12-4-2006

Characterizations in Domination Theory

Andrew Robert Plummer

CHARACTERIZATIONS IN DOMINATION THEORY

by

ANDREW ROBERT PLUMMER

Under the Direction of Johannes H. Hattingh

ABSTRACT

Let $G = (V, E)$ be a graph. A set $R \subseteq V$ is a restrained dominating set (total restrained dominating set, resp.) if every vertex in $V - R$ (V) is adjacent to a vertex in R and (every vertex in $V - R$) to a vertex in $V - R$. The restrained domination number of G (total restrained domination number of G), denoted by $\gamma_r(G)$ ($\gamma_{tr}(G)$), is the smallest cardinality of a restrained dominating set (total restrained dominating set) of G. If T is a tree of order n, then $\gamma_r(T) \geq \lceil \frac{n+2}{3} \rceil$. We show that $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil$. Moreover, we show that if $n \equiv 0 \bmod 4$, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil + 1$. We then constructively characterize the extremal trees achieving these lower bounds. Finally, if G is a graph of order $n \geq 2$ such that both G and $\overline{G}$ are not isomorphic to P_3, then $4 \leq \gamma_r(G) + \gamma_r(\overline{G}) \leq n + 2$. We provide a similar result for total restrained domination and characterize the extremal graphs G of order n achieving these bounds.

INDEX WORDS: Domination, Restrained Domination, Total Restrained Domination, Nordhaus-Gaddum, Dominating Set

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A Thesis Submitted in Partial Fulfillment of Requirements for the Degree of

Master of Science

in the College of Arts and Sciences

Georgia State University

2006

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Electronic Version Approved:

Office of Graduate Studies
College of Arts and Sciences
Georgia State University
December 2006

The lion and the eagle, to them this thesis is dedicated.

Pasquale Visconti

and

Christopher Michael Plummer

ACKNOWLEDGEMENTS

Foremost, I wish to express my deepest feelings of appreciation for my family; my father Jude for passing on his intellectual curiosity; my mother Patricia for passing on her artistic prowess greatly needed for mathematical imagination; my sister Elizabeth for instilling in me the very essence of critical analysis; my sister Kathleen for reminding me of the world outside of school; my sister Tanya and her family, Glenn, Emily and Matthew, for their stellar example of familial perseverance and countless hours of puerile fun; my grandparents Pasquale and Mildred for inculcating me with incurable workaholism; and my uncle Robert for prompting me to chart my own course. To my extended family I wish to express my appreciation for all of the intangible yet indelible virtues they have infixed in me.

I also wish to thank Professor Johannes Hattingh for helping me find my confidence and direction through many hours of discussion, Professor Guantao Chen for encouraging me to undertake difficult challenges, and Professor George Davis for his invaluable assistance in navigating the labyrinthine arena that is graduate school. Furthermore, I wish to thank Larry Fitzgerald, Brian Cook, Janine Du Plessis, Jingwu He, Crystal Gordon and Paul Channel for establishing a sense of camaraderie among the graduate students. Finally, I wish to extend my gratitude to the faculty and staff of the Department of Mathematics and Statistics, the Department of Computer Science and the Department of Philosophy for all of their kindness in shepherding me through the byzantine bureaucracy of Georgia State University.

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Chapter 1

Proem

In the first section of this chapter we present the notation and basic definitions that will be used throughout this thesis. In Section 1.2, we précis the provenance and development of the concept of domination, and of the variants restrained domination and total restrained domination. We then give formal definitions of these concepts and state several results previously established in this field of research. Finally, in Section 1.3, we delineate the scope of the remainder of this thesis.

1.1 Definitions

A *graph* G consists of a finite nonempty set of *vertices* (singular *vertex*) and a (possibly empty) set of unordered pairs of distinct vertices of G called *edges*. The *vertex set* of G is denoted by $V(G)$ (or simply V if the context is clear), while the *edge set* of G is denoted by $E(G)$ (or simply E). The number of vertices in $V(G)$ is denoted by $n(G)$ which is also known as the *order* of the graph G. A graph G is *trivial* if $n(G) = 1$ and *non-trivial* if $n(G) \geq 2$. Unless otherwise specified, the symbol $n(G)$ (or simply n) will be reserved exclusively for the order of a graph G. We write $G = (V, E)$ to mean that the graph G has vertex set V and edge set E.

The edge $e = uv$ is said to *join* the vertices u and v. If $e = uv$ is an edge of G, then u and v are *adjacent vertices*, while u and e are *incident*, as are v and e. A graph G is called *complete* if every two vertices of G are adjacent. We denote a complete graph of order n by K_n. The *degree* of a vertex v in G is the number of edges incident with v and is denoted $\deg_G(v)$ (or simply $\deg(v)$ if the context is clear). The *minimum degree* (respectively, *maximum degree*) among the vertices of G is denoted by $\delta(G)$ (respectively, $\Delta(G)$). If there is a vertex $v \in V(G)$ such that $\deg(v) = 0$, then v is called an *isolated vertex*, if $\deg(v) = 1$, then v is called an *endvertex*.

A *path* of G is a finite, alternating sequence $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$ of vertices and edges, beginning with vertex v_0 and ending with vertex v_n, such that $e_i = v_{i-1}v_i$ and $v_i \neq v_j$ for $i, j = 1, 2, \ldots, n$ and $i \neq j$. The number n (the number of occurrences of edges) is called the *length* of the path. For convenience, we omit the edge and comma syntax and instead write $v_0v_1v_2 \ldots v_n$ to indicate a path, unless otherwise specified. A graph of order n that is a path is denoted by P_n. Therefore, $P_n = v_1v_2 \ldots v_n$ indicates a path of order n on the vertices $v_1, v_2, \ldots, v_n$. A *cycle* of G is a path $v_1v_2 \ldots v_n$ $(n \geq 3)$ with the additional edge v_nv_1. A graph of order n that is a cycle is denoted by C_n. Therefore, $C_n = v_1v_2 \ldots v_nv_1$ indicates a cycle of order n on the vertices $v_1, v_2, \ldots, v_n$.

Let u and v be distinct vertices of G. The *distance* between vertices u and v, denoted by $d_G(u, v)$ (or simply $d(u, v)$ if the context is clear) is the length of a shortest path $u \ldots v$, if such a path exist in G. We call a path of maximum length in G a *diametrical path* in G. If there exists a path $u \ldots v$ in G we say that u is *connected* to v. The graph G is itself *connected* if u is connected to v for every pair u, v of vertices of G. A graph that is not connected is called *disconnected*. The trivial graph, then, is connected. A *subgraph* H of a graph G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is a *component* of G if H is a maximal connected subgraph of G.

For a graph $G = (V, E)$, let $v \in V$ and let $S \subseteq V$. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E\}$ (or simply $N(v)$) and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$ (or simply $N[v]$). A vertex $u \in N(V)$ is called a *neighbor* of v. The *open neighborhood* of S is defined by $N_G(S) = \cup_{v \in S} N_G(v)$ (or simply $N(S)$), and the *closed neighborhood* of S by $N_G[S] = N_G(S) \cup S$ (or simply $N[S]$). For a vertex v (respectively, an edge e) of G we denote by $G - v$ (respectively, $G - e$) the graph obtained from G by deleting the vertex v (respectively, the edge e).

A *tree* is a connected graph which has no cycles. We refer to a vertex of degree 1 in a tree T as a *leaf* of T. A vertex adjacent to a leaf we call a *remote vertex* of T. A *star* is a tree of order n comprising exactly $n - 1$ leaves. The trivial graph, then, is a star, and is also called the *trivial star*. A star of order $n \geq 2$ is called a *non-trivial* star. The vertex of a non-trivial star which is not a leaf is called the *center* of the star. For consistency, we consider P_2 a star on two vertices with the center chosen arbitrarily. A *galaxy* is a graph whose components are stars. A *double star* is a tree of order n comprising exactly $n - 2$ leaves.

For a vertex v of a tree T, we shall use the expression, *attach a P_m at v*, to refer to the operation of taking the union of T and a path P_m and joining one of the ends of this path to v with an edge. For $v \in V(T)$ and a leaf ℓ of T, the path $vx_1 \ldots x_k\ell$ is called a $v - L$ *endpath* if $\deg x_i = 2$ for each i. If the vertex v need not be specified, a $v - L$ path is also called an *endpath*.

1.2 Perlustration

The concept of domination is quite natural and appears in many situations in which one desires an optimal covering of some sort. Lore has it that domination in graphs derives from strategies in the game of chess, where one desires to cover (or dominate) the squares of a chessboard using certain chess pieces. In 1862 de Jaenisch [6] considered the problem of determining the minimum number of queens (with standard movement rules) that can be placed on a chessboard such that each square is either occupied by a queen or is occupiable by a queen in a single move.

The parallel between de Jaenisch's chessboard problem and domination in graphs is patent. Consider a standard chessboard and let the 64 squares comprise the vertex set of a graph G. Let two vertices (squares) be adjacent in G if each square is occupiable in a single move by a queen stationed on the other square. The graph G defined as such is called the *queen's graph*. Choosing a set of vertices that dominates G is tantamount to positioning queens on the chessboard as to either occupy or potentially occupy (in one move) each square. The domination number of G is the minimum number of queens required to achieve the desired covering (see Figure 1.1).

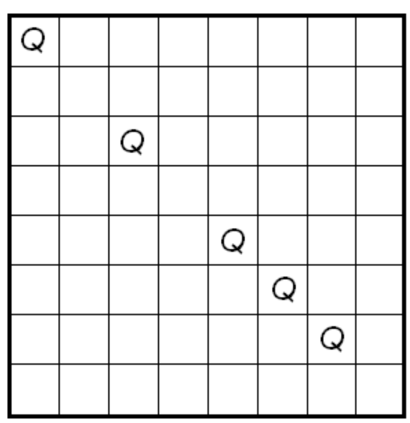

Figure 1.1 The minimum number of queens that dominate the squares of a standard chessboard.

Domination in graphs was formalized by Berge (see [1], p. 40) in 1958, and Ore [17] in 1962. We now provide the elements of the theory. A vertex v in a graph G *dominates* itself and each of its neighbors. Hence, v dominates the vertices in $N[v]$. A set $S \subseteq V$ is a *dominating set* if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of cardinality $\gamma(G)$ will be called a $\gamma(G)$*-set*. A *minimal dominating set* is a dominating set that contains no dominating set as a proper subset. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The recent book of Chartrand and Lesniak [3] includes a chapter on domination. A thorough study of domination appears in [10, 11]. In demonstrating the development of the theory, we give several known results.

Theorem 1.1 (Ore [17]) *Let D be a dominating set of a graph G. Then D is a minimal dominating set of G if and only if each $v \in D$ has at least one of the following two properties.*
P1: There exists a vertex $w \in V(G) - D$ such that $N(w) \cap D = \{v\}$;
P2: The vertex v is adjacent to no other vertex of D.

Theorem 1.2 (Bollobás and Cockayne [2]) *If G is a graph with no isolated vertex, then there exists a minimum dominating set D of vertices of G in which every vertex has property P1.*

Theorem 1.3 (Ore [17]) *If G is a graph with no isolated vertex and D is a minimal dominating set of G, then $V(G) - D$ is a dominating set of G.*

Corollary 1.4 (Ore [17]) *If G is a graph of order n with no isolated vertex, then* $\gamma(G) \leq \frac{n}{2}$.

Theorem 1.5 (Payan [18]) *Let G be a graph of order n with minimum degree $\delta \geq 2$. Then $\gamma(G) \leq \frac{n(1+\ln(\delta+1))}{\delta+1}$.*

The most commonly addressed application of domination in graphs is that of networking. Consider a network of transceivers and let G be a graph with a vertex set comprising the transceivers in the network. Two vertices (transceivers) are adjacent in G if each transceiver is capable of receiving transmissions broadcasted by the other (e.g. Figure 1.2).

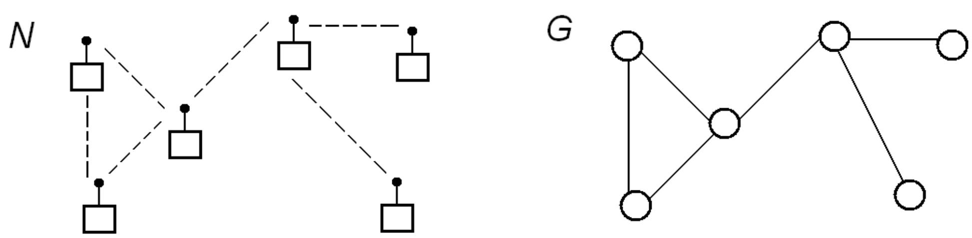

Figure 1.2 A network N of transceivers and its corresponding graph G.

Choosing a set of vertices that dominates G is tantamount to selecting transceivers in the network such that every transceiver is either broadcasting a signal or receiving a signal broadcast (see Figure 1.3). The domination number of G is the minimum number of transceivers required to achieve the desired covering.

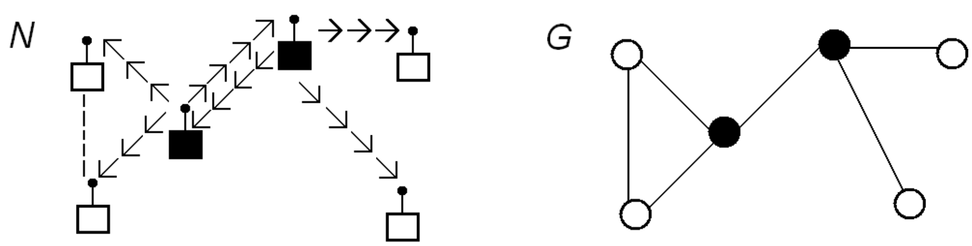

Figure 1.3 The darkened transceivers cover N with a signal broadcast, as the darkened vertices dominate G.

Now, suppose the network contains transceivers that broadcast and receive both a primary signal and an auxiliary signal. Moreover, no transceiver broadcasts both signals simultaneously, and a transceiver not broadcasting the primary signal must broadcast the auxiliary signal. Again, let G be a graph with a vertex set comprising the transceivers in the network with two vertices (transceivers) adjacent in G if each transceiver is capable of receiving transmissions broadcasted by the other. Choosing a set of vertices that dominates G is tantamount to selecting transceivers in the network such that every transceiver is either broadcasting the primary signal or receiving a primary signal broadcast. The domination number of G is the minimum number of transceivers required to achieve the desired covering.

Consider the transceivers not broadcasting the primary signal. We now require that these transceivers also receive the auxiliary signal. This is the concept of restrained domination. Choosing a set of vertices that dominates G with restraint is tantamount to selecting transceivers in the network such that every transceiver is either broadcasting the primary signal or receiving a primary signal broadcast, and each transceiver not broadcasting the primary signal also receives an auxiliary signal broadcast (e.g. Figure 1.4). The restrained domination number of G is the minimum number of transceivers required to achieve the desired covering.

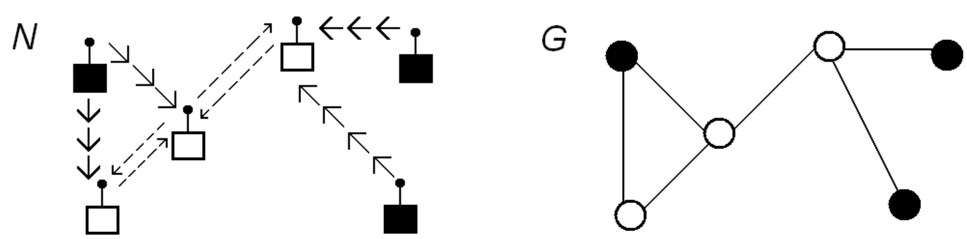

Figure 1.4 The darkened transceivers cover N with a primary signal broadcast while the remaining transceivers receive the auxiliary signal. The darkened vertices constitute a restrained dominating set of G.

