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# Three Topics in Analysis: (I) The Fundamental Theorem of Calculus Implies that of Algebra, (II) Mini Sums for the Riesz Representing Measure, and (III) Holomorphic Domination and Complex Banach Manifolds Similar to Stein Manifolds

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THREE TOPICS IN ANALYSIS: (I) THE FUNDAMENTAL THEOREM OF CALCULUS IMPLIES THAT OF ALGEBRA, (II) MINI SUMS FOR THE RIESZ REPRESENTING MEASURE, AND (III) HOLOMORPHIC DOMINATION AND COMPLEX BANACH MANIFOLDS SIMILAR TO STEIN MANIFOLDS

by

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Under the direction of Dr. Imre Patyi

ABSTRACT

We look at three distinct topics in analysis. In the first we give a direct and easy proof that the usual Newton–Leibniz rule implies the fundamental theorem of algebra that any nonconstant complex polynomial of one complex variable has a complex root. Next, we look at the Riesz representation theorem and show that the Riesz representing measure often can be given in the form of mini sums just like in the case of the usual Lebesgue measure on a cube. Lastly, we look at the idea of holomorphic domination and use it to define a class of complex Banach manifolds that is similar in nature and definition to the class of Stein manifolds.

INDEX WORDS: Fundamental theorem of calculus, Fundamental theorem of algebra, Riesz representation theorem, Regular measure, Holomorphic domination, Complex Banach manifolds, Stein manifolds.

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## DEDICATION

I dedicate this dissertation to my Ammichi and Appachan, my brother Jose, and my wife Diana.

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# CHAPTER 1. INTRODUCTION.

Given the author's interest in education and his background as a full-time high school mathematics teacher, we treat two topics of great import in the teaching of mathematics at the undergraduate level (the fundamental theorem of algebra) and at the graduate level (the Riesz representation theorem for positive linear functionals), and a third topic of high research value (holomorphic domination and complex Banach manifolds). The first two of the above themes are thoroughly classical and play vitally important roles in many parts of mathematics. We believe that we add here valuable insights and novel approaches to these venerable theorems. Our third theme is a modern one, right at the forefront of current research in a newly rejuvenated confluence of several complex variables and functional analysis, namely, the theory of complex Banach manifolds.

We provide some introduction, background, and preliminary material as and when we need them (mostly in the form of references, and not explicit development here), but only sparingly lest it should obscure and overshadow our main points, which we rather prefer to keep in sharp focus. Accordingly, our treatment will be densely concentrated, and brutally to the point.

# CHAPTER 2. THE FUNDAMENTAL THEOREM OF CALCULUS IMPLIES THAT OF ALGEBRA.

In this chapter we take a look at the fundamental theorem of algebra, and present two minimalistic proofs of it. The theorem itself goes back at least to the doctoral dissertation of the young Gauss in the late 1700s in his teens. He gave several proofs of it also later in his life as the "princeps mathematicorum."(*) There are many known proofs of the theorem and the book [FR], entirely devoted to the topic, lists a large number of them. Almost any "principle" of complex analysis (such as the maximum principle, the minimum principle, the argument principle, the averaging principle of circular means, Cauchy's integral formula, the Cauchy estimates, the global residue theorem, the hyperbolicity principle (Liouville's theorem), and Rouché's theorem) give instant proofs. So do also methods of algebraic topology (winding numbers (also called indices of curves), index of a vector field, and homotopy invariance of degrees of maps from the circle to the circle or from the Riemann sphere to the Riemann sphere). There are also largely algebraic proofs that involve quadratic extensions, real-closed fields, and the fact that a real polynomial of odd degree has a real root (a well known fact from calculus).

We present here two short proofs of that require only minimal background. The first one only uses the fundamental theorem of calculus from first semester calculus (and thus may perhaps reach the ultimate simplicity and earliest possible point of introduction to students of a proof of the fundamental theorem of algebra), while the second relies on the inverse function theorem and basic notions of point-set topology, and thus could provide for a discussion in a class of analysis when teaching the inverse function theorem or in a class of point-set topology when teaching about continuity and compactness.

## 2.1. THE FIRST PROOF.

In this section we look at the fundamental theorem of algebra, that states that any polynomial $p(z) \in \mathbb{C}[z]$ of degree $m \geq 1$ with complex coefficients has at least one complex

(*) Prince of mathematicians.

root $z_0 \in \mathbb{C}$ with $p(z_0) = 0$. We show that this follows from the fundamental theorem of calculus, i.e., from the Newton–Leibniz rule that $g(b) - g(a) = \int_a^b g'(x)\,dx$ if $g\colon [a,b] \to \mathbb{C}$ is continuously differentiable on a segment $[a,b] \subset \mathbb{R}$.

As a preparation, recall the following error estimate of the difference between a Riemann sum and the corresponding Riemann integral.

**Proposition.** *If $g\colon [0,1] \to \mathbb{C}$ is continuously differentiable and $n \geq 1$ is an integer, then the difference*

$$E = \frac{1}{n}\sum_{\tau=1}^{n} g(\tau/n) - \int_0^1 g(t)\,dt = \sum_{\tau=1}^{n} \int_{\frac{\tau-1}{n}}^{\frac{\tau}{n}} (g(\tau/n) - g(t))\,dt$$

*satisfies that $|E| \leq \frac{M}{2n}$, where $M$ is an upper bound of the derivative $g'$ on $[0,1]$.*

Proof. For $\frac{\tau-1}{n} \leq t \leq \frac{\tau}{n}$ the Newton–Leibniz rule implies that $g(\tau/n) - g(t) = \int_t^{\tau/n} g'(s)\,ds$; so $|g(\tau/n) - g(t)| \leq M(\frac{\tau}{n} - t)$, and

$$\begin{aligned}
|E| &\leq \sum_{\tau=1}^{n} \int_{\frac{\tau-1}{n}}^{\frac{\tau}{n}} |g(\tau/n) - g(t)|\,dt \leq \sum_{\tau=1}^{n} \int_{\frac{\tau-1}{n}}^{\frac{\tau}{n}} M(\tfrac{\tau}{n} - t)\,dt \\
&\leq \sum_{\tau=1}^{n} -\tfrac{1}{2}M\left[(\tfrac{\tau}{n} - t)^2\right]_{t=\frac{\tau-1}{n}}^{\frac{\tau}{n}} \leq \tfrac{1}{2}M\sum_{\tau=1}^{n}\frac{1}{n^2} \leq \frac{M}{2n}
\end{aligned}$$

as claimed.

To prove now that the Newton–Leibniz rule implies the fundamental theorem of algebra, suppose for a contradiction that $p(z) \neq 0$ for all $z \in \mathbb{C}$ and define $f(z) = 1/p(z)$ for $z \in \mathbb{C}$. Then the rational functions $f, f', f''$ are all defined and bounded on $\mathbb{C}$ and $f$ vanishes to order $m$ at $\infty$. Hence there is a constant $0 < M < \infty$ with $|f(z)| < M/|z|^m$ for $z \neq 0$, $|f'(z)| < M$ and $|f''(z)| < M$ for all $z \in \mathbb{C}$.

For $r > 0$ and $n \geq 1$ integer, consider the discretized average

$$J_n(r) = \frac{1}{n} \sum_{\tau=1}^{n} f(re^{2\pi i\,\tau/n})$$

of $f$ on the circle $|z| = r$ in the $z$-plane $\mathbb{C}$, rewrite by the Newton–Leibniz rule the summand $f(re^{2\pi i\,\tau/n}) - f(0)$ of the difference $I_n(r) = J_n(r) - f(0)$ as the integral of its derivative, obtaining

$$I_n(r) = \frac{1}{n} \sum_{\tau=1}^{n} \int_0^r f'(se^{2\pi i\,\tau/n})\, e^{2\pi i\,\tau/n}\, ds = \int_0^r \frac{1}{n} \sum_{\tau=1}^{n} f'(se^{2\pi i\,\tau/n})\, e^{2\pi i\,\tau/n}\, ds,$$

and introduce the continuous (Riemann integral) version $K_n(r)$ of the Riemann sum $I_n(r)$ as an iterated integral by

$$K_n(r) = \int_0^r \left\{ \int_0^1 f'(se^{2\pi i\,t})\, e^{2\pi i\,t}\, dt \right\} ds.$$

Then $|J_n(r)| \leq M/r^m$, and $K_n(r) = \int_0^r \left[ f(se^{2\pi i\,t})/(2\pi i\, s) \right]_{t=0}^{1} ds = \int_0^r 0\, ds = 0$ for $n \geq 1$ and $r > 0$ by the Newton–Leibniz rule.

Write the difference $I_n(r) = I_n(r) - K_n(r)$ as

$$I_n(r) = \int_0^r \left\{ \frac{1}{n} \sum_{\tau=1}^{n} f'(se^{2\pi i\,\tau/n})\, e^{2\pi i\,\tau/n} - \int_0^1 f'(se^{2\pi i\,t})\, e^{2\pi i\,t}\, dt \right\} ds,$$

and apply the Proposition to the integrand with $g(t) = f'(se^{2\pi i\,t})\, e^{2\pi i\,t}$ for each fixed $s$, noting that $|g'(t)| \leq 2\pi M(s+1)$, to find the bound

$$|I_n(r)| \leq \int_0^r \frac{2\pi M(s+1)}{2n}\, ds \leq \frac{\pi M}{2n}(r+1)^2$$

for $r > 0$ and $n \geq 1$ integer. The estimate

$$0 < |f(0)| \leq |J_n(r) - I_n(r)| \leq |J_n(r)| + |I_n(r)| \leq \frac{M}{r^m} + \frac{\pi M}{2n}(r+1)^2$$

leads to the contradiction $|f(0)| < |f(0)|$ if we fix $r > 0$ first so large that $M/r^m < |f(0)|/2$ and second we fix $n \geq 1$ so large that $\pi M(r+1)^2/(2n) < |f(0)|/2$. This contradiction completes the proof that the fundamental theorem of calculus implies that of algebra.

In conclusion we note that the above proof from the author's paper [MP] uses only first or second semester calculus and a hallmark calculus trick of writing zero in a funny way. Our argument is obtained by removing all 'advanced' concepts from one of the usual proofs of Liouville's theorem (that states that a bounded entire function $f$ is constant), and involves manipulating the Cauchy integral $f(0) = \frac{1}{2\pi i}\int_{|z|=r} f(z)\, z^{-1}\, dz$, of which our quantity $J_n(r)$ is a discretization as a Riemann sum. We can similarly prove that the fundamental theorem of calculus implies that the spectrum of any element of any complex Banach algebra is not empty. We hope that the above development can be incorporated in a class of calculus, perhaps as a project or outlook section for some of the better or honors students. It also makes the presentation of partial fraction decomposition and integration of rational functions in terms of elementary functions self-contained in first-year calculus.

## 2.2. THE SECOND PROOF.

In this section we give a second proof of the fundamental theorem of algebra, this time around largely based on point-set topology.

