Georgia State University
ScholarWorks @ Georgia State University

Mathematics Theses | Department of Mathematics and Statistics

6-9-2006

Algebraic Concepts in the Study of Graphs and Simplicial Complexes

Christopher Michael Zagrodny

ALGEBRAIC CONCEPTS IN THE STUDY OF GRAPHS AND SIMPLICIAL COMPLEXES

by

CHRISTOPHER MICHAEL ZAGRODNY

Under the Direction of Florian Enescu

ABSTRACT

This paper presents a survey of concepts in commutative algebra that have applications to topology and graph theory. The primary algebraic focus will be on Stanley-Reisner rings, classes of polynomial rings that can describe simplicial complexes. Stanley-Reisner rings are defined via square-free monomial ideals. The paper will present many aspects of the theory of these ideals and discuss how they relate to important constructions in commutative algebra, such as finite generation of ideals, graded rings and modules, localization and associated primes, primary decomposition of ideals and Hilbert series. In particular, the primary decomposition and Hilbert series for certain types of monomial ideals will be analyzed through explicit examples of simplicial complexes and graphs.

INDEX TERMS: Commutative Algebra, Graph Theory, Stanley-Reisner Rings

ALGEBRAIC CONCEPTS IN THE STUDY OF GRAPHS AND SIMPLICIAL COMPLEXES

by

Christopher Michael Zagrodny

A Thesis Submitted in Partial Fulfilment of the Requirements for the Degree of

Master of Science

in the College of Arts and Sciences

Georgia State University

2006

ALGEBRAIC CONCEPTS IN THE STUDY OF GRAPHS AND SIMPLICIAL COMPLEXES

by

CHRISTOPHER MICHAEL ZAGRODNY

Major Professor: Florian Enescu

Committee Members: Mihaly Bakonyi

Guantao Chen

Electronic Version Approved:

Office of Graduate Studies

College of Arts and Science

Georgia State University

May 2006

Acknowledgements

I would like to thank Dr. Enescu for his help and advisement and Dr. Bakonyi and Dr. Chen for proofreading the final draft.

I would like also to thank my parents for their support and encouragement in pursuing this degree.

Contents

Acknowledgements **iv**

1 Introduction **1**

2 Fundamental Concepts **3**

2.1 Conventions 3

2.2 Polynomials, Monomials, and Ordering 4

2.2.1 Ordering of monomials 4

2.3 Monomial Ideals 5

2.4 Modules 11

2.5 Graded rings and modules 13

3 Prime Ideals, Localization, and Associated Primes **19**

3.1 Local Rings 19

3.2 Associated Primes 25

3.3 Support 31

4 Stanley-Reisner Rings and Graph Ideals **34**

4.1 Stanley-Reisner rings 34

4.2 Support 36

4.3 Graph Ideals 39

5 Primary Decomposition **43**

5.1 Irreducible Ideals 43

5.2 Primary Decomposition 44

6 Hilbert Polynomials of Stanley-Reisner Rings **53**

6.1 Stanley-Reisner rings 53

6.2 Hilbert Series of Graph Rings 56

6.2.1 Special types of graphs 56

6.2.2 Graph Rings of dimension 1 and 2 60

List of Figures

4.1 Simplicial Complex (Example 4.1.7) 35

4.2 Simplicial Complex Decomposed (Example 4.2.4) 38

4.3 Graph to Simplicial Complex 41

6.1 Graphs with Rings of Dimension 2 63

Chapter 1

Introduction

The goal of this paper is to survey the relationship between some central notions in commutative algebra and certain topological and combinatorial objects such as simplicial complexes and graphs. We will show that it is natural and convenient to describe graphs and simplicial complexes in algebraic terms. This will be done by associating a finitely generated polynomial algebra over a field to a simplicial complex such that the topological nature of the complex is captured by the algebraic properties of the algebra.

The aim of this paper will be the study of primary ideals, associated primes, Hilbert polynomials in view of applying them to Stanley-Reisner rings. These are certain polynomial algebras over a field that are generally associated to simplicial complexes, but can also be associated to graphs with no multiple edges. The particular properties of the Stanley-Reisner rings and their connection to algebraic combinatorics are derived from the fact that they are quotients of polynomials rings by square free monomial ideals which by themselves carry combinatorial features. The theory of Stanley-Reisner rings and graph ideals is a very active area of research in combinatorial commutative algebra. Among the mathematicians who brought major contributions to the subject are Stanley, Hochster, Reisner, Simis, Villarreal to mention only a few. Some of their contributions are surveyed in this thesis.

Some relevant intermediary concepts will need to be covered in Chapters 2 and 3. Properties of monomial ideals and more general homogenous ideals in graded rings will be addressed in Chapter 2. Some necessary background in modules and localization will be covered in chapter 3. Chapter 4 is introduces the main objects of study in this thesis, Stanley-Reisner rings and graph ideals, and investigate their minimal primes. These investigations will later lead to primary decomposition, which will be addressed in detail in Chapter 5, along with its applications to Stanley-Reisner rings.

The final topic to be discussed is Hilbert Series of a graded ring. Their characteristics and composition for particular graphs and simplicial complexes will be compared in Chapter 6.

Chapter 2

Fundamental Concepts

In this chapter, we will introduce the basic concepts from modern algebra needed in the main part of the paper. We will discuss the theory of monomial ideals, modules, special classes of modules such as Noetherian, Artinian and graded modules as well as the Hilbert series of a graded module. Some of the theorems will be presented with proofs.

2.1 Conventions

In the following chapters, unless otherwise stated, all rings will be commutative with a multiplicative identity. Also, if not defined by the context, R and A will be rings, F and k are fields, and M will be an R-module.

Given a mapping, $i : S \to T$, from a set S to a set T, and subsets $A \subseteq S$ and $U \subseteq T$, we define $A \cap U = \{a \in S : i(a) \in U\}$. If $S \subset T$, then this is the normal intersection of subsets.

Also, for the purpose of this paper, $\mathbb{N}$ will represent the set of natural numbers plus zero.

2.2 Polynomials, Monomials, and Ordering

Let A be a commutative ring with identity. The notation $A[X_1, X_2, \ldots, X_n]$ will be used to refer to the ring of polynomials over a commutative ring A with indeterminates $X_1, X_2, \ldots, X_n$. In many cases A will be a field k. A monomial $X_1^{a_1} X_2^{a_2} \ldots X_n^{a_n} \in A[X_1, X_2, \ldots, X_n]$ will be denoted by X^α with $\alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ being the multidegree of the monomial.

2.2.1 Ordering of monomials

Definition 2.2.1. A *monomial ordering* on the set of monomials of $A[X_1, X_2, \ldots, X_n]$ is a relation "$\leq$" between the multidegrees satisfying the following conditions:

1. It is a total ordering.
2. Given $\alpha, \beta \in \mathbb{N}^n$, $\alpha \leq \beta \Leftrightarrow \alpha + \rho \leq \beta + \rho, \forall \rho \in \mathbb{N}^n$.
3. It is well-ordered.

The following definitions provide examples of monomial orderings.

Definition 2.2.2. *Lexicographical ordering, '$\leq_{lex}$',* compares the individual degrees of the indeterminates. Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$ be elements of $\mathbb{N}^n$, $\alpha \leq_{lex} \beta$ if the left most non-zero entry of $\beta - \alpha$ is positive. Also for all X^α, X^β,

$$X^\alpha \leq_{lex} X^\beta$$

if and only if

$$\alpha \leq_{lex} \beta$$

For example, $XY^4 \leq_{lex} X^2$, and $X^2Y^2 \leq_{lex} X^3Y$.

Definition 2.2.3. *Graded lexicographical ordering, '$\leq_{grlex}$', is similar to lexicographical ordering, but it first compares total degrees. Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$ be elements of $\mathbb{N}^n$, $\alpha \leq_{grlex} \beta$ if $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$, or $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and $\alpha \leq_{lex} \beta$.*

For example, $X^2 \leq_{grleX} XY^4$ and $X^2Y^2 \leq_{grleX} X^3Y$.

Definition 2.2.4. *Graded reverse lexicographical ordering, '$\leq_{grrevlex}$', is the same as graded lexicographical ordering, but comparing degrees from right to left. Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$ be elements of $\mathbb{N}^n$, $\alpha \leq_{grrevlex} \beta$ if $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$, or $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the right most non-zero entry of $\beta - \alpha$ is positive.*

For example, $X^2 \leq_{grleX} XY^4$ and $X^3Y \leq_{grlex} X^2Y^2$.

Definition 2.2.5. Let '$\leq$' be a monomial ordering on $R = A[X_1, X_2, \ldots, X_n]$ and let $f \in R$ be a polynomial such that $f = \sum_{i=1}^m a_i X^{\alpha_i}$, where $a_i \in A$ for all $i = 1, \ldots, n$. We define the *multidegree* of f to be $multidegree(f) = \max\{\alpha_i | a_i \neq 0\}$, the largest multidegree of its terms. The *leading term* of f is $LT(f) = a_i X^{\alpha_i}$ where $\alpha_i = multidegree(f)$. Also, $LC(f) = a_i$ is called the *leading coefficient* of f and $LM(f) = X^{\alpha_i}$ is the *leading monomial* of f.

2.3 Monomial Ideals

Definition 2.3.1. Let I be an ideal of a polynomial ring, $k[X_1, X_2, \ldots, X_n]$. Then I is called a *monomial ideal* if, for any $f = \sum_{i=1}^m a_i X^{\alpha_i} \in I$,where all a_i are in k, then $X^{\alpha_i} \in I$ for $i = 1, \ldots m$.

An equivalent definition follows from the next theorem.

Theorem 2.3.2. *Let Λ be a subset of $\mathbb{N}^n$. Then, the ideal I generated by $I = (X^\alpha : \alpha \in \Lambda)$ of $R = k[X_1, X_2, \ldots, X_n]$ is a monomial ideal. Conversely, any monomial ideal I of R can be written as $I = (X^\alpha : \alpha \in \Lambda)$ for some Λ subset of $\mathbb{N}^n$.*

Proof. Given $\Lambda \subseteq \mathbb{N}^n$, let $I = (a_\alpha X^\alpha : \alpha \in \Lambda, a_\alpha \in A)$. Then any $f \in I$ can be written as

$$f = \sum a_i X^{\alpha_i}, a_i \in R.$$

Since each $a_i = \sum_{j=1}^m h_{ij} X^{\beta_{ij}}$ with $h_{ij} \in A$, we can expand out f, and then simply observe that each term in the decomposition of f is divisible by some $X^{\alpha_i} \in I$. So the terms that appear in the decomposition are in I. Therefore I is a monomial ideal by definition.

The opposite is proved by finding a subset of $\mathbb{N}^n$ for any given monomial ideal. Let $I \leq R = k[X_1, X_2, \ldots, X_n]$ be a monomial ideal and Λ be the set of all multidegrees of all monomials in each member of I. Letting $J = (X^\alpha : \alpha \in \Lambda, X^\alpha \in I)$, we can show $I = J$. If $f \in I$ then each term of f is in I and so its multidegree is in Λ. So f is a sum of monomials of the form $\{X^\alpha : \alpha \in \Lambda\}$. And so $f \in J$ and $I \subseteq J$. If $f \in J$, then f is of the form $\sum a_i X^{\alpha_i}$ with $\alpha_i \in \Lambda$ and $a_i \in R$. Since $X^{\alpha_i} \in I$ as well as J, X^{α_i} is in I and the summation f, is in I as well. $\square$

Theorem 2.3.3. *Let $I = (X^\alpha : \alpha \in \Lambda)$ be a monomial ideal. A monomial X^β belongs to I if and only if there exists $X^\alpha, \alpha \in \Lambda$, such that $X^\alpha \mid X^\beta$. This implies that there exists a $\gamma \in \mathbb{N}^n$ such that $\alpha + \gamma = \beta$.*

Proof. If $X^\beta \in I \subseteq R$ then X^β can be written as

$$X^\beta = \sum_{i=1}^m a_i X^{\alpha_i},$$

for all $a_i \in R$ and $\alpha_i \in \Lambda$. However each $a_i \in R$ can be also be written as a sum,

$a_i = \sum_{j=1}^{m_i} h_{ij} X^{\rho_{ij}}$, with $h_{ij} \in R$ and $\rho_{ij} \in \mathbb{N}^n$. If we substitute the sums for the a_i's into the original summation and expand out, we have

$$X^\beta = \sum_{i,j} h_{ij} X^{\rho_{ij}} X^{\alpha_i} = \sum_{i,j} h_{ij} X^{\rho_{ij}+\alpha_i}.$$

Since $h_{ij} = 0$ when $\rho_{ij} + \alpha_i \neq \beta$, what we have left is $X^\beta = \sum_{i,j} h_{ij} X^{\rho_{ij}} X^{\alpha_i}$, where $\rho_{ij} + \alpha_i = \beta$. And so X^β is then divisible by some X^{α_i} where α_i is in Λ.

If X^β is divisible by an $X^\alpha \in I$ then we can write $X^\beta = X^\alpha f$, for some $f \in R$. Since X^α is in I, $X^\alpha f$ is in I, so X^β is in I as well.

□

Theorem 2.3.4 (Division Algorithm). *Given $f, g_i \in R = k[X_1, X_2, \ldots, X_n]$, $i = 1 \ldots s$, there exists $a_i \in R$ and $r \in R, g_i \nmid r, \forall i = 1 \ldots s$ such that*

$$f = r + \sum_{i=1}^{s} a_i g_i.$$

Proof. We will prove the theorem by construction of the a_i's and r. Initially we assume all $a_i = r = 0$ for all $i = 1, \ldots, n$ and define $f' = f$, where f' will represent an intermediate result such that f equals f' minus the multiples of the g_i's subtracted from it in the previous steps.

First we check to see if $LT(g_1)$ divides $LT(f')$, if it doesn't, we check $LT(g_2) \mid LT(f')$, and continue checking the g_i's in order until $LT(g_i) \mid LT(f')$. Then, if we found an appropriate g_i, let $q = \frac{LT(f')}{LT(g_i)}$ and then subtract qg_i from f' and add q to a_i. If no g_i exists such that $LT(g_i)$ divides $LT(f')$, we instead subtract $LT(f')$ from f' and add $LT(f')$ to r.

We then repeat the process, taking what remains of f' and checking the g_i's in order, to find one whose leading term divides the leading term of f'. We then subtract

$g_i \frac{LT(f')}{LT(g_i)}$ from f' and add $\frac{LT(f')}{LT(g_i)}$ to a_i. If no g_i is found the leading term of f' is discarded and added to the remainder r

Then process is continued until f' reaches 0, a result guaranteed by the fact that the degree of f' is decreases as its leading terms are subtracted out. When finished r will consist of terms not divisible by any leading term of the g_i's. $\square$

Note that if the remainder is zero, then $f = a_1 g_1 + \ldots + a_s g_s \in (g_1, \ldots, g_s)$. So, given a set of generators of an ideal in a polynomial ring, we can use the division algorithm to test membership in the ideal. However, this test only gives a sufficient condition, it is not necessary for the remainder to be zero for f to be in the ideal generated by the g_i's.

Example 2.3.5. *Let $R = k[x, y]$ and set ordering to be graded lexicographical. Given $f = x^2 + xy + x - y$, $g_1 = x^2 + 1$, and $g_2 = y + 1$. First we compare the leading terms starting with g_1. Since $LT(g_1) = LT(f) = x^2$, $LT(g_1) \mid LT(f)$ and so set $q = \frac{x^2}{x^2} = 1$. Then $f' = x^2 + xy + x - y - 1(x^2 + 1) = xy + x - y - 1$ and $a_1 = 1$. Next since $g_1 \nmid f'$, we look to g_2. $q = \frac{LT(f'}{LT(g_2)} = x$, $f' = f' - xg_2 = xy + x - y - 1 - x(y + 1) = -y - 1$, and $a_2 = a_2 + q = x$. Finally, $g_1 \nmid f'$ again, so $q = \frac{-y}{y} = -1$, $f' = f' - (-1)(y + 1) = 0$, and $a_2 = a_2 + q = x - 1$. Since $f' = 0$, we finish with $f = a_1 g_1 + a_2 g_2 + r = 1(x^2 + 1) + (x - 1)(y + 1) + 0$.*

Theorem 2.3.6 (Dickson)**.** *Given a monomial ideal $I = (X^\alpha : \alpha \in \Lambda \subseteq \mathbb{N}^n)$, of $R = k[X_1, \ldots, X_n]$ there exists a finite set $\Lambda' \subset \Lambda$ such that $I = (X^\alpha : \alpha \in \Lambda')$. In particular, this means that any monomial ideal is finitely generated.*

Proof. In the case of $I \leq k[X]$ the ideal is generated by the monomial of the smallest degree. To show this, assume the monomial with the smallest degree in I is X^m. If X^m is not a generator of I then there must be an a generator $X^t \in I, t > 0$, such that $X^m = X^t f$ for some $f \in R$ with degree $s > 0$. Then $m = degree(X^m) = degree(X^t) + degree(f) = t + s$ so $m = t + s$ and $m > t$ so X^t has a degree less then X^m, which is a contradiction of our assumption.