Notice that a transceiver that receives transmissions from at most one other transceiver cannot receive both the primary signal and the auxiliary signal. Hence, each transceiver receiving transmissions from at most one other transceiver must broadcast the primary signal. That is, each vertex with degree less than or equal to one must be included in every restrained dominating set.

The concept of restrained domination was introduced by Telle and Proskurowski [19], albeit indirectly, as a vertex partitioning problem. Here conditions are imposed on a set S, the complementary set $V-S$ and on edges between the sets S and $V-S$. For example, if we require that every vertex in $V-S$ should be adjacent to some other vertex of $V-S$ (the condition on the set $V-S$) and to some vertex in S (the condition on edges between the sets S and $V-S$), then S is a restrained dominating set.

Restrained domination in graphs was formalized by Domke et al. [8] in 1999. We now provide the elements of the theory. Let $G=(V,E)$ be a graph. A set $S \subseteq V$ is a *restrained dominating set* (abbreviated **RDS**) if every vertex not in S is adjacent to a vertex in S and to a vertex in $V-S$. Every graph has a **RDS**, since $S=V$ is such a set. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a **RDS** of G. A **RDS** of cardinality $\gamma_r(G)$ will be called a $\gamma_r(G)$*-set*. In demonstrating the development of the theory, we give several known results.

Theorem 1.6 (Domke et al. [8]) *Let G be a graph of order n. Then $\gamma_r(G)=n$ if and only if G is a disjoint union of stars.*

Let $T_{\gamma_r} = \{T \mid T$ is obtained from P_4, P_5 or P_6 by attaching P_1 at v, where v is an remote vertex of the path$\}$. Let $C_{\gamma_r} = \{C \mid C$ is C_4 or C_5 or C can be obtained from C_3 by attaching P_1 at no more than two of the vertices of the cycle$\}$. Finally, let $\mathcal{F} = \{F \mid F$ is one of the bad graphs described in [7], p.240$\}$.

Theorem 1.7 (Domke et al. [8]) *Let G be a graph of order n. Then $\gamma_r(G)=n-2$ if and only if exactly one of the components of G is isomorphic to a graph $G' \in T_{\gamma_r} \cup C_{\gamma_r}$.*

Theorem 1.8 (Domke et al. [7]) *Let G be a connected graph of order $n \geq 3$ with $\delta(G) \geq 2$. If $G \notin \mathcal{F}$, then $\gamma_r(G) \leq \frac{n-1}{2}$.*

Theorem 1.9 (Domke et al. [7]) *Let G be a graph of order n with minimum degree $\delta \geq 2$. Then $\gamma_r(G) \leq n(1 + (\frac{1}{\delta})^{\frac{\delta}{\delta-1}} - (\frac{1}{\delta})^{\frac{1}{\delta-1}})$.*

Now, consider once again the network containing transceivers that broadcast and receive both a primary signal and an auxiliary signal. Recall that no transceiver broadcasts both signals simultaneously, and a transceiver not broadcasting the primary signal must broadcast the auxiliary signal. And again, let G be a graph with a vertex set comprising the transceivers in the network with two vertices (transceivers) adjacent in G if each transceiver is capable of receiving transmissions broadcasted by the other.

Suppose we desire to build redundancy into the network by requiring that all transceivers in the network receive the primary signal, that is, reception of the primary signal is total among the network of transceivers. Choosing a set of vertices that totally dominates G with restraint is tantamount to selecting transceivers in the network such that every transceiver is receiving a primary signal broadcast, and each transceiver not broadcasting the primary signal also receives an auxiliary signal broadcast (e.g. Figure 1.5). The total restrained domination number of G is the minimum number of transceivers required to achieve the desired covering.

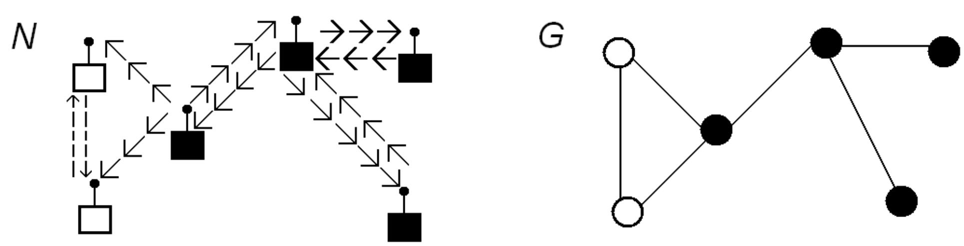

Figure 1.5 The darkened transceivers totally cover N with a primary signal broadcast while the remaining transceivers receive the auxiliary signal. The darkened vertices constitute a total restrained dominating set of G.

Notice that a transceiver that receives transmissions from no other transceiver receives neither the primary signal nor the auxiliary signal. Thus, we require that each transceiver receives transmissions from at least one other transceiver. That is, total restrained domination is well-defined only on graphs with minimum degree at least one.

Moreover, a transceiver that receives transmissions from exactly one other transceiver cannot receive both the primary signal and the auxiliary signal. Hence, each transceiver receiving transmissions from exactly one other transceiver must broadcast the primary signal, and receive the primary signal from the one other transceiver. That is, each vertex with degree equal to one, and its neighbor, must be included in every total restrained dominating set.

We note that the concept of total restrained domination was introduced by Telle and Proskurowski [19], albeit indirectly, as a vertex partitioning problem. Here conditions are imposed on a set S, the complementary set $V - S$ and on edges between the sets S and $V - S$. For example, if we require that every vertex in $V - S$ should be adjacent to some other vertex of $V - S$ (the condition on the set $V - S$) and to some vertex in S (the condition on edges between the sets S and $V - S$), and every vertex in S is also adjacent to some vertex in S (the condition on edges among vertices of S), then S is a total restrained dominating set.

Total restrained domination in graphs was formalized by Chen, Ma and Sun [4] in 2005, and further studied by Zelinka [20] and Maritz [13]. We now provide the elements of the theory. Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a *total restrained dominating set* (abbreviated **TRDS**) if every vertex is adjacent to a vertex in S and every vertex in $V - S$ is also adjacent to a vertex in $V - S$. Every graph without isolated vertices has a total restrained dominating set, since $S = V$ is such a set. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a **TRDS** of G. A **TRDS** of cardinality $\gamma_{tr}(G)$ will be called a $\gamma_{tr}(G)$*-set*. In demonstrating the development of the theory, we give several known results.

Theorem 1.10 (Chen et al. [4]) *Let T be a tree of order $n \geq 2$. Then $\gamma_{tr}(T) \geq \Delta(T) + 1$. Furthermore, $\gamma_{tr}(T) = \Delta(T) + 1$ if and only if T is a star.*

Theorem 1.11 (Maritz [13]) *If G is a connected graph of order $n \geq 4$, maximum degree Δ where $\Delta \leq n-2$, and minimum degree at least 2, then $\gamma_{tr}(G) \leq n - \frac{\Delta}{2} - 1$; and this bound is sharp.*

Theorem 1.12 (Maritz [13]) *If G is a connected bipartite graph of order $n \geq 5$, maximum degree Δ where $3 \leq \Delta \leq n-2$, and minimum degree at least 2, then $\gamma_{tr}(G) \leq n - \frac{2}{3}\Delta - \frac{2}{9}\sqrt{3\Delta - 8} - \frac{7}{9}$; and this bound is sharp.*

1.3 Purview

The unifying theme of this thesis is the characterization of extremal graphs corresponding to bounds on the graphical parameters restrained domination and total restrained domination. The characterizations are novel and simple, and the proof techniques employed remain viable in other areas of domination theory. Thus, the purpose of this thesis is to further the study of restrained domination and total restrained domination in graphs by presenting original results in these fields, and by so doing, circulate the proof techniques utilized herein.

In Chapter 2, we discuss restrained domination in trees. It is established in [9] that if T is a tree of order n, then $\gamma_r(T) \geq \lceil \frac{n+2}{3} \rceil$, and a characterization of trees achieving this bound is given. We recount the characterization given in [9] and conclude by giving a simpler, constructive characterization of the extremal trees T of order n achieving this lower bound.

In Chapter 3, we discuss total restrained domination in trees. We prove that if T is a tree of order n, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil$. We then give a constructive characterization of the extremal trees T of order n achieving this lower bound. Next, we show that if T is a tree of order $n \equiv 0 \bmod 4$, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil + 1$. We again constructively characterize the extremal trees T of order n achieving this lower bound.

Finally, in Chapter 4, we discuss Nordhaus-Gaddum results for restrained domination and total restrained domination in graphs. We bound the sum of the total restrained domination numbers of a graph and its complement, and provide characterizations of the extremal graphs achieving these bounds. It is known (see [8]) that if G is a graph of order $n \geq 2$ such that both G and $\overline{G}$ are not isomorphic to P_3, then $4 \leq \gamma_r(G) + \gamma_r(\overline{G}) \leq n + 2$. We also provide characterizations of the extremal graphs G of order n achieving these bounds.

Chapter 2

Restrained Domination in Trees

2.1 Introduction

In this chapter, we continue the study of a variation of the domination theme, namely that of restrained domination [7, 8, 9, 12, 19]. Recall that a set $S \subseteq V$ is a *restrained dominating set* (abbreviated **RDS**) if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a **RDS** of G. A **RDS** of cardinality $\gamma_r(G)$ will be called a $\gamma_r(G)$*-set*.

It is known [9] that if T is a tree of order n, then $\gamma_r(T) \geq \lceil \frac{n+2}{3} \rceil$. For $n \geq 1$, let $\mathcal{T}_n = \{T \mid T \text{ is a tree of order } n \text{ such that } \gamma_r(T) = \lceil \frac{n+2}{3} \rceil \}$. A constructive characterization of the extremal trees T of order n achieving this lower bound was obtained in [9]. For the purpose of stating this characterization, we define a **type (1)** operation on a tree T as attaching a P_2 at v where v is a vertex of T not belonging to some minimum **RDS** of T, and a **type (2)** operation as attaching a P_3 at v where v belongs to some minimum **RDS** of T. For $i = 1, 2$, let T_i be the tree obtained from $K(1, 3)$ by subdividing i edges once.

Let $\mathcal{C}_{3k} = \{T \mid T$ is a tree of order $3k$ which can be obtained from the tree T_2 by a finite sequence of operations of type (2)$\}$. Let $\mathcal{C}_{3k+1} = \{T \mid T$ is a tree of order $3k+1$ which can be obtained from P_4 by a finite sequence of operations of type (2)$\}$. Finally, let $\mathcal{C}_{3k+2} = \{T \mid T$ is a tree of order $3k+2$ which can be obtained from P_5 or from the tree T_1 by a finite sequence of operations of type (2)$\} \cup \{T \mid T$ is a tree of order $3k+2$ which can be constructed from the tree T_2 by a finite sequence of operations of type (2), followed by one operation of type (1) and then by a finite sequence of operations of type (2)$\}$. It was established in [9] that

Theorem 2.1 *For $n \geq 4$, $\mathcal{T}_n = \mathcal{C}_n$.*

The purpose of this chapter is to provide a simpler constructive characterization of the extremal trees T of order n achieving this lower bound. The technique employed in proving the characterization involves a diametrical argument that will be utilized again in the next chapter.

2.2 Extremal trees T with $\gamma_r(T) = \lceil \frac{n+2}{3} \rceil$

Let $\mathcal{T}$ be the class of all trees T of order n such that $\gamma_r(T) = \lceil \frac{n+2}{3} \rceil$. We will constructively characterize the trees in $\mathcal{T}$. In order to state the characterization, we define three simple operations on a tree T.

O1. Join a leaf or a remote vertex, or a vertex v or x of T on an endpath $vxyz$ to a vertex of K_1, where $n(T) \equiv 1 \bmod 3$.

O2. Join a remote vertex, or a vertex v of T which lies on an endpath vxz to a leaf of P_2, where $n(T) \equiv 0 \bmod 3$ or $n(T) \equiv 1 \bmod 3$.

O3. Join a leaf of T to ℓ disjoint copies of P_3 for some $\ell \geq 1$.

Let $\mathcal{C}$ be the class of all trees obtained from P_2 or P_4 by a finite sequence of Operations **O1**– **O3**. We will show that $T \in \mathcal{T}$ if and only if $T \in \mathcal{C}$. Let S be a $\gamma_r(T')$-set of T' throughout the proofs of the following lemmas.

Lemma 2.2 *Let $T' \in \mathcal{T}$ be a tree of order $n \equiv 1 \bmod 3$. If T is obtained from T' by Operation* **O1***, then $T \in \mathcal{T}$.*

Proof. Let u be a leaf or a remote vertex, or a vertex w or x on an endpath $wxyz$ of T', and suppose T is formed by attaching the singleton v to u. Then $S \cup \{v\}$ is a **RDS** of T, and so $\lceil \frac{n+3}{3} \rceil \leq \gamma_r(T) \leq \lceil \frac{n+2}{3} \rceil + 1$. Since $n \equiv 1 \bmod 3$, we have $\gamma_r(T) = \lceil \frac{n(T)+2}{3} \rceil$. Thus, $T \in \mathcal{T}$. $\Box$

Lemma 2.3 *Let $T' \in \mathcal{T}$ be a tree of order $n \equiv 0 \bmod 3$ or $n \equiv 1 \bmod 3$. If T is obtained from T' by Operation* **O2***, then $T \in \mathcal{T}$.*

Proof. Suppose v is a remote vertex or v lies on the endpath vxz and T is obtained from T' by adding the path vyz'.

We show that $v \notin S$. First consider the case when v is a remote vertex adjacent to a leaf z. Suppose $v \in S$. Then $S' = S - \{z\}$ is a **RDS** of $T'' = T' - z$, and so $\lceil \frac{n+1}{3} \rceil \leq \gamma_r(T'') \leq \lceil \frac{n+2}{3} \rceil - 1$, which is a contradiction when $n \equiv 0 \bmod 3$ or $n \equiv 1 \bmod 3$. Thus, $v \notin S$. In the case when v lies on the endpath vxz, one may show, as in the previous paragraph, that $x \notin S$. But then $v \notin S$, as required.

In both cases, the set $S \cup \{z'\}$ is a **RDS** of T, and so $\lceil \frac{n+4}{3} \rceil \leq \gamma_r(T) \leq \lceil \frac{n+2}{3} \rceil + 1$. However, as $n \equiv 0 \bmod 3$ or $n \equiv 1 \bmod 3$, we have $\gamma_r(T) = \lceil \frac{n+4}{3} \rceil = \lceil \frac{n(T)+2}{3} \rceil$. Thus, $T \in \mathcal{T}$. The proof is complete. $\Box$

Lemma 2.4 *Let $T' \in \mathcal{T}$ be a tree of order n. If T is obtained from T' by the Operation* **O3***, then $T \in \mathcal{T}$.*

Proof. Let S be a $\gamma_r(T')$-set of T', and suppose v is a leaf of T'. Then $v \in S$. Let T be the tree which is obtained from T by adding the paths $vx_iy_iz_i$ for $i = 1, \ldots, \ell$. Then $S \cup_{i=1}^{\ell} \{z_i\}$ is a **RDS** of T, and so $\lceil \frac{n+3\ell+2}{3} \rceil \leq \gamma_r(T) \leq \lceil \frac{n+2}{3} \rceil + \ell$. Consequently, $\gamma_r(T) = \lceil \frac{n(T)+2}{3} \rceil$, and so $T \in \mathcal{T}$. □

We are now in a position to prove the main result of this section.

