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Rational Realizations of the Minimum Rank of a Sign Pattern Matrix

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Under the Direction of Frank J. Hall and Zhongshan Li

ABSTRACT

A sign pattern matrix is a matrix whose entries are from the set $\{+, -, 0\}$. The minimum rank of a sign pattern matrix A is the minimum of the rank of the real matrices whose entries have signs equal to the corresponding entries of A. It is conjectured that the minimum rank of every sign pattern matrix can be realized by a rational matrix. The equivalence of this conjecture to several seemingly unrelated statements are established. For some special cases, such as when A is entrywise nonzero, or the minimum rank of A is at most 2, or the minimum rank of A is at least $n-1$, (where A is $m \times n$), the conjecture is shown to hold. Connections between this conjecture and the existence of positive rational solutions of certain systems of homogeneous quadratic polynomial equations with each coefficient equal to either -1 or 1 are explored. Sign patterns that almost require unique rank are also investigated.

Keywords: sign pattern matrix; minimum rank; maximum rank; rational matrix

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Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

Master of Science

in the College of Arts and Sciences

Georgia State University

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Office of Graduate Studies

College of Arts and Sciences

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December 2005

ACKNOWLEDGEMENTS

The results contained in this thesis are part of an overall research project involving Dr. Frank J. Hall, Dr. Zhongshan Li, Dr. Marina Arav, Dr. Bhaskara Rao, and Selcuk Koyuncu.

The author wishes to gratefully acknowledge the assistance of Dr. Frank J. Hall, Dr. Zhongshan Li and Dr. Marina Arav, without whose guidance this thesis would not have been possible.

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1. Introduction and Preliminaries

In qualitative and combinatorial matrix theory, we study properties of a matrix based on combinatorial information, such as the signs of entries in the matrix. A matrix whose entries are from the set $\{+, -, 0\}$ is called a *sign pattern matrix* (or sign pattern, pattern). We denote the set of all $n \times n$ sign pattern matrices by Q_n. For a real matrix B, $sgn(B)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-$, 0). For a sign pattern matrix A, the *sign pattern class of* A is defined by

$$Q(A) = \{ B : \operatorname{sgn}(B) = A \}.$$

The sign pattern $I_n \in Q_n$ is the diagonal pattern of order n with $+$ diagonal entries.

A sign pattern matrix P is called a *permutation pattern* if exactly one entry in each row and column is equal to $+$, and all other entries are 0. Two sign pattern matrices A_1 and A_2 are said to be *permutationally equivalent* if there are permutation patterns P_1 and P_2 such that $A_2 = P_1 A_1 P_2$.

A *signature (sign) pattern* is a diagonal sign pattern all of whose diagonal entries are nonzero. Two sign pattern matrices A_1 and A_2 are said to be *signatorily equivalent* if there are signature patterns S_1 and S_2 such that $A_2 = S_1 A_1 S_2$.

A sign pattern $A \in Q_n$ is said to be *sign nonsingular* if every matrix $B \in Q(A)$ is nonsingular. It is well known that A is sign nonsingular if and only if $\det A = +$ or $\det A = -$, that is, in the standard expansion of $\det A$ into $n!$ terms, there is at least one nonzero term, and all the nonzero terms have the same sign. A is said to be *sign singular* if every matrix $B \in Q(A)$ is singular, or equivalently, if $\det A = 0$.

A sign pattern matrix A is said to be an *L-matrix* (see [3]) if every real matrix $B \in Q(A)$ has linearly independent rows. It is known that A is an L-matrix if and only if for every nonzero diagonal pattern D, DA has a unisigned column (that is, a nonzero column that is nonnegative or nonpositive). For a sign pattern matrix A, the *minimum rank* of A, denoted $\mathrm{mr}(A)$, is defined as

$$\mathrm{mr}(A) = \min_{B \in Q(A)} \{\mathrm{rank}\ B\},$$

while the *maximum rank* of A, denoted $\mathrm{MR}(A)$, is defined as

$$\mathrm{MR}(A) = \max_{B \in Q(A)} \{\mathrm{rank} B\}.$$

The maximum rank of a sign pattern A is the same as the term rank of A, which is the maximum number of nonzero entries which lie in distinct rows and in distinct columns of A. However, determination of the minimum rank of a sign pattern matrix in general is a longstanding open problem (see [1, 9]) in combinatorial matrix theory. Recently, there have been some papers concerning this topic, for example [2, 4, 5, 6, 7, 8, 10, 11]. In particular, as indicated in [5], matrices realizing the minimum rank of a sign pattern have applications in the study of neural network. In [2, 8, 10, 11], the author allow a free diagonal. We consider a fixed sign pattern so that the diagonal entries have prescribed signs. In our research we raise the following basic conjecture. For any $m \times n$ sign pattern matrix A with $\mathrm{mr}(A) = k$, there exists a rational matrix (equivalently, an integer matrix) $B \in Q(A)$ such that rank $B = k$. We know that the conjecture holds in certain cases, but we do not know the complete answer. In Section 2, we give several statements equivalent to this original conjecture, and in section 3 and 4 we exhibit some cases for which the conjecture or some equivalent statement holds. Finally, in Section 5, we consider connections between this conjecture and the existence of positive rational solutions of certain systems of quadratic homogeneous polynomial equations with each coefficient equal to either -1 or 1.

2. Equivalent Conjectures

We recall the original conjecture, which we now refer to as Conjecture 1.

Conjecture 1. For every $m \times n$ sign pattern matrix A with $\mathrm{mr}(A) = k$, there exists a rational matrix (equivalently, an integer matrix) $B \in Q(A)$ such that $\mathrm{rank} B = k$.