**Theorem.** *Any complex polynomial $f(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$, $n \geq 1$, has a complex root.*

Proof. Let $S = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, and extend $f$ continuously to $f\colon S \to S$ by putting $f(\infty) = \infty$. Let $S_1 = f(S)$ be the image; $S_1$ is a compact subset of the sphere $S$. Let $K = \{\infty, z \in \mathbb{C}: f'(z) = 0\}$ be the set of critical points; $K$ is a finite set of at most $n$ points. Let $K_1 = f(K)$ be its image; $K_1$ is the set of critical values and it is a finite set of at most $n$ points. The set $U = S \setminus K$ is a non-empty open subset of $\mathbb{C}$, and its image $U_1 = f(U)$ is also an open subset of $\mathbb{C}$ by the inverse function theorem for continuously differentiable mappings of plane domains to the plane. As $S_1 = U_1 \cup K_1$ is compact, we see that the closure $\overline{U_1}$ in $S$ is the union of $U_1$ and finitely many points, or, the boundary $\partial U_1$ in $S$ is a finite set. We claim that $\overline{U_1}$ is the whole sphere $S$. Indeed, suppose for a contradiction that $\overline{U_1}$ is a proper closed subset of $S$. Then $G = S \setminus \overline{U_1}$ is a non-empty open subset of $S$ with the same boundary as $\partial U_1$, i.e., a finite set. Stereographic projection from a point in $U_1$ converts $G$ into a bounded open subset $G_1$ of the plane $\mathbb{C}$ whose boundary is a finite set, but such a set $G_1$ does not exist, because the first point on each ray issued from a fixed interior point of our bounded open set $G_1$ is a boundary point of $G_1$. The above contradiction shows that $S_1$ is the whole sphere $S$, because $S_1$ contains $\overline{U_1}$, i.e., $f$ is surjective onto the sphere; in particular, it assumes in $\mathbb{C}$ the value 0, too. The proof is complete.

This ends our discussion of the fundamental theorem of algebra.

## CHAPTER 3. MINI SUMS FOR THE RIESZ REPRESENTING MEASURE.

In this chapter we look at the dual of the space of continuous functions on a compact Hausdorff space and identify its members as integration against certain type of measures.

The Riesz representation theorem for positive linear functionals $\xi \in C(X)^*$ on the space $C(X)$ of all continuous functions $f: X \to \mathbb{R}$ endowed with the supremum norm $\|f\| = \sup\{f(x): x \in X\}$ on a compact Hausdorff space $X$ states that there is a regular

Borel measure $\mu$ on $X$ with $\xi(f) = \int_X f\, d\mu$ for $f \in C(X)$. This theorem was given first for the unit interval $X = [0,1]$ by F. Riesz in around 1909, and in the above generality by Markov and Kakutani perhaps in the 1930s. The standard proofs of the theorem involve, explicitly or implicitly, "contents" and define $\mu(A)$ for $A \subset X$ as

$$\mu(A) = \inf_{A \subset G} \sup_{f} \xi(f),$$

where $G$ runs through all open sets $G$ in $X$ with $A \subset G$, and $f$ runs through all functions $f \in C(X)$ such that $f \geq 0$ on $X$, $f \leq 1$ on $G$, and the support $\operatorname{supp}(f) \subset G$ lies inside $G$.

If we take the simplest case $X = [0,1]$ and $\xi(f) = \int_0^1 f(x)\, dx$ given by the Riemann integral of $f \in C(X)$, then $\mu$ is the usual Lebesgue measure on $X$. The above displayed formula for $\mu(A)$ is quite different from the classical formula

$$\mu(A) = \inf_{(I_n)} \sum_{n=1}^{\infty} |I_n|$$

of Lebesgue, where $I_n \subset X$ is an interval of length $|I_n|$ for $n \geq 1$, and $A \subset \bigcup_{n=1}^{\infty} I_n$.

Another approach to the Riesz representation theorem is based on monotone limits, the Daniell integral, and Stone's axiom and theorem. Such a presentation was initiated by Riesz himself, can be read in Loomis's book [Ls], and is repeated in the notes of Sternberg [Sg]. An extension of the theorem involving "tight" positive linear functionals is used by probabilists in connection with defining measures on "large" spaces as in the Kolmogorov extension theorem or the Wiener process, and is treated by Stroock in [Sk].

Traditionally, at least in almost all the books that we consulted, measures are built from outer measures by the Carathéodory method of extension. In [Ls] and the newer work [Sk] we find that the Daniell method gives a quick proof of the Carathéodory extension theorem and treats functions (as we prefer) rather than sets directly. It seems to us that very basic measure theory could be based on an axiomatic notion of outer integral rather than on an outer measure, and could proceed by constructing a Daniell integral out of an outer integral similarly but alternatively to the way Carathéodory constructs a measure out of an outer measure. While we do not try here to introduce an axiomatic notion of an outer integral in general, we do so implicitly in the case of the Riesz representation theorem below. We imagine the outer integral $\xi'$ arising from a positive linear functional $\xi \in C(X)^*$ as a mini sum, just like in the case of the classical Lebesgue outer measure, given by

$$\xi'(f) = \inf_{(f_n)} \sum_{n=1}^{\infty} \xi(f_n),$$

where $f\colon X \to [0,\infty]$ is any function, and the infimum is taken for all sequences of functions $(f_n)$ such that $f_n\colon X \to [0,\infty)$ is continuous for $n \geq 1$, and $f(x) \leq \sum_{n=1}^{\infty} f_n(x)$ for all $x \in X$. As for $f_n(x) = n$ we have $f \leq \infty = \sum_n f_n$, the infimum is not taken over the empty set of sequences. We define $\mu(A)$ for $A \subset X$ by

$$\mu(A) = \xi'(1_A),$$

where $1_A$ is as usual the indicator function of $A$ in $X$.

If $X = [0,1]$ and $\xi(f) = \int_0^1 f(x)\,dx$ is the Riemann integral of $f \in C(X)$, then it is easy to see that $\xi'(f)$ given by the mini sum above equals $\xi'(f) = \int_0^1 f(x)\,dx$ the Lebesgue integral of $f$ if $f \geq 0$ lies in $L_1(X)$.

In what follows we show that this definition of $\mu$ works at least when $X$ is a compact metric space (which is the most directly useful case anyway). We need a technical condition.

We say that a topological space $X$ *satisfies condition (R)* if $1_G(x) = \sum_{f\in F} f(x)$ for $x \in X$ for each open set $G$ in $X$ and some subset $F \subset C(X)^+ = C(X, [0,\infty))$ with the support $\mathrm{supp}(f) \subset G$ inside $G$ for $f \in F$.

If $X$ is a normal Hausdorff space, and every open subset $G$ of $X$ is $\sigma$-compact, then $X$ satisfies condition (R) by Urysohn's separation theorem. This is the case when $X$ is a compact metric space.

**Theorem 1.** (Riesz representation theorem) *Let $X$ be a compact Hausdorff space satisfying condition (R), $C(X)$ the space of all continuous real functions $f\colon X \to \mathbb{R}$ endowed with the supremum norm $\|f\| = \sup\{|f(x)|\colon x \in X\}$, and $\xi\colon C(X) \to \mathbb{R}$ a positive linear functional, i.e., $\xi$ is a real linear functional such that $\xi(f) \geq 0$ if $f \geq 0$ in $C(X)$. Then there is a measure $\mu$ on the Borel sets of $X$ such that $\xi(f) = \int_X f\, d\mu$ for all $f \in C(X)$, $\xi(1) = \mu(X)$, $\mu(A) = \inf_{A\subset G} \mu(G)$, and $\mu(G) = \sup_{K\subset G} \mu(K)$, where $K$ is compact, $G$ is open, and $A$ is arbitrary in $X$.*

Recall that the sum $\sum_{x\in A} x$ for $A \subset [0,\infty)$ is defined as $\sup_{F\subset A} \sum_{x\in F} x$, where $F$ in the supremum runs through all finite subsets of $A$. For $A \subset X$ define

$$\mu(A) = \inf\Big\{ \sum_{f\in F} \xi(f)\colon \sum_{f\in F} f \geq 1_A \Big\},$$

where $F$ in the infimum runs through all subsets of $C(X,[0,\infty)) = C(X)^+$ that satisfy $\sum_{f\in F} f(x) \geq 1$ for $x \in A$. Note that the function $\sum_{f\in F} f\colon X \to [0,\infty]$, while possibly infinite at some points, is not arbitrarily ugly: it is lower semicontinuous, since $\sum_{f\in F} f = \sup_{F'\subset F} \sum_{f\in F'} f$ is the supremum of the continuous functions $\sum_{f\in F'} f$ as $F'$ runs through all finite subsets $F'$ of $F$.

**Proposition 2.** *The following hold.*

(a) $\mu(X) \leq \xi(1)$.

(b) $\mu(X) \geq \xi(1)$.

(c) $\mu(\emptyset) = 0$.

(d) $\mu(A) \leq \mu(B)$ *if* $A \subset B \subset X$.

(e) $\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n)$ *if* $A = \bigcup_{n=1}^{\infty} A_n \subset X$.

(f) $\mu(T \cap (K_1 \cup K_2)) = \mu(T \cap K_1) + \mu(T \cap K_2)$ *if* $T, K_1, K_2 \subset X$ *with* $K_1$ *and* $K_2$ *disjoint compact sets.*

(g) *If* $K \subset X$ *is compact, and* $\varepsilon > 0$, *then there is a continuous function* $k\colon X \to [0,1]$ *with* $1_K \leq k$ *(in fact,* $k$ *can be taken to be equal to* $1$ *on an open neighborhood of* $K$ *in* $X$*) and* $\mu(K) \leq \xi(k) < \mu(K) + \varepsilon$.

(h) *If* $G \subset X$ *is open, then* $\mu(G) = \sup \mu(K)$, *where* $K$ *in the supremum runs through all compacts sets in* $X$ *with* $K \subset G$.

(i) *If* $A \subset X$ *is any subset, then* $\mu(A) = \inf \mu(G)$, *where* $G$ *in the infimum runs through all open sets in* $X$ *with* $A \subset G$.

(j) $\mu(T) = \mu(T \cap K) + \mu(T \cap K')$ *for all* $T, K \subset X$, *where* $K$ *is compact and* $K' = X \setminus K$ *is its complement.*

Proof. (a) Consider $F = \{1\}$, and note that $1_X = 1 \leq 1$, so $\mu(X) \leq \xi(1)$.