Let us now treat the case of more than one variable. For the following, first fix the monomial ordering on $k[X_1, X_2, \ldots, X_n]$, $n > 1$. We will prove the theorem by induction on n, the number of indeterminates.

Assume that any monomial ideal of $n-1$ indeterminates is finitely generated by monomials. We must then prove that an monomial ideal $I \leq R = k[X_1, X_2, \ldots, X_n]$ is finitely generated. For the following paragraphs, monomials in $k[X_1, X_2, \ldots, X_n]$ will be written in the form $X^\alpha X_n^m$ or $X^\beta X_n^m$, where $\alpha, \beta \in \mathbb{N}^{n-1}$ and $m \in \mathbb{N}$.

Given a monomial ideal $I \leq R = k[X_1, X_2, \ldots, X_n]$, let $J \leq k[X_1, X_2, \ldots, X_{n-1}]$ be the monomial ideal generated by monomials X^α such that $X^\alpha X_n^m \in I$, for some $m \geq 0$. Then let $(X^{\alpha(1)}, \ldots, X^{\alpha(s)})$ be a finite set of generators for J, which must exist according to the induction hypothesis.

By the definition of J, for each $X^{\alpha(i)}$ there is an $m_i \geq 0, i = 1, \ldots, n-1$ such that $X^{\alpha(i)} X_n^{m_i} \in I$. If $m = max(m_i)$, then for each $p = 0, \ldots, m-1$, define $J_p \leq k[X_1, X_2, \ldots, X_{n-1}]$ as the ideal generated by monomials X^β, where $X^\beta X_n^p \in I$. By the induction hypothesis, each J_p is generated by a finite set of monomials $(X^{\alpha_p(1)}, \ldots, X^{\alpha_p(s_p)})$.

Now we show that I is generated by the union of the sets of monomials $\{X^{\alpha(i)}\}$, $i = 1, ..., s$ (the generators of J) and $\{X^{\alpha_p(i)}\}$, $p = 0, ..., m-1$ and $i = 1, ..., s_p$ (the combined set of generators of the ideals J_p, $p = 1, \ldots, m-1$). Given any monomial $X^\alpha X_n^q \in I$, consider two cases:

$q < m$: $X^\alpha X_n^q$ is divisible by some $X^{\alpha_q(j)} X_n^q, X^{\alpha_q(j)} \in J_q$, by construction of J_q, so it is divisible by $X^{\alpha_q(j)} \in J_q$.

$q \geq m$: $X^\alpha X_n^q$ is divisible by some $X^{\alpha(j)} X_n^m, X^{\alpha(j)} \in J$, by construction of J, so it is divisible by $X^{\alpha(j)} \in J$

This shows that I is generated by the set of monomials described. Relabeling these monomial generators, we can write that $I = (X^{\beta(1)}, \ldots, X^{\beta(r)})$ where r is a positive integer and $\beta(i) \in \Lambda$ for $1 \leq i \leq r$.

Now, using the set of generators just found, we will show that there is finite set $\Lambda' \subset \Lambda$ such that $I = (X^\alpha : \alpha \in \Lambda')$.

Since $X^{\beta(i)} \in I$, $i = 1, \ldots, r$, there exist $X^{\alpha(1)}, \ldots, X^{\alpha(r)}, \alpha(i) \in \Lambda$, such that $X^{\alpha(i)} \mid X^{\beta(i)}$, by Theorem 2.3.3. We must now show that $I = (X^{\beta(1)}, \ldots, X^{\beta(s)})$ equals $I' = (X^{\alpha(1)}, \ldots, X^{\alpha(s)})$.

To show this, we first note that each $f \in I$ can be written as $f = \sum a_i X^{\beta(i)}$, where $a_i \in R$. Since for all $\beta(i)$, $X^{\alpha(i)} \mid X^{\beta(i)}$, f can be written alternatively as $f = \sum c_i X^{\alpha_i}$, where $c_i = a_i X^{\beta(i) - \alpha(i)}$. So, $f \in I'$, and $I \subseteq I'$.

Also, the generators of I', $X^{\alpha(i)}$, $i = 1 \ldots s$, are in I, since each $\alpha(i)$ is in Λ, therefore $I' \subseteq I$ and $I = I'$. $\square$

Example 2.3.7. *Let I be a monomial ideal of $k[X_1, X_2]$ and let $I = (X^\alpha : \alpha \in \Lambda)$ where $\Lambda = \{(a, b) | b \geq 3 \text{ if } a = 2, b \geq 1 \text{ if } a \geq 3\}$. We will use the method outlined above to find a finite set of monomial generators for I. (Note that $I = (X_1^3 X_2, X_1^2 X_2^3)$, so $X_1^3 X_2$ and $X_1^2 X_2^3$ are the generators we should find.)*

Let $J = (X_1^t : X_1^t X_2^m \in I, \text{ for some } m \in \mathbb{N})$, so J is the monomial ideal containing all powers of X_1 found in elements of I. We can write $J = \{X_1^t : t \geq 2\}$, or $J = (X_1^2)$, since $X_1^2 X_2 \in I$. Next we define J_1 and J_2, by $J_1 = \{X_1^t : X_1^t X_2 \in I\} = \{X_1^t : t \geq 3\} = (X_1^3)$ and $J_2 = \{X_1^t : X_1^t X_2^2\} = \{X_1^t : t \geq 3\} = (X_1^3)$. We then have the following set of generators for I:

From J: $X_1^2 X_2^3$.

From J_1: $X_1^3X_2$.

From J_2: $X_1^3X_2^2$.

So we have a new set of generators, $I = (X_1^2X_2^3, X_1^3X_2, X_1^3X_2^2)$ for the monomial ideal I. It should be noted that this gives us one more generator, $X_1^3X_2^2$, which is redundant.

Theorem 2.3.8 (Hilbert). *Let k be a field and $R = k[X_1, X_2, \ldots, X_n]$ be a ring of polynomials over k. Any non-zero ideal of R is finitely generated.*

Proof. Fix a monomial ordering on R. For any non-zero ideal $I \leq R$, we define $LT(I)$ as the ideal generated by $\{LT(f) : f \in I\}$, the leading terms of I. Using Theorem 2.3.6, we can find a basis for $LT(I)$, using the leading terms from a finite number of polynomials $g_i \in I, i = 1 \ldots m$. So $(LT(g_1), LT(g_2), \ldots, LT(g_m)) = LT(I)$. Now by the division algorithm, there exists $a_i \in R$ such that $\sum_{i=1}^{m} a_i g_i + r = f$ where either $r = 0$ or r is a linear combination of monomials, none of which is divisible by the leading terms of the g_i's. However, since $f \in I$ and $g_i \in I, i = 1 \ldots, n$, then $f - \sum a_i g_i = r \in I$ and so $LT(r)$ must be in $LT(I)$. From Theorem 2.3.3, this implies that either $LT(g_i)$ divides $LT(r)$ for some g_i or $r = 0$. Since $LT(g_i) \nmid LT(r)$ for $i = 1, \ldots, m$, r must be 0. Then we have $f = \sum a_i g_i$ for any f in I and so I generated by the set $\{g_1, \ldots, g_m\}$. □

2.4 Modules

Definition 2.4.1. A *module, M,* is a commutative group associated with a ring, R, and a function $f : R \times M \mapsto M$ with the properties:

$$f(r, m_1 + m_2) = f(r, m_1) + f(r, m_2)$$

$f(r_1 + r_2, m) = f(r_1, m) + f(r_2, m)$

$f(r_1, f(r_2, m)) = f(r_1 r_2, m)$

$f(1, m) = m$

where $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$. If necessary to avoid confusion, a module M defined as above will be referred to as a R-module. For the rest of the paper $f(r, m)$ will be written as rm.

It should be noted that an ideal I of a ring R can be viewed as an R-module with $f(r, i) = r \cdot i$.

Example 2.4.2. *Let R be a ring. Factor rings of the form R/I where I is an ideal of R are examples of R-modules. As an illustration, the factor ring $k[x, y]/(x^2 - y^3)$ is a module over the ring $k[x, y]$.*

The following notation will be used to describe multiplication of rings and modules. Let M be a module and R a ring. If $x \in M$ and $r \in R$ then $rM = \{f : f = \sum rx_i, x_i \in M\}$ and $Rx = \{f : f = \sum r_i x, r_i \in R\}$.

Definition 2.4.3. A *Noetherian ring, R,* is a ring that satisfies the *ascending chain condition,* that is, every ascending chain of ideals,

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots,$$

will eventually terminate, that is, there exists a natural number n such that $I_n = I_{n+1} = \ldots$. A *Noetherian module* is a module that satisfies the ascending chain condition with respect to submodules.

Theorem 2.4.4. *Given a ring (or module) the following are equivalent.*

1. *The ring (or module) is Noetherian.*

2. *All ideals (or submodules) are finitely generated.*

3. *Any set of ideals (submodules) has a maximal element by set inclusion.*

The proof of this is omitted but can be found in [Lan71]. Note that from our previous work, a polynomial ring over a ring has all ideals finitely generated and so it is Noetherian.

Definition 2.4.5. An *Artinian ring, R,*is a ring that satisfies the *descending chain condition,* that is, every descending chain of ideals,

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots,$$

will eventually end, that is, there exists a natural number n such that $I_n = I_{n+1} = \dots$. An *Artinian module* is a module that satisfies the descending chain condition with respect to submodules.

Artinian rings (or modules) are in general more special than Noetherian rings (or modules). For example, it is know, by a result of Akizuki that Artinian rings are Noetherian (see [Mat86]), but the converse is not true. A polynomial ring with finitely many indeterminates over a field is Noetherian, but not Artinian.

2.5 Graded rings and modules

Definition 2.5.1. A ring R is said to be *graded* if it can be written as a direct sum of abelian groups,

$$R = \oplus_{i \in G} R_i,$$

where G is an abelian semigroup with identity and the indexed groups have the property that $R_iR_j \subseteq R_{i+j}$. The ring R is then said to be *G-graded.* An R-module, M, is *G-graded* if it can be written similar as a direct sum of abelian groups

$$M = \oplus_{i \in G} M_i,$$

where $R_iM_j \subseteq M_{i+j}$ for all $i, j \in G$.

The definition implies that each R_i and M_i are in fact R_0-modules.

Example 2.5.2. *Generally, $G = \mathbb{N}$, $G = \mathbb{Z}$, or $G = \mathbb{N}^n$ are used to define the gradings in the examples that are to follow. A simple example would be $k[X]$ with $\mathbb{N}$-grading.*

Definition 2.5.3. Given a graded module $M = \oplus_{i \in G} M_i$, a *homogenous element,* $m \in M$ is one such that $m \in M_i$ for some $i \in G$, and i is then called the *degree* of m. Given an element $x \in M$, its *homogenous components* are the homogenous elements $x_i \in M_i$ such that $x = \oplus_{i \in G} x_i$.

Definition 2.5.4. A *homogenous submodule (or ideal),* is one that is generated only by homogenous elements.

With the following example we observe that the homogenous submodules have similar properties to those shown to hold for monomial ideals.

Example 2.5.5. *Let k be a field. Define the natural $\mathbb{N}^n$-grading of $R = k[X_1, \ldots X_n]$, where $R = \oplus_{\alpha \in \mathbb{N}^n} R_\alpha$ and $R_\alpha = \{aX^\alpha : a \in k\}$. Then the homogenous elements of R are exactly the monomials of R. And so the homogenous ideals of a polynomial ring with the given grading are the monomial ideals.*

Theorem 2.5.6. *Let M be a G-graded module and N be a homogenous submodule of M.*

- *If $x \in N$ then each homogenous component of x is in N*
- $N = \oplus_{i \in G}(N \cap M_i)$, *that is, N is the direct sum of its projections onto the M_i's*

Given N a homogenous submodule of M, we will use the notation $N_i = N \cap M_i$. Also, if we are using an $\mathbb{N}$-grading, then we define the ideal $R_+ = \sum_{n>0} R_n$ and note that $R/R_+ \simeq R_0$.

Since for any $i \in G$, $R_0 R_0 \subset R_0$, $R_0 M_i \subset M_i$, we see that $R_0 \subset R$ is a subring of R and all M_i's are R_0-modules.

Definition 2.5.7. Given a ring R, *an ideal filtration* is defined as a descending chain of ideals, $R = J_0 \supset J_1 \supseteq J_2 \supseteq \ldots$, such that $J_n J_m \subset J_{n+m}$ for all n, m.

Definition 2.5.8. The graded ring, *$gr(R)$,* associated with a filtration of R, $R = J_0 \supseteq J_1 \supseteq \ldots$ is the $\mathbb{N}$-graded ring where its n^{th}-graded component is defined by $gr_n(R) := J_n/J_{n+1}$. Given $x \in J_m$ and $y \in J_n$, we define the product of $gr_m(R)$ and $gr_n(R)$,

$$(x + J_{m+1})(y + J_{n+1}) = (xy + J_{m+n+1}) \in gr_{m+n}(R).$$

This multiplication is well-defined and gives us a graded ring, $gr(R) = \oplus_{i \in \mathbb{N}} gr_n(R)$.

Given a particular filtration, $R \supseteq I \supseteq I^2 \supseteq \ldots$ where I is an ideal of R, we use the notation $gr_I(R)$ in place of $gr(R)$.

Theorem 2.5.9. *Let $R = \oplus_{i \in \mathbb{N}} R_i$ be an $\mathbb{N}$-graded ring. Then R is Noetherian if and only if R_0 is Noetherian and R is finitely generated module over R_0.*

Proof. First, if R is finitely generated over a Noetherian ring, R_0, then it is a Noetherian ring itself since it can be written as a factor ring of a polynomial ring over R_0 in finitely many indeterminates, so the reverse implication holds.

Now, if R is Noetherian, and R_+ is an ideal in R, the quotient $R_0 \simeq R/R_+$ is Noetherian as well. Also, let R be generated by a finite set of elements, which we denote by $x_1, \dots, x_r$. Now we show that each R_n is a finitely generated over R_0. If $n = 0$, this is trivial. Assume that for all $m < n$, R_m is a finitely generated over R_0, and we only need to show that R_n must be as well. Given any $y \in R_n \subset R_+$, let $y = s_1x_1 + s_2x_2 + \ldots + s_nx_r$, where $s_i \in R$, $i = 1, \dots, n$. For each x_i, let $d_i \in \mathbb{N}$ be the degree of x_i, that is $x_i \in R_{d_i}$. Note that for all degrees of x_i's, $d_i \geq 1$, and so $n - d_i < n$. For all $i = 1, \dots, r$, set $t_i \in R_{n-d_i}$ to be the homogenous component of s_i of degree $n - d_i$, with $t_i = 0$ when $d_i > n$. Note that, by the induction hypothesis, each R_{n-d_i} is finitely generated over R_0. Now $y = t_1x_1 + \ldots + t_rx_r$, is a finite sum of homogenous elements and in fact this shows that

$$R_n \subset \sum x_i R_{n-d_i},$$

where all R_{n-d_i} are finitely generated over R_0. So, we can conclude that R_n is finitely generated over R_0 as well. $\square$

Definition 2.5.10. The *length,* $\ell(M)$ of a module M is the length n of the longest ascending chain of its submodules: $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$. If there is no longest chain of submodules, then the length is defined as infinite.

Note that it is known that the length of a module M is finite if and only if M is both Artinian and Noetherian.

Theorem 2.5.11. *If k is a field, the length of a k-module M equals the dimension of M as a vector space over k.*

Example 2.5.12. *If we consider the grading of $R = k[X_1, \ldots X_r]$ by total degree of its monomials, we can compute the length of R_n by counting the number of distinct* *monomials with total degree n. That is*

$$\ell(R_n) = \binom{r+n-1}{r-1}.$$

Definition 2.5.13. Given a $\mathbb{Z}$-graded R-module M, the *Hilbert function of M, $H(M,n)$,* is equal to the length of the n-graded component regarded as an R_0-module. That is,

$$H(M,n) = \ell_{R_0}(M_n).$$

For a polynomial ring with only non-negative degrees, which would be a $\mathbb{N}$ graded module, we can consider the function equal to zero for all negative degrees. In other words, $H(M,n) = 0$, for all $n < 0$.