Theorem 2.5 *$T \in \mathcal{C}$ if and only if $T \in \mathcal{T}$.*

Proof. Suppose $T \in \mathcal{C}$. We show that $T \in \mathcal{T}$, by using induction on $c(T)$, the number of operations required to construct the tree T. If $c(T) = 0$, then $T = P_2$ or $T = P_4$, both of which are in $\mathcal{T}$. Assume, then, for all trees $T' \in \mathcal{C}$ with $c(T') < k$, where $k \geq 1$ is an integer, that T' is in $\mathcal{T}$. Let $T \in \mathcal{C}$ be a tree with $c(T) = k$. Then T is obtained from some tree T' by one of the Operations $\mathbf{O1} - \mathbf{O3}$. But then $T' \in \mathcal{C}$ and $c(T') < k$. Applying the inductive hypothesis to T', T' is in $\mathcal{T}$. Hence, by Lemmas 2.2,2.3 or 2.4, $T \in \mathcal{T}$.

To show that $T \in \mathcal{C}$ for a nontrivial $T \in \mathcal{T}$, we use induction on n, the order of the tree T. If $n = 2$, then $T = P_2 \in \mathcal{C}$. If $n = 3$, then $T \notin \mathcal{T}$. If $n = 4$, then either $T = P_4$ or T is a star. If T is a star then $T \notin \mathcal{T}$. If $T = P_4$ then $T \in \mathcal{C}$. Let $T \in \mathcal{T}$ be a tree of order $n \geq 5$, and assume for all trees $T' \in \mathcal{T}$ of order $4 \leq n' < n$, that $T' \in \mathcal{C}$. Since $n(T) \geq 5$ and no stars are in $\mathcal{T}$, $\operatorname{diam}(T) \geq 3$.

If $\operatorname{diam}(T) = 3$, then T is a double star of order 5, has a remote vertex adjacent to two leaves, and is therefore constructible from P_4 by **O1**, whence $T \in \mathcal{C}$. Thus, we may assume $\operatorname{diam}(T) \geq 4$. Throughout, S will be used to denote a $\gamma_r(T)$-set of T.

Claim 2.6 *Suppose z is a leaf of T. If $S - \{z\}$ is a* **RDS** *of $T' = T - z$, then $n(T') \equiv 1$ mod 3 and $T' \in \mathcal{C}$.*

Proof. Suppose $S-\{z\}$ is a **RDS** of T'. Then $\lceil\frac{n-1+2}{3}\rceil \leq \gamma_r(T') \leq \lceil\frac{n+2}{3}\rceil - 1$. This yields a contradiction when $n \equiv 0 \bmod 3$ or $n \equiv 1 \bmod 3$. Hence, $n \equiv 2 \bmod 3$, and $\gamma_r(T') = \frac{n+1}{3} = \lceil\frac{n(T')+2}{3}\rceil$. Thus, $T' \in \mathcal{T}$, with $n(T') = n-1 \equiv 1 \bmod 3$. By the induction assumption, $T' \in \mathcal{C}$. $\diamond$

Suppose vxz or vz is an endpath of T. If $v, x \in S$, then $S-\{z\}$ is a **RDS** of $T' = T-z$. By Claim 2.6, the tree $T' = (T-z) \in \mathcal{C}$ and T can be constructed from T' by Operation **O1**. Thus, if vxz or vz is an endpath of T, we may assume $v, x \notin S$.

Suppose v is a remote vertex adjacent to at least two leaves, and let z be a leaf adjacent to v. Then $S-\{z\}$ is a **RDS** of $T' = T-z$. By Claim 2.6, the tree $T' = (T-z) \in \mathcal{C}$ and T can be constructed from T' by Operation **O1**. Thus, we may assume that every remote vertex is adjacent to exactly one leaf.

Let T be rooted at a leaf r of a longest path. Let v be any vertex on a longest path at distance $\text{diam}(T)-2$ from r. Suppose v lies on the endpath vyz'. Then, by the above remark, $v, y \notin S$.

Suppose $\deg(v) \geq 3$ and first assume v is a remote vertex adjacent to a leaf u. Since diam(T) ≥ 4, v has a parent vertex v_0. Suppose $v_0 \in S$. Moreover, suppose $\deg(v) \geq 4$. By Claim 2.6, v is adjacent to one leaf only, x is on an endpath vxz where $x \notin S$. Since $v_0 \in S$, it follows that $S' = S-\{u, z\}$ is a **RDS** for $T' = T-u-x-z$. Hence, $\lceil\frac{(n-3)+2}{3}\rceil \leq \gamma_r(T') \leq \lceil\frac{n+2}{3}\rceil - 2$, which is a contradiction. Hence $\deg(v) = 3$.

Consider $T' = T-u$. The vertex v in T' is on the endpath v_0vyz'. Since $v_0 \in S$, it follows that $S' = S-\{u\}$ is a **RDS** for T'. Thus, by Claim 2.6, $T' \in \mathcal{C}$ and T can be constructed from T' by Operation **O1**, whence $T \in \mathcal{C}$. Therefore, we may suppose $v_0 \notin S$. Then $S' = S-\{z'\}$ is a **RDS** for $T' = T-y-z'$. Hence, $\lceil\frac{(n-2)+2}{3}\rceil \leq \gamma_r(T') \leq \lceil\frac{n+2}{3}\rceil - 1$, which is a contradiction when $n \equiv 1 \bmod 3$. Hence $n \equiv 0 \bmod 3$ or $n \equiv 2 \bmod 3$ and $\gamma_r(T') = \lceil\frac{n}{3}\rceil = \lceil\frac{n(T')+2}{3}\rceil$. Thus, $T' \in \mathcal{T}$, with $n(T') = n-2 \equiv 0 \bmod 3$ or

$n(T') = n - 2 \equiv 1 \bmod 3$. By the induction assumption, $T' \in \mathcal{C}$. The tree T can now be constructed from T' by applying Operation **O2**, whence $T \in \mathcal{C}$.

We now assume that v is not a remote vertex. Thus, v lies on the endpaths vxz and vyz'. It follows that $S' = S - \{z'\}$ is a **RDS** for $T' = T - y - z'$. Hence, by reasoning similar to that in the previous paragraph, the tree T can be constructed from T' by applying Operation **O2**, whence $T \in \mathcal{C}$. Thus, we assume each vertex on a longest path at distance $\mathrm{diam}(T) - 2$ or $\mathrm{diam}(T) - 1$ from r has degree two.

Let v be any vertex on a longest path at distance $\mathrm{diam}(T) - 3$ from r. Let $vx_1y_1z_1$ be an endpath of T. Then $x_1, y_1 \notin S$, and so $v \in S$. Suppose $\deg(v) \geq 3$. If v is on an endpath vxz, it follows that $x, z \in S$, and by the remark following Claim 2.6, $T \in \mathcal{C}$. Suppose v is a remote vertex adjacent to a leaf u. By Claim 2.6, u is the only leaf adjacent to v. Moreover, $S' = S - \{u\}$ is a **RDS** for $T' = T - u$. Thus, by Claim 2.6, $T' \in \mathcal{C}$ and T can be constructed from T' by Operation **O1**, whence $T \in \mathcal{C}$.

We may assume that v lies only on endpaths $vx_iy_iz_i$, for $i = 1, \ldots, \ell$. Let e be the edge that joins v with its parent, and let $T(v)$ be the component of $T - e$ that contains v. Then $T(v)$ consists of ℓ disjoint paths $x_iy_iz_i$ ($i = 1, \ldots, \ell$) with v joined to x_i for $i = 1, \ldots, \ell$. Let $i \in \{1, \ldots, \ell\}$. Since $x_iy_iz_i$ is an endpath of T, we have $x_i \notin S$, $y_i \notin S$ and $v \in S$. Then $S - \cup_{i=1}^{\ell}\{z_i\}$ is a **RDS** of $T' = T - (T(v) - \{v\})$, and so $\lceil \frac{n-3\ell+2}{3} \rceil \leq \gamma_r(T') \leq \lceil \frac{n+2}{3} \rceil - \ell$, whence $\gamma_r(T') = \lceil \frac{n(T')+2}{3} \rceil$. Thus, $T' \in \mathcal{T}$, and by the induction assumption, $T' \in \mathcal{C}$. Note that v is a leaf of T'. The tree T can now be constructed from T' by applying Operation **O3**, whence $T \in \mathcal{C}$. $\square$

Chapter 3

Total Restrained Domination in Trees

3.1 Introduction

In this chapter, we continue the study of a variation of the domination theme, namely that of total restrained domination [4, 13, 20]. Recall that a set $S \subseteq V$ is a *total restrained dominating set* (abbreviated **TRDS**) if every vertex is adjacent to a vertex in S and every vertex of $V - S$ is adjacent to a vertex in $V - S$. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the smallest cardinality of a **TRDS** of G. A **TRDS** of cardinality $\gamma_{tr}(G)$ will be called a $\gamma_{tr}(G)$*-set*.

We show that if T is a tree of order n, then $\gamma_{tr}(T) \geq \lceil (n+2)/2 \rceil$. Moreover, we constructively characterize the extremal trees T of order n achieving this lower bound. Lastly, we show that if T is a tree of order $n \equiv 0 \bmod 4$, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil + 1$, and also constructively characterize the extremal trees T of order n achieving this lower bound.

3.2 The Lower Bound

The following result was established in [4], using a more cumbersome proof. As we shall see, this result will be useful in establishing a sharp lower bound on the total restrained domination number of a tree.

Proposition 3.1 *If $n \geq 2$ is an integer, then $\gamma_{tr}(P_n) = n - 2\lfloor \frac{n-2}{4} \rfloor$.*

Proof. Suppose S is a **TRDS** of P_n, whose vertex set is $V = \{v_1, \ldots, v_n\}$. Note that $v_1, v_2 \in S$. Moreover, any component of $V - S$ is of size exactly two. Each component is adjacent to a vertex of S, which, in turn, is adjacent to another vertex of S. Suppose there are m such components. Then $2m + 2m + 2 \leq n$ and so $m \leq \lfloor \frac{n-2}{4} \rfloor$. Thus $|S| = n - 2m \geq n - 2\lfloor \frac{n-2}{4} \rfloor$. On the other hand, $V - \{v_i \,|\, i \in \{3, 4, 7, 8, \ldots, 4\lfloor \frac{n-2}{4} \rfloor - 1, 4\lfloor \frac{n-2}{4} \rfloor\}\}$ is a **TRDS** of P_n, whence $\gamma_{tr}(P_n) = n - 2\lfloor \frac{n-2}{4} \rfloor$. $\square$

Corollary 3.2 *If $n \geq 2$ is an integer, then $\gamma_{tr}(P_n) \geq \lceil \frac{n+2}{2} \rceil$.*

Proof. Since $n - 2\lfloor \frac{n-2}{4} \rfloor \geq \lceil \frac{n+2}{2} \rceil$, the result follows from Proposition 3.1. $\square$

Let $T = (V, E)$ be a tree and $v, a, b \in V$ such that $\deg v \geq 3$ and $a, b \in N(v)$. Let ℓ_b be a leaf of the component of $T - v$ that contains b. Then the tree T' which arises from T by deleting the edge va and joining a to ℓ_b is called a (v, a, b)-*pruning* of T.

Theorem 3.3 *If T is a tree of order $n \geq 2$, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil$.*

Proof. We use induction on n. It is easy to check that the result is true for all trees T of order $n \leq 8$. Suppose, therefore, that the result is true for all trees of order less than n, where $n \geq 9$. Let $\gamma_{tr} = \min\{\gamma_{tr}(T) \mid T \text{ is a tree of order } n\}$. We will show that $\gamma_{tr} \geq \lceil \frac{n+2}{2} \rceil$.

Let $\mathcal{T} = \{T \mid T$ is a tree of order n such that $\gamma_{tr}(T) = \gamma_{tr}\}$. Among all trees in $\mathcal{T}$, let T be chosen so that the sum $s(T)$ of the degrees of its vertices of degree at least 3 is minimum. With respect to this, let T be chosen such that the number of leaves of T is minimum. If $s(T) = 0$, then $T \cong P_n$, and so $\gamma_{tr} = \gamma_{tr}(P_n) \geq \lceil \frac{n+2}{2} \rceil$. Suppose, therefore, that $s(T) \geq 1$. Since $s(T) \geq 1$, there exists a vertex v such that $\deg(v) \geq 3$. Let S be a $\gamma_{tr}(T)$-set of T.

Claim 3.4 *If v is a vertex of degree at least 3, then*
(i) $v \notin S$,
(ii) v is adjacent to exactly one vertex of S,
(iii) $\deg(v) = 3$.

Proof. Suppose $v \in S$. Then there exist $a, b \in N(v)$ such that $b \in S$. Let T' be a (v, a, b)-pruning of T. Then S is a **TRDS** of T', and so, by definition of γ_{tr}, we have that $\gamma_{tr} \leq \gamma_{tr}(T') \leq |S| = \gamma_{tr}$. Hence, $T' \in \mathcal{T}$. However, as T' has fewer leaves than T, we obtain a contradiction.

Thus, assume $v \notin S$ and let $a, b \in N(v)$ such that $a \notin S$ and $b \in S$. If $c \in N(v) - \{a, b\}$ is in S, then, by considering the (v, b, c)-pruning of T, we obtain a contradiction as before. We therefore assume that b is the only vertex in S which is adjacent to v.

Suppose $\deg(v) \geq 4$, let $\{c_1, \ldots, c_{\deg(v)-2}\} = N(v) - \{a, b\}$, let $c = c_1$ and let ℓ_b be a leaf of the component of $T - v$ that contains b. Let T' be the tree which arises from T by deleting the edges vc_i for $i = 1, \ldots, \deg(v) - 2$ and joining c to $\ell_b, c_2, \ldots, c_{\deg(v)-2}$. Note that $\deg_{T'}(v) = \deg_{T'}(\ell_b) = 2, \deg_{T'}(c) = \deg(c) + \deg(v) - 3 \geq \deg(c) + 1 \geq 3$, while all other vertices have the same degree in T' as in T. On the one hand, if $\deg(c) = 2$, then $s(T') = s(T) - \deg(v) + \deg_{T'}(c) = s(T) - 1$. On the other hand, if $\deg(c) \geq 3$, then $s(T') = s(T) - \deg(v) + \deg(v) - 3 = s(T) - 3$. Then S is a **TRDS** of T'. As $T' \in \mathcal{T}$ and $s(T') < s(T)$, we obtain a contradiction in both cases. Thus, $\deg(v) = 3$. $\diamond$

Claim 3.5 *No two vertices of degree* 3 *are adjacent.*

Proof. Using the notation employed in Claim 3.4, b is the only neighbor of v in S. By Claim 3.4, $\deg(b) \leq 2$. If $\deg(c) = 3$, then, by Claim 3.4, c is adjacent to a vertex in $V - S$ (other than v). Let T' be the (v, c, b)-pruning of T. Then S is a **TRDS** of T', and so, by definition of γ_{tr}, we have that $\gamma_{tr} \leq \gamma_{tr}(T') \leq |S| = \gamma_{tr}$. Hence, $T' \in \mathcal{T}$. However, as T' has fewer leaves than T, we obtain a contradiction. $\diamond$

Using the notation employed in the proof of Claim 3.4, the vertex $b \in S$ and, as it must be adjacent to another vertex in S, $\deg(b) = 2$ (cf. Claim 3.4). Let $b' \in S$ be the vertex adjacent to b and suppose b' is not a leaf. Then, by Claim 3.4, $\deg(b') = 2$. Let b'' be the neighbor of b' different from b. Then S is a **TRDS** of a tree T' obtained from T by deleting the edge $b'b''$ and joining the vertex b'' to some leaf of the component of $T - v$ containing c. Thus $T' \in \mathcal{T}$ and b' is a leaf of T'. Hence we may assume that b' is a leaf of T.