Conjecture 2. For a real matrix $B = \begin{bmatrix} I_r & C \\ D & 0 \end{bmatrix}$, where $r = \mathrm{rank}\ B$, there exists a rational matrix F such that $\mathrm{sgn}(F) = \mathrm{sgn}(B)$ and $\mathrm{rank}(F) = r$.

Conjecture 3. For real matrices D and C with $DC = 0$, there are rational matrices D^* and C^* such that $\mathrm{sgn}(D^*) = \mathrm{sgn}(D)$, $\mathrm{sgn}(C^*) = \mathrm{sgn}(C)$, and $D^*C^* = 0$.

Conjecture 4. For real matrices D, C, and E, with $DC = E$, there are rational matrices D^*, C^*, and E^* such that $\mathrm{sgn}(D^*) = \mathrm{sgn}(D)$, $\mathrm{sgn}(C^*) = \mathrm{sgn}(C)$, $\mathrm{sgn}(E^*) = \mathrm{sgn}(E)$, and $D^*C^* = E^*$.

Theorem 2.1. The above Conjecture 1–4 are equivalent.

Proof. First, assume that Conjecture 1 holds, and consider a real matrix $B = \begin{bmatrix} I_r & C \\ D & 0 \end{bmatrix}$, where $r = \mathrm{rank} B$. Set $A := \mathrm{sgn}(B)$ and $k := \mathrm{mr}(A)$. We then have $k \leq r$. However, from the form of the matrix B, it is clear that $r \leq k$. Hence,$\mathrm{mr}(A) = r$, and from Conjecture 1, we have a rational matrix F such that $\mathrm{sgn}(F) = \mathrm{sgn}(B)$ and rank $F = r$. We have thus proved the implication Conjecture 1 $\Rightarrow$ Conjecture 2.

Next, observe that the matrix $B = \begin{bmatrix} I_r & C \\ D & 0 \end{bmatrix}$ is row equivalent to the matrix $\begin{bmatrix} I_r & C \\ 0 & -DC \end{bmatrix}$. Hence, $\mathrm{rank} B = r$ if and only if $DC = 0$. Therefore, Conjecture 2 $\Leftrightarrow$ Conjecture 3.

To prove the implication Conjecture 3 $\Rightarrow$ Conjecture 4, assume that Conjecture 3 holds. Consider real matrices D, C, and E, with DC =E, and let t be the number of rows of E. We have $DC - E = O$, or,

$$\begin{bmatrix} D & I_t \end{bmatrix} \begin{bmatrix} C \\ -E \end{bmatrix} = 0.$$

From Conjecture 3, we obtain rational matrices $\begin{bmatrix} D^* & D_t^* \end{bmatrix}$ and $\begin{bmatrix} C^* \\ -E^* \end{bmatrix}$ with

$$\begin{bmatrix} D^* & D_t^* \end{bmatrix} \begin{bmatrix} C^* \\ -E^* \end{bmatrix} = 0$$

such that $\text{sgn}(D^*) = \text{sgn}(D)$, $\text{sgn}(C^*) = \text{sgn}(C)$, $\text{sgn}(E^*) = \text{sgn}(E)$, $\text{sgn}(D_t^*) = \text{sgn}(I_t)$ (that is, D_t^* is diagonal matrix with positive diagonal entries). Then, $D^*C^* - D_t^*E^* = 0$, or , $D^*C^* = D_t^*E^*$. With $\text{sgn}(D_t^*) = \text{sgn}(I_t)$, we have $\text{sgn}(D_t^*E^*) = \text{sgn}(E^*) = \text{sgn}(E)$. Thus, Conjecture 3 $\Rightarrow$ Conjecture 4.

Finally, to prove the implication Conjecture 4 $\Rightarrow$ Conjecture 1, assume that Conjecture 4 holds. Consider any $m \times n$ sign pattern matrix A with $\text{mr}(A) = k$. We have a real matrix $C \in Q(A)$ such that $\text{rank}C = k$. Let $C = LR$ be a full-rank factorization of C, so that L and R have dimensions $m \times k$, and $k \times n$, respectively, and rank L = rank $R = k$. From Conjecture 4, we have rational matrices C^*, L^*, and R^* with $C^* = L^*R^*$, Where L^* and R^* have dimensions $m \times k$ and $k \times n$,respectively, and $\text{sgn}(C^*) = \text{sgn}(C) = A$. Now, rank C^* = rank $L^*R^* \leq$ rank $L^* \leq k$, that is, rank $C^* \leq k$. But, with $C^* \in Q(A)$, and $\text{mr}(A) = k$, we get rank $C^* \geq k$. Hence, rank $C^* = k$. Thus, Conjecture 4 $\Rightarrow$ Conjecture 1. ■

As a special case, we can show that Conjecture 3 holds true if C has one column or D has one row.

Proposition 2.2. For real matrices D and C with $DC = 0$, where C has one column or D has one row, there are rational matrices D^* and C^* such that $\text{sgn}(D^*) = \text{sgn}(D)$, $\text{sgn}(C^*) = \text{sgn}(C)$, and $D^*C^* = 0$.

Proof. Suppose D and C are real matrices with $DC = 0$. Without loss of generality, assume that C has one column, with say n entries. We then have

$$c_1d_1 + c_2d_2 + ... + c_nd_n = 0,$$

where d_i denotes the i^{th} column of D and c_i denotes the i^{th} entry of C. So, we can write

$$a_1(|c_1|d_1) + a_2(|c_2|d_2) + ... + a_n(|c_n|d_n) = 0, \text{ where } a_i = \begin{cases} -1 & , \ c_i < 0 \\ 0 & , \ c_i = 0 \\ 1 & , \ c_i > 0 \end{cases}$$

By inspecting each coordinate separately, we can replace the respective columns $(|c_i|d_i)$ by integer vectors to obtain an integer matrix D^*, along with a $(1, -1, 0)$-vector C^*, such that $\text{sgn}(D^*) = \text{sgn}(D)$, $\text{sgn}(C^*) = \text{sgn}(C)$, and $D^*C^* = 0$. ■

Lemma 2.3. Let A be an $m \times n$ sign pattern matrix. Then there exists a rational matrix $H \in Q(A)$ such that rank $H = \text{MR}(A)$.