Definition 2.5.14. The *Hilbert series* of a graded $\mathbb{N}$ module M is defined as

$$HS(M,t) = \sum_{n=0}^{\infty} \ell(M_n)t^n,$$

which is in fact the generating function of the length of M_n.

The following theorem, stated without proof, demonstrates that the Hilbert series can be described as a rational function. In fact, this is the form we will be working with primarily.

Theorem 2.5.15. *Let R be an $\mathbb{N}$-graded Noetherian ring and M be a graded Noetherian R-module. If $x_1, \ldots, x_r$ generate R over R_0, denoted as $R = R_0[x_1, \ldots, x_r]$ with d_i being the degree of x_i, $i = 1, \ldots r$, then:*

$$HS(M,t) = \frac{f(t)}{\prod_{i=1}^{r}(1-t^{d_i})}.$$

Here $f(t)$ is a polynomial with integer coefficients.

The proof of this theorem can be found in [Mat86].

Theorem 2.5.16. *Given R, M, and x_i, as in Theorem 2.5.15 with all the x_i's of degree one, then the length of the R_0-module M_n can be defined as a polynomial function of n, $P(n)$, for n sufficiently large.*

Proof. Since all generators are of degree one, we have $HS(M,t) = \frac{f(t)}{(1-t)^1 \ldots (1-t)^1} = \frac{f(t)}{(1-t)^r}$. Then if $(1-t)$ divides $f(t)$ we can reduce this to $HS(M,t) = (1-t)^{-d} f'(t)$ where d is r minus the multiplicity of 1 in f. Note that $d \geq 0$ and since $f'(t)$ has no term of $(t-1)$, $d > 0$ implies $f(1) \neq 0$.

Let $f'(t) = \sum_{i=0}^{s} a_i t^i$ where $a_i \in \mathbb{Z}$. Then

$$HS(M,t) = (1-t)^d f'(t) = [\sum_{j=0}^{\infty} \binom{d+j-1}{d-1} t^j][\sum_{i=0}^{s} a_i t^i].$$

For large n, the coefficients of t^n in the expansion are

$$\ell(M_n) = \sum_{i=0}^{s} \binom{d+(n-i)-1}{d-1} a_i.$$

This can then be rewritten in terms of a polynomial of degree $d-1$ in the variable n.

$$\ell(M_n) = H(M,n) = \frac{a_0 + a_1 + \ldots + a_s}{(d-1)!} n^{(d-1)} + g(n) = \frac{f(1)}{(d-1)!} n^{(d-1)} + g(n),$$

where $g(n)$ is a polynomial of degree $d-2$. $\square$

Definition 2.5.17. The polynomial $HP(X) = H(M,X)$ such that $\ell(M_n) = H(M,n)$ for n sufficiently large is called the *Hilbert polynomial of M.*

Chapter 3

Prime Ideals, Localization, and Associated Primes

This chapter will discuss some basic concepts of commutative algebra, such as primes ideals, rings of fractions and the theory of associated primes, which will be relevant for the main part of the paper.

3.1 Local Rings

Definition 3.1.1. A *local ring* is a ring with only one maximal ideal. Some authors assume that a local ring is also Noetherian, but we will not make this assumption, although the rings discussed in this paper will generally be Noetherian.

In this chapter, we will discuss the localization of a ring R at a multiplicative set S. The outcome is a new ring $S^{-1}R$ and the elements of R map canonically to $S^{-1}R$ such that elements of S are sent, under this mapping, to invertible elements of $S^{-1}R$. The main example to keep in mind is the localization of $\mathbb{Z}$ at the set of nonzero integers. In this case we obtain the field of rationals, $\mathbb{Q}$, that naturally contains $\mathbb{Z}$.

When the multiplicative set is the complement of a prime ideal, then we obtain a local ring, a setting which usually offers us more tools of investigation.

We will first remind the reader the following fact about prime and maximal ideals.

Definition 3.1.2. An ideal I of a ring R is *prime* if $I \neq R$ and for any $a, b \in R$, $ab \in I$ implies that a or b is in I.

Theorem 3.1.3. *Given a ring R and prime ideals $P, P_i \leq R, i = 1 \ldots n$, such that $\cap_{i=1}^{n} P_i \subseteq P$, there exists i such that $P_i \subseteq P$.*

Proof. Assume that for every P_i, $P_i \nsubseteq P$. Then for each P_i there exists $x_i \in P_i$ such that $x_i \notin P$. However, $x = \prod_{i=1}^{n} x_i$ must be contained in every P_i and so $x \in \cap_{x=1}^{n} P_i \subseteq P$. Then $x \in P$, but since P is prime this results in a contradiction. $\square$

The following result is well-known and is included here without a proof.

Theorem 3.1.4. *Given a prime ideal, P, and a maximal ideal, M, of a ring R, the factor rings R/P and R/M, are an integral domain and a field, respectively. Also, since a field is an integral domain, a maximal ideal is prime.*

It is easy to check that a ring is an integral domain if and only if the zero ideal is prime. It is worth mentioning that maximal ideals (hence prime ideals) do exist in any ring (integral or not). This is due to the following result by Krull.

Theorem 3.1.5. *Let R be a commutative ring and I a proper ideal in R. Then I is contained in a maximal ideal of R.*

Definition 3.1.6. A subset $S \subset R$ containing the identity and closed under multiplication is called a *multiplicative subset.*

Examples of multiplicative subsets include:

1. The set of units of R.

2. The complement of a prime ideal P, in R. Let $S = R \setminus P$. For any $a, b \in S$, a and b are not in P. Therefore ab must not be in P, and so it is in S.

3. The set $S = \{1, x, x^2, \ldots\} = \{x^n : n \in \mathbb{N}\}$ for all non-zero $x \in R$.

Theorem 3.1.7. *Given a multiplicative subset S of a ring R, with $0 \notin S$, the set $R \setminus S$ contains a maximal ideal of R which is prime.*

Proof. Zorn's lemma guarantees the existence of a maximal element, I, in the set of ideals contained in $R \setminus S$. If this ideal is not prime, there exists $a, b \in R$ but not in I, such that $ab \in I$. Then (a, I) and (b, I) are ideals containing I. But since I is maximal in $R \setminus S$, (a, I) and (b, I) contain elements of S. Then there exists $x, y \in S$ such that $x \in (I, a)$ and $y \in (I, b)$ such that

$$x = am_1 + i_1n_1 \text{ and } y = bm_2 + i_2n_2,$$

where $m_1, m_2, n_1, n_2 \in R$ and $i_1, i_2 \in I$. Then, by the fact that S is a multiplicative subset,

$$S \ni xy = abm_1m_2 + i_2n_2am_1 + i_1n_1bm_2 + i_1i_2n_1n_2.$$

However, each of the terms is a multiple of I, so $xy \in I$ which contradicts $I \subseteq R \setminus S$. Therefore, a or b must be an element of I and so I is prime. $\square$

Theorem 3.1.8. *Let R be a ring and $\mathfrak{m}$ a maximal ideal of R. Then the following assertions are equivalent:*

1. R is local:

2. $R \setminus \mathfrak{m}$ consists of units of R;

3. *the set of non-units equals* $\mathfrak{m}$*;*

4. *if* a, b *are non-units, then* $a + b$ *is a non-unit.*

Proof.

- (1) $\Rightarrow$ (2) For any $b \in R \backslash \mathfrak{m}$, assume b is a non-unit. Then (b) is a proper ideal in R and thus is a subset of $\mathfrak{m}$, the sole maximal ideal. This implies $b \in \mathfrak{m}$ which gives a contradiction.

- (2) $\Rightarrow$ (3) If $\mathfrak{m}$ contains a unit of R then $\mathfrak{m} = R$ which contradicts m being maximal. Therefore all units must be in $R \setminus \mathfrak{m}$ and so $\mathfrak{m}$ is the ideal composed of all non-units of R.

- (3) $\Rightarrow$ (4) This follows from the definition of an ideal as a commutative subgroup of R.

- (4) $\Rightarrow$ (1) Assume $\mathfrak{n} \neq \mathfrak{m}$ is another maximal ideal in R. Take an element $b \in \mathfrak{n}$ such that $b \notin \mathfrak{m}$ and note that it is a non-unit of R and $(b) \leq \mathfrak{n}$. Then $\mathfrak{m} \subset (\mathfrak{m}, b) = R$ since $\mathfrak{m}$ is maximal. Therefore there must be some element of $(\mathfrak{m}, b)$ that is a unit of R. However, this would imply that there exists $a \in \mathfrak{m}$ and $r \in R$ such that $a + rb$ is a unit which contradicts (4). Therefore $\mathfrak{m}$ is the only maximal ideal in R.

$\square$

Example 3.1.9. *The ring* $\mathbb{C}[[x]]$*, the power series with complex coefficients, is local with maximal ideal* (x)*. We can show* (x) *is the only maximal ideal in* $\mathbb{C}[[x]]$ *by checking that the set of all non-invertible elements is equal to* (x)*.*

Definition 3.1.10. The *equivalence relation between elements of fraction ring,* $S^{-1}R$*,* is defined for any $a, c \in R$ and $b, d \in S$ as: $(a, b) \sim (c, d)$ if and only if there exists an $s \in S$ such that $s(ad - bc) = 0$. The equivalence class of the pair (a, b) will be denoted as a/b.

Note that given a multiplicative subset S of R, and an ideal $I \leq R$, if $x \in S \cap I$ then both $\frac{x}{1}$ and its inverse, $\frac{1}{x}$, are in $IS^{-1}R = S^{-1}I$. Therefore $S^{-1}I$ is not a proper ideal in $S^{-1}R$.

Definition 3.1.11. Given a prime ideal P, let $S = R \setminus P$. The *localization* of R at P is defined by the ring of fractions $S^{-1}R = \{\widehat{(r,s)} = r/s : r \in R, s \in S\}$. $S^{-1}R$ will be denoted as R_P and elements $\widehat{(a,b)}$ will be written as $\frac{a}{b}$. R is imbedded in R_P by the inclusion mapping $\iota(r) = \frac{r}{1}$.

It should be noted that R is not necessarily an integral domain and this is why equivalency between two elements, $\frac{a}{b}$ and $\frac{c}{d}$, is defined by the relationship $s(ad-bc) = 0$ rather then $ad - bc = 0$, where s is an element of S.

Similarly, for an R-module M, the fraction ring $S^{-1}M$ is the $S^{-1}R$-module of equivalence classes defined as for rings. That is for any $x, y \in M$ and $s, t \in S$, $\frac{x}{s} = \frac{y}{t}$ iff $u(tx - sy) = 0$ for some $u \in S$. Also for a prime ideal P, $M_P = (R \setminus P)^{-1}M$.

We define the inclusion mapping, $\iota : R \hookrightarrow S^{-1}R$, as $\iota(r) = \frac{r}{1}$ for every r in R.

Theorem 3.1.12. *The inclusion mapping is a homomorphism of rings and, if S contains no zero divisors, it is injective.*

Proof. For any $a, b \in R$ and any $n \in \mathbb{N}$,

1. $\iota(a) + \iota(b) = \frac{a}{1} + \frac{b}{1} = \frac{a1+b1}{1} = \frac{a+b}{1} = \iota(a+b)$
2. $\iota(a)\iota(b) = \frac{a}{1}\frac{b}{1} = \frac{ab}{1} = \iota(ab)$
3. $n\iota(a) = n\frac{a}{1} = \frac{na}{1} = \iota(na)$

Also, if an element $a \in R$ is in the kernel of ι, then $\frac{0}{1} = \iota(a) = \frac{a}{1}$ and so for some $s \in S$, $0 = s(a - 0) = sa$ so either $a = 0$ or s is a zero-divisor. But, if S has no zero-divisors, then the kernel of ι must be zero, and ι is injective. $\square$

We will be referring to the mapping of ideals from R to $S^{-1}R$ in a similar manner as we have for elements. However, when we write $I' = \iota(I)$ for a given ideal $I \leq R$, I' will be the ideal generated by I's corresponding elements in $S^{-1}R$ rather then just the mapped elements. That is,

$$\iota(I) = (\frac{i}{1} : i \in I).$$

Theorem 3.1.13. *Given a prime ideal P and the multiplicative subset $S = R \setminus P$, for any ideal I in R, the image of I in R_P is an ideal of R_P, specifically $\iota(I) = IR_P = S^{-1}I$.*

Proof. First of all, for all ideals $I \leq R$, $IR_P = \{\iota(i)\frac{r}{s} = \frac{ir}{s} = \frac{i'}{s} : i, i' \in I, r \in R, s \in S\} = S^{-1}I$. Also, given any $\frac{a}{s_1} \in R_P$ and $\frac{i}{s_2} \in S^{-1}I$, $\frac{a}{s_1}\frac{i}{s_2} = \frac{ai}{s_1 s_2} \in S^{-1}I$. □

Definition 3.1.14. The *spectrum* of a ring R, denoted $Spec(R)$ is the set of all prime ideals in R.

The following result describes the spectrum of R_P.

Theorem 3.1.15. *All proper ideals Q of R_P are of the form $Q = \{\frac{i}{s} : i \in I\} = S^{-1}I$, where I is an ideal contained in P.*

In particular, R_P is a local ring with PR_P as the maximal ideal.

Proof. First we show that if Q is an ideal of R_P, then $I = Q \cap R$ is an ideal of R. Let a and b be elements of I, then $\frac{a}{1}, \frac{b}{1} \in Q$ and $\frac{a+b}{1} = \frac{a}{1} + \frac{b}{1} \in Q$. So $a + b \in Q \cap R = I$ and we have that I is a subgroup. Now for any $r \in R$, $\frac{ra}{1} = \frac{r}{1}\frac{a}{1} \in Q$, so $ra \in I$ and $I = Q \cap R$ is an ideal of R.

Since the units of R_P are its invertible elements, they are of the form $\frac{s}{s'}, s, s' \in S = R \setminus P$. Therefore, an ideal in R_P with any elements of this form, that is with a numerator not in P, is equal to R_P and so not a proper ideal. □

3.2 Associated Primes

Definition 3.2.1. The *radical* of an ideal $I \leq R$ is the set of all elements in R that have a power in I. In other words, $rad(I) = \sqrt{I} = \{a \in R : a^n \in I, \text{ for some } n \in \mathbb{N}\}$.

Note that the radical of an ideal is itself an ideal.

Definition 3.2.2. Given an ideal I of a ring R, I is a *primary ideal* if for any $a, b \in R$, if $b \notin I$, $ab \in I$ implies $a^n \in I$ for some $n \geq 1$. In other words, a must lie in the radical of I, if b is not in I.

Definition 3.2.3. An ideal $I \leq R$ is called the *annihilator* of an element $m \in M$ if $I = \{i \in R : im = 0\}$. This is denoted by $I = Ann_R(m)$. If N is a submodule of M, the *annihilator of* N is the ideal I of elements that annihilate every element of N, that is $i \in I$ implies $in = 0$ for all elements $n \in N$. We denote $I = Ann_R(N)$.

Definition 3.2.4. Given a ring R and a module M over R, a prime ideal $P \leq R$ is called an associated prime if there exists some $m \in M$ such that $P = Ann_R(m)$. The set of all associated primes in M is denoted by $Ass_R(M)$.

Note that in terms of rings, an associated prime of R is defined the same way if it is viewed as a module over itself.

Example 3.2.5. *Let $R = k[x_1, x_2]/(x_1x_2)$ be a polynomial ring. Then (x_1) is an associated prime of R.*

Example 3.2.6. *We can view the cyclic group of ab elements, where a and b are prime, $\mathbb{Z}_{ab}$ as a $\mathbb{Z}$-module. Then (a) and (b) are prime ideals, and $(a) = ann_{\mathbb{Z}}(b)$ and $(b) = ann_{\mathbb{Z}}(a)$. So a and b generate associated primes of $\mathbb{Z}_{ab}$.*

Theorem 3.2.7. *A prime $P \in R$ is an associated prime of a module M if and only if there exists an R-linear, linear as a transformation of R-modules, injection, $R/P \hookrightarrow M$.*

Proof. Start with the assumption that P is an associated prime of M. We must now define an injection $i : R/P \to M$. We claim that the function $i(\hat{a}) = am$ satisfies this, where $m \in M$ is an element annihilated by P. We first show that the mapping is well defined over the cosets of R/P. Given two equivalent cosets $\hat{a}$ and $\hat{b}$ in R/P, where $a \neq b$, we have $\hat{a} = \hat{b}$ which implies $a - b \in P$. Then $a - b$ must annihilate m, so $(a-b)m = 0$ and so $am = bm$. Next we must show this is one to one. Given $a, b \in R$ we must show if $ma = mb$ then $\hat{a} = \hat{b}$. Since $ma = mb$ implies $0 = ma - mb = m(a-b)$, $a - b$ annihilates m. Therefore $(a - b) \in P$ and so $\hat{a} = \hat{b}$.