By Claim 3.5, $\deg(a) = \deg(c) = 2$. Let $a'(c'$, respectively) be the neighbor of a (c, respectively) which is different from v. Necessarily, $a', c' \in S$. Then $\deg(a') = \deg(c') = 2$ (cf. Claim 3.4). As each vertex in S is adjacent to another vertex of S, there exist vertices a'' and c'' in S which are adjacent to a' and c' respectively. We may assume, as we did for b', that a'' is a leaf of T.

If $n = 9$, then $\gamma_{tr}(T) = 6 = \lceil \frac{n+2}{2} \rceil$. Suppose, therefore, that $n \geq 10$. Let T' be the component of $T - cc'$ containing c'. Then $S \cap V(T')$ is a **TRDS** of T', so that $|S \cap V(T')| \geq \gamma_{tr}(T')$. Hence, $|S| \geq 4 + \gamma_{tr}(T')$. Applying the inductive hypothesis to the tree T' of order $n - 7$, we have $\gamma_{tr}(T') \geq \lceil \frac{n-5}{2} \rceil$, and so $\gamma_{tr}(T) = |S| \geq \lceil \frac{n+3}{2} \rceil \geq \lceil \frac{n+2}{2} \rceil$. $\square$

3.3 Extremal trees T with $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil$

Let $\mathcal{T}$ be the class of all trees T of order $n(T)$ such that $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil$. We will constructively characterize the trees in $\mathcal{T}$. In order to state the characterization, we define four simple operations on a tree T.

O1. Join a leaf or a remote vertex of T to a vertex of K_1, where $n(T)$ is even.

O2. Join a vertex v of T which lies on an endpath vxz to a leaf of P_3, where $n(T)$ is even.

O3. Join a vertex v of T which lies on an endpath vx_1x_2z to a leaf of P_3, where $n(T)$ is even.

O4. Join a remote vertex or a leaf of T to a leaf of each of ℓ disjoint copies of P_4 for some $\ell \geq 1$.

Let $\mathcal{C}$ be the class of all trees obtained from P_2 by a finite sequence of Operations **O1- O4**. We will show that $T \in \mathcal{T}$ if and only if $T \in \mathcal{C}$.

Lemma 3.6 *Let $T' \in \mathcal{T}$ be a tree of even order $n(T')$. If T is obtained from T' by one of the Operations* **O1-O3**, *then $T \in \mathcal{T}$.*

Proof. Let S be a $\gamma_{tr}(T')$-set of T' throughout the proof of this result.

Case 1. T is obtained from T' by Operation **O1**.

Let u be a leaf or a remote vertex of T', and suppose T is formed by attaching the singleton v to u. Then $S \cup \{v\}$ is a **TRDS** set of T, and so $\lceil \frac{n(T')+3}{2} \rceil \leq \gamma_{tr}(T) \leq \lceil \frac{n(T')+2}{2} \rceil + 1$. Since $n(T')$ is even, we have $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil$. Thus, $T \in \mathcal{T}$.

Case 2. T is obtained from T' by Operation **O2** or Operation **O3**.

Suppose v lies on the endpath vxz or vx_1x_2z and T is obtained from T' by adding the path y_1y_2z' to T' and joining y_1 to v. We show that $v \notin S$. First consider the case when v lies on the endpath vxz. Suppose $v \in S$. Then $S' = S - \{z\}$ is a **TRDS** of $T'' = T' - \{z\}$, and so $\lceil \frac{n(T')+1}{2} \rceil \leq \gamma_{tr}(T'') \leq \lceil \frac{n(T')+2}{2} \rceil - 1$. However, as $n(T')$ is even, we have $\frac{n(T')+2}{2} \leq \gamma_{tr}(T'') \leq \frac{n(T')+2}{2} - 1$, which is a contradiction. Thus, $v \notin S$. In the case when v lies on the endpath vx_1x_2z, one may show, as in the previous paragraph, that $x_1 \notin S$. But then $v \notin S$, as required.

In both cases, the set $S \cup \{y_2, z'\}$ is a **TRDS** of T, and so $\lceil \frac{n(T')+5}{2} \rceil \leq \gamma_{tr}(T) \leq \lceil \frac{n(T')+2}{2} \rceil + 2$. However, as $n(T')$ is even, we have $\gamma_{tr}(T) = \frac{n(T')+6}{2} = \lceil \frac{n(T)+2}{2} \rceil$. Thus, $T \in \mathcal{T}$. The proof is complete. $\Box$

Lemma 3.7 *Let $T' \in \mathcal{T}$ be a tree of order $n(T')$. If T is obtained from T' by the Operation* **O4**, *then $T \in \mathcal{T}$.*

Proof. Let S be a $\gamma_{tr}(T')$-set of T', and suppose v is a remote vertex or a leaf of T'. Then $v \in S$. Let T be the tree which is obtained from T' by adding the paths $u_ix_iy_iz_i$ to T' and joining u_i to v for $i = 1, \ldots, \ell$. Then $S \cup_{i=1}^{\ell} \{y_i, z_i\}$ is a **TRDS** of T, and so $\lceil \frac{n(T')+4\ell+2}{2} \rceil \leq \gamma_{tr}(T) \leq \lceil \frac{n(T')+2}{2} \rceil + 2\ell$. Therefore, $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil$, and so $T \in \mathcal{T}$. $\Box$

We are now in a position to prove the main result of this section.

Theorem 3.8 *T is in $\mathcal{C}$ if and only if T is in $\mathcal{T}$.*

Proof. Assume $T \in \mathcal{C}$. We show that $T \in \mathcal{T}$, by using induction on $c(T)$, the number of operations required to construct the tree T. If $c(T) = 0$, then $T = P_2$, which is in $\mathcal{T}$. Assume, then, for all trees $T' \in \mathcal{C}$ with $c(T') < k$, where $k \geq 1$ is an integer, that T' is in $\mathcal{T}$. Let $T \in \mathcal{C}$ be a tree with $c(T) = k$. Then T is obtained from some tree T' by one of the Operations **O1** $-$ **O4**. But then $T' \in \mathcal{C}$ and $c(T') < k$. Applying the inductive hypothesis to T', T' is in $\mathcal{T}$. Hence, by Lemma 3.6 or Lemma 3.7 , T is in $\mathcal{T}$.

To show that $T \in \mathcal{C}$ for a nontrivial $T \in \mathcal{T}$, we use induction on n, the order of the tree T. If $n = 2$, then $T = P_2 \in \mathcal{C}$. Let $T \in \mathcal{T}$ be a tree of order $n \geq 3$, and assume for all trees $T' \in \mathcal{T}$ of order $2 \leq n(T') < n$, that $T' \in \mathcal{C}$. Since $n(T) \geq 3$, $\mathrm{diam}(T) \geq 2$. If $\mathrm{diam}(T) = 2$, then T is a star with exactly two leaves, which can be constructed from P_2 by applying Operation **O1**. Thus, $T \in \mathcal{C}$. Since no double star is in $\mathcal{T}$, we may assume $\mathrm{diam}(T) \geq 4$. Throughout S will be used to denote a $\gamma_{tr}(T)$-set of T.

Claim 3.9 *Let z be a leaf of T. If $S - \{z\}$ is a* **TRDS** *of $T' = T - z$, then $T \in \mathcal{C}$.*

Proof. Assume $S - \{z\}$ is a **TRDS** of T'. Then $\lceil \frac{n-1+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 1$. This yields a contradiction when n is even. Hence, n is odd, and $\gamma_{tr}(T') = \frac{n+1}{2} = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{T}$, with $n(T') = n - 1$ even. By the induction assumption, $T' \in \mathcal{C}$. The tree T can now be constructed from T' by applying Operation **O1**, whence $T \in \mathcal{C}$. $\diamond$

Claim 3.9 implies that if vxz is an endpath of T, then we may assume $v \notin S$, since otherwise the tree is constructible. Claim 3.9 also implies that every remote vertex of T is adjacent to exactly one leaf, since otherwise it is constructible.

Claim 3.10 *If u is a leaf of T and v is either another leaf of T or the remote vertex adjacent to u, then $S' = S - \{u, v\}$ is not a* **TRDS** *of $T' = T - u - v$.*

Proof. Suppose, to the contrary, that S' is a **TRDS** of T'. Then $\lceil \frac{n-2+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 2$. Thus, $\lceil \frac{n}{2} \rceil + 2 \leq \lceil \frac{n+2}{2} \rceil$, which yields a contradiction. $\diamond$

Let T be rooted at a leaf r of a longest path. Let v be any vertex on a longest path at distance $\mathrm{diam}(T) - 2$ from r. Suppose v lies on the endpath vyz'. Then, by the remark above, $v \notin S$, which implies that v is not adjacent to a leaf. If v also lies on the endpath vxz, then $S - \{x, z\}$ is a **TRDS** of $T - x - z$, which is a contradiction by Claim 3.10. Thus, we assume each vertex on a longest path at distance $\mathrm{diam}(T) - 2$ or $\mathrm{diam}(T) - 1$ from r has degree two.

Let v be any vertex on a longest path at distance $\mathrm{diam}(T) - 3$ from r. Let vy_1y_2z' be an endpath of T. Then $y_1 \notin S$, and so $v \notin S$, which means all neighbors of v have degree at least 2.

Assume v also lies on the path vxz, where z is a leaf. Then, since each remote vertex is adjacent to exactly one leaf, vxz is an endpath. If v is dominated by a vertex other than x, then $S - \{x, z\}$ is a **TRDS** of $T' = T - x - z$, which is a contradiction (cf. Claim 3.10). Hence, v is dominated only by x. Then $S' = S - \{y_2, z'\}$ is a **TRDS** of $T' = T - y_1 - y_2 - z'$ and so $\lceil \frac{n-3+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 2$. This yields a contradiction when n is even. Hence, n is odd and $\gamma_{tr}(T') = \frac{n-1}{2} = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{T}$, with $n(T') = n - 3$ even. By the induction assumption, $T' \in \mathcal{C}$. The tree T can now be constructed from T' by applying Operation **O2**, whence $T \in \mathcal{C}$.

Assume v lies on the path vx_1x_2z. Since x_1 (x_2, respectively) is on a longest path at distance $\mathrm{diam}(T) - 2$ ($\mathrm{diam}(T) - 1$, respectively) from r, we have $\deg(x_1) = 2$ ($\deg(x_2) = 2$, respectively). This implies that vx_1x_2z is an endpath, and so $x_1 \notin S$. But then $S' = S - \{x_2, z\}$ is a **TRDS** of $T' = T - x_1 - x_2 - z$. Thus, $\lceil \frac{n-3+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 2$. This yields a contradiction when n is even. Hence, n is odd and $\gamma_{tr}(T') = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{T}$, with $n(T') = n - 3$ even. By the induction assumption, $T' \in \mathcal{C}$ and T can now be constructed from T' by applying Operation **O3**, whence $T \in \mathcal{C}$. Thus, we assume each vertex on a longest path at distance $\mathrm{diam}(T) - 3$ from r has degree two.

Let v be any vertex on a longest path at distance $\mathrm{diam}(T) - 4$ from r. As $P_5 \notin \mathcal{T}$, $v \neq r$ and $\mathrm{diam}(T) \geq 5$. Assume $\deg_T(v) \geq 3$. Let $vy_1y_2y_3z'$ be an endpath of T. But then, as y_2y_3z' is an endpath of T, it follows that $y_2 \notin S$, which implies $y_1 \notin S$ and $v \in S$. Moreover, $S' = S - \{y_3, z'\}$ is a **TRDS** of $T' = T - y_1 - y_2 - y_3 - z'$. Thus, $\lceil \frac{n-4+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 2$, whence $\gamma_{tr}(T') = \lceil \frac{n(T')+2}{2} \rceil$. We conclude that $T' \in \mathcal{T}$, and by the induction assumption, $T' \in \mathcal{C}$. If $\deg_T(v) = 2$ or when v is a remote vertex, then T can be constructed from T' by applying Operation **O4**.

We now assume that $\deg_T(v) \geq 3$ and that v is not adjacent to a leaf. If v also lies on the path vxz, where z is a leaf, then $v \notin S$, which is a contradiction. We therefore assume that v lies on the path vx_1x_2z, where z is a leaf. Since x_2 is a remote vertex, we have $\deg(x_2) = 2$. As x_1x_2z is an endpath of T, it follows that $x_1 \notin S$. As x_1 must be adjacent to another vertex in $V - S$, vertex x_1 lies on a path x_1, u_1, u_2, z''. But then x_1, with $\deg(x_1) \geq 3$, is a vertex at distance $\mathrm{diam}(T) - 3$ on a longest path from r, which is a contradiction.

Let e be the edge that joins v with its parent, and let $T(v)$ be the component of $T - e$ that contains v. Then $T(v)$ consists of ℓ disjoint paths $u_ix_iy_iz_i$ $(i = 1, \ldots, \ell)$ with v joined to u_i for $i = 1, \ldots, \ell$. Let $i \in \{1, \ldots, \ell\}$. Since $x_iy_iz_i$ is an endpath of T, we have $x_i \notin S$, $u_i \notin S$ and $v \in S$. Then $S - \cup_{i=1}^{\ell}\{y_i, z_i\}$ is a **TRDS** of $T' = T - (T(v) - v)$, and so $\lceil \frac{n-4\ell+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 2\ell$, whence $\gamma_{tr}(T') = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{T}$, and by the induction assumption, $T' \in \mathcal{C}$. Note that v is a leaf of T'. The tree T can now be constructed from T' by applying Operation **O4**, whence $T \in \mathcal{C}$. $\square$

Theorem 3.11 *Let T be a tree of order $n(T)$. If $n(T) \equiv 0 \bmod 4$, then $\gamma_{tr}(T) \geq \lceil \frac{n(T)+2}{2} \rceil + 1$.*

Proof. We will show that every tree T in $\mathcal{T} = \mathcal{C}$ has $n(T) \not\equiv 0 \bmod 4$, by using induction on $s(T)$, the number of operations required to construct the tree T. If $s(T) = 0$, then $T = P_2$, and $2 \not\equiv 0 \bmod 4$. Assume, then, for all trees $T' \in \mathcal{C}$ with $s(T') < k$, where $k \geq 1$ is an integer, that $n(T') \not\equiv 0 \bmod 4$. Let $T \in \mathcal{C}$ be a tree with $s(T) = k$. Then T is obtained from some tree T' by one of the Operations **O1** $-$ **O4**. Then $T' \in \mathcal{C}$, and by the induction hypothesis, $n(T') \not\equiv 0 \bmod 4$. If T is obtained from T' by one of the Operations **O1** $-$ **O3**, then $n(T') \equiv 2 \bmod 4$, and, since either a path of order one or a path of order three is attached to T' to form T, $n(T) \not\equiv 0 \bmod 4$. Moreover, $n(T) = n(T') + 4$ if T is obtained from T' by Operation **O4**, whence $n(T) \not\equiv 0 \bmod 4$. The result now follows. $\square$

3.4 Extremal trees T with $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil + 1$

Let $\mathcal{T}^* = \{T \mid T$ is a tree of order $n(T) \equiv 0 \bmod 4$ such that $\gamma_{tr}(T) = \lceil \frac{n+2}{2} \rceil + 1\}$. In order to constructively characterize the trees in $\mathcal{T}^*$, we define the following operations on a tree T:

O5. Join a leaf or a remote vertex v of T to a vertex of K_1, where $n(T) \equiv 3 \bmod 4$.

