(B) \bigcap \mathrm{col}(C) = \{0\}$, for all $B \in Q\left(\begin{bmatrix} X \\ Z \end{bmatrix}\right)$ and $C \in Q\left(\begin{bmatrix} Y \\ 0 \end{bmatrix}\right)$,

or

(b). $\mathrm{MR}(Z) = \mathrm{mr}(Z) + d$, $[X\ Y]$ is an L-matrix, and

$\mathrm{row}(B) \bigcap \mathrm{row}(C) = \{0\}$, for all $B \in Q\left([X\ Y]\right)$ and $C \in Q\left([X\ 0]\right)$.

Then $\mathrm{MR}(A) = \mathrm{mr}(A) + d$.

Proof. The proof is similar to the proof of Theorem 5.6. ■

If $d \geq 2$, then there are $(d+1)(d+2)/2$ choices for (d_1, d_2). Hence, there are no straightforward generalizations of Theorem 5.7.

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