Proof. Let $t = \text{MR}(A)$ and $q = \min\{m, n\}$. By assigning values of q or $-q$ to entries on some generalized diagonal of lenght t, while assigning values of 1 or -1 to the other nonzero entries of A, we obtain an integer matrix $H \in Q(A)$ with a $t \times t$ submatrix that has a strictly dominant generalized diagonal. Since this submatrix must be nonsingular,

$$t \leq \text{rank}\, H \leq \text{MR}(A) = t.$$

Thus, rank $H = t$. ■

Lemma 2.4. Let A be an $m \times n$ sign pattern matrix, and let C be a rational matrix in $Q(A)$. Then, for each positive integer 1 satisfying rank $C \leq 1 \leq \text{MR}(A)$, there exists a rational matrix $C_1 \in Q(A)$ with rank $C_1 = 1$.

Proof. From Lemma 2.3, we have a rational matrix $H \in Q(A)$ such that rank $H = \text{MR}(A)$. By successively replacing only one entry of C by the corresponding entry of H, we obtain a sequence of rational matrices in $Q(A)$,

$$F_0 = C, F_1, F_2, ..., F_s = H,$$

Where F_i and F_{i-1} differ in at most one entry, $i = 1, ..., s$. Hence, rank F_i and rank F_{i-1} are either the same or differ by 1. It follows that the set $\{\text{rank}\, F_i\}$ runs through all the integer values between rank C and rank H. In particular, there is some i such that $\text{rank}\, F_i = 1$. The matrix $C_1 := F_i$ is rational with rank $C_1 = 1$, and $C_1 \in Q(A)$. ■

We can now give two other statements equivalent to Conjecture 1.

Theorem 2.5. Let A be an $m \times n$ sign pattern matrix. Then, the following statements are equivalent:

(i) For any $m \times n$ sign pattern matrix A with $\text{mr}(A) = k$, there exists a rational matrix (equivalently, an integer matrix) $B \in Q(A)$ such that $\text{rank} B = k$.

(ii) For each positive integer l such that there exists a matrix $B \in Q(A)$ with $\text{rank} B = l$, there exists a rational matrix $C \in Q(A)$ with $\text{rank}\, C = l$.

(iii) For each positive integer j, $\text{mr}(A) \leq j \leq \text{MR}(A)$, there exists a rational matrix $C_j \in Q(A)$ with $\text{rank}\, C_j = j$.

Proof. Clearly, the implication (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) hold. Conversely, assume that (i) holds, so that there is a rational matrix in $Q(A)$ whose rank is $\text{mr}(A)$. Then, by Lemma 2.4, the statement (ii) and (iii) are true. ■

3. Special Cases

In this section, we first show that the above conjectures are true when $\text{mr}(A)$ is $1, 2, n-1$, or n.

Proposition 3.1. For any $m \times n$ sign pattern matrix A with $\text{mr}(A) = 1$, there exists a rational matrix $B \in Q(A)$ such that $\text{rank} B = 1$.

Proof. Since $\mathrm{mr}(A) = 1$, each nonzero column of A is a fixed sign pattern vector or its negation. Thus the $(1, -1, 0)$ matrix $B \in Q(A)$ has rank 1. ■

Proposition 3.2. For any $m \times n$ sign pattern matrix A with $\mathrm{mr}(A) = n$, there exists a rational matrix $B \in Q(A)$ such that $\mathrm{rank}B = n$.

Proof. Since $\mathrm{mr}(A) = n$, every matrix in Q(A) has rank n, so the result follows. Note that in this case, A^T is an L- matrix. ■

Lemma 3.3. For any $m \times n$ sign pattern matrix A with $\mathrm{mr}(A) \leq n-1 \leq \mathrm{MR}(A)$, there exists a rational matrix $B \in Q(A)$ such that $\mathrm{rank}B = n - 1$.

Proof. Since $\mathrm{mr}(A) \leq n - 1 \leq \mathrm{MR}(A)$, there is a real matrix $D \in Q(A)$ with $\mathrm{rank}D \leq n - 1$. So, the columns of D are linearly dependent, and some nonzero linear combination of these columns is zero. Hence, $DC = 0$ for some nonzero $n \times 1$ matrix C. Then, from Proposition 2.2, we have rational matrices D^* and C^* such that $\mathrm{sgn}(D^*) = \mathrm{sgn}(D) = A$, $\mathrm{sgn}(C^*) = \mathrm{sgn}(C)$, and $D^*C^* = 0$. So, some nontrivial linear combination of the columns of D^* is zero, and hence $\mathrm{rank}D^* \leq n - 1$. Thus, from Lemma 2.4, there exists a rational matrix $B \in Q(A)$ such that rank $B = n-1$. ■

The following result follows immediately from Lemma 3.3.

Theorem 3.4. For any $m \times n$ sign pattern matrix A with $\mathrm{mr}(A) = n - 1$, there exists a rational matrix $B \in Q(A)$ such that $\mathrm{rank}B = n - 1$.

We are now ready to establish one of the main result of this paper settling the case of minimum rank 2.

Theorem 3.5. For any $m \times n$ sign pattern matrix A with $\mathrm{mr}(A) = 2$, there exists a rational matrix $B \in Q(A)$ such that $\mathrm{rank}B = 2$.