O6. Join a vertex v of T which lies on an endpath vxz to a vertex of K_2, where $n(T) \equiv 2 \bmod 4$.

O7. Join a vertex v of T which lies on an endpath vx_1x_2z to a vertex of K_2, where $n(T) \equiv 2 \bmod 4$.

O8. Join a vertex v of T which lies on an endpath vxz to a leaf of P_3, where $n(T) \equiv 1 \bmod 4$.

O9. Join a vertex v of T which lies on an endpath vx_1x_2z to a leaf of P_3, where $n(T) \equiv 1 \bmod 4$.

Let $\mathcal{I} = \{T \mid T$ is a tree obtained by applying one of the Operations **O5** $-$ **O9** to a tree $T' \in \mathcal{C}$ exactly once$\}$. Let $\mathcal{C}^* = \{T \mid T$ is a tree obtained from a tree $T' \in \mathcal{I}$ by applying Operation **O4** to T' zero or more times$\}$. We will show that $\mathcal{T}^* = \mathcal{C}^*$.

Lemma 3.12 *Let $T' \in \mathcal{C}$ be a tree of order $n(T') \equiv 3$ mod 4. If T is obtained from T' by Operation* **O5***, then $T \in \mathcal{T}^*$.*

Proof. Let u be a leaf or a remote vertex of T', and suppose T is formed by attaching the singleton v to u. Let S be a $\gamma_{tr}(T')$-set of T'. Then $S \cup \{v\}$ is a **TRDS** set of T, and so, since $n(T) \equiv 0 \bmod 4$, $\lceil \frac{n(T)+2}{2} \rceil + 1 \leq \gamma_{tr}(T) \leq |S| + 1 = \lceil \frac{n(T')+2}{2} \rceil + 1 = \lceil \frac{n(T)+1}{2} \rceil + 1$. Hence, $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil + 1$, and so $T \in \mathcal{T}^*$. $\square$

Lemma 3.13 *Let $T' \in \mathcal{C}$ be a tree of order $n(T') \equiv 2$ mod 4. If T is obtained from T' by either Operation* **O6** *or Operation* **O7***, then $T \in \mathcal{T}^*$.*

Proof. Let $\{u, v\}$ be the vertex set of K_2 and let S be a $\gamma_{tr}(T')$-set. The set $S \cup \{u, v\}$ is a **TRDS** of T, and so, since $n(T) \equiv 0$ mod 4, $\lceil \frac{n(T)+2}{2} \rceil + 1 \leq \gamma_{tr}(T) \leq |S| + 2 = \lceil \frac{n(T')+2}{2} \rceil + 2 = \lceil \frac{n(T)}{2} \rceil + 2$. Hence, $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil + 1$, and so $T \in \mathcal{T}^*$. $\Box$

Lemma 3.14 *Let $T' \in \mathcal{C}$ be a tree of order $n(T') \equiv 1$ mod 4. If T is obtained from T' by either Operation* **O8** *or Operation* **O9***, then $T \in \mathcal{T}^*$.*

Proof. Let S be a $\gamma_{tr}(T')$-set of T'. Assume v lies on the endpath vxz or vx_1x_2z and T is obtained from T' by adding the path y_1y_2z' to T' and joining y_1 to v. We show that $v \notin S$. First consider the case when v lies on the endpath vxz. Suppose $v \in S$. Then $x, z \in S$, and $S - \{z\}$ is **TRDS** of $T'' = T' - z$. Since $n(T'') \equiv 0$ mod 4, $\lceil \frac{n(T'')+2}{2} \rceil + 1 \leq \gamma_{tr}(T'') \leq |S| - 1 = \lceil \frac{n(T')+2}{2} \rceil - 1 = \lceil \frac{n(T'')+3}{2} \rceil - 1$, and so $\frac{n(T'')+4}{2} \leq \frac{n(T'')+2}{2}$, which is a contradiction. Thus, $v \notin S$. In the case when v lies on the endpath vx_1x_2z, one may show, as in the previous paragraph, that $x_1 \notin S$. But then $v \notin S$, as required. In both cases, the set $S \cup \{y_2, z'\}$ forms a **TRDS** of T, so that $\lceil \frac{n(T)+2}{2} \rceil + 1 \leq \gamma_{tr}(T) \leq |S| + 2 = \lceil \frac{n(T')+2}{2} \rceil + 2 = \lceil \frac{n(T)-1}{2} \rceil + 2$. Hence, $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil + 1$, and so $T \in \mathcal{T}^*$. $\Box$

The proof of the following result is similar to that of Lemma 3.7.

Lemma 3.15 *If T is obtained from $T' \in \mathcal{T}^*$ by Operation* **O4***, then $T \in \mathcal{T}^*$.*

Lemma 3.16 *If T is in $\mathcal{I}$, then T is in $\mathcal{T}^*$.*

Proof. Assume $T \in \mathcal{I}$. Then T is obtained from $T' \in \mathcal{C}$ by applying one of the Operations **O5** $-$ **O9** exactly once. Then, by Lemmas 3.12, 3.13 and 3.14, $T \in \mathcal{T}^*$. $\Box$

Theorem 3.17 *T is in $\mathcal{C}^*$ if and only if T is in $\mathcal{T}^*$.*

Proof. Assume $T \in \mathcal{C}^*$. We show that $T \in \mathcal{T}^*$, by using induction on $c(T)$, the number of operations required to construct the tree T. If $c(T) = 0$, then $T \in \mathcal{I}$, and the result follows from Lemma 3.16. Assume, then, for all trees $T' \in \mathcal{C}^*$ with $c(T') < k$, where $k \geq 1$ is an integer, that T' is in $\mathcal{T}^*$. Let $T \in \mathcal{C}^*$ be a tree with $c(T) = k$. Then T is obtained from some tree T' by applying Operation **O4**. But then $T' \in \mathcal{C}^*$ and $c(T') < k$. Applying the inductive hypothesis to T', T' is in $\mathcal{T}^*$. Hence, by Lemma 3.15, T is in $\mathcal{T}^*$.

To show that $T \in \mathcal{C}^*$ for a nontrivial $T \in \mathcal{T}^*$, we employ induction on $4n$, the order of the tree T. Suppose $n = 1$. Then $T \cong K_{1,3}$ or $T \cong P_4$, and T can be constructed from $P_3 \in \mathcal{C}$ by applying Operation **O5**. Let $T \in \mathcal{T}^*$ be a tree of order $4n$, where $n \geq 2$, and suppose $T' \in \mathcal{C}^*$ for all trees $T' \in \mathcal{T}^*$ of order $4n'$ where $n' < n$. The only trees T with $\mathrm{diam}(T) \leq 3$ which are in $\mathcal{T}^*$ are $K_{1,3}$ and P_4. As $4n \geq 8$, it follows that $\mathrm{diam}(T) \geq 4$. Throughout S will be used to denote a γ_{tr}-set of T, i.e. $|S| = \lceil \frac{n+2}{2} \rceil + 1$.

Claim 3.18 *If u and v are vertices of T such that $T' = T - u - v$ is a tree and $S' = S - \{u, v\}$ is a* **TRDS** *of T', then $n(T') \equiv 2$ mod 4 and $T' \in \mathcal{C}$.*

Proof. As $\lceil \frac{n-2+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil + 1 - 2$, we have $\gamma_{tr}(T') = \lceil \frac{n-2+2}{2} \rceil = \lceil \frac{n(T')+2}{2} \rceil$, and so $T' \in \mathcal{C}$. $\diamond$

Claim 3.19 *Let z be a leaf of T. If $S - \{z\}$ is a* **TRDS** *of $T' = T - z$, then $T \in \mathcal{C}^*$.*

Proof. Assume $S - \{z\}$ is a **TRDS** of T'. Then $\lceil \frac{n-1+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil + 1 - 1 = \lceil \frac{n+2}{2} \rceil$. Hence, $n - 1 \equiv 3$ mod 4 and $\gamma_{tr}(T') = \lceil \frac{n+1}{2} \rceil = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{C}$. The tree T can now be constructed from T' by applying Operation **O5**, whence $T \in \mathcal{C}^*$. $\diamond$

Claim 3.19 implies that if vxz is an endpath of T, then we may assume $v \notin S$, since otherwise the tree is constructible. Claim 3.19 also implies that every remote vertex of T is adjacent to exactly one leaf, since otherwise it is constructible.

Let T be rooted at a leaf r of a longest path. Let v be any vertex on a longest path at distance $\operatorname{diam}(T)-2$ from r. Suppose v lies on the endpath vyz'. Then, by the remark above, $v \notin S$, which implies that v is not adjacent to a leaf. If v also lies on the endpath vxz, then $S-\{x,z\}$ is a **TRDS** of $T-x-z$ and so $T' \in \mathcal{C}$ (cf. Claim 3.18), whence $T \in \mathcal{C}^*$ (as it can be constructed from T' by applying Operation **O6**). Thus, we assume each vertex on a longest path at distance $\operatorname{diam}(T)-2$ or $\operatorname{diam}(T)-1$ from r has degree two.

Let v be any vertex on a longest path at distance $\operatorname{diam}(T)-3$ from r. Let vy_1y_2z' be an endpath of T. Then $y_1 \notin S$, and so $v \notin S$, which means all neighbors of v have degree at least 2.

Assume v also lies on the path vxz, where z is a leaf. Then, since each remote vertex is adjacent to exactly one leaf, vxz is an endpath. If v is dominated by a vertex other than x, then $S-\{x,z\}$ is a **TRDS** of $T'=T-x-z$ and so $T' \in \mathcal{C}$ (cf. Claim 3.18), whence $T \in \mathcal{C}^*$ (as it can be constructed from T' by applying Operation **O7**). Hence, v is dominated only by x. Then $S'=S-\{y_2,z'\}$ is a **TRDS** of $T'=T-y_1-y_2-z'$ and so $\lceil \frac{n-3+2}{2} \rceil \le \gamma_{tr}(T') \le \lceil \frac{n+2}{2} \rceil - 1$. But then $\gamma_{tr}(T') = \lceil \frac{n-1}{2} \rceil = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{C}$. The tree T can now be constructed from T' by applying Operation **O8**.

Assume v lies on the path vx_1x_2z. Since x_1 (x_2, respectively) is on a longest path at distance $\operatorname{diam}(T)-2$ ($\operatorname{diam}(T)-1$, respectively) from r, we have $\deg(x_1)=2$ ($\deg(x_2)=2$, respectively). This implies that vx_1x_2z is an endpath, and so $x_1 \notin S$. But then $S'=S-\{x_2,z\}$ is a **TRDS** of $T'=T-x_1-x_2-z$. Thus, $\lceil \frac{n-3+2}{2} \rceil \le \gamma_{tr}(T') \le \lceil \frac{n+2}{2} \rceil - 1$. But then $\gamma_{tr}(T') = \lceil \frac{n-1}{2} \rceil = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{C}$ and so T can now be constructed from T' by applying Operation **O9**. Thus, we assume each vertex on a longest path at distance $\operatorname{diam}(T)-3$ from r has degree two.

Let v be any vertex on a longest path at distance $\operatorname{diam}(T)-4$ from r. As $P_5 \notin \mathcal{T}^*$, $v \neq r$ and $\operatorname{diam}(T) \ge 5$. Assume $\deg_T(v) \ge 3$. Let $vy_1y_2y_3z'$ be an endpath of T. But then, as y_2y_3z' is an endpath of T, it follows that $y_2 \notin S$, which implies $y_1 \notin S$ and $v \in S$. Moreover, $S' = S - \{y_3, z'\}$ is a **TRDS** of $T' = T - y_1 - y_2 - y_3 - z'$. Thus, $\lceil \frac{n-4+2}{2} \rceil + 1 \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 1$, whence $\gamma_{tr}(T') = \lceil \frac{n(T')+2}{2} \rceil + 1$. We conclude that $T' \in \mathcal{T}^*$, and by the induction assumption, $T' \in \mathcal{C}^*$. If $\deg_T(v) = 2$ or when v is a remote vertex, then T can be constructed from T' by applying Operation **O4**, whence $T \in \mathcal{C}^*$.

We therefore assume that $\deg_T(v) \geq 3$ and that v is not adjacent to a leaf. If v also lies on the path vxz, where z is a leaf, then $v \notin S$, which is a contradiction. We now asume that v lies on the path vx_1x_2z, where z is a leaf. Then, since x_2 is a remote vertex, we have $\deg(x_2) = 2$. As x_1x_2z is an endpath of T, it follows that $x_1 \notin S$. As x_1 must be adjacent to another vertex in $V - S$, vertex x_1 lies on a path x_1, u_1, u_2, z''. But then x_1, with $\deg(x_1) \geq 3$, is a vertex at distance $\mathrm{diam}(T) - 3$ on a longest path from r, which is a contradiction.

Let e be the edge that joins v with its parent, and let $T(v)$ be the component of $T - e$ that contains v. Then $T(v)$ consists of ℓ disjoint paths $u_ix_iy_iz_i$ $(i = 1, \ldots, \ell)$ with v joined to u_i for $i = 1, \ldots, \ell$. Let $i \in \{1, \ldots, \ell\}$. Since $x_iy_iz_i$ is an endpath of T, we have $x_i \notin S$, $u_i \notin S$ and $v \in S$. Then $S - \cup_{i=1}^{\ell}\{y_i, z_i\}$ is a **TRDS** of $T' = T - (T(v) - v)$, and so $\lceil \frac{n-4\ell+2}{2} \rceil + 1 \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 2\ell + 1$, whence $\gamma_{tr}(T') = \lceil \frac{n(T')+2}{2} \rceil + 1$. Thus, $T' \in \mathcal{T}^*$, and by the induction assumption, $T' \in \mathcal{C}^*$. Note that v is a leaf of T'. The tree T can now be constructed from T' by applying Operation **O4**, whence $T \in \mathcal{C}^*$. $\square$

Chapter 4

Nordhaus-Gaddum Results for Restrained Domination and Total Restrained Domination in Graphs

4.1 Introduction

In this chapter, we continue the study of restrained domination and total restrained domination in graphs. Recall that a set $S \subseteq V$ is a *restrained dominating set* (abbreviated **RDS**), if every vertex in $V - S$ is adjacent to a vertex in S and a vertex in $V - S$. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a **RDS** of G. A **RDS** of cardinality $\gamma_r(G)$ will be called a $\gamma_r(G)$-*set*. A set $S \subseteq V$ is a *total restrained dominating set*, (abbreviated **TRDS**) if every vertex is adjacent to a vertex in S and every vertex in $V - S$ is also adjacent to a vertex in $V - S$. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a **TRDS** of G. A **TRDS** of cardinality $\gamma_{tr}(G)$ will be called a $\gamma_{tr}(G)$-*set*.