Now assume $i : R/P \hookrightarrow M$ is an injection, using the same mapping defined above. Now we must demonstrate P is an associated prime of M. Note that since i is injective, $i(\hat{a}) = ma = 0$ if and only if $\hat{a} = 0$, that is $ma = 0$ if and only if $a \in P$. Therefore P annihilates m and $P \in Ass_R(M)$. □

Lemma 3.2.8. *Given $b \in R$ and $m \in M$ such that $bm \neq 0$, $Ann(m) \subseteq Ann(bm)$.*

Proof. If $a \in Ann(m)$, then $am = 0$ and $a(bm) = (ab)m = (ba)m = b(am) = b0 = 0$. Therefore $a \in Ann(bm)$. □

For the following theorem, we define $\mathcal{P} = \{Ann(m) : m \in M\}$ as the set of annihilators of each element of M.

Theorem 3.2.9. *The maximal elements of $\mathcal{P}$ are prime.*

Proof. If I is an ideal that is maximal in $\mathcal{P}$, then there exists $m \in M$ such that $I = Ann(m)$. Assume that there exists elements $a, b \in R$ such that $ab \in I$ and $b \notin I$. Then $Ann(m) \subseteq Ann(bm)$, according to the above lemma. Since $I = Ann(m)$ is maximal in $\mathcal{P}$, $Ann(m) = Ann(bm)$. □

Theorem 3.2.10. *If M is a nonempty module over a Noetherian ring R, then $Ass_R(M) \neq \emptyset$. Also, every zero divisor of M lies in an associated prime of M.*

That is:

$$ZD(M) = \bigcup_{P \in Ass_R(M)} P$$

where $ZD(M)$ is the set of all zero divisors of M.

Proof. If $a \in R$ is a zero divisor of M, then there exists an $I \leq R$ such that $a \in I = Ann(m)$. Since M is Noetherian, there exists a $J \leq R$ maximal in $\mathcal{P}$ that contains I, which, by the last theorem, is prime. Then $a \in I \subseteq J \subseteq \bigcup_{P \in Ass_R(M)} P$.

Conversely, if $a \in \bigcup_{P \in Ass_R(M)} P$, then a is an element of some associated prime in R. Therefore there exists some $0 \neq m \in M$ such that $am = 0$. So $a \in ZD(M)$. $\square$

Definition 3.2.11. A diagram of homomorphisms, $A \rightarrow^i B \rightarrow^j C$, is called an *exact sequence* if the image of i is equal to the kernel of j. A *short exact sequence,* $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, is a set of three exact sequences where the map from 0 to A maps to the zero elements of a and the map from C to 0 maps all of C to zero. This implies the homomorphism from A to B is injective, and the homomorphism from B to C is surjective.

Theorem 3.2.12. *If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a short exact sequence, then $Ass(M') \subseteq Ass(M) \subseteq Ass(M') \cup Ass(M'')$. Also, if $M = M' \oplus M''$ then $Ass(M) = Ass(M') \cup Ass(M'')$.*

Proof. Let $i : M' \rightarrow M$ be the mapping from M' to M and $j : M \rightarrow M''$ be the mapping from M to M''. Given $I \in Ass(M')$, there exist an $m \in M'$ such that $Im = 0$. Note that $i(m) \in M$ and $Ii(m) = i(Im) = i(0) = 0 \in M$ so $I \in Ass(M)$.

For the second inclusion, assume $I \in Ass(M)$. Note that since i is an injection, we can consider $M' \subseteq M$. Then we have two cases, $I \in Ass(M')$ or $I \in Ass(M) \setminus Ass(M')$. If $I \in Ass(M')$, then $I \in Ass(M') \cup Ass(M'')$ and we are done. On the other hand, if $I \in Ass(M) \setminus Ass(M')$, we must show that there is a non-zero element of M'' that I annihilates. This can be found by looking in the inverse image of j, that is, by finding an element not in the kernel of j, whose projection is annihilated by I. Consider the factor group $M/kerj \simeq M/M'$ and an element $m \in M \setminus M'$ that I annihilates. Then $0 \neq \hat{m} \in M/M'$ is mapped into M'' by

$$j(\hat{m}) = j(m + M') = j(m) + j(M') = j(m) + j(kerj) = j(m).$$

Then $Ij(m) = j(Im) = j(0) = 0 \in M''$. So $I \in Ann(M'') \subseteq Ann(M') \cup Ann(M'')$.

The second part can be proved by noting that if $M = M' \oplus M''$, then the order of the exact sequence can be reversed. That is,

$$0 \to M' \to M' \oplus M'' \to M'' \to 0 \Longleftrightarrow 0 \to M'' \to M' \oplus M'' \to M' \to 0$$

So, $Ass(M'')$ is also a subset of $Ass(M)$ which means $Ass(M'') \cup Ass(M') \subseteq Ass(M) \cup Ass(M) = Ass(M)$. Therefore $Ass(M) = Ass(M') \cup Ass(M'')$. $\square$

Theorem 3.2.13. *If M is a finitely generated $R-$ module, and R is Noetherian, then there exists a chain of submodules,*

$$0 = M_0 \subseteq M_1 \ldots \subseteq M_n = M$$

where $M_i/M_{i-1} \simeq R/P_i$, $P_i \in Spec(R)$.

Proof. Since R is Noetherian, R must have at least one associated prime P_1. Then $R/P_1 \hookrightarrow M/M_0 = M$. We then define M_1 as the image of the injection, and so $M_1/M_0 \simeq R/P_1$. Note that M/M_1 is Noetherian, therefore there is a $P_2 \in Ass(M/M_1) \subseteq Spec(R)$. And so $R/P_2 \hookrightarrow M/M_1$ and let M_2 be the image of this injection. Continuing this for each subsequent i, pick $P_i = Ass(M/M_{i-1})$ and let M_i

be the image of R/P_i in M/M_{i-1}. This ascending chain of submodules will eventually terminate since M is Noetherian. $\square$

Lemma 3.2.14. *Let $S \subseteq R$ be a multiplicative subset and $P \leq R$ a prime ideal. Then $P \cap S \neq \emptyset$ if and only if $S^{-1}R = S^{-1}P$. In other words, $S^{-1}P$ is a proper ideal if and only if $P \subseteq S^C$.*

Proof. Given a prime $P \in Ass_R(N)$, if $P \cap S \neq \emptyset$, then there exists a $p \in P \cap S$ such that $\frac{p}{1} \cdot \frac{1}{p} = \frac{1}{1}$ and so $S^{-1}P = S^{-1}R$. Also if $S^{-1}P = S^{-1}R$, then $\frac{1}{1} = \frac{p}{s} \in S^{-1}P$ for some p in P and s in S. Then there exists $u \in S$ such that, $u(s-p) = 0$, which implies u or $(s-p)$ is in P and so u or s is in $P \cap S$. This implies $P \cap S$ is non-empty. $\square$

Theorem 3.2.15. *Given given a multiplicative set $S \subseteq R$ and a $S^{-1}R$-module N, $Ass_R(N) \rightleftharpoons Ass_{S^{-1}R}(N)$. That is,*

$$P \in Ass_R(N) \Rightarrow PS^{-1}R = S^{-1}P \in Ass_{S^{-1}R}(N),$$

and conversely,

$$P \in Ass_{S^{-1}R}(N) \Rightarrow R \cap P \in Ass_R(N)$$

Proof. We first show that a prime in $Ass_{S^{-1}R}(N)$ corresponds to a prime in $Ass_R(N)$. Take an ideal $P \in Ass_{S^{-1}R}(N)$, then P annihilates some non-zero $x \in N$. We claim that the annihilator of x in R is $R \cap P = \{r \in R : \frac{r}{1} \in P\}$. Let $r \in Ann_R(x)$, then $rx = 0$ and so $\frac{r}{1}x = \frac{rx}{1} = 0$. So $\frac{r}{1} \in P$ and $r \in R \cap P$. On the other hand, if $r \in R \cap P$, $\frac{r}{1} \in P$. Then $\frac{r}{1}x = 0$ and so $rx = 0$. Therefore, r is in $Ann_R(x)$ and $R \cap P = Ann_R(x)$.

We next show that a prime in $Ass_R(N)$ corresponds to a prime in $Ass_{S^{-1}R}(N)$. Assume now that $P \in Ass_R(N)$, then there exists $x \in N$ such that $P = Ann_R(x)$.

Since $P \in Ass_R(N) \subseteq Spec(R)$ then $S^{-1}P \in Spec(S^{-1}R)$. Also, given $\frac{p}{1} \in S^{-1}R$, $\frac{p}{1}x = \frac{px}{1} = 0$. So $S^{-1}P$ is in $Ann_{S^{-1}R}(x)$. Therefore $S^{-1}P$ is in $Ass_{S^{-1}R}(N)$. $\square$

Theorem 3.2.16. *If R is Noetherian, S a multiplicative subset of R, and M is an R-module, then $Ass_R(S^{-1}M) = Ass_R(M) \cap Spec(S^{-1}R)$. That is, the set of associated primes in R of $S^{-1}M$ is equal to the set associated primes of M that map to prime ideals in $S^{-1}R$.*

Proof. Let $P \in Ass_R(S^{-1}M)$. We must show that $P \in Ass_R(M)$ and $S^{-1}P$ is prime in $S^{-1}R$. Since $P \in Ass_R(S^{-1}M)$, there exists a $y \in S^{-1}R$ such that $P = Ann_R(y)$. Let $y = \frac{x}{s}$ where x is in M and s is in S. Note that $P = Ann(\frac{x}{1})$ since for all $p \in P$, $0 = p\frac{x}{s} \Leftrightarrow$ there exists $u \in S$ such that $u(px - s \cdot 0) = upx = 0 \Leftrightarrow 0 = p\frac{x}{1}$. Then for any $p \in P$, $ps \in P$, so $0 = ps\frac{x}{s} = p\frac{sx}{s} = p\frac{x}{1}$. This implies there exist a $u \in S$ such that $upx = 0$. Since R is Noetherian, $P = (p_1, p_2, \ldots, p_n)$. Using the previous result we define u_i's such that $u_i p_i x = 0$. We can show that if $u = \prod_{i=1}^n u_i$, P is the annihilator of $ux \in M$. First, given $p = \sum_{i=1}^n r_i p_i \in P$, with the r_i's in R, $pux = ux\sum_{i=1}^n r_i p_i = \sum_{i=1}^n r_i(u_i p_i x)u/u_i = \sum_{i=1}^n r_i 0u/u_i = 0$. So $P \subseteq Ann_R(ux)$. Next, if $q \in Ann_R(ux)$, then $qux = 0$ and $q\frac{x}{1} = \frac{qx}{1} = \frac{uqx}{u} = 0$. So $q \in Ann_R(\frac{x}{1}) = P$.

Now let $P = Ass_R(M) \cap Spec(S^{-1}R)$. Then $PS^{-1}R = S^{-1}P$ is a prime ideal in $S^{-1}R$, which also implies $S \cap P = \emptyset$. Also, there exists an $x \in M$ such that $P = Ann_R(x)$. We then have $\frac{x}{1} \in S^{-1}M$, and for any $p \in P$, $p\frac{x}{1} = \frac{px}{1} = \frac{0}{1} = 0$. So P is a subset of $Ann_R(\frac{x}{1})$. Now we show that for any $q \in Ann_R(\frac{x}{1})$, q is in P. Since $q\frac{x}{1} = \frac{qx}{1} = 0$, there exists a $u \in S$ such that $u(qx - 0) = uqx = 0$. And so $uq \in Ann_R(x) = P$, a prime ideal. This means that since $u \notin P$, $q \in P$ and therefore $Ann_R(\frac{x}{1}) \subseteq P$. $\square$

3.3 Support

Definition 3.3.1. Given a prime $P \leq R$, P is said to be the *support* of an R-module M if $M_P \neq \emptyset$. We define $Supp(M)$ as the set of all primes supporting M.

Theorem 3.3.2. *If R is Noetherian, and M a finitely generated R-module, then the following are true:*

1. $Ass(M) \neq \emptyset$.

2. $Ass(M) \subseteq Supp(M)$.

3. $min(Ass(M)) = min(Supp(M))$*, where* $min(\mathcal{P})$ *is the set of elements of* $\mathcal{P}$ *that are the minimal elements by set inclusion.*

Proof. The first part was shown earlier. For the second part, let $S = R \setminus P$ and note $M_P = S^{-1}M \neq 0$ if only if for some $m \in M$, $Ann(x) \subseteq P$. This is true since for all $x \in M$, $\frac{x}{s} = 0$ if and only if $ux = 0$ for some $u \in S$. This means if P is an associated prime of M, then it must support M, since there exists some $x \in M$ such that $\frac{x}{1} \in S^{-1}R$ is non-zero.

For the last part, if $P \in Ass(M)$ and is minimal in $Supp(M)$ then it is must be minimal in $Ass(M)$, since $Ass(M) \subseteq Supp(M)$. So we must show that $P \in min(Supp(M))$ is in $Ass(M)$. Since P supports M, $M_P \neq \emptyset$. Then there exists $Q \subseteq P$ such that $QR_P \in Ass_{R_P}(M_P)$. So Q supports M is well and since P is minimal, $P = Q$. $\square$

We now give the necessary and sufficient conditions for an ideal to be in the support of M.

Theorem 3.3.3. *If M is a Noetherian R-module then its support is the set of all* *primes in R that contain the annihilator of M. That is,*

$$Supp(M) = \{P : P \in Spec(R), Ann(M) \subseteq P\}.$$

Proof. First we take $Q \leq R$, a prime containing the annihilator of M. Then $S = R \backslash Q$ does not contain any element of the annihilator. Suppose $M_Q = 0$, then for all $x \in M$, $0 = \frac{0}{1} = \frac{x}{1} \in M_Q$. And so for each $x \in M$ there exists a $u \in S$ such that $0 = u(x - 0) = ux$. Since M is Noetherian, there is a finite set of generators of M, $\{x_1, \ldots, x_n\}$, and therefore every $x \in M$ can be written as a linear combination of these generators. That is,

$$\forall x \in M, x = \sum_{i=1}^{n} r_i x_i,$$

where, for $i = 1, \ldots, n$, $r_i \in R$. Since for each x_i there is a $u_i \in S$ that annihilates it, we can construct an element of $Ann(M)$ by taking the product of the u_i's, $u = \prod_{i=1}^{n} u_i$, and noting that it annihilates every term in the above summation. However, u is also a finite product of elements of S, a multiplicative set, and therefore must also be in S. This contradicts the fact that S contains no element of the annihilator and so $M_Q \neq 0$ and $Q \in Supp(M)$.

Now we show that if $Q \in Supp(M)$, then it must be prime and contain the annihilator of M. Let $S = R \setminus Q$. Since $M_Q \neq 0$, there exists an $x \in M$ such that for all $s \in S$, $sx \neq 0$. Therefore $Ann(x) \cap S = \emptyset$ and so, $Ann(x) \subseteq Q$. Since $Ann(M) \subseteq Ann(x)$ we then have $Ann(M) \subseteq Q$. Also, note that if Q is not prime, then S is not a multiplicative set, since for some $a, b \in S$, $ab \in Q = R \setminus S$.

□

We will close this chapter with the following set of definitions.

Definition 3.3.4. The *height* of a prime ideal P in R, is the length n of the longest chain of prime ideals, $P_0 \subset P_1 \subset \ldots \subset P_n = P$, that it contains. The height of an arbitrary ideal I is defined as the infimum over the heights of all prime ideal P containing I.

Definition 3.3.5. The *(Krull) dimension* of a ring R, is the supremum of the heights of all prime ideals P in R. When R is local, the dimension of R equals the height of its maximal ideal.

Further information can be found in [Mat86] as well as most of the other texts in bibliography.

Chapter 4

Stanley-Reisner Rings and Graph Ideals

4.1 Stanley-Reisner rings

Definition 4.1.1. A *simplicial complex* Δ, is a collection of subsets, called faces or simplices, of a finite set of vertices, $p_1 \ldots, p_n$, with the property that if one subset is in the complex, then all sets contained in that subset are in the complex. That is, $\tau \in \Delta \Rightarrow \forall \sigma \subseteq \tau$, $\sigma \in \Delta$.

A simplex (face) with vertices $\tau = \{p_{i_1}, \ldots, p_{i_m}\}$ will be alternately identified as $(i_1, \ldots, i_m)$, an element of $\mathbb{N}^n$. We also will associate it with a monomial, X^τ, in $k[X_1, \ldots, X_n]$ such that $X^\tau = \prod_{i \in \tau} X_i$.