Proof. Let $A = [a_{ij}]$ be an $m \times n$ sign pattern with $\mathrm{mr}(A) = 2$. Then, there is a real matrix $C = [c_{ij}] \in Q(A)$ with rank C=2. Let $C = LR$ be a full rank factorization of C, where L is $m \times 2$ and R is $2 \times n$. Writing $L = [u_0\ u]$ and $R = \begin{bmatrix} v_0^T \\ v^T \end{bmatrix}$, we have

$$C = LR = [u_0\ u] \begin{bmatrix} v_0^T \\ v^T \end{bmatrix} = u_0 v_0^T + uv^T. \tag{*}$$

The minimum rank of a sign pattern is invariant under signatory and permutational equivalence. Hence, by pre-and post-multiplication of the two sides of the above equation by suitable nonsingular diagonal matrices and permutation matrices (and thus replacing A by an equivalent sign pattern with minimum rank 2), we may assume that u_0 and v_0 are (0, 1) matrices and, further, that

$$u_0 v_0^T = \begin{bmatrix} J_{k_1 \times k_2} & 0 \\ 0 & 0 \end{bmatrix}$$

for some positive integers k_1 and k_2, where $J_{k_1 \times k_2}$ is the all 1's $k_1 \times k_2$ matrix.

Write $u = [u_1\ u_2 \ldots u_m]^T = [v_1\ v_2 \ldots v_n]^T$. In order to construct a rational matrix in the same sign pattern class as $C \in Q(A)$, we are going to perturb the entries of u and v on the right side of (*). Thus we introduce $\tilde{u} = [x_1\ x_2 \ldots x_m]^T$, $\tilde{v} = [y_1\ y_2 \ldots y_n]^T$ and treat the entries of $\tilde{u}$ and $\tilde{v}$ as variables such that $\mathrm{sgn}(x_i) = \mathrm{sgn}\ (u_i)$ for $1 \le i \le m$, and $\mathrm{sgn}(y_j) = \mathrm{sgn}\ (v_j)$ for $1 \le j \le n$.

Note that some entries of u or v may be zero and thus that the corresponding variables x_i or y_j must be constantly zero. Also note that if $i > k_1$ or $j > k_2$, then for all real (in particular, rational) values x_i and y_j such that $\mathrm{sgn}(x_i) = \mathrm{sgn}\ (u_i)$ and $\mathrm{sgn}(y_j) = \mathrm{sgn}\ (v_j)$, the perturbed right side of (*), namely

$$f(\tilde{u}, \tilde{v}) = u_0 v_0^T + \tilde{u}\tilde{v}^T = u_o v_0^T + [x_1\ x_2 \ldots x_m][y_1\ y_2 \ldots y_n]^T,$$

has the (i, j) entry with the same sign as c_{ij}, since the (i, j) entry of $u_0 v_0^T$ is zero and $\mathrm{sgn}(x_i y_j) = \mathrm{sgn}(u_i v_j) = \mathrm{sgn}\ (c_{ij}) = a_{ij}$.

Therefore, we now consider the (i, j) entries of the perturbed right side of (*) for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$; such entries are equal to $1 + x_i y_j$. In order that the perturbed right side of (*) is still in the sign pattern class of A, we must have

$$1 + x_i y_j > 0 \text{ for all } (i, j) \text{ with } c_{ij} > 0; \tag{1}$$

$$1 + x_i y_j < 0 \text{ for all } (i, j) \text{ with } c_{ij} < 0; \tag{2}$$

$$1 + x_i y_j = 0 \text{ for all } (i, j) \text{ with } c_{ij)} = 0; \tag{3}$$

Observe that rank $f(\tilde{u}, \tilde{v}) \leq 2$, since $f(\tilde{u}, \tilde{v})$ is the sum of two rank one matrices. Hence, if $f(\tilde{u}, \tilde{v})$ is in $Q(A)$, we must have rank $f(\tilde{u}, \tilde{v}) = 2$, since $\mathrm{mr}(A) = 2$. Therefore, to complete the proof, it suffices to construct a rational matrix $f(\tilde{u}, \tilde{v})$ in $Q(A)$.

Of course, if $[x_1 \ \ldots x_m] = u^T$ and $[y_1 \ \ldots y_n] = v^T$, then all the conditions (1)–(3) are satisfied. Note that by continuity, there is a positive number ϵ such that all the inequalities in the conditions (1) or (2) remain valid as long as the values of the nonzero x_i (respectively, y_j) are in neighborhoods of u_i (respectively, v_j) of radius ϵ, for all i and j. We may further assume that ϵ is less than the minimum absolute value of nonzero entries of $u = [u_1 \ u_2 \ldots u_m]^T$ and $v = [v_1 \ v_2 \ldots v_n]^T$. Thus it remains to find a suitable rational solution of the system of equations (3), so that each x_i (respectively, y_j) is in an ϵ-neighborhood of u_i (respectively, v_j). In case the system (3) is empty, we think of it to be satisfied by all values of the variables. Thus we may assume that (3) is nonempty. To find a desired rational solution to the system of equations (3), we consider the bipartite graph G with vertex set $\{1, 2, \ldots k_1\} \cup \{1', 2', \ldots, k_2'\}$ such that (i, j') is an edge iff $1 + x_i y_j = 0$ is in the system (3), or equivalently, iff $c_{ij} = 0$. Since $1 + x_i y_j = 0$ can be written as $x_i y_j = -1$, or $y_j = -1/x_i$, it is clear that if i_1 and i_2 have a common neighbor in G, then $x_1 = y_{j_2}$. Thus we identify x_{i_1} with x_{i_2} iff i_1 and i_2 have a common neighbor in G. Similarly, we identify y_{j_1} with y_{j_2} iff j'_1 and j'_2 have a common neighbor in G. Such identifications correspond to contractions of vertices in G; two vertices are replaced by one iff they have a common neighbor. After all such variable identifications, the system (3) is reduced to a system $(\hat{3})$ of independent equations, in which each variable occurs in at most one equation. suppose that $1 + x_i y_j = 0$ is in $(\hat{3})$. Then $y_j = -1/x_i$, so the value of y_j is determined by that of x_i. Further, $1 + u_i v_j = 0$ and there is a positive number $\delta_i < \epsilon$ such that whenever x_i is in the δ_i-neighborhood of u_i, then $y_j = -1/x_i$ is in the ϵ-neighborhood of v_j. Let $1 + x_{i_1} y_{j_1} = 0, \ldots, 1 + x_{i_t} y_{j_t} = 0$ be the equations in $(\hat{3})$.