Recall that a set $S \subseteq V$ is a *dominating set* (abbreviated herein as **DS**) if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a **DS** of G. A **DS** of cardinality $\gamma(G)$ will be called a $\gamma(G)$*-set*. Nordhaus and Gaddum present best possible bounds on the sum of the chromatic number of a graph and its complement in [16]. The corresponding result for the domination number of a graph is presented by Jaeger and Payan in [15]: If G is a graph of order $n \geq 2$, then $\gamma(G) + \gamma(\overline{G}) \leq n + 1$. A best possible bound on the sum of the restrained domination numbers of a graph and its complement is obtained in [8]:

Theorem 4.1 *If G is a graph of order $n \geq 2$ such that both G and $\overline{G}$ are not isomorphic to P_3, then $4 \leq \gamma_r(G) + \gamma_r(\overline{G}) \leq n + 2$.*

A best possible bound on the sum of the total restrained domination numbers of a graph and its complement is obtained in [4]:

Theorem 4.2 *If G is a graph of order $n \geq 2$ such that neither G nor $\overline{G}$ contains isolated vertices or has diameter two, then $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 4$.*

Let K be the graph obtained from K_3 by matching the vertices of $\overline{K}_2$ to distinct vertices of K_3. Note that K is self-complementary, K nor $\overline{K}$ contains isolated vertices or has diameter two, while $\gamma_{tr}(K) + \gamma_{tr}(\overline{K}) = 2 \times 5 = 10 > n(K) + 4$. Thus, Theorem 4.2 is incorrect.

We will show, in Section 4.2, that if G is a graph of order $n \geq 2$ such that neither G nor $\overline{G}$ contains isolated vertices or is isomorphic to K, then $4 \leq \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 4$. Moreover, we will characterize the graphs G of order n for which $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$ and also characterize those graphs G for which $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = 4$. In Section 4.3, we characterize the graphs G of order n for which $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$ as well as those graphs G for which $\gamma_r(G) + \gamma_r(\overline{G}) = 4$.

4.2 Total Restrained Domination

In this section, we provide bounds on the sum of the total restrained domination numbers of a graph and its complement, and provide characterizations of the extremal graphs achieving these bounds.

Let $n \geq 5$ be an integer and suppose $\{x, y, u, v\}$ and X are disjoint sets of vertices such that $|X| = n-4$. Let $\mathcal{L}$ be the family of graphs G of order n where $V(G) = \{x, y, u, v\} \cup X$ and with the following properties:

P1: x and y are non-adjacent, while u and v are adjacent,

P2: each vertex in $\{x, y\} \cup X$ is adjacent to some vertex of $\{u, v\}$,

P3: each vertex in $\{u, v\} \cup X$ is non-adjacent to some vertex of $\{x, y\}$,

P4: each vertex in $\{x, y\} \cup X$ is adjacent to some vertex of $\{x, y\} \cup X$,

P5: each vertex in $\{u, v\} \cup X$ is non-adjacent to some vertex of $\{u, v\} \cup X$.

Theorem 4.3 *If G be a graph of order $n \geq 2$ such that neither G nor $\overline{G}$ contains isolated vertices, then $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = 4$ if and only if $G \in \mathcal{L}$.*

Proof. Suppose G is a graph such that neither G nor $\overline{G}$ contains isolated vertices, and suppose $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = 4$. Then $\gamma_{tr}(G) = \gamma_{tr}(\overline{G}) = 2$. Let $S = \{u, v\}$ ($S' = \{x, y\}$, respectively) be a **TRDS** of G ($\overline{G}$, respectively). Then x is non-adjacent to y, while u is adjacent to v, and Property **P1** holds. Clearly, $S \neq S'$. Suppose $u = x$ with $v \neq y$. Since $\{u, v\}$ is a **DS** of G and y is non-adjacent to $x = u$, the vertex y must be adjacent to v. But then v is not dominated by S' in $\overline{G}$, which is a contradiction. Thus, $S \cap S' = \emptyset$. Let $X = V(G) - \{x, y, u, v\}$. Then $|X| = n - 4$, and since S (S', respectively) is a **TRDS** of G ($\overline{G}$, respectively), Properties **P2 − P5** hold for G. Thus, $G \in \mathcal{L}$. The converse clearly holds as $\{u, v\}$ ($\{x, y\}$, respectively) is a **TRDS** of G ($\overline{G}$, respectively). $\square$

Let $\operatorname{diam}(G)$ denote the diameter of G, and let u, v be two vertices of G such that $d(u, v) = \operatorname{diam}(G)$. The set of vertices at distance i from u, $0 \leq i \leq \operatorname{diam}(G)$, will be denoted by V_i, and the sets $V_0, \ldots, V_{\operatorname{diam}(G)}$ will then be called the *level decomposition of G with respect to u*. To facilitate argumentation we use the following definition given by Cockayne, Dawes and Hedetniemi [5]. A *total dominating set* (abbreviated **TDS**) of G is a set $S \subseteq V$ such that every vertex of G is adjacent to a vertex of S.

Let $\mathcal{U} = \{G \mid G$ is a graph of order n which can be obtained from a P_4 with consecutive vertices labeled u, v_1, v_2, v by joining vertices v_1 and v_2 to each vertex of K_{n-4} where $n \geq 6\}$.

Theorem 4.4 *Let G be a graph of order $n \geq 2$ such that neither G nor $\overline{G}$ contains isolated vertices or is isomorphic to K. Then $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 4$. Moreover, $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$ if and only if $G \in \mathcal{U}$ or $\overline{G} \in \mathcal{U}$ or $G \cong P_4$.*

Proof. If G is disconnected, then $\gamma_{tr}(\overline{G}) = 2$. Hence $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 2$. Thus, without loss of generality, assume both G and $\overline{G}$ are connected. Let u and v be vertices such that $d(u, v) = \operatorname{diam}(G)$ and let $V_0, \ldots, V_{\operatorname{diam}(G)}$ be the level decomposition of G with respect to u. We consider the following cases:

Case 1. $\operatorname{diam}(G) \geq 5$.

We claim that $\{u, v\}$ is a **TRDS** of $\overline{G}$. The vertex u is non-adjacent to all vertices in V_i where $2 \leq i \leq \operatorname{diam}(G)$, while the vertex v is non-adjacent to all vertices in V_i where $0 \leq i \leq \operatorname{diam}(G) - 2$. Moreover, every vertex in $V(G) - \{u, v\}$ is non-adjacent to some vertex of $V(G) - \{u, v\}$. Thus, $\gamma_{tr}(\overline{G}) = 2$, and so $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 2$.

Case 2. $\operatorname{diam}(G) = 4$.

Suppose u, v_1, v_2, v_3, v is a diametrical path. If $|V_4| \geq 2$, then $\{u, v\}$ is a **TRDS** of $\overline{G}$, and the result follows. Thus, $V_4 = \{v\}$. Let $V_{21} = \{x \in V_2 \mid$ there exists a vertex in

$V_1 \cup V_2 \cup V_3$ that is not adjacent to $x\}$ and let $V_{22} = V_2 - V_{21}$. The set $\{u, v\} \cup V_{22}$ is a **TRDS** of $\overline{G}$. So we have that $\gamma_{tr}(\overline{G}) \leq 2+|V_{22}|$. If $|V_{22}| \leq 1$, then $\gamma_{tr}(G)+\gamma_{tr}(\overline{G}) \leq n+3$. Hence $|V_{22}| \geq 2$.

Let $t \in V_{22}$ such that $t \neq v_2$. Suppose $|V_1 \cup V_{21} \cup V_3| \geq 4$. Let $s \in V_1 \cup V_{21} \cup V_3 - \{v_1, v_2, v_3\}$. Then $V_1 \cup V_{21} \cup V_3 \cup \{u, v, t\} - \{s\}$ is a **TRDS** of G and so $\gamma_{tr}(G)+\gamma_{tr}(\overline{G}) \leq n-(|V_{22}|-1)-1+|V_{22}|+2 \leq n+2$. Hence $|V_1| = 1$, $|V_{21}| \leq 1$ and $|V_3| = 1$. Therefore, $V(G) - V_{22}$ is a **TRDS** of G and so $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n - |V_{22}| + 2 + |V_{22}| \leq n + 2$.

Case 3. $\mathrm{diam}(G) = 3$.

Let u, v_1, v_2, v be a diametrical path. Suppose $t \in V_3 - \{v\}$. We define $V_{21} = \{x \in V_2 \,|\, \text{there exists a vertex in } V_1 \cup V_2 \cup V_3 - \{t\} \text{ that is not adjacent to } x\}$ and let $V_{22} = V_2 - V_{21}$. The set $\{u, t\} \cup V_{22}$ is a **TRDS** of $\overline{G}$ and so $\gamma_{tr}(\overline{G}) \leq 2+|V_{22}|$. If $|V_{22}| = 1$, then surely $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 3$. Hence $|V_{22}| \geq 2$. The vertex t is adjacent to some vertex $s \in V_2$. If $s \in V_{22}$, then the set $\{u, s\} \cup V_1 \cup V_{21} \cup V_3 - \{v\}$ is a **TRDS** of G. If $s \notin V_{22}$, then the set $\{u, w\} \cup V_1 \cup V_{21} \cup V_3 - \{v\}$ is a **TRDS** of G, where $w \in V_{22}$. In both cases, $\gamma_{tr}(G) \leq n - |V_{22}|$, and so $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n - |V_{22}| + 2 + |V_{22}| = n + 2$. Thus, $V_3 = \{v\}$.

Define $V_{11} = \{x \in V_1 \,|\, \text{there exists a vertex in } V_1 \cup V_2 \text{ that is not adjacent to } x\}$ and let $V_{12} = V_1 - V_{11}$. Moreover, let $V_{21} = \{x \in V_2 \,|\, \text{there exists a vertex in } V_1 \cup V_2 \text{ that is not adjacent to } x\}$ and let $V_{22} = V_2 - V_{21}$. Then $\{u, v\} \cup V_{12} \cup V_{22}$ is a **TRDS** of $\overline{G}$, whence $\gamma_{tr}(\overline{G}) \leq 2 + |V_{12}| + |V_{22}|$.

Case 3.1 $|V_{12}| + |V_{22}| \leq 2$.

Clearly $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 4$. We now investigate when, in this case, $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$. As $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$, we must have that $|V_{12}| + |V_{22}| = 2$. We first show that $\deg(u) = \deg(v) = 1$.

Suppose, to the contrary, $\{v_1, w\} \subseteq N(u)$, and let $t \in V_{12} \cup V_{22} - \{w\}$. Then t is adjacent to every vertex of $V_1 \cup V_2$, and so $V(G) - \{u, w\}$ is a **TRDS** of G. It now follows that $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n - 2 + 4 = n + 2$, which is a contradiction. Thus, $\deg(u) = 1$, and $\deg(v) = 1$ follows similarly. Hence $V_1 = V_{12} = \{v_1\}$, and the set V_{22} consists of exactly one vertex, say w.

Suppose $w \neq v_2$. If $|V_2| = 2$, then $G \cong K$, which is not allowable. So, let $w' \in V_2 - \{v_2, w\}$. Then w and w' are adjacent, and $V(G) - \{w, w'\}$ is a **TRDS** of G. As before, we obtain a contradiction. We conclude $w = v_2$. If $V_{21} = \emptyset$, then $G \cong P_4$. If $V_{21} \neq \emptyset$, then surely $|V_{21}| \geq 2$. If two vertices, say t and t', of V_{21} are adjacent in G, then $V(G) - \{t, t'\}$ is a **TRDS** of G, and we obtain a contradiction as before. Thus, V_{21} is independent, and so $\overline{G} \in \mathcal{U}$.

Case 3.2 $|V_{12}| + |V_{22}| \geq 3$.

If we can show that G has a **TRDS** of size at most $s := n - |V_{12}| - |V_{22}| + 1$, then $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n - |V_{12}| - |V_{22}| + 1 + 2 + |V_{12}| + |V_{22}| = n + 3$. First consider the case when $v_1 \in V_{11}$. Choose $w = v_2$ if $v_2 \in V_{22}$, otherwise choose $w \in V_{12} \cup V_{22}$. In both situations, $\{u, v, w\} \cup V_{11} \cup V_{21}$ is a **TRDS** of G of size s. Thus, $v_1 \notin V_{11}$. If $v_2 \in V_{21}$, then $\{u, v_1, v\} \cup V_{11} \cup V_{21}$ is a **TRDS** of G of size s. Thus, $v_2 \notin V_{21}$. We conclude that $v_1 \in V_{12}$, while $v_2 \in V_{22}$.

Suppose u is adjacent to a vertex w which is distinct from v_1. If $w \in V_{12}$, then $\{v_1, v_2, v\} \cup V_{11} \cup V_{21}$ is a **TRDS** of size s. If $w \in V_{11}$, then $\{v_1, v_2, v\} \cup (V_{11} - \{w\}) \cup V_{21}$ is a **TRDS** of size $s - 1$. Thus, $\deg(u) = 1$, and $\deg(v) = 1$ follows similarly.

Suppose $V_{22} = \{v_2\}$. If $V_{21} = \emptyset$, then $G \cong P_4$ and $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$. If $V_{21} \neq \emptyset$, then surely $|V_{21}| \geq 2$. If two vertices, say t and t', of V_{21} are adjacent in G, then $\{u, v_1, v_2, v\} \cup (V_{21} - \{t, t'\})$ is a **TRDS** of G of size $s - 1$. Thus, V_{21} is independent, $\overline{G} \in \mathcal{U}$ and $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$.

Therefore, $|V_{22}| \geq 2$. If $V_{21} = \emptyset$, then V_{22} induces a clique. If $|V_{22}| = 2$, then $G \cong K$, which is not allowable. If $|V_{22}| \geq 3$, then $G \in \mathcal{U}$ and $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$. Thus, $V_{21} \neq \emptyset$, and so $|V_{21}| \geq 2$. Let $\{t, t'\} \subseteq V_{21}$. Then $\{u, v_1, v_2, v\} \cup (V_{21} - \{t, t'\})$ is a **TRDS** of G of size $s - 1$.

Case 4. $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 2$.

Note that $\delta(G) \geq 2$ and $\delta(\overline{G}) \geq 2$, since otherwise G or $\overline{G}$ will have isolated vertices.

Case 4.1 $\delta(G) = 2$ or $\delta(\overline{G}) = 2$.

Without loss of generality, assume $\delta(G) = 2$ and suppose u is a vertex of minimum degree in G. Let $N(u) = \{v, w\}$. Let $N_{v,w} = \{x \in V(G) - \{u, v, w\} \,|\, x$ is adjacent to both v and $w\}$, let $N_{v,\overline{w}} = \{x \in V(G) - \{u, v, w\} \,|\, x$ is adjacent to v but not to $w\}$, and let $N_{w,\overline{v}} = \{x \in V(G) - \{u, v, w\} \,|\, x$ is adjacent to w but not to $v\}$. Moreover, let $N_1 = \{x \in N_{u,v} \,|\, N(x) = \{v, w\}\}$ and let $N_2 = N_{v,w} - N_1$.

Now, if $N_1 = \emptyset$, then $\{u, v, w\}$ is a **TRDS** of G and so $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 3$. Thus, $N_1 \neq \emptyset$. If $N_{v,\overline{w}} = \emptyset$ ($N_{w,\overline{v}} = \emptyset$, respectively), then $\{u, w\}$ ($\{u, v\}$, respectively) is a **TRDS** of G, whence $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 2$. Thus, $N_{v,\overline{w}} \neq \emptyset$ and $N_{w,\overline{v}} \neq \emptyset$.