Definition 4.1.2. A monomial is said to be *supported* on a set S if its indeterminates correspond to the points of the set S.

This is distinct from the support of a module and should be clear from the context.

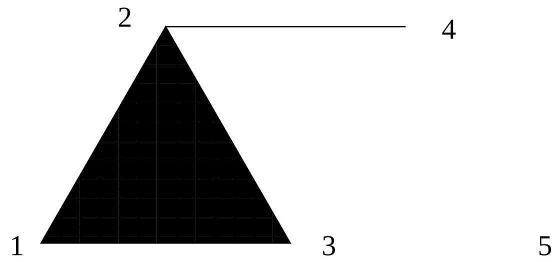


Figure 4.1: Simplicial Complex (Example 4.1.7)

Example 4.1.3. *The simplex* $\tau = \{v_1, v_4, v_5\}$ *supports the monomials* $X_1X_4X_5$, $X_1^3X_4X_5^2$, *and* $X_1^{\alpha_1}X_4^{\alpha_4}X_5^{\alpha_5}$ *where* $\alpha_1, \alpha_4, \alpha_5 \geq 1$.

Definition 4.1.4. A *facet* of a complex is a maximal face of the complex.

Definition 4.1.5. The *dimension, dim*(τ), of a simplex of p points is $p-1$. In geometric terms, it is the smallest dimension that can contain p linearly independent points. The *dimension of a simplicial complex* is the dimension of its largest facet.

We now define a ring associated with a given simplicial complex.

Definition 4.1.6. The *Stanley-Reisner ideal* I_Δ, of a complex of n points, Δ, is the ideal in $k[X_1, \ldots, X_n]$ generated by the monomials associated with faces not in Δ. That is, $I_\Delta = (X^\sigma : \sigma \notin \Delta)$. The *Stanley-Reisner ring,* $k[\Delta]$, is the factor ring $k[X_1, \ldots, X_n]/I_\Delta$.

Example 4.1.7. *Given a collection of points,* $S = \{p_1, p_2, p_3, p_4, p_5\}$, *define* Δ *as a collection of subsets of* S *such that*

$$\Delta = \{\{p_1,p_2,p_3\},\{p_1,p_2\},\{p_2,p_3\},\{p_1,p_3\},\{p_2,p_4\},\{p_1\},\{p_2\},\{p_3\},\{p_4\},\{p_5\}\}.$$

Then $I_\Delta = (X_1X_5, X_1X_4, X_2X_5, X_3X_4, X_3X_5, X_4X_5)$.

4.2 Support

The following theorems provide the conditions for an element to be in the support of a Stanley-Reisner ring.

Theorem 4.2.1. *I_Δ can be written as the intersection of prime ideals corresponding to the complements of the facets of Δ. That is,*

$$I_\Delta = \bigcap_{\tau \in facet(\Delta)} (X_i : i \in \bar{\tau}).$$

Proof. First we show that I_Δ is contained in every ideal $(X_i : i \in \bar{\tau})$, $\tau \in \Delta$. Given a monomial, X^α in I_Δ, if its corresponding set of points, α, is contained entirely in a facet of Δ then it must represent a face of Δ which would contradict the definition of I_Δ. Therefore for each facet, τ of Δ, α has at least one point, say p_i, not in it. And so X^α is in the ideal $(X_i) \subseteq (X_j : j \in \bar{\tau})$ for every facet τ of Δ.

If X^α is in $\bigcap_{\tau \in Facet(\Delta)}(X_i : i \in \bar{\tau})$, then for each $\tau \in \Delta$ there must be a point p_j such that $X_j \mid X^\alpha$ and X_j is a generator of $(X_i : i \in \bar{\tau})$. In other words, for every facet of Δ, α has at least one point outside of it. If $\tau_1, \ldots, \tau_m$ is the set of facets of Δ, then we define a set of points $q_1, \ldots, q_m$ such that each q_i is a point in α that is not in τ_i. Let $\beta \subseteq \Delta$ represent the unique q_i's, and note that $X^\beta \mid X^\alpha$. We now must show X^β is in I_Δ by showing that $\beta \notin \Delta$. If β is in Δ then it must be contained in some facet of Δ, and so all points in β are part of that facet which contradicts the construction of β. Therefore $X^\beta \in I_\Delta$ and consequently $X^\alpha \in I_\Delta$. $\square$

Note that each ideal in the intersection above is generated by monomials of order 1. Therefore the ideals are prime in $k[X_1, \ldots, X_p]$, where p is the number of points in the complex.

Theorem 4.2.2. *The ideals in the intersection above, $\mathcal{P}^{\bar{\tau}} = (X_i : i \in \bar{\tau})$, where τ is a facet of Δ, correspond to the minimal primes of $k[\Delta]$.*

Proof. Assume that there is a $P = \mathcal{P}^{\bar{\tau}}$ for some facet τ and a prime, Q such that $I_\Delta \leq Q \leq P$. Since I_Δ is an intersection of primes of the form $\mathcal{P}^{\bar{\sigma}}$, σ being a facet, Theorem 3.1.3 states that there exists a facet τ' such that $\mathcal{P}^{\bar{\tau'}} \leq Q$. Then either τ contains τ' or $\tau = \tau'$. The former would contradict the definition of facets as maximal faces, and so we have $\tau = \tau'$ and $P = \mathcal{P}^{\bar{\tau}} = \mathcal{P}^{\bar{\tau'}}$. Then by contraction, $Q = P$, and so P must be minimal prime containing I_Δ.

Note as well that any prime $Q \leq k[X_1, \ldots X_n]$ where $I \subset Q$ must contain some $\mathcal{P}^{\bar{\tau}}$ for a facet τ. Therefore the primes $\mathcal{P}^{\bar{\tau}}/I_\Delta$ represent all minimal primes in $k[\Delta]$

□

Theorem 4.2.3. $Ass(k[\Delta]) = min(Spec(k[\Delta]))$.

Proof. We show that if P/I is an associated prime of $k[\Delta]$, then it is one of the prime ideals in the intersection described in Theorom4.2.1.

If $P/I \in Ass(k[\Delta])$ then there exists $\hat{x} \neq \hat{0}$ in $k[\Delta]$ such that $P/I = Ann(\hat{x})$. Then $(P/I)\hat{x} = \hat{0}$ and $(P/I)(x + I) = \frac{P(x+I)}{I} = 0 + I$. This further simplifies to $Px + I = I$ and so $Px \in I$. Since $I_\Delta = \cap_{\tau \in facet(\Delta)} \mathcal{P}^{\bar{\tau}}$, Px is then contained in all primes in the intersection. Note however that x cannot be contained in all of these primes, since x would then be in I_Δ and so $\bar{x} = \bar{0}$. Then there must be a $\mathcal{P}^{\bar{\tau}}$ that does not contain x but does contain Px. Since $\mathcal{P}^{\bar{\tau}}$ is prime, it must then contain P, and since it is minimal among primes containing I_Δ, we have $P = \mathcal{P}^{\bar{\tau}}$. □

Example 4.2.4. *Using the same simplicial complex as before, we can write I_Δ as the intersection of its minimal primes,*

$$I_\Delta = (X_4, X_5) \cap (X_1, X_3, X_5) \cap (X_1, X_2, X_3, X_4).$$

We refer to this as a primary decomposition of I_Δ.

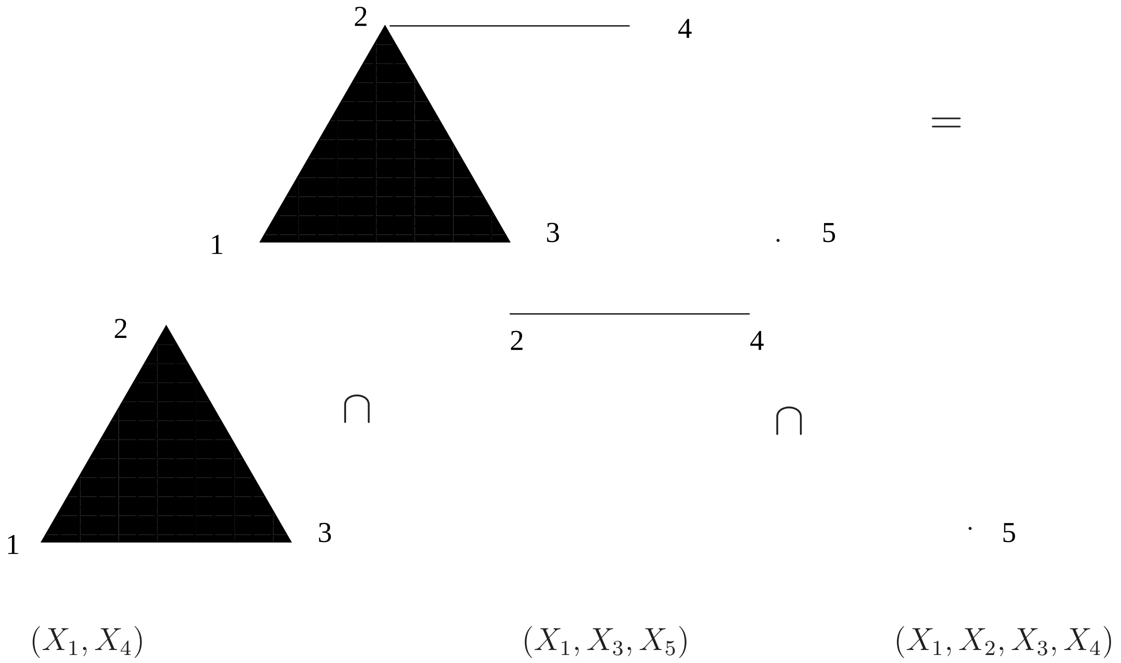


Figure 4.2: Simplicial Complex Decomposed (Example 4.2.4)

The Krull dimension of a Stanley-Reisner ring can be computed via the simplicial complex as explained in the following result.

Theorem 4.2.5. *Given a Stanley-Reisner ring, its Krull dimension is one more than the dimension of its associated simplicial complex.*

Proof. First remember that the Stanley-Reisner ideal $I[\Delta]$, is equal to the intersection of its minimal primes. But each minimal prime, $P = (X_{i_1}, \dots, X_{i_t})$, is generated by the indeterminates associated with a set of points not contained in a particular facet, $\{v_{i_1}, \dots, v_{i_t}\} = \bar{\tau}, \tau \in \Delta$. Therefore the height of a maximal chain of primed ideals in $k[\Delta]$ equals to the number of vertices in its facets. Then the maximal height giving the dimension of the ring is equal to the maximal number of vertices in a facet, which by definition is one more then the dimension of the complex, and so $dim_{Krull}(k[\Delta]) = dim(\Delta) + 1$. $\square$

4.3 Graph Ideals

We now look at ideals of graphs.

Definition 4.3.1. We refer to a *graph, G,* as a collection of points $V(G) = \{p_1, \ldots, p_n\}$, and edges, $E(G) = \{e_i = \{p_{i_1}, p_{i_2}\} : p_{i_1}, p_{i_2} \in V(G)\}$.

Definition 4.3.2. A *vertex cover* of a graph G, is a subset U of $V(G)$ such that every edge in $E(G)$ has at least one vertex in U. That is,

$$\forall e \in E(G), e \cap U \neq \emptyset.$$

A *minimal vertex cover* is a vertex cover with the property that no subset of it is a vertex cover. The *vertex covering number* is the minimal number of vertices needed to cover the edges, that is, $\nu(G) = \min\{|U| : U \subseteq V(G), U \text{ covers } G\}$

While we will only consider graphs with no duplicate edges. That is, each element of $E(G)$ is unique.

Like simplicial complexes, we can construct a unique square-free monomial ideal for a given graph.

Definition 4.3.3. Given a graph, G, the *graph ideal of* G*,* $I(G) \leq k[X_1, \ldots, X_n]$,where $n = |G|$ and k is a field, is given by:

$$I(G) = (X_i X_j : \{p_i, p_j\} \in E(G)).$$

This also means that any square free, monomial ideal that can be generated only by monomials of degree 2 is a graph ideal. The associated graph is the unique graph with vertices corresponding to each indeterminate and edges corresponding to each monomial.

Note that $I(G)$ is a square free monomial ideal and so we can find a simplicial complex associated with it.

Theorem 4.3.4. *If G is a graph and $I(G)$ the graph associated with it, there exists a simplicial complex, Δ, such that $I_\Delta = I(G)$.*

Proof. We prove this by constructing the simplicial complex corresponding to any given G. First, for each $v_i \in V(G)$, let p_i be a point in Δ corresponding to v_i. Next, note that if $\{v_i, v_j\}$ is an edge in G, then X_iX_j is a monomial in $I(G)$. Therefore we must construct Δ so that $\{p_i, p_j\}$, or any face containing it, is not in Δ.

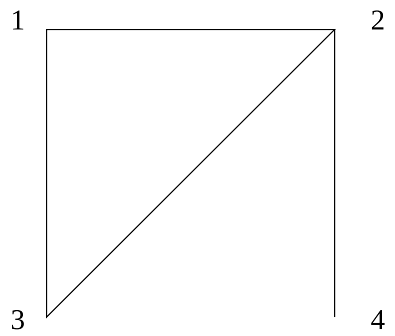
We begin our construction of Δ by first assuming every possible face is in it. We then iterate through each edge, $\{v_i, v_j\}$ of G eliminating all faces σ such that $\{p_i, p_j\} \subseteq \sigma$. And so, for each $\{v_i, v_j\}$ in $G(E)$, X_iX_j is in I_Δ as well. $\square$

However the opposite is not true, for example, consider the simplicial complex composed of three points and only the 0 and 1 dimensional faces. Then the Stanley-Reisner ideal is equal to $(X_1X_2X_3)$ which is not generated by monomials of order two. And so we can't associate a graph with this complex. We generalize this observation by noting that for a complex, Δ, if $\sigma \notin \Delta$ but $\tau \in \Delta$ for all $\tau \subsetneq \sigma$ then X^σ is a generator of I_Δ. If $|\sigma| \geq 3$, then I_Δ is not a graph ideal.

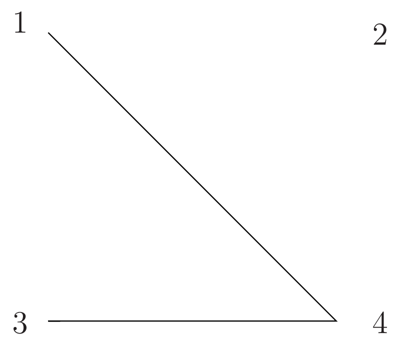
Note that since a graph ideal is also a Stanley-Reisner ideal, the properties found for the latter can also describe the former. For the following, let Δ be the simplicial complex associated with G, such that $I_\Delta = I(G)$.

Theorem 4.3.5. *$Ass(k[G]) = min(Spec(k[G]))$.*

Proof. Given $k[G]$ is a Stanley-Reisner ring, the proof follows directly from 4.2.3 $\square$



Graph



Complex

$I(G) = (X_1X_2, X_1X_3, X_2X_3, X_2X_4)$ $\qquad$ $I_\Delta = (X_1X_2, X_1X_3, X_2X_3, X_2X_4)$

Figure 4.3: Graph to Simplicial Complex

Theorem 4.3.6. *The minimal primes of $k[G]$ correspond to the minimal vertex covers of G. That is, given $U \subseteq V(G)$,*

$$\mathbb{P} \in min(Spec(k[G])) \Leftrightarrow \; U \in min(VC(G))$$

where $\mathbb{P} = (X_i : v_i \in U)$.

Proof. First we show that $I(G) \subseteq \mathbb{P}$ if and only if U is a vertex cover of G. Given $U = \{v_{U_1}, \ldots, v_{U_s}\}$, a vertex cover of G, then every edge $e = \{v_i, v_j\}$ contains at least one point in U. And so for every monomial, J, in $I(G)$ there exists a $v_J \in U$ such that $X_J|J$. Therefore $I(G) \subseteq \mathbb{P} = (X_i : v_i \in U)$.

If however $I(G) \subseteq \mathbb{P} = (X_i : v_i \in U)$ then every edge in G has a monomial in $\mathbb{P}$ so there exists $v \in U$ that is contained in the edge. And so U is a vertex cover.

Now note that $k[G]$ is Stanley-Reisner ring and so minimal primes are of the form $(X_i : i \in I)$. Therefore if U is a minimal vertex cover, then $\mathbb{P} = (X_i : v_i \in U)$ must be a minimal prime containing $I(G)$.