(b) For $\varepsilon > 0$ consider a family $F \subset C(X)^+$ with $\mu(X) \leq \sum_{f \in F} \xi(f) < \mu(X) + \varepsilon$ and $\sum_{f \in F} f(x) \geq 1$ for all $x \in X$. Then $\varepsilon + \sum_{f \in F'} f(x) > 1$ for a finite subset $F' \subset F$ at each point $x \in X$. As the above open type inequality of continuous functions also persists in an open neighborhood of $x$ in $X$, and the compact space $X$ is covered by these open sets, there is a finite set $F'' \subset F$ with $\varepsilon + \sum_{f\in F''} f(x) > 1$ for all $x \in X$. Then $\xi(1) \le \xi(\varepsilon + \sum_{f\in F''} f) = \varepsilon\xi(1) + \sum_{f\in F''} \xi(f) \le \varepsilon\xi(1) + \sum_{f\in F} \xi(f) \le \varepsilon\xi(1) + \mu(X) + \varepsilon$. So $\xi(1) \le \varepsilon\xi(1) + \mu(X) + \varepsilon$, making in which $\varepsilon \to +0$ yields $\xi(1) \le \mu(X)$.

(c) Consider $F = \{0\}$, and note that $1_\emptyset = 0 \le 0$, so $\mu(\emptyset) \le \xi(0) = 0$.

(d) For $\varepsilon > 0$ consider a family $F \subset C(X)^+$ with $\mu(B) \le \sum_{f\in F} \xi(f) < \mu(B) + \varepsilon$ and $\sum_{f\in F} \xi(f) \ge 1_B$. As $1_A \le 1_B$, we get $\mu(A) \le \sum_{f\in F} \xi(f) \le \mu(B)+\varepsilon$. Making $\varepsilon \to +0$ yields $\mu(A) \le \mu(B)$.

(e) For $\varepsilon > 0$ and $n \ge 1$ consider a family $F_n \subset C(X)^+$ with $\mu(A_n) \le \sum_{f\in F_n} \xi(f) < \mu(A_n)+\varepsilon/2^n$ and $\sum_{f\in F_n} f \ge 1_{A_n}$. With $F = \bigcup_{n=1}^{\infty} F_n$ note as $1_A \le \sup_{n\ge 1} 1_{A_n}$ that $1_A \le \sum_{f\in F} f$ and $\sum_{f\in F} \xi(f) \le \sum_{n=1}^{\infty} \sum_{f\in F_n} \xi(f) \le \sum_{n=1}^{\infty}(\mu(A_n) + \varepsilon/2^n) \le \varepsilon + \sum_{n=1}^{\infty} \mu(A_n)$, i.e., $\mu(A) \le \varepsilon + \sum_{n=1}^{\infty} \mu(A_n)$, making in which $\varepsilon \to +0$ yields $\mu(A) \le \sum_{n=1}^{\infty} \mu(A_n)$.

(f) The subadditivity in (e) implies that $\mu(T \cap K) \le \mu(T \cap K_1) + \mu(T \cap K_2)$ for $K = K_1 \cup K_2$. For $\varepsilon > 0$ there is a family $F \subset C(X)^+$ with $\mu(T \cap K) \le \sum_{f\in F} \xi(f) < \mu(T \cap K) + \varepsilon$ and $\sum_{f\in F} f \ge 1_{T\cap K}$. By Urysohn's separation theorem in the normal Hausdorff space $X$, the disjoint closed sets $K_1$ and $K_2$ can be separated by a continuous function, i.e., there is a continuous function $\chi\colon X \to [0,1]$ with $\chi = i-1$ on $K_i$ for $i = 1, 2$. Then $1_{T\cap K_1} \le (1-\chi)1_{T\cap K} \le \sum_{f\in F}(1-\chi)f$, so $\mu(T \cap K_1) \le \sum_{f\in F} \xi((1-\chi)f)$. Also $1_{T\cap K_2} \le \chi 1_{T\cap K} \le \sum_{f\in F} \chi f$, and so $\mu(T \cap K_2) \le \sum_{f\in F} \xi(\chi f)$. Then $\mu(T \cap K_1) + \mu(T \cap K_2) \le \sum_{f\in F}(\xi((1-\chi)f) + \xi(\chi f)) = \sum_{f\in F} \xi(f) \le \mu(T \cap K) + \varepsilon$, making in which $\varepsilon \to +0$ yields $\mu(T \cap K_1) + \mu(T \cap K_2) \le \mu(T \cap K)$, and together with the first sentence in this paragraph it also yields $\mu(T \cap K_1) + \mu(T \cap K_2) = \mu(T \cap K)$.

(g) Just like in (b) for $\eta > 0$ consider a family $F \subset C(X)^+$ such that $\mu(K) \leq \sum_{f\in F} \xi(f) < \mu(K) + \eta$ and $\sum_{f\in F} f \geq 1_K$. Then $\eta + \sum_{f\in F'} f(x) > 1$ for a finite set $F' \subset F$ at each point $x \in K$. As the above open type inequality of continuous functions persists in an open neighborhood of $x$ in $X$, and the compact set $K$ is covered by these open sets, there is a finite set $F'' \subset F$ with $\eta + \sum_{f\in F''} f > 1_K$ on $X$. Then $k = 1 \wedge (\eta + \sum_{f\in F''} f) \in C(X)^+$ satisfies that $1_K \leq k \leq \eta + \sum_{f\in F''} f$, thus $\mu(K) \leq \xi(k) \leq \eta\xi(1) + \sum_{f\in F''} \xi(f) \leq \eta\xi(1) + \sum_{f\in F} \xi(f) \leq \eta\xi(1) + \mu(K) + \eta < \mu(K) + \varepsilon$ if $0 < \eta < \varepsilon/(\xi(1) + 1)$, i.e., $\mu(K) \leq \xi(k) < \mu(K) + \varepsilon$. Note that the function $k$ equals the constant 1 on an open neighborhood of $K$ in $X$.

(h) Clearly, $\mu(K) \leq \mu(G)$ by the monotonicity in (d). It remains for $\varepsilon > 0$ that we find a compact set $K$ in $X$ with $K \subset G$ and $\mu(G) \leq \mu(K) + \varepsilon$. As $X$ satisfies condition (R), there is a family $\Phi \subset C(X)^+$ such that $\mathrm{supp}(\varphi) \subset G$ for $\varphi \in \Phi$ and $1_G = \sum_{\varphi\in\Phi} \varphi$.

There is a family $F \subset C(X)^+$ with $\mu(G) \leq \sum_{f\in F} \xi(f) < \mu(G) + \varepsilon$ and $\sum_{f\in F} f \geq 1_G$. Then $1_G \leq \sum_{\varphi\in\Phi, f\in F} \varphi f$, and $\mu(G) \leq \sum_{\varphi\in\Phi, f\in F} \xi(\varphi f) \leq \sum_{f\in F} \xi(f) < \mu(G) + \varepsilon$, since if $\Phi' \subset \Phi$ and $F' \subset F$ are finite, then we have that $\sum_{\varphi\in\Phi', f\in F'} \xi(\varphi f) = \xi((\sum_{\varphi\in\Phi'} \varphi)(\sum_{f\in F'} f)) \leq \xi(1 \sum_{f\in F'} f) = \sum_{f\in F'} \xi(f) \leq \sum_{f\in F} \xi(f)$. There is a finite family $\Phi' \subset \Phi$ with $\mu(G) - \varepsilon < \sum_{\varphi\in\Phi'} \xi(\varphi) < \mu(G) + \varepsilon$. Let $K$ be the union of the supports of $\varphi \in \Phi'$, which is a compact set $K \subset G$. There is by (g) a continuous function $k\colon X \to [0, 1]$ with $1_K \leq k$ and $\mu(K) \leq \xi(k) < \mu(K) + \varepsilon$. Then $\sum_{\varphi\in\Phi'} \varphi \leq 1_K \leq k$, hence $\mu(G) - \varepsilon < \sum_{\varphi\in\Phi'} \xi(\varphi) \leq \xi(k) < \mu(K) + \varepsilon$. Thus $\mu(G) < \mu(K) + 2\varepsilon$, and so $\mu(G) = \sup \mu(K)$ as $K$ runs through all compact subsets of $G$.

(i) For $\varepsilon > 0$ there is a family $F \subset C(X)^+$ with $1_A \leq \sum_{f\in F} f$ and $\mu(A) \leq$

$\sum_{f\in F}\xi(f) < \mu(A)+\varepsilon$. Look at $1_A < \varepsilon + \sum_{f\in F} f$, and let $G = \{\varepsilon + \sum_{f\in F} f > 1\}$, which is an open set being the level set of the given lower semicontinuous function. Then $A \subset G$, $1_G < \varepsilon + \sum_{f\in F} f$ and $\mu(G) \leq \varepsilon\xi(1) + \sum_{f\in F}\xi(f)$, i.e., $\mu(G) \leq \varepsilon\xi(1) + \mu(A) + \varepsilon$. As $\mu(A) \leq \mu(G)$ by the subadditivity in (e), we have $\mu(A) = \inf_{A\subset G}\mu(G)$ as $G$ runs through all open sets in $X$ with $A \subset G$.

(j) Due to the subadditivity in (e) it is enough to prove that $\mu(T\cap K)+\mu(T\cap K') \leq \mu(T)$. For $\varepsilon > 0$ there is by (i) an open set $G$ in $X$ with $T \subset G$, $\mu(G) \leq \mu(T)+\varepsilon$, and there is by (h) a compact set $L$ in the open set $G \cap K'$ with $\mu(G\cap K') \leq \mu(L)+\varepsilon$. Then $\mu(T\cap K) + \mu(T\cap K') \leq \mu(G\cap K) + \mu(G\cap K') \leq \mu(G\cap K) + \mu(G\cap L) + \varepsilon = \mu(G\cap(K\cup L)) + \varepsilon \leq \mu(G)+\varepsilon \leq \mu(T)+2\varepsilon$, where we applied the monotonicity in (e), the choice of $L$, the additivity in (f), and the choice of $G$. Thus $\mu(T\cap K)+\mu(T\cap K') \leq \mu(T)+2\varepsilon$, making in which $\varepsilon \to +0$ and taking into account the first sentence of this paragraph we obtain that $\mu(T\cap K) + \mu(T\cap K') = \mu(T)$. QED.

Proof of Theorem 1. By Proposition 2 our $\mu$ is a finite outer measure on the power set of $X$, and compact and thus open sets in $X$ are all Carathéodory measurable with respect to $\mu$. Thus by Carathéodory's theorem every Borel set in $X$ is measurable with respect to $\mu$ and $\mu$ is a measure when restricted to the Borel $\sigma$-algebra. It remains to prove that the integration against $\mu$ represents the given positive linear functional $\xi$. Note that any Borel set in $X$ is measurable with respect to $\mu$, and thus any bounded Borel function (such as a continuous function) on $X$ is summable with respect to $\mu$.