Notice that the set $\{u, v, w\} \cup N_1$ is a **TRDS** of G. Let $Y = V(G) - \{u\} - N_1$. Since all vertices in $N_{v,\overline{w}}$ dominate all vertices in $N_1 \cup \{u\}$ in $\overline{G}$, and since $N_1 \cup \{u\}$ is a clique in $\overline{G}$, we have that Y is a **RDS** of $\overline{G}$. If Y is total, we have that $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq 3 + |N_1| + n - 1 - |N_1| = n + 2$ and we are done.

Assume, therefore, that Y is not total. As w (v, respectively) is non-adjacent to every vertex of $N(v, \overline{w})$ ($N(w, \overline{v})$, respectively), the set $N_2 \neq \emptyset$, since otherwise Y is a **TRDS** of $\overline{G}$. Moreover, Y will also be a **TRDS** of $\overline{G}$ if every vertex of N_2 is non-adjacent to some vertex of Y. Hence, there exists a vertex $y \in N_2$ which is adjacent to every vertex of $Y - \{y\}$.

Notice that the set $\{v, y\}$ is a **TDS** of G. If $\{v, y\}$ is also a **RDS**, we have that $\gamma_{tr}(G)+\gamma_{tr}(\overline{G}) \leq n+2$. Moreover, the set $\{w, y\}$ is also a **TDS** of G, and if it is a **RDS**, we are done. Thus, there exist vertices $v' \in N_{v,\overline{w}}$ and $w' \in N_{w,\overline{v}}$ such that $N(v') = \{v, y\}$ and $N(w') = \{w, y\}$.

We now show that $Z = \{u, v', w'\}$ is a **TRDS** of $\overline{G}$. Notice that Z is a **TDS** of $\overline{G}$. Indeed, the vertex v' dominates w in $\overline{G}$, the vertex w' dominates v in $\overline{G}$, while the vertex u dominates $V(G) - \{u, v, w, v', w'\}$ in $\overline{G}$. Moreover, the vertex u dominates $\{v', w'\}$ in $\overline{G}$. Now, suppose to the contrary that Z is not a **RDS** of $\overline{G}$. Hence, there exists a vertex $z \notin Z$ such that z is adjacent to every vertex of $V(G) - Z - \{z\}$ in G. As $\deg(\overline{G}) \geq 2$, the vertex z is adjacent in $\overline{G}$ to at least two vertices of Z. We consider the following cases:

Case 4.1.1 The vertex z is adjacent in $\overline{G}$ to u and at least one of the vertices v' and w'.

Without loss of generality assume that z is adjacent in $\overline{G}$ to the vertex v'. As z is non-adjacent to u in G, it follows that $z \notin \{v, w\}$. As z is adjacent to both of the vertices v and w in G, we have $z \in N_1 \cup N_2$. If $z \in N_1$, then it is not adjacent to y in G, which contradicts the fact that z is adjacent to every vertex of $V(G) - Z - \{z\}$. If $z \in N_2$, then since $N_1 \neq \emptyset$, there exists a vertex $z' \in N_1$ such that z is not adjacent to z' in G, which is again a contradiction.

Case 4.1.2 The vertex z is adjacent in $\overline{G}$ to v' and w', but not to u.

In this case, $z \in \{v, w\}$. Without loss of generality, assume $z = v$. Then v is adjacent in $\overline{G}$ to both v' and w', which is a contradiction. Therefore, the set $Z = \{u, v', w'\}$ is a **TRDS** of $\overline{G}$ and so $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 3$.

Case 4.2 $\delta(G) \geq 3$ and $\delta(\overline{G}) \geq 3$.

Let u be a vertex of minimum degree in G. Suppose $N(u) = \{u_1, \ldots, u_\delta\}$ where $\delta = \delta(G)$. Suppose the sets $N[u]$ and $N[u] - \{u_i\}$ for $i \in \{1, \ldots, \delta\}$ are not total restrained dominating sets of G. Let $N_1 = \{x \in V(G) - N[u] \,|\, N(x) = N(u)\}$ and let $N_2 = V(G) - N[u] - N_1$. As $N[u]$ is a **TDS** of G, but not a **RDS** of G, the set $N_1 \neq \emptyset$. If $N_2 = \emptyset$, then $\{u, u_1\}$ is a **TRDS** of G, whence $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq 2 + n$. Thus, $N_2 \neq \emptyset$.

Suppose $N[u] - \{u_i\}$ is a **DS** for some $i \in \{1, \ldots, \delta\}$. If a vertex $x \in N_2$ is adjacent to vertices in $N(u) - \{u_i\}$ only, then $\deg(x) \leq \delta - 1$, which is impossible. Thus, $N[x] - \{u_i\}$ is a **TRDS** of G, which is contrary to our assumption. Hence, for each $i \in \{1, \ldots, \delta\}$, there exists $u_i' \in N_2$ such that $N(u_i') \cap N(u) = \{u_i\}$.

We claim that $X = \{u, u_1', u_2'\}$ is a **TRDS** of $\overline{G}$. The vertex u_1' dominates all vertices in $N(u) - \{u_1\}$ in $\overline{G}$. Similarly, u_2' dominates all vertices in $N(u) - \{u_2\}$ in $\overline{G}$. The vertex u dominates all vertices in $V(G) - N[u]$ in $\overline{G}$, and so X is a **TDS**. Suppose X is not a **RDS** of $\overline{G}$. Thus, there exists a vertex $x \notin X$ such that x is adjacent in G to each of the vertices in $V(G) - X - \{x\}$. As $\delta(\overline{G}) \geq 3$, the vertex x is not adjacent to each of the vertices in X. Hence, $x \in N_1 \cup N_2$. If $x \in N_1$, then since $|N_2| \geq \delta \geq 3$, there exists a vertex $x' \in N_2 - \{u_1', u_2'\} \subset V(G) - X - \{x\}$ such that x is not adjacent to x' in G, which is a contradiction. Similarly, if $x \in N_2 - \{u_1', u_2'\}$, then, since $N_1 \neq \emptyset$, there exists a vertex $x' \in N_1 \subset V(G) - X - \{x\}$ such that x is not adjacent to x' in G, which is a contradiction. Hence X is a **TRDS** of $\overline{G}$ and so $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 3$.

We may therefore assume that $N_G[u]$ or $N_G[u] - \{u_i\}$ is a **TRDS** of G for some $i \in \{1, \ldots, \delta\}$. Similarly, if v is a minimum degree vertex in $\overline{G}$ and $N_{\overline{G}}(v) = \{v_1, \ldots, v_{\delta(\overline{G})}\}$, we assume that $N_{\overline{G}}[v]$ or $N_{\overline{G}}[v] - \{v_j\}$ is a **TRDS** of $\overline{G}$ for some $j \in \{1, \ldots, \delta(\overline{G}\}$. Hence $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq \delta(G) + 1 + \delta(\overline{G}) + 1 = \delta(G) + 1 + n - \Delta(G) - 1 + 1 = n + \delta(G) - \Delta(G) + 1 \leq n + 1$.

Clearly, if $G \in \mathcal{U}$ or $\overline{G} \in \mathcal{U}$ or $G \cong P_4$, then $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$. $\square$

4.3 Restrained Domination

In this section, we provide bounds on the sum of the restrained domination numbers of a graph and its complement, and provide characterizations of the extremal graphs achieving these bounds. Let $\mathcal{H}$ be the family of graphs G of order n where G or $\overline{G}$ is one of the following four types:

Type 1. $V(G) = \{x, y, z\} \cup X$. Moreover:

P1.1: x is adjacent to each vertex of $\{y, z\} \cup X$,

P1.2: each vertex of $\{y, z\} \cup X$ is adjacent to some vertex of $\{y, z\} \cup X$,

P1.3: each vertex of X is non-adjacent to some vertex of $\{y, z\}$ and non-adjacent to some vertex in X.

Type 2. $V(G) = \{x, y\} \cup X$. Moreover:

P2.1: each vertex of X is adjacent to exactly one vertex of $\{x, y\}$ and also non-adjacent to exactly one vertex of $\{x, y\}$,

P2.2: each vertex of X is non-adjacent to some vertex of X,

P2.3: each vertex of X is adjacent to some vertex of X.

Type 3. $V(G) = \{u, v, y\} \cup X$. Moreover:

P3.1: each vertex of $X \cup \{y\}$ is adjacent to some vertex of $\{u, v\}$,

P3.2: each vertex of $X \cup \{u\}$ is non-adjacent to some vertex of $\{v, y\}$,

P3.3: each vertex of $X \cup \{y\}$ is adjacent to some vertex of $X \cup \{y\}$,

P3.4: each vertex of $X \cup \{u\}$ is non-adjacent to some vertex of $X \cup \{u\}$.

Type 4. $V(G) = \{x, y, u, v\} \cup X$. Moreover:

P4.1: each vertex in $\{x, y\} \cup X$ is adjacent to some vertex of $\{u, v\}$,

P4.2: each vertex in $\{u, v\} \cup X$ is non-adjacent to some vertex of $\{x, y\}$,

P4.3: each vertex in $\{x, y\} \cup X$ is adjacent to some vertex of $\{x, y\} \cup X$,

P4.4: each vertex in $\{u, v\} \cup X$ is non-adjacent to some vertex of $\{u, v\} \cup X$.

Theorem 4.5 *If G be a graph of order $n \geq 2$, then $\gamma_r(G) + \gamma_r(\overline{G}) = 4$ if and only if G or $\overline{G} \in \mathcal{H}$.*

Proof. Suppose G is a graph such that $\gamma_r(G) + \gamma_r(\overline{G}) = 4$. Then $\gamma_r(G) = 1$ and $\gamma_r(\overline{G}) = 3$ or $\gamma_r(\overline{G}) = 1$ and $\gamma_r(G) = 3$ or $\gamma_r(G) = \gamma_r(\overline{G}) = 2$.

Case 1. $\gamma_r(G) = 1$ and $\gamma_r(\overline{G}) = 3$ or $\gamma_r(\overline{G}) = 1$ and $\gamma_r(G) = 3$.

Suppose $\gamma_r(G) = 1$ and $\gamma_r(\overline{G}) = 3$. Let $\{x\}$ be a **RDS** of G. Then x is adjacent to every other vertex of G, and so x is isolated in $\overline{G}$ and is therefore in every **RDS** of $\overline{G}$ - let $\{x, y, z\}$ be a **RDS** of $\overline{G}$. Let $X = V(G) - \{x, y, z\}$. It now follows that Properties **P1.1 - P1.3** hold for G. Thus, G is a graph of Type 1. If $\gamma_r(\overline{G}) = 1$ and $\gamma_r(G) = 3$, then $\overline{G}$ is also of Type 1.

Case 2. $\gamma_r(G) = 2$ and $\gamma_r(\overline{G}) = 2$.

Let $\{u, v\}$ ($\{x, y\}$, respectively) be a **RDS** of G ($\overline{G}$, respectively). Let $X = V(G) - \{u, v, x, y\}$.

Case 2.1 Suppose $u = x$ and $v = y$.

If some vertex $w \in X$ is adjacent to both u and v, then w is not dominated by $\{u, v\}$ in $\overline{G}$, which is a contradiction. As $\{u, v\}$ is a **DS** of G, each vertex $w \in X$ is adjacent to at least one vertex in $\{u, v\}$. Thus, G satisfies Property **P2.1**. Moreover, Properties **P2.2** and **P2.3** hold for G. Thus, G is a graph of Type 2.

Case 2.2 Suppose $u \neq y$ and $x = v$.

Clearly, in this case G is a graph of Type 3.

Case 2.3 $\{u, v\} \cap \{x, y\} = \emptyset$.

It is easy to see, that P4.1 - P4.4 hold, so G is a graph of Type 4.

For the converse, suppose $G \in \mathcal{H}$. For a graph of Type 1 we have $\gamma_r(G) = 1$ and $\gamma_r(\overline{G}) \leq 3$. For Types 2, 3 or 4 we obtain $\gamma_r(G) \leq 2$ and $\gamma_r(\overline{G}) \leq 2$. Hence, in all cases $\gamma_r(G) + \gamma_r(\overline{G}) \leq 4$. It is known (see [3]) that $\gamma_r(G) + \gamma_r(\overline{G}) \geq 4$. Therefore, $\gamma_r(G) + \gamma_r(\overline{G}) = 4$. $\square$

We will now characterize graphs G of order n for which $\gamma_r(G) + \gamma_r(\bar{G}) = n + 2$.

Let $\mathcal{B} = \{P_3, \overline{P}_3\}$, and let $\mathcal{G} = \{G \,|\, G \text{ or } \overline{G} \text{ is a galaxy of non-trivial stars}\}$.

Let $\mathcal{S} = \{G \,|\, G \text{ or } \overline{G} \cong K_1 \cup S \text{ where } S \text{ is a star and } |S| \geq 3\}$.

Lastly, let $\mathcal{E} = \mathcal{G} \cup \mathcal{S}$.

Lemma 4.6 *If $G \in \mathcal{E} - \mathcal{B}$, then $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$.*

Proof. Suppose $G \in \mathcal{G}$ has order n and, without loss of generality, suppose G is a galaxy of non-trivial stars $S_1, S_2, \ldots, S_k$, for $k \geq 2$. Then $\gamma_r(G) = n$. Let $s \in V(S_1)$ and $t \in V(S_2)$. Since S_i is non-trivial for $i \in \{1, \ldots, k\}$, it follows that $R = \{s, t\}$ is a **RDS** of $\overline{G}$. Suppose $\{v\}$ is a **RDS** of $\overline{G}$. Then $\deg_G(v) = 0$, which is a contradiction. Hence $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$. Now, suppose $k = 1$. That is, G is a non-trivial star S such that $S \neq P_3$. The result follows immediately if $|S| = 2$. Thus we may assume $|S| \geq 4$. Then $\gamma_r(G) = n$. Let s be the center of S and let $t \in N_G(s)$. Notice that $\langle V(G) - \{s\} \rangle \cong K_{n-1}$ in $\overline{G}$. Thus $R = \{s, t\}$ is a **RDS** of $\overline{G}$. Suppose $\{v\}$ is a **RDS** of $\overline{G}$. Then $\deg_G(v) = 0$, which is a contradiction.

Suppose $G \in \mathcal{S}$ and, without loss of generality, let $G = K_1 \cup S$ where S is a star and $|S| \geq 3$. Then $\gamma_r(G) = n$. Let s be the center of S and let $\langle u \rangle$ be the second component of G. Then $R = \{s, u\}$ is a **RDS** of $\overline{G}$. Suppose $\{v\}$ is a **RDS** of $\overline{G}$. Then $\deg_G(v) = 0$, and $v = u$, which is a contradiction as $\{u\}$ is not a **RDS** of $\overline{G}$. Hence $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$. $\square$

Theorem 4.7 *Let $G = (V, E)$ be a graph of order $n \geq 2$ such that $G \notin \mathcal{B}$. Then $\gamma_r(G) + \gamma_r(\overline{G}) \leq n + 2$. Moreover, $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$ if and only if $G \in \mathcal{E}$.*

Proof. Let $G = (V, E)$ be a graph of order n such that $G \notin \mathcal{B}$. Notice that either G or $\overline{G}$ must be connected. Without loss of generality, suppose $\overline{G}$ is connected. Note that G may also be connected. Let G be composed of the components $G_1, G_2, \ldots, G_\ell$ with ℓ possibly equal to one. Without loss of generality, let G_1 be a component of G with longest diameter.