Let x_{i_k} be a rational number in the δ_{i_k}-neighborhood of u_{i_k}, for $1 \leq k \leq t$. Let $y_{j_k} = -1/x_{i_k}$, for $1 \leq k \leq t$. Set the other variables occurring in (3) through identification mentioned above. For all the remaining nonzero x_i (corresponding to nonzero u_i), let x_i be any rational number in the ϵ-neighborhood of u_i. Similarly, for all the remaining nonzero y_j (corresponding to nonzero v_j), let y_j be any rational number in the ϵ- neighborhood of v_j. It is clear that with such choices of the values of the variables, all the conditions in (1)–(3) are satisfied and we arrive at a rational matrix $B = f(\tilde{u}, \tilde{v})$ in $Q(A)$, which completes the proof. ■

From the above results, we can now see that our original conjecture holds true if A has no more than four rows or no more than four columns.

Corollary 3.6. For any $m \times n$ sign pattern matrix A, where $m \leq 4$ or $n \leq 4$, there exists a rational matrix $B \in Q(A)$ such that rank $B = \mathrm{mr}(A)$.

For a sign pattern A which admits a matrix $B \in Q(A)$ with a certain structure, we can show that the original conjecture holds.

Theorem 3.7. If $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$, where B_1 is $r \times r$, rank$B$ = rank$B_1 = r$, and B_4 is entrywise nonzero, then there is a rational matrix F such that $\text{sgn}(F) = \text{sgn}(B)$ and $\text{rank}(F) = r$.

Proof. Observe that the matrix B is row equivalent to the matrix $\begin{bmatrix} B_1 & B_2 \\ 0 & B_4 - B_3B_1^{-1}B_2 \end{bmatrix}$. Now, $\text{rank}B = r$ implies that $B_4 = B_3B_1^{-1}B_2$. Since each entry of $B_3B_1^{-1}B_2$ depends continuously on the entries of B_1, B_2 and B_3, and $B_4 = B_3B_1^{-1}B_2$ is entrywise nonzero, there is a positive number ϵ (less than the smallest absolute value of the nonzero entries of B_1, B_2 and B_3) such that if the nonzero entries of B_1, B_2 and B_3 are perturbed within ϵ-neighborhoods of their original values, then any resulting perturbed matrices $\tilde{B}_1, \tilde{B}_2$ and $\tilde{B}_3$ satisfy that $\tilde{B}_1$ is invertible, $\text{sgn}(\tilde{B}_i)$=$\text{sgn}(B_i)$ for $1 \leq i \leq 3$, and $\text{sgn}(\tilde{B}_3\tilde{B}_1^{-1}\tilde{B}_2)$=sgn (B_4). We may choose $\tilde{B}_1, \tilde{B}_2$ and $\tilde{B}_3$ to be rational perturbations of B_1, B_2 and B_3, respectively, such that each nonzero entry is within the ϵ-neighborhood of the original value. Then $F = \begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 \\ \tilde{B}_3 & \tilde{B}_3\tilde{B}_1^{-1}\tilde{B}_2 \end{bmatrix}$ is a rational matrix with sgn $(F) = \text{sgn}(B)$ and rank $F = r$. ■

As a consequence of Theorem 3.7, we have

Theorem 3.8. If A is an entrywise nonzero sign pattern, then there is a rational matrix $F \in Q(A)$ such that $\text{rank}F = \text{mr}(A)$.

Of course, an important question is what happens when the matrix B_4 in Theorem 3.7 has some zero entries.

4. Partial Results

In this section, we give some partial results concerning the four conjectures posed in Section 2. In particular, we show that Conjecture 3 is true when one of the two matrices D or C is a rational matrix. To this end, we first establish the following fundamental useful result.

Theorem 4.1. Let V be a subspace of $\mathbb{R}^n$. If V has a rational basis and V contains a positive vector, then V contains a positive rational vector.

Proof. Let $\{v_1, v_2, ..., v_k\}$ be a rational basis for V and suppose that there is a positive vector $x_0 \in V \subseteq \mathbb{R}^n$. Now, $x_0 = c_1v_1 + c_2v_2 + \cdots + c_kv_k$ for some real scalars c_i. For each $1 \leq i \leq k$, choose a rational number t_i within an ϵ-neighborhood of c_i, for some sufficiently small constant $\epsilon > o$ so that $x = t_1v_1 + t_2v_2 + \cdots + t_kv_k$ is a positive, rational vector in V. ■

Corollary 4.2 Suppose that D is a rational matrix and x is a real vector such that $Dx = 0$. Then there is a rational vector x^* such that $\text{sgn}(x^*) = \text{sgn}(x)$ and $Dx^* = 0$.

Proof. Since $Dx = 0$, $x \in N(D)$, the null space of D. If $x = 0$, then the result is clearly true. If x has some nonpositive coordinate, we can delete or negate entries of x and the corresponding columns of D so that $Dx = 0$ and $x > 0$. Hence, we assume that $x > 0$. Since D is a rational matrix, $N(D)$ has a rational basis. By Theorem 4.1, there is a positive rational vector $x^* \in N(D)$, and thus $Dx^* = 0$.
■

Proposition 4.3. Let D be a rational matrix and let C be a real matrix such that $DC = 0$. Then there is a rational matrix C^* such that $\text{sgn}(C^*) = \text{sgn}(C)$ and $DC^* = 0$.