We claim that if $f\colon X \to [0,1]$ is continuous, then $\int_X f\,d\mu \geq \xi(f)$. Indeed, for $n \geq 1$ consider the step function $g_n = \frac{1}{n}\sum_{i=1}^{n} 1_{K_{ni}}$, whose steps are the compact sets

$K_{ni} = \{f \geq \frac{i}{n}\}$. Note that if $f(x) = j/n$ for an integer $j$, then $f(x) \geq i/n$ for $i = 1, \ldots, j$, i.e., $g_n(x) = j/n = f(x)$. If $j/n < f(x) < (j+1)/n$ for an integer $j$, then $f(x) \geq i/n$ for $i = 1, \ldots, j$, i.e., $g_n(x) = j/n < f(x) < g_n(x) + 1/n$. Thus $g_n \leq f \leq g_n + \frac{1}{n}$ on $X$. For the compact set $K_{ni}$ there is by (g) a continuous function $k_{ni}\colon X \to [0,1]$ with $1_{K_{ni}} \leq k_{ni}$ and $\mu(K_{ni}) \leq \xi(k_{ni}) < \mu(K_{ni}) + \frac{1}{n}$. Look at the function $h_n = \frac{1}{n} + \frac{1}{n}\sum_{i=1}^{n} k_{ni}$, which is continuous on $X$, and $f \leq \frac{1}{n} + g_n \leq h_n$ on $X$. Thus $\xi(f) \leq \xi(h_n)$, and $\int_X f\, d\mu \leq \int_X (\frac{1}{n} + g_n)\, d\mu = \frac{1}{n}\mu(X) + \frac{1}{n}\sum_{i=1}^{n} \mu(K_{ni}) \leq \frac{1}{n}\mu(X) + \frac{1}{n}\sum_{i=1}^{n} \xi(k_{ni}) = \xi(h_n) \leq \frac{1}{n}\mu(X) + \frac{1}{n}\sum_{i=1}^{n}(\mu(K_{ni}) + \frac{1}{n}) = \frac{1}{n} + \int_X (\frac{1}{n} + g_n)\, d\mu \leq \frac{1}{n} + \int_X (\frac{1}{n} + f)\, d\mu = \frac{1}{n}(1 + \mu(X)) + \int_X f\, d\mu$, i.e., $\int_X f\, d\mu \leq \xi(h_n) \leq \frac{1}{n}(1+\mu(X)) + \int_X f\, d\mu$. In other words, $\xi(h_n) \to \int_X f\, d\mu$ as $n \to \infty$. As $\xi(h_n) \geq \xi(f)$ for all $n \geq 1$, we get in the limit that $\int_X f\, d\mu \geq \xi(f)$.

Let now $f_n = \frac{1}{2} \pm \frac{1}{n} g$ for $g \in C(X)$ and $n \geq 1$, and note that $f_n \to \frac{1}{2}$ uniformly on $X$ as $n \to \infty$. Thus for $n$ large enough $0 \leq f_n \leq 1$, and so $\xi(f_n) \leq \int_X f_n\, d\mu$, i.e., $\frac{1}{2}\mu(X) \pm \frac{1}{n}\xi(g) \leq \frac{1}{2}\mu(X) \pm \frac{1}{n}\int_X g\, d\mu$, which simplifies to $\pm\xi(g) \leq \pm\int_X g\, d\mu$, i.e., $\xi(g) = \int_X g\, d\mu$ for all $g \in C(X)$. QED.

One advantage of our proof of the Riesz representation theorem above is that it is short and direct, and it avoids dealing with contents, partitions of unity, monotone families, Dini's lemma, and envelopes called "philtre" by Bourbaki (at least in French). One drawback is that perhaps condition (R) may not be satisfied by all compact Hausdorff spaces. If we restrict ourselves to defining the representing measure $\mu$ on the Baire $\sigma$-algebra (generated by the $\sigma$-compact open sets), then we do not need the condition (R) and we may use countable families, i.e., ordinary series, in the definition of $\mu(A)$.

Varadarajan [V] gave another proof of the Riesz representation theorem that is largely based on point-set topology. When the underlying space $X$ is a compact metric space, it uses a continuous surjection $C \to X$ from the Cantor set $C = \{0,1\}^{\mathbb{N}}$, and reduces the proof to the case when $X = C$. Then the representing measure is easily constructed due to the existence of many continuous indicator functions, since the Cantor set $C$ is totally disconnected. The same idea was also used by Garling [G], Hartig [H] and Sunder [Sr].

# CHAPTER 4. ON HOLOMORPHIC DOMINATION.

In this chapter we look at a device that makes up for the lack of compact exhaustions in pseudoconvex open sets in a Banach space. It is useful in many questions of complex analysis on Banach spaces, e.g., involving analytic cohomology.

Let $X$ be a separable Banach space and $u\colon X \to \mathbb{R}$ locally upper bounded. We show that there are a Banach space $Z$ and a holomorphic function $h\colon X \to Z$ with $u(x) < \|h(x)\|$ for $x \in X$. As a consequence we find that the sheaf cohomology group $H^q(X, \mathcal{O})$ vanishes if $X$ has the bounded approximation property (i.e., $X$ is a direct summand of a Banach space with a Schauder basis), $\mathcal{O}$ is the sheaf of germs of holomorphic functions on $X$, and $q \geq 1$. As another consequence we prove that if $f$ is a $C^1$-smooth $\bar{\partial}$-closed $(0,1)$-form on the space $X = L_1[0,1]$ of summable functions, then there is a $C^1$-smooth function $u$ on $X$ with $\bar{\partial} u = f$ on $X$.

## 4.1. INTRODUCTION.

The ideas of plurisubharmonic domination and holomorphic domination along with some of their applications appeared in [L3] by Lempert. Following him we say that *plurisub-*

*harmonic domination* is possible on a complex Banach manifold $M$ if for every locally upper bounded $u: M \to \mathbb{R}$ there is a continuous plurisubharmonic $\psi: M \to \mathbb{R}$ with $u(x) < \psi(x)$ for all $x \in M$. If $\psi$ can be taken in the form $\psi(x) = \|h(x)\|$ for $x \in M$, where $h: X \to Z$ is a holomorphic function to a Banach space $Z$, then we say that *holomorphic domination* is possible in $M$.

One tool to achieve holomorphic domination is the following Runge approximation property of a Banach space $X$.

**Hypothesis 4.1.1.** [L3, Hypothesis 1.5] *There is a constant $0 < \mu < 1$ such that if $Z$ is any Banach space, $\varepsilon > 0$, and $f: B_X \to Z$ is holomorphic on the open unit ball $B_X$ of $X$, then there is a holomorphic function $g: X \to Z$ with $\|f(x) - g(x)\| < \varepsilon$ for $\|x\| < \mu$.*

Lempert and Meylan proved the following theorem involving the above.

**Theorem 4.1.2.** (a) (Lempert, [L2]) *If $X$ is a Banach space with an unconditional basis, then Hypothesis 4.1.1 above holds for $X$.*

(b) (Meylan, [M]) *If $X$ is a Banach space with an unconditional finite dimensional Schauder decomposition, then Hypothesis 4.1.1 holds for $X$.*

(c) (Lempert, [L3]) *If $X$ is a Banach space with a Schauder basis (or a direct summand of one) and Hypothesis 1.1 holds for $X$, then holomorphic domination is possible in every pseudoconvex open subset of $X$.*

Our main goal in this paper is to find a route to holomorphic domination that bypasses Hypothesis 4.1.1 above. Our main results are Theorems 4.1.3, 4.1.4, 4.1.5, and 4.6.1 below.

**Theorem 4.1.3.** *If $X$ is a separable Banach space, then holomorphic domination is possible* (a) *in $X$, and* (b) *in every convex open $\Omega \subset X$.*

As a consequence of Theorem 4.1.3 we get cohomology vanishing as follows.

**Theorem 4.1.4.** *Let $X$ be a Banach space with the bounded approximation property, $\Omega \subset X$ pseudoconvex open, $M \subset \Omega$ a closed split complex Banach submanifold of $\Omega$, $S \to M$ a cohesive sheaf, $E \to \Omega$ a holomorphic Banach vector bundle, and $I \to \Omega$ the sheaf of germs of holomorphic sections of $E$ over $\Omega$ that vanish on $M$. If plurisubharmonic domination is possible in $\Omega$ (which is guaranteed by Theorem 4.1.3 if $\Omega \subset X$ is convex open), then the following hold.*

(a) *The cohesive sheaf $S \to M$ admits a complete resolution over $M$.*

(b) *The sheaf cohomology group $H^q(M, S)$ vanishes for all $q \geq 1$.*

(c) *The sheaf $I$ is cohesive over $\Omega$, $H^q(\Omega, I) = 0$ for $q \geq 1$, and any holomorphic section $f \in \mathcal{O}(M, E)$ extends to a holomorphic section $F \in \mathcal{O}(\Omega, E)$ with $F(x) = f(x)$ for $x \in M$.*

(d) *If $\Omega \subset X$ is convex open, then $E$ is holomorphically trivial over $\Omega$.*

As a consequence of Theorem 4.1.4 we get the following Theorem 4.1.5 on the $\bar{\partial}$-equation.

**Theorem 4.1.5.** *Let $X$ be an $\mathcal{L}_1$-space with the bounded approximation property (e.g., $X = L_1[0, 1]$), $\Omega \subset X$ pseudoconvex open, $E \to \Omega$ a holomorphic Banach vector bundle, and $f \in C^1_{0,1}(\Omega, E)$ a $C^1$-smooth $\bar{\partial}$-closed $(0, 1)$-form with values in $E$. If plurisubharmonic domination is possible in $\Omega$ (which is guaranteed by Theorem 4.1.3 if $\Omega \subset X$ is convex* *open), then there is a $C^1$-smooth section $u \in C^1(\Omega, E)$ of $E$ with $\bar\partial u = f$ over $\Omega$.*

Our strategy is to imitate the relevant parts of [L3] closely, but refrain from using Runge approximation for functions unbounded on balls. The reader is assumed to have a copy of [L3] along side this paper. In our §§ 4.2-4.4 we adopt without comment the notation of [L3, §§ 2-4].

## 4.2. BACKGROUND.

In this section we recall some material useful later. The paper [L3] uses a particular exhaustion $\Omega_N\langle\alpha\rangle$, $N \ge 1$, of any pseudoconvex open subset $\Omega$ of any Banach space $X$ with a bimonotone Schauder basis, and there are numerous other sets used there to help out with the analysis of the said exhaustion. In our case all the sets involved will be convex open in $X$ or in the span of finitely many of its basis vectors. The infinite dimensional ones among the sets that we need are all of the form $D \times B$, where $D$ is a convex open set in the span of the first few basis vectors and $B$ is a ball in the closed span of the rest of the basis vectors. As we shall need very little of the properties of the many sets discussed in [L3] we just help ourselves directly to the results there and skip any of their details (even their definitions) here.

In a Banach space $X$, put $B_X(x_0, r) = \{x \in X \colon \|x - x_0\| < r\}$ for the open ball of radius $r$ centered at $x_0 \in X$, and write $B_X = B_X(0, 1)$ for the unit ball. Denote by $\mathcal{O}(M_1, M_2)$ the set of holomorphic functions $M_1 \to M_2$ from one complex Banach manifold $M_1$ to another $M_2$.