On the other hand, if $(X_i : v_i \in U)$ is a minimal prime containing $I(G)$ then there is no v_i such that $U \setminus \{v_i\}$ is a vertex cover of G. $\square$

We now can rephrase Theorem 4.2.1 to relate to graph ideals specifically.

Theorem 4.3.7.

$$I(G) = \bigcap_{U \in \min(VC(G))} (X_i : v_i \in U)$$

Proof. From 4.3.6, the minimal primes of $I(G)$ are of the form $(X_i : v_i \in U)$ where U is a minimal vertex cover. Then the equality follows from Theorems 4.2.1 and 4.2.2. $\square$

We can now construct minimal primes of $k[X_1, \ldots, X_n]/I(G)$ by finding minimal vertex covers of G.

Theorem 4.3.8. *Repeat the following for all permutations, V, of the vertices in G*

1. *Let $U = \emptyset$ and $E = E(G)$.*
2. *Select a vertex, v from V.*
3. *Remove all edges, e, from E satisfying $v \in e$*
4. *If any edge was removed let $U = U \cup v$ and $V = V \setminus v$.*
5. *If $E \neq \emptyset$ then repeat steps two through four.*

This produces U, a minimal vertex cover for G for each permutation of the vertices of G. We can then construct the minimal prime $\mathbb{P} = (X_i : v_i \in U)$.

The drawback to this method is that it requires iterating over all the possible permutations of vertices to find all minimal primes. This can lead to long execution times, of order $O(n!)$, and many duplicates. This can be easily streamlined however by the right order of consideration of each permutation and implementing a means to stop consideration of similar inputs that will result in a duplication of past output.

Chapter 5

Primary Decomposition

We have showed that the Stanley-Reisner and graph ideals both can be written as intersection of prime ideals. This feature can be generalized to describe submodules of Noetherian modules as intersections of primary submodules. In particular, one can show that any ideal in a Noetherian ring can be written as the intersection of primary ideals. In this chapter, we will survey the theory of primary decomposition for ideals in a Noetherian ring and show how this applied to monomial ideals in general. This is relevant to our paper since Stanley-Reisner and graph ideals are square-free monomial ideals.

5.1 Irreducible Ideals

Definition 5.1.1. A *reducible ideal* is an ideal that can be written as the intersection of two other, distinct, ideals. An *irreducible ideal* is an ideal which is not reducible.

Theorem 5.1.2. *An irreducible ideal in a Noetherian ring is primary.*

Proof. Let $I \leq R$ be an irreducible ideal. Then there no two ideals exist, $I < I_1$ and $I < I_2$, such that $I = I_1 \cap I_2$. Now assume there exist $a, b \in R, b \notin I$ such that $ab \in I$.

We must show that there exists an $n \geq 1$ such that $a^n \in I$. This is equivalent to showing $\hat{a}^n = \hat{0}$ in the factor ring R/I.

Note that an ascending chain of ideals can be formed with the annihilators of powers of $\hat{a}$.

$$\hat{b} \in Ann(\hat{a}) \leq Ann(\hat{a}^2) \leq Ann(\hat{a}^3) \ldots.$$

Since R/I is Noetherian there exists $m \geq 0$ such that $Ann(\hat{a}^m) = Ann(\hat{a}^{m+1}) = \ldots$. Now given $c \in (\hat{a}^m) \cap (\hat{b})$, if we can show that c must be $\hat{0}$, then we have $I = (\hat{0}) = (\hat{a}^m) \cap (\hat{b})$. Furthermore since $b \notin I$, $I = (\hat{0}) < (\hat{b})$, we will have shown $(\hat{0}) = (\hat{a}^m)$, since the zero ideal is irreducible in R/I, as I is irreducible in R. Now, if $\hat{c} \in (\hat{b})$ then $\hat{c} = \hat{r}\hat{b}$ for some $r \in R$, and then $\hat{c}\hat{a} = \hat{r}\hat{b}\hat{a} = \hat{0}$ (remember that $\hat{b}\hat{a} = \hat{0}$). Since $\hat{c} \in (\hat{a}^m)$, we obtain $\hat{c} = \hat{s}\hat{a}^m \Rightarrow \hat{0} = \hat{c}\hat{a} = \hat{s}\hat{a}^{m+1}$ which shows that $\hat{s} \in Ann(\hat{a}^{m+1})$. But this implies that $\hat{s} \in Ann(\hat{a}^m)$, hence

$$\hat{0} = \hat{s}\hat{a}^m = \hat{c} \Rightarrow \hat{c} = 0.$$

Therefore $I = (a^m R + I) \cap (bR + I)$, and so we have that $I = a^m R + I$ since I is irreducible, which gives $a^m \in I$ and we are done. $\square$

5.2 Primary Decomposition

Definition 5.2.1. A *primary decomposition* of an ideal, I, in a Noetherian ring R, is a finite set of primary ideals, $I_1, \ldots, I_n$ such that

$$I = \cap_{i=1}^n I_i$$

The following theorems will prove the existence and describe the nature of primary decompositions of ideals.

Theorem 5.2.2. *Given R a Noetherian ring, any ideal I, is reducible to a finite intersection of irreducible ideals containing I. That is, any ideal $I \leq R$ can be written as $I = \cap_{i=1}^{n} Q_i$, where the Q_i's are irreducible.*

Proof. Let S be the set of ideals that can not be written as a finite intersection of irreducible ideals. For the theorem to be true, S must be empty. Assume $S \neq \emptyset$, then there must be a maximal element I of S. Note that I must be reducible, otherwise it would be trivial intersection of an irreducible ideal (itself). Then $I = I_1 \cap I_2$, where $I < I_1$ and $I < I_2$, but since I is maximal on S, I_1 and I_2 are not in S and so can be written as,

$$I_1 = \cap_{i=0}^{r} Q_i, I_2 = \cap_{j=0}^{s} P_j,$$

where the P_i's and Q_j's are irreducible. And so

$$I = \cap_{i=0}^{r} Q_i \cap \cap_{j=0}^{s} P_j,$$

a finite intersection of irreducible ideals. This contradicts I being in S, therefore $S = \emptyset$. □

Note that a primary decomposition is not unique. However, different decompositions of an ideal can be shown to have similar characteristics.

Lemma 5.2.3. *Given a ring R, a prime ideal P is equal to it's radical. That is,*

$$P \in SpecR \Rightarrow \sqrt{P} = P.$$

Proof. It is trivial to show $P \subseteq \sqrt{P}$. To show $\sqrt{P} \subseteq P$, let $p \in \sqrt{P}$. Then $p^m \in P$ for some $m \geq 1$, $m \in \mathbb{N}$. Assume m is the smallest value such that $p^m \in P$, then

$p^{m-1} \notin P$. Therefore, since $pp^{m-1} \in P$, $p \in P$. $\square$

Definition 5.2.4. The *nilradical* of a ring is the collection of all nilpotent elements of the ring, in other words it is the radical of the zero ideal, $\sqrt{(0)}$.

Theorem 5.2.5. *The nilradical of a ring is equal to the intersection of its primes.*

$$\sqrt{(0)} = \cap_{P \in Spec(R)} P.$$

Proof. Given $a \in \sqrt{0}$, then for all primes P, $a^m \in (0) \leq P$ for some $m \geq 1$. So $a^m \in P$, which implies $a \in P$. Therefore $a \in \cap_{P \in Spec(R)} P$.

Assume there exists $a \in \cap_{P \in Spec(R)} P$ and $a \notin \sqrt{(0)}$. Then $S = \{1, a, a^2, \ldots\}$ is a multiplicative subset of R. So there is a prime ideal Q, such that $Q \cap S = \emptyset$. But a must be in Q as well as S, and so we have a contradiction. $\square$

Theorem 5.2.6. *The radical of an ideal is equal to the intersection of prime ideals that contain it. That is,*

$$\sqrt{I} = \cap_{P \in Spec(R), I \subseteq P} P.$$

Proof. Let $A = R/I$, then I maps canonically to the zero ideal in A and so $\sqrt{I}$ maps to the nilradical. Since the primes in R containing I have a one to one mapping into the primes of A, this is the same as the nilradical of A being equal to the intersection of its primes. $\square$

Lemma 5.2.7. *The radical of a primary ideal is prime.*

Proof. Let $Q \leq R$ be a primary ideal of R, and P the radical of Q. If $a, b \in P$, $b \notin P$, then $(ab)^m \in Q$ for some $m \in \mathbb{N}$. So $a^m b^m \in Q$, but since $b \notin P = \sqrt{Q}$, a^m must be in Q, and so a is in P. $\square$

If $P = \sqrt{Q}$ we say that Q is P-primary.

Theorem 5.2.8. *If Q_1 and Q_2 are P-primary, then $Q_1 \cap Q_2$ is P-primary.*

Proof. If $P = \sqrt{Q_1}$ and $P = \sqrt{Q_2}$, then for every $p \in P$ there exists $m, n \in \mathbb{N}$ such that $p^m \in Q_1$ and $p^n \in Q_2$. Then $p^{\max(m,n)} \in Q_1 \cap Q_2$. So $P \subseteq \sqrt{Q_1 \cap Q_2}$.

On the other hand, if $p \in \sqrt{Q_1 \cap Q_2}$ there exists $r \in \mathbb{N}$ such that $p^r \in Q_1 \cap Q_2$. Then $p^r \in Q_1$ and $p^r \in Q_2$, so $p \in \sqrt{Q_1} = P$. □

Definition 5.2.9. A ring is called *co-primary* if it has exactly one associated prime.

Theorem 5.2.10. *Let R be a Noetherian ring and Q an ideal of R. Then $Q \leq R$ is primary if and only if R/Q is co-primary. Furthermore, if $I = Ann_R(R/Q)$ and $\{P\} = Ass_R(R/Q)$, then I is primary and $\sqrt{I} = P$.*

Proof. Assume R/Q is co-primary and $\{P\} = Ass_R(R/Q)$. Let $I = Ann_R(R/Q)$. Since P is the sole associated prime of R/Q, the support of R/Q consists of P and all primes containing P. Note that since the primes containing P are also the primes containing I,

$$P = \bigcap_{P' \supseteq P \supseteq I} P' = \sqrt{I}.$$

Now let $\hat{a} \in R/Q$ be a zero divisor of R/Q so that $ar \in Q$ for some $r \notin Q$ in R. Then from Theorem 3.2.10, $a \in P = \sqrt{I}$ so Q is primary.

Now assume Q is primary and $P \in Ass_R(R/Q)$. Then for any $a \in P$ there exists an $\hat{0} \neq \hat{r} \in R/Q$ such that $a\hat{r} = \hat{0}$. So a is a zero divisor of R/Q and so since Q is a primary submodule of R, $a \in \sqrt{Ann(R/Q)} = \sqrt{I}$. Therefore $P \subseteq \sqrt{I}$ but since $I \subseteq P$ and consequently $\sqrt{I} \subseteq P$, $\sqrt{I} = P$. So P is unique and equal to the radical of the annihilator in R of R/Q. Also, this means if we have $b, c \in R$, $c \notin I$, such that $bc \in I$ then $bc(R/Q) = 0$ but $c(R/Q) \neq 0$. Then b annihilates elements of $c(R/Q)$ and so it is a zero divisor of R/Q. And so $b \in P = \sqrt{I}$ and I is primary. □

Definition 5.2.11. Given a decomposition of an ideal $I = \cap_{i=1}^n Q_i$, it is called a *minimal, or irredundant, primary decomposition* if it has the property $P_i \neq P_j$, if $i \neq j$, where $\{P_i\} = Ass(R/Q_i)$.

Theorem 5.2.12. *Given an irredundant primary decomposition of an ideal $I = \cap_{i=1}^n Q_i$ in a Noetherian ring R, the union of associated primes of the factor rings R/Q_i equals the set of associated primes of R/I. That is,*

$$Ass(R/I) = \cup_{i=1}^n Ass(R/Q_i).$$

Proof. Without loss of generality, we can assume $I = \cap_{i=1}^n Q_i = 0$ and so $R = R/(0) = R/(I)$. Note that R as an R-module is isomorphic to a submodule of the direct sum of R-modules $\oplus_{i=1}^n R/Q_i$. Then $Ass(R) \subseteq Ass(\bigoplus_{i=1}^n R/Q_i) = \bigcup_{i=1}^n Ass(R/Q_i) = \{P_1, \ldots, P_n\}$.

We must show that any given P_i is an associated prime of R. Without loss of generality, we can focus on showing $P_1 \in Ass(R)$. Since the decomposition was irredundant, there exists a non-zero $x \in Q_2 \cap \ldots \cap Q_n$. Now $Ann(x) = Q_1 : x$ and since $P_1 = \sqrt{Q_1}$, there exits $v > 0$ such that for all $u > v$, $P_1^u x = 0$. Let m be the smallest integer such that $P_1^m x \neq 0$ but $P_1^{m+1} x = 0$. Choose a non-zero $y \in P_1^m x$ and note that $P_1 y = 0$, so $P_1 \subseteq Ann(y)$. Since y is in $Q_2 \cap \ldots \cap Q_n$ but $y \neq 0$, $y \notin Q_1$. Now given $a \in Ann(y)$, we have $ay = 0 \in Q_1$, therefore $a^t \in Q_1$ so $a \in \sqrt{Q_1} = P$. And so $Ann(y) \subseteq P_1$ and $P_1 = Ann(y)$. Therefore $P_1 \in Ass(R)$. $\square$

Definition 5.2.13. The *P-primary component* of a decomposition, $I = \cap_{i=1}^n Q_i$ is the component Q_i such that $P = Ass(R/Q_i)$.

Lemma 5.2.14. *If S is a multiplicative subgroup of a ring R and I and I' are ideals of R, then $(I \cap I')_S = I_S \cap I'_S$.*

Proof. If $\frac{i}{s} \in (I \cap I')_S$ then $i \in I \cap I'$ and $s \in S$. Then $i \in I_S$ and $i \in I'$, and so $\frac{i}{s} \in I_S \cap I'_S$.

On the other hand, if $\frac{i}{s} \in I_S \cap I'_S$, then $\frac{i}{s} \in I_S \cap I'_S$. So $s \in S$ and $i \in I \cap I'$. Then $\frac{i}{s} \in (I \cap I')_S$. □

Lemma 5.2.15. *Let I be an ideal of R and S a multiplicative subset of R, then $(R/I)_S \simeq R_S/I_S$*

Proof. Let $\theta : (R/I)_S \to R_S/I_S$ be defined as the mapping that, given $x \in R$ and $s \in S$, sends $\frac{\hat{x}}{s} \in (R/I)_S$ to $\frac{x}{s}(mod\ I_S)$. This is a ring homomorphism with the kernel being $\{\frac{\hat{0}}{1} = \frac{0+I}{1}\}$. So θ is an isomorphism between $(R/I)_S$ and R_S/I_S

□

Theorem 5.2.16. *Let R be a Noetherian ring with a proper ideal I. The ideal I admits a minimal primary decomposition. Given a minimal associated prime $P \leq R$ of R/I, the P-primary component of the decomposition is $\theta^{-1}(I_P)$, where θ is the localization map ($\theta(r) = \frac{r}{1}$) from R to R_P.*

Proof. The first part is derived from Theorem 5.2.2. Let $I = \bigcap_{i=1}^{n} Q_i$ be a minimal primary decomposition. Given $P \in min(Ass_R(R/I))$ and Q_j it's P-primary component, we must show that $Q_j = \theta^{-1}(I_P)$. Let $P_i = Ass(R/Q_i)$ for all $i = 1, \ldots, n$ and so $P = P_j = Ass(R/Q_j)$. For every $i \neq j$, $P = P_j \notin Ass(R/Q_i)$ since the decomposition is irredundant. Then $(R/Q_i)_P = 0$ and so $R_P/(Q_i)_P = 0$ implying that $R_P = (Q_i)_P$. Note that $I_P = \bigcap_{i=1}^{n}(Q_i)_P$, and so $I_P = (Q_j)_P$. Then $\theta^{-1}(I_P) = \theta^{-1}((Q_j)_P)$

□

The following theorem summarizes some results concerning primary decompositions of monomial ideals.

Theorem 5.2.17. *Let k be field, $R = k[X_1, \dots, X_n]$ a polynomial ring over k, and $I \leq R$ a monomial ideal in R.*

1. *I has a unique minimal generating set, S. That is $I = (X^\alpha : X^\alpha \in S)$ and for any $X^\alpha, X^\beta \in S$, $X^\alpha \nmid X^\beta$ and $X^\beta \nmid X^\alpha$*
2. *$\sqrt{I}$ is generated by the set of monomials $\{X_{i_1} \dots X_{i_k} : \exists \alpha_{i_1}, \dots, \alpha_{i_k} \in \mathbb{N}$ such that $X_{i_1}^{\alpha_{i_1}} \dots X_{i_k}^{\alpha_{i_k}} \in S\}$*
3. *For any X_i, $I : X_i = \sum_{m \in S}((m) : X_i)$. That is, any polynomial f such that $X_i f \in I$ can be written as the sum of polynomials, f_m, where $m \in S$ and $f_m X_i \in (m)$.*

For the following, a generator of an ideal will be assumed to be from a minimal generating set of that ideal.