Claim 4.8 *If G_1 contains a path uv_1v_2v and $\ell \geq 3$, then $\gamma_r(G) + \gamma_r(\overline{G}) \leq n$.*

Proof. Let uv_1v_2v be a path in G_1. Notice that $V(G) - \{v_1, v_2\}$ is a **RDS** of G. Hence $\gamma_r(G) \leq n - 2$. Let $x \in V(G_1)$ and $w \in V(G_2)$. Since $\ell \geq 3$ it follows that $\{x, w\}$ is a **RDS** of $\overline{G}$ and $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$. $\diamond$

Claim 4.9 *If $\ell \geq 3$ and there exists $i \in \{1, \ldots, \ell\}$ such that $G_i \cong K_1$, then $\gamma_r(G) + \gamma_r(\overline{G}) \leq n + 1$.*

Proof. Trivial. $\diamond$

By Claim 4.8, for cases in which $\mathrm{diam}(G_1) \geq 3$, we may immediately assume that $\ell \leq 2$. Note that for the following two cases $V(G_2)$ may or may not be empty. Let u and v be vertices such that $d(u, v) = \mathrm{diam}(G)$. As before, the sets $V_0, \ldots, V_{\mathrm{diam}(G)}$ will denote the level decomposition of G with respect to u

Suppose $\mathrm{diam}(G_1) \geq 5$. Let $uv_1v_2 \ldots v_{\mathrm{diam}(G_1)}$ be a diametrical path in G_1. Notice that $V(G) - \{v_1, v_2\}$ is a **RDS** of G. Hence $\gamma_r(G) \leq n - 2$. Moreover, notice that

$R' = \{u, v_5\}$ is a **RDS** of $\overline{G}$, as R' is clearly a dominating set of $\overline{G}$, $v_1 \in V(\overline{G}) - R'$ is adjacent to $V_3 \cup V_4 \cup \ldots \cup V_{\mathrm{diam}(G)}$, and $v_4 \in V(\overline{G}) - R'$ is adjacent to $V_1 \cup V_2 \cup V(G_2)$. Hence $\gamma_r(\overline{G}) \leq 2$ and we have that $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$.

Now, suppose $\mathrm{diam}(G_1) = 4$. Let $uv_1v_2v_3v_4$ be a diametrical path in G_1. Notice that $V(G) - \{v_1, v_2\}$ is a **RDS** of G. Hence $\gamma_r(G) \leq n - 2$. Suppose $|V_4| \geq 2$. Then there exists a vertex $t \in V_4 - \{v_4\}$. Notice that $R' = \{u, v_4\}$ is a **RDS** of $\overline{G}$, as R' is clearly a dominating set of $\overline{G}$, $v_1 \in V(\overline{G}) - R'$ is adjacent to $V_3 \cup V_4$, and $t \in V(\overline{G}) - R'$ is adjacent to $V_1 \cup V_2 \cup V(G_2)$. Hence $\gamma_r(\overline{G}) \leq 2$ and we have that $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$.

Thus we may assume that $|V_4| = 1$. Let $V_{21} = \{x \in V_2 \mid$ there exists $y \in V_1 \cup V_2 \cup V_3$ such that $\{x, y\} \notin E(G_1)\}$ and let $V_{22} = V_2 - V_{21}$. Consider $R' = \{u, v_4\} \cup V_{22}$. Notice that R' is a dominating set of $\overline{G}$, $v_1 \in V(\overline{G}) - R'$ is adjacent to V_3, and $v_3 \in V(\overline{G}) - R'$ is adjacent to $V_1 \cup V(G_2)$. If $V_{21} = \emptyset$, then $V_2 = V_{22} \subseteq R'$ and R' is a **RDS** of $\overline{G}$. If $V_{21} \neq \emptyset$, then by definition, for each $x \in V_{21}$ there exists a $y \in V_1 \cup V_{21} \cup V_3$ such that $xy \notin E(G_1)$. Hence R' is a **RDS** of $\overline{G}$. In either case we have that $\gamma_r(\overline{G}) \leq 2 + |V_{22}|$.

If $|V_{22}| \leq 1$, then $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 + |V_{22}| \leq n + 1$. Thus we may assume that $|V_{22}| \geq 2$. Hence there exists a vertex $t \in V_{22} - \{v_2\}$. Then $R = \{u, v_4, t\} \cup V(G_2)$ is a **RDS** of G, as R clearly dominates G, and a vertex $w \in V_{22} - \{t\}$ is adjacent to every vertex of $V(G) - R$. Thus, $\gamma_r(G) \leq 3 + |V(G_2)|$ and so $\gamma_r(G) + \gamma_r(\overline{G}) \leq 3 + |V(G_2)| + 2 + |V_{22}| = 1 + (4 + |V_{22}| + |V(G_2)|) = 1 + (|\{u, v_1, v_3, v_4\}| + |V_{22}| + |V(G_2)|) = 1 + |\{u, v_1, v_3, v_4\} \cup V_{22} \cup V(G_2)| \leq 1 + |V(G)| = 1 + n$.

Now, suppose $\mathrm{diam}(G_1) = 3$. Let $uv_1v_2v_3$ be a diametrical path in G_1. Notice that $V(G) - \{v_1, v_2\}$ is a **RDS** of G. Suppose that $V(G_2) \neq \emptyset$. If $V(G_2) = \{v\}$, then $\{v\}$ is a **RDS** of $\overline{G}$, whence $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 1 = n - 1$. Thus we may assume that $|V(G_2)| \geq 2$. Let $v \in V(G_2)$. Then $\{u, v\}$ is a **RDS** of $\overline{G}$ and so $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$.

Thus $V(G_2) = \emptyset$ and both $G_1 = G$ and $\overline{G}$ are connected. Suppose $|V_3| \geq 2$ and let $t \in V_3 - \{v_3\}$. Let $V_{21} = \{x \in V_2 |$ there exists $y \in (V_1 \cup V_2 \cup V_3) - \{t\}$ such that $xy \notin E(G)\}$ and let $V_{22} = V_2 - V_{21}$. Consider $R' = \{u, t\} \cup V_{22}$. By reasoning similar to that in the case for $\mathrm{diam}(G_1) = 4$, R' is a **RDS** of $\overline{G}$ and $\gamma_r(\overline{G}) \leq 2 + |V_{22}|$. If $|V_{22}| \leq 1$, then $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 + |V_{22}| \leq n + 1$.

Thus we may assume that $|V_{22}| \geq 2$. Hence there exists a vertex $z \in V_{22} - \{v_2\}$. Consider $R = \{u, t, z\}$. By reasoning similar to that in the case for $\mathrm{diam}(G_1) = 4$, R is a **RDS** of G and so $\gamma_r(G) + \gamma_r(\overline{G}) \leq 3 + 2 + |V_{22}| = 1 + (4 + |V_{22}|) = 1 + (|\{u, v_1, v_3, t\}| + |V_{22}|) = 1 + |\{u, v_1, v_3, t\} \cup V_{22}| \leq 1 + |V(G)| = 1 + n$.

So we may assume that $|V_3| = 1$. Let $V_{11} = \{x \in V_1 |$ there exists $y \in V_1 \cup V_2$ such that $xy \notin E(G)\}$ and let $V_{12} = V_1 - V_{11}$. Also, let $V_{21} = \{x \in V_2 |$ there exists $y \in V_1 \cup V_2$ such that $xy \notin E(G)\}$ and let $V_{22} = V_2 - V_{21}$. Then $\{u, v_3\} \cup V_{12} \cup V_{22}$ is a **RDS** of $\overline{G}$ and $\gamma_r(\overline{G}) \leq 2 + |V_{12}| + |V_{22}|$.

If $|V_{12}| + |V_{22}| \leq 1$, then $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 + |V_{12}| + |V_{22}| \leq n + 1$. So we may assume that $|V_{12}| + |V_{22}| \geq 2$. Since $v_1 v_3 u v_2$ is a path in $\overline{G}$, it follows that $V(\overline{G}) - \{v_3, u\}$ is a **RDS** of $\overline{G}$, whence $\gamma_r(\overline{G}) \leq n - 2$.

Now, suppose $|V_{12}| \geq 2$ and let $z \in V_{12} - \{v_1\}$. Then $\{z, v_3\}$ is a **RDS** of G, and so $\gamma_r(G) + \gamma_r(\overline{G}) \leq 2 + n - 2 = n$. Thus $|V_{12}| \leq 1$. Suppose $V_{12} = \{z\}$. Then $\{u, v_3, z\}$ is a **RDS** of G except when $G = P_4$, in which case $\{u, v_3\}$ is a **RDS** of G. In both cases $\gamma_r(G) \leq 3$. Hence, $\gamma_r(G) + \gamma_r(\overline{G}) \leq 3 + n - 2 = n + 1$. Thus $V_{12} = \emptyset$ and so $|V_{22}| \geq 2$. Let $z \in V_{22} - \{v_2\}$. Then $\{u, v_3, z\}$ is a **RDS** of G. Therefore, $\gamma_r(G) \leq 3$. Hence, $\gamma_r(G) + \gamma_r(\overline{G}) \leq 3 + n - 2 = n + 1$.

Thus we may assume $\mathrm{diam}(G_1) \leq 2$, and by a similar argument, $\mathrm{diam}(\overline{G}) \leq 2$. As $n \geq 2$, $\mathrm{diam}(\overline{G}) \geq 1$. Suppose $\mathrm{diam}(\overline{G}) = 1$. Then $\overline{G} \cong K_i$ for some $i \geq 2$. If $i \geq 3$, then $\gamma_r(G) + \gamma_r(\overline{G}) \leq n + 1$. Thus, $\overline{G} \cong K_2$, and so $G \in \mathcal{G}$ and $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$.

Thus, $\mathrm{diam}(\overline{G}) = 2$. Suppose $\mathrm{diam}(G_1) = 0$. Then $G \cong nK_1$ and $\overline{G} \cong K_n$, which is a contradiction as $\mathrm{diam}(\overline{G}) = 2$.

Suppose $\mathrm{diam}(G_1) = 1$. Then $G_1 \cong K_i$ where $2 \leq i \leq n$. Since we assumed that $\overline{G}$ is connected, $\ell \neq 1$. Suppose $\ell = 2$. If $G_2 \cong K_1$, then $i \neq 2$, as $G \notin \mathcal{B}$. Thus $i \geq 3$, so $G \in \mathcal{G}$ and $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$. Thus $G_2 \cong K_j$ where $2 \leq j \leq n - i$. If $i = j = 2$, then $G \in \mathcal{G}$ and we are done. Without loss of generality, suppose $i \geq 3$. Let $V(G_1) = \{v_1, v_2, \ldots, v_i\}$ and let $z \in V(G_2)$. Since $i \geq 3$, $V(G) - \{v_2, v_3\}$ is a **RDS** of G and $\{v_1, z\}$ is a **RDS** of $\overline{G}$. Hence $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$. Thus $\ell \geq 3$. By Claim 4.9, $G_k \ncong K_1$ for all $k \in \{1, \ldots, \ell\}$. Suppose $G_k \cong K_2$ for all k. Then $G \in \mathcal{G}$ and we are done. Thus, by relabeling if necessary, we may assume that $G_1 \cong K_i$ for $i \geq 3$. Let $V(G_1) = \{v_1, v_2, \ldots, v_i\}$ and let $z \in V(G_2)$. Since $i \geq 3$, $V(G) - \{v_2, v_3\}$ is a **RDS** of G and $\{v_1, z\}$ is a **RDS** of $\overline{G}$. Hence $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$.

Thus we may assume $\mathrm{diam}(G_1) = 2$. Suppose $\ell \geq 3$. By Claim 4.9, $G_k \ncong K_1$ for all $k \in \{1, \ldots, \ell\}$. If G is a galaxy of non-trivial stars, then $G \in \mathcal{G}$, and we are done. Thus at least one component, say G_1, contains a cycle containing an edge v_1v_2, say. Let $z \in V(G_2)$. Then $V(G) - \{v_1, v_2\}$ is a **RDS** of G, while $\{v_1, z\}$ is a **RDS** of $\overline{G}$, whence $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$.

Suppose $\ell = 2$ and first suppose $G_2 \ncong K_1$. If G_1 and G_2 are stars, then $G \in \mathcal{G}$ and we are done. Thus at least one component contains a cycle containing the edge v_1v_2. Let z be an arbitrary vertex in the other component of G. Then $V(G) - \{v_1, v_2\}$ is a **RDS** of G, while $\{v_1, z\}$ is a **RDS** of $\overline{G}$, whence $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$.

So we may assume that $G_2 \cong K_1$. Let $V(G_2) = \{z\}$. If $\Delta(G_1) \leq n - 3$, then $\{z\}$ is a **RDS** of $\overline{G}$ and so $\gamma_r(G) + \gamma_r(\overline{G}) \leq n + 1$. Thus $\Delta(G_1) = n - 2$, and there exists a vertex $u \in V(G_1)$ such that $\deg(u) = n - 2$. Let L be the set of leaves in G_1 and let $X = N(u) - L$. If $L = \emptyset$, then $\{u, z\}$ is a **RDS** of G. Since $\mathrm{diam}(G_1) = 2$, there exist nonadjacent vertices $x, y \in V(G_1)$. Then $V(\overline{G}) - \{x, y\}$ is a **RDS** of $\overline{G}$ and

$\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$. Thus $L \neq \emptyset$. Let $v \in L$ and consider $\{u, v\}$. Since $\mathrm{diam}(G_1) = 2$, it follows that $\deg(u) \geq 2$. Thus $\{u, v\}$ is a **RDS** of $\overline{G}$. Suppose $X \neq \emptyset$ and let $s \in X$. Since $s \notin L$, s is adjacent to a vertex $t \in N(v)$. Hence $t \notin L$, so $t \in X$ and thus $|X| \geq 2$. Moreover, $V(G) - X$ is a is a **RDS** of G, and so $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$. Thus $X = \emptyset$ and so G_1 is a non-trivial star of order $n - 1 \geq 3$. Therefore $G \in \mathcal{S}$ and we are done.

Thus $G \cong G_1$, and $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$. Let uv_1v_2 be a diametrical path in G. If v_2 is a leaf of G, then every vertex $v \in V_1 - \{v_1\}$ is adjacent to v_1, whence $\deg(v_1) = n - 1$, which is a contradiction as $\overline{G}$ is connected. Moreover, if some vertex $v \in V_1$ is a leaf, then $\mathrm{diam}(G) \geq d(v, v_2) = 3$, which is a contradiction. Lastly, if u is a leaf, then v_1 is adjacent to every vertex of V_2, whence $\deg(v_1) = n - 1$, which is a contradiction. Thus we may assume that $\delta(G) \geq 2$. A similar argument shows that $\delta(\overline{G}) \geq 2$. Let $\mathcal{F}$ be the collection of graphs described in [7]. It is known (see [7]) that if $G \notin \mathcal{F}$ is a connected graph with order $n \geq 3$ and $\delta(G) \geq 2$, then $\gamma_r(G) \leq \frac{n-1}{2}$. It follows immediately that $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 1$, provided that $G, \overline{G} \notin \mathcal{F}$. Without loss of generality, suppose $G \in \mathcal{F}$. It is easily verified that $\gamma_r(G) + \gamma_r(\overline{G}) \leq n + 1$ and we are done.

Finally, recounting the argument, we have that $\gamma_r(G) + \gamma_r(\overline{G}) \leq n + 1$ in all cases, save when $G \in \mathcal{E}$. Hence, if $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$ it follows that $G \in \mathcal{E}$. This observation together with Lemma 4.6 implies that $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$ if and only if $G \in \mathcal{E}$. $\square$

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