Let $X$ be a Banach space, $A \subset X$, and $u \colon A \to \mathbb{R}$. We say that $u$ can be *dominated* *by entire functions with values in Banach spaces* on $A$ if there are a Banach space $Z$ and an entire holomorphic function $h \in \mathcal{O}(X, Z)$ with $u(x) \le \|h(x)\|$ for all $x \in A$.

If $T$ is any set, then denote by $\ell_\infty(T)$ the Banach space of bounded functions $f\colon T \to \mathbb{C}$ with the sup norm $\|f\| = \sup\{\|f(t)\| : t \in T\}$.

## 4.3. DOMINATION ON THE WHOLE SPACE.

In this section we show that if a function $u$ can be dominated on every ball of a fixed radius, then $u$ can be dominated on the whole space as well.

Let $X$ be Banach space with a Schauder basis. Fix the norm and the Schauder basis of $X$ so as to make a bimonotone Schauder basis of $X$. Fix $N \ge 1$ and write $\pi$ for the Schauder projection onto the span of the first $N+1$ basis vectors, $\varrho = 1 - \pi$ for the complementary projection, and $Y = \varrho X$ for the complementary space.

**Proposition 4.3.1.** *If $X$ is a Banach space with a bimonotone Schauder basis, $0 < R < \infty$, $u\colon X \to [1, \infty)$ is continuous, and $u$ can be dominated by entire functions with values in Banach spaces on every ball $B_X(x_0, R)$ of radius $R$ and centered at any $x_0 \in X$, then $u$ can be dominated by entire functions with values in Banach spaces on $X$.*

The proof of Proposition 4.3.1 will occupy us for a while.

**Proposition 4.3.2.** (Cf. [L3, Lemma 4.1]) *Let $A_2 \subset\subset A_3$ be relatively open bounded convex subsets of $\pi(X) \cong \mathbb{C}^{N+1}$, $A_1$ a compact convex subset of $A_2$, and $0 < r_1 < r_2 < r_3 < \infty$ constants. If $Z$ is a Banach space and $g \in \mathcal{O}(X, Z)$ is an entire function, then there are a Banach space $W$ and an entire function $h \in \mathcal{O}(X, W)$ with*

(i) $\|h(x)\|_W \le 1$ *for* $x \in A_1[r_1]$ *and*

(ii) $\|h(x)\|_W \geq \|g(x)\|_Z$ *for* $x \in A_3(r_3) \setminus A_2(r_2)$.

Proof. Consider the bounded convex sets $H_1, H_2, H_3$ in $\pi(X) \times \mathbb{C} \cong \mathbb{C}^{N+2}$ given by $H_1 = \{(s,\lambda) \in A_1 \times \mathbb{C} : |\lambda| \leq r_1\}$, $H_i = \{(s,\lambda) \in A_i \times \mathbb{C} : |\lambda| < r_i\}$ for $i = 2,3$. Since $H_1$ is compact convex in $\mathbb{C}^{N+2}$ there are a finite set $J$ and polynomials $\varphi_j \in \mathcal{O}(\pi(X) \times \mathbb{C})$ for $j \in J$ such that $|\varphi_j(s,\lambda)| \leq \frac{1}{4}$ for $(s,\lambda) \in H_1$ and for every $(s,\lambda) \in H_3 \setminus H_2$ there is a $j \in J$ with $|\varphi_j(s,\lambda)| \geq 4$. Denote by $L = \overline{B_{Y^*}}$ the set of all linear functionals $l \in Y^*$ with $\|l\| \leq 1$, and by $V = \ell_\infty(L \times J)$. Define $\varphi \in \mathcal{O}(X,V)$ by $\varphi(x)(l,j) = \varphi_j(\pi x, l\varrho x)$ for $x \in X$, $l \in L$, and $j \in J$.

The rest of the proof of Proposition 4.3.2 is the same word for word as that of [L3, Lemma 4.1] starting with "Going back" near [L3, (4.1)].

**Proposition 4.3.3.** (Cf. [L3, Proposition 4.2]) *Let* $0 < \mu < 1$, $N \geq 1$, *and* $2^4\beta < \alpha < 2^{-8}\mu$. *If $Z$ is a Banach space and* $g \in \mathcal{O}(X,Z)$ *is an entire function, then there are a Banach space $W$ and an entire function* $h \in \mathcal{O}(X,W)$ *such that*

(i) $\|h(x)\|_W \leq 1$ *for* $x \in \Omega_N\langle\beta\rangle$ *and*

(ii) $\|h(x)\|_W \geq \|g(x)\|_Z$ *for* $x \in \Omega_{N+1}\langle\alpha\rangle \setminus \Omega_N\langle\alpha\rangle$.

Proof. In Proposition 4.3.3 the sets $\Omega_N\langle\beta\rangle$, etc, refer to those constructed in [L3, §3] for $\Omega = X$. Proposition 4.3.3 follows from Proposition 4.3.2 in the same way as [L3, Proposition 4.2] does from [L3, Lemma 4.1] only more simply.

Proof of Proposition 4.3.1. On replacing $u$ by $u(Rx/2)$ we may assume that $R = 2$. Let $\Omega = X$, fix $0 < \mu < 1$ and $0 < \alpha < 2^{-8}\mu$. First, we construct a Banach space $Z_N$ and an entire function $g_N \in \mathcal{O}(X, Z_N)$ for each $N \geq 1$. The set $A = \overline{\Omega_N\langle\alpha\rangle} \cap \pi_N(X)$ is compact and if $t \in A$, then $\Omega_N\langle\alpha\rangle \cap \pi_N^{-1}(t) \subset B_X(t,\alpha)$. Hence $t$ has an open neighborhood $U \subset \pi_N(X)$ with $\Omega_N\langle\alpha\rangle \cap \pi_N^{-1}(U) \subset B_X(t,2\alpha)$. Therefore

$$\Omega_N\langle\alpha\rangle \subset \bigcup_{t\in T} B_X(t,2\alpha) \tag{4.3.1}$$

for some finite $T \subset A$. Let $B_t = B_X(t,2\alpha/\mu)$, the radius of which is less than 2. By our assumption that $u$ can be dominated by entire functions with values in Banach spaces on $B_X(x_0,2)$ for every $x_0 \in X$, there are a Banach space $V_t$ and an entire function $f_t \in \mathcal{O}(X,V_t)$ with $u(x) \le \|f_t(x)\|_{V_t}$ for $x \in B_t$, $t \in T$. Let $Z_N$ be the $\ell_\infty$-sum of the finitely many Banach spaces $V_t$ for $t \in T$ and $g_N \in \mathcal{O}(X,Z_N)$ the map whose components are the $f_t$ for $t \in T$. We see from (4.3.1) that $u(x) \le \|g_N(x)\|_{Z_N}$ for $x \in \Omega_N\langle\alpha\rangle$.

The rest of the proof of Proposition 4.3.1 is the same as that of [L3, Proposition 2.1] starting with "In the second step" on page 368 there.

## 4.4. DOMINATION ON A BALL.

In this section we show that if a function $u$ can be dominated on every ball of half the radius of a ball $B$ and centered at any point of $B$, then $u$ can be dominated on $B$ itself.

**Proposition 4.4.1.** *If $X$ is a Banach space with a bimonotone Schauder basis, $0 < R < \infty$, $u\colon X \to [1,\infty)$ is continuous, and $u$ can be dominated by entire functions with values in Banach spaces on every ball $B_X(x_0,R/2)$ of radius $R/2$ and centered at any $x_0 \in B = B_X(y_0,R)$, then there is continuous function $\tilde{u}\colon X \to [1,\infty)$ such that $\tilde{u}(x) \le u(x)$ for all $x \in X$, $\tilde{u}(x) = u(x)$ for $x \in B$, and $\tilde{u}$ can be dominated by entire functions with values in Banach spaces on every ball $B_X(x_0,R/8)$ of radius $R/8$ centered at any $x_0 \in X$.*

*Proof.* Let $\chi\colon [0,\infty) \to [0,1]$ be a cutoff function

$$\chi(t) = \begin{cases} 1 & 0 \le t \le R \\ 1 - \frac{4}{R}(t-R) \quad \text{if} & R \le t \le \frac{5}{4}R \\ 0 & t \ge \frac{5}{4}R \end{cases},$$

and define $\tilde{u}$ by $\tilde{u}(x) = \chi(\|x-y_0\|)u(x) + 1 - \chi(\|x-y_0\|)$ for $x \in X$. As $\tilde{u}(x) - u(x) = (1-\chi(\|x-y_0\|))(1-u(x)) \le 0$, being the product of a nonnegative number by a nonpositive number, we get that $\tilde{u}(x) \le u(x)$ for all $x \in X$. Hence $\tilde{u}$ can be dominated by entire functions with values in Banach spaces on any set on which $u$ can.

If $x_0 \in X$ satisfies that $\|x_0 - y_0\| \ge \frac{11}{8}R$, then $B_X(x_0, \frac{1}{8}R)$ lies outside $B_X(y_0, \frac{5}{4}R)$ since the distance $\|x_0 - y_0\|$ of their centers exceeds the sum of their radii $\frac{5}{4}R + \frac{1}{8}R = \frac{11}{8}R$. Hence $\tilde{u} = 1$ on $B_X(x_0, \frac{1}{8}R)$, and so $\tilde{u}$ can be dominated by entire functions with values in Banach spaces on $B_X(x_0, \frac{1}{8}R)$.

If $\|x_0 - y_0\| < R$, then $x_0 \in B_X(y_0, R)$ and $B_X(x_0, \frac{1}{8}R) \subset B_X(x_0, \frac{1}{2}R)$.

If $R \le \|x_0 - y_0\| < \frac{11}{8}R$, then choose a value $0 < R' < R$ with $\|x_0 - y_0\| < \frac{11}{8}R'$, and let $z_0 = y_0 + R'\frac{x_0 - y_0}{\|x_0 - y_0\|}$. Then $\|z_0 - x_0\| = R' < R$ so $z_0 \in B_X(y_0, R)$ and we claim that $B_X(x_0, \frac{1}{8}R) \subset B_X(z_0, \frac{1}{2}R)$. To that end we must show that the distance $\|z_0 - x_0\|$ of the centers is less than the difference of the radii, i.e., $\|z_0 - x_0\| < \frac{1}{2}R - \frac{1}{8}R = \frac{3}{8}R$. Indeed, $\|z_0 - x_0\| = \|y_0 - x_0 + R'\frac{x_0 - y_0}{\|x_0 - y_0\|}\| = \|x_0 - y_0\| - R' < \frac{11}{8}R' - R' = \frac{3}{8}R' < \frac{3}{8}R$. The proof of Proposition 4.4.1 is complete.