Proof. Apply Corollary 4.2 to the columns of C. ■

It is easy to see that any subspace of $\mathbb{R}^n$ that has a rational basis can be viewed as the null space of a rational matrix. Hence, Corollary 4.2 may be used to obtain the following generalization (which is of independent interest) of Theorem 4.1.

Theorem 4.4. Let V be a subspace of $\mathbb{R}^n$. If V has a rational basis and $x \in V$, then V contains a rational vector x^* such that $\operatorname{sgn}(x^*) = \operatorname{sgn}(x)$.

We now consider another special situation for which Conjecture 3 holds.

Proposition 4.5. Suppose that D and C are real matrices such that $DC = 0$. If

$$\min\{\operatorname{rank} D, \operatorname{rank} C\} \ \leq \ 1,$$

then there are rational matrices C^* and D^* such that $\operatorname{sgn}(C^*) = \operatorname{sgn}(C)$, $\operatorname{sgn}(D^*) = \operatorname{sgn}(D)$ and $D^*C^* = 0$.

Proof. If rank D or rank C is zero, then the result is clearly true. Next, without loss of generality, assume that rank $C = 1$. Let v be a nonzero column of C. Then, any other nonzero column of C is a positive or negative multiple of v. Hence, we may assume that C has one (nonzero) column. The result then follows from Proposition 2.2. ■

Corollary 4.6. Suppose that D and C are real matrices such that $DC = 0$. If D has dimensions $m \times k$ where $k \leq 3$, then there are rational matrices C^* and D^* such that $\operatorname{sgn}(C^*) = \operatorname{sgn}(C)$, $\operatorname{sgn}(D^*) = \operatorname{sgn}(D)$ and $D^*C^* = 0$.

Proof. Since the row space of D is orthogonal to the column space of C, we have

$$\operatorname{rank} D + \operatorname{rank} C \ \leq k \ \leq \ 3.$$

Hence,

$$\min\{\operatorname{rank} D, \operatorname{rank} C\} \ \leq \ 1,$$

and the result follows from Proposition 4.5. ■

Corollary 4.7. Suppose that D and C are real matrices such that $DC = 0$. If D has dimensions $m \times k$ and

$$\max\{\operatorname{rank} D, \operatorname{rank} C\} \ \geq \ k - 1,$$

then there are rational matrices C^* and D^* such that $\operatorname{sgn}(C^*) = \operatorname{sgn}(C)$, $\operatorname{sgn}(D^*) = \operatorname{sgn}(D)$ and $D^*C^* = 0$.

Proof. The inequality

$$\max\{\operatorname{rank} D, \operatorname{rank} C\} \;\geq\; k - 1,$$

forces the inequality

$$\min\{\operatorname{rank} D, \operatorname{rank} C\} \;\leq\; 1.$$

Hence, we say again use Proposition 4.5 to obtain our result. ■

5. Connection with Systems of Polynomial Equations

Suppose that D and C are real matrices such that $DC = 0$, as in Conjecture 3. By representing the positive entries of D and C by some independent variables (indeterminate) and representing the negative entries of D and C by the negatives of some other independent variables, we obtain a matrix equation $\tilde{D}\tilde{C} = 0$. For example, starting with

$$\begin{bmatrix} 1 & 1 & \sqrt{2} & -1 \\ 1 & -3 & -\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ -\sqrt{2} & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

we arrive at the following matrix equation

$$\begin{bmatrix} x_1 & x_2 & x_3 & -x_4 \\ x_5 & -x_6 & -x_7 & 0 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \\ -y_5 & 0 \\ 0 & y_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

By comparing the corresponding entries of the two sides of the above equation, we get a system of homogeneous quadratic polynomial equations in the variables where each coefficient is either -1 or 1. The assumptions on D and C imply that this system has a positive solution (a solution with all the variables positive). Conjecture 3 amounts to saying that every such system of homogeneous quadratic polynomial equations has a positive rational solution.

By allowing D and C in the matrix equation $DC = 0$ to be generic matrices, each of whose nonzero entries is represented by a distinct variables or the negative of a distinct variables, we arrive at an equivalent, polynomial version of Conjecture 3:

Conjecture 5. Let $D_{m\times k}, C_{k\times n}$ be matrices each of whose nonzero entries is represented by a distinct variables or the negative of a distinct variables. If the system of homogeneous quadratic equations arising from $DC = 0$ has a positive solution, then it has a positive rational solution.

A natural, more general question is:

Question 5.1. Given a system of homogeneous quadratic polynomial equations where each nonzero term involves the product of two distinct variables and each coefficient in every equation is either -1 or 1. Suppose that the system has a positive solution. Does it necessarily have a positive rational solution?

Obviously, if the answer to Question 5.1 is yes, then Conjecture 5, and hence Conjecture 3, is true. However, as the following examples shows the answer to Question 5.1 turns out to be negative.

Example 5.2. The system of homogeneous quadratic polynomial equations

$$xy + xz - yw = 0 \tag{1}$$

$$xw + yz - zw = 0 \tag{2}$$

$$yz - xz - yw = o \tag{3}$$

has a positive solution. But it does not have any positive rational solution.