Lemma 5.2.18. *[Swa03] If I is a monomial ideal of a polynomial ring R over a field k, then $\sqrt{I}$ is a monomial ideal.*

Proof. Given f, a member of $\sqrt{I}$ we need to show that any monomial summand of f is in $\sqrt{I}$. Let X^α be any monomial summand of f, then f can be written as $f = aX^\alpha + h$, where $a \in k$ and $h \in R$. Now there exists r such that $f^k \in I$ and so,

$$f^r = (aX^\alpha + h)^r = a^r (X^\alpha)^r + \sum_{i=1}^{r} \binom{r}{i} X^{\alpha(r-i)} h^i \in I.$$

Then $(X^\alpha)^r \in I$ and so X^α must be in $\sqrt{I}$. Therefore for any element in $\sqrt{I}$, every one of its monomials is in $\sqrt{I}$ and so $\sqrt{I}$ is a monomial ideal. $\square$

Theorem 5.2.19. *[Swa03] Let $I \leq R = k[X_1, \dots, X_n]$ be a monomial ideal, then I is primary if and only if every X_i that divides a generator of I is in $\sqrt{I}$.*

Proof. Assume I is primary, then every X_i that divides a generator of I is a factor of that generator and so is in $\sqrt{I}$ by the definition of a primary ideal.

Now assume that every X_i that divides a generator of I must be in $\sqrt{I}$. Since $\sqrt{I}$ is a monomial ideal, we can describe a generator as a monomial, $X_{i_1} \ldots X_{i_r}$. Then there exist $a_1, \ldots, a_r \in \mathbb{N}$ such that $m = X_{i_1}^{a_1} \ldots X_{i_r}^{a_r}$ is a generator in I. But since X_{i_1} divides m, $X_{i_1}^{a_1}$ is in I, so $r = 1$ and $\sqrt{I}$ generated by monomials of order 1 and so it is prime. We must show that $\sqrt{I}$ is the unique associated prime of I. Assume $\sqrt{I}$ is not the sole associated prime of I. Let X_i be an indeterminate contained in an associated prime but not in $\sqrt{I}$. Then X_i divides some generator of I, and so $X_i \in \sqrt{I}$ by our initial assumption. But this contradicts $X_i \notin \sqrt{I}$ and so $\sqrt{I}$ is the only associated prime of I and so I is primary.

$\square$

We now present a method to find the primary decomposition of a monomial ideal.

Theorem 5.2.20. *[Swa03] Let I be an ideal in $R = k[X_1, \ldots, X_n]$ and assume that it's not primary. Then there exists an X_i and $m \in S$, S being the unique minimal set of generators of I, such that $X_i \mid m$ but $X_i \notin \sqrt{I}$. We can now decompose I as follows:*

$$I = (I + (X_i^n)) \cap (I : X_i^n)$$

for any n such that $I : X_i^n = X_i^{n+1}$, which exists since R is Noetherian.

By iterating through these steps, we can achieve a primary decomposition of I.

Proof. First we must show $I < (I + (X_i^n))$ and $I < (I : X_i^n)$.

Now consider $s \in (I + (X_i^n)) \cap (I : X_i^n)$. We can write $s = f + rX^n$ where $f \in I$ and $r \in R$. Then, since $s \in (I : X_i^n)$, we have $X^n s = fX^n + rX^{2\alpha} \in I$. So $rX^{2n} \in I$, but, by how we defined n, this means $X^n \in I$. Therefore, $s \in I$ and so $(I + (X_i^n)) \cap (I : X_i^n) \subseteq I$.

Since R is Noetherian, any chain of ascending ideals, such as $I \leq (I + X_i^{\alpha}) \leq (I + X_i^{\alpha} + X_j^{\beta}) \leq \ldots$ or $I \leq (I : X_i^{\alpha}) \leq ((I : X_i^{\alpha}) : X_j^{\beta}) \leq \ldots$, will terminate. And so eventually this procedure will terminate into irreducible, and so primary, ideals. $\square$

Example 5.2.21. Let $I = (x^2, xy, yz)$ be an ideal of the ring $R = k[x, y, z]$. Note that I is not primary since y divides the generators xy, and yz but $y^m \notin I$ for any $m \geq 1$. Since y^2 does not divide any generator of I, we have the decomposition

$$I = ((x^2, xy, yz) + (y)) \cap ((x^2, xy, yz) : y) = (x^2, xy, yz, y) \cap (x, z) = (x^2, y) \cap (x, z).$$

Since (x^2, y) is primary and (x, z) is prime we are done.

Chapter 6

Hilbert Polynomials of Stanley-Reisner Rings

In the following chapter, we present the general facts on the Hilbert polynomial of a Stanley-Reisner ring. We show how the combinatorial features of the simplicial complex play a crucial role in the computation of the Hilbert polynomials of the rings mentioned above. At the end of the chapter, we discuss how these results apply to graph ideals by analyzing a few special classes of graphs. Much of this material as well as further information can be found in [Vil01].

6.1 Stanley-Reisner rings

In what follows, we regard the Stanley-Reisner rings as graded rings with $\mathbb{N}$-grading based on total degree.

Definition 6.1.1. Given a p-dimensional simplicial complex, the *f-vector, $f = (f_0, f_1, \ldots f_p)$,* is defined such that for $i = 0, \ldots, p$, f_i is the number of i-dimensional faces in the complex. Note that f_0 is the number of vertices (0-dimensional simplexes) in the complex. Also, when necessary, f_{-1} is defined as 1.

Note that given a Stanley-Reisner ring over a field, k, graded by total degree, its homogenous components are generated by monomials supported by faces of the complex, that is with indeterminates matching to vertices.

Before we describe the Hilbert function for a Stanley-Reisner ring we need to count the number of possible monomials supported by a simplex. The following lemma derives from the combinatorics problem of finding all possible nonnegative partitions of an integer.

Lemma 6.1.2. *Given a simplex of dimension $r-1$, $\tau = \{v_1, \ldots, v_r\}$, the number of monomials of degree n supported by τ is equal to $\binom{n-1}{r-1}$.*

Proof. Let $X_{i_1}^{\alpha_1} \ldots X_{i_r}^{\alpha_r}$ be a monomial of degree n supported by τ. Then $\alpha_1 + \ldots + \alpha_r = n$. We need to count the number of distinct combinations of values for $\{\alpha_1, \ldots, \alpha_r\}$ with $\alpha_i \geq 1$ for $i = 1, \ldots, r$. It is convenient to consider the equivalent problem, $\beta_1 + \ldots + \beta_r = n - r$, $\beta_i = \alpha_i - 1, \beta_i \geq 0$ for $i = 1, \ldots, n$.

We can view this computation as the problem of finding places to put $r-1$ dividers among $n-r$ separate items. The number of possible positions for the first divider is the total number of items and dividers, $(n-r)+(r-1) = n-1$. Then there are $\binom{n-1}{r-1}$ ways to place $r-1$ dividers, and so on. Hence, $n-r$ can be partitioned into r terms as above in $\binom{n-1}{r-1}$ ways. □

Theorem 6.1.3. *Let $R = k[\Delta]$ be the Stanley-Reisner ring of the p-dimensional simplicial complex Δ and let $H_R(n)$ be the Hilbert function of R. Then*

$$H_R(n) = \sum_{i=0}^{p} \binom{n-1}{i} f_i, \forall n \geq 1,$$

where the f_i's are the components of the f-vector.

Proof. We are looking for the length of R_n, which is equal to the dimension of R_n as a vector space over $R_0 = k$. In other words, we need to determine size of the basis of R_n, which is the number of unique monomials of degree n supported by a face of the complex.

If we look at the index i in the summation, we note that it corresponds to the a set of faces of a particular dimension in the complex, or for that matter, the number of indeterminates in the monomials corresponding to them. Then each term in the sum is equal to number of monomials with i distinct indeterminates of degree n. Since there are f_i faces of dimension i and $\binom{n-1}{i}$ unique monomials of power n over i indeterminates, we have $\binom{n-1}{i} f_i$ monomials corresponding to all the faces of dimension i. □

We can now use the previous result to define the Hilbert series for a Stanley-Reisner-Ring.

Theorem 6.1.4.

$$HS(R,t) = \sum_{i=-1}^{p} \frac{f_i t^{i+1}}{(1-t)^{i+1}}.$$

Proof. First note that $(1-t)^{-s} = \sum_{n=0}^{\infty} \binom{s+n-1}{s-1} t^n$. Since $HS(R,t)$ is the generating function of $\ell(R_n) = H_R(n)$ we have,

$$HS(R,t) = \sum_{n=0}^{\infty} H_R(n) t^n = 1 + \sum_{n=1}^{\infty} \sum_{i=0}^{p} \binom{n-1}{i} f_i t^n.$$

This is then equivalent to

$$HS(R,t) = 1 + \sum_{i=0}^{p} f_i \sum_{n=1}^{\infty} \binom{n-1}{i} t^n.$$

We now consider the inner sum, $S = \sum_{n=1}^{\infty} \binom{n-1}{i} t^n$. Note that if $n \leq i$, $\binom{n-1}{i} = 0$, and so $S = \sum_{n=i+1}^{\infty} \binom{n-1}{i} t^n$. We now can factor t^{i+1} out of the sum and replace n with $m + (i+1)$ so that $S = t^{i+1} \sum_{m=0}^{\infty} \binom{m+i}{i} t^m$. From the earlier identity, we have $S = t^{i+1}(1-t)^{-(i+1)}$. And so, since f_{-1} was defined as 1,

$$HS(R,t) = \sum_{i=-1}^{p} \frac{f_i t^{i+1}}{(1-t)^{i+1}}.$$

$\square$

6.2 Hilbert Series of Graph Rings

We conclude this paper by computing the Hilbert series for rings associated to special classes of graph ideals

6.2.1 Special types of graphs

Proposition 6.2.1. *Let D_n be the (disconnected) graph with n vertices and no edges. Then, if $R = k[D_n]$,*

$$HS_R(t) = \sum_{d=0}^{\infty} \binom{d+n-1}{n-1} t^d = (1-t)^{-n}.$$

Proof. First note that $I(D_n) = (0)$ and $R = k[X_1, \ldots, X_n]/(0) = k[X_1, \ldots, X_n]$. Therefore, for $n \geq 0$, $H_R(d) = \ell_{R_0}(R_d) = \binom{d+n-1}{n-1}$ and $HS_R(t) = \sum_{d=0}^{\infty} \binom{d+n-1}{n-1} t^d = (1-t)^{-n}$. $\square$

Proposition 6.2.2. *Let $\mathbf{K_n}$ be the complete graph on n vertices, that is, the graph with a complete set of edges. Then, $HS(t) = 1 + nt + nt^2 + nt^3 + \ldots = 1 + nt(1 + t + t^2 + \ldots) = 1 + \frac{nt}{1-t} = \frac{1+(n-1)t}{1-t}$.*

Proof. The graph ideal is defined as $I(\mathbf{K_n}) = (X_1X_2, X_1X_3, \ldots, X_1X_n, X_2X_3, \ldots, X_{n-1}X_n)$, the ideal generated by all monomials consisting of exactly two distinct indeterminates. Then the non-zero elements of $R[\mathbf{K_n}]$ are only those containing less then two indeterminates. For degree 0, we have $H_R(0) = \ell_{R_0}(R_0) = 1$. For degree $d > 0$, the basis for R_d as a vector space over $R_0 = k$ is exactly $\{X_1^d, \ldots X_n^d\}$. So $H_R(d) = \ell_{R_0}(R_d) = n$, the number of elements of degree d in one indeterminate. And so, $HS(t) = 1 + nt + nt^2 + nt^3 + \ldots = 1 + nt(1 + t + t^2 + \ldots) = 1 + \frac{nt}{1-t} = \frac{1+(n-1)t}{1-t}$. □

Proposition 6.2.3. *A bipartite graph, $\mathbf{K_{m,n}}$, is a graph consisting of two disconnected graphs, A and B, $A \cap B = \emptyset$, $|A| = m$ and $|B| = n$, with edges covering every pair of vertices $\{a, b\}$, where $a \in A$ and $b \in B$. The Hilbert series of a complete bipartite graph, $\mathbf{K_{m,n}}$ is*

$$HS_R = (1-t)^{-m} + (1-t)^{-n}.$$

Proof. We can divide the graph into two sets of vertices, A and B, of size m and n respectively. We denote the indeterminates supported by A as $X_1, \ldots, X_m$ and those supported by B as $Y_1, \ldots, Y_n$. Then the ideal is generated by monomials containing indeterminates from both A and B, i.e. X_1Y_2, X_3Y_2, etc. The ring as a vector space has its basis the monomials supported only by one set or the other. We can then split the basis into two groups, one supported by the m vertices in A and the other by the n vertices in B. Note that this implies the vector space can be split into two separate vector spaces, each corresponding to an induced subgraph with vertices in either A or B. These subgraphs are then disconnected and so their associated rings are $R' = k[X_1, \ldots, X_m]$ and $R'' = k[Y_1, \ldots, Y_n]$. We can compute the length of R_d as the sum of lengths of R'_d and R''_d. Therefore, $H_R(d) = \binom{d+m-1}{m-1} + \binom{d+n-1}{n-1}$ and $HS_R = \sum_{d=0}^{\infty}[\binom{d+m-1}{m-1} + \binom{d+n-1}{n-1}]$. We can simplify this by breaking up the summation in two, $\sum_{d=0}^{\infty} \binom{d+m-1}{m-1} + \sum_{d=0}^{\infty} \binom{d+n-1}{n-1} = (1-t)^{-m} + (1-t)^{-n}$. □

Proposition 6.2.4. *Let $\mathbf{P_n}$ be a path in n vertices, with $V(\mathbf{P_n}) = \{v_1, \ldots, v_n\}$ and $E(\mathbf{P_n}) = \{v_1v_2, v_2v_3, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_n\}$. Then the Hilbert series is*

$$1 + \sum_{d=1}^{\infty}[\sum_{s=1}^{d} \binom{n-s+1}{s}\binom{d-1}{s-1}]t^d.$$

Proof. The graph ideal of $\mathbf{P_n}$ is $I(\mathbf{P_n}) = (X_1X_2, X_2X_3, \ldots, X_{n-1}X_n)$. Let $R = k[X_1, \ldots, X_n]/I(\mathbf{P_n})$ and set the grading by total degree, as before. Then note that the basis of each R_d is composed of monomials supported by non-consecutive vertices.

For any $d > 0$, we can describe the length of R_d as a sum of a series of terms, with each term equaling the number of elements in the basis with a particular number of vertices in it's support. To count the number of basis elements supported by s vertices and with degree d, we need to multiply the number of ways to choose s non-consecutive vertices by the number of different monomials of degree d supported by a particular set of s vertices. We have shown the latter to be $\binom{d-1}{n-1}$. It remains to find the number of ways to choose s non-consecutive vertices from the set $\{v_1, \ldots, v_n\}$. We will denote this as $t(n, s)$.

Note that $t(m, 1) = m$ for any $m \geq 1$, and $t(m, k) = 0$ for $m < 2k - 1$. We prove by induction on m and k that $t(m, k) = \binom{m-k+1}{k}$. First note that if $m < 2k - 1$ then $m - k + 1 < k$ and $\binom{m-k+1}{k} = 0$. Also $\binom{m-1+1}{1} = \binom{m}{1} = m$. Now assume $t(m', k') = \binom{m'-k'+1}{k'}$ for $m' < m$ and $k' < k$. Now if v_1 is already chosen, then there is $t(m-3, k-1) = \binom{m-2-(k-1)+1}{k-1} = \binom{m-k}{k-1}$ ways to choose the remaining vertices. In general, there are $t(m-j-1, k-1) = \binom{m-j-1-(k-1)+1}{k-1} = \binom{m-k-j+1}{k-1}$ possible sets of nonconsecutive vertices if v_j is the smallest ordered vertex in the set.