**Proposition 4.4.2.** *If $X$ is a Banach space with a bimonotone Schauder basis, $0 < R < \infty$, $u\colon X \to [1,\infty)$ is continuous, and $u$ can be dominated by entire functions with values in Banach spaces on every ball $B_X(x_0, R/2)$ of radius $R/2$ centered at any $x_0 \in B = B_X(y_0, R)$, then $u$ can be dominated by entire functions with values in Banach spaces* *on the ball $B$.*

Proof. Proposition 4.4.1 gives us a $\tilde{u}$ that can be dominated by entire functions with values in Banach spaces on every ball of radius $R/8$ in $X$. Proposition 4.3.1 gives us a Banach space $Z$ and an entire function $h \in \mathcal{O}(X, Z)$ with $\tilde{u}(x) \leq \|h(x)\|$ for all $x \in X$. As $u(x) = \tilde{u}(x) \leq \|h(x)\|$ for $x \in B$, the proof of Proposition 4.4.2 is complete.

## 4.5. PREPARATION.

This section is preparatory to the proofs of Theorems 4.1.3, 4.1.4, and 4.1.5.

Recall the following theorem of Pełczyński's.

**Theorem 4.5.1.** (Pełczyński, [P]) *A Banach space $X$ has the bounded approximation property if and only if $X$ is isomorphic to a direct summand of a Banach space $Y$ with a Schauder basis, i.e., there are a Banach space $Y$ with a Schauder basis and a direct decomposition $Y = Y_1 \oplus Y_2$ of Banach spaces such that $X \cong Y_1$.*

In most of our proofs we can avoid dealing with Banach spaces with the bounded approximation property, and only work with Banach spaces with a Schauder basis.

**Proposition 4.5.2.** *Let $X$ be a Banach space with the bounded approximation property, and $\Omega \subset X$ pseudoconvex open. If plurisubharmonic domination is possible in $\Omega$, then so is holomorphic domination.*

Proof. It is enough by Theorem 4.5.1 to prove this when $X$ has a Schauder basis, in which case it follows from the argument of [L3], only more simply.

**Proposition 4.5.3.** *If $M_0$ is a closed complex Banach submanifold of a complex Banach* *manifold $M$, and holomorphic domination is possible on $M$, then holomorphic domination is possible on $M_0$, too.*

Proof. Let $u_0 \colon M_0 \to \mathbb{R}$ be the locally upper bounded function to be dominated. Define $u \colon M \to \mathbb{R}$ by setting $u(x) = u_0(x)$ for $x \in M_0$ and $u(x) = 0$ otherwise. Clearly, $u$ is locally upper bounded, $M_0$ being a closed subset of $M$. If $Z$ is Banach space and $h \in \mathcal{O}(M, Z)$ dominates $u$ on $M$, then the restriction $h_0$ of $h$ to $M_0$ is holomorphic and dominates $u_0$ in $M_0$. The proof of Proposition 4.5.3 is complete.

**Proposition 4.5.4.** *If $M$ is a separable complex Banach manifold that is biholomorphic to a closed Banach submanifold of a Banach space $X$, then $M$ can be embedded in a separable Banach space as a closed complex Banach submanifold.*

Proof. It is easy to see that the closed linear span of a separable subset of any Banach space is itself separable. It is a standard theorem that any separable Banach space is isomorphic to a closed linear subspace of the space $Y = C[0, 1]$ of continuous functions, and $Y$ has a Schauder basis. Thus $M$ is biholomorphic to a closed complex Banach submanifold of $Y$, completing the proof of Proposition 4.5.4.

**Proposition 4.5.5.** *Let $X$ be a Banach space, and $\Omega \subset X$ open. If one of* (a), (b), (c) *below holds, then $\Omega$ is biholomorphic to a closed complex Banach submanifold $M$ of a Banach space $Y$.*

(a) *$\Omega$ is convex.*

(b) *There is a direct decomposition $X = X_1 \oplus X_2$ of Banach spaces with* $\dim_{\mathbb{C}}(X_1) < \infty$*, and $\Omega$ is of the form $\Omega = \{(x_1, x_2) \in D \times X_2 \colon \|x_2\| < R(x_1)\}$, where $D \subset X_1$ is pseu-*

*doconvex (relatively) open, $R: D \to (0,\infty)$ is continuous and $-\log R$ is plurisubharmonic on $D$.*

(c) *$\Omega$ is of the form $\Omega = \{x \in \Omega' : \|f(x)\| < 1\}$, where $\Omega' \subset X$ is open, the closure $\overline{\Omega} \subset \Omega'$, and $f \in \mathcal{O}(\Omega', Z_1)$ is holomorphic with values in a Banach space $Z_1$.*

Proof. In each case we define a Banach space $Z$ and a holomorphic function $h \in \mathcal{O}(\Omega, Z)$ with $\liminf_{\Omega \ni x \to x_0} \|h(x)\| = \infty$ for each boundary point $x_0 \in \partial\Omega$. Then the graph $M \subset Y = X \times Z$ of $h$ defined by $M = \{(x,z) \in \Omega \times Z : z = h(x)\}$ does the job.

(a) (See also [Pt1, Proposition 8.2].) Assume as we may that $0 \in \Omega$. Let $p: X \to \mathbb{R}$ be the Minkowski functional $p(x) = \inf\{\lambda > 0 : \frac{x}{\lambda} \in \Omega\}$ of the convex open set $\Omega$, and $K = \{\xi \in X^* : \mathrm{Re}(\xi x) \le p(x) \text{ for all } x \in X\}$. Then $K \neq \emptyset$ is a convex subset of the dual space $X^*$ of $X$. We endow $K$ with the weak star topology, in which $K$ is compact.

Let $Z = C([0,2\pi] \times K, \mathbb{C})$ be the usual Banach space with the sup norm, $g(t) = 1/(1-t)$ for $t \in B_{\mathbb{C}}$, and define for $x \in \Omega$ a function $h(x) \in Z$ by $h(x)(\theta,\xi) = g(e^{i\theta}e^{\xi x - 1})$. Then we have $\|h(x)\| = \sup\{|h(x)(\theta,\xi)| : \theta \in [0,2\pi],\ \xi \in K\} \le \sup_{\theta,\xi} g(|e^{i\theta}e^{\xi x-1}|) \le \sup_{\theta,\xi} g(e^{\mathrm{Re}(\xi x)-1}) \le g(e^{p(x)-1})$. For every $x \in X$ the Hahn–Banach theorem gives a $\xi \in K$ with $\mathrm{Re}(\xi x) = p(x)$. On choosing $\theta \in [0,2\pi]$ so that $e^{i\theta}e^{\xi x-1} = e^{\mathrm{Re}(\xi x)-1} = e^{p(x)-1}$, we find that $\|h(x)\| = g(e^{p(x)-1})$. Hence, $h \in \mathcal{O}(\Omega, Z)$, and $\|h(x)\| = 1/(1-e^{p(x)-1}) \to \infty$ as $x \in \Omega$ tends to point $x_0 \in X$ with $p(x_0) = 1$, in particular, to any boundary point $x_0 \in \partial\Omega$.

(b) Let $\omega = \{(x_1,\lambda) \in D \times \mathbb{C} : x_1 \in D, |\lambda| < R(x_1)\}$. As $\omega$ is pseudoconvex open in the complex Euclidean space $X_1 \times \mathbb{C}$, there is a proper holomorphic embedding $j: \omega \to \mathbb{C}^N$

for $N$ high enough. Let $K$ be the closed unit ball of the dual space $X_2^*$ of $X_2$ endowed with the weak star topology, and for $(x_1, x_2) \in \Omega$ define $h(x_1, x_2) \in Z = C(K, \mathbb{C})$ (endowed with the sup norm) by $h(x_1, x_2)(\xi_2) = j(x_1, \xi_2 x_2)$ for $\xi_2 \in K$. Note that $\|h(x_1, x_2)\| = \sup_{|\lambda| \le \|x_2\|} \|j(x_1, \lambda)\| \ge \|j(x_1, \|x_2\|)\|$ by the Hahn–Banach theorem, and the last tends to $\infty$ if $(x_1, \|x_2\|)$ tends to a boundary point of $\omega$, in particular, when $(x_1, x_2)$ tends in $\Omega$ to a boundary point of $\Omega$ in $X$.

(c) Let $K$ be the closed unit ball of the dual space $Z_1^*$ of $Z_1$ endowed with the weak start topology, and $Z = C(K, \mathbb{C})$ with the sup norm. For $x \in \Omega$ define $h(x) \in Z$ by $h(x)\zeta = g(\zeta f(x))$, where $\zeta \in K$ and $g(t) = 1/(1-t)$ for $t \in B_{\mathbb{C}}$ as in (a). Then $\|h(x)\| = g(\|f(x)\|)$ for $x \in \Omega$ by the Hahn–Banach theorem, and $h \in \mathcal{O}(\Omega, Z)$ is holomorphic. If $x \in \Omega$ tends to a boundary point $x_0 \in \partial\Omega$, then $x_0 \in \overline{\Omega} \subset \Omega'$, hence $x_0 \in \Omega'$ and $f(x) \to f(x_0)$, i.e., $\|f(x)\| \to \|f(x_0)\| = 1$, and $\|h(x)\| = 1/(1 - \|f(x)\|) \to \infty$.

The proof of Proposition 4.5.5 is complete.

## 4.6. THE PROOFS OF THEOREMS 4.1.3, 4.1.4, AND 4.1.5.

In this section we complete the proof of Theorems 4.1.3 on holomorphic domination, 4.1.4 on vanishing and Banach vector bundles, and 4.1.5 on the $\bar{\partial}$-equation.

Proof of Theorem 4.1.3(a). Without loss of generality we may assume by Theorem 4.5.1 that $X$ has a bimonotone Schauder basis. Let $u\colon X \to \mathbb{R}$ be the locally upper bounded function to be dominated. By paracompactness of $X$ there is a continuous function $u_1\colon X \to [1, \infty)$ with $u(x) \le u_1(x)$ for $x \in X$. Replacing $u$ by $u_1$, let us assume that $u \ge 1$ is continuous on $X$.

Suppose for a contradiction that $u$ cannot be dominated by entire functions with values in Banach spaces on $X$. The hypothesis of Proposition 4.3.1 must then be false. Hence there is a ball $B_0 = B_X(x_0, 1)$ on which $u$ cannot be dominated by entire functions with values in Banach spaces. The hypothesis of Proposition 4.4.2 must then also be false. So there is a ball $B_1 = B_X(x_1, 1/2)$ with $x_1 \in B_0$ such that $u$ cannot be dominated by entire functions with values in Banach spaces on $B_1$. Again, the hypothesis of Proposition 4.4.2 must be false and there is a ball $B_2 = B_X(x_2, 1/4)$ with $x_2 \in B_1$ such that $u$ cannot be dominated by entire functions with values in Banach spaces on $B_2$. Proceeding in this way we get a sequence of balls $B_n = B_X(x_n, 1/2^n)$ with $x_{n+1} \in B_n$ such that $u$ cannot be dominated by entire functions with values in Banach spaces on $B_n$ for $n \geq 0$.