Proof. Consider any nontrivial solution of the system with $y \neq 0$. Since the system is homogeneous, then by dividing the value of each variable by the value of y, we get a solution with $y = 1$. Thus, without loss of generality, we assume that $y = 1$. Substituting 1 for y in the equations (1)–(3), we obtain

$$x + xz - w = 0 \tag{4}$$

$$xz + z - zw = 0 \tag{5}$$

$$z - xz - w = 0 \tag{6}$$

From equation (4) and (6), we have $x - z + 2xz = 0$, or $z = \frac{x}{1-2x}$. Substituting $w = x + xz$ (obtained from (4) into (5), we get

$$x(x + xz) + z - z(x + xz) = 0, \text{ namely,}$$

$$x^2 + x^2 z + z - xz - xz^2 = 0 \tag{7}$$

By substituting $z = \frac{x}{1-2x}$ into (7) and simplifying the resulting equation, we obtain

$$x(2x^3 - 2x^2 - 2x + 1) = 0. \tag{8}$$

Hence, every solution of the system with $y = 1$ is given by

$$(x, y, z, w) = \left(x, 1, \frac{x}{1-2x}, \frac{x(1-x)}{1-2x}\right),$$

where x satisfies (8). Such a solution is positive if and only if $0 < x < 1/2$. By Intermediate Value Theorem, (8) has a solution in the open interval $(0, 1/2)$, which yields a positive solution of the homogeneous system. However, it can be easily verified that (8) has no rational solution in the open interval $(0, 1/2)$, and hence, the homogeneous system has no positive rational solution with $y = 1$. It follows that the homogeneous system has no positive rational solution. ■

Note that a system of homogeneous quadratic polynomial equations that can arise from a matrix equation of the form $DC = 0$ is quite restrictive. In particular, such a system must satisfy that

(i) each coefficient in any equation is either -1 or 1,

(ii) each nonzero term in any equation involves the product of two distinct variables,

(iii) each variable can occur in at most one term of any of the equations in the system, and

(iv) the set of variables may be partitioned into $X \cup Y$ such that each term in any equation involves a product of a variable in X and a variable in Y.

Since the system in Example 5.2 does not satisfy (iii), it can not arise from a matrix equation $DC = 0$.

If only positive solutions are concerned, a system of homogeneous quadratic polynomial equations with some square terms can be transformed into an equivalent system of homogeneous quadratic polynomial equations without square terms. For instance, a square term x^2 may be replaced by xx_1 after adding an equation such as $y(x - x_1) = 0$. We illustrate this idea with the following example, which also provides a simpler example than Example 5.2.

Example 5.3. If only the positive solutions are concerned, the system of homogeneous quadratic polynomial equations

$$x^2 - y^2 = 0 \tag{9}$$

$$x^2 + y^2 - z^2 = 0 \tag{10}$$

is equivalent to the following system of homogeneous quadratic polynomial equations without square terms

$$xx_1 - yy_1 = 0 \tag{11}$$

$$xx_1 + yy_1 - zz_1 = 0 \tag{12}$$

$$y(x - x_1) = 0 \tag{13}$$

$$z(y - y_1) = 0 \tag{14}$$

$$x(z - z_1) = 0 \tag{15}$$

Furthermore, it is easy to see that the system (9)-(10) (and hence, the system (11)-(15)) has a positive solution, but it does not have a positive rational solution.

■

Note that the system (11)-(15) can not arise from a matrix equation $DC = 0$, since condition (iii) is not satisfied.

It is clear that one homogeneous quadratic polynomial equation satisfying the conditions (i)–(iv) can arise from $DC = 0$ with $D_{1\times k}$ and $C_{k\times 1}$. Therefore, by Proposition 2.2, such an equation has a positive rational solution if and only if it has a positive solution.

Of course, a system of homogeneous quadratic polynomial equations in standard form with a positive solution must satisfy the condition:

(v) each equation contains a positive term and a negative term.

It is apparent that to have a system of homogeneous quadratic polynomial equations satisfying (i)–(v), the number of variables must be at least 4. In fact, in the case of 4 or 5 variables (denoted $x_1, \ldots, x_5$), every equation of a system of homogeneous quadratic polynomial equations satisfying (i)–(v) must be of the form $x_i x_j - x_k x_l = 0$, and hence, setting all the variables to be 1 yields a positive rational solution.

6. Sign Patterns That Almost Require Unique Rank

In this section, we study the sign patterns A such that

$$\mathrm{MR}(A) = \mathrm{mr}(A) + 1.$$

This can be rephrased as " sign patterns that almost require a unique rank ". Sign patterns A that require fixed rank, namely

$$\mathrm{MR}(A) = \mathrm{mr}(A),$$

are characterized in the following result, proved by D.Hershkowitz and H.Schneider (see [7])

Theorem 6.1. A sign pattern A requires a fixed rank r if and only if A is permutationally equivalent to a sign pattern of the form

$$\begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix},$$

where X is $k \times (r-k)$, $0 \leq k \leq r$, and Y and Z^T are $L-$ matrices. ■

As a first step towards characterizing patterns that almost require a unique rank, we consider whether a rank one adjustment of a sign pattern that requires a unique rank has this property. More specifically, suppose that $\mathrm{mr}(A) = \mathrm{MR}(A)$, $\mathrm{mr}(A_1) = \mathrm{MR}(A_1) = 1$, and $\tilde{A} = A + A_1$, where the sum is unambiguous. It is always true that

$$\mathrm{MR}(\tilde{A}) \quad \leq \mathrm{mr}(\tilde{A}) + 1$$

holds. Note that it can be easily seen that

$$\mathrm{MR}(\tilde{A}) \quad \leq \mathrm{MR}(A) + \mathrm{MR}(A_1) = \mathrm{MR}(A) + 1,$$

and

$$\mathrm{mr}(A) - \mathrm{MR}(A_1) \leq \mathrm{mr}(\tilde{A}).$$

Thus we clearly have

$$\mathrm{MR}(\tilde{A}) \leq \mathrm{mr}(\tilde{A}) + 2.$$

Can equality occur in this last expression?

As a special case, consider the case when A_1 has only one nonzero entry. Write $A = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix}$, as in Theorem 6.1. Note that if any modification occurs within the X block only, then $\mathrm{mr}(\tilde{A}) = \mathrm{MR}(\tilde{A}) = \mathrm{mr}(A)$.