We then have $t(m, k) = \sum_{j=1}^{m-2k+2} \binom{m-k-j+1}{k-1} = \binom{m-k}{k-1} + \binom{m-k-1}{k-1} + \ldots + \binom{k+1}{k-1} + \binom{k}{k-1} + \binom{k-1}{k-1}$. The last term of this series is equal to $1 = \binom{k}{k}$, and by summing this with the preceding term we have $\binom{k}{k} + \binom{k}{k-1} = \binom{k+1}{k}$. We continue with $\binom{k+1}{k-1} + \binom{k}{k-1} +$

$\binom{k-1}{k-1} = \binom{k+1}{k-1} + \binom{k+1}{k} = \binom{k+2}{k}$. By subsequent additions, the series is contracted to $\binom{m-k}{k-1} + \binom{m-k-1}{k-1} + \binom{m-k-1}{k} = \binom{m-k}{k-1} + \binom{m-k}{k} = \binom{m-k+1}{k}$. Therefore there are $\binom{m-k+1}{k}$ ways to choose k non-consecutive vertices out of m.

We can now define the length of R_d for $d \geq 1$, as $\ell(R_d) = \sum_{s=1}^{min\{\lfloor \frac{n-1}{2} \rfloor, d\}} \binom{n-s+1}{s}\binom{d-1}{s-1}$. The Hilbert Series is then $1 + \sum_{d=1}^{\infty}[\sum_{s=1}^{min\{\lfloor \frac{n-1}{2} \rfloor, d\}} \binom{n-s+1}{s}\binom{d-1}{s-1}]t^d$. □

Proposition 6.2.5. *The Hilbert Series for a star-shaped graph, a connected graph with all edges having one vertex in common, is*

$$HS(t) = \frac{t}{1-t} + \frac{1}{(1-t)^{n-1}}.$$

Proof. For n vertices, we define the graph by declaring the edges to be $E(G) = \{v_1v_n, v_2v_n, \ldots, v_{n-1}v_n\}$. The ideal is then $I(G) = (X_1X_n, X_2X_n, \ldots, X_{n-1}X_n) = (X_1, \ldots, X_{n-1}) \cap (X_n)$. Then the corresponding complex consists of an isolated 0-dimensional vertex, v_n, along with a $n-1$-dimensional simplex, $\{v_1, \ldots, v_{n-1}\}$. To find the Hilbert series, we first determine the f-vector. We know f_0 is n, and as for f_1 to f_{n-1} we determine the values to be the same as for a simplex of $n-1$ vertices, $f_k = \binom{n-1}{k}$, $k = 2, \ldots, n-1$. Then the Hilbert Series is

$$HS(t) = 1 + \frac{nt}{1-t} + \sum_{i=2}^{n-1} \frac{\binom{n-1}{i}t^i}{(1-t)^i}.$$

This can be rewritten as

$$HS(t) = 1 + \frac{nt}{1-t} - 1 - \frac{(n-1)t}{1-t} + \sum_{i=0}^{n-1} \frac{\binom{n-1}{i}t^i}{(1-t)^i}$$

$$= \frac{t}{1-t} + \sum_{i=0}^{n-1} \frac{\binom{n-1}{i}t^i}{(1-t)^i}.$$

Note that summation is equivalent to a Hilbert series with f-vector equal to

$$(\binom{n-1}{1}, \binom{n-1}{2}, \ldots, \binom{n-1}{n-1}).$$

This is the same as an f-vector of a complete simplex of $n-1$ points. Since a Stanley-Reisner ring of a complete simplex is the same as a graph ring with no edges, we can use our result for a disconnected graph to simplify the summation. And so we have $\sum_{i=0}^{n-1} \frac{\binom{n-1}{i} t^i}{(1-t)^i} = \frac{1}{(1-t)^{n-1}}$. Therefore, $HS(t) = \frac{t}{1-t} + \frac{1}{(1-t)^{n-1}}$. □

6.2.2 Graph Rings of dimension 1 and 2

The Krull dimension of a Stanley-Reisner ring can be easily computed from its primary decomposition. This fact makes describing the Hilbert series for small dimensions manageable.

Theorem 6.2.6. *If the Stanley-Reisner ring of a graph G has dimension d then the size of any minimal vertex cover is at least $|V(G)| - d$. For that matter, at least one minimal vertex cover is of size $|V(G)| - d$.*

Proof. Let $n = |V(n)|$ and $R = k[X_1, \ldots, X_n]$. If $I(G) = P_1 \cap \ldots \cap P_m$ is a minimal primary decomposition then for every $i = 1, \ldots, m$, $P_i = (X_{i_1}, \ldots, X_{i_{r_i}})$ corresponds to vertex cover, $vc_i = \{v_{i_1}, \ldots, v_{i_{r_i}}\}$. Also, if P_i is minimal, then vc_i is minimal. Now any maximal ascending chain of primes in $k(G)$ corresponds to a chain of primes, $(X_{j_1}, \ldots, X_{j_r}) = P \subset Q_1 \subset \ldots \subset Q_t = (X_1, \ldots X_n)$ in R, where P is minimal in the decomposition of $I(G)$. Then the length of the chain is $n - r$ which must be equal to or less then d. Therefore $n - d \leq r$. □

Proposition 6.2.7. *The Hilbert Series of a graph ideal of Krull dimension 1 is $HS(t) = \frac{1+(n-1)t}{1-t}$.*

Proof. This follows if and only if the Stanley-Reisner ring of dimension 1 is a complete graph. Note that for a graph whose ring is dimension 1, any minimal vertex cover of G must be have exactly $n-1$ vertices. For a complete graph this is true since any set of $n-1$ can cover a graph, but a set with $n-2$ can not cover the edge belonging to the missing vertices. However, if we removed any edge from the complete graph, both of its vertices become redundant and the dimension is then greater than 1. Therefore a graph ring of dimension 1 is a complete graph and its Hilbert series follows. $\square$

It is somewhat harder to categorize graphs of dimension 2 Stanley-Reisner rings, since they can have minimal vertex covers of size $|V|-1$ and $|V|-2$, while the vertex covering number is $|V|-2$. In general, for rings of higher dimensions, it is convenient to use characteristics of both the graph and the related simplicial complex.

First, we present an example of a graph with two minimal vertex covers of different lengths.

Example 6.2.8. Let G be the graph defined by $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{v_1v_2, v_2v_3, v_2v_4, v_3v_4\}$. The graph ideal is then $I(G) = (X_1X_2, X_2X_3, X_2X_4, X_3X_4) = (X_1, X_3, X_4) \cap (X_2, X_3) \cap (X_2, X_4)$. So the lengths of the chains of primes in $k[G]$ are 1,2,and 2, as measured from the minimal primes of $I(G)$. It follows that $k[G]$ has Krull dimension 2 and two minimal vertex covers are of size $4-1$ (given by the vertices $\{v_1, v_3, v_4\}$) and $4-2$ (with vertices $\{v_2, v_3\}$ and $\{v_2, v_4\}$).

The following proposition describes the Hilbert series for a graph ring of dimension 2 if the number of edges in the graph is known.

Proposition 6.2.9. *Given a graph G with n vertices and e edges and whose Stanley-Reisner ring is of dimension 2, the Hilbert series is*

$$HS(t) = 1 + \frac{nt}{1-t} + \frac{(n^2-n-2e)t^2}{2(1-t)^2}.$$

Furthermore we can bound e by the inequality $n-2 \leq e \leq \frac{(n-2)(n+1)}{2}$.

Proof. First, the corresponding simplicial complex of the graph ideal has dimension $2-1=1$, so our f-vector has only two components f_0, f_1. By construction, the number of 1-dimensional faces in the complex is equal to the total possible number of 1-dimensional faces minus the number of edges in the original graph, so $f_1 = \binom{n}{2} - e$. So the f-vector is $(n, \frac{n(n-1)}{2} - e)$ and the Hilbert Series is $\sum_{i=-1}^{1} \frac{f_i t^{i+1}}{(1-t)^{i+1}} = 1 + \frac{nt}{1-t} + \frac{(n^2-n-2e)t^2}{2(1-t)^2}$.

The maximum number of edges a graph of n vertices can have is $\frac{n(n-1)}{2}$. However we've shown that a graph with this exact number of edges has a ring of dimension one. But if we remove any one edge from a complete graph, then the resulting graph can be covered with one less vertex, and so the ring of this graph would be of dimension two. Remember for a complete graph, any vertex cover must have at least $n-1$ points, since if more than one vertex was missing, we could not cover the edges in between the missing points. However, if one edge was removed, then both points contained by it could then be removed from the cover, since any other edge incident to it is incident to some other vertex.

Now note that any graph must have at least as many edges as it has vertices in any minimal vertex cover. And so, since the minimal vertex covers of the graph are least $n-2$, this must be the least number of edges possible in the graph. $\square$

The following examples illustrate various graphs with graph rings of dimension 2.

Example 6.2.10. Let G be a graph with vertices $V(G) = \{v_1, v_2, v_3\}$ and edges $E(G) = \{v_1v_2, v_2v_3\}$. We have minimal vertex covers of 1 and 2 and so the dimension of the ring is $3-1=2$. The Hilbert series is $1 + \frac{3t}{1-t} + \frac{t^2}{1-t}$.

Example 6.2.11. Let G be a graph with four vertices and two edges, such that $E(G) = \{v_1v_3, v_2v_4\}$. Then the minimal vertex covers are $\{v_1, v_2\}$, $\{v_1, v_4\}$, $\{v_3, v_2\}$,

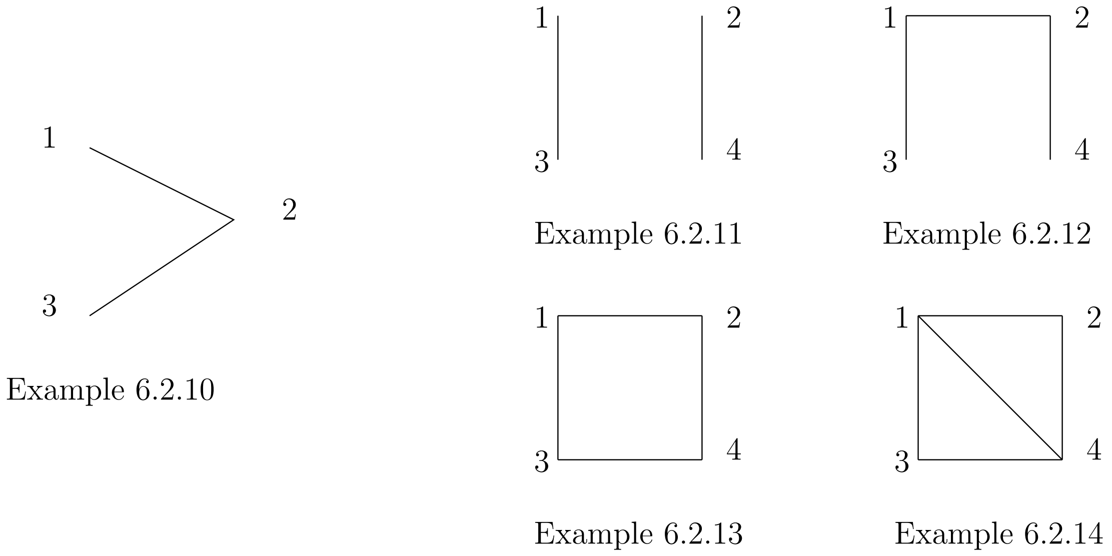


Figure 6.1: Graphs with Rings of Dimension 2

and $\{v_3, v_4\}$, and $\dim(k[G]) = 2$. The graph ideal is,

$$I(G) = (X_1X_3, X_2X_4)$$

$$= (X_1, X_2) \cap (X_1, X_4) \cap (X_3, X_2) \cap (X_3, X_4).$$

The Hilbert series is $1 + \frac{4t}{1-t} + \frac{4t^2}{(1-t)^2}$.

Example 6.2.12. Let G be a graph with four vertices and three edges, such that $E(G) = \{v_1v_3, v_2v_4, v_1v_2\}$. Then the minimal vertex covers are $\{v_1, v_2\}, \{v_1, v_4\}$, and $\{v_3, v_2\}$, and $\dim(k[G]) = 2$. The graph ideal is,

$$I(G) = (X_1X_3, X_2X_4, X_1X_2)$$

$$= (X_1, X_2) \cap (X_1, X_4) \cap (X_3, X_2).$$

The Hilbert series is $1 + \frac{4t}{1-t} + \frac{3t^2}{(1-t)^2}$.

Example 6.2.13. Let G be a graph with four vertices and five edges such that $E(G) = \{v_1v_2, v_2v_4, v_4v_3, v_3v_1\}$ then the minimal vertex covers are $\{v_1, v_4\}$, and $\{v_3, v_2\}$, and the Krull dimension is 2. The graph ideal is,

$$I(G) = (X_1X_3, X_1X_2, X_2X_4, X_3X_4)$$

$$= (X_1, X_4) \cap (X_3, X_2).$$

The Hilbert series is $1 + \frac{4t}{1-t} + \frac{2t^2}{(1-t)^2}$.

Example 6.2.14. Let G be a graph with four vertices and five edges, such that $E(G) = \{v_1v_2, v_2v_4, v_4v_3, v_3v_1, v_1v_4\}$. Then the minimal vertex covers are $\{v_1, v_4\}$, $\{v_1, v_2, v_3\}$, and $\{v_2, v_3, v_4\}$, and $\dim(k[G]) = 2$. The graph ideal is,

$$I(G) = (X_1X_2, X_1X_3, X_1X_3, X_2X_4, X_3X_4)$$

$$= (X_1, X_4) \cap (X_1, X_2, X_3) \cap (X_2, X_3, X_4).$$

The Hilbert series is $1 + \frac{4t}{1-t} + \frac{t^2}{(1-t)^2}$.

For graphs with a Stanley-Reisner ring of dimension three, we need to know additional information, such as vertex degrees, and the number of triangles formed by edges in G.

Example 6.2.15. *[Vil01] Let G be a graph with n vertices and Stanley-Reisner ring $R = k[G]$ of dimension 3. The Hilbert series is then:*

$$HS_R(t) = \frac{1 + gt + (\binom{g+1}{2} - q)t^2 + \frac{1}{6}(2g + 3g^2 + g^3 - 6q - 6gq - 6N_t + 3v)}{(1-t)^3}.$$

Where g is the height of the ideal $I(G)$, N_t is the number of triangles (i.e. sets of *edges* $\{v_{i_1}v_{i_2}, v_{i_2}v_{i_3}, v_{i_3}v_{i_1}\}$*), and* v *is the sum of the squares of the vertex degrees.*

The proof given in the text involves finding the f-vector of the corresponding complex and then substituting them into the formula we have for a simplicial complex.

In the general case, if $n = |V(G)|$ and $e = |E(G)|$ we know $f_{-1} = 1$, $f_0 = n$, and f_1 equals the number of edges (1-dimensional faces) in a complete complex of n vertices, $\binom{n}{2}$ minus e. Then the first three terms of the Hilbert Series are

$$1 + \frac{nt}{(1-t)} + \frac{(\frac{n(n-1)}{2} - e)t^2}{(1-t)^2}.$$

Higher order terms depend on other characteristics of the graph such as the vertex degrees and the number of complete subgraphs of a certain number of vertices, such as the number of triangles, contained within it.

Bibliography

[Cox97] David A. Cox. *Ideals, varieties, and algorithms : an introduction to computational algebraic geometry and commutative algebra*. Springer, 1997.

[Eis95] David Eisenbud. *Commutative algebra with a view toward algebraic geometry*. Graduate texts in mathematics. Springer-Verlag, 1995.

[Ene03] Florian Enescu. A first semester in commutative algebra. Lecture notes from course taught at the University of Utah, 2003.

[Hun80] Thomas W. Hungerford. *Algebra*. Springer, 1980.

[Jac85a] Nathan Jacobson. *Basic Algebra I*. W. H. Freeman, 1985.

[Jac85b] Nathan Jacobson. *Basic Algebra II*. W. H. Freeman, 1985.

[Lan71] Serge Lang. *Algebra*. Addison-Wesley Pub. Co., 1971.

[Mat80] Hideyuki Matsumura. *Commutative algebra*. Benjamin/Cummings Pub. Co., 1980.

[Mat86] Hideyuki Matsumura. *Commutative ring theory*. Cambridge University Press,, 1986.

[Sch03] Hal Schenck. *Computational Algebraic Geometry*. Number 58 in London Mathematical Society Student Texts. Cambridge University Press, 2003.

[Swa03] Irena Swanson. International conference on commutative algebra and combinatorics. In *Primary Decompositions*, page 42, Allahabad, India, 2003. http://www.reed.edu/ iswanson/primdec.ps.

[Vil01] Rafael H. Villarreal. *Monomial Algebras*. Marcel Dekker, Inc., New York, 2001.

[Wae91] B. L. van der Waerden. *Algebra*, volume I. Springer, 1991.