As $x_{n+1} \in B_n$ we see that $\|x_{n+1} - x_n\| < 1/2^n$ and $\sum_{n=0}^{\infty}(x_{n+1} - x_n)$ is an absolutely convergent series in the Banach space $X$. Thus there is a limit $x_n \to x \in X$ as $n \to \infty$. Let $r > 0$ be so small that $u$ is upper bounded on the ball $B_X(x, r)$. Choose $n \geq 0$ so large that $B_n \subset B_X(x, r)$. Hence $u$ can be dominated by entire functions with values in Banach spaces on $B_n$ after all, being upper bounded there. This contradiction completes the proof of (a).

**Theorem 4.6.1.** (a) *If $M$ is as in Proposition 4.5.4, then holomorphic domination is possible in $M$.*

(b) *In particular, if $X$ is a separable Banach space, and $\Omega \subset X$ open is as in Proposition 4.5.5, then holomorphic domination is possible in $\Omega$.*

Proof. Part (a) follows from Theorem 4.1.3(a) via Proposition 4.5.3 upon embedding $M$ in $C[0, 1]$ as a closed complex Banach submanifold. Part (b) follows from (a) by

Proposition 4.5.5. The proof of Theorem 4.6.1 is complete, and as Theorem 4.1.3(b) is a special case of (b), the proof of Theorem 4.1.3 is also complete.

**Theorem 4.6.2.** *Let $X$ be a Banach space with the bounded approximation property, $\Omega \subset X$ pseudoconvex open, and $S \to \Omega$ a cohesive sheaf. If plurisubharmonic domination is possible in $\Omega$, then*

(a) *the cohesive sheaf $S$ admits a complete resolution over $\Omega$, and*

(b) *the sheaf cohomology group $H^q(\Omega, S)$ vanishes for all $q \geq 1$.*

Proof. Without loss of generality we may assume by Theorem 4.5.1 that $X$ has a bimonotone Schauder basis. An inspection of the proof of the analogous Theorem 9.1 in [LP] reveals that therein it is enough to have plurisubharmonic domination in $\Omega$ and in those subsets of $\Omega$ to which Proposition 4.5.5 applies, and thus in which plurisubharmonic domination holds by Theorem 4.6.1. The proof of Theorem 4.6.2 is complete.

Proof of Theorem 4.1.4. Parts (a) and (b) follow directly from Theorem 4.6.2, (c) from [LP, § 10] and Theorem 4.6.2, while (d) follows from [Pt2, Theorem 1.3(f)], completing the proof of Theorem 4.1.4.

Proof of Theorem 4.1.5. As the $\bar\partial$-equation $\bar\partial u = f$ can be solved locally on balls in $\Omega$ by a theorem of Defant and Zerhusen [DZ] (based upon the earlier work [L1] of Lempert) a standard step in one of the usual proofs of the Dolbeault isomorphism together with Theorem 4.1.4(c) completes the proof of Theorem 4.1.5.

Further applications of Theorems 4.1.4 and 4.1.5 can also be made, e.g., as in [DPV] or [LP].

CHAPTER 5. ON COMPLEX BANACH MANIFOLDS

SIMILAR TO STEIN MANIFOLDS.

In this chapter we give an abstract definition, similar to the axioms of a Stein manifold, of a class of complex Banach manifolds in such a way that a manifold belongs to the class if and only if it is biholomorphic to a closed split complex Banach submanifold of a separable Banach space.

Stein manifolds can be characterized among complex manifolds in various ways, including the two ways (I) and (II) below. A paracompact second countable Hausdorff complex manifold $M$ of pure dimension is a Stein manifold if and only if one and hence both of the following equivalent conditions (I) and (II) below hold.

(I) (a) $M$ is holomorphically convex, i.e., if $K \subset M$ is compact, then its $\mathcal{O}(M)$ holomorphic hull $\hat{K}$ is compact in $M$. (b) If $x \neq y$ in $M$, then there is an $f \in \mathcal{O}(M)$ with $f(x) \neq f(y)$. (c) If $x \in M$, then there are an integer $n \geq 0$ and a holomorphic function $g \in \mathcal{O}(M, \mathbb{C}^n)$ that is a biholomorphism from an open neighborhood $W$ of $x$ in $M$ to an open neighborhood $g(W)$ of $g(x)$ in $\mathbb{C}^n$.

(II) There is an $n \geq 1$ such that $M$ is biholomorphic to a closed complex submanifold $M'$ of $\mathbb{C}^n$.

Let $X$ be a separable Banach space, and $M$ a paracompact second countable Hausdorff complex Banach manifold modelled on $X$. We call $M$ a *linear complex Banach manifold modelled on* $X$ if (i-iv) below hold.

(i) Holomorphic domination is possible in $M$, i.e., if $u\colon M \to \mathbb{R}$ is any locally upper bounded function, then there are a Banach space $Z$ and a holomorphic function $h: M \to Z$ with $u(x) < \|h(x)\|$ for all $x \in M$.

(ii) There are open sets $U_n, V_n \subset M$, and holomorphic functions $f_n \in \mathcal{O}(M)$, $n \geq 1$, such that $\bigcup_{n=1}^{\infty}(U_n \times V_n) = (M \times M) \setminus \Delta_M$, where $\Delta_M = \{(x,x): x \in M\}$ is the diagonal of $M \times M$, and $f_n(U_n)$ and $f_n(V_n)$ are disjoint sets in $\mathbb{C}$ for all $n \geq 1$.

(iii) There are open sets $W_n \subset M$ and holomorphic maps $g_n \in \mathcal{O}(M, X)$ for $n \geq 1$ such that $\bigcup_{n=1}^{\infty} W_n = M$ and $g_n|W_n$ is a biholomorphism from $W_n$ onto an open set $g_n(W_n)$ in $X$.

(iv) There are open sets $G_k \subset M$, $k \geq 1$, with $\bigcup_{k=1}^{\infty} G_k = M$ and $\sup_{x \in G_k}(|f_n(x)| + \|g_n(x)\|) < \infty$ for all $k, n \geq 1$, where $(f_n)$ and $(g_n)$ are as in (ii) and (iii).

If $M$ is finite dimensional, then it is easy to see that (i-iii) together are equivalent to (I), and (iv) is vacuous, since if $G_k$, $k \geq 1$, is an exhaustion of $M$ by precompact open sets $G_k$, then any continuous function $|f_n(x)| + \|g_n(x)\|$ on $M$ is bounded on $G_k$ for $k, n \geq 1$. Thus if $M$ is finite dimensional, then (i-iv) together are equivalent to $M$ being a Stein manifold. The word 'linear complex Banach manifold' is a complex geometric analog of the word 'affine manifold' in algebraic geometry.

**Theorem 1.** *Let $X$ be a separable Banach space, and $M$ a paracompact second countable Hausdorff complex Banach manifold modelled on $X$. Then $M$ is a linear complex Banach manifold modelled on $X$ if and only if there is a separable Banach space $X'$ such that $M$ is biholomorphic to a closed split complex Banach submanifold $M'$ of $X'$.*

Here Banach manifolds and Banach submanifolds are understood in terms of biholomorphically related charts, and a Banach submanifold is called *split* if each of its tangent spaces has a direct complement in the ambient Banach space. Clearly, a complex Banach submanifold $M$ of $X$ is split if and only if near each point $x_0 \in M$ it is possible to split $X$ as a direct sum $X = X' \times X''$ of closed linear subspaces $X', X''$ of $X$ such that with $x_0 = (x_0', x_0'')$ and $x = (x', x'')$ we can write $M$ as the graph $x'' = m(x')$ of a holomorphic function $m$ from an open neighborhood of $x_0'$ in $X'$ to $X''$, where $x_0'' = m(x_0')$.

Proof. Suppose first that $M$ is biholomorphic to an $M'$ and verify that $M$ satisfies (i-iv). It is enough to show that $M'$ does.

As holomorphic domination is possible in $X'$ by Theorem 4.1.3(b), and thus also in $M'$, since $M'$ is closed in $X'$, (i) is true. We define some linear functions $f_n : X' \to \mathbb{C}$ and $g_n : X' \to X$ for $n \geq 1$ whose restrictions to $M'$ will do the job. For linear functions (iv) is automatic: we can let $G_k$ be the intersection of $M'$ with the open ball $\|x\| < k$ in $X'$ and write $|f_n(x)| + \|g_n(x)\| \leq (\|f_n\| + \|g_n\|)\|x\| \leq (\|f_n\| + \|g_n\|)k < \infty$ for $x \in G_k$ and $n, k \geq 1$.

If $x \neq y$ in $M'$, then $x - y \neq 0$ in $X'$ and the Hahn–Banach theorem gives us a complex linear functional $f_{xy} \in (X')^*$ of norm 1 with $\operatorname{Re} f_{xy}(x-y) = \|x-y\| > 0$. Let $U_{xy} = \{z \in X' : -\frac{1}{2}\|x-y\| + \operatorname{Re} f_{xy}(x) < \operatorname{Re} f_{xy}(z)\}$ and $V_{xy} = \{z \in X' : \operatorname{Re} f_{xy}(z) < \frac{1}{2}\|x-y\| + \operatorname{Re} f_{xy}(y)\}$. Then $x \in U_{xy}$, $y \in V_{xy}$, and their images $f_{xy}(U_{xy})$ and $f_{xy}(V_{xy})$ are disjoint since they are the half planes $-\frac{1}{2}\|x-y\| + \operatorname{Re} f_{xy}(x) < \operatorname{Re} w$, $\operatorname{Re} w < \frac{1}{2}\|x-y\| + \operatorname{Re} f_{xy}(y)$, which are clearly disjoint since $-\frac{1}{2}\|x-y\| + \operatorname{Re} f_{xy}(x) = \frac{1}{2}\|x-y\| + \operatorname{Re} f_{xy}(y)$

Fix any point $x_0 \in M'$ and denote its complex tangent space $T_{x_0}M'$ by $X$ and regard it as a closed linear subspace of $X'$. If $x \in M'$, then the complex tangent space

$T_xM'$ and $X$ are linearly isomorphic via a bounded linear map $i_x: T_xM' \to X$, and there is a bounded linear projection $p_x: X' = T_xX' \to T_xM'$. Thus the linear map $g_x: X' \to X$ given by $g_x(y) = i_x(p_x(y))$ for $y \in X'$ satisfies that $(dg_x)(x)y = i_x(y)$ for $y \in T_xM'$, i.e., $(dg_x)(x)$ is a linear isomorphism from $T_xM'$ onto $X$. By the inverse function theorem $g_x$ is biholomorphic from an open neighborhood $W_x$ of $x$ in $M'$ to an open neighborhood $g_x(W_x)$ of $g_x(x) = 0$ in $X$.

By Lindelöf's theorem in the second countable (separable metric) spaces $(M' \times M') \setminus \Delta_{M'}$ and $M'$ the open coverings $U_{xy} \times V_{xy}$, $(x, y) \in (M' \times M') \setminus \Delta_{M'}$, and $W_x$, $x \in M'$, can be reduced to countable subcoverings $U_n \times V_n$, $W_n$, where $U_n = U_{x_ny_n}$, $V_n = V_{x_ny_n}$, and $W_n = W_{x'_n}$ for $n \geq 1$. Thus the functions $f_n = f_{x_ny_n}$, $g_n = g_{x'_n}$, $n \geq 1$, do the job.