Theorem 6.2. Suppose $A = \begin{bmatrix} X & Y \\ Z & O \end{bmatrix}$ (where Y and Z^T are $L-$ matrices) requires a unique rank and A_1 is a unit sign pattern (namely, a sign pattern that has only one nonzero entry) with the same size as A. Suppose that $\tilde{A} = A + A_1$ is unambiguously defined. Then

(i) If the nonzero entry of A_1 occurs at a position in X, then

$$\mathrm{mr}(\tilde{A}) = \mathrm{MR}(\tilde{A}) = \mathrm{mr}(A).$$

(ii) If the nonzero entry of A_1 occurs at a position in Y or Z, then

$$\mathrm{MR}(\tilde{A}) \quad \leq \mathrm{mr}(\tilde{A}) + 1 \;\text{ and }\; \mathrm{MR}(\tilde{A}) = \mathrm{MR}(A).$$

(iii) If the nonzero entry of A_1 occurs at a position in the zero block O of A, then

$$\mathrm{MR}(\tilde{A}) \quad \leq \mathrm{mr}(\tilde{A}) + 2.$$

The following example shows that equality can be attained in part (3).

Example 6.3. Let

$$A = \begin{bmatrix} + & 0 & + & + & + \\ 0 & 0 & 0 & + & + \\ + & 0 & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\mathrm{MR}(A) = \mathrm{mr}(A) = 4$, $\mathrm{MR}(A + A_1) = 5$, $\mathrm{mr}(A + A_1) = 3$.

Theorem 6.3. Let A be a sign pattern matrix such that

$$\mathrm{MR}(A) \quad \leq \mathrm{mr}(A) + 1.$$

Then there is a rational matrix $B \in Q(A)$ attaining the minimum rank of A.

Proof. Let $r = \mathrm{MR}(A)$. Since r is the rank of A, A is permutationally equivalent to

$$\begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix}$$

where X is $k \times (r-k)$ for some k ($0 \leq k \leq r$). If A requires a unique rank r, then certainly every rational matrix $B \in Q(A)$ attains the minimum rank of A.(In this case, Y and Z^T are $L-$ matrices, as in Theorem 6.1.) Suppose that A does not require unique rank. Then Y or Z^T is not an $L-$ matrix. Assume that Y is not an $L-$ matrix. Then it is well-known that there is a rational matrix $\tilde{A} \in Q(Y)$ such that $\mathrm{rank}(\tilde{Y}) \quad \leq k-1$ (since the rows of $\tilde{A}$ are linearly independent). Let $\begin{bmatrix} X' \\ Z' \end{bmatrix}$ be any rational matrix in $Q(\begin{bmatrix} X \\ Z \end{bmatrix})$. Then

$$\mathrm{rank}(\begin{bmatrix} X' & Y' \\ Z' & 0 \end{bmatrix}) \quad \leq \mathrm{rank}(\begin{bmatrix} X' \\ Z' \end{bmatrix}) + \mathrm{rank}(\begin{bmatrix} Y' \\ 0 \end{bmatrix}) \leq (r-k) + (k-1) = r-1.$$

However, since $r - 1 = \mathrm{MR}(A) - 1 \quad \leq \mathrm{mr}(A)$, we have $\mathrm{rank}(\begin{bmatrix} X' & Y' \\ Z' & 0 \end{bmatrix}) = r-1$.
■

Theorem 6.4.

$$\mathrm{mr}(\begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix}) \quad \geq \mathrm{mr}(Y) + \mathrm{mr}(Z)$$

.

Proof. If Y' has r_1 linearly independent columns and Z' has r_2 linearly independent columns, then the corresponding $r_1 + r_2$ columns of $\begin{bmatrix} X' \\ Z' \end{bmatrix}$ and $\begin{bmatrix} Y' \\ 0 \end{bmatrix}$ are easily seen to be linearly independent. ■

Alternatively, this can be seen from the fact that if B_1 and B_2 are nonsingular matrices of order r_1 and r_2, respectively, then every matrix containing $\begin{bmatrix} B_1 & * \\ 0 & B_2 \end{bmatrix}$ a submatrix has rank at least $r_1 + r_2$. Theorem 6.4 can be used to obtain lower

bounds for $\mathrm{mr}(A)$ when A has a lot of zero entries. Trivially, equality in Theorem 6.4 holds if A has no zero entries. Of course, we also have

$$\mathrm{mr}(\begin{bmatrix} X & Y \\ Z & O \end{bmatrix}) \quad \geq \mathrm{mr}(\begin{bmatrix} X & Y \end{bmatrix})$$

and

$$\mathrm{mr}(\begin{bmatrix} X & Y \\ Z & O \end{bmatrix}) \quad \geq \mathrm{mr}(\begin{bmatrix} X \\ Z \end{bmatrix}).$$

Note that there is counterpart of Theorem 6.4 for MR.

$$\mathrm{MR}(\begin{bmatrix} X & Y \\ Z & O \end{bmatrix}) \quad \geq \mathrm{MR}(Y) + \mathrm{MR}(Z),$$

as can be seen when $X \neq 0$ while $Y = 0$ and $Z = 0$. Nevertheless, in the term-rank partition

$$A = \begin{bmatrix} X & Y \\ Z & O \end{bmatrix}$$

where $Y_{k\times s}$ has k entries in distinct rows and columns while $Z_{t\times(r-k)}$ has $r-k$ entries in distinct rows and columns, we do have

$$\mathrm{MR}(A) = \mathrm{MR}(Y) + \mathrm{MR}(Z).$$

Acknowledgement

I would like to thank Drs. Florian Enescu, Carlos D'Andrea, and Jia-yu Sha for valuable discussions leading to Example 5.2. and 5.3.

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