Conversely, assume that $M$ satisfies (i-iv) and embed $M$ biholomorphically as a closed split Banach submanifold $M'$ into a separable Banach space $X'$.

If $i \geq 1$, then let $C_i = L_i = 1 + \sup\{|f_n(x)| + \|g_n(x)\|: 1 \leq k, n \leq i, x \in G_k\}$. So if $k, n \geq 1$, and $x \in G_k$, then $|f_n(x)| + \|g_n(x)\| \leq C_kL_n$. Thus upon replacing $f_n$ by $f_n/(L_n2^n)$ and $g_n$ by $g_n/(L_n2^n)$, we obtain new functions again to be called $f_n$, $g_n$ that satisfy (ii), (iii), and the slightly strengthened version $\sup_{x\in G_k}(|f_n(x)|+\|g_n(x)\|) < C_k/2^n$, $k, n \geq 1$, of (iv).

The covering $W_n$, $n \geq 1$, of the paracompact space $M$ has a locally finite refinement, which by Lindelöf's theorem can be taken to be countable, and can be shrunk since a paracompact Hausdorff space $M$ is normal. There are open sets $M_n \subset M$, $n \geq 1$, with $\bigcup_{n=1}^{\infty} M_n = M$, and for each $n \geq 1$ there is an index $j(n) \geq 1$ with the closure $\overline{M_n} \subset W_{j(n)}$. Define $u: M \to \mathbb{R}$ by $u(x) = \inf\{n \geq 1: x \in M_n\}$. Then $u$ is locally upper bounded on $M$

since $u \le n$ on the open set $M_n$.

By assumption (i) on holomorphic domination there are a Banach space $Z$ and a holomorphic function $h \in \mathcal{O}(M, Z)$ with $u(x) < \|h(x)\|$ for $x \in M$. As $Z' = \overline{\mathrm{span}}\{h(x): x \in M\}$ is a separable Banach space, and as any separable Banach space can be embedded into $C[0,1]$ we can replace the Banach space $Z$ by the separable space $Z = C[0,1]$ endowed with the sup norm.

Define a Banach space $X'$ by $X' = Z \times \ell_1 \times \ell_1(X)$, where $\ell_1$ and $\ell_1(X)$ denote the spaces of summable sequences in $\mathbb{C}$ and in $X$. Let us write the variable $y$ in $X'$ as $y = (y', y'', y''')$, where $y'' = (y''_n) \in \ell_1$ and $y''' = (y'''_n) \in \ell_1(X)$. Clearly, $X'$ is a separable Banach space, being the product of three such spaces.

Define the map $\Phi: M \to X'$ given by $y = \Phi(x)$, where

$$\begin{cases} y' &= h(x) \\ y''_n &= f_n(x)\ , \qquad n \ge 1. \\ y'''_n &= g_n(x) \end{cases}$$

If $k \ge 1$ and $x \in G_k$, then we have $\sum_{n=1}^{\infty}(|f_n(x)| + \|g_n(x)\|) \le \sum_{n=1}^{\infty} C_k/2^n \le C_k < \infty$. Thus $\Phi$ is holomorphic.

Our $\Phi$ is injective, since if $x \ne y$ in $M$, then there is an index $n \ge 1$ with $x \in U_n$ and $y \in V_n$, so $f_n(x) \ne f_n(y)$, and even more so $\Phi(x) \ne \Phi(y)$.

We claim that the set $M' = \Phi(M)$ is closed in $X'$. Indeed, suppose that $y_i = \Phi(x_i)$, $x_i \in M$, converges $y_i \to y$ in the norm in $X'$ as $i \to \infty$ to an element $y \in X'$, we must show that there is an $x \in M$ with $y = \Phi(x)$, i.e., $y \in M'$. As $y'_i = h(x_i)$, $i \ge 1$, is bounded, being convergent, there is an index $b \ge 1$, with $\|h(x_i)\| \le b$ for $i \ge 1$, i.e., $u(x_i) < \|h(x_i)\| \le b$, or, $x_i \in M_1 \cup \ldots \cup M_b$ for $i \ge 1$. By the pigeon hole principle there are an index $a$ with

$1 \le a \le b$ and an infinite set $I$ of indices $i$ such that $x_i \in M_a$ for all $i \in I$. As $\overline{M_a} \subset W_{j(a)}$, and $g_{j(a)}$ is biholomorphic on $W_{j(a)}$, we see that $\Phi(x_i)$, $i \in I$, may converge only if the $x_i$, $i \in I$, converge in $\overline{M_a}$ to one of its elements $x \in \overline{M_a} \subset W_{j(a)}$. Thus $y_i = \Phi(x_i) \to \Phi(x) = y$ as $i \to \infty$ in $I$. As $M'$ contains $y$, it is closed.

If $y_0 = \Phi(x_0)$ in $M'$, then there is an index $n \ge 1$ with $x_0 \in W_n$. So $y_n''' = g_n(x)$ is biholomorphic from a connected open set $W_n'$ with $x_0 \in W_n' \subset \overline{W_n'} \subset W_n$ to a connected open set $g_n(W_n')$ in $X$. Then the connected component of the set $M' \cap \{y_n''' \in g_n(W_n')\}$ that contains the point $y_0$ equals the graph

$$\begin{cases} y_n''' = y_n''' \\ y' \ \ = h(g_n^{-1}(y_n''')) \\ y_n'' \ = f_n(g_n^{-1}(y_n''')) \ , \qquad \nu \neq n, \\ y_\nu'' \ = f_\nu(g_n^{-1}(y_n''')) \\ y_\nu''' = g_\nu(g_n^{-1}(y_n''')) \end{cases}$$

of a holomorphic map $y_n''' \mapsto (y', y_n'', y_n''', y_\nu'', y_\nu''')$, $\nu \neq n$, from $W_n'$ to the Banach space $X' \cap \{y_n''' = 0\}$.

Thus $M'$ is a closed split complex Banach submanifold of $X'$ and $\Phi\colon M \to M'$ is a biholomorphism. QED.

The most substantial part of the above proof is to show that holomorphic domination is possible on a separable Banach space. That was done in Chapter 4 above based upon the work of Lempert in [L3]. It might be possible to weaken the axioms (i-iv) perhaps by dropping (iv) and replacing (i), that stands in for holomorphic convexity, by plurisubharmonic domination, i.e., by requiring a continuous plurisubharmonic function $\psi\colon M \to \mathbb{R}$ that dominates the given locally upper bounded function $u\colon M \to \mathbb{R}$. Nevertheless, axioms (i-iv) represent perhaps the ultimate axioms for "Stein Banach manifolds" since any other system for which the desirable Theorem 1 holds must be equivalent with (i-iv). Most known methods of plurisubharmonic domination also yield holomorphic domination, and a 'constructive' procedure for building the functions $f_n, g_n$ in (ii) and (iii) is likely to produce functions that also satisfy (iv). The author doubts whether a successful "Stein theory" could be built up for nonseparable Banach spaces and Banach manifolds. Even for separable Banach manifolds it would be better to restrict attention to the ones modelled on separable Banach spaces with the bounded approximation property (there are virtually no practical separable Banach spaces that do not satisfy the bounded approximation property). If $M$ is a linear complex Banach manifold modelled on such a Banach space, then the sheaf cohomology group $H^q(M,S)$ vanishes if $q \geq 1$ and $S \to M$ is a so-called cohesive sheaf defined in [LP] by Lempert et al. The question arises whether $M$ is a linear complex Banach manifold if $H^q(M,S) = 0$ for all $q \geq 1$ and all cohesive sheaves $S \to M$. If $M$ is an open subset of a separable Banach space with the bounded approximation property, then the answer is Yes.

## References


[DZ] Defant, A., Zerhusen, A. B., *Local solvability of the $\bar{\partial}$-equation on $\mathcal{L}_1$-spaces*, Arch. Math., **90**, (2008), 545–553.

[DPV] Dineen, S., Patyi, I., Venkova, M., *Inverses depending holomorphically on a parameter in a Banach space*, J. Funct. Anal., **237** (2006), no. 1, 338–349.

[FR] Fine, B., Rosenberger, G., *The fundamental theorem of algebra*, Springer, (1997), New York.

[G] Garling, D. J. H., *A "short" proof of the Riesz representation theorem*, Proc. Cambridge Philos. Soc., **73** (1973), 459–460.

[H] Hartig, D. G., *The Riesz representation theorem revisited*, Amer. Math. Monthly, **90** (1983), no. 4, 277–280.

[L1] Lempert, L., *The Dolbeault complex in infinite dimensions, II*, J. Amer. Math. Soc., **12** (1999), 775–793.

[L2] ______, *Approximation of holomorphic functions of infinitely many variables, II*, Ann. Inst. Fourier Grenoble **50** (2000), 423–442.

[L3] ______, *Plurisubharmonic domination*, J. Amer. Math. Soc., **17** (2004), 361–372.

[LP] ______, Patyi, I., *Analytic sheaves in Banach spaces*, Ann. Sci. École Norm. Sup., 4e série, **40** (2007), 453–486.

[Ls] Loomis, L. H., *An introduction to abstract harmonic analysis*, D. Van Nostrand Co., Inc., (1953), New York.

[MP] Mathew, P. J., Patyi, I., *The fundamental theorem of calculus implies that of algebra*, The Mathematical Gazette, to appear.

[M] Meylan, F., *Approximation of holomorphic functions in Banach spaces admitting a Schauder decomposition*, Ann. Scuola Norm. Sup. Pisa, (5) **5** (2006), no. 1, 13–19.

[Pt1] Patyi, I., *On complex Banach submanifolds of a Banach space*, Contemp. Math., **435** (2007), 343–354.

[Pt2] ______, *On holomorphic Banach vector bundles over Banach spaces*, Math. Ann., **341** (2008), no. 2, 455–482.

[P] Pełczyński, A., *Projections in certain Banach spaces*, Studia Math., **19** (1960), 209–228.

[R] Riesz, F., *Sur les opérations fonctionelles linéaires*, C. R. Acad. Sci., Paris, **149**, (1909), 974–977.

[Sg] Sternberg, S., *Theory of functions of a real variable*, May 10, 2005, notes from the author's website at Harvard.

[Sk] Stroock, D. W., *A concise introduction to the theory of integration*, 3rd ed., (1999), Birkhäuser, Boston.

[Sr] Sunder, V. S., *The Riesz representation theorem*, April 10, 2008, manuscript from the author's website at the Institute of Mathematical Sciences, Madras, India.

[V] Varadarajan, V. S., *On a theorem of F. Riesz concerning the form of linear functionals*, Fund. Math., **46**, (1959), 